Proposition: Suppose $A_1, A_2, \ldots, A_n$ is a partition of a set A and a relation R is defined on A by $(x, y) \in R$ if and only if x and y belong to the same block of the partition. Then R is an equivalence relation.

Proof: Every element $a \in A$ is in the same block as itself, giving $(a, a) \in R$. This makes R reflexive. If $(a, b) \in R$, then a and b belong to the same block of the partition. However, this means b and a belong to the same block of the partition and $(b, a) \in R$. This makes R symmetric. If $(a, b) \in R$ and $(b, c) \in R$, then a and b belong to the same block of the partition and b and c belong to the same block of the partition. However, this means all three belong to the same block of the partition and $(a, c) \in R$. This makes R transitive.

Definition: If R is an equivalence relation on a set A and $a \in A$ we define the equivalence class of a to be the subset of A given by

$$E_a = \{b \in A \mid (a, b) \in R\}.$$

Example: Let A be a set of first year DCU students taking CA, BS or AL, A_1 be the set of CA1 students in A, A_2 be the set of BS1 students in A and A_3 be the set of AL1 students in A. Define the relation R by aRb if b is in the same programme as a. If a is a CA1 student then $E_a = A_1$.

Example: Let $A = Z$ and say aRb if $b - a$ is divisible by 3. Then

$$E_5 = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$$

$$E_9 = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$$

Example: Let A be the set of ratios p/q where p and q are integers and $q \neq 0$. Define the relation R on A by $(p/q)R(s/t)$ if $pt = qs$. Then

$$E_{3/2} = \{\ldots. -6/(-4), -3/(-2), 3/2, 6/4, 9/6, \ldots\}$$

The equivalence classes are the usual rational numbers $\mathbb{Q}$.

Theorem: Let R be an equivalence relation on a non-empty set A. Then the collection of distinct equivalence classes form a partition of A.

Proof: For each $a \in A$ we have $a \in E_a$ since R is reflexive. This means the equivalence classes are nonempty and the union of the equivalence classes gives all of A. Next we show that $(a, b) \in R$ means $E_a = E_b$. (Recall that we show sets X and Y are equal by showing $X \subseteq Y$ and $Y \subseteq X$.) Suppose

$c \in E_a$. This means $(a, c) \in R$. We know $(a, b) \in R$, so that $(b, a) \in R$ by symmetry, and $(b, c) \in R$ by transitivity. So $c \in E_b$ and $E_a \subseteq E_b$. Similarly, $E_b \subseteq E_a$ and $E_a = E_b$. Finally suppose $E_a \cap E_b \neq \emptyset$. So there is a c in the intersection. This means $(a, c) \in R$ and $(b, c) \in R$. Again symmetry and transitivity give $(a, b) \in R$ and $E_a = E_b$. Thus the distinct equivalence classes are disjoint.