Extension of pigeonhole principle: Suppose A and B are finite sets with $k|B| < |A|$ and $f : A \to B$ is a function. Then at least one element of B must be the image of at least $k+1$ elements of A under f.

Example: How many different surnames must be in a telephone directory in order to ensure that at least four surnames have the same first and the same second letter?

Here we need at least $3(26)^2 + 1$ surnames.

Example: Show that, in any group of six people, there are either three who all know each other or three complete strangers.

Let x be any one of the people and set A equal to the set consisting of the other 5. Set $B = \{0, 1\}$ and define $f : A \to B$ by

$$f(y) = \begin{cases} 1 & \text{if } x \text{ knows } y \\ 0 & \text{if } x \text{ does not know } y \end{cases}$$

Since $|A| = 5 > 4 = 2|B|$ at least one of the numbers 0 or 1 arises three times as $f(y)$. So either x knows three of the others or x does not know three of the others. In the first case, suppose x knows y_1, y_2 and y_3. If any two of these know each other then we can add x to this pair to get a set of three who all know each other. If no two of these know each other then we have a set of three complete strangers. The second case is similar.

COUNTING AND COMBINATORICS

Note: If A and B are disjoint sets with n and m elements respectively then $A \cup B$ has $n+m$ elements. This follows from our rule

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Saying A and B are disjoint means $A \cap B = \emptyset$ so that $|A \cap B| = 0$.

Addition Principle of Counting: If an event E_1 can occur in n different ways and an event E_2 can occur in m different ways and both events cannot occur at the same time then E_1 or E_2 can occur in $n+m$ different ways.

Rule of thumb: 'or' $\leftrightarrow$ addition

Example: If a phone shop stocks 5 models of phone of brand 1 and 6 models of phone of brand 2, then it stocks 11 different models of phone.

Example: There are 33 numbers between 1 and 100 inclusive which are multiples of three and 50 which are multiples of 2. There are not 83 numbers in this range which are multiples of 3 or multiples of 2. Here A, the set of multiples of 3, and B, the set of multiples of 2, are not disjoint since $A \cap B$ contains all the multiples of 6.

Note: Of course, this principle can be extended to more than 2 events: If event E_i can occur in n_i different ways for $i = 1, 2, \ldots, k$ and no two of the events can occur at the same time then E_1 or E_2 or ... or E_k can occur in $n_1 + \ldots + n_k$ different ways.

Note: If E is the compound event that exactly one of E_1 or E_2 or ... or E_k occurs, then E is partitioned by the sets $E_1, E_2, \ldots, E_k$.

Example: Binary strings of length at least 4 consist of strings of length 0, 1, 2, 3 or 4 and a string cannot have two different lengths. If B_i is the set of binary strings of length i then the total number is

$$|B_0| + |B_1| + |B_2| + |B_3| + |B_4| = 1 + 2 + 4 + 8 + 16 = 31.$$

Example: The number of integers between 100 and 299 which are divisible by 5 can be expressed as the disjoint union of 4 sets:

$$S_1 = \{\text{strings of form [1][k][0]}\}$$

$$S_2 = \{\text{strings of form [1][k][5]}\}$$
$$S_3 = \{\text{strings of form [2][k][0]}\}$$
$$S_4 = \{\text{strings of form [2][k][5]}\}$$

Each S_i has 10 elements (10 choices for k). Now the number we want is

$$n = |S_1| + |S_2| + |S_3| + |S_4| = 10 + 10 + 10 + 10 = 40.$$

Difference Rule: If A is a finite set and B is a subset of A then

$$|A - B| = |A| - |B|.$$

Proof: The sets $B, A - B$ partition A. So $|A| = |B| + |A - B|$.

Example: How many binary strings of length 5 are not divisible by 4?

A string $abcde$ is divisible by 4 if and only if $d = e = 0$. (Remember that $abcde = a2^4 + b2^3 + c2^2 + d2 + e = 2^2(a2^2 + b2 + c) + d2 + e$.) So a string of length 5 which is divisible by 4, $abc00$, is a string of length 3 followed by two 0's. The number of strings divisible by 4 is $2^3 = 8$ and the number not divisible by 4 is

$$2^5 - 2^3 = 32 - 8 = 24.$$

Here A is the set of strings of length 5. B is the set of strings of length 5 which are divisible by 4. We want $|A - B|$.

Note: If A and B are sets with n and m elements respectively then their Cartesian product $A \times B$ has nm elements.

Product Principle of Counting: If event E_1 can occur in n ways and event E_2 can occur in m ways, then the number of ways the events can occur in the order E_1 followed by E_2 is nm.

Rule of thumb: 'and' $\leftrightarrow$ multiplication

Note: Of course, this principle can be extended to more than 2 events: If event E_i can occur in n_i different ways for $i = 1, 2, \ldots, k$ then E_1 followed by E_2 followed by ... followed by E_k can occur in $n_1 n_2 \ldots n_k$ different ways.

Example: How many license plates are there with three letters followed by three digits? The first digit cannot be zero. Answer: (26)(26)(26)(9)(10)(10). E_1 is choice of first letter, E_2 is choice of second letter, E_3 is choice of third letter, E_4 is choice of first non-zero digit, E_5 is choice of second digit, E_6 is choice of last digit.