# MS121: IT Mathematics

## Limits & Continuity

## Rates of Change & Tangents

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## Outline





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## Is there a limit to how fast a man can run?

### 3 Recent World Records

- Jamaica's Asafa Powell clocked 9.74 seconds for the Men's 100m sprint on September 9, 2007 in Rieti, Italy.
- This was followed by Usain Bolt's showboating Olympic run of 9.69 seconds in August 2008.
- His more recent assault on the track and field record book running 9.58 on 16 August 2009, when he reached a top speed of nearly 28 mph, has again raised the question if humans are fast approaching their physical limit?

## The Men's 100m World Record in 2008 & 2009






## Is there a limit to how fast a man can run?

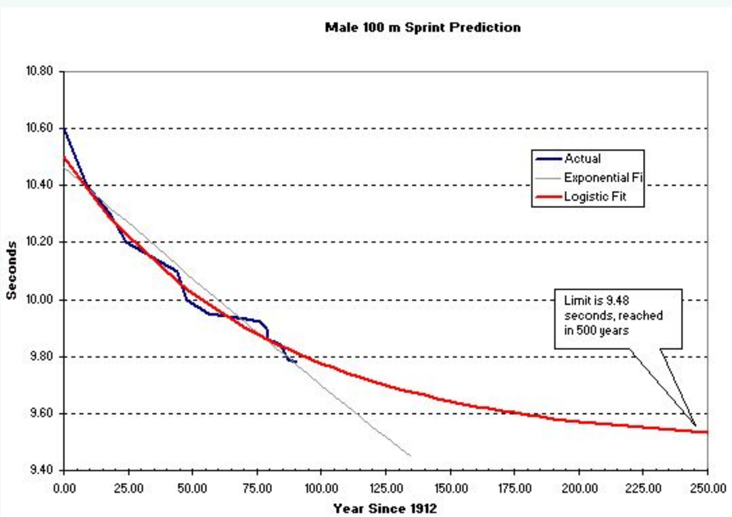
### Some Observations

- A linear regression least-square error best-fit line over 90 years shows a slope of 7.7 millisecond fall per year.
- The correlation is 0.97, which is considered good for some applications (a best fit exponential gives virtually identical results).
- Even if this straight line trend were correct, it would imply that, in about 1,300 years, the 100m record would be run literally in no time flat (the record would be 0 seconds).
- More sensibly, a logistic equation reaches a limit of 9.48 seconds in 2412.





## Men's 100m World Record in 2412?








## Is there a limit to human performance?

### Empirical Studies

- A 2008 study by the French Institute of Sport concluded that athletics will finally hit the ceiling in 2060. After that, no more world records.
- The institute analysed all 3,260 world records set since the first modern Olympics in 1896, and says that athletes are nudging their physiological limits.
- It estimates that athletes were operating at 75% of their potential in 1896, while in 2008, they were at 99%. By 2027, the athletes in about half of the events will have reached 100%, and by 2060 they all will.

## Outline







## Rates of Change

Calculus, to a great extent, is the study of the rate at which quantities change. We would like to answer questions like:

- how fast is a population growing?
- how fast is a car travelling?
- how fast will a ball travel if dropped from a height?





## Rates of Change

**Example**

A rock breaks free from the top of the cliff. What is the average speed during the first two seconds of the fall?

- Experiments have shown that the rock will fall

$$s = f(t) = 16t^2$$

  feet in the first $t$ seconds.
- The average speed in the first 2 seconds is the distance travelled divided by 2 *(length of time interval)*:

$$u = \frac{16 \star 2^2}{2} = \frac{64}{2} = 32\text{ft/sec}$$





## Instantaneous Speed

**What is the instantaneous speed at time $t = 2$ (secs)?**

We can estimate it by working out the average speed over a shorter time period (including $t = 2$):

- from $t = 1$ to $t = 2$ is

$$\frac{f(2) - f(1)}{2 - 1} = \frac{16(2^2) - 16(1^2)}{1} = 48 ft/sec;$$

- from $t = 1.9$ to $t = 2$ is $\frac{f(2) - f(1.9)}{2 - 1.9} = 62.4 ft/sec;$
- from $t = 1.99$ to $t = 2$ is $\frac{f(2) - f(1.99)}{2 - 1.99} = 63.84 ft/sec.$

In fact, the average speed is tending towards the instantaneous speed of $64 ft/sec$ at $t = 2$

## Instantaneous Speed

### MS121 Perspective

This example encapsulates what we will do in calculus:

- The limit of the estimates is $64 ft/sec$.
- The speed is obtained from the derivative of $f$.
- The function

$$f(t) = 16t^2$$

  is obtained using integration, using the fact that gravity causes the rock to accelerate at a rate of $32 ft/sec$





## To Calculate the Instantaneous Speed

### Using Average Spped

- We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$ having length $\Delta t = h$, as

$$\frac{\Delta s}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$$

- We cannot use this formula to calculate the *instantaneous* speed at $t_0$ by substituting $h = 0$ as we cannot divide by zero.
- As before, we calculate average speeds over increasingly short time intervals.





## Instantaneous Speed

### Increasingly shorter time intervals

| $h$ | 1 | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|---|
| Av.speed | 80 | 65.6 | 64.16 | 64.016 | 64.0016 |

$$\frac{\Delta s}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h}$$

$$= \frac{64h + 16h^2}{h} = 64 + 16h$$

### Observation

The average speed has the limiting value 64 as $h$ approaches zero, at $t_0 = 2$ secs.

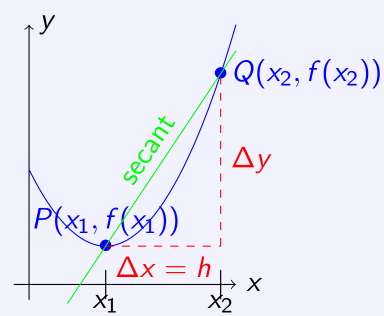




## Average Rate of Change

The *average rate of change* of $y = f(x)$ with respect to $x$ over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0$$




A line joining two points of a curve is a *secant* to the curve — the average rate of change of $f$ from $x_1$ to $x_2$ is the slope of secant $PQ$.

## Average Rate of Change

**Examples**

(i) Find the average rate of change of $f(x) = \sqrt{4x+1}$ on the interval $[0, 2]$.

$$\frac{\Delta y}{\Delta x} = \frac{f(2) - f(0)}{2 - 0} = \frac{3-1}{2} = 1$$

(ii) Find the average rate of change of $f(x) = x^3 - 4x^2 + 5x$ on the interval $[1, 2]$.

$$\frac{\Delta y}{\Delta x} = \frac{f(2) - f(1)}{2 - 1} = 2 - 2 = 0$$





## Outline

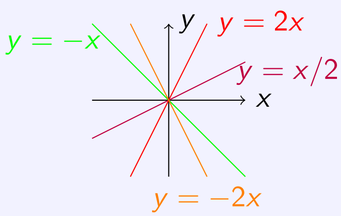






## Slope

The graph of the function $y = mx + c$ is a straight line. The number $m$ is called the slope and it tells us which way and by how much the line is tilted, i.e. it tells us whether $y$ is *increasing or decreasing* as $x$ increases



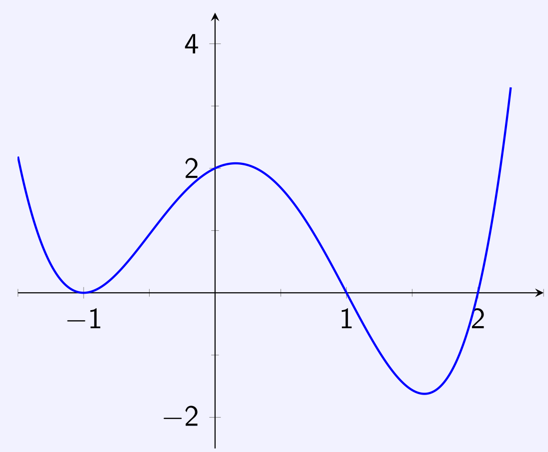

and how rapidly this occurs.

- $m = \frac{1}{2}$: $y$ increases as $x$ increases;
- $m = 2$: $y$ increases more rapidly as $x$ increases;
- $m = -1$: $y$ decreases as $x$ increases;
- $m = -2$: $y$ decreases more rapidly as $x$ increases.





## Example: $f(x) = (x+1)^2(x-1)(x-2)$



## Slope

**Behaviour of $f(x) = (x+1)^2(x-1)(x-2)$**

- When $x = -0.5$, $y$ is increasing
- When $x = 2.2$, $y$ is increasing more rapidly;
- When $x = 1$, $y$ is decreasing;
- When $x = -1.5$, $y$ is decreasing more rapidly.

The major difference here is that the "slope" may change from point to point.





## Tangents

**Definition**

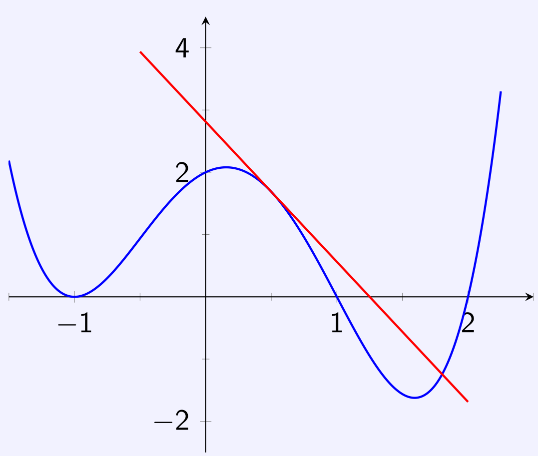
- The *tangent* to a curve $\mathcal{C}$ at the point $P$ on the curve is a straight line which touches $\mathcal{C}$ at $P$ and may cross the curve at $P$ but otherwise does not cross the curve near $P$.
- The *slope* of the curve at $x = c$ is equal to the slope of the tangent line $T$ at $x = c$.





## Tangent to $f(x) = (x+1)^2(x-1)(x-2)$ at $x = 0.5$

**Slope of the Tangent $= -\frac{9}{4}$**

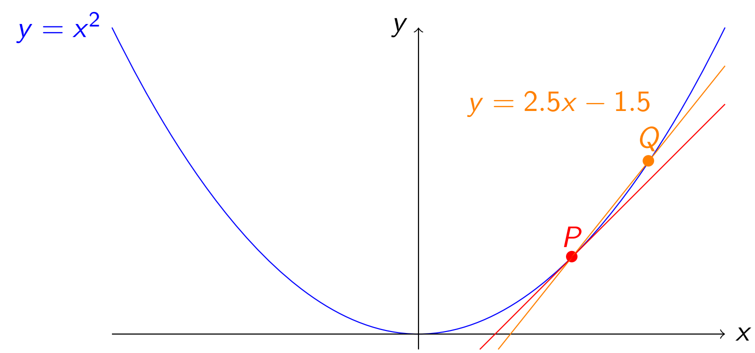







## How to find Tangents

- We want to find the tangent to $y = x^2$ at $P = (1, 1)$
- Take a point $Q$ on the curve near $P$: i.e. $Q = (1.5, 2.25)$
- The line through $P$ and $Q$ is $y = 2.5x - 1.5$
- The line segment $PQ$ is called a secant.

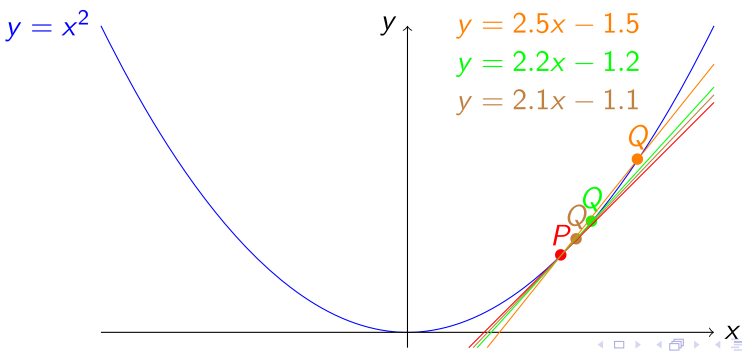


## Find the Tangent: Let $Q$ get closer to $P$

- Suppose that $Q = (1 + h, (1 + h)^2)$, where $h$ is small and positive.
- The slope of the secant $PQ$ is

$$\frac{\Delta y}{\Delta x} = \frac{(1+h)^2 - 1}{1 + h - 1} = \frac{2h + h^2}{h} = 2 + h.$$

So as $h$ gets smaller the slope approaches $2$.





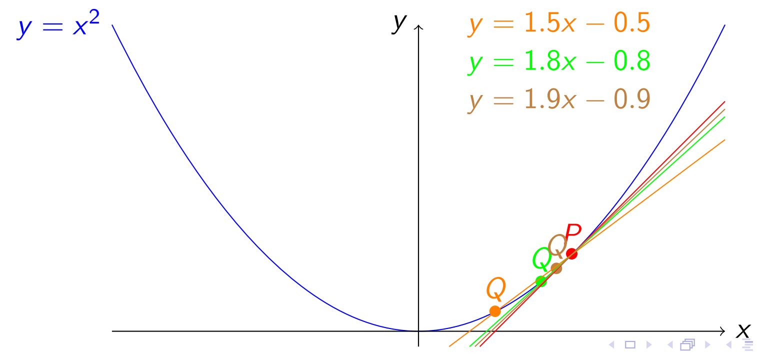



## Let $Q$ approaching $P$ from the other direction

Assume that $Q = (1 + h, (1 + h)^2)$, where $h$ is small and negative. The slope of the secant $PQ$ is

$$\frac{\Delta y}{\Delta x} = \frac{(1+h)^2 - 1}{1 + h - 1} = \frac{2h + h^2}{h} = 2 + h.$$

So as $h$ gets smaller the slope approaches $2$.








## Tangent to $y = x^2$: Conclusion

- If $Q$ is close to $P$, then $Q = (1 + h, (1 + h)^2) = (1 + h, 1 + 2h + h^2)$ where $h$ is small.
- If $h > 0$, then $Q$ lies above and to the right of $P$.
- If $h < 0$, then $Q$ lies below and to the left of $P$.
- As $Q$ approaches $P$, $h \to 0$.
- The slope of the secant $PQ$ is

$$\frac{1 + 2h + h^2 - 1}{1 + h - 1} = \frac{2h + h^2}{h} = 2 + h.$$

- The slope of the tangent to $y = x^2$ at $x = 1$ is $2$.
- The tangent passes through the point $P = (1, 1)$.
- The point-slope equation of the tangent is

$$y = 1 + 2(x - 1) = 2x - 1.$$