Proposition: The probability of exactly j successes in n Bernoulli trials is

$$\binom{n}{j} p^j (1-p)^{n-j} \quad \text{for} \quad 0 \leq j \leq n$$

where p is the probability of success in any individual trial.

Proof: The sample space Ω is the set of all sequences of length n in the symbols S (success) and F (failure). There are 2^n such sequences, but they are not equally likely; for instance $SS \ldots S$ has probability p^n while $FFS \ldots S$ has probability $p^2(1-p)^{n-2}$. However all sequences with exactly j successes have the same probability, $p^j(1-p)^{n-j}$, irrespective of the location of the j successes in the sequence. So the probability of the event A_j: exactly j successes is $p^j(1-p)^{n-j}|A_j|$.

To count the number of outcomes in A_j we note that such outcomes are completely specified by the location of the j successful trials; for instance

$$SSFSFFFSFFSF$$

has five successes out of twelve, located at trials $1, 2, 4, 8, 11$. But choosing the location of j successes amounts to choosing j numbers out of n, a problem we have seen before. Hence

$$|A_j| = \binom{n}{j},$$

proving the result.

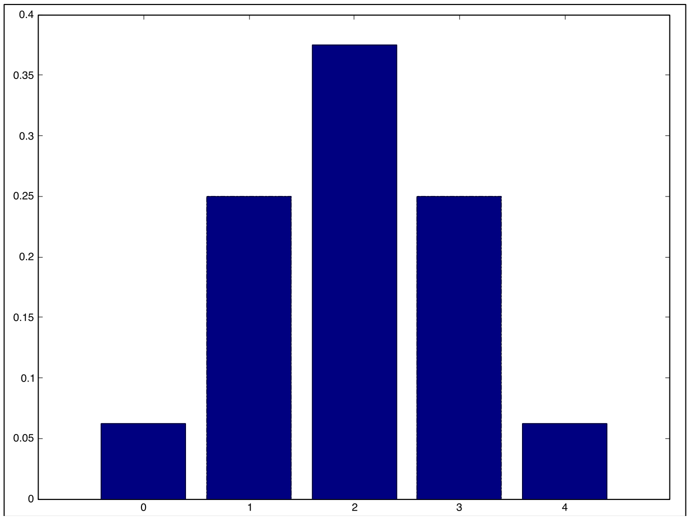
Definition: The assignment of probabilities

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

to the sample points $\{0, 1, 2, 3, \ldots, n\}$ is called the binomial distribution.

Example: Four coins are tossed and the number of heads showing is noted. The sample space is $\{0, 1, 2, 3, 4\}$ and the probabilities are

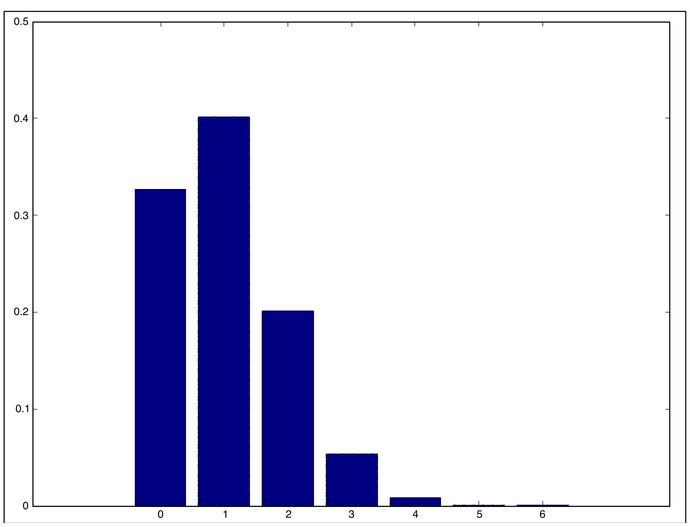
Sample point	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$



Here $p = q = 1/2$ and $n = 4$.

Example: Six dice are tossed and the number of 1's showing is noted. The sample space is $\{0, 1, 2, 3, 4, 5, 6\}$ and the probabilities are

Sample point	0	1	2	3	4	5	6
Probability	$\frac{15625}{46656}$	$\frac{18750}{46656}$	$\frac{9375}{46656}$	$\frac{2500}{46656}$	$\frac{375}{46656}$	$\frac{30}{46656}$	$\frac{30}{46656}$



Here $p = 1/6$, $q = 5/6$ and $n = 6$.

Example: A company produces microchips and one microchip in 1000 is defective. What is the probability that a sample of 200 microchips contains at most two defective chips?

Setting $p = 1/1000 = 0.001$ and $n = 200$ the desired probability is

$$\binom{n}{0} p^0(1-p)^{n-0} + \binom{n}{1} p^1(1-p)^{n-1} + \binom{n}{2} p^2(1-p)^{n-2}$$

since the sample will have at most 2 defective items if it has 0, 1 or 2 defective items. Computing this quantity gives approximately 0.99887.

Question 3 from last year's repeat paper:

QUESTION 3

(a) A committee of 4 is to be chosen from a group of 10 people.
(i) In how many ways can this be done?
(ii) Suppose 5 of the group are women and 5 are men. How many of the committees have at least one woman and at least one man? Explain your answer.

[8 marks]

(b) A fair die (6-sided) is rolled 30 times. Use the Binomial distribution to calculate the probability that the number of times side 1 shows is two or less.

[9 marks]

(b) The probability that side 1 shows on any particular roll of the die is 1/6. The probability that side 1 does not show on any particular roll is 5/6. The probability that side 1 shows twice or less in 30 rolls of the die is

$$\mathbb{P}(0;30,1/6) + \mathbb{P}(1;30,1/6) + \mathbb{P}(2;30,1/6)$$

which we compute using $\mathbb{P}(k;n,p) = \binom{n}{k} p^k(1-p)^{n-k}$ to be

$$\begin{aligned} & \binom{30}{0}(1/6)^0(5/6)^{30} + \binom{30}{1}(1/6)^1(5/6)^{29} + \binom{30}{2}(1/6)^2(5/6)^{28} \\ = \;& (1)(1)(5/6)^{30} + (30)(1/6)(5/6)^{29} + (15)(29)(1/6)^2(5/6)^{28} \\ \approx \;& 0.103 \end{aligned}$$

(a) (i) Choose 4 from 10 in $\binom{10}{4} = 210$ ways.

(ii) The number of committees which do not have at least one woman and at least one man is the number with no women or no men. This number is obtained by choosing all men or all women giving

$$\binom{5}{4}\binom{5}{0} + \binom{5}{0}\binom{5}{4} = 5 + 5 = 10.$$

Thus the number we want is obtained using the subtraction law to be
$210 - 10 = 200$.