MS121: IT Mathematics

Functions

Domain & Range

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Outline

1. Sets & Inequalities
2. Functions and their Graphs
3. Domain & Range
4. Which Curves are Graphs of Functions?

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2. Functions and their Graphs
3. Domain & Range
4. Which Curves are Graphs of Functions?

Sets & Numbers

Set

- A set is a collection of objects and the objects in the set are called elements.
- Sets are usually denoted by upper-case letters; if x belongs to some set S, this is written as $x \in S$, and if x does not belong to S, this is written as $x \notin S$, e.g.

$$S = \{1, 2, 3, 4, 5\}, \qquad 5 \in S, \qquad 7 \notin S.$$

- Set A is a subset of set B if every element in set A is also in set B, written $A \subset B$. For example,

$$\{1, 3, 17\} \subset \{-4, 0, 1, 2, 3, 13, 17\}$$

Special Number Sets

N	Set of natural (positive whole) numbers: $1, 2, 3, \ldots$
Z	Set of integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
Q	Set of rational numbers (fractions): $\frac{a}{b}$, $a, b \in Z$, $b \neq 0$.
R	Set of all real numbers, consisting of all rational numbers and irrational numbers.
C	Set of all complex numbers.
$\emptyset$	Empty (or null) set (the set which has no elements).

It is easy to see that $N \subset Z$ and, more generally, that

$$N \subset Z \subset Q \subset R$$

Set Operations

Intersection

The intersection of sets A and B is the set of elements which belong to both A and B and is denoted by $A \cap B$. For example:

$$\{1, 2, 10\} \cap \{-3, -1, 0, 2, 6\} = \{2\}$$

whereas

$$\{1, 2, \} \cap \{3, 4\} = \emptyset,$$

the empty (or null) set.

Set Operations (Cont'd)

Union

The union of sets A and B is the set of elements which belong to either A or B and is denoted by $A \cup B$. For example:

$$\{0, 2, 4, 5\} \cup \{1, 3, 5\} = \{0, 1, 2, 3, 4, 5\}$$

and

$$\{1, 2\} \cup \{1, 2, 3\} = \{1, 2, 3\}$$

Set Operations (Cont'd)

Disjoint Sets

The sets A and B are disjoint is they have no elements in common, i.e. if

$$A \cap B = \emptyset$$

The set $A \setminus B$ is "set A less set B" and denotes the set of elements in A that are not in set B. If

$$A = \{1, 3, 10, 19\} \qquad B = \{-1, 1, 14\}$$

then

$$A \setminus B = \{3, 10, 19\}$$

Example

Question

What is the set defined by:

$$\{x \in Z \mid 3 < x \leq 6\}$$

Answer

The expression $x \in Z$ means that x must be an integer whereas $3 < x \leq 6$ means that x must be greater than 3 and less than or equal to 6. Putting these together, we obtain:

$$\{x \in Z \mid 3 < x \leq 6\} = \{4.5, 6\}$$

Example

Question

What is the set defined by:

$$\left\{x \in R \mid \sqrt{x} \in N \text{ and } x < 30\right\}$$

Answer

- We are looking for real numbers (since $x \in R$) less than 30 (since $x < 30$) whose squares must be positive whole numbers (since $\sqrt{x} \in N$).
- The set of numbers less than 30 having whole number square roots are $\sqrt{1} = 1$, $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{25} = 5$.
- Therefore

$$\left\{x \in R \mid \sqrt{x} \in N \text{ and } x < 30\right\} = \{1, 4, 9, 16, 25\}$$

Intervals

What is an "Interval"

Intervals are subsets of the real line without gaps, i.e.

$$\{x \in R \mid -2 < x < 7\}$$

is an interval while

$$\{1, 4, 9\}$$

is not.

Intervals

Open Interval

An interval is <u>open</u> if it does NOT include the end-points. The interval

$$\{x \in R \mid -2 < x < 7\}$$

is an open interval and is written as $(-2, 7)$ (round brackets).

Intervals (Cont'd)

Closed Interval

An interval is closed if it does include the end-points. The interval

$$\{x \in R \mid -1 \leq x \leq 0\}$$

is a closed interval and is written as $[-1, 0]$ (square brackets).

Intervals (Cont'd)

Half-open (or Half-closed) Interval

An interval is called half-open (or half-closed) if it includes only one end-point. The interval

$$\{x \in R \mid -10 \leq x < -2\}$$

is a half-open interval and is written as $[-10, -2)$.

Intervals (Cont'd)

Infinite Intervals

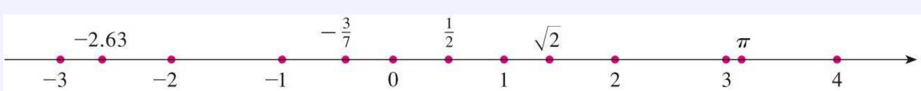
Intervals can extend out to plus and minus infinity (i.e. to $+\infty$ and $-\infty$). The interval

$$\{x \in R \mid x > 2\}$$

really means

$$\{x \in R \mid 2 < x < \infty\}$$

and is written $(2, \infty)$. The entire real line, R, is written as $(-\infty, \infty)$.



Example

Question

Express the following as intervals:

1. $\{x \in R \mid -10 < x \leq 10\}$
2. $\{x \in R \mid x \geq -4\}$
3. $\{x \in R \mid x \leq 2\}$

Answer

1. $(-10, 10]$
2. $[-4, \infty)$
3. $(-\infty, 2]$

Example

Question

Re-write the following sets:

1. $\{x \in Z \mid x^2 < 25\}$
2. $\{x \in R \mid \frac{x}{3} \in N \text{ and } x < 10\}$

Answer

1. $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
2. $\{3, 6, 9\}$

Outline

Functions and their Graphs

Rationale

Functions are the key to describing the real world in mathematical terms.

- The interest paid on an investment depends on the length of time the money is invested.
- The distance an object travels at a fixed speed depends on the elapsed time.
- The area of the circle depends on the radius of the circle.
- The run time of an algorithm depends on the length of the input.

Function

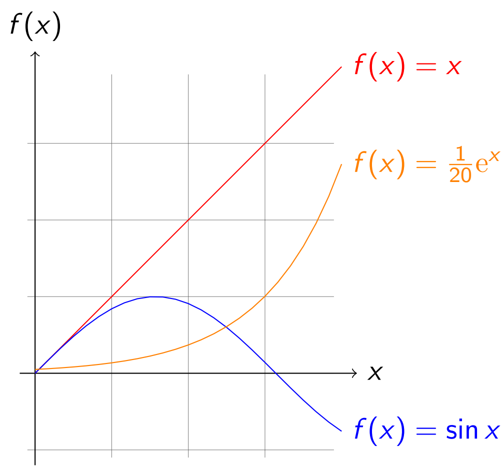
Definition

- A function/map from a set D (domain) to a set R (range) is a rule which assigns to each element in D a unique (single) element in R.
- The value of one (dependent) variable, say y, depends on the value of another (independent) variable, say x: we say that y is a function of x and write

$$y = f(x).$$

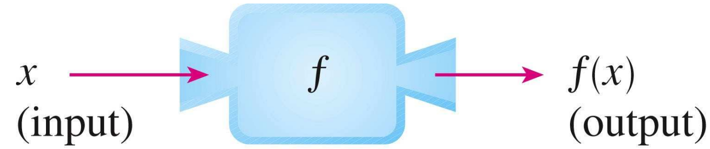
- The set D of all possible input values is called the domain of f.
- The set of all values of $f(x)$ as x varies throughout D is called the range of f.
- We write $f : D \to \mathbb{R}$.
- The graph of f is the set $\{(x, f(x)) \mid x \in D\}$.

Different Types of Functions



The graph of f is the set $\{(x, f(x)) \mid x \in D\}$.

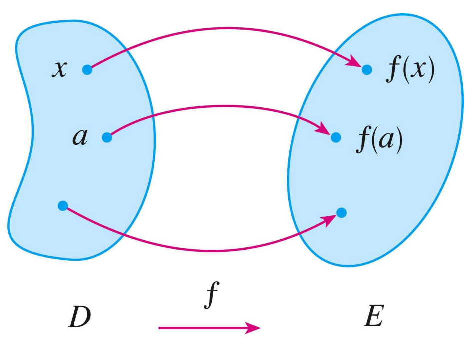
It may be helpful to think of a function as a machine:



If x is in the domain of the function f, then when x enters the machine, it is accepted as an input and the machine produces an output $f(x)$ according to the rule of the function.

Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

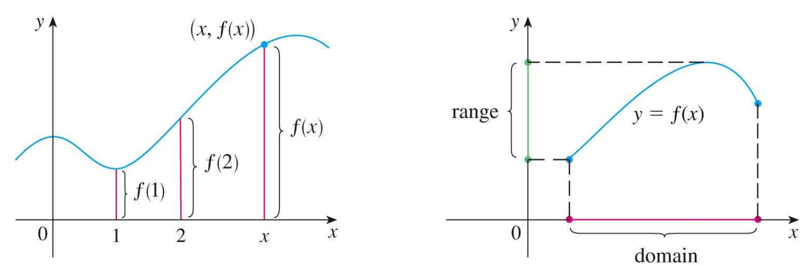
Another way to picture a function is by an arrow diagram



Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, f (a) is associated with a, and so on.

The most common method for visualizing a function is its graph. If f is a function with domain D, then its graph is the set of ordered pairs

$$\{(x,\ f(x)) \mid x \in D\}$$



- In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.
- The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis

Outline

Domain & Range

- We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the domain of the function.
- The number $f(x)$ is the value of f at x and is read "f of x".
- The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.
- A symbol that represents an arbitrary number in the domain of a function f is called an independent variable.
- A symbol that represents a number in the range of f is called a dependent variable.
- In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.
- The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis.

Domain & Range Example

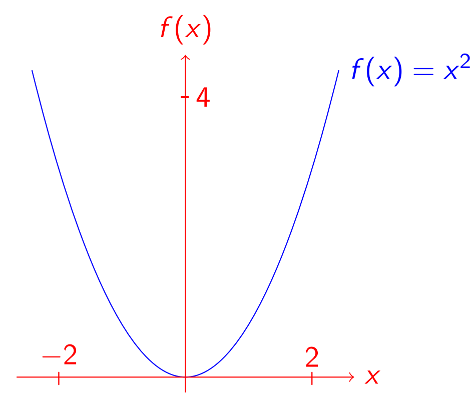
Question

Consider the function $f(x) = x^2$.

Analysis

- The natural domain is simply R, since any real number can be squared.
- However, the range of $f(x)$ is $[0, \infty)$ since the square of any number (i.e. x^2) cannot be negative.

Domain & Range of $f(x) = x^2$



Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $[0, \infty)$

Note that in this graph, we have restricted our domain to $[-2, 2]$ so that the range is $[0, 4]$.

Domain & Range Example

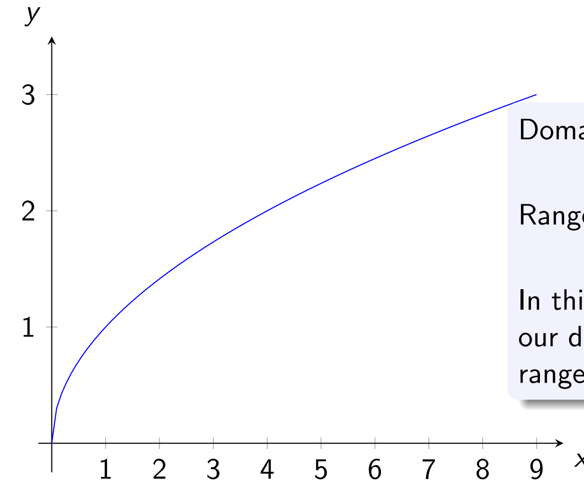
Question

Consider the positive square-root function $f(x) = +\sqrt{x}$.

Analysis

- We cannot take the square root of a negative number so the domain must be restricted to $[0, \infty)$ since the function is the positive square-root.
- Note that, although every positive number has two square-roots, $+\sqrt{}$ and $-\sqrt{}$, the operation $f(x)$ would not be a function unless we defined it uniquely.

Domain & Range of $f(x) = \sqrt{x}$

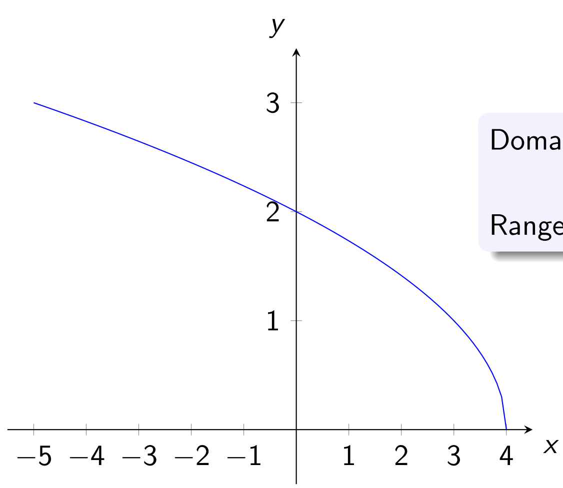


Domain: $[0, \infty)$

Range: $[0, \infty)$

In this picture, we have restricted our domain to $[0, 9]$ and so the range is $[0, 3]$.

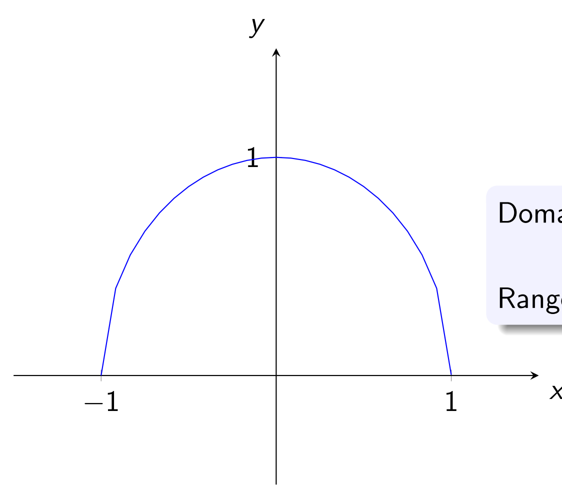
Domain & Range of $f(x) = \sqrt{4 - x}$



Domain: $(-\infty, 4]$

Range: $[0, \infty)$

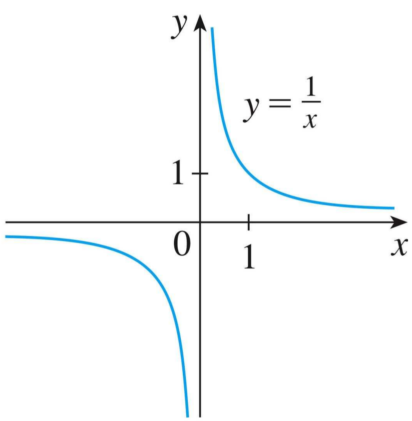
Domain & Range of $f(x) = \sqrt{1 - x^2}$



Domain: $D = (-1, 1)$

Range: $R = [0, 1]$

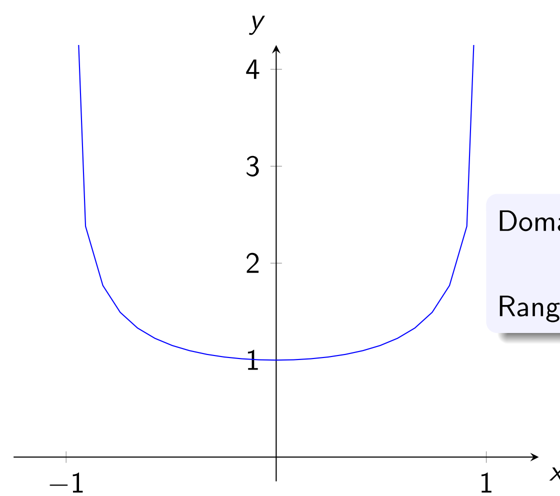
Domain & Range of $f(x) = \frac{1}{x}$



Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

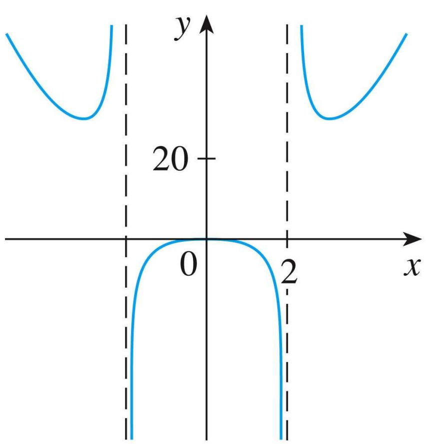
Domain & Range of $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$



Domain: $D = (-1, 1)$

Range: $R = [1, \infty)$

Domain & Range of a Rational Function

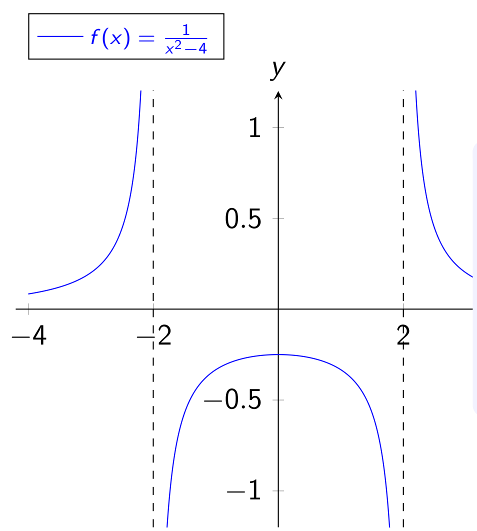


$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

Domain: $\{x \mid x \neq \pm 2\}$

Range: $(-\infty, \infty)$

Domain & Range of a Rational Function



$$f(x) = \frac{1}{x^2 - 4}$$

Domain: $\{x \mid x \neq \pm 2\}$

Range: $(-\infty, -\frac{1}{4}] \cup (0, \infty)$

Domain & Range Example

Question

Consider the function $f(x) = \dfrac{1}{x-3}$.

Analysis

- It is not defined for $x = 3$ since we would be attempting to divide by zero at that point. So the domain is $R \setminus \{3\}$, i.e. all the real numbers except 3.
- The range is also restricted as follows.
- The function $\frac{1}{x-3}$ can produce any real number except 0 since no real value for x will make $\frac{1}{x-3} = 0$.
- Therefore, the range is $R \setminus \{0\}$.

Domain & Range Example

Question

Consider the function $f(x) = \dfrac{1}{\sqrt{x-19}}$.

Analysis

- There are two potential problems in finding the natural domain here.
- We cannot divide by zero and we cannot take the square root of a negative number.
- Thus, we need
$$x - 19 > 0 \quad \Rightarrow \quad x > 19$$
and so the natural domain is $(19, \infty)$.
- To determine the range, note that the square root is positive so the range must be positive, i.e. $(0, \infty)$.

Domain & Range Example

Question

Determine the natural domain of $f(x) = \dfrac{1}{x^2 - x - 2}$.

Answer

We need to avoid dividing by zero and so the natural domain will be all real numbers except those which make $x^2 - x - 2$ equal to zero. In order to eliminate these points, we need to solve

$$x^2 - x - 2 = 0$$

giving the two roots $x = 2$ and $x = -1$. Hence, the natural domain of $f(x)$ is $R \setminus \{-1, 2\}$.

Domain & Range Example

Solving $x^2 - x - 2 = 0$.

Roots of a Quadratic: By factorisation

$$x^2 - x - 2 = (x-2)(x+1) = 0$$

giving the two roots $x = 2$ and $x = -1$.

Domain & Range Example

Solving $x^2 - x - 2 = 0$.

Roots of a Quadratic: Using the formula

We will use

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a, b and c are the coefficients of the quadratic equation

$$ax^2 + bx + c = 0.$$

With $a = 1$, $b = -1$ and $c = -2$, we have

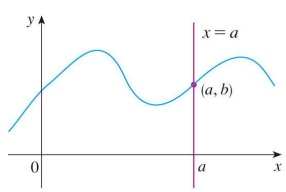
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

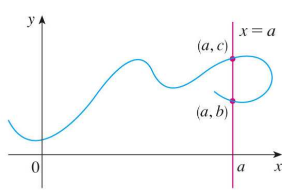
giving the two values $x = 2$ and $x = -1$ (as before).

Outline

Which Curves are Graphs of Functions?

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions?

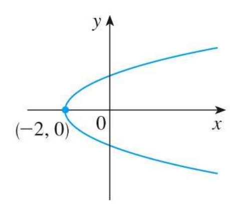




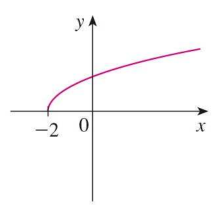
This is answered by the following test.

Vertical Line Test

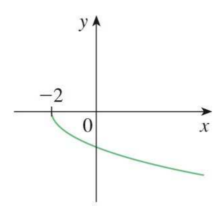
If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But, if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve cannot represent a function because a function cannot assign two different values to a.



(a) $x = y^2 - 2$



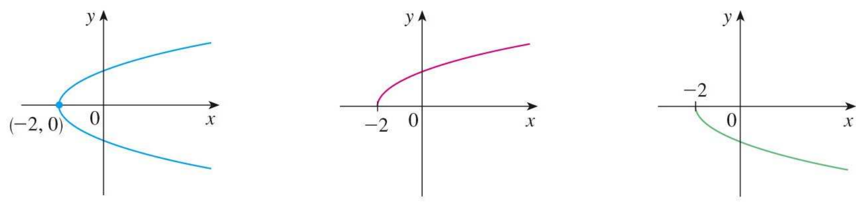
(b) $y = \sqrt{x+2}$



(c) $y = -\sqrt{x+2}$

The parabola $x = y^2 - 2$ is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice.

The parabola, however, does contain the graphs of two functions of x.



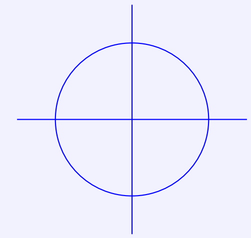
(a) $x = y^2 - 2$ (b) $y = \sqrt{x+2}$ (c) $y = -\sqrt{x+2}$

Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x+2}$.

Thus the upper and lower halves of the parabola are the graphs of the functions $y = \sqrt{x+2}$ and $y = -\sqrt{x+2}$.

Vertical Line Test

A function f can have only one value $f(x)$, for each x in its domain, so no vertical line can intersect the graph of a function more than once.

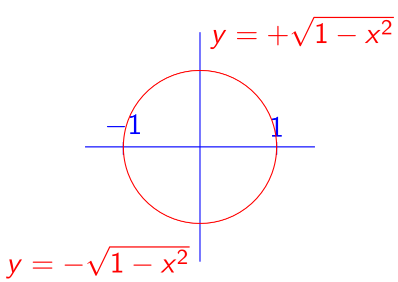


A circle is not the graph of a function as the vertical line $x = 0$ (y-axis) cuts it twice.

Vertical Line Test

However, the unit circle centred on $(0, 0)$ has equation

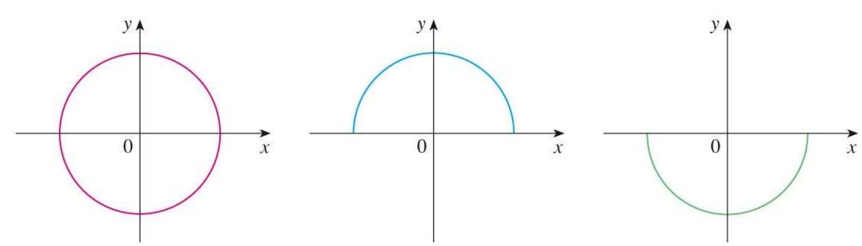
$$x^2 + y^2 = 1 \iff y^2 = 1 - x^2 \iff y = \pm\sqrt{1-x^2}.$$



The upper semicircle is the graph of a function $f(x) = \sqrt{1-x^2}$.

The lower semicircle is the graph of a function $g(x) = -\sqrt{1-x^2}$

The Circle and 2 Functions



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$