MS121: IT Mathematics

Applications of Differentiation 1

Curve Sketching

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Outline

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Optimization Problems

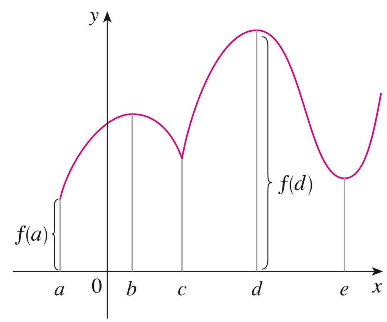
Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function.

Outline

Maximum & Minimum Values



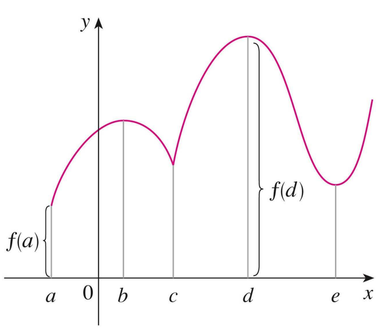
Definition

Let c be a number in the domain D of a function f. Then f is the

- absolute maximum value of f on D if $f(c) \geq f(x)$ for all $x \in D$.
- absolute minimum value of f on D if $f(c) \leq f(x)$ for all $x \in D$.

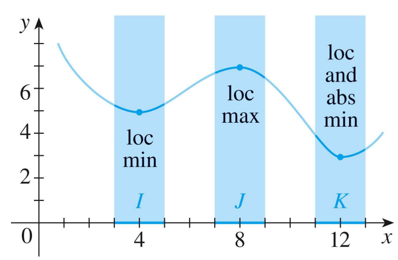
An absolute maximum or minimum is sometimes called a global maximum or minimum. The maximum and minimum values of f are called extreme values of f.

Maximum & Minimum Values



- $f(a)$ is the absolute minimum on $[a, e]$.
- $f(d)$ is the absolute maximum on $[a, e]$.
- $f(c)$ is the local minimum on $[b, d]$.
- $f(b)$ is the absolute maximum on $[a, c]$.

Local Maximum & Minimum Values

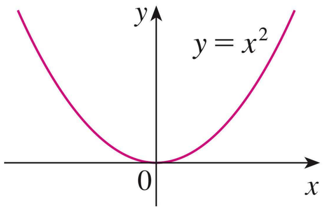


Definition

The number $f(c)$ is a

- local maximum value of f if $f(c) \geq f(x)$ when x is near c.
- local minimum value of f if $f(c) \leq f(x)$ when x is near c.

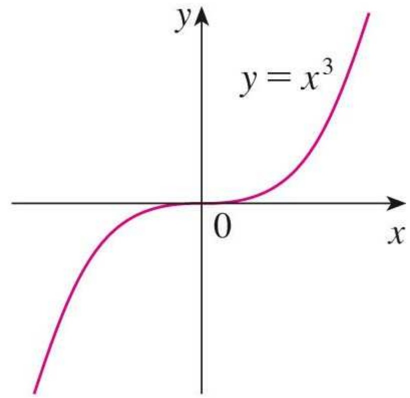
Global & Local Minimum Value



Absolute & Local Mimiumum

- If $f(x) = x^2$, then $f(x) \geq f(0)$ because $x^2 \geq 0$ for all x.
- Therefore $f(0) = 0$ is the absolute (and local) minimum value of f. This corresponds to the fact that the origin is the lowest point on the parabola.
- However, there is no highest point on the parabola and so this function has no maximum value.

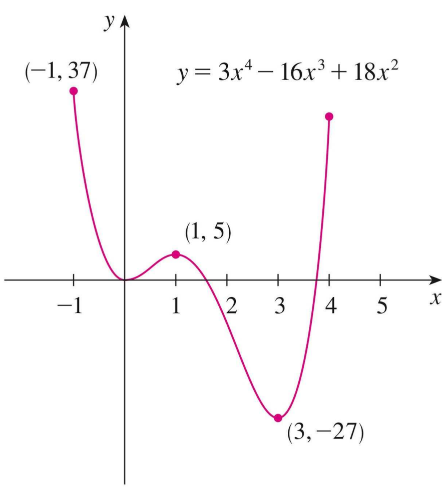
No Local/Global Maximum or Minimum Values



No "Extreme" Values

- From the graph of the function, we see that this function has neither an absolute maximum value nor an absolute minimum value.
- In fact, it has no local extreme values either.

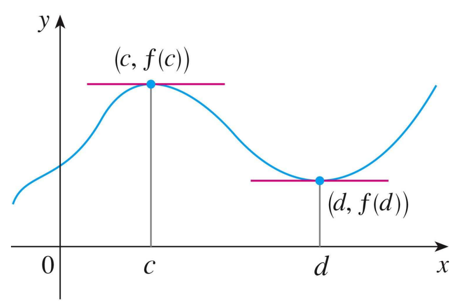
Local v Global Maximum Values



Both "Extreme" Values

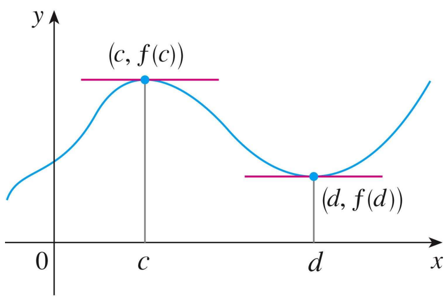
- $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1) = 37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.)
- Also, $f(0) = 0$ is a local minimum and $f(3) = -27$ is both a local and an absolute minimum.
- Note f that has neither a local nor an absolute maximum at $x = 4$.

When does a Max/Min Occur?



- The graph of a function f shows a local maximum at c and a local minimum at d.
- It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0.
- We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$.

When does a Max/Min Occur?

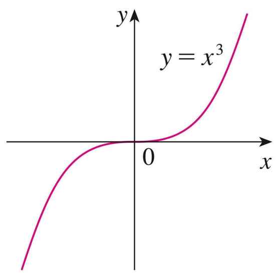


Fermat's Theorem

If f has a local maximum or minimum at $f'(c) = 0$, and if $f'(c)$ exists, then

$$f'(c) = 0$$

A Word of Caution on Fermat's Theorem



The Converse is False!

- If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$.
- But f has no maximum or minimum at 0, as you can see from its graph.
- The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at $(0, 0)$.
- Instead of having a maximum or minimum at $(0, 0)$, the curve crosses its horizontal tangent there.

Outline

Finding the Critical Points

Definition: A Critical Number (Point)

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's Theorem: Re-Stated

If f has a local maximum or minimum at c, then c is a critical number of f.

The Closed Interval Method: Absolute Max/Min Values

3-Step Procedure

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example

Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \le x \le 4$$

Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method:

$$\begin{aligned} f(x) &= x^3 - 3x^2 + 1 \\ f'(x) &= 3x^2 - 6x = 3x(x-2) \end{aligned}$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $\left(-\frac{1}{2}, 4\right)$.

Example (Cont'd)

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \le x \le 4$$

The critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. The values of at these critical numbers are

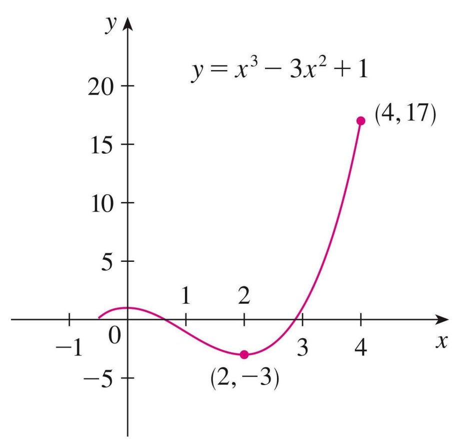
$$f(0) = 1 \qquad f(2) = -3$$

The values of at the endpoints of the interval are

$$f\left(-\frac{1}{2}\right) = \frac{1}{8} \qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Example (Cont'd)

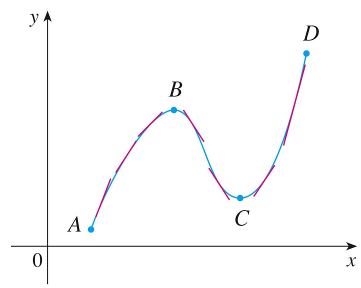


Note

In this example, the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number.

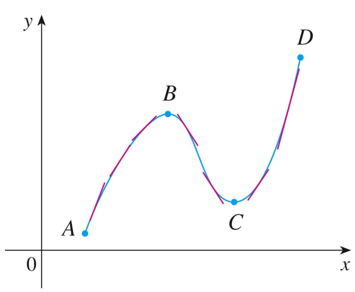
Outline

Increasing/Decreasing Test



- Between A and B and between C and D, the tangent lines have positive slope and so $f'(x) > 0$.
- Between B and C, the tangent lines have negative slope and so $f'(x) < 0$.
- Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative.

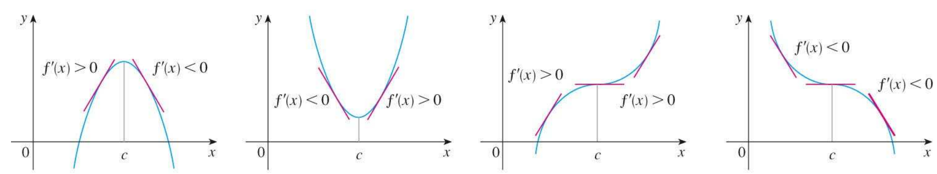
Increasing/Decreasing Test



Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test



(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maximum or minimum

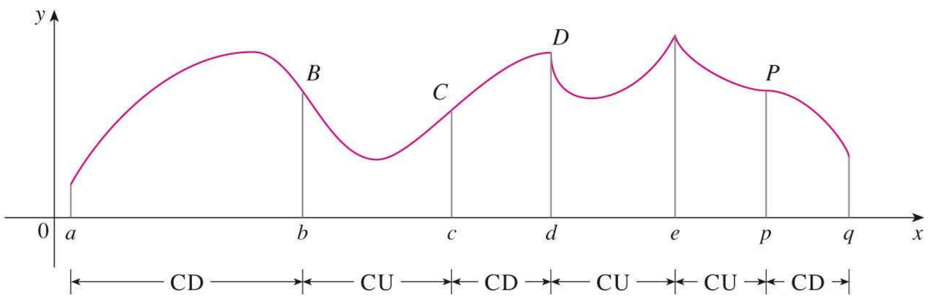
Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

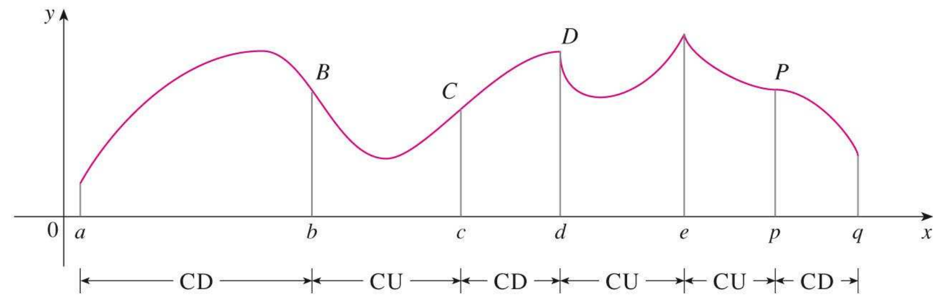
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Using the 2nd Derivative



- The slope of the tangent decreases from a to b, so f' decreases and therefore f'' is negative.
- The slope of the tangent increases from b to c, and this means that the derivative is an increasing function and therefore its derivative (i.e. f'') is positive.

Concavity Test



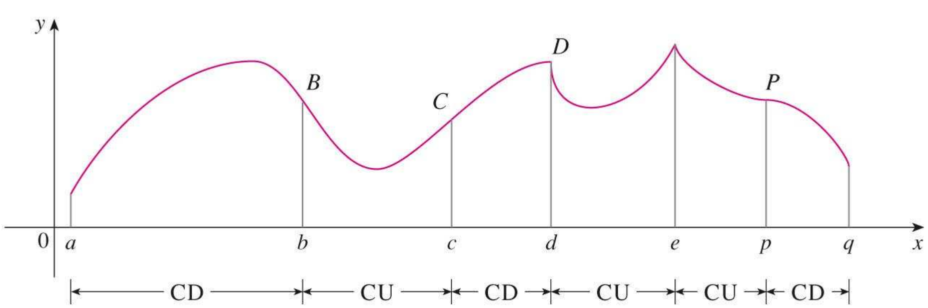
(a) If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

Point of Inflection

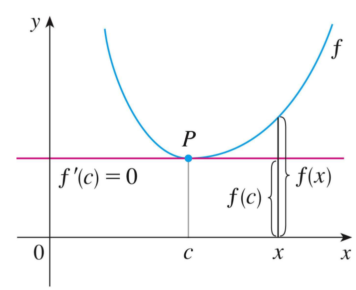
Definition

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.



B, C, D and P are the points of inflection.

Identifying the Maximum/Minumum Points



Second Derivative Test

Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

A Practical Example

- Discuss the curve
$$y = x^4 - 4x^3$$
with respect to concavity, points of inflection, and local maxima and minima.
- Use this information to sketch the curve.

Note

$$\begin{aligned} f(x) &= x^4 - 4x^3 \\ f'(x) &= 4x^3 - 12x^2 = 4x^2(x-3) \\ f''(x) &= 12x^2 - 24x = 12x(x-2) \end{aligned}$$

A Practical Example (Cont'd)

$$\begin{aligned} f(x) &= x^4 - 4x^3 \\ f'(x) &= 4x^3 - 12x^2 = 4x^2(x-3) \\ f''(x) &= 12x^2 - 24x = 12x(x-2) \end{aligned}$$

Solution

$$\begin{aligned} f'(x) = 0 \quad &\Rightarrow \quad x = 0 \text{ and } x = 3 \\ f''(x) = 0 \quad &\Rightarrow \quad x = 0 \text{ and } x = 2 \end{aligned}$$

Furthernore, note that $f''(0) = 0$ and $f''(3) = 36 > 0$. Since $f'(3) = 0$ and $f''(3) > 0$, we have that $f(3) = -27$ is a local minimum.

Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0.

A Practical Example (Cont'd)

$$\begin{aligned} f(x) &= x^4 - 4x^3 \\ f'(x) &= 4x^3 - 12x^2 = 4x^2(x-3) \\ f''(x) &= 12x^2 - 24x = 12x(x-2) \end{aligned}$$

$f'(0) = 0$ and $f''(0) = 0$

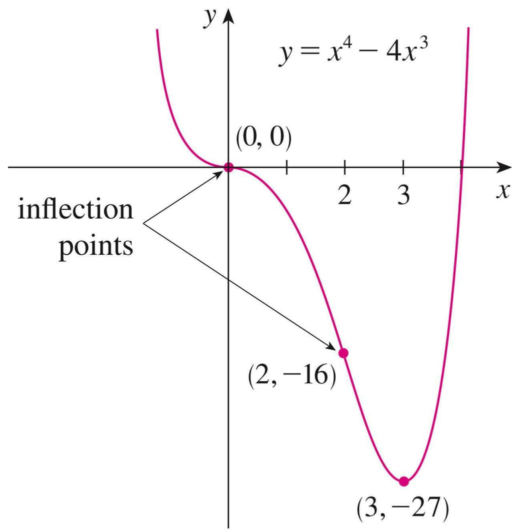
- But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that does not have a local maximum or minimum at 0.
- Since $f'(0) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

A Practical Example (Cont'd)

Interval	$f''(x) = 12x(x-2)$	Concavity
$(-\infty, 0)$	+	Upward
$(0, 2)$	-	Downward
$(2, \infty)$	+	Upward

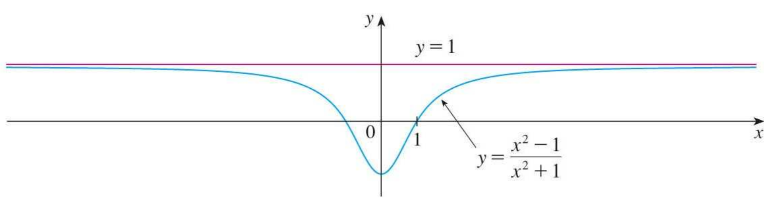
- The point $(0, 0)$ is an inflection point since the curve changes from concave upward to concave downward there.
- Also $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

A Practical Example (Concluded)



Outline

A Reminder about Limits



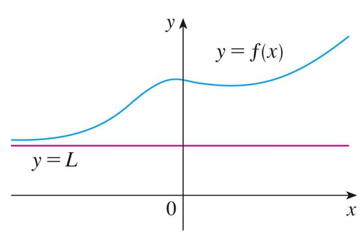
Definition

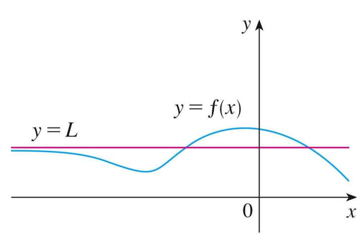
Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of f can be made arbitrarily close to L by taking x sufficiently large.

Horizontal Asymptote





Definition

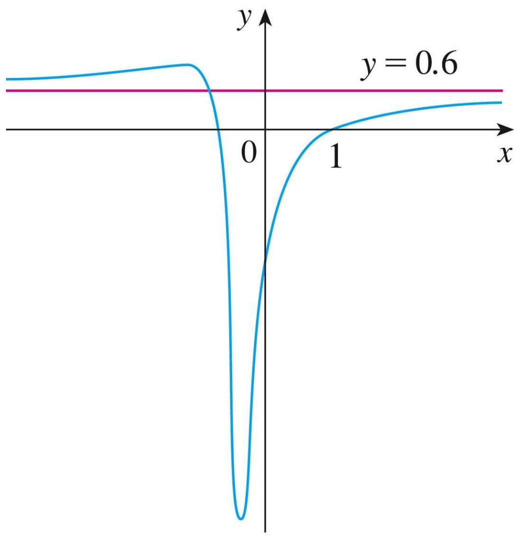
The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L$$

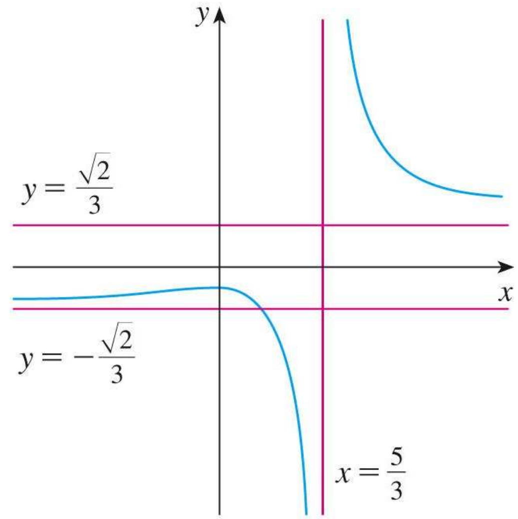
or

$$\lim_{x \to -\infty} f(x) = L$$

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$



$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\sqrt{2}}{3}$$



Outline

8 Steps for Example 1

Requirement

Identify the critical points and asymptotes of

$$y = \frac{2x^2}{x^2 - 1}$$

Sketch the curve.

Steps 1 & 2

1. Domain:
   The domain is

$$\{x \mid x^2 - 1 \neq 0\} \quad = \quad \{x \mid x \neq \pm 1\}$$

2. Intercepts:
   The x- and y- intercepts are both 0.

8 Steps for Example 1 Cont'd

$$y = \frac{2x^2}{x^2 - 1}$$

Steps 3 & 4

3 Symmetry:
Since $f(-x) = f(x)$, the function is even. The curve is symmetric about the y-axis

4 Asymptotes:

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-\frac{1}{x^2}} = 2$$

Therefore, the line $y = 2$ is a horizintal asymptote.

8 Steps for Example 1 Cont'd

$$y = \frac{2x^2}{x^2 - 1}$$

Step 4 Cont'd

4 Asymptotes:
Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \infty \qquad \lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty$$

$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty \qquad \lim_{x\to -1^-} \frac{2x^2}{x^2-1} = +\infty$$

Therefore the lines $x = -1$ and $x = +1$ are vertical asymptotes.

8 Steps for Example 1 Cont'd

$$y = \frac{2x^2}{x^2 - 1}$$

Steps 5 & 6

5 Intervals of Increase or Decrease:

$$f'(x) = \frac{(x^2-1)(4x) - (2x^2)(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

6 Local Maximum or Minimum Values:
The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

8 Steps for Example 1 Cont'd

$$y = \frac{2x^2}{x^2 - 1}$$

Step 7

7 Concavity & Points of Inflection:

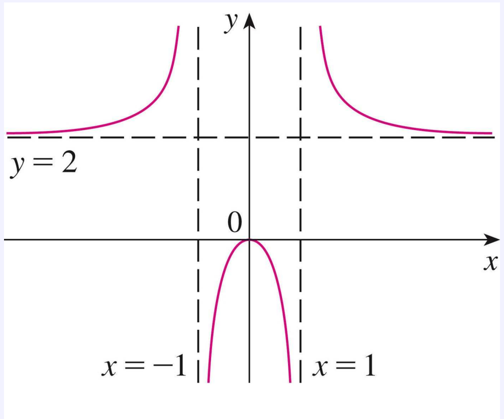
$$f''(x) = \frac{\left(x^2-1\right)^2(-4) + 8x\left(x^2-1\right)(2x)}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have
$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$ and
$f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and and $(1, \infty)$and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

8 Steps for Example 1 Concluded

$$y = \frac{2x^2}{x^2 - 1}$$

Step 8: Sketch



Curve Sketching: A Final Example

Problem Statement

For the function

$$y = \frac{x^2 + 2x + 4}{2x}$$

find and classify all the critical points, determine the asymptotes and hence sketch the curve.

Curve Sketching — Final Example: $y = \frac{x^2 + 2x + 4}{2x}$

Identify the Critical Points

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$:

$$\begin{aligned}
y &= \frac{x^2 + 2x + 4}{2x} \\
\frac{dy}{dx} &= \frac{2x(2x+2) - (x^2 + 2x + 4)2}{4x^2} \\
&= \frac{2x^2 - 8}{4x^2} = \frac{x^2 - 4}{2x^2} \\
\frac{d^2y}{dx^2} &= \frac{2x^2(2x) - (x^2 - 4)4x}{4x^4} \\
&= \frac{16x}{4x^4} = \frac{4}{x^3}
\end{aligned}$$

Curve Sketching — Final Example: $y = \frac{x^2 + 2x + 4}{2x}$

Identify the Critical Points (Cont'd)

Since

$$\frac{dy}{dx} = \frac{x^2 - 4}{2x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{4}{x^3}$$

we see that

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x^2 - 4 = 0 \quad \Rightarrow \quad x = \pm 2$$

and since

$$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = -\frac{1}{2} < 0 \qquad \left.\frac{d^2y}{dx^2}\right|_{x=+2} = \frac{1}{2} > 0$$

there is a local maximum at $(-2, -1)$ and a local minimum at $(2, 3)$.

Curve Sketching — Final Example: $y = \dfrac{x^2+2x+4}{2x}$

Identify the Asymptotes

Note also that the denominator of y is zero when $x = 0$ and,

$$\lim_{x\to 0^-} \frac{x^2+2x+4}{2x} = -\infty \qquad \lim_{x\to 0^+} \frac{x^2+2x+4}{2x} = +\infty$$

there is a vertical asymptote at $x = 0$ as $x \to 0^-$ and as $x \to 0^+$. Furthermore, by division, we find that

$$\frac{x^2+2x+4}{2x} = \frac{x}{2} + 1 + \frac{2}{x}$$

and, since

$$\lim_{x\to -\infty} \frac{2}{x} = \lim_{x\to +\infty} \frac{2}{x} = 0$$

There is an asymptote in the line $y = \frac{x}{2} + 1$ as $x \to \pm\infty$.

Curve Sketching — Final Example — The Graph