MS121: IT Mathematics

Limits & Continuity

Limits at Infinity

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Outline

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Limits as $x \to \infty$

Introduction

- So far, we have only considered limits as $x \to c$ where c is some finite value.
- We now examine what happens to functions as x becomes infinitely large, i.e. as $x \to \infty$.

Limits as $x \to \infty$

Simple Illustration

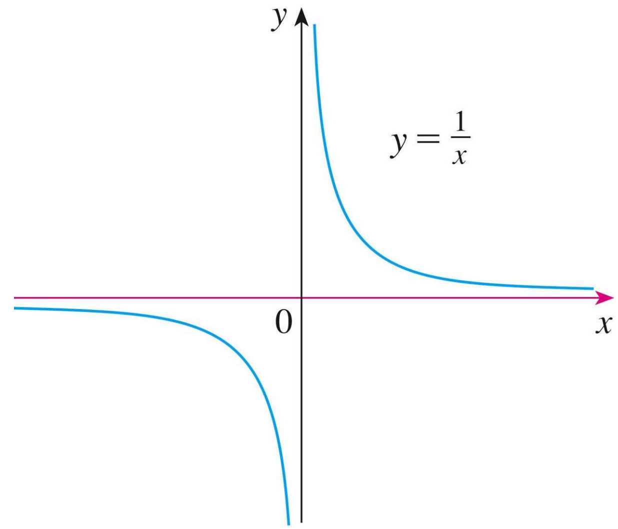
One example to consider first is $f(x) = \frac{1}{x}$.

x-value	$f(x)$-value
10	0.1
100	0.01
1000	0.001
1000000	0.000001

As x becomes infinitely large, then $\frac{1}{x}$ becomes smaller and smaller, and we write:

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \qquad \lim_{x \to \infty} \frac{1}{x} = 0$$



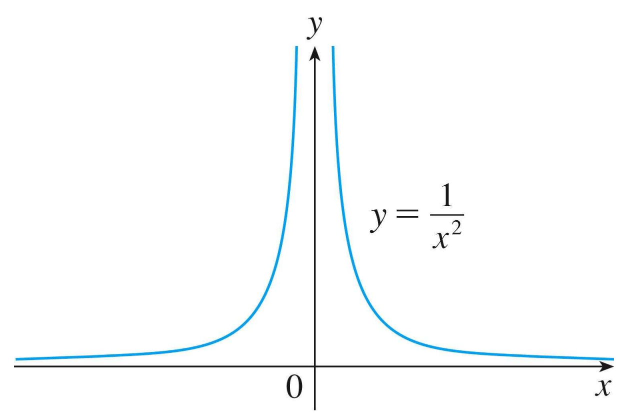
Limits as $x \to \infty$

A Second (Comparative) Illustration

- Consider the evaluation of $\lim_{x \to \infty} \frac{1}{x^2}$.
- As x becomes large, then $\frac{1}{x}$ becomes small but x^2 is even larger than x and so $\frac{1}{x^2}$ is even smaller than $\frac{1}{x}$.
- Hence, we conclude that

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x^2} = 0 \qquad \lim_{x \to \infty} \frac{1}{x^2} = 0$$



Limits as $x \to \infty$

More Generally

- The more general result
$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$
holds for any $n > 0$.
- Note that n can be less than 1, for example $n = \frac{1}{2}$ when we may write
$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Limits as $x \to \infty$

Making Comparisons

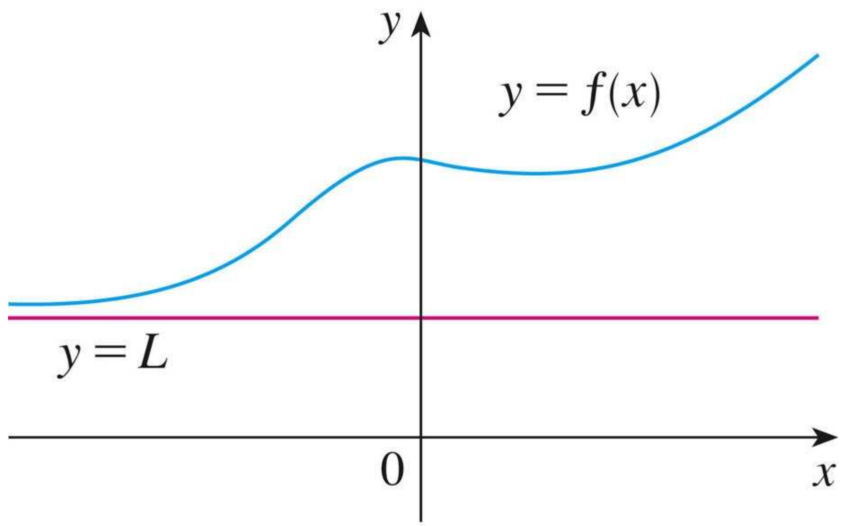
- If, instead, we require $\lim_{x\to\infty} \frac{1}{x-1}$, we can reason as follows:
- If x is infinitely large, then $x - 1$ is also infinitely large, and dividing by an infinitely large number produces an infinitely small number, and so that the limit must be zero.
- We could also proceed as follows: divide above and below by x:
$$\frac{1}{x-1} = \frac{\frac{1}{x}}{1 - \frac{1}{x}}$$
- Now, take the limit as $x \to \infty$:
$$\lim_{x\to\infty} \frac{1}{x-1} = \frac{\lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x}} = \frac{0}{1-0} = 0$$

Outline

Finite Limits as $x \to \pm\infty$

- The function f has a real limit L as x tends to ∞ if, however small a distance we choose, $f(x)$ gets closer than this distance to L and stays closer, no matter how large x becomes and we write
$$\lim_{x\to\infty} f(x) = L \text{ or } f(x) \to L, \text{ as } x \to \infty$$
- The function f has a real limit L as x tends to $-\infty$ if, however small a distance we choose, $f(x)$ gets closer than this distance to L and stays closer, no matter how large x and negative becomes and we write
$$\lim_{x\to-\infty} f(x) = L \text{ or } f(x) \to L, \text{ as } x \to -\infty$$

$\lim_{x \to -\infty} f(x) = L$



Rules for Limits as $x \to \pm\infty$

If L, M and k are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L, \qquad \lim_{x \to \pm\infty} g(x) = M \qquad \text{then}$$

(i) Sum Rule: $\lim_{x \to \pm\infty} f(x) + g(x) = L + M.$

(ii) Difference Rule: $\lim_{x \to \pm\infty} f(x) - g(x) = L - M.$

(iii) Product Rule: $\lim_{x \to \pm\infty} f(x)g(x) = LM.$

(iv) Constant Multiple Rule: $\lim_{x \to \pm\infty} kf(x) = kL.$

(v) Quotient Rule: If $M \neq 0$, then $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}.$

(vi) Power Rule: If r and s are integers with no common factors and $s \neq 0$, then $\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$, provided that $L^{r/s}$ is a real number.

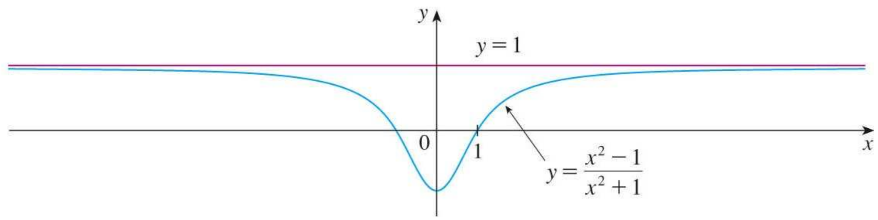
Outline

Rational Functions:Limits as $x \to \pm\infty$

Simple Rational Function

- Let $f(x) = x^2 - 1$ and $g(x) = x^2 + 1$.
- Therefore,
$$\lim_{x \to \infty} f(x) = \infty = \lim_{x \to \infty} g(x)$$
- Set
$$h(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 1}{x^2 + 1}$$
- What is
$$\lim_{x \to \infty} h(x)?$$

Plot of $\dfrac{x^2-1}{x^2+1}$



$\displaystyle\lim_{x\to\infty} \frac{x^2-1}{x^2+1}$

Rational Functions

- As we are assuming x is large (and hence non-zero), we can divide through by x^2 (the highest power of x occurring in the denominator) to get:

$$h(x) = \frac{x^2-1}{x^2+1} = \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}$$

- As $x \to \infty$, $\dfrac{1}{x^2} \to 0$ and $1 \to 1$.
- Therefore

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1}{1} = 1$$

Outline

Problem Solving

General Approach

We will evaluate some limits at infinity in the following examples by dividing above and below by the highest power of x in the original expression.

Example 1

To evaluate

$$\lim_{x\to\infty} \frac{1}{x^2-4}$$

we divide above and below by x^2 and take limits as follows:

$$\lim_{x\to\infty} \frac{1}{x^2-4} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{1-\frac{4}{x^2}} = \frac{0}{1-0} = 0$$

Example 2

To evaluate

$$\lim_{x\to\infty} \frac{x}{x^2-4}$$

we again divide above and below by x^2 and take limits as follows:

$$\lim_{x\to\infty} \frac{x}{x^2-4} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{4}{x^2}} = \frac{0}{1-0} = 0$$

Note

Although the numerator x becomes infinitely large, the denominator x^2-4 was still infinitely larger than the numerator and so the overall ratio was zero in the limit.

Example 3

Consider

$$\lim_{x\to\infty} \frac{x^3+27x^2+1}{x^4+6}$$

Solution

We divide above and below by the highest power which, in this case, is x^4:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3+27x^2+1}{x^4+6} &= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{27}{x^2}+\frac{1}{x^4}}{1+\frac{6}{x^4}} \\ &= \frac{0+0+0}{1+0} \\ &= 0\end{aligned}$$

Example 4

Consider

$$\lim_{x\to\infty} \frac{2x^3+1}{3x^3+4x^2+2}$$

Solution

We divide above and below by the highest power which, in this case, is x^3, to obtain:

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^3+1}{3x^3+4x^2+2} &= \lim_{x\to\infty} \frac{2+\frac{1}{x^3}}{3+\frac{4}{x}+\frac{2}{x^3}} \\ &= \frac{2+0}{3+0+0} \\ &= \frac{2}{3}\end{aligned}$$

Example 5

To evaluate

$$\lim_{x\to\infty} \frac{2x+4x^2+x^5}{1+x^4}$$

note that the highest power is x^5 and so we obtain:

$$\begin{aligned}\lim_{x\to\infty} \frac{2x+4x^2+x^5}{1+x^4} &= \lim_{x\to\infty} \frac{\frac{2}{x^4}+\frac{4}{x^3}+1}{\frac{1}{x^5}+\frac{1}{x}} \\ &= \frac{0+0+1}{0+0} \\ &= \frac{1}{0} = \infty\end{aligned}$$

Example 6

Consider

$$\lim_{x\to\infty} \frac{\sqrt{x}+1}{x}$$

Solution

We divide above and below by the highest power, namely x, to obtain:

$$\lim_{x\to\infty} \frac{\sqrt{x}+1}{x} = \lim_{x\to\infty} \frac{\frac{1}{x^{\frac{1}{2}}}+\frac{1}{x}}{1} = \frac{0+0}{1} = 0$$

Note that

$$\frac{\sqrt{x}}{x} = \frac{x^{\frac{1}{2}}}{x} = \frac{1}{x^{\frac{1}{2}}}$$

Example 7

Consider

$$\lim_{x\to\infty} \frac{x^{\frac{5}{2}}+x}{3x^{\frac{5}{2}}+2x^2+1}$$

Solution

We divide above and below by the highest power, $x^{\frac{5}{2}}$, to obtain:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^{\frac{5}{2}}+x}{3x^{\frac{5}{2}}+2x^2+1} &= \lim_{x\to\infty} \frac{1+\frac{1}{x^{\frac{3}{2}}}}{3+\frac{2}{x^{\frac{1}{2}}}+\frac{1}{x^{\frac{5}{2}}}} \\ &= \frac{1+0}{3+0+0} = \frac{1}{3}\end{aligned}$$

Example 8

Consider

$$\lim_{x\to\infty} \frac{|x|}{x}$$

Solution

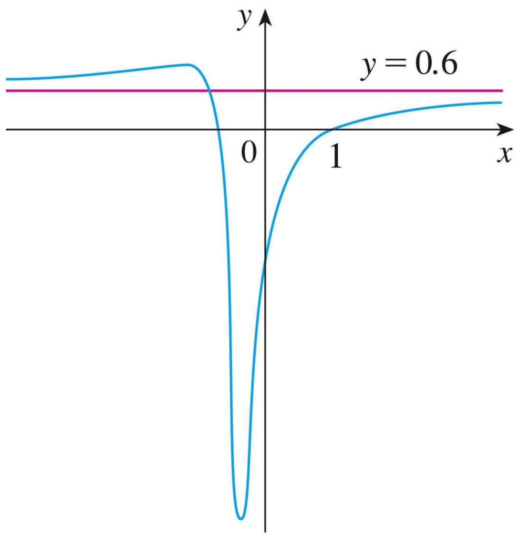
As $x \to +\infty$, then certainly $x > 0$ and, when $x > 0$, we have

$$\frac{|x|}{x} = 1$$

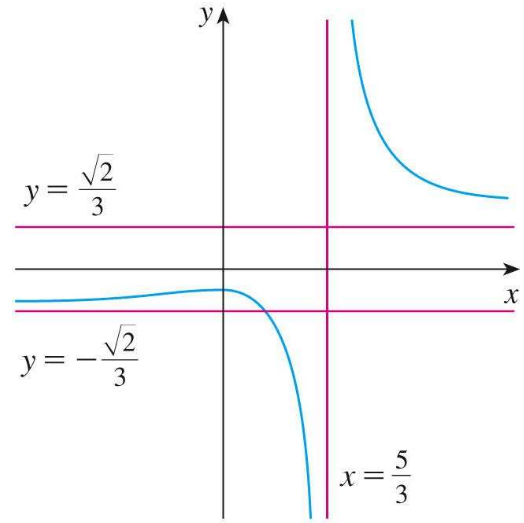
Hence, the limit which we require must be the limit of the constant value 1, i.e.

$$\lim_{x\to\infty} \frac{|x|}{x} = \lim_{x\to\infty} 1 = 1$$

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

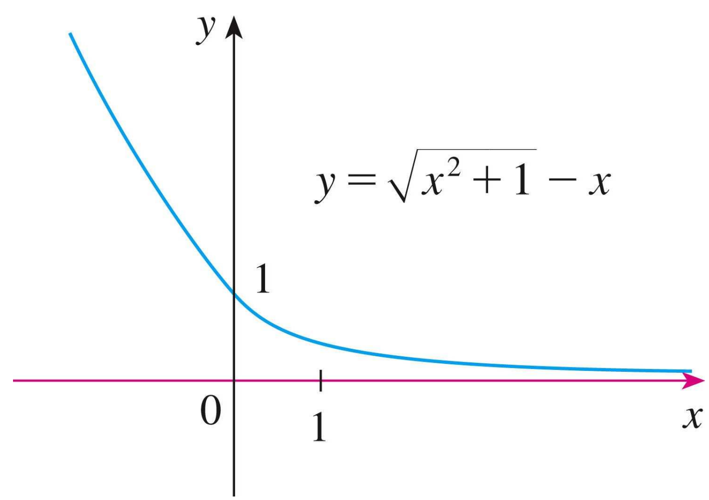


$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x - 5} = \frac{\sqrt{2}}{3}$$



Outline

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = 0$$



Show that $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=0$

Rationalize the Numerator

$$
\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\star\left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right]\\
&= \lim_{x\to\infty}\frac{\left(x^2+1\right)-x^2}{\sqrt{x^2+1}+x}\\
&= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}\\
&= 0
\end{aligned}
$$