The Principle of Mathematical Induction: Let P(n) be a predicate that is defined for all integers $n \geq 1$. Suppose that

1. P(1) is true, and
2. $\forall k \geq 1$, (P(k) $\Rightarrow$ P($k+1$)) is true.

Then P(n) is true for all $n \geq 1$.

The reasoning is that P(1) is true by 1 and, by 2 applied repeatedly, P(2) is true, P(3) is true, P(4) is true, etc.

Example: Show that, for all integers $n \geq 1$:

$$\mathrm{P}(n) : 1 + 3 + 5 + \ldots + (2n-1) = n^2.$$

Base case: ($n = 1$) LHS is $1 = 1$ while RHS is $(1)^2 = 1$ also.

Inductive step: Prove (P(k) $\Rightarrow$ P($k+1$)). So assume P(k) is true, that is

$$1 + 3 + \ldots + (2k-1) = k^2$$

and prove that P($k+1$)) follows. Note that P($k+1$)) states

$$1 + 3 + \ldots + (2k-1) + (2(k+1)-1) = (k+1)^2$$

and the LHS of P($k+1$)) contains the LHS of P(k)). This is the key to this proof.

$$\begin{aligned} 1 + 3 + \ldots + (2k-1) + (2(k+1)-1) &= k^2 + (2(k+1)-1) \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

However this is the RHS of P($k+1$).

Note: In order to prove the inductive step, we must find some relationship between the statement $P(k)$ and the statement $P(k+1)$. You must use $P(k)$ in your proof of $P(k+1)$. Otherwise it is not an inductive proof.

Example: If a is a positive real number with $a \neq 1$ then

$$1 + a + a^2 + a^3 + \ldots + a^{n-1} = \frac{1-a^n}{1-a}.$$

Proof: Base case: $n = 1$, LHS is 1 (only one term) while RHS is $(1-a)/(1-a) = 1$.

Inductive step: Assume $P(k)$, that is,

$$1 + a + a^2 + a^3 + \ldots + a^{k-1} = \frac{1 - a^k}{1 - a}.$$

and try to deduce $P(k+1)$, that is

$$1 + a + a^2 + a^3 + \ldots + a^{k-1} + a^k = \frac{1 - a^{k+1}}{1 - a}.$$

Here we notice again that the LHS of $P(k+1)$ contains the LHS of $P(k)$ and we can substitute for this the RHS of $P(k)$. Thus

$$\begin{aligned} 1 + a + a^2 + a^3 + \ldots + a^{k-1} + a^k &= (1 + a + a^2 + a^3 + \ldots + a^{k-1}) + a^k \\ &= \frac{1 - a^k}{1 - a} + a^k \\ &= \frac{1 - a^k + a^k(1 - a)}{1 - a} \\ &= \frac{1 - a^{k+1}}{1 - a} \end{aligned}$$

which is the RHS of $P(k+1)$.

Note: In the next example, we define a sequence of numbers. Rather than call them

the first number, the second number , the third number, ...

we will write

x_1, x_2, x_3, ...

where x denotes a number and the subscript tells us where in the sequence the number is. This allows to write

'each term in the sequence is obtained by dividing the previous term by the previous term plus three'

as

$$x_{k+1} = \frac{x_k}{x_k + 3}$$

but we have to be careful to distinguish between x_{k+1} which is the value of the term immediately after the kth term and $x_k + 1$ which is the sum of the value of the kth term and the number 1.

Example: If a sequence of numbers $x_1, x_2, \ldots, x_n$ is defined recursively by

$$x_1 = 1 \text{ and } \; x_{k+1} = \frac{x_k}{x_k + 3}$$

compute the first 4 terms and prove, by induction, that

$$x_n = \frac{2}{3^n - 1}$$

For the first part we compute

$$\begin{aligned}
x_1 &= 1 \\
x_2 &= \frac{x_1}{x_1 + 3} \\
&= \frac{1}{4} \\
x_3 &= \frac{x_2}{x_2 + 3} \\
&= \frac{1/4}{1/4 + 3} \\
&= \frac{1}{13} \\
x_4 &= \frac{x_3}{x_3 + 3} \\
&= \frac{1/13}{1/13 + 3} \\
&= \frac{1}{40}
\end{aligned}$$

Base step: $x_1 = 1$ and, from the formula $x_n = 2/(3^n - 1)$ with $n = 1$ we get $2/(3^1 - 1) = 2/2 = 1$ also.

Inductive step: We assume

$$\mathrm{P}(k) : x_k = 2/(3^k - 1)$$

and we want to prove

$$\mathrm{P}(k+1) : x_{k+1} = 2/(3^{k+1} - 1)$$

using the recursion $x_{k+1} = x_k/(x_k + 3)$.

$$
\begin{aligned}
x_{k+1} &= \frac{x_k}{x_k+3} \\
&= \frac{2/(3^k-1)}{2/(3^k-1)+3} \text{ (by P}(k)) \\
&= \frac{2}{2+3(3^k-1)} \text{ (multiplying above and below by } 3^k-1) \\
&= \frac{2}{2+3^{k+1}-3} \\
&= \frac{2}{3^{k+1}-1}
\end{aligned}
$$

Note: If the manipulation of fractions causes concern do it slowly

$$
\frac{2/(3^k-1)}{2/(3^k-1)+3} = \frac{2/(3^k-1)}{(2+3(3^k-1))/(3^k-1)} = \frac{2}{3^k-1} \cdot \frac{3^k-1}{2+3(3^k-1)}
$$

where we have used the rule

$$
\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}
$$