MS121: IT Mathematics

Integration

An Important Application of Integration

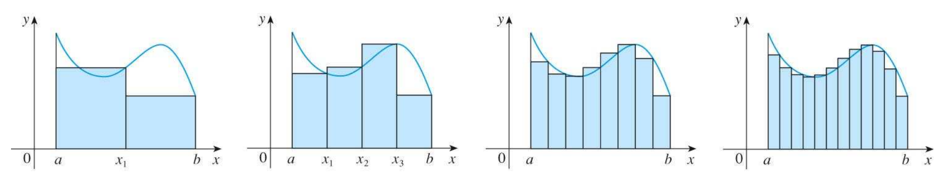
John Carroll
School of Mathematical Sciences

Dublin City University

Outline

Outline

The Area Problem: General Formulation



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

We saw that the approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

Definition

The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x\right]$$

The Area Problem: The Definite Integral

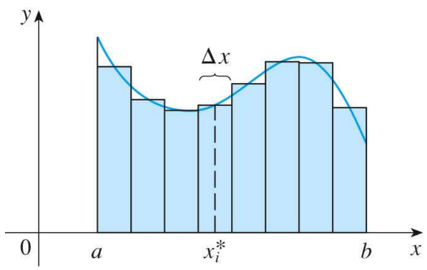


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

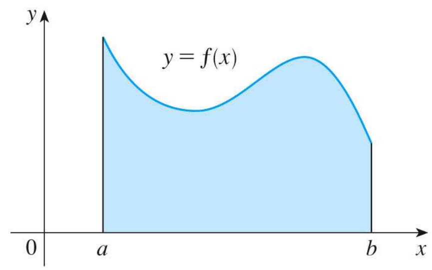


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

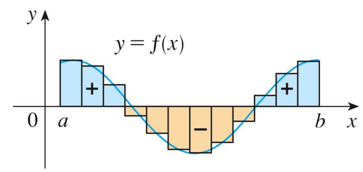


FIGURE 3
$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area.

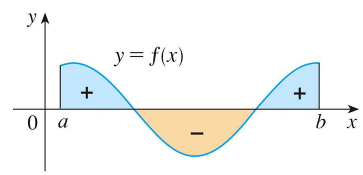


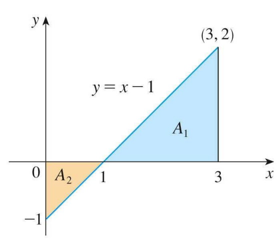
FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

- If $f(x)$ takes on both positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis.
- A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

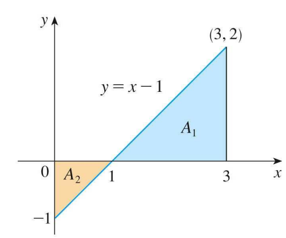
Positive & Negative Areas: Example 1



No need to perform integration. Simple calculations yield:

$$\begin{aligned}\int_0^3 (x-1)\,dx &= A_1 - A_2\\ &= \frac{1}{2}(2 \star 2) - \frac{1}{2}(1 \star 1)\\ &= \frac{3}{2}\end{aligned}$$

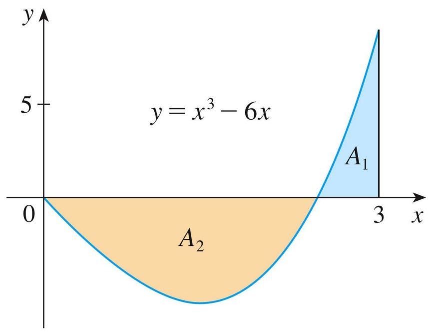
Positive & Negative Areas: Example 1 Re-Visited



Alternatively, integration yields:

$$\begin{aligned}\int_0^3 (x-1)\,dx &= \left[\frac{x^2}{2} - x\right]_0^3\\ &= \left(\frac{3^2}{2} - 3\right) - \left(\frac{0^2}{2} - 0\right)\\ &= \frac{3}{2}\end{aligned}$$

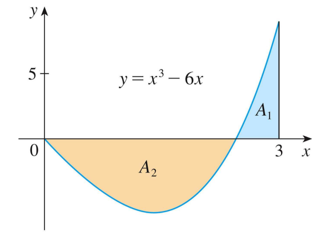
Positive & Negative Areas: Example 2



$A1$ is the area of the region above the x-axis and below the graph of f

$A2$ is the area of the region below the x-axis and above the graph of f.

Positive & Negative Areas: Example 2



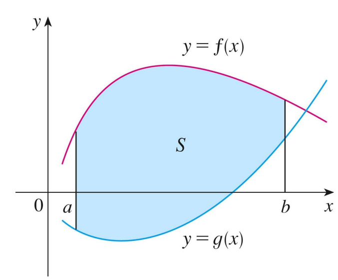
Calculations yield:

$$\begin{aligned}\int_0^3 (x^3 - 6x)\,dx &= \left[\frac{x^4}{4} - 3x^2\right]_0^3 \\ &= \left(\frac{3^4}{4} - 3(3^2)\right) - \left(\frac{0}{4} - 3(0^2)\right) \\ &= -\frac{27}{4}\end{aligned}$$

Outline

Area Between Curves: Developing The Formula

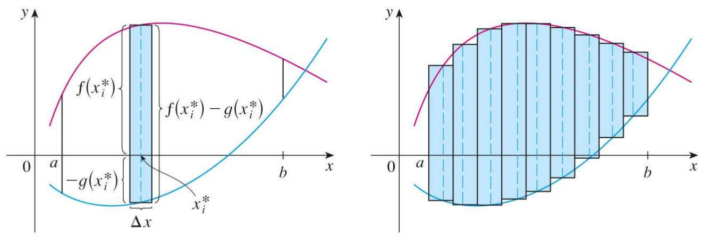
Earlier, we defined and calculated areas of regions that lie under the graphs of functions. Here we use integrals to find areas of regions that lie between the graphs of two functions.



Consider the region that lies between two curves $f(x)$ and $f(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all $x \in [a, b]$.

Area Between Curves: Developing The Formula

We divide S into n strips of equal width and then we approximate the i-th strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$:



(a) Typical rectangle (b) Approximating rectangles

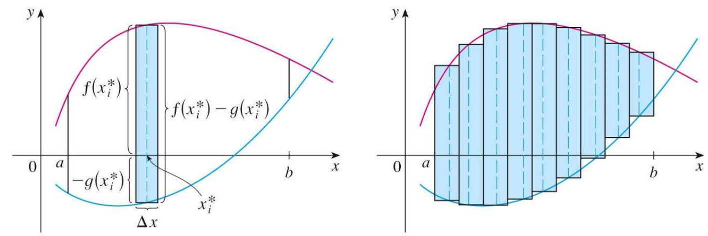
The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\ \Delta x$$

is an approximation to what we intuitively think of as the area of S.

Area Between Curves: Developing The Formula

This approximation appears to become better and better as $n \to \infty$.



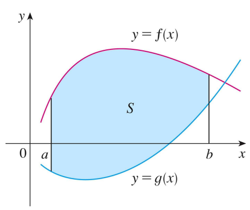
(a) Typical rectangle (b) Approximating rectangles

Therefore we define the area A of the region S as the limiting value of the sum of the areas of these approximating rectangles:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\ \Delta x$$

Area Between Curves: Definition

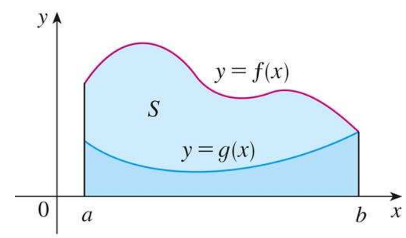
We recognize this limit as the definite integral of $f - g$. Therefore, we have the following formula for area.



The area A of the region bounded by the curves $f(x)$ and $f(x)$, and the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all $x \in [a, b]$ is

$$A = \int_a^b [f(x) - g(x)]\ dx$$

Area Between Curves: The Formula



In the case where both f and g are positive, you can see from the diagram why this is true.

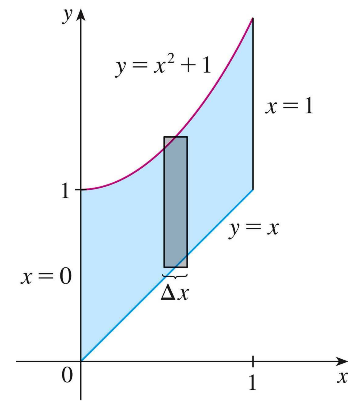
$$\begin{aligned} A &= [\text{area under } y = f(x)] - [\text{area under } y = g(x)] \\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\ dx \end{aligned}$$

Outline

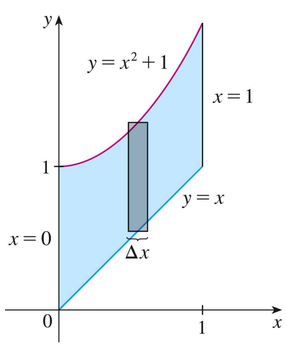
Area Between Curves: Worked Examples

Problem Statement

Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



Area Between Curves: Worked Examples



The area of the shaded region is

$$\begin{aligned} A &= \int_0^1 \left[\left(x^2 + 1\right) - x \right] \, dx \\ &= \int_0^1 \left(x^2 - x + 1\right) \, dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} + x \right]_0^1 \\ &= \frac{5}{6} \end{aligned}$$

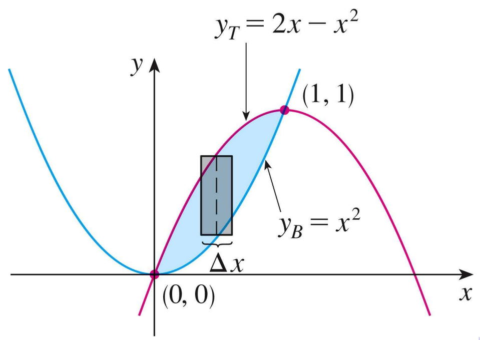
Outline

Concluding Special-Case Examples

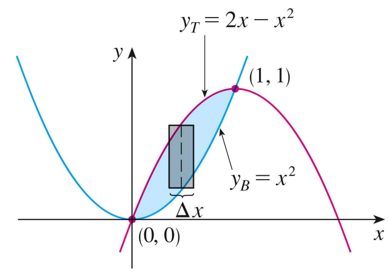
Problem Statement

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

We first have to find the points of intersection of the parabolas by solving their equations simultaneously.



Concluding Special-Case Examples



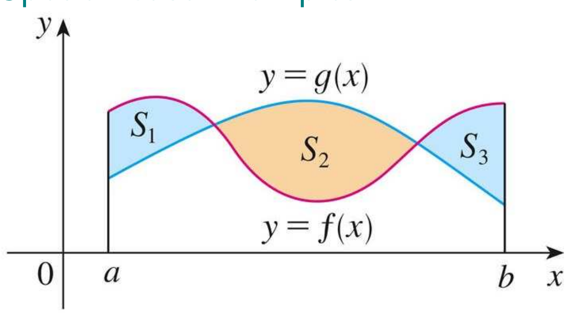
Solving the equations simultaneously:

$$x^2 = 2x - x^2 \Rightarrow 2x(x - 1) \Rightarrow x = 0,\ 1$$

so the region lies between $x = 0$ and $x = 0$ and the total area is

$$\begin{aligned} A &= \int_0^1 (2x - 2x^2)\ dx \\ &= \left[2\left(\frac{x^2}{2}\right) - 2\left(\frac{x^3}{3}\right) \right]_0^1 \\ &= \frac{1}{3} \end{aligned}$$

Concluding Special-Case Examples

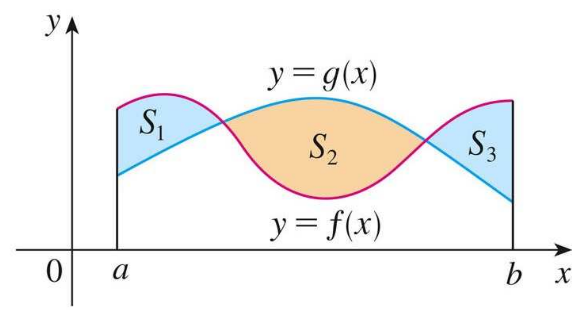


Issue

If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x. Note that

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases}$$

Concluding Special-Case Examples



Definition

The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

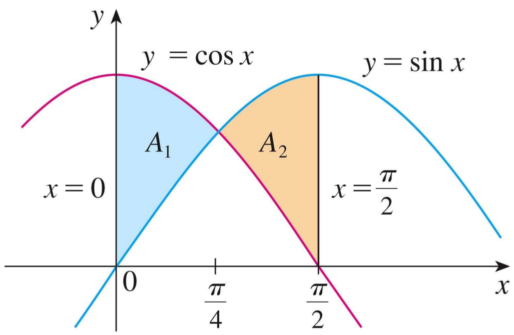
$$A = \int_a^b |f(x) - g(x)|\ dx$$

Concluding Special-Case Examples

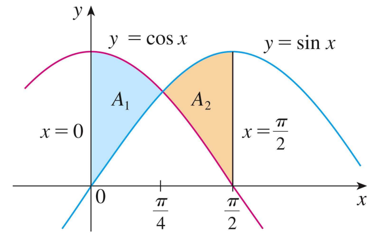
Problem Statement

Find the area of the region bounded by the curves $y = \sin(x)$, $y = \cos(x)$, $x = 0$ and $x = \pi$.

We first have to find the point of intersection of the curves:



Concluding Special-Case Examples

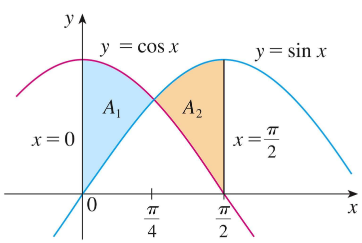


Solution

The points of intersection of the curve is when $\sin(x) = \cos(x)$ which is equivalent to $\tan(x) = 1$ and occurs when $x = \frac{\pi}{4}$. The required area is

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} |\cos(x) - \sin(x)|\ dx \\ &= \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\ dx + \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\ dx \end{aligned}$$

Concluding Special-Case Examples

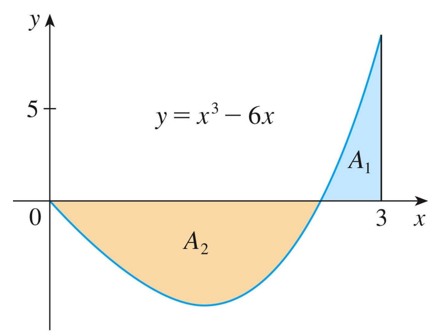


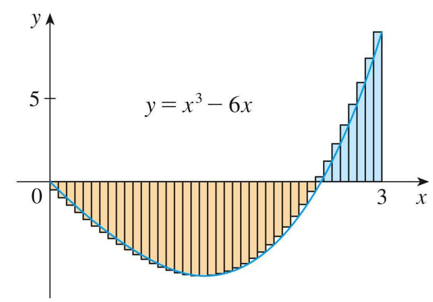
Solution (Cont'd)

The required area is

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\ dx + \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\ dx \\ &= [\sin(x) + \cos(x)]_0^{\frac{\pi}{4}} + [-\cos(x) - \sin(x)]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= 2\sqrt{2} - 2 \end{aligned}$$

Using Absolute Values: Summary Example

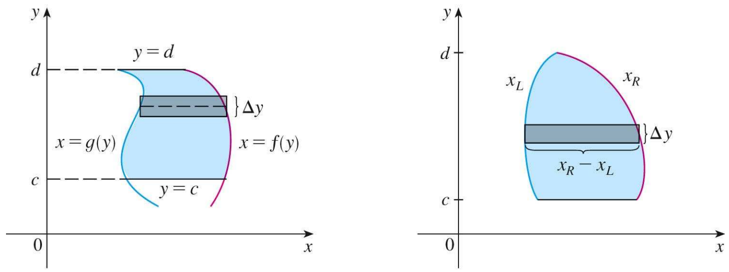




- $\int_0^{\sqrt{6}} (x^3 - 6x)\ dx = -9$ $\quad$ $\int_{\sqrt{6}}^{3} (x^3 - 6x)\ dx = \frac{9}{4}$
- $\int_0^{3} (x^3 - 6x)\ dx = -\frac{27}{4}$ $\quad$ Area: $|A_2| + A_1 = 9 + \frac{9}{4} = \frac{45}{4}$

Area Between Curves: Writing x as a function of y

Some regions are best treated by regarding x as a function of y.

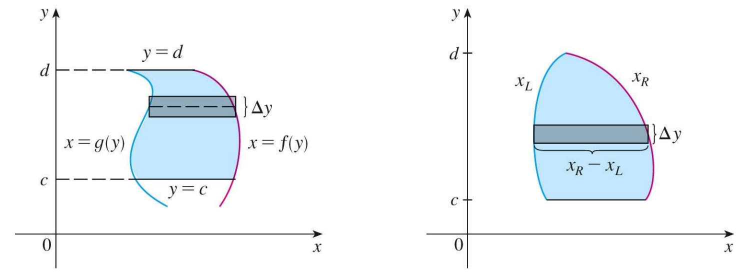


If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then its area is

$$A = \int_c^d [f(y) - g(y)]\, dy$$

Area Between Curves: Writing x as a function of y

$$A = \int_c^d [f(y) - g(y)]\, dy$$



If we write x_R for the right boundary and x_L for the left boundary, then we have

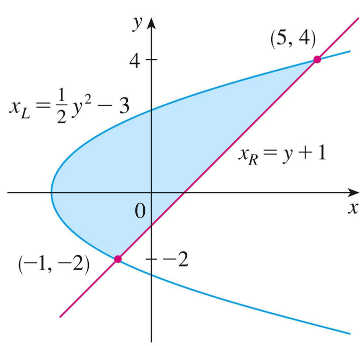
$$A = \int_c^d (x_R - x_L)\, dy$$

Area Between Curves: Writing x as a function of y

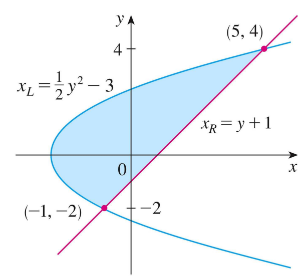
Problem Statement

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

By solving the two equations, we find that the points of intersection are $(-1, -2)$ and $(5, 4)$.



Area Between Curves: Writing x as a function of y



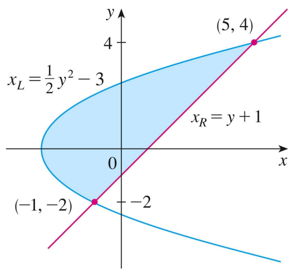
Solution

We solve the equation of the parabola for x and notice the left- and right-boundary curves are

$$x_L = \frac{1}{2}y^2 - 3 \quad \text{and} \quad x_R = y + 1$$

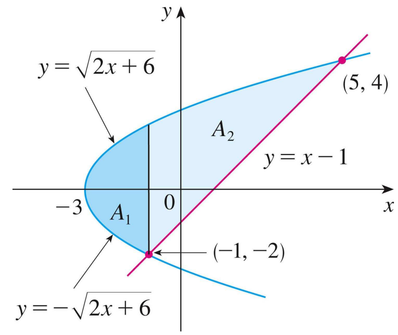
We must integrate between the appropriate y-values, $y = -2$ and $y = 4$.

Area Between Curves: Writing x as a function of y



$$
\begin{aligned}
A &= \int_{-2}^{4} (x_R - x_L)\, dy = \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3 \right) \right] dy \\
&= \int_{-2}^{4} \left(\frac{1}{2}y^2 + y + 4 \right) dy \\
&= \left[\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + y \right]_{-2}^{4} = 18
\end{aligned}
$$

Area Between Curves: Writing x as a function of y



Footnote to the Example

We could have found the area by integrating with respect to x instead of y, but the calculation is much more involved. It would have meant splitting the region in two and computing the areas labelled A_1 and A_2 (as shown). The method that we have just used is much easier.