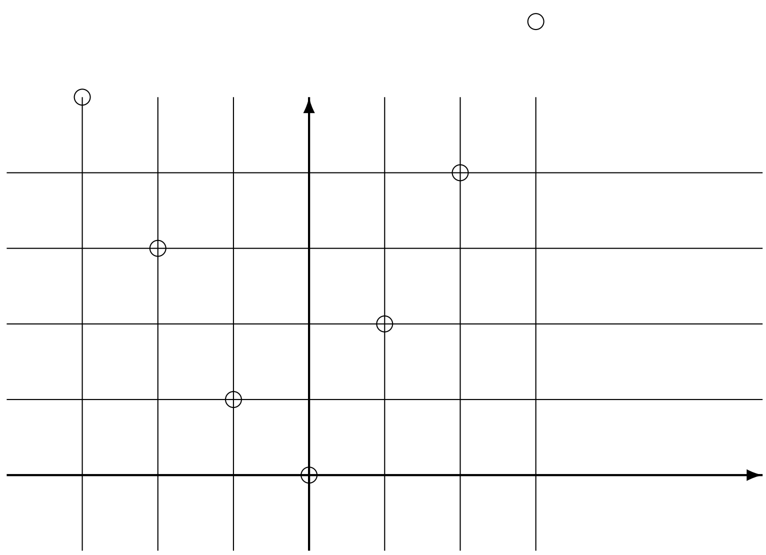
Example: The function $f : \mathbb{R} \to \mathbb{R} : x \mapsto 3x - 5$ is both injective and surjective and is hence a bijection. To show injectivity suppose $f(a) = f(b)$. Then $3a - 5 = 3b - 5$ and $a = b$. To show surjectivity suppose $y \in \mathbb{R}$ and look for $x \in \mathbb{R}$ with $f(x) = y$. Then $3x - 5 = y$ and we can always solve to get $x = (1/3)(y + 5)$.

Example: A function from $\mathbb{Z} \to \mathbb{Z}_{\geq 0}$ which is both surjective and injective!

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}$$



We note that $f(n)$ is even if $n \geq 0$ while $f(n)$ is odd if $n < 0$. To establish surjectivity, suppose $m \in \mathbb{Z}_{\geq 0}$. If m is even $m = 2k = f(k)$. If m is odd, $m = 2l - 1 = f(-l)$. To establish injectivity, suppose $f(a) = f(b) = m$. If m is even, then a and b are non-negative with $2a = 2b = m$ so that $a = b$. If m is odd, then a and b are negative with $-2a - 1 = -2b - 1 = m$ so that $a = b$ also.

Definition: A function f from a set A to a set B is called invertible if the inverse relation $f^{-1} \subseteq B \times A$ is a function.

Example: $A = \{a, b, c\}$, $B = \{p, q, r\}$ with

$$f = \{(a, q), (b, r), (c, p)\}.$$

The inverse relation is

$$f^{-1} = \{(p, c), (q, a), (r, b)\}.$$

This is a function from B to A so f is invertible.

Example: Recall the function $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ given by

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n-1 & \text{if } n < 0 \end{cases}$$

As a relation

$$f = \{\ldots, (-3,5), (-2,3), (-1,1), (0,0), (1,2), (2,4), (3,6), \ldots\}.$$

The inverse relation is

$$f^{-1} = \{(0,0), (1,-1), (2,1), (3,-2), (4,2), (5,-3) \ldots\}.$$

This inverse relation is a function so f is invertible.

Theorem: A function $f : A \to B$ is invertible if and only if f is bijective.

Proof: Suppose $f : A \to B$ is invertible. We will show f is bijective. Start with surjectivity. If $b \in B$ then $(b,a) \in f^{-1}$ for some $a \in A$ since f^{-1} is a function (property 1). Thus $(a,b) \in f$ and $f(a) = b$. Next consider injectivity. Suppose $f(a_1) = f(a_2) = b$. Then $(a_1, b) \in f$ and $(a_2, b) \in f$ so that $(b, a_1) \in f^{-1}$ and $(b, a_2) \in f^{-1}$. But f^{-1} is a function so that $a_1 = a_2$ (property 2).

For the converse, suppose $f : A \to B$ is bijective. We will show f is invertible by showing f^{-1} is a function. Let $b \in B$. Since f is surjective $(a,b) \in f$ for some $a \in A$. This gives $(b,a) \in f^{-1}$ and f^{-1} satisfies property 1 of a function. Next suppose $(b, a_1) \in f^{-1}$ and $(b, a_2) \in f^{-1}$. Then $(a_1, b) \in f$ and $(a_2, b) \in f$ so that $f(a_1) = b$ and $f(a_2) = b$. But f is injective, so $a_1 = a_2$ and f^{-1} satisfies property 2 of a function.