Example: How many different ways can a group of 12 students be arranged into three teams of 4 if

(a) the teams are ordered, and

(b) the teams are not ordered.

For (a) the number is

$$\binom{12}{4,4,4} = \frac{12!}{4!4!4!} = 34,650$$

For (b), we can permute the 3 teams and the number is

$$\frac{1}{3!}\binom{12}{4,4,4} = \frac{12!}{3!4!4!4!} = 5,775.$$

Example: In how many ways can 52 cards be dealt out evenly among 4 people?

Here the number is

$$\binom{52}{13,13,13,13} = \frac{52!}{13!13!13!13!} = 5.364 \times 10^{28}$$

Example: How many eleven-letter words can be formed using the letters of MISSISSIPPI?

Here the number is

$$\binom{11}{1,4,4,2} = \frac{11!}{1!4!4!2!} = 34,650$$

Past exam question on counting

QUESTION 3

(a) A committee of 3 is to be chosen from a group of 9 politicians.
(i) In how many ways can this be done?
(ii) Suppose two of the group belong to Party I, three of the group belong to Party II and four of the group belong to Party III. How many of the committees have at least two members from the same party? Explain your answer.

[8 marks]

(i) Here the answer is the number of ways of choosing 3 people from 9. We are picking without replacement and the order is unimportant. Thus the number is

$$\binom{9}{3} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 3 \times 4 \times 7 = 84.$$

(ii) Here we should use the subtraction rule and count the number with one from each party which is

$$\binom{2}{1}\binom{3}{1}\binom{4}{1} = 2 \times 3 \times 4 = 24$$

Thus the answer to (ii) is $84 - 24 = 60$.

Incidently, we can count the number in (ii) directly but it takes a lot longer. The number of 3-selections from 3

$$\binom{3 + (3-1)}{3-1} = \binom{5}{2} = 10.$$

In stars and bars notation these are

$$* * *|| \qquad * *| * | \qquad * *|| * \qquad *| * *| \qquad * | * |*$$

$$*|| * * \qquad | * * * | \qquad | * *| * \qquad | * | * * \qquad || * **$$

translating into 'three from Party I', 'two from Party I, one from Party II', etc. Of these, the first is excluded since Party I has only two members and the fourth is excluded since that does not have at least two in the same party. The counts for the remaining eight possibilities sum to

$$
\begin{aligned}
&\binom{2}{2}\binom{3}{1}\binom{4}{0}+\binom{2}{2}\binom{3}{0}\binom{4}{1}+\binom{2}{1}\binom{3}{2}\binom{4}{0}\\
&+\binom{2}{1}\binom{3}{0}\binom{4}{2}+\binom{2}{0}\binom{3}{3}\binom{4}{0}+\binom{2}{0}\binom{3}{2}\binom{4}{1}\\
&+\binom{2}{0}\binom{3}{1}\binom{4}{2}+\binom{2}{0}\binom{3}{0}\binom{4}{3}\\
&=(1\times 3\times 1)+(1\times 1\times 4)+(2\times 3\times 1)+(2\times 1\times 6)\\
&+(1\times 1\times 1)+(1\times 3\times 4)+(1\times 3\times 6)+(1\times 1\times 4)\\
&=3+4+6+12+1+12+18+4=60.
\end{aligned}
$$

Probability

Probability theory is the mathematical study of chance. As a branch of mathematics, but it is perhaps unique in that it deals with questions that we encounter in everyday life.

Examples: (i) What is the probability of winning the jackpot in the game of lotto?
(ii) In a class of 42 students, what is the probability that two or more have a common birthday?
(iii) What is the probability that a hand of poker contains 4 cards of the same type?

We will learn shortly how to solve these problems and many more related ones. It will also be useful to consider simpler problems involving tossing coins and throwing dice, if only to illustrate the definitions.

At the other end of the spectrum, probability theory is also capable of handling much more complicated questions, like calculating the probability that a corporate stock will fall below a certain price at any stage during the next year.

Because probability theory is dealing with relatively concrete problems, we can have an intuitive idea as to what the answer should be. This can be useful because it can help us find the answer. However it can also be misleading because our intuition is often wrong.

Example: (The Monty Hall problem) In a game show a contestant is told that a prize is behind one of three doors. After the contestant picks one door the host (Monty) opens another door revealing no prize. The contestant is invited to switch his/her choice to the remaining door or not. What should he/she do?

Note: Here are two views of the Monty Hall problem which may influence your intuition:
(a) The prize is either behind the door you picked or behind another door. You would expect the probability that it is behind your door to be 1/3 and therefore the probability that it is behind another door to be 2/3. However, if you change doors you will get the prize if it is behind another door.
(b) Imagine there were 100 doors. You pick a door, say 17, and Monty opens

all the other doors except one. He opens

$$1, 2, \ldots, 15, 16, 18, 19, \ldots, 60, 61, 63, 64, \ldots, 99, 100$$

Would you now stick with your door or switch to this very, very special door 62?