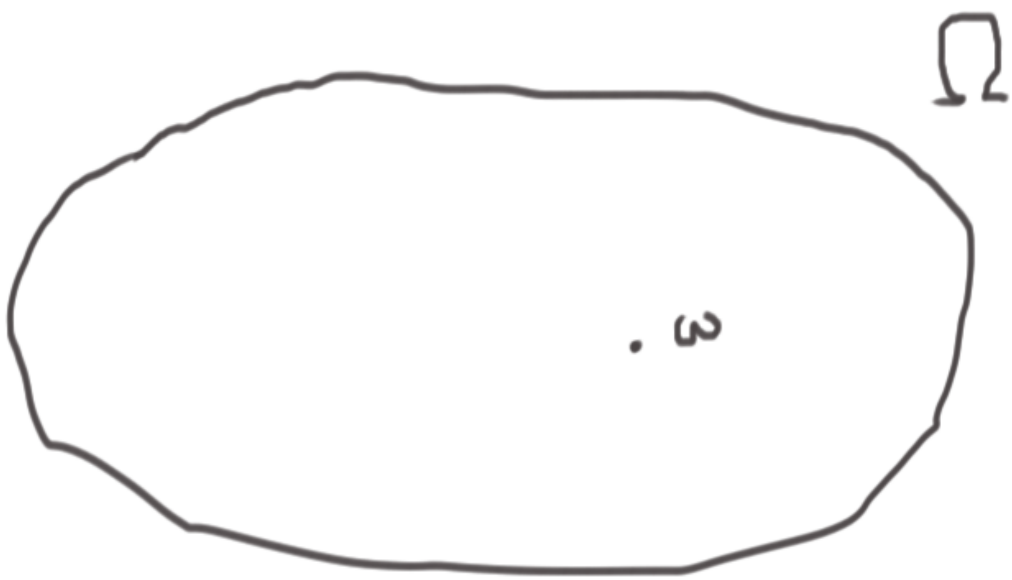
We think of an experiment involving randomness as an experiment with many possible outcomes. In order to build a mathematical model of the experiment we must first know the set of all possible outcomes of the experiment. This is called the *sample space* of the experiment and is denoted Ω (Capital omega). The individual outcomes, denoted ω (little omega) are the elements of the set Ω.



Example: The experiment involves tossing a coin. It can fall 'heads' or 'tails' so that the sample space is

$$\Omega = \{H, T\}.$$

Example: The experiment involves tossing a coin three times. Since each toss can fall 'heads' or 'tails' an outcome is a string of three symbols with each one a 'H' or a 'T'.

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Example: The experiment involves throwing a die. The upper side of the die can show any of the numbers $1, 2, 3, 4, 5, 6$ so that the sample space is

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

Example: In the Lottery a machine draws successively 6 balls out of a pool of 42 numbered balls (disregard bonus ball). Although the balls are drawn in a specific order, the rules of the lottery are such that this order is irrelevant. So the outcome is a subset of size 6 taken from the set $\{1, 2, \ldots, 42\}$. The sample space is

$$\Omega = \{\{n_1, n_2, n_3, n_4, n_5, n_6\} \mid 1 \leq n_1 < n_2 < n_3 < n_4 < n_5 < n_6 \leq 42, n_j \in \mathbb{N}\}.$$

Example: A standard deck of cards has 52 cards, identified by their value

$$V = \{2, 3, 4, 5, \ldots, 9, 10, J, Q, K, A\}$$

and their suits

$$S = \{C(\clubsuit), D(\diamondsuit), H(\heartsuit), S(\spadesuit)\}$$

In the game of poker each player is dealt 5 cards, so the appropriate sample space is

$$\Omega = \{\{h_1, h_2, h_3, h_4, h_5\} \mid h_i \in V \times S\}.$$

Example: The Birthday problem. To analyse the probability of coincident birthdays in a group of n people, we conduct the following experiment. Order the people in some way, say alphabetically, and record the birthday of each person. An outcome is an ordered sequence

$$(d_1, d_2, \ldots, d_n) \quad \text{with} \quad d_j \in \{1, 2, \ldots, 365\}$$

(we disregard leap years). This is an n-sample from 365.

Sometimes, the question that we want to answer is 'what is the probability of a specific outcome?' (e.g., lotto: want probability of your grid being selected by the machine). But much more often, the relevant question is of the type 'what is the probability of the outcome being one of a specific set?'. Subsets of the sample space are called *events*.

Example: In the experiment where a coin is tossed three times with sample space

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

the event 'exactly two heads occur' is

$$A = \{HHT, HTH, THH\},$$

while the event 'tails come up for the first time at the second throw' is

$$B = \{HTH, HTT\}.$$

Example: In the lottery problem, the event 'all the winning numbers are even' is

$$A = \{\{n_1, n_2, \ldots, n_6\} \mid 2 \leq n_1 < n_2 < \ldots < n_6 \leq 42, n_j \in \{2, 4, 6, \ldots, 42\}\}.$$

Example: In the birthday problem, the event 'at least one common birthday' is

$$A = \{(d_1, d_2, \ldots, d_n) \mid d_i = d_j \quad \text{for some} \quad i \neq j\}.$$

Because events are subsets of the sample space Ω, we can use the tools of set theory to operate on them. The following summarises the probabilistic meanings of the various set operations:

Notation	set terminology	event terminology
$A \subset \Omega$	A is a subset	A is an event
$A \cup B$	the union of A and B	A or B
$A \cap B$	the intersection of A and B	A and B
$A \cap B = \emptyset$	A and B are disjoint	A and B are incompatible
$A \subseteq B$	A is included in B	A implies B
$\bar{A}$	the complement of A	the converse event of A
$\emptyset$	the empty set	the impossible event
Ω	the entire set	the certain event

Exercise: Review the set theoretic identities, see what they mean in probabilistic terms and convince yourself that they are true.

Probabilities.

We will be considering mostly experiments with a finite number of possible outcomes. For these the sample space is a finite set

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}.$$

As before we use the notation $|\Omega| = n$ to signify that Ω has n elements.

Note: Recall that such a sample space has 2^n events, i.e., 2^n distinct subsets, including $\emptyset$ and Ω itself.

To complete our model, it remains to construct a *probability measure*, i.e., a function $\mathbb{P}$ which associates to every event A a number $\mathbb{P}[A]$ with $0 \leq \mathbb{P}[A] \leq 1$. This is done as follows:
(a) To each outcome ω_j associate a number $p_j \geq 0$ in such a way that

$$p_1 + p_2 + \ldots + p_n = 1.$$

(Then clearly $p_j \leq 1$, for each j); the number p_j is called the *probability of the outcome* ω_j.
Next we define the probability of an event to be the sum of the probabilities of the outcomes contained in the event.

$$\mathbb{P}[A] = p_{j_1} + p_{j_2} + \ldots + p_{j_k}, \text{ where } A = \{\omega_{j_1}, \omega_{j_2}, \ldots, \omega_{j_k}\}.$$

Note: We use the convention $\mathbb{P}[\emptyset] = 0$, and we note that by construction

$$\mathbb{P}[\Omega] = p_1 + p_2 + \ldots + p_n = 1.$$

Note: The theory does not tell you how to choose the numbers p_j; it is for you to decide on an appropriate choice of p_j's to model the specific experiment.

Example: The experiment is to toss a coin. Here $\Omega = \{H, T\}$. Since there are only two outcomes, we need only choose two non-negative numbers p_H and p_T with $p_H + p_T = 1$. If the coin is fair, we must have $p_H = p_T$ and hence $p_H = p_T = 1/2$. Here

$$\mathbb{P}[\Omega] = 1, \mathbb{P}[H] = 1/2, \mathbb{P}[T] = 1/2, \mathbb{P}[\emptyset] = 0.$$

Other choices model a biased coin (e.g., $p_H = 2/3$, $p_T = 1/3$.)

Example: The experiment is to toss a coin three times. Here

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Again a fair coin leads to giving equal probability 1/8 to each outcome. We can compute the probability of any events associated to this sample space. If A is the event 'exactly two H's come up' then

$$A = \{HHT, HTH, THH\}$$

and we get

$$\mathbb{P}[A] = p_{HHT} + p_{HTH} + p_{THH} = 1/8 + 1/8 + 1/8 = 3/8.$$

If B is the event 'at least one H comes up' then

$$B = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$$

and we get

$$\begin{aligned} \mathbb{P}[B] &= p_{HHH} + p_{HHT} + p_{HTH} + p_{HTT} + p_{THH} + p_{THT} + p_{TTH} \\ &= 1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 \\ &= 7/8. \end{aligned}$$

In general, $\mathbb{P}[C] = (1/8)|C|$, for this example.

Note: In those examples where the outcomes are all equally likely, their common value p must satisfy

$$1 = \mathbb{P}[\Omega] = p + p + p + \ldots + p = |\Omega| \cdot p$$

and p must be $1/|\Omega|$. In such a case, calculating the probability of an event amounts to counting the number of outcomes in it:

$$\mathbb{P}[A] = p_{j_1} + p_{j_2} + \ldots + p_{j_k} = |A| \cdot p = \frac{|A|}{|\Omega|}.$$

Example: What is the probability that a hand of poker gives a 'straight flush' (i.e., 5 cards of consecutive value, all of the same suit, such as $8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit$)?

To find the answer, we come up with a procedure for constructing a generic straight flush:

Step 1	choose suit	4 choices
Step 2	Choose card of lowest value	10 choices

(Note: Ace can serve as highest card or as lowest card so that lowest card can belong to set $\{A, 2, 3, \ldots, 10\}$.)

This procedure determines the hand fully and uniquely; moreover all straight flushes can be constructed in this way. Therefore, there are 40 different hands of poker that give a straight flush and the probability is $40/\begin{pmatrix} 52 \\ 5 \end{pmatrix} = 1/(64974) = 0.00015$.