Definition: A relation R on a set A is called transitive if whenever we have $(a, b) \in R$ and $(b, c) \in R$ we also have $(a, c) \in R$.

Note: If, in the digraph of a transitive relation R, one can get from a to b following arrows then there must be an arrow from a to b. (If we see two edges following each other the third side of the directed triangle must also be present.)

Examples: The R in Example 1 is not transitive. $(3, 2) \in R$ and $(2, 1) \in R$ but $(3, 1) \notin R$. Here $a = 3$, $b = 2$ and $c = 1$. Note that we also have $(3, 2) \in R$ and $(2, 3) \in R$ but $(3, 3) \notin R$. (a and c could be equal.) The R in Example 2 is not transitive. If I link to your webpage, I do not necessarily link to the crazy sites that you link to. The R in Example 3 is transitive. If $(a, b) \in R$ and $(b, c) \in R$ then $b = ka$ and $c = lb$ for some positive integers k and l. However this means $c = lb = l(ka) = (lk)a$ so that $(a, c) \in R$. The R in Example 4 is transitive. If $(x, y) \in R$ and $(y, z) \in R$ then $y-x = 2k$ and $z-y = 2l$ for some integers k and l. However this gives $z-x = z-y+(y-x) = 2l+2k = 2(l+k)$ and $(x, z) \in R$. The R in Example 5 is transitive. If student A is in the same programme as student B and student B is in the same programme as student C then all three are in the same programme and student A is in the same programme as student C.

Definition: If A is a set and R is a relation on A which is not reflexive we define the reflexive closure of R to be the relation consisting of R together with $\{(a, a) \mid a \in A\}$. (Recall that this is just a union of sets.)

Definition: If A is a set and R is a relation on A which is not symmetric we define the symmetric closure of R to be the relation consisting of R together with $\{(a, b) \mid (b, a) \in R\}$.

Definition: If A is a set and R is a relation on A which is not transitive we define the transitive closure of R to be the relation R^* on A defined by aR^*b if and only if there is a sequence of elements of A, $a_1, a_2, \ldots, a_k$ with $a = a_1$, $b = a_k$ and a_iRa_{i+1}.

Example: Transitive closures of relations are the most applicable. If A is a set of websites and R is the relation define by 'is linked to', then the transitive closure of R is the relation 'is connected to by a sequence of links'.

Example: If $A = \{1, 2, 3\}$ and R is the relation $\{(1, 3), (2, 1), (2, 2), (2, 3), (3, 2)\}$, then R is not reflexive, not symmetric and not transitive. What are the corresponding closures?

The reflexive closure is

$$\{(1,1),(1,3),(2,1),(2,2),(2,3),(3,2),(3,3)\}$$

Here we had to add $(1,1)$ and $(3,3)$ to get $(a,a) \in R^*$ for all $a \in A$.

The symmetric closure is

$$\{(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2)\}$$

Here we had to add $(1,2)$ and $(3,1)$ to get $(b,a) \in R^*$ for all $(a,b) \in R$.

The transitive closure is

$$\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),\}$$

Here the construction is more complicated. We see that $1R3$ and $3R2$ so that transitivity of R^* will force $(1,2) \in R^*$. However $1R^*2$ and $2R^*1$ will force $(1,1) \in R^*$. Thus 1 is related to every element of A by R^*. Continue in this way for 2 and 3.