Example: Compute the probability of winning the jackpot in the lotto by playing a single grid.

An outcome of the lotto is a subset of 6 numbers from the set $\{1, 2, \ldots, 42\}$; there are

$$\binom{42}{6} = \frac{42!}{6!36!} = \frac{42 \times 41 \times 40 \times 39 \times 38 \times 37}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 5,245,786$$

such subsets. So the probability that a specific grid wins the jackpot is

$$\frac{1}{\binom{42}{6}} = \frac{1}{5,245,786} \simeq 0.19 \times 10^{-6}$$

Example: (where outcomes are not equally likely) Suppose a blue die and a red die are rolled together, and the numbers of dots that occur face up on each are added. What would reasonable probabilities be for the outcomes?

The sample space is $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and the probabilities are $p_2 = 1/36$, $p_3 = 2/36$, $p_4 = 3/36$, $p_5 = 4/36$, $p_6 = 5/36$, $p_7 = 6/36$, $p_8 = 5/36$, $p_9 = 4/36$, $p_{10} = 3/36$, $p_{11} = 2/36$, $p_{12} = 1/36$. We arrive at these by considering the related sample space

$$\Omega' = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

of possible pairs (i, j) where i is the number showing on the blue die and j is the number showing on the red die. If the dice are fair it is reasonable to assume each outcome in Ω' is equally likely and has probability $1/36$. Now for outcomes in the first sample space we add. For example, 6 can arise in 5 ways: $(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$ so that we assign probability $5/36$.

Example: A sample space for the Monty Hall problem could be the set of 27 triples (A, B, C), where $A, B, C \in \{1, 2, 3\}$ and triple (A, B, C) is the outcome that
(a) The prize is behind door A
(b) You picked door B and
(c) The host opened door C, knowing that the prize was behind door A and that you had picked door B.

Now assign probabilities to each triple using these rules:
If C is either A or B (or both) the probability of the triple is 0 (Host will not

open the prize door, nor will he open the door you have chosen)
The probabilities for $(A,B,1)$, $(A,B,2)$ and $(A,B,3)$ should add to $1/9$ (We expect all pairs (prize door, chosen door) to be equally likely)
The probabilities for (A,A,B) and (A,A,C) should be equal (If the host has a choice of two doors to open they should be equally likely)

triple	prob	triple	prob	triple	prob
$(1,1,1)$	0	$(2,1,1)$	0	$(3,1,1)$	0
$(1,1,2)$	1/18	$(2,1,2)$	0	$(3,1,2)$	1/9
$(1,1,3)$	1/18	$(2,1,3)$	1/9	$(3,1,3)$	0
$(1,2,1)$	0	$(2,2,1)$	1/18	$(3,2,1)$	1/9
$(1,2,2)$	0	$(2,2,2)$	0	$(3,2,2)$	0
$(1,2,3)$	1/9	$(2,2,3)$	1/18	$(3,2,3)$	0
$(1,3,1)$	0	$(2,3,1)$	1/9	$(3,3,1)$	1/18
$(1,3,2)$	1/9	$(2,3,2)$	0	$(3,3,2)$	1/18
$(1,3,3)$	0	$(2,3,3)$	0	$(3,3,3)$	0

Now, if E is the event that you win a prize by changing doors then E consists of those triples (A,B,C) with A and B distinct. This event has probability $2/3$.

Proposition: (The addition rule for probabilities) Suppose $A \cap B = \emptyset$. Then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B].$$

Proof: If $A = \{\omega_{j_1}, \omega_{j_2}, \ldots, \omega_{j_k}\}$ and $B = \{\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_s}\}$, then $A \cup B = \{\omega_{j_1}, \omega_{j_2}, \ldots, \omega_{j_k}, \omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_s}\}$ and there is no repetition since $A \cap B = \emptyset$. From the definition

$$\begin{aligned} \mathbb{P}[A \cup B] &= p_{j_1} + \ldots p_{j_k} + p_{r_1} + \ldots + p_{r_s} \\ &= \mathbb{P}[A] + \mathbb{P}[B]. \end{aligned}$$

Corollary: (The complement rule) $\mathbb{P}[\bar{A}] = 1 - \mathbb{P}[A]$.

Proof: Note that $\Omega = A \cup \bar{A}$, $A \cap \bar{A} = \emptyset$ and $\mathbb{P}[\Omega] = 1$, so that

$$1 = \mathbb{P}[\Omega] = \mathbb{P}[A] + \mathbb{P}[\bar{A}]$$

Note: As in the case of the subtraction rule for counting, the complement rule is useful in cases where $\bar{A}$ is a simpler event than A. Then it is better to calculate $\mathbb{P}[\bar{A}]$ first in order to get $\mathbb{P}[A]$.

Example: For example, when a coin is tossed three times and where B is the event 'at least one H comes up', it is clear that $\bar{B}$ is the event 'no H comes up' and $\bar{B} = \{TTT\}$, giving $\mathbb{P}[\bar{B}] = 1/8$ and

$$\mathbb{P}[B] = 1 - \mathbb{P}[\bar{B}] = 1 - 1/8 = 7/8.$$

Example: What is the probability that a four digit PIN has at least one repeated digit?

Here $|\Omega| = 10^4$ (4 samples from 10). Suppose A is the event that there is at least one repeated digit. Then $\bar{A}$ is the event that no digits are repeated (permutations of 4 numbers from 10). Thus $|\bar{A}| =^{10} P_4$ and $\mathbb{P}[\bar{A}] =^{10} P_4/10^4 = 0.504$. Hence $\mathbb{P}[A] = 1 - 0.504 = 0.496$.

Example: The birthday problem. We want to calculate the probability that, in a class of n students, two or more have a common birthday. The sample space is

$$\Omega = \{(d_1, d_2, \ldots, d_n) \mid d_i \text{ integers}, 1 \leq d_i \leq 365\}$$

and the event 'at least one common birthday' is

$$A = \{(d_1, d_2, \ldots, d_n) \in \Omega \mid d_i = d_j \text{ for at least one pair } i \neq j\}$$

Assume that all birthdays are equally likely, so that all outcomes in Ω are equally likely. Then $\mathbb{P}[A] = |A|/|\Omega|$ and the problem reduces to finding the number of elements in Ω and the number of elements in A. We start with Ω. This is just the set of n-samples from 365. (Replacement is used and order is important.) $|\Omega| = (365)^n$ and the probability of any specific outcome is

$$p = \frac{1}{365^n}.$$

Next we look at A. Since this involves single repeated dates, dates repeated multiple times, separate dates repeated etc., we consider the converse event $\bar{A}$. All the outcomes in $\bar{A}$ have no repeated date. So we will compute $\mathbb{P}[\bar{A}]$ and then get $\mathbb{P}[A] = 1 - \mathbb{P}[\bar{A}]$. Now $\mathbb{P}[\bar{A}] = |\bar{A}|/|\Omega|$. However outcomes in $\bar{A}$ are simply permutations of n dates from 365. So

$$|\bar{A}| =^{365} P_n = 365 \times 364 \times \ldots \times (365 - n + 1)$$

and

$$\mathbb{P}[\bar{A}] = \frac{365 \times 364 \times \ldots \times (365 - n + 1)}{365^n} = \frac{365}{365}\frac{364}{365}\frac{363}{365}\cdots\frac{365 - n + 1}{365}.$$

Note: The value for n is obtained from the value for $n - 1$ by multiplying by the fraction $(365 - n + 1)/(365)$. This can be programmed easily enough. Here's a python version:

```
>>> a, b = 1, 365
>>> while b > 340:
...  print a
...  a, b = (a*b)/(365.0), b-1
...
1
1.0
0.997260273973
0.991795834115
0.983644087533
0.9728644263
0.959537516351
0.943764296904
0.925664707648
0.905376166111
0.883051822289
0.858858621678
0.832975211162
0.805589724768
0.776897487995
0.747098680236
0.716395994747
0.684992334703
0.653088582128
0.620881473968
0.588561616419
0.556311664835
0.524304692337
0.492702765676
0.461655742085
```

The variable a is the current probability while b is $365 - n$ so initialise a at 1 and b at 365. I just do 25 steps, hence the $b > 340$ condition. In the loop the probability has to be printed, b has to decrease by 1 and the probability has to be multiplied by $b/365$. The 365.0 forces the output to be a decimal. We see a $50 - 50$ chance of shared birthdays as soon as n reaches 22.