Problem Sheet 10

MS121 Semester 2 IT Mathematics

Exercise 1.

Use an integration by parts to evaluate the following integrals:

(a) $\int xe^{-x}\ \mathrm{d}x$,

(b) $\int t\sin(3t)\ \mathrm{d}t$,

(c) $\int x\ln(x)\ \mathrm{d}x$,

(d) $\int_0^1 (s^2-1)e^s\ \mathrm{d}s$,

(e) $\int_1^2 (x^3-2x)\ln(x)\ \mathrm{d}x$,

$\oslash$

Solution 1.

(a) $\int xe^{-x}\ \mathrm{d}x = x(-e^{-x}) - \int -e^{-x}\mathrm{d}x = -xe^{-x} - e^{-x}$,

(b) $\int t\sin(3t)\ \mathrm{d}t = t(-\frac{1}{3}\cos(3t)) - \int -\frac{1}{3}\cos(3t)\mathrm{d}t = -\frac{1}{3}t\cos(3t) + \frac{1}{9}\sin(3t)$,

(c) $\int x\ln(x)\ \mathrm{d}x = (\frac{1}{2}x^2)\ln(x) - \int \frac{1}{2}x^2\cdot x^{-1}\ \mathrm{d}x = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2$,

(d) We integrate by parts twice to find:

$$\begin{aligned}\int_0^1 (s^2-1)e^s\ \mathrm{d}s &= \left[(s^2-1)e^s\right]_0^1 - \int_0^1 2se^s\ \mathrm{d}s\\ &= 1 - [2se^s]_0^1 + \int_0^1 2e^s\ \mathrm{d}s\\ &= 1 - 2e + [2e^s]_0^1 = 1 - 2e + 2e - 2 = -1,\end{aligned}$$

(e) $\int_1^2 (x^3-2x)\ln(x)\ \mathrm{d}x = \left[(\frac{1}{4}x^4 - x^2)\ln(x)\right]_1^2 - \int_1^2 (\frac{1}{4}x^4 - x^2)x^{-1}\ \mathrm{d}x = 0 - \left[\frac{1}{16}x^4 - \frac{1}{2}x^2\right]_1^2 = -(1-2) + (\frac{1}{16} - \frac{1}{2}) = \frac{9}{16}$,

$\diamondsuit$

Exercise 2.

Use a substitution to evaluate the following integrals:

(a) $\int \frac{\ln(x)}{x}\ \mathrm{d}x$,

(b) $\int t\sqrt{t-5}\ \mathrm{d}t$,

(c) $\int_0^{\pi^2} \frac{\cos(\sqrt{x})}{\sqrt{x}}\ \mathrm{d}x$,

(d) $\int_0^1 \frac{x^3}{x^4+5}\ \mathrm{d}x$,

(e) $\int_0^1 te^{-t^2}\mathrm{d}t$,

(f) $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{\sin(t)}\cos(t)\mathrm{d}t$.

$\oslash$

Solution 2.

(a) $\int \frac{\ln(x)}{x}\ \mathrm{d}x$ can be computed with

$$y = \ln(x), \quad x = e^y, \quad \frac{\mathrm{d}x}{\mathrm{d}y} = e^y$$

as

$$\int \frac{y}{e^y}\ e^y\mathrm{d}y = \int y\mathrm{d}y = \frac{1}{2}y^2 = \frac{1}{2}(\ln(x))^2.$$

(b) $\int\ t\sqrt{t-5}\ \mathrm{d}t$ can be computed with

$$s = t-5, \quad t = s+5, \quad \frac{\mathrm{d}t}{\mathrm{d}s} = 1$$

as

$$\int\ (s+5)\sqrt{s}\ \mathrm{d}s = \int\ s^{\frac{3}{2}} + 5s^{\frac{1}{2}}\ \mathrm{d}s = \frac{2}{5}s^{\frac{5}{2}} + 5\cdot\frac{2}{3}s^{\frac{3}{2}} = \frac{2}{5}(t-5)^{\frac{5}{2}} + 3\frac{1}{3}(t-5)^{\frac{3}{2}}.$$

(c) $\int_0^{\pi^2}\ \frac{\cos(\sqrt{x})}{\sqrt{x}}\ \mathrm{d}x$ can be computed with

$$y = \sqrt{x}, \quad x = y^2, \quad \frac{\mathrm{d}x}{\mathrm{d}y} = 2y$$

as

$$\int_0^{\pi}\ \frac{\cos(y)}{y}\ 2y\mathrm{d}y = \int_0^{\pi}\ 2\cos(y)\ \mathrm{d}y = [2\sin(y)]_0^{\pi} = 0.$$

(d) $\int_0^1\ \frac{x^3}{x^4+5}\ \mathrm{d}x$ can be computed with

$$y = x^4+5, \quad x = \sqrt[4]{y-5}, \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{4}(y-5)^{-\frac{3}{4}} = \frac{1}{4}x^{-3}$$

as

$$\int_5^6\ \frac{(y-5)^{\frac{3}{4}}}{y}\ \frac{1}{4}(y-5)^{-\frac{3}{4}}\mathrm{d}y = \frac{1}{4}\int_5^6\ y^{-1}\ \mathrm{d}y = \frac{1}{4}\left[\ln(y)\right]_5^6 = \frac{1}{4}\ln\left(\frac{6}{5}\right).$$

(e) $\int_0^1\ te^{-t^2}\mathrm{d}t$ can be computed with

$$s = -t^2, \quad t = \sqrt{-s}, \quad \frac{\mathrm{d}t}{\mathrm{d}s} = -\frac{1}{2}(-s)^{-\frac{1}{2}}$$

as

$$\int_0^{-1}\ \sqrt{-s}e^s\ \left(-\frac{1}{2}\right)(-s)^{-\frac{1}{2}}\mathrm{d}s = \frac{1}{2}\int_{-1}^0\ e^s\ \mathrm{d}s = \frac{1}{2}\left[e^s\right]_{-1}^0 = \frac{1}{2}(1-e^{-1}).$$

(f) $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\ e^{\sin(t)}\cos(t)\mathrm{d}t$ can be computed with

$$x = \sin(t), \quad t = \sin^{-1}(x), \quad \frac{\mathrm{d}t}{\mathrm{d}x} = (1-x^2)^{-\frac{1}{2}}$$

as

$$\int_{-1}^1\ e^x\sqrt{1-x^2}\ (1-x^2)^{-\frac{1}{2}}\mathrm{d}x = [e^x]_{-1}^1 = e - e^{-1}.$$

$\diamondsuit$

Exercise 3.

Find the following areas:

(a) the area above the x-axis and below the curve $y = 9 - x^2$,

(b) the area between the curves $y = x^2$ and $y = x$,

(c) the area above the curve $y = (x-3)^2$ and below the curve $y = 9 + 2x - x^2$,

(d) the area above the line $y = 1$ and below the curve $y = \frac{6x}{x^2+5}$.

$\oslash$

Solution 3.

(a) the parabola intersects the x-axis at $x = -3$ and $x = 3$, so the desired area is

$$\int_{-3}^{3} 9 - x^2 \, \mathrm{d}x = \left[9x - \frac{1}{3}x^3\right]_{-3}^{3} = (27-9) - (-27+9) = 36,$$

(b) noting that the curves $y = x^2$ and $y = x$ intersect at $x = 0$ and at $x = 1$, and that $x > x^2$ in between, we find the area

$$\int_0^1 x - x^2 \, \mathrm{d}x = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

(c) the curves intersect where $0 = (9 + 2x - x^2) - (x-3)^2 = -2x^2 + 8x$, i.e. at $x = 0$ and $x = 4$, and in between we have $9 + 2x - x^2 > (x-3)^2$, so the desired area is

$$\int_0^4 (9 + 2x - x^2) - (x-3)^2 \, \mathrm{d}x = \int_0^4 8x - 2x^2 \, \mathrm{d}x = \left[4x^2 - \frac{2}{3}x^3\right]_0^4 = 64 - 64 \cdot \frac{2}{3} = 21\frac{1}{3},$$

(d) noting that the curves intersect when $x^2 + 5 - 6x = 0$, i.e. when $x = 1$ and $x = 5$, we find the area

$$\int_1^5 \frac{6x}{x^2+5} - 1 \, \mathrm{d}x = \int_6^{30} \frac{3}{y} \, \mathrm{d}y - [x]_1^5 = [3\ln(y)]_6^{30} - 4 = 3\ln\left(\frac{30}{6}\right) - 4 = 3\ln(5) - 4,$$

where we substituted $y = x^2 + 5$, $x = \sqrt{y-5}$, $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{2}(y-5)^{-\frac{1}{2}}$ in the first term.

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Exercise 4.

For a continuous function $f : \mathbb{R} \to \mathbb{R}$ and any number $a > 0$ we consider the integral

$$\int_{-a}^{a} f(x) \, \mathrm{d}x.$$

Show that this integral is 0 when f is an odd function, i.e. when $f(x) = f(-x)$. (Hint: use the substitution $y = -x$.) $\oslash$

Solution 4.
Using $f(x) = -f(-x)$ and the substitution $y = -x$ with $\frac{\mathrm{d}x}{\mathrm{d}y} = -1$ we find

$$\begin{aligned}\int_{-a}^{a} f(x)\ \mathrm{d}x &= \int_{-a}^{a} -f(-x)\ \mathrm{d}x \\ &= \int_{a}^{-a} f(y)\ \mathrm{d}y \\ &= -\int_{-a}^{a} f(y)\ \mathrm{d}y \\ &= -\int_{-a}^{a} f(x)\ \mathrm{d}x.\end{aligned}$$

where the reversal of the boundaries incurs a sign in the third line and we just renamed the (dummy) variable in the last line. It now follows that the integral is a number $I \in \mathbb{R}$ such that $I = -I$, which means that $2I = 0$ and hence $I = 0$. (Geometrically, any area below (resp. above) the curve in the region $x > 0$ is compensated for by an area above (resp. below) the curve in the region $x < 0$.) $\diamondsuit$