MS121: IT Mathematics

Integration

Introduction — The Area Problem

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Outline

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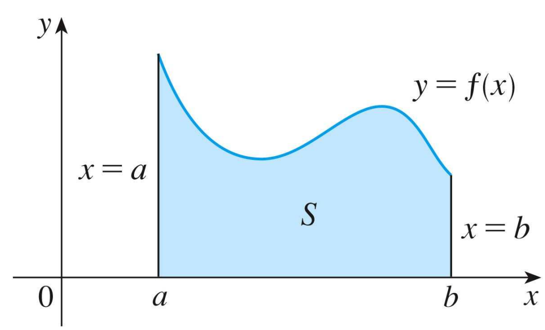
The Area Problem

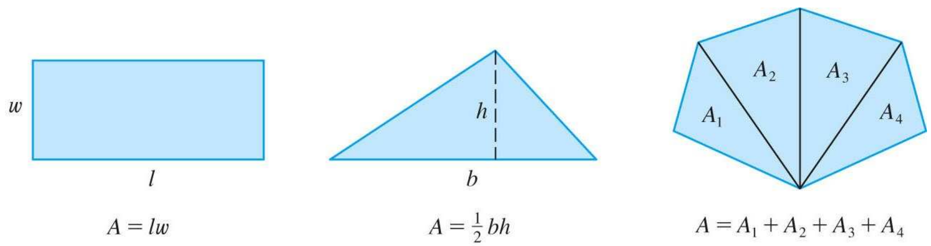
We begin by attempting to solve the area problem: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.



This means that S is bounded by the graph of a continuous function f, where $f(x) \geq 0$, the vertical lines $x = a$ and $x = b$, and the x-axis.

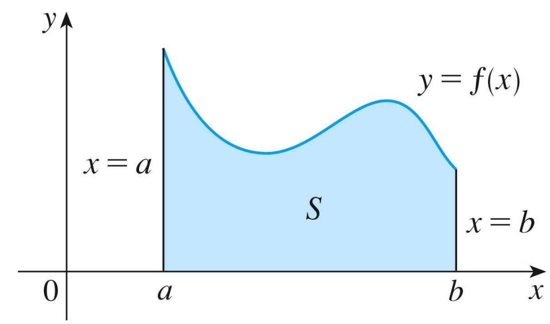
Area Problems that can be Solved

For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height.



The area of a polygon is found by dividing it into triangles and adding the areas of the triangles.

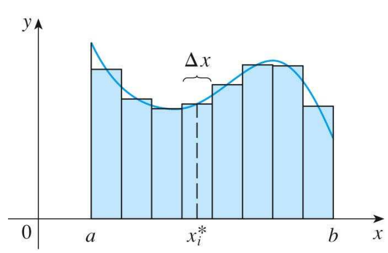
The Area Problem

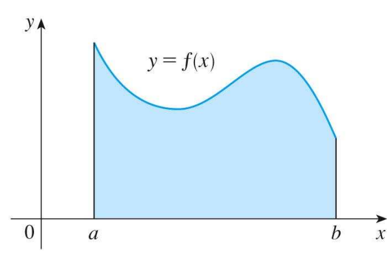


- It is not so easy to find the area of a region with curved sides.
- We all have an intuitive idea of what the area of a region is.
- Part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

The Area Problem: Strategy

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations.

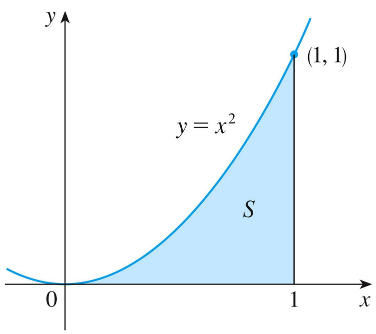




We pursue a similar idea for areas. We first approximate the shaded region by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

Outline

The Area Problem: An Illustration

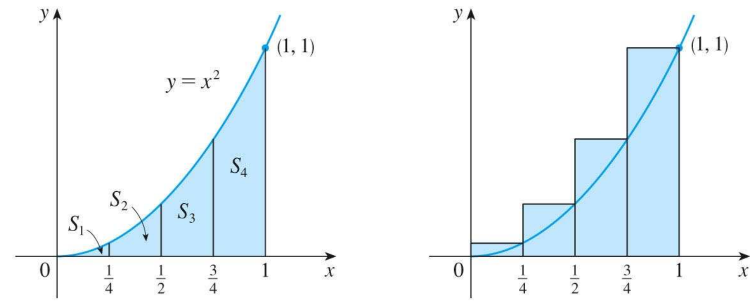


The Challenge

Use rectangles to estimate the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$ (the parabolic region S illustrated above).

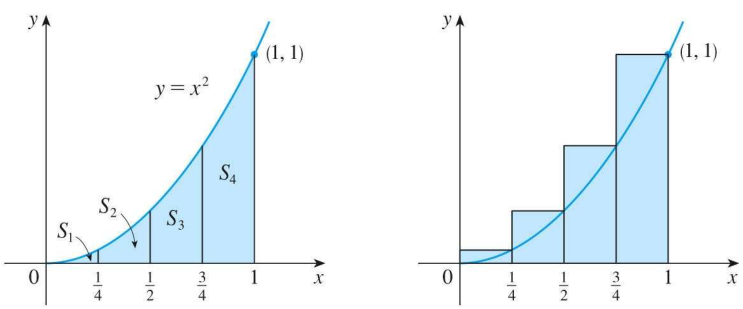
The Area Problem: An Illustration

Suppose we divide S into four strips $S1$, $S2$, $S3$ and $S4$ by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$ and $x = \frac{3}{4}$.



We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip.

The Area Problem: An Illustration

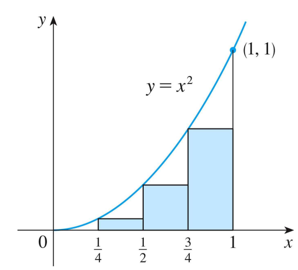


If we let R_4 be the sum of the areas of these approximating rectangles, we obtain

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot (1)^2 = 0.46875$$

From the diagram, we can see that the area A of S is less than R_4, so $A < 0.46875$.

The Area Problem: An Illustration



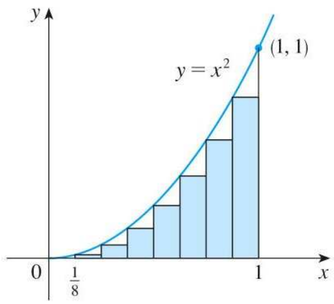
Instead of using the larger rectangles in the previous slide, we could use the smaller rectangles:

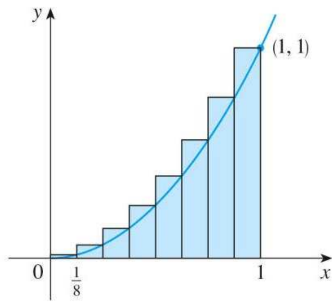
$$L_4 = \frac{1}{4} \cdot (0)^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A: $0.21875 < A < 0.46875$.

The Area Problem: An Illustration

We can repeat this procedure with a larger number of strips (8).

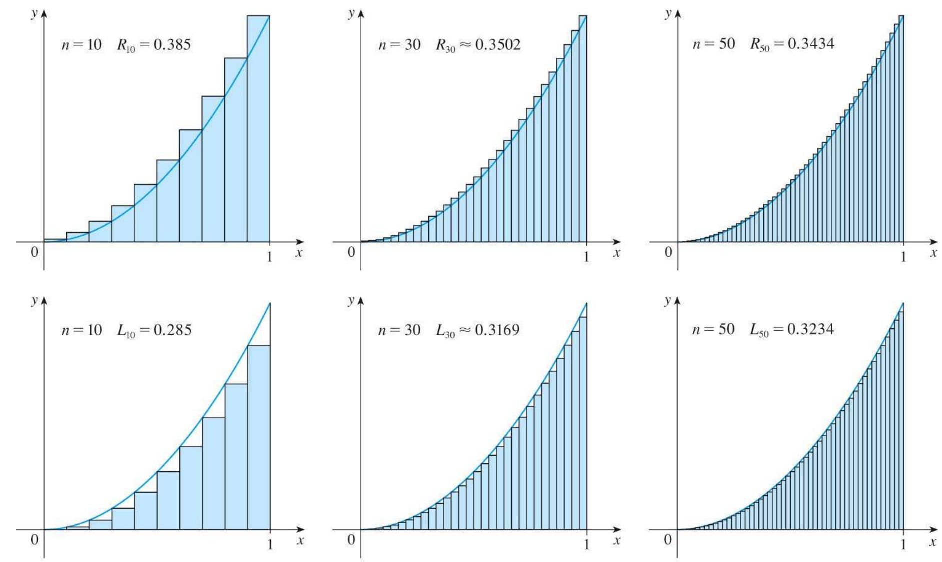




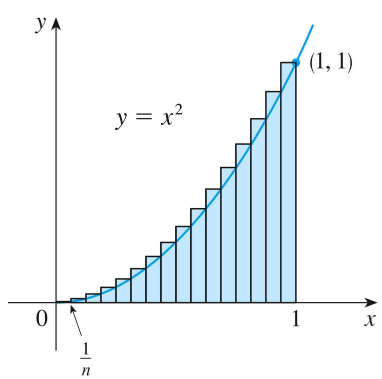
By computing the sum of the areas of the smaller rectangles L_8 and the sum of the areas of the larger rectangles R_8, we obtain better lower and upper estimates for A: $0.2734375 < A < 0.3984375$.

The Area Problem: An Illustration

Repeating this procedure with a even larger number of strips:

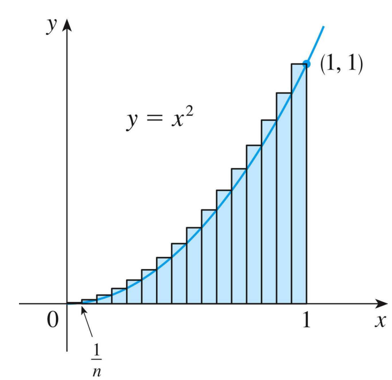


The Area Problem: An Illustration (Exact Calculation)



$$
\begin{aligned}
R_n &= \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{3}{n}\right)^2 + \cdots + + \frac{1}{n}\cdot\left(\frac{n}{n}\right)^2 \\
&= \frac{(n+1)(2n+1)}{6n^2}
\end{aligned}
$$

The Area Problem: An Illustration (Exact Calculation)

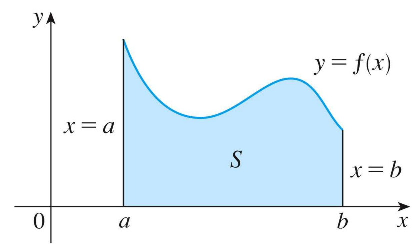


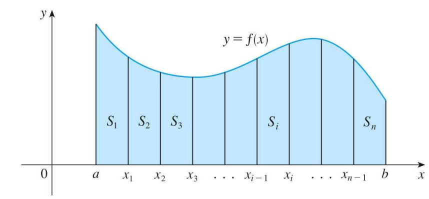
$$
\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{3}
\end{aligned}
$$

Outline

The Area Problem: General Formulation

We will apply the idea of the last example ($y = x^2$) to the more general region S:

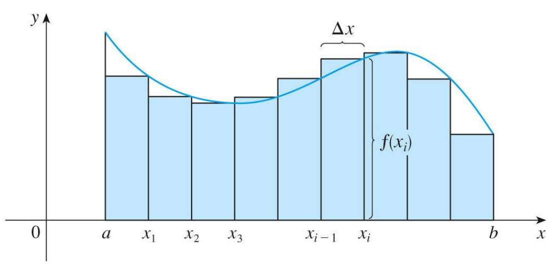




We subdivide S into n strips S_1, S_2, . . ., S_n of equal width.

The Area Problem: General Formulation

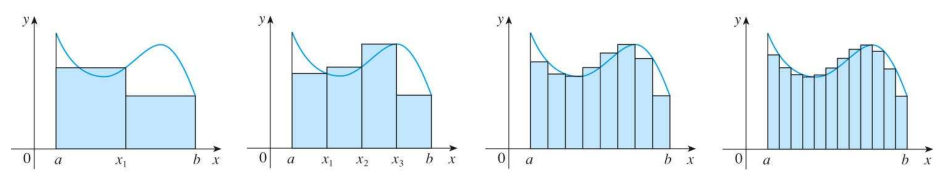
We approximate the i-th strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint.



Then the area of the i-th rectangle is $f(x_i)\,\Delta x$. The area of S is approximated by the sum of the areas of these rectangles:

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

The Area Problem: General Formulation



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

Notice that the approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

Definition

The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

Outline

Graphical Explanation

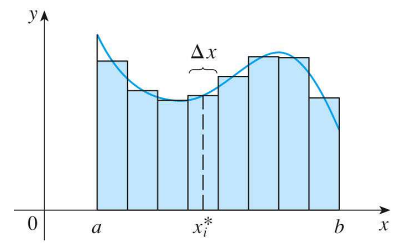


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

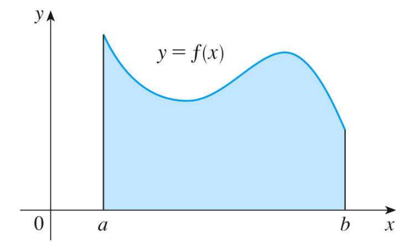
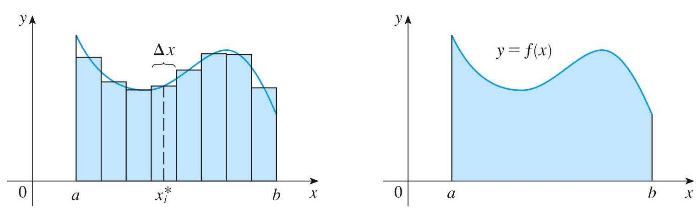


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

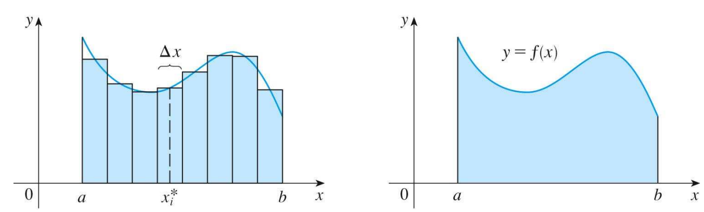
A More Mathematical Explanation



Definition (Slide 1/2)

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a$, x_1, . . ., $x_n = b$ be the endpoints of these subintervals and we let x_1^*, x_2^*, . . ., x_n^* be any sample points in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$.

A More Mathematical Explanation



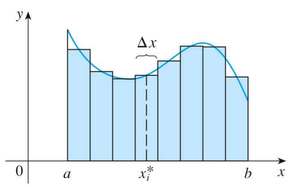
Definition (Slide 2/2)

Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

A More Mathematical Explanation



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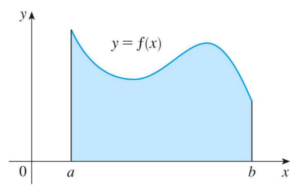


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

- The symbol $\int$ was introduced by Leibniz and is called an integral sign.
- It is an elongated S and was chosen because an integral is a limit of sums.
- $f(x)$ is called the integrand and a and b are called the limits of integration; a is the lower limit and b is the upper limit.

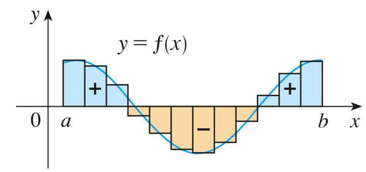


FIGURE 3
$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area.

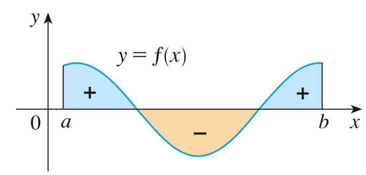


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

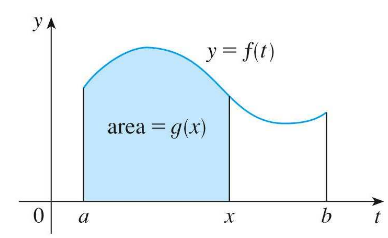
- If $f(x)$ takes on both positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis.
- A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

Outline

Differentiation & Integration as Inverse Processes



The Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is $F' = f$.