Definition: If A and B are sets their Cartesian product, denoted $A \times B$, is the set whose elements are all possible ordered pairs (a, b), where $a \in A$ and $b \in B$. That is,

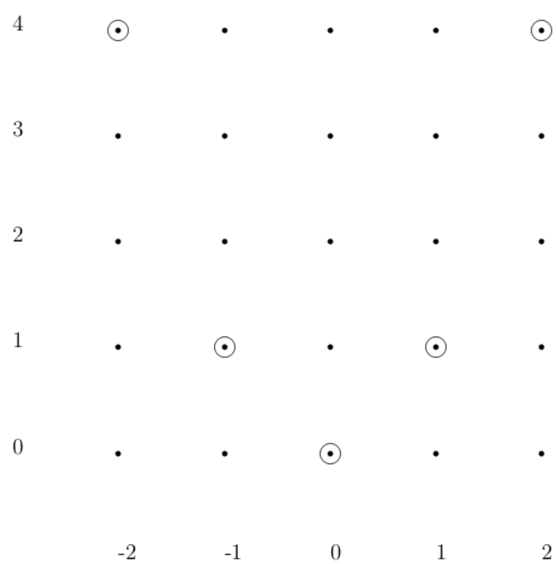
$$A \times B = \{(a, b) \mid a \in A \textbf{ and } b \in B\}$$

Note: In the next two sections we will use Cartesian products to define relations and functions. A relation between sets A and B will simply be a subset of $A \times B$ and a function will be a special type of relation.

Example: Suppose $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4\}$. So $A \times B$ is a 25-element set. It contains the subset

$$S = \{(x, y) \in A \times B \mid y = x^2\}$$

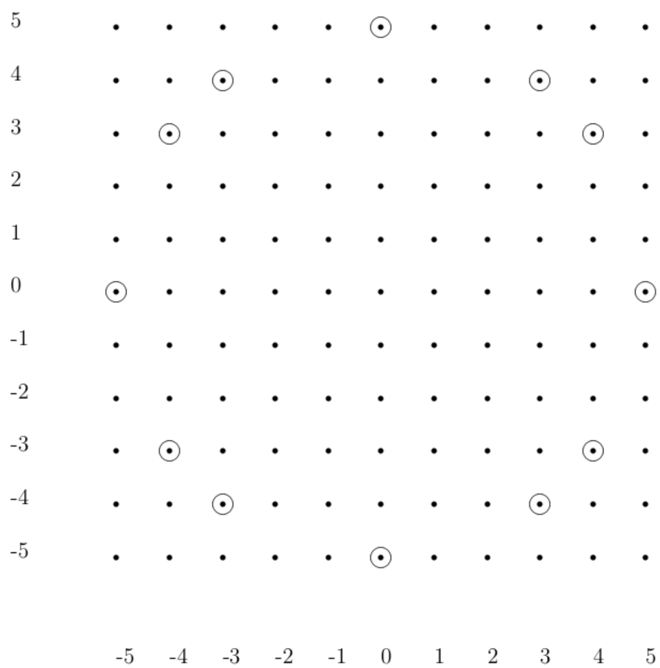
which, in the grid picture, looks like (part of) the graph of the function $y = x^2$.



Example: Suppose $A = B = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. So $A \times B$ is a 121-element set. It contains the subset

$$S = \{(x, y) \in A \times B \mid x^2 + y^2 = 25\}$$

which, in the grid picture, looks like (part of) the graph of the circle with equation $x^2 + y^2 = 25$. (Recall that $3^2 + 4^2 = 5^2$.)



Chapter 3: Relations.

Here is a table (spreadsheet) describing the processes used in the making of products.

Product	Process
A	2
A	4
B	1
B	4
C	2
C	3

If process i is used in the making of product p we see a row with p in the first column and i in the second column. The table simply lists the elements

of the subset
$\{(A,2),(A,4),(B,1),(B,4),(C,2),(C,3)\}$
of the set $\{A,B,C\} \times \{1,2,3,4\}$.

Relations, from a mathematical point of view, are simply subsets of product sets. The reason they are interesting is that they describe precisely what we mean when we say that some elements of one set are related to some elements of another. For example, if A is the set of students in a university and B is the set of modules taught in the university, then the huge set $A \times B$ contains the much smaller subset

$$R = \{(a,b) \in A \times B \mid \text{student } a \text{ is registered for module } b\}$$

and the subset R is a relation. It completely describes the relationship 'is registered for' between students and modules. In web advertising such relationships are very important. When web user A clicks on link L then that relationship sparks exposure to particular advertising.

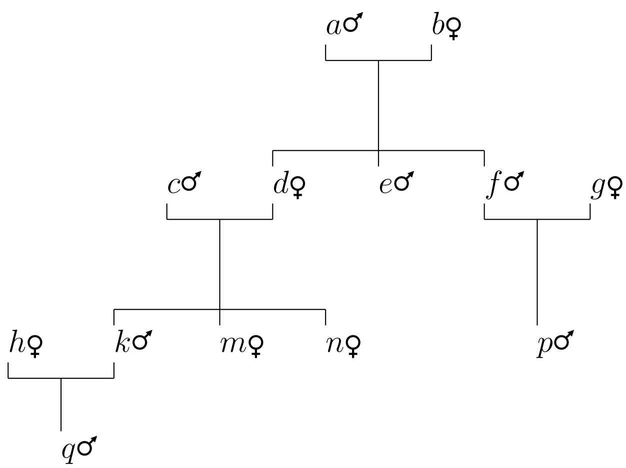
Definition: A binary relation between two sets A and B is a subset R of the Cartesian product $A \times B$.

Note: We could have $A = B$ in which case we refer to R as a relation on A.

Example: If $A = \{1,2\}$ and $B = \{p,q,r\}$, then $A \times B$ has 6 elements $(1,p),(1,q),(1,r),(2,p),(2,q),(2,r)$, so that $A \times B$ has $2^6 = 64$ subsets. This means there are 64 different relations between A and B, ranging from $\emptyset$, where no elements of A are related to elements in B to $A \times B$, where every element of A is related to every element of B.

In between are the other 62 relations, including 6 where exactly one element of A is related to one element of B, 2 where one element of A is related to all elements of B, 3 where all elements of A are related to a single element of B, etc.,

Example: In English, we think of the word relation as most closely connected to family relationships. Here's a family tree.



What are the relations
(a) R_1: Is an uncle of,
(b) R_2: Is a grandchild of?

(a) $R_1 = \{(e, k), (e, m), (e, n), (e, p), (f, k), (f, m), (f, n)\}$

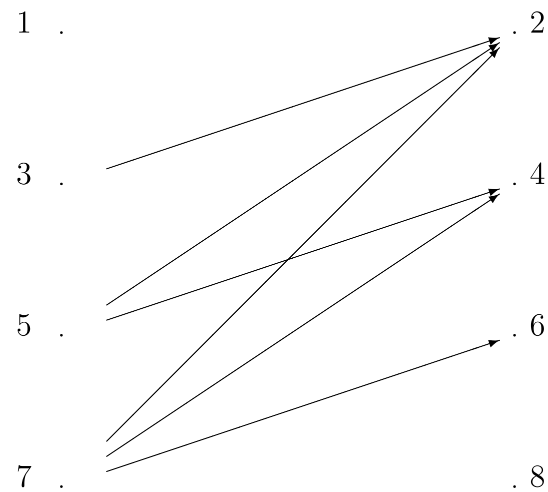
(b) $R_2 = \{(k, a), (k, b), (m, a), (m, b), (n, a), (n, b), (p, a), (p, b), (q, c), (q, d)\}$

Example: For the sets of integers $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ what are the relations
(a) $R_1 = \{(x, y) \in A \times B \mid x > y\}$
(b) $R_2 = \{(x, y) \in A \times B \mid x - y \text{ is even }\}$.
For (a): $R_1 = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$. For (b) $R_2 = \emptyset$.

Note: For small examples, there are devices which help us visualise relations.

Definition: If R is a binary relation between sets A and B the digraph of the relation is a graph with vertex set $A \cup B$ and an edge joining a to b if $(a, b) \in R$.

Example: For R_1 in the last example, the digraph is


1
2
3
4
5
6
7
8