MS121: IT Mathematics

Differentiation

Introduction

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Outline

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Differentiation

What is Differentiation

- Differentiation is a mathematical technique for analysing the way in which functions change.
- In particular, it determines how rapidly a function is changing at any specific point.
- Differentiation also allows us to find maximum and minimum values of a function which can be useful in determining optimum values of key variables.

Differentiation from First Principles

Slope of a Line

Recall how we find the slope of a straight line connecting two points (x_1, y_1) and (x_2, y_2), given by

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

We could also write this as

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{dy}{dx}$$

Differentiation from First Principles

Slope of a Curve?

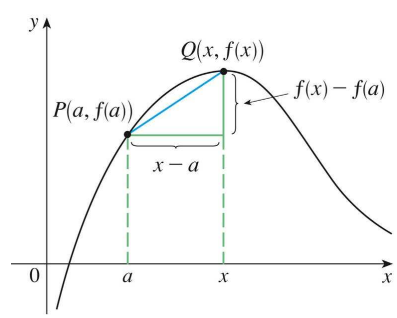
How do we define the slope of a curve? We define it in terms of the slope of a straight line which we already know.

Finding the Slope of a Curve

- Take any curve and the point at which the slope is required.
- At this point, draw the tangent to the curve.
- The tangent is a straight line which, at the point in question, appears to be parallel to the curve.
- The slope of the curve at this point is simply the slope of the tangent.

Outline

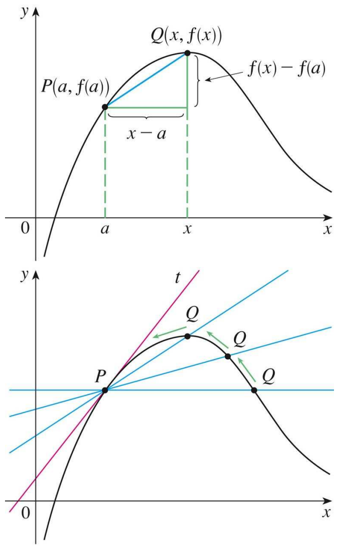
Differentiation: An Introduction — Tangents



If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

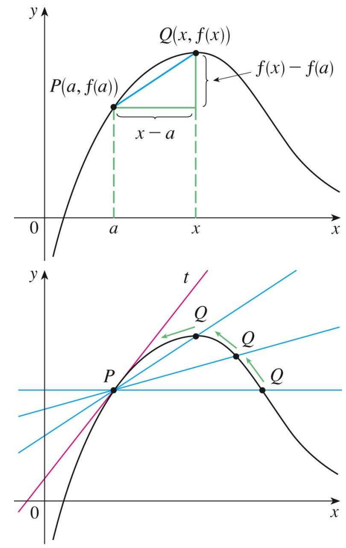
Differentiation: An Introduction — Tangents



$$m_{\text{PQ}} = \frac{f(x) - f(a)}{x - a}$$

- Then we let Q approach P along the curve C by letting x approach a.
- If m_{PQ} approaches a number m, then we define the tangent t to be the line through P with slope m.
- The tangent line is the limiting position of the secant line PQ as Q approaches P.

Differentiation: An Introduction — Tangents



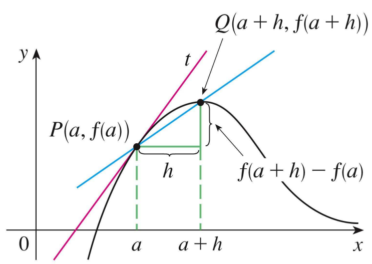
Definition

The tangent line to the curve $y = f(x)$ at the point at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Differentiation: An Introduction — Tangents



Easier Expression

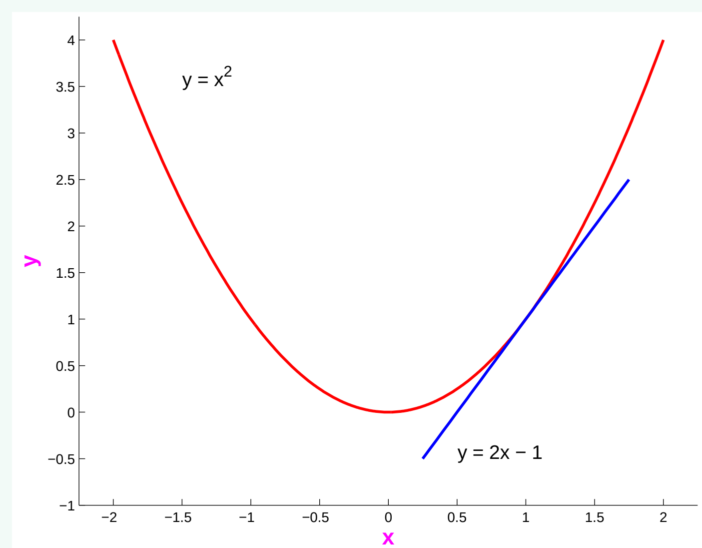
- If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{\text{PQ}} = \frac{f(a + h) - f(a)}{h}$$

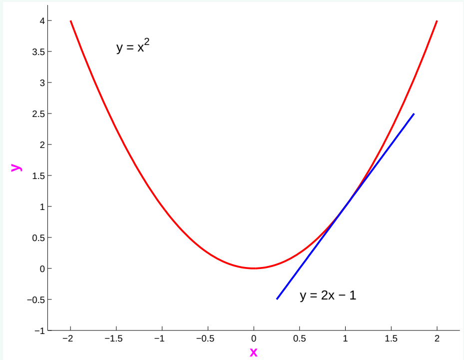
- As x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line becomes

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Tangent to $y = x^2$ at the point $x = 1$



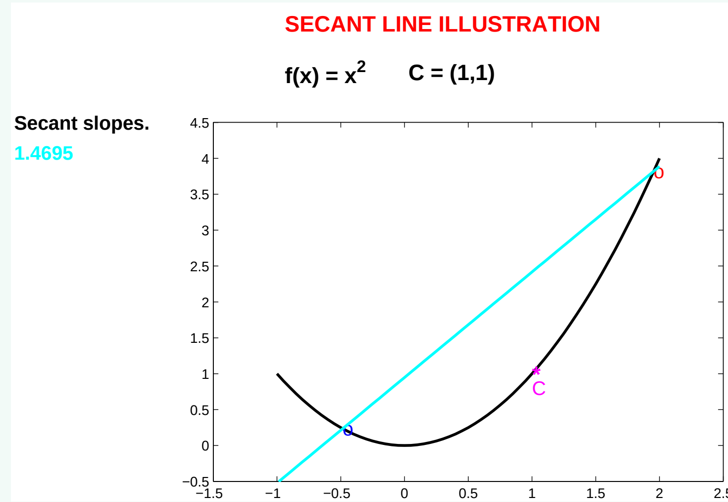
Differentiation from First Principles



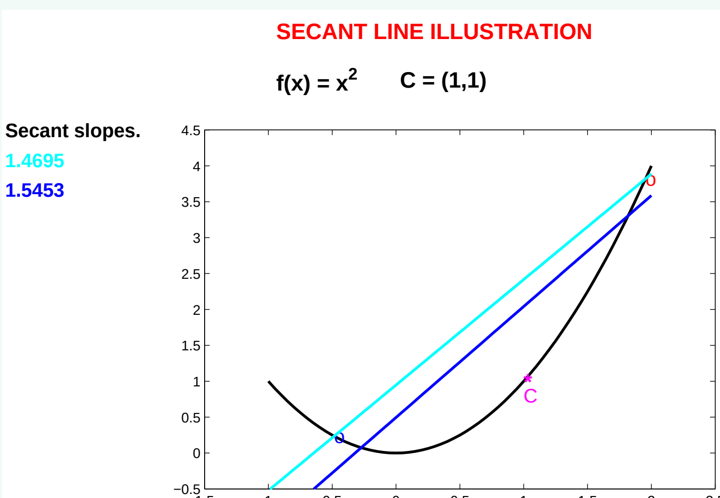
Slope of a Curve

As you will see from the graph, the slopes at different points on the curve will be different (unlike the straight line where the slope is always the same, no matter where it is evaluated).

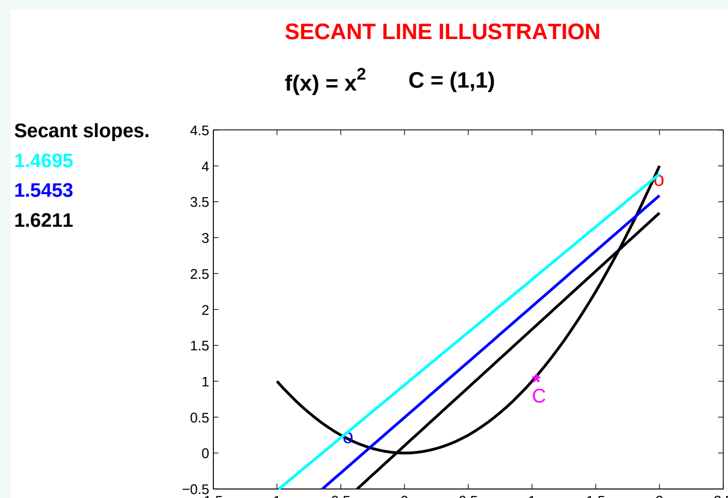
Limit of Secant Lines: $y = x^2$ at the point $x = 1$



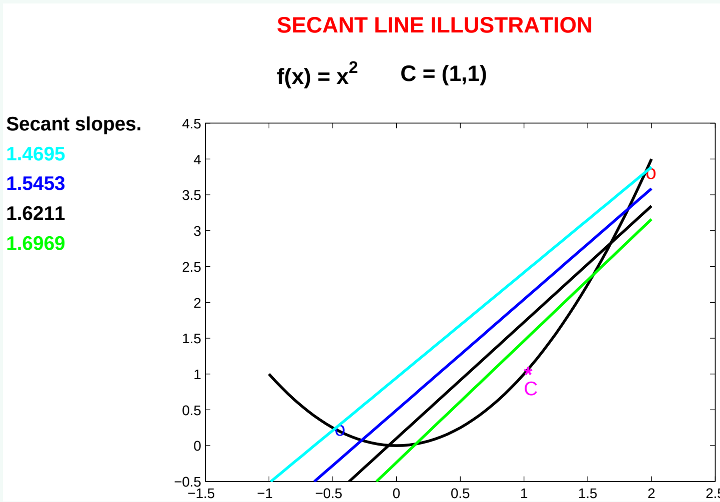
Limit of Secant Lines: $y = x^2$ at the point $x = 1$



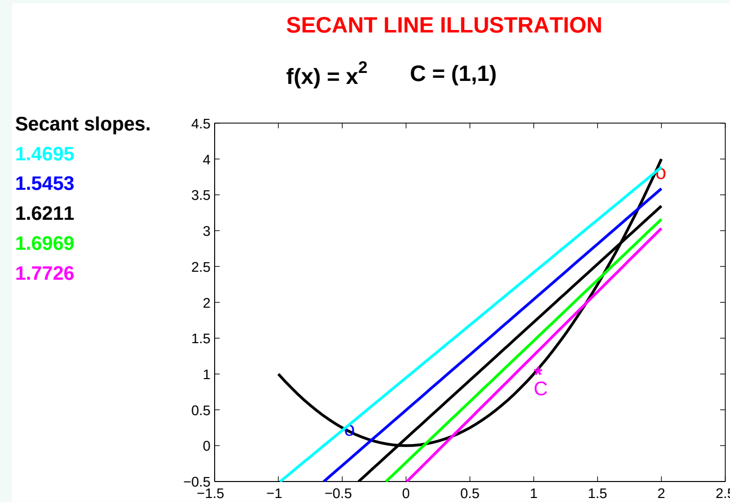
Limit of Secant Lines: $y = x^2$ at the point $x = 1$



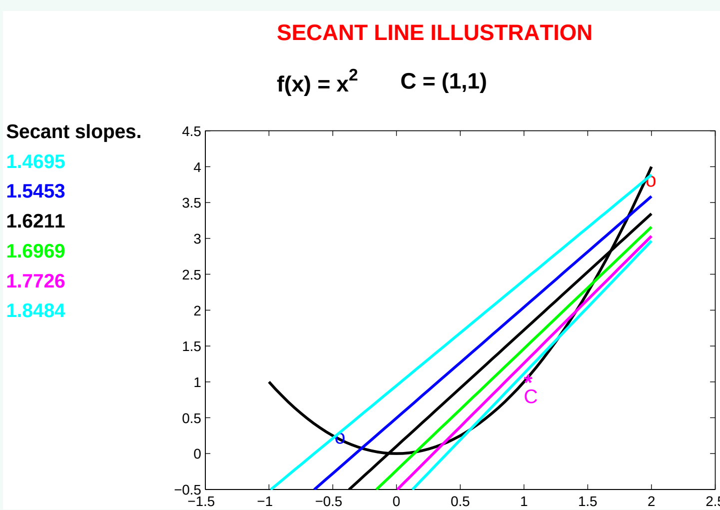
Limit of Secant Lines: $y = x^2$ at the point $x = 1$



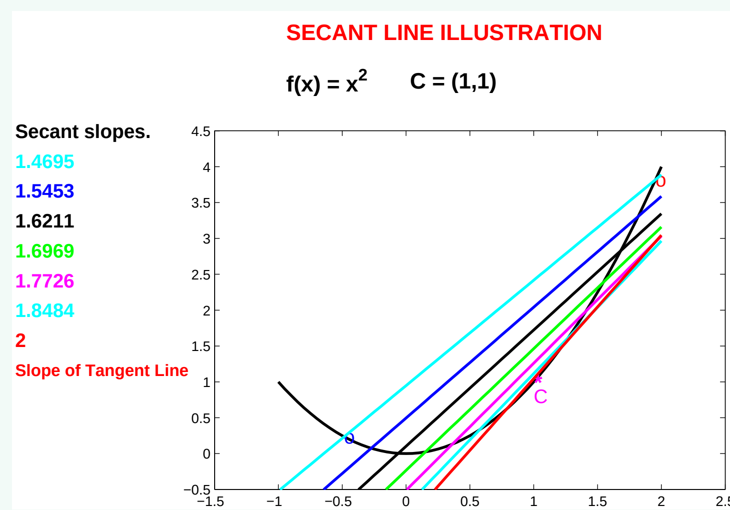
Limit of Secant Lines: $y = x^2$ at the point $x = 1$

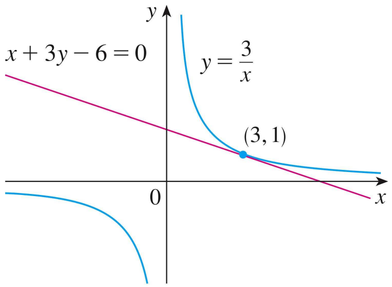


Limit of Secant Lines: $y = x^2$ at the point $x = 1$



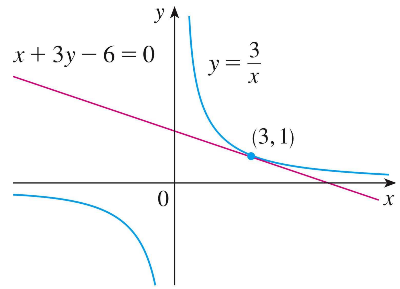
Limit of Secant Lines: $y = x^2$ at the point $x = 1$





Example

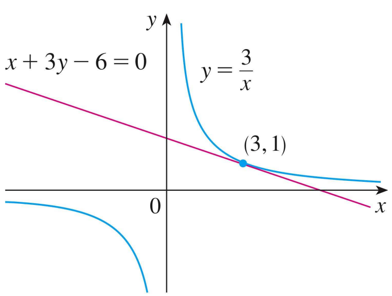
Find an equation of the tangent line to the hyperbola $y = \frac{3}{x}$ at the point $(3, 1)$.



Solution (1/2): Slope of Tangent

Let $f(x) = \frac{3}{x}$. Then the slope of the tangent at $(3, 1)$ is:

$$
\begin{aligned}
m &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} \\
&= \lim_{h \to 0} \frac{-1}{3+h} \\
&= -\frac{1}{3}
\end{aligned}
$$



Solution (2/2): Equation of Tangent

Therefore, an equation of the tangent at the point $(3, 1)$ is

$$y - 1 = -\frac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$

Outline

Differentiation from First Principles

Method of Differentiation

- We do not need to keep drawing tangents to find the slopes — we can use differentiation.
- The derivative (obtained by differentiation) of a curve at a point is the slope of the curve at that point.
- We shall first differentiate functions from first principles.

The Derivative of a Function

Definition

The derivative of the function $f : R \to R$ at a value $x = a$, if it exists, is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Differentiation from First Principles

Basic Approach

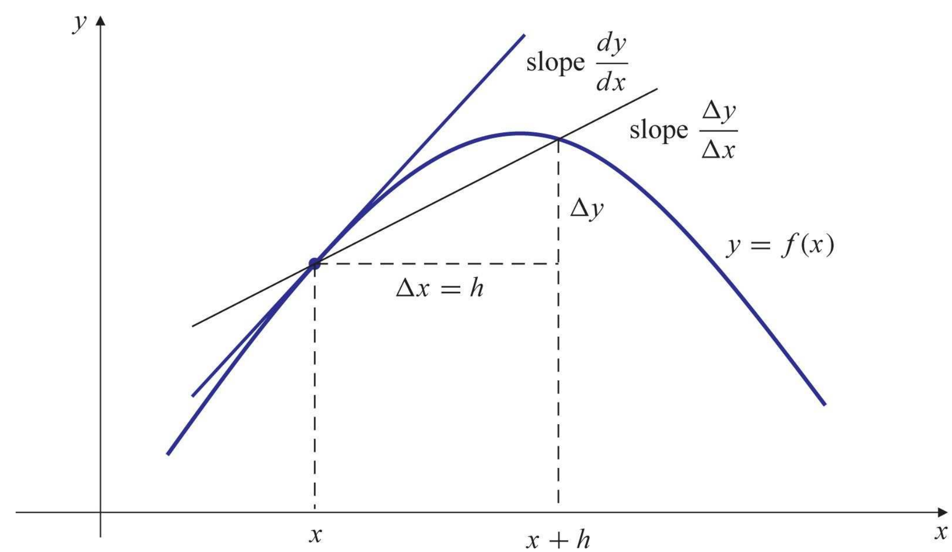
- The idea is to consider the line joining the two points $(a, f(a))$ and $(a+h, f(a+h))$ which, for h small, will approximate the tangent to the curve at the point $x = a$.
- Compute the slope of this line:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

and then take the limit as h tends to zero, giving

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Differentiation from First Principles



Outline

Example 1

We will find the derivative from first principles of $f(x) = x^2$ at $x = a$.

Solution

We will find the derivative from first principles of $f(x) = x^2$ at $x = a$. First compute

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{(a+h)^2 - a^2}{h} \\ &= \frac{(a^2+2ah+h^2)-a^2}{h} \\ &= \frac{2ah+h^2}{h} \\ &= 2a+h\end{aligned}$$

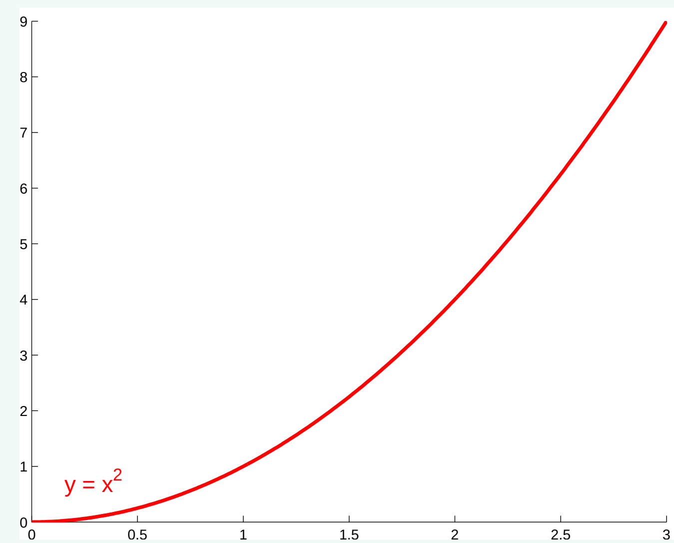
$$f(x) = x^2$$

Example 1 (Cont'd)

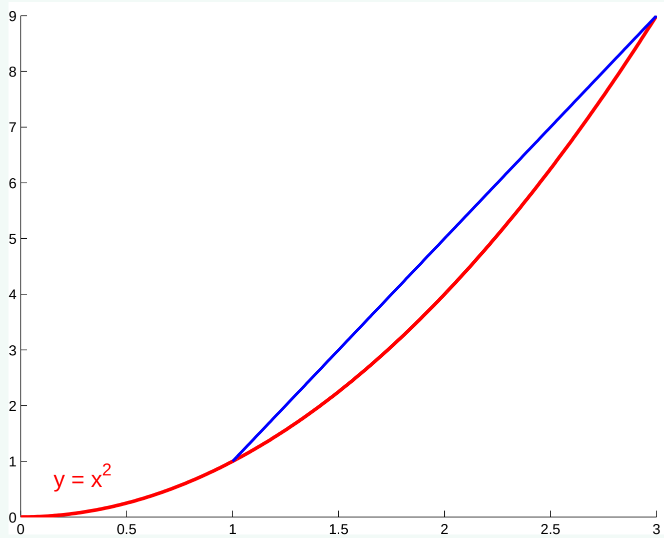
Then take limits to obtain

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(2a+h) = 2a$$

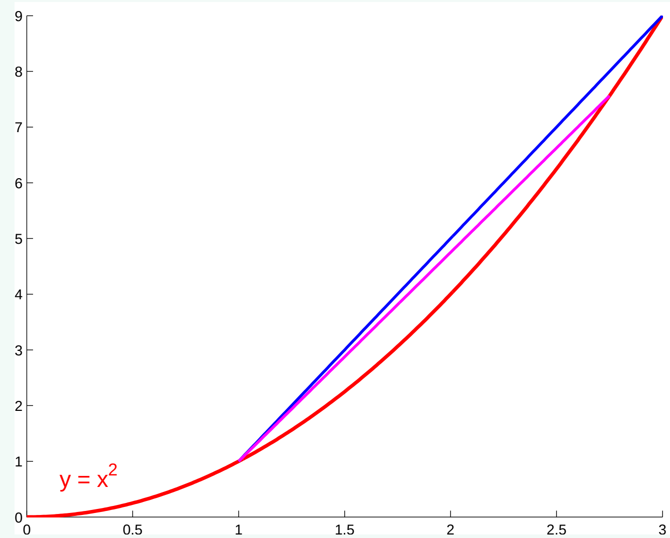
The function $y = x^2$ on the interval $[0, 3]$



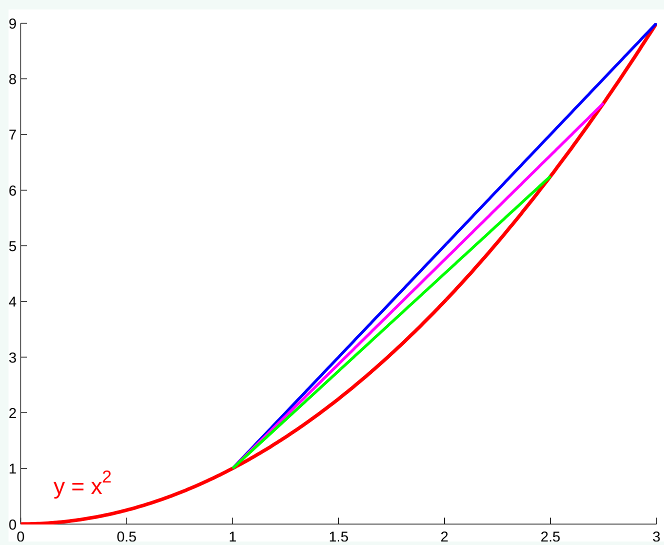
Function $y = x^2$ 1st Secant Line with $h = 2$



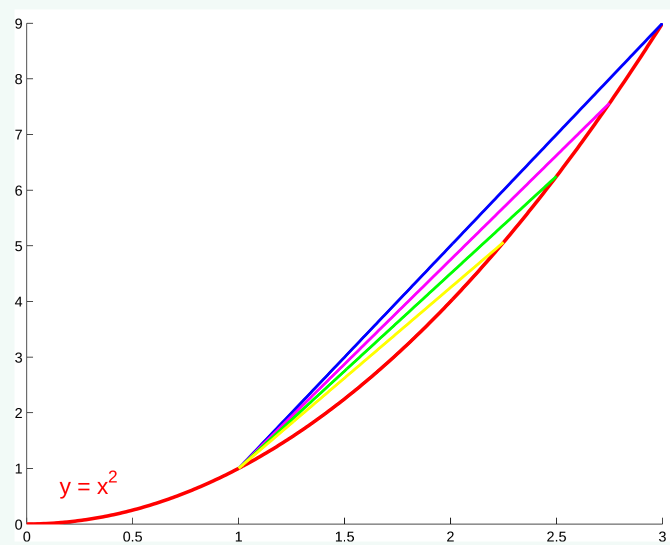
Function $y = x^2$ 2nd Secant Line with $h = 1.75$



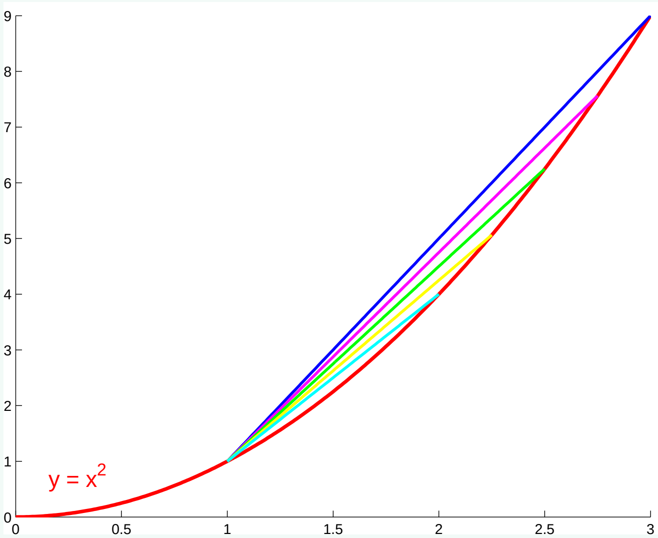
Function $y = x^2$ 3rd Secant Line with $h = 1.5$



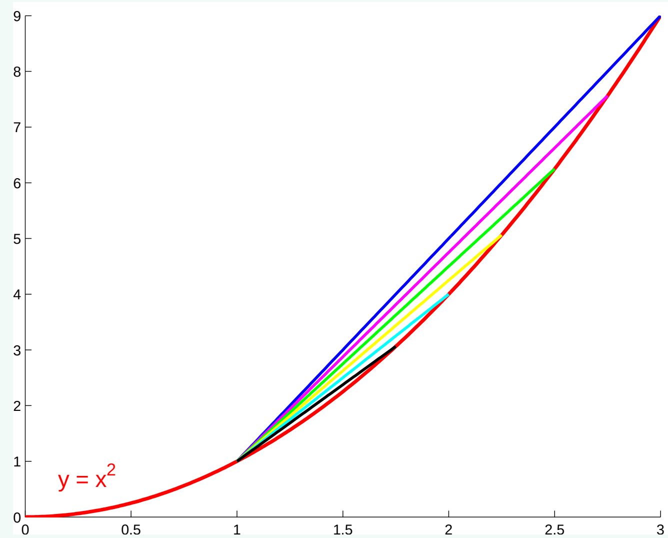
Function $y = x^2$ 4th Secant Line with $h = 1.25$



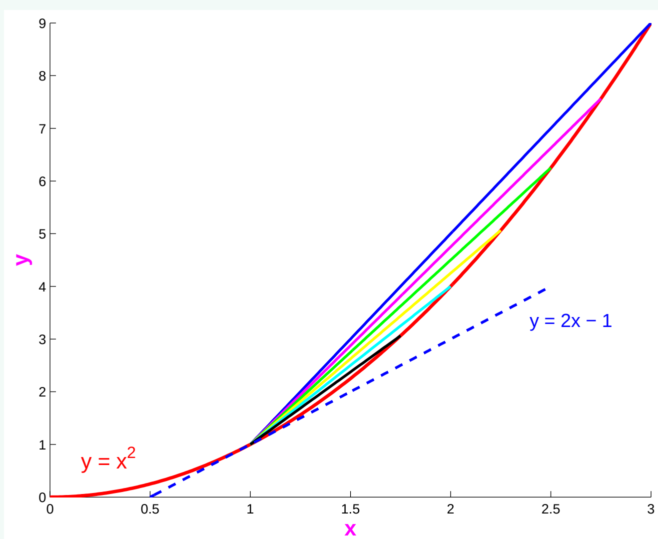
Function $y = x^2$ 5th Secant Line with $h = 1$



Function $y = x^2$ 6th Secant Line with $h = 0.75$



Function $y = x^2$ Limit as $h \to 0$



Differentiation

Notation

The notation we use is as follows:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \left.\frac{df(x)}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

where the vertical line (above) denotes "evaluated at".

Differentiation

Other Notation

It is important to note at this point that an alternative (“Δx”) notation is also widely used, namely:

$$\lim_{\Delta x \to 0} \frac{(y+\Delta y)-y}{(x+\Delta x)-x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

We illustrate the use of the “Δ” in the following example.

Example 2

Differentiate the function $y = x^3 + 1$ from first principles at $x = a$.

Solution

First compute the quotient $\frac{\Delta y}{\Delta x}$:

$$\begin{aligned} y &= x^3 + 1 \\ y + \Delta y &= (x+\Delta x)^3 + 1 \\ \Delta y &= (x+\Delta x)^3 - x^3 \\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 \\ &= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \\ \frac{\Delta y}{\Delta x} &= 3x^2 + 3x\Delta x + (\Delta x)^2 \end{aligned}$$

$$y = x^3 + 1$$

Example 2 (Cont'd)

We then take the limit as Δx tends to zero:

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= 3x^2 + 3x\Delta x + (\Delta x)^2 \\ \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= 3x^2 \\ \Rightarrow \left.\frac{dy}{dx}\right|_{x=a} &= 3a^2 \end{aligned}$$

Example 3

As a simpler example, we will find the derivative from first principles of $f(x) = x$.

Solution

We compute

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)-a}{h} = \frac{h}{h} = 1$$

so that, irrespective of h, we obtain

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} 1 = 1$$

This is as expected since $y = x$ is just the straight line with constant slope $(= 1)$.

Example 4

Differentiate from first principles the function $y = \sqrt{x}$ for $x > 0$.

Solution

Since

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\left[\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right]\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

we obtain

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

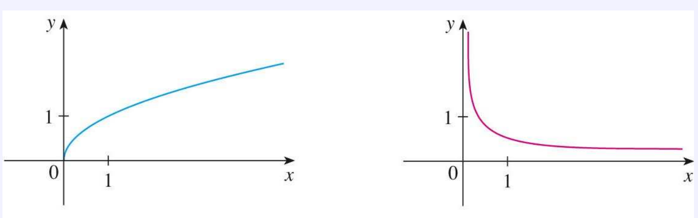
$$y = \sqrt{x} \qquad\qquad x > 0$$

Solution (Cont'd)

We then obtain

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$y = \sqrt{x} \qquad\qquad x > 0$$



(a) $f(x) = \sqrt{x}$ (b) $f'(x) = \frac{1}{2\sqrt{x}}$

Footnote to Example 4

- Note that, in order to simplify the expression $\sqrt{x+h}-\sqrt{x}$, we multiplied above and below by the "conjugate" expression $\sqrt{x+h}+\sqrt{x}$.
- We are making use of the identity

$$A^2 - B^2 = (A-B)(A+B)$$

and on the basis that it may sometimes be easier to deal with $A^2 - B^2$ than $A - B$ (as in this case).

Example 5

What is the slope of the curve $y = \frac{1}{x}$ at the point $x = 2$?

Solution

As before, we compute:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{1}{x+h}-\frac{1}{x}}{h}\\ &= \frac{x-(x+h)}{x(x+h)h}\\ &= \frac{-h}{x(x+h)h}\\ &= \frac{-1}{x(x+h)}\end{aligned}$$

$$y = \frac{1}{x}$$

Example 5 (Cont'd)

Then take limits to obtain

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{-1}{x(x+0)} = -\frac{1}{x^2}.$$

Finally, evaluate at $x = 2$ to obtain

$$\left.\frac{dy}{dx}\right|_{x=2} = -\left.\frac{1}{x^2}\right|_{x=2} = -\frac{1}{4}.$$

Outline

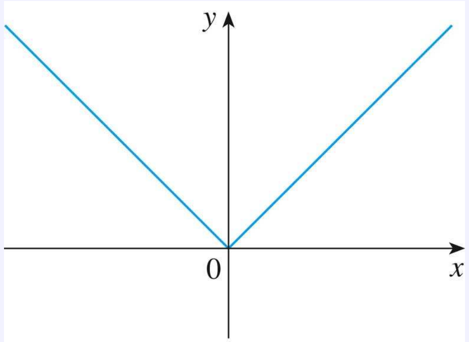
When is a Function Differentiable?

Definition

- A function is differentiable at a if $f'(a)$ exists.
- It is differentiable on an open interval (a, b) if it is differentiable at every number in the interval.

When is a Function Differentiable?

Example: $f(x) = |x|$



(a) $y = f(x) = |x|$

When is a Function Differentiable?

Example: $f(x) = |x|$: Case $x > 0$

If $x > 0$, then $|x| = x$ and we can choose h small enough so that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h\to 0} \frac{x+h-x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 = 1 \end{aligned}$$

and so f is differentiable for any $x > 0$.

When is a Function Differentiable?

Example: $f(x) = |x|$: Case $x < 0$

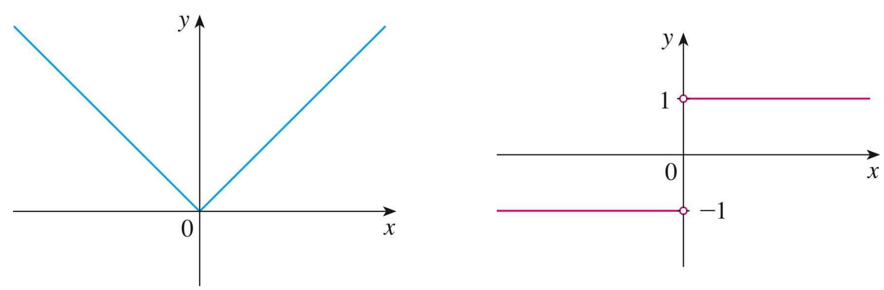
If $x < 0$, then $|x| = -x$ and we can choose h small enough so that $x + h < 0$ and hence $|x + h| = -(x + h)$. Therefore, for $x < 0$, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h\to 0} \frac{-x-h+x}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} \\ &= \lim_{h\to 0} -1 = -1 \end{aligned}$$

and so f is differentiable for any $x < 0$.

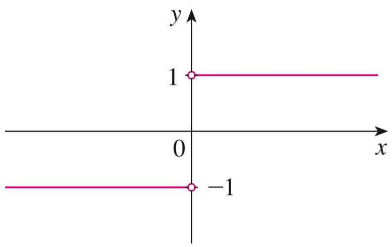
When is a Function Differentiable?

$f'(0)$ does not exist.



(a) $y = f(x) = |x|$ (b) $y = f'(x)$

When is a Function Differentiable?



(b) $y = f'(x)$

A formula for f' is given by

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases}$$

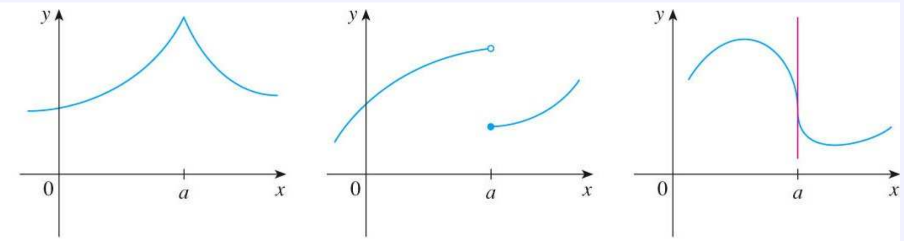
Note

$f(x) = |x|$ is differentiable at all x except 0.

How Can a Function Fail to Be Differentiable?

If the graph of a function f has a "corner" or "kink" in it, then the graph of has no tangent at this point and is not differentiable there.

The Three Cases



(a) A corner (b) A discontinuity (c) A vertical tangent