Problem Sheet 7

MS121 Semester 2 IT Mathematics

Exercise 1.

Find the tangent lines to the following curves at the indicated points:

(a) $y = \sin(t)$ at $t = 0$,

(b) $y = \sqrt{x}$ at $x = 1$,

(c) $y = \frac{1}{x+1}$ at $x = 2$,

(d) $y = 2^t$ at $t = 3$.

$\oslash$

Solution 1.

(a) $y = \sin(t)$ at $t = 0$ has $y = 0$ and $\frac{dy}{dt}(0) = \cos(0) = 1$, so the tangent line is $y = t$.

(b) $y = \sqrt{x}$ at $x = 1$ has $y = 1$ and $\frac{dy}{dx}(1) = \frac{1}{2} \cdot 1^{-\frac{1}{2}} = \frac{1}{2}$, so the tangent line is $y - 1 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}(x + 1)$.

(c) $y = \frac{1}{x+1}$ at $x = 2$ has $y = \frac{1}{3}$ and $\frac{dy}{dx}(2) = \frac{-1}{(2+1)^2} = -\frac{1}{9}$, so the tangent line is $y - \frac{1}{3} = -\frac{1}{9}(x-2)$ or $y = -\frac{1}{9}x + \frac{5}{9}$.

(d) $y = 2^t$ at $t = 3$ has $y = 2^3 = 8$ and $\frac{dy}{dt}(3) = \ln(2) \cdot 2^3 = 8\ln(2)$, so the tangent line is $y - 8 = 8\ln(2)(x - 3)$ or $y = 8\ln(2)x + 8 - 24\ln(2)$.

$\diamondsuit$

Exercise 2.

Differentiate the following functions with respect to t:

(a) $f(t) = t^\pi$,

(b) $f(t) = t^{\frac{5}{2}}\cos(t)$,

(c) $f(t) = t^2\ln(t)$,

(d) $f(t) = e^t + \frac{5t}{(t+1)^3}$,

(e) $f(t) = \frac{\sqrt{t^2+1}}{3+\sqrt{t}}$,

(f) $f(t) = \sqrt{\frac{t-9}{t^2+7}}$,

(g) $f(t) = \left(t^2 - t^{-2}\right)^{-5}$.

$\oslash$

Solution 2.

(a) $f(t) = t^\pi$ has $f'(t) = \pi t^{\pi-1}$,

(b) $f(t) = t^{\frac{5}{2}}\cos(t)$ has $f'(t) = \frac{5}{2}t^{\frac{3}{2}}\cos(t) - t^{\frac{5}{2}}\sin(t)$,

(c) $f(t) = t^2\ln(t)$ has $f'(t) = 2t\ln(t) + t$ (on $t > 0$),

(d) $f(t) = e^t + \frac{5t}{(t+1)^3}$ has $f'(t) = e^t + \frac{5(t+1)-15t}{(t+1)^4}$,

(e) $f(t) = \frac{\sqrt{t^2+1}}{3+\sqrt{t}}$ has

$$f'(t) = \frac{(3+\sqrt{t})t(t^2+1)^{-\frac{1}{2}} - \frac{1}{2}\sqrt{t^2+1}\,t^{-\frac{1}{2}}}{(3+\sqrt{t})^2},$$

(f) $f(t) = \sqrt{\frac{t-9}{t^2+7}}$ has

$$f'(t) = \frac{1}{2}\sqrt{\frac{t^2+7}{t-9}}\,\frac{(t^2+7)-2(t-9)t}{(t^2+7)^2},$$

(g) $f(t) = \left(t^2 - t^{-2}\right)^{-5}$ has

$$f'(t) = -5\left(t^2 - t^{-2}\right)^{-6}(2t + 2t^{-3}).$$

$\diamondsuit$

Exercise 3.
For the following functions, find all critical points:

(a) $f(x) = x^3 - 3x^2 - 9x$,

(b) $f(x) = x^4 - 2x^3 + 5$,

(c) $f(x) = (x^3 - 27)^8$,

(d) $f(x) = \frac{7x}{x^2+1}$.

$\oslash$

Solution 3.

(a) $f(x) = x^3 - 3x^2 - 9x$ has $f'(x) = 3x^2 - 6x - 9$ and $f''(x) = 6x - 6$. The critical points $f'(x) = 0$ are at $x = -1$ and $x = 3$.

(b) $f(x) = x^4 - 2x^3 + 5$ has $f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$ and $f''(x) = 12x^2 - 12x$. The critical points $f'(x) = 0$ are at $x = 0$ (double) and $x = \frac{3}{2}$.

(c) $f(x) = (x^3 - 27)^8$ has $f'(x) = 8(x^3 - 27)^7 \cdot 3x^2$. The critical points $f'(x) = 0$ are at $x = 0$ (double) and $x = 3$ (7-fold).

(d) $f(x) = \frac{7x}{x^2+1}$ has $f'(x) = \frac{7(x^2+1) - 14x^2}{(x^2+1)^2}$. The critical points $f'(x) = 0$ are at $x = -1$ and $x = +1$.

$\diamondsuit$

Exercise 4.
Assume that the function f is differentiable at $a \in \mathbb{R}$. Argue that f must be continuous at $a \in \mathbb{R}$. Hint: First write down the assumption in terms of limits. Then show that $\lim_{x \to a} f(x) - f(a) = 0$.

$\oslash$

Solution 4.
From

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

we conclude that

$$\begin{aligned}
\lim_{x \to a}(f(x) - f(a)) &= \lim_{x \to a}(x-a)\frac{f(x) - f(a)}{x - a} \\
&= \left(\lim_{x \to a}(x - a)\right) \cdot \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right) \\
&= 0 \cdot f'(a) = 0\ .
\end{aligned}$$

Because $f(a)$ is constant (independent of x) it follows that $\lim_{x \to a} f(x) = f(a)$, which means that f is continuous at $a \in \mathbb{R}$. $\diamondsuit$