MS121: IT Mathematics

Functions

Catalogue of Essential Functions

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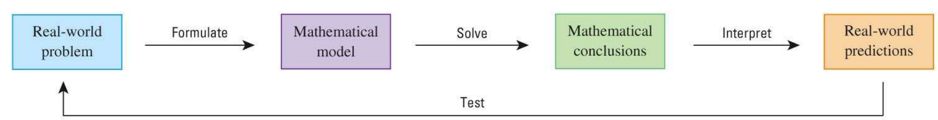
Outline

Outline

Mathematical Model

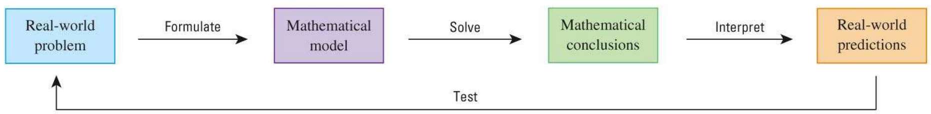


A mathematical model is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as

- the size of a population,
- the demand for a product,
- the speed of a falling object,
- the concentration of a product in a chemical reaction,
- the life expectancy of a person at birth, or
- the cost of emission reductions.

The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

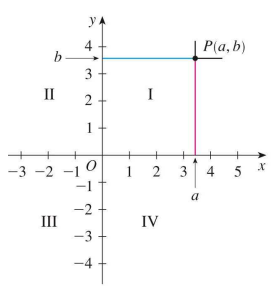
Mathematical Model

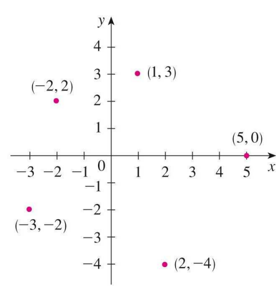


- A mathematical model is never a completely accurate representation of a physical situation – it is an idealization.
- A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.
- It is important to realize the limitations of the model.
- In the end, Mother Nature has the final say

Outline

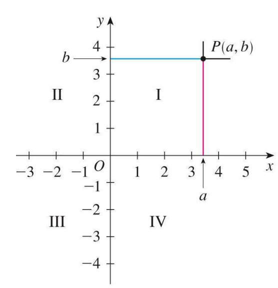
Coordinate Geometry & Lines

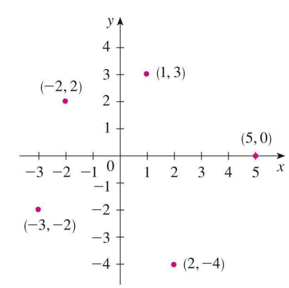




- Any point in the plane can be located by a unique ordered pair of numbers as follows:
- Draw lines through perpendicular to the x- and y-axes.
- These lines intersect the axes in points with coordinates a and b as shown above.

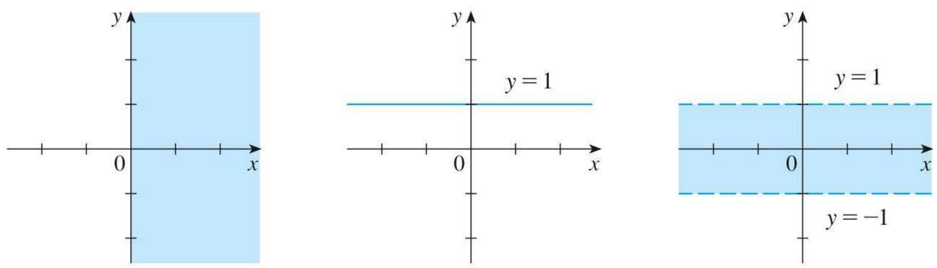
Coordinate Geometry & Lines (Cont'd)





- Then the point P is assigned the ordered pair (a, b).
- The first number a is called the x-coordinate of P; the second number b is called the y-coordinate of P.
- We say that P is the point with coordinates (a, b), and we denote the point by the symbol $P(a, b)$. Several points are labeled with their coordinates in the second plot above.

Regions in the Plane

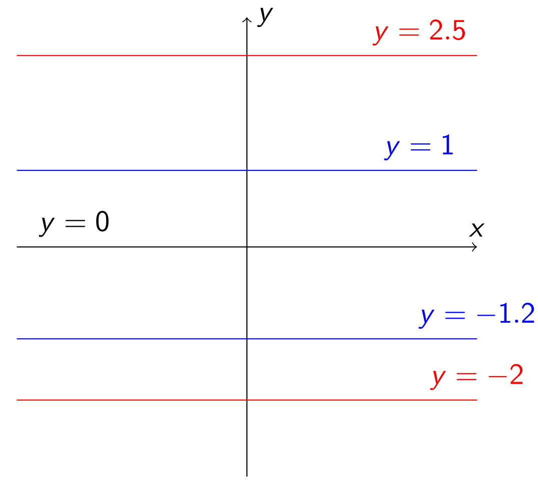


(a) $x \geqslant 0$ (b) $y = 1$ (c) $|y| < 1$

Formally

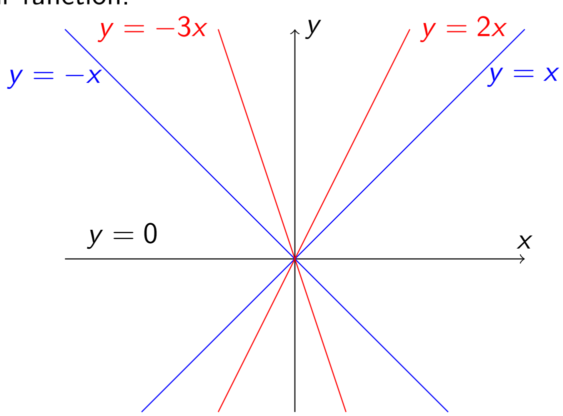
- $\{(x, y) \mid x \geq 0\}$
- $\{(x, y) \mid y = 1\}$
- $\{(x, y) \mid |y| < 1\}$

Constant Functions: Slope $m = 0$



Linear Functions

A function of the form $y = f(x) = mx + c$, where m and c are constants, is called a linear function.



Case 1: $c = 0$

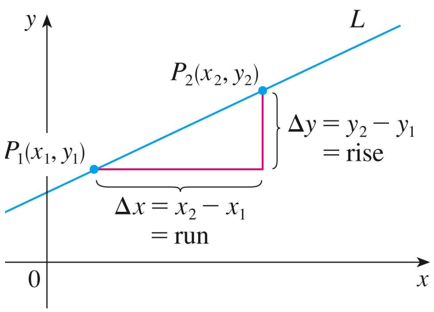
Linear Functions

- When we say that y is a linear function of x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + c$$

  where m is the slope of the line and c is the y-intercept.
- A characteristic feature of linear functions is that they grow at a constant rate.

Slope of a Straight Line

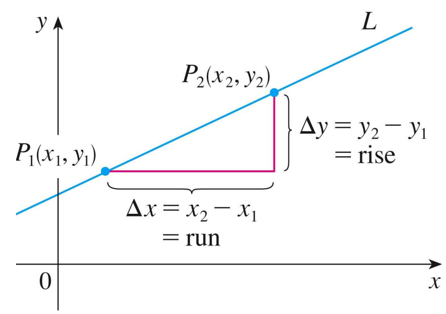


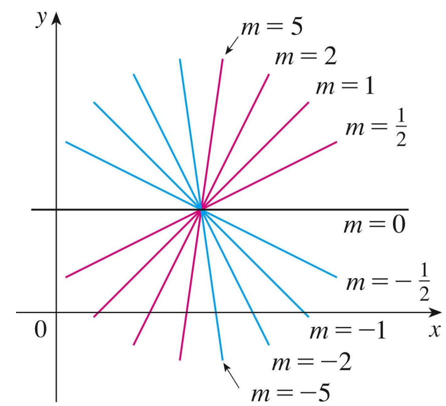
The slope of a nonvertical line which passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

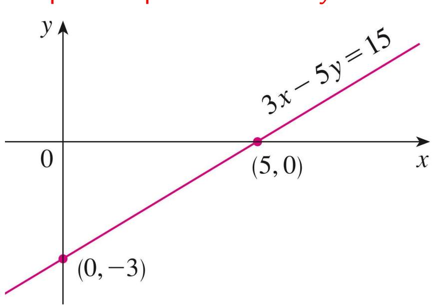
Slope of a Straight Line (Cont'd)



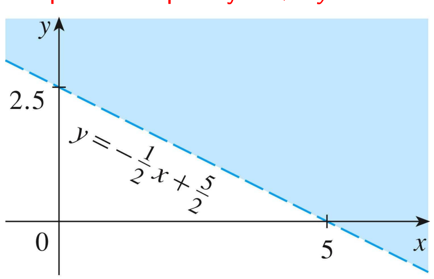


Straight Line: Equation & Inequality

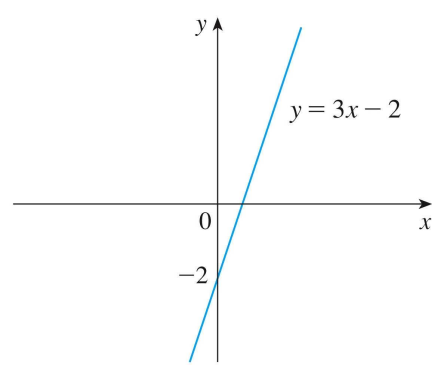
Graph of equation $3x - 5y = 15$



Graph of inequality $x + 2y > 5$



Slope as the Rate of Change



- Consider the graph of the linear function

$$y = f(x) = 3x - 2$$

- You will notice that whenever x increases by 1, the value of y increases by 3.
- So y increases three times as fast as x.
- Thus the slope of the graph, namely 3, can be interpreted as the rate of change of y with respect to x.

Polynomials

Definition

- A function is called a polynomial if

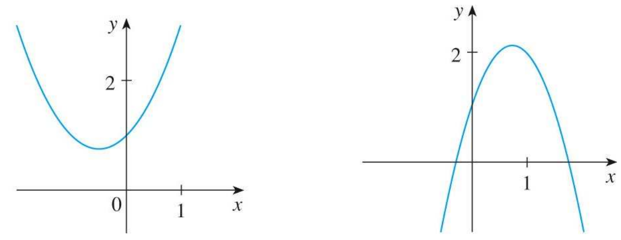
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

  where n is a nonnegative integer and the numbers $a_0, a_1, \ldots, a_n$ are constants called the coefficients of the polynomial.
- The domain of any polynomial $R = (-\infty, +\infty)$.
- If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.
- For example, the function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x + \sqrt{2}$$

  is a polynomial of degree 6.

Polynomial of Degree 2: Quadratic Function



(a) $y = x^2 + x + 1$ (b) $y = -2x^2 + 3x + 1$

- A polynomial of degree 2 is of the form

$$P(x) = ax^2 + bx + c$$

  and is called a quadratic function.
- Its graph is always a parabola obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$.

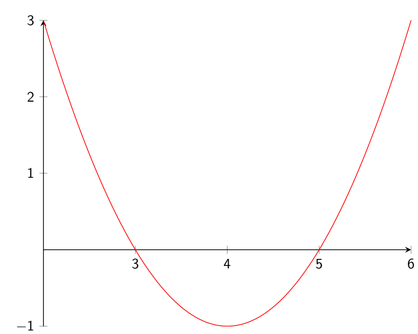
Quadratic functions

$$f(x) = ax^2 + bx + c$$

Questions to ask

- Where is $f(x) = (x-3)(x-5)$ positive?
- Where is $f(x) = (x-a)(x-b)$, when $a < b$, positive?
- Where is $g(x) = x^2 - b^2 = (x-b)(x+b)$, when $0 < b$, positive?
- Where is $h(x) = b^2 - x^2 = (b-x)(b+x) = -g(x)$, when $0 < b$, positive?

Where is $(x-3)(x-5) = x^2 - 8x + 15 > 0$?



- The roots of this function are the solutions to the quadratic equation

$$(x-3)(x-5) = 0$$

$$\text{i.e. } x = 3 \text{ and } x = 5.$$

- So

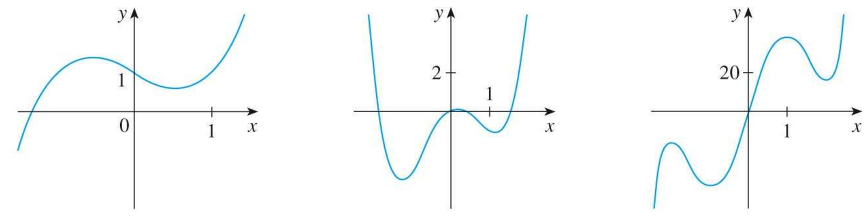
$$f(x) > 0 \quad \Leftrightarrow x < 3 \text{ or } x > 5,$$
$$f(x) < 0 \quad \Leftrightarrow 3 < x < 5.$$

Quadratic functions

Some Answers

- The quadratic function $f(x) = (x-a)(x-b)$, where $a < b$, is
  - zero when $x = a$ and $x = b$;
  - positive if $x < a$ or $x > b$;
  - negative when $a < x < b$.
- The quadratic function $g(x) = x^2 - b^2 = (x-b)(x+b)$, where $0 < b$, is
  - zero when $x = \pm b$;
  - positive if $x < -b$ or $x > b$;
  - negative when $-b < x < b$.
- The quadratic function $h(x) = b^2 - x^2 = (b-x)(b+x) = -g(x)$, where $0 < b$, is
  - zero when $x = \pm b$;
  - positive if $-b < x < b$.
  - negative when $x < -b$ or $x > b$;

Polynomial Functions of Degree n



(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d, \qquad a \neq 0$$

and is called a cubic function.

Polynomials: Recap

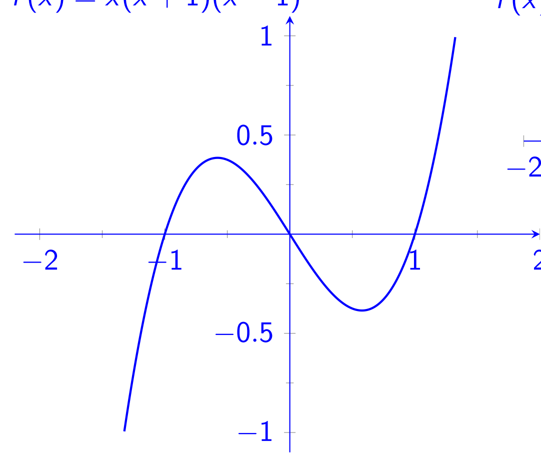
A function f is a polynomial if

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_o$$

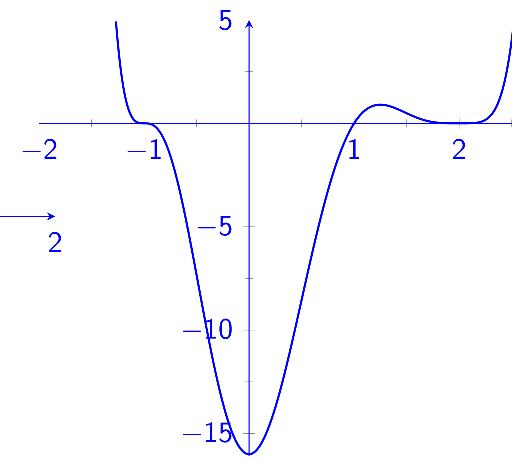
where n is a non negative integer, called the degree of the polynomial, and $a_0, a_1, \ldots, a_n$ are real constants, called coefficients, with $a_n \neq 0$

Examples of Polynomials

$f(x) = x(x+1)(x-1)$



$f(x) = (x-2)^4(x+1)^3(x-1)$



Polynomials

Properties

All polynomials have domain $\mathbb{R} = (-\infty, \infty)$.

Linear functions- $y = mx + b$ (with $m \neq 0$) are polynomials of degree 1.

Quadratic functions $y = ax^2 + bx + c$ (with $a \neq 0$) are polynomials of degree 2.

Cubic functions $y = ax^3 + bx^2 + cx + d$ (with $a \neq 0$) are polynomials of degree 3.

If n is even, then f has even degree and if n is odd, then f has odd degree.

Examples of Polynomials

- A ball is thrown into the air with an initial velocity of $160 ft/sec$. It reaches a height, h, after t seconds:

$$h(t) = 160t - 16t^2 \text{ ft.}$$

- The supply and demand functions for a commodity are

$$P = 2Q_S^2 + 9Q_S + 150$$

$$P = -2Q_D^2 - 12Q_D + 275$$

where P is the price per unit.

Evaluating Polynomials

Let $p(x) = 7\,x^3 - 12\,x^2 - 8\,x + 8$. p is a polynomial of degree 3 (a cubic).

We can, for example, evaluate at $x = 0$ to find

$$p(0) = 7\,(0)^3 - 12\,(0)^2 - 8\,(0) + 8 \;=\; 8$$

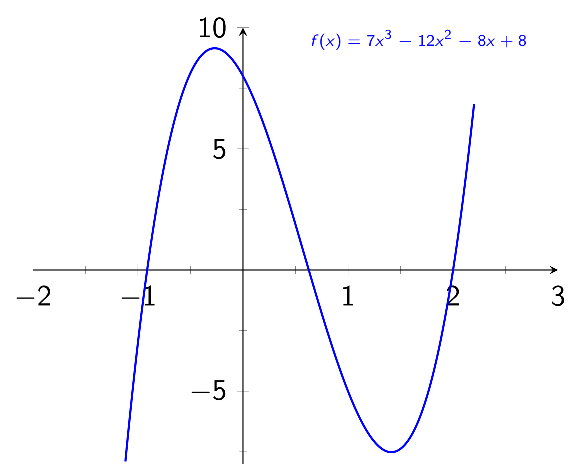
While evaluating at $x = 2$ gives

$$p(2) = 7\,(2)^3 - 12\,(2)^2 - 8\,(2) + 8 \;=\; 0$$

In this case, $x = 2$ is called a root of the polynomial p. Note that

$$p(x) = (x - 2)(7x^2 + 2x - 4).$$

Graph of $p(x) = 7\,x^3 - 12\,x^2 - 8\,x + 8$



Roots of a Polynomial

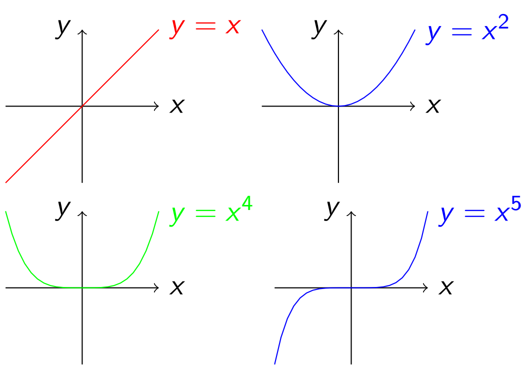
- In general, given any function f, we call a solution of the equation
$$f(x) = 0$$
a root or zero of the function f.
- If $x = a$ is a root of a polynomial p, then $p(x) = (x - a)q(x)$, for some polynomial q.
- We say that $(x - a)$ is a factor of p.
- A root $x = a$ is called a repeated root of the polynomial p, if $p(x) = (x - a)^m s(x)$, for some $m \geq 2$ and some polynomial $s(x)$.
- If p is a polynomial of degree n, then p has at most n real roots.

Outline

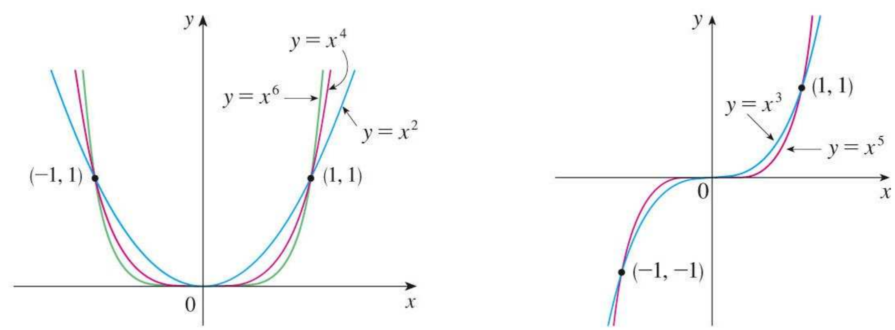
Power Functions

A function of the form $y = f(x) = x^n$, where n is a constant, is called a power function.

The graphs for $n = 1, 2, 3, 4, 5$ are displayed below:

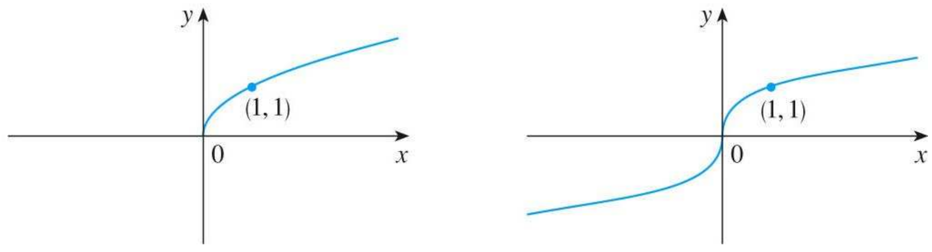


Families of Power Functions



- The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.
- Note that, as n increases, the graph of $y = x^n$ becomes flatter near $x = 0$ and steeper when $x \geq 1$.

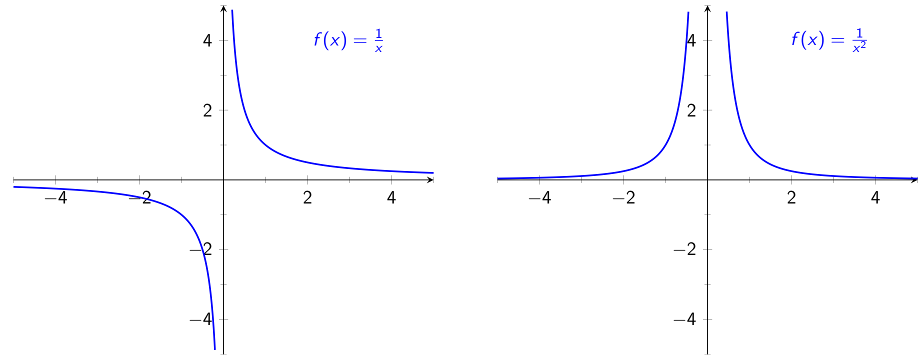
Power Functions: $x^{\frac{1}{n}}$, n a positive integer



(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

- Note that the domain of $\sqrt{x}$ is $[0, \infty)$ whose graph is the upper half of the parabola $x = y^2$.
- The domain of $\sqrt[3]{x}$ is R (every real number has a cube root).

Power Functions: Two Further Examples



Outline

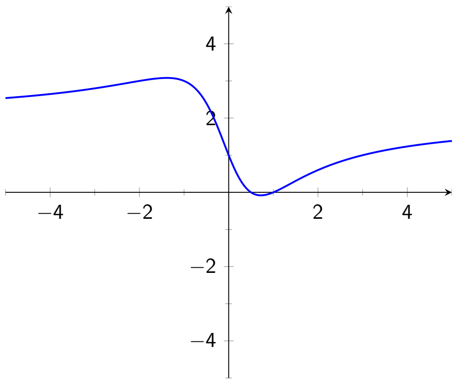
Rational Functions

A rational function is a quotient or ratio of two polynomial functions:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. The domain of a rational function is the set of all x for which $q(x) \neq 0$.

Example of a Rational Function

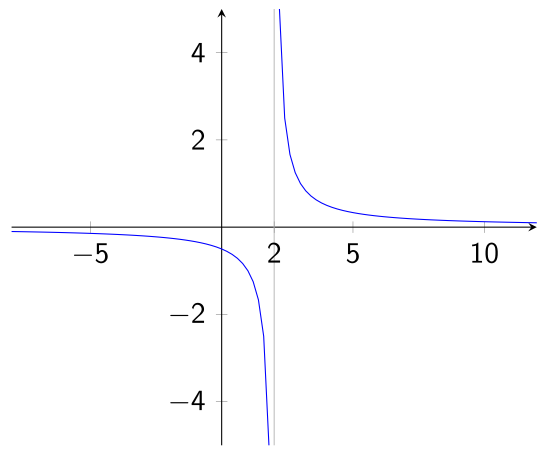


The function

$$f(x) = \frac{2x^2 - 3x + 1}{x^2 + 1}$$

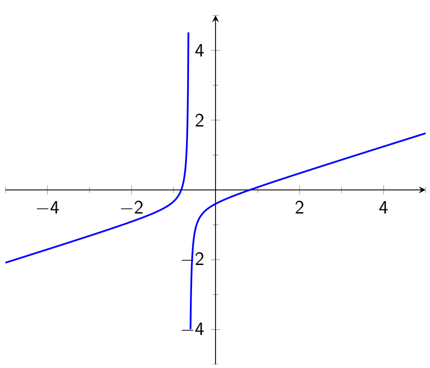
is a rational function with domain all real numbers.

Rational Function with a Discontinuity



$$f(x) = \frac{1}{x - 2}$$

Rational Function with a Discontinuity



The function

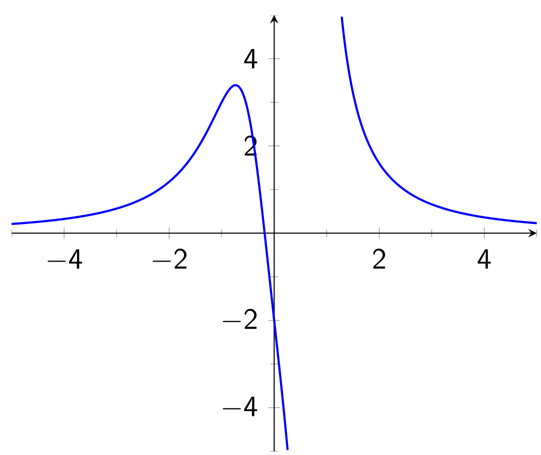
$$f(x) = \frac{3x^2 - 2}{8x + 5}$$

is a rational function with domain all real numbers except when the denominator is zero:

$$8x + 5 = 0 \Longleftrightarrow x = -\frac{5}{8}.$$

So the domain is $\mathbb{R} \backslash \left\{ -\frac{5}{8} \right\} = \left(-\infty, -\frac{5}{8} \right) \cup \left(-\frac{5}{8}, \infty \right)$.

Rational Function with a Discontinuity



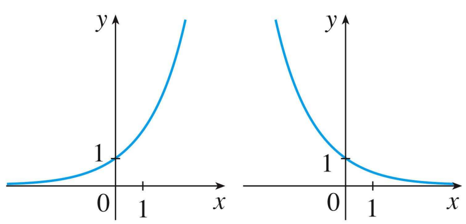
$$f(x) = \frac{11x + 2}{2x^3 - 1}$$

Outline

Exponential Functions

- Functions of the form
$$y = a^x,$$
where $a > 0$ are called exponential functions, a is called the base.
- All exponential functions have domain $\mathbb{R}$.
- If $a \neq 1$, then the range of $y = a^x$ is $(0, \infty)$.

Exponential Function: $f(x) = a^x$

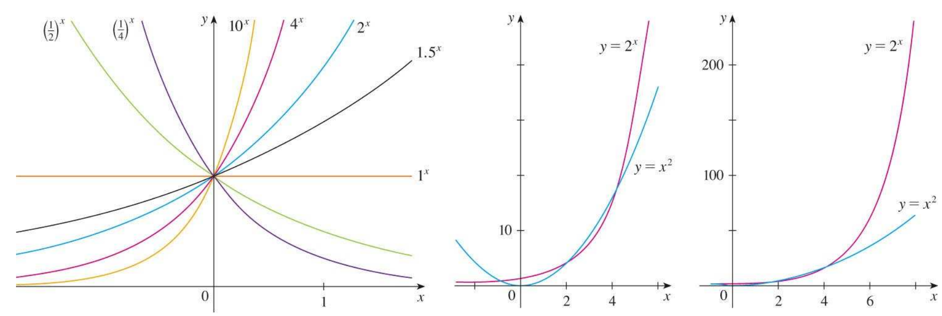


(a) $y = 2^x$ (b) $y = (0.5)^x$

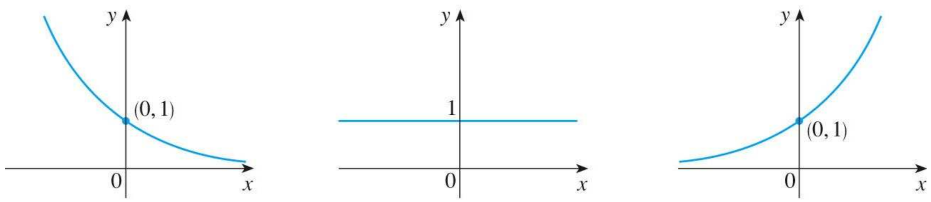
In both cases,

- the domain is $(-\infty, \infty)$, and
- the range is $(0, \infty)$

Exponential Functions: Family Members



The 3 kinds of Exponential Functions $y = a^x$



(a) $y = a^x,\ 0 < a < 1$ (b) $y = 1^x$ (c) $y = a^x,\ a > 1$

Exponential Functions

Application

- Suppose that €10, 000 euro is invested in an account with an interest rate of 2% per annum.
- The amount of money in the account after t years is

$$P(t) = 10,000(1.02)^t.$$

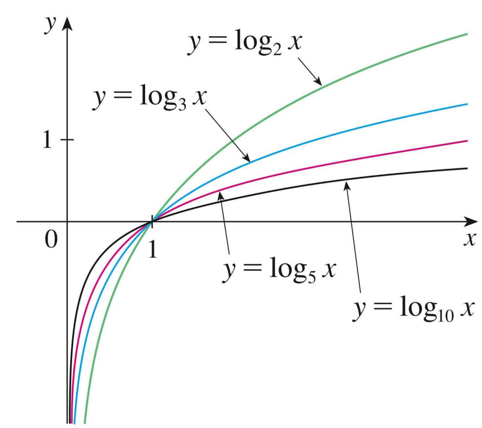
Logarithmic Functions

- Logarithmic functions are functions of the form

$$y = \log_a x$$

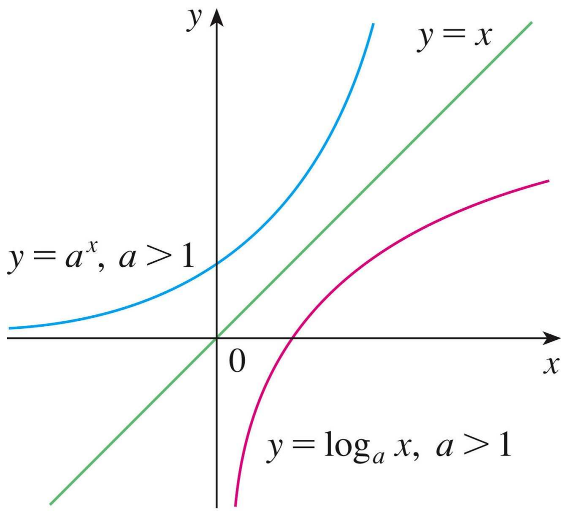
  where the base $a \neq 1$ is a positive constant, are the inverse functions of the exponential functions.
- All logarithmic functions have domain $(0, \infty)$ and range $\mathbb{R}$.

Logarithmic Function: $f(x) = \log_a x$

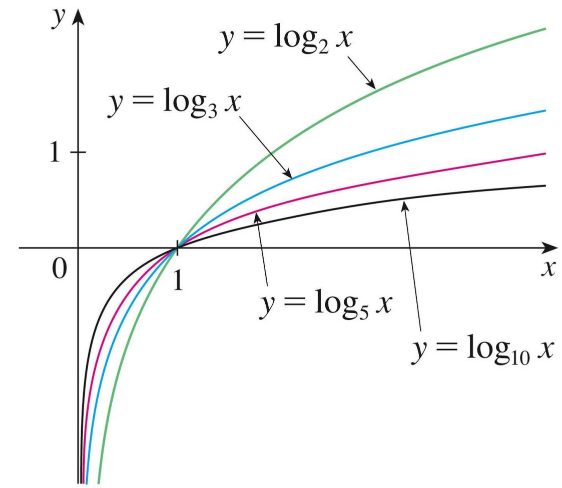


- The domain is $(0, \infty)$, and
- the range is $(-\infty, \infty)$

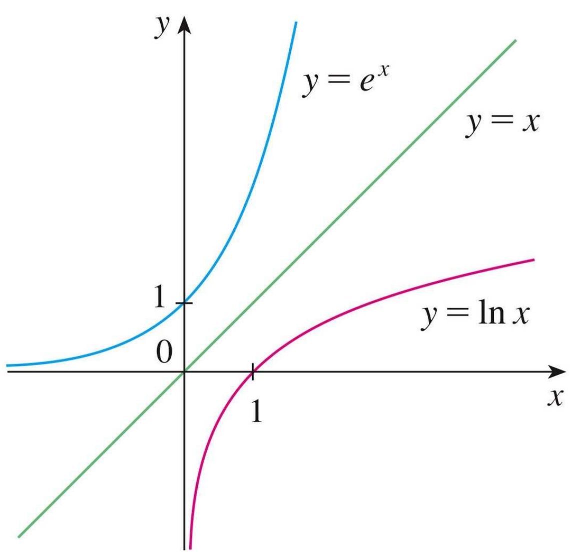
Exponential & Logarithmic Functions $a > 1$



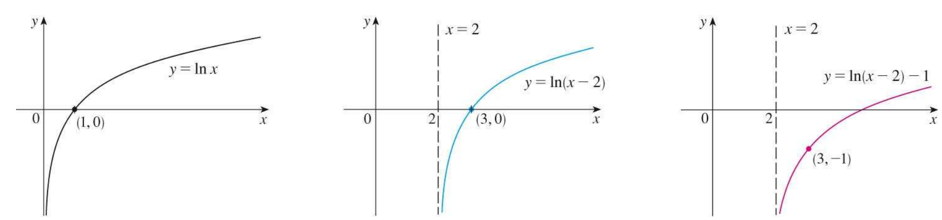
Logarithmic Functions Base a



Natural Logarithmic Function $y = \ln x$



Logarithmic Function $y = \ln(x - 2) - 1$

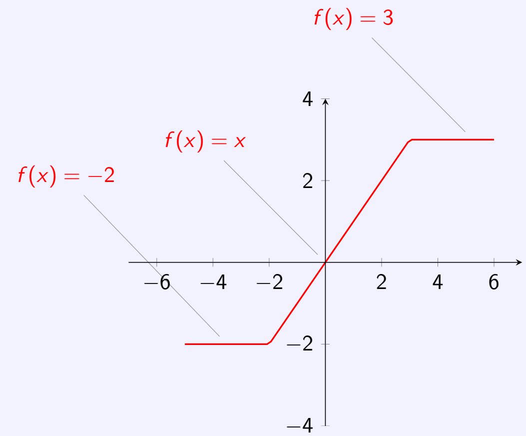


Outline

Piecewise-Defined Functions

Piecewise-defined functions are functions which are defined by different formulae in different parts of their domains.

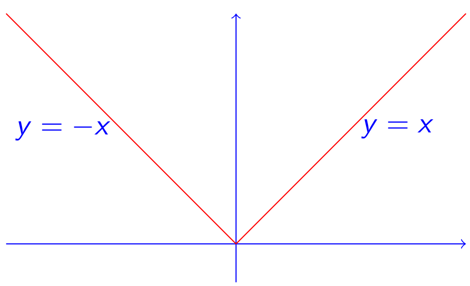
Simple Example



Piecewise-Defined Functions

Another example of a function which is described by using different formulae on different parts of its domain is the absolute value function:

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

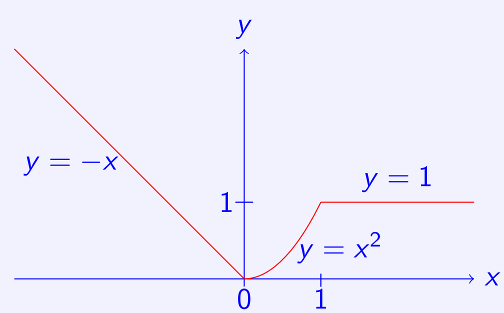


Graphing Piecewise-Defined Functions

The function

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x^2, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } x > 1. \end{cases}$$

is defined on the real line but has values given by different formulas depending on the position of x.



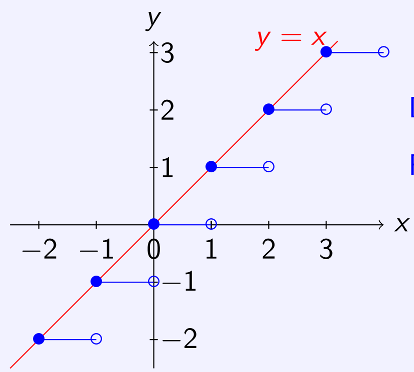
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Piecewise-Defined Functions

The Greatest Integer function is the function whose value at x is the *greatest integer less than or equal to x* it is also called the integer floor function and is denoted $\lfloor x \rfloor$. For example

$$\lfloor 3.2 \rfloor = 3 = \lfloor 3 \rfloor = \lfloor 3.99 \rfloor, \quad \lfloor -3.7 \rfloor = -4 = \lfloor -3.0001 \rfloor.$$



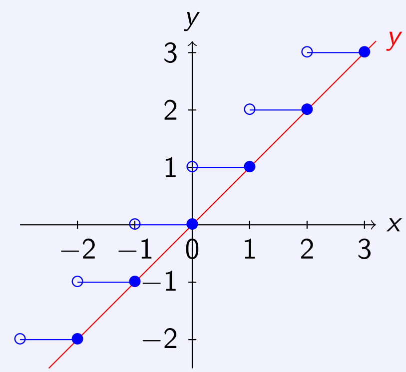
Domain: $(-\infty, \infty)$

Range: $\mathbb{Z}$

Piecewise-Defined Functions

The Least Integer function is the function whose value at x is the *smallest integer greater than or equal to x* it is also called the integer ceiling function and is denoted $\lceil x \rceil$. For example

$$\lceil 3.2 \rceil = 4 = \lceil 4 \rceil = \lceil 3.0001 \rceil, \quad \lceil -3.7 \rceil = -3 = \lceil -3.999999 \rceil.$$



Domain: $(-\infty, \infty)$

Range: $\mathbb{Z}$