MS121: IT Mathematics

Differentiation

Rules for Differentiation: Part 2

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Outline

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The Chain Rule

Context

The Chain Rule is used to differentiate the composition of functions.

Composition of 2 Functions

The function

$$y = \ln(\sin x)$$

is the composition $f \circ g$ where

$$f(\cdot) = \ln(\cdot) \qquad \text{and} \qquad g(\cdot) = \sin(\cdot).$$

The Chain Rule

Composition of 2 Functions

The function

$$y = \ln(\sin x)$$

is the composition $f \circ g$ where $f(\cdot) = \ln(\cdot)$ and $g(\cdot) = \sin(\cdot)$.

The Chain Rule

We can differentiate $y = f \circ g$ as

$$\frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

where f is written in terms of g and we only differentiate f with respect to its argument, namely g. Since g is written in terms of x, this differentiation is simply with respect to x.

The Chain Rule

Composition of 2 Functions

$$y = \ln(\sin x)$$

Using the Chain Rule

We obtain:

$$\begin{aligned} g &= \sin x \Rightarrow \frac{dg}{dx} = \frac{d}{dx}\sin x = \cos x \\ f &= \ln(g) \Rightarrow \frac{df}{dg} = \frac{d}{dg}\ln g = \frac{1}{g} = \frac{1}{\sin x} \\ \Rightarrow \frac{dy}{dx} &= \frac{df}{dg} \cdot \frac{dg}{dx} = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} = \cot x \end{aligned}$$

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Example 1: Composition of 2 Functions

$$y = \cos\left(2x^2\right)$$

Solution

$$u = 2x^2 \quad \Rightarrow \quad \frac{du}{dx} = 4x$$

$$y = \cos(u) \quad \Rightarrow \quad \frac{dy}{du} = -\sin(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u) \cdot 4x = -4x\sin\left(2x^2\right)$$

Note

Note that you must always express your answer wholly in terms of x (just as the question was expressed in the first place).

Example 2: Composition of 2 Functions

$$y = \tan\left(e^x\right)$$

Solution

$$u = e^x \quad \Rightarrow \quad \frac{du}{dx} = e^x$$

$$y = \tan(u) \quad \Rightarrow \quad \frac{dy}{du} = \sec^2(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2(u) \cdot e^x = e^x \sec^2\left(e^x\right)$$

Example 3: Composition of 2 Functions

$$y = \sin\left(\ln(x)\right)$$

Solution

$$u = \ln(x) \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$y = \sin(u) \quad \Rightarrow \quad \frac{dy}{du} = \cos(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \cdot \frac{1}{x} = \frac{1}{x} \cos\left(\ln(x)\right)$$

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Example 4: Composition of 3 Functions

$$y = \ln\left(\sin\left(e^x\right)\right)$$

Solution

$$u = e^x \quad \Rightarrow \quad \frac{du}{dx} = e^x$$

$$v = \sin(u) \quad \Rightarrow \quad \frac{dv}{du} = \cos(u)$$

$$y = \ln(v) \quad \Rightarrow \quad \frac{dy}{dv} = \frac{1}{v}$$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = \frac{1}{v} \cdot \cos(u) \cdot e^x = \frac{1}{\sin\left(e^x\right)} \cos\left(e^x\right) e^x = e^x \cot(x)$$

Example 5: Composition of 3 Functions

$$y = \ln\left(\ln\left(\ln(x)\right)\right)$$

Solution

$$u = \ln(x) \quad \Rightarrow \quad \frac{\mathrm{du}}{\mathrm{dx}} = \frac{1}{x}$$

$$v = \ln(u) \quad \Rightarrow \quad \frac{\mathrm{dv}}{\mathrm{du}} = \frac{1}{u}$$

$$y = \ln(v) \quad \Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dv}} = \frac{1}{v}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dv}} \cdot \frac{\mathrm{dv}}{\mathrm{du}} \cdot \frac{\mathrm{du}}{\mathrm{dx}} = \frac{1}{v} \cdot \frac{1}{u} \cdot \frac{1}{x} = \frac{1}{\ln\left(\ln(x)\right)} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

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Example 6: Composition of Functions & Product Rule

$$y = e^{x \sin x}$$

Solution

$$u = x \sin x \quad \Rightarrow \quad \frac{\mathrm{du}}{\mathrm{dx}} = \sin x + x \cos x$$

$$y = e^{u} \quad \Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{du}} = e^{u}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{du}} \cdot \frac{\mathrm{du}}{\mathrm{dx}} = e^{u} \cdot (\sin x + x \cos x) = e^{x \sin x}(\sin x + x \cos x)$$

Example 7: Composition of Functions & Quotient Rule

$$y = \frac{\sqrt{x}}{e^{\tan x}}$$

Observation

At first sight, this appears only to be an example involving the quotient rule with

$$u = \sqrt{x}, \qquad v = e^{\tan x}.$$

However, we must first use the chain rule to determine $\frac{dv}{dx}$.

Example 7: Step 1 — Evaluate $\frac{dv}{dx}$

$$v = e^{\tan x}$$

Solution

$$w = \tan x \quad \Rightarrow \quad \frac{\mathrm{dw}}{\mathrm{dx}} = \sec^2 x$$

$$v = e^w \quad \Rightarrow \quad \frac{\mathrm{dv}}{\mathrm{dw}} = e^w$$

$$\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{dv}}{\mathrm{dw}} \cdot \frac{\mathrm{dw}}{\mathrm{dx}} = e^w \cdot \sec^2(x) = e^{\tan x} \sec^2 x$$

Example 7: Step 2 — Evaluate $\frac{dy}{dx}$

$$y = \frac{\sqrt{x}}{e^{\tan x}}$$

Solution

$$u = \sqrt{x} \quad \Rightarrow \quad \frac{\mathrm{du}}{\mathrm{dx}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$v = e^{\tan x} \quad \Rightarrow \quad \frac{\mathrm{dv}}{\mathrm{dx}} = e^{\tan x} \sec^2 x$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{e^{\tan x} \cdot \frac{1}{2\sqrt{x}} - \sqrt{x} \cdot e^{\tan x} \sec^2 x}{\left(e^{\tan x}\right)^2} = \frac{\frac{1}{2\sqrt{x}} - \sqrt{x} \sec^2 x}{e^{\tan x}}$$