Definition: If A and X are sets we define the complement of A in X to be

$$X - A = \{x \mid x \in X \textbf{ and } x \notin A\}.$$

That is the set consisting of those elements in X which are not also in A.

Note: In applications there is usually a universal set in the background denoted U. This could be the set of all the people in the world, the set of all real numbers, etc.,

Definition: If there is a universal set U, we define the complement of A to be

$$\sim A = U - A = \{x \mid \textbf{not } (x \in A)\}.$$

Note: Since
inclusions, unions, intersections and complements
are defined using
$\Rightarrow$, **or**, **and** and **not**
many identities about sets follow from logical equivalences.

Example: We know **not** (P **and** Q) $\equiv$ (**not** P) **or** (**not** Q). Letting P and Q stand for the statements
P: $x \in A$
Q: $x \in B$

gives **not** $(x \in A$ **and** $x \in B) \equiv$ (**not** $x \in A$) **or** (**not** $x \in B$)

which is the same as

$$\sim(A \cap B) = (\sim A) \cup (\sim B)$$

Here are more properties:

$$A \cup \sim A = U, \quad A \cap \sim A = \emptyset.$$

If A is not necessarily a subset of X,

$$X - \emptyset = X, \quad X - X = \emptyset, \quad A \cap (X - A) = \emptyset.$$

De Morgan's Laws

$$X - (A \cup B) = (X - A) \cap (X - B), \quad X - (A \cap B) = (X - A) \cup (X - B).$$

De Morgan's Laws for $X = U$

$$\sim(A \cup B) = (\sim A) \cap (\sim B), \ \ \sim(A \cap B) = (\sim A) \cup (\sim B).$$

Definition: The symmetric difference of two sets is defined by

$$A \triangle B = \{x \mid (x \in A \textbf{ and } x \notin B) \textbf{ or } (x \notin A \textbf{ and } x \in B)\}.$$

Example: If $A = \{1, 2, 3\}$ and $B = \{0, 2, 5\}$ then

$$A \triangle B = \{0, 1, 3, 5\}.$$

Cardinality

Definition: If A is a finite set we write $|A|$ for the number of elements in A and call $|A|$ the cardinality of A.

Proposition: If A and B are finite sets then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof: From a Venn diagram we see

$$|A \cup B| = |A \cap \sim B| + |A \cap B| + |\sim A \cap B|$$

The first two terms give $|A|$ while the last term gives $|B| - |A \cap B|$.

Example: In a survey of 100 phoneowners 55 have iphones, 40 have Samsungs and 3 have both. (!!!) How many have neither?

Solution: Let I be the set of iphone owners in the survey and S be the set of Samsung phone owners in the survey. Here the universal set U is the set of people surveyed. The set of surveyed people with neither an iphone nor a Samsung is $\sim(I \cup S)$. So we want to find $|\sim(I \cup S)|$. We know

$$|\sim(I \cup S)| = |U| - |I \cup S| = 100 - |I \cup S|$$

so we should find $|I \cup S|$ and the last proposition gives

$$|I \cup S| = |I| + |S| - |I \cap S|.$$

Thus we can say

$$\begin{aligned} |I \cup S| &= |I| + |S| - |I \cap S| \\ &= 55 + 40 - 3 = 92 \end{aligned}$$

so that the number with neither phone is $100 - 92 = 8$.

Proposition: If A, B and C are finite sets then

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad -|A \cap B| - |A \cap C| - |B \cap C| \\ &\quad +|A \cap B \cap C| \end{aligned}$$

Note: This proposition and the earlier one for two sets are examples of the Inclusion-Exclusion Principle. See its Wikipedia page.

Proof: Look at Venn diagram and record how often regions are counted. I'll write $\sim D$ as D' and $D \cap E$ as DE to save space.

Summand	ABC	ABC'	$AB'C$	$AB'C'$	$A'BC$	$A'BC'$	$A'B'C$	$A'B'C'$
$+\|A\|$	1	1	1	1	0	0	0	0
$+\|B\|$	1	1	0	0	1	1	0	0
$+\|C\|$	1	0	1	0	1	0	1	0
$-\|AB\|$	-1	-1	0	0	0	0	0	0
$-\|AC\|$	-1	0	-1	0	0	0	0	0
$-\|BC\|$	-1	0	0	0	-1	0	0	0
$+\|ABC\|$	1	0	0	0	0	0	0	0
Total	1	1	1	1	1	1	1	0

Example: A group of 100 families are surveyed about where they shop. They all shop at at least one of Tesco, Supervalu or Dunnes Stores. Suppose 45 shop at Tesco, 50 shop at Supervalu and 60 shop at Dunnes, while 22 shop at Tesco and Supervalu, 17 shop at Tesco and Dunnes and 19 shop at Supervalu and Dunnes. How many shop only at Tesco? Only at Supervalu? Only at Dunnes?

Idea: $100 = |T \cup S \cup D|$. Use the proposition to find $|T \cap S \cap D|$. Fill in the rest of the numbers.

$$\begin{aligned} |T \cup S \cup D| &= |T| + |S| + |D| \\ &\quad -|T \cap S| - |T \cap D| - |S \cap D| \\ &\quad +|T \cap S \cap D| \end{aligned}$$

This gives

$$100 = 45 + 50 + 60 - 22 - 17 - 19 + |T \cap S \cap D|$$

so that $|T \cap S \cap D| = 3$. So

$$\begin{aligned}
|T \cap S \cap \sim D| &= |T \cap S| - 3 = 19 \\
|T \cap \sim S \cap D| &= |T \cap D| - 3 = 14 \\
|\sim T \cap S \cap D| &= |D \cap S| - 3 = 16 \\
|T \cap \sim S \cap \sim D| &= |T| - |T \cap S \cap \sim D| - |T \cap \sim S \cap D| - 3 = 9 \\
|\sim T \cap S \cap \sim D| &= |S| - |T \cap S \cap \sim D| - |\sim T \cap S \cap D| - 3 = 12 \\
|\sim T \cap \sim S \cap D| &= |D| - |T \cap S \cap \sim D| - |\sim T \cap S \cap D| - 3 = 27
\end{aligned}$$