Proposition: The number of r element subsets of a set with $n \geq r$ elements is $\left(\begin{array}{c} n \\ r \end{array} \right)$.

Proof: Each such subset arises when we pick a first element followed by a second element up to an rth element. The number of such choices is ${}^nP_r$. But this process counts each subset $r!$ times, one for each permutation of the subset.

Example: A committee of 4 is to be chosen from a group of 10 people. In how many ways can this be done? If there are 6 men and 4 women in the group, how many of the possible committees will have 2 men and 2 women? No women? Suppose two of the group refuse to serve on a committee together. How many committees are now possible?

The first number is $\left(\begin{array}{c} 10 \\ 4 \end{array} \right) = (10)(9)(8)(7)/(4)(3)(2) = (10)(3)(7) = 210$. The second situation involves choosing 2 of the 6 men **and** 2 of the four women. The product principle applies to give the number as

$$\left(\begin{array}{c} 6 \\ 2 \end{array} \right) \left(\begin{array}{c} 4 \\ 2 \end{array} \right) = \frac{(6)(5)}{(2)(1)} \frac{(4)(3)}{(2)(1)} = (15)(6) = 90.$$

If there are to be no women on the committee we must choose 4 from the 6 men giving $\left(\begin{array}{c} 6 \\ 4 \end{array} \right) = (6)(5)(4)(3)/(4)(3)(2)(1) = 15$. In the last part it is easier to count the number of committees with the disagreeable two and apply the subtraction principle to get

$$\text{number} = \left(\begin{array}{c} 10 \\ 4 \end{array} \right) - \left(\begin{array}{c} 2 \\ 2 \end{array} \right) \left(\begin{array}{c} 8 \\ 2 \end{array} \right) = 210 - 28 = 182.$$

Here the number of committees including the difficult pair are chosen by picking those 2 in 1 way and picking another 2 from the remaining 8.

Theorem: (Binomial Theorem) The coefficient of $a^r b^{n-r}$ in the expansion of $(a+b)^n$ is $\left(\begin{array}{c} n \\ r \end{array} \right)$.

Example: For $n = 6$ we get

$$(a+b)^6 = (1)a^6 + (6)a^5 b + (15)a^4 b^2 + (20)a^3 b^3 + (15)a^2 b^4 + (6)ab^5 + (1)b^6$$

For example the 15 terms involving a^4b^2 are

$$aaaabb, aaabab, aaabba, aabaab, aababa,$$

$$aabbaa, abaaab, abaaba, ababaa, abbaaa,$$

$$baaaab, baaaba, baabaa, babaaa, bbaaaa,$$

one for each choice of 4 places from 6 for the a's.

Proof: (of Binomial Theorem) Expand $(a+b)^n$ into 2^n terms where we keep track of the order the terms, that is, do not replace ba by ab. Now collect terms. The number contributing to $a^r b^{n-r}$ is equal to the number of ways of picking r places from n for the a's, and thus is equal to $\begin{pmatrix} n \\ r \end{pmatrix}$.

Example: The coefficient of x^5 in the expansion of $(1+x)^8$ is $\begin{pmatrix} 8 \\ 5 \end{pmatrix} = 56$.

Example: The coefficient of x^5 in the expansion of $(2+x)^8$ is $2^3 \begin{pmatrix} 8 \\ 5 \end{pmatrix} = 448$.

Pascal's Triangle. Pascal's Triangle is the name given to an arrangement of all the binomial coefficients in a triangular pattern with the numbers $\begin{pmatrix} n \\ r \end{pmatrix}$ centred on the nth row in order of increasing r from left to right.

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & & \vdots & & & \vdots & & & \vdots & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & \vdots & & & \vdots & & & \vdots & &
\end{array}
$$