Example: If $A = \{0, 1\}$ then A^4 is the set of binary strings of length 4. Using $0 < 1$ we can totally order A^4.

The correct order is

$$\begin{array}{c} 0000 \\ 0001 \\ 0010 \\ 0011 \\ 0100 \\ 0101 \\ 0110 \\ 0111 \\ 1000 \\ 1001 \\ 1010 \\ 1011 \\ 1100 \\ 1101 \\ 1110 \\ 1111 \end{array}$$

Definition: The inverse, R^{-1}, of a relation $R \subseteq A \times B$ between a set A and a set B is the relation between B and A given by

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}.$$

Example: If A and B are sets of people and R is the relation 'is a parent of', then R^{-1} is the relation 'is a child of'.

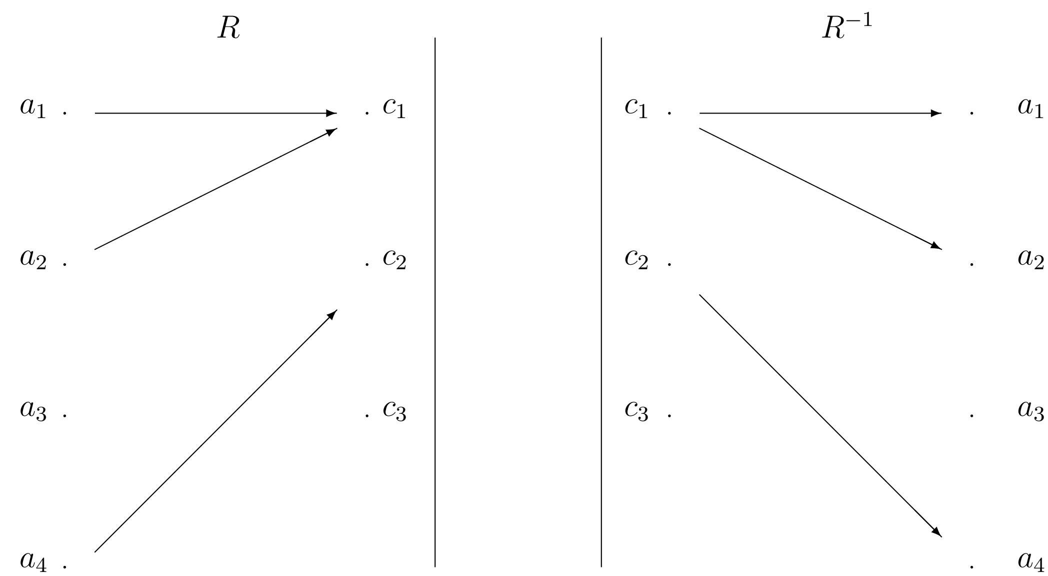
Example: If $A = B$ is a set of positive integers and R is the relation 'is a divisor of', then R^{-1} is the relation 'is a multiple of'.

Note: The digraph of R^{-1} is obtained from the digraph of R by reversing the direction of each edge. The matrix of R^{-1} is obtained from the matrix of R by reflecting it along its diagonal.

Example: For $A = \{a_1, a_2, a_3, a_4\}$ and $C = \{c_1, c_2, c_3\}$ with

$$R = \{(a_1, c_1), (a_2, c_1), (a_4, c_2)\}$$

the digraphs of R and R^{-1} are



and the matrices are

R	c_1	c_2	c_3
a_1	T	F	F
a_2	T	F	F
a_3	F	F	F
a_4	F	T	F

R^{-1}	a_1	a_2	a_3	a_4
c_1	T	T	F	F
c_2	F	F	F	T
c_3	F	F	F	F

Note: Suppose R is a relation on a set A. If R is reflexive, then R^{-1} is also reflexive. (Interchanging the entries of (a, a) gives (a, a).) If R is symmetric, then R^{-1} is the same relation as R. (If $(a, b) \in R$ then $(b, a) \in R$ by symmetry. This means $(a, b) \in R^{-1}$. Similarly $R^{-1} \subseteq R$.) If R is transitive, then R^{-1} is also transitive. (If (a, b) and (b, c) are pairs in R^{-1}, then (b, a) and (c, b) are pairs in R. However R transitive means $(c, a) \in R$ so that $(a, c) \in R^{-1}$.)

Definition: If R is a relation between a set A and a set B and S is a relation between B and a set C then the composition of S with R, written $S \circ R$, is the relation between A and C given by

$$S \circ R = \{(a, c) \in A \times C \mid \text{ for some } b \in B, \ [((a, b) \in R) \textbf{ and } ((b, c) \in S)]\}.$$

Example: If A is a set of men, R is the relation 'is the father of' and S is the relation 'is a brother of', then $S \circ R$ is the relation 'is an uncle of' while $R^2 = R \circ R$ is the relation 'is a grandfather of'.

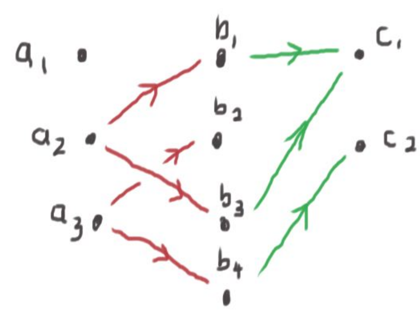
Note: The digraph of the composition of two relations can be read from the digraphs of the relations.

Example: Suppose $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3, b_4\}$ and $C = \{c_1, c_2\}$ and the relations R and S are given by

$$R = \{(a_2, b_1), (a_2, b_3), (a_3, b_2), (a_3, b_4)\} \subseteq A \times B$$

$$S = \{(b_1, c_1), (b_3, c_1), (b_4, c_2)\} \subseteq B \times C$$

draw the corresponding digraphs, deduce the digraph for $S \circ T$ and express $S \circ T$ as a set of ordered pairs.



For $a_1 \in A$ and $c_1 \in C$ we see no $(a_1, b) \in R$ so $(a_1, c_1) \notin S \circ R$. Similarly, $(a_1, c_2) \notin S \circ R$. For $a_2 \in A$ and $c_1 \in C$ we see $(a_2, b_1) \in R$ and $(b_1, c_1) \in S$, so $(a_2, c_1) \in S \circ R$. (We could also go via b_3.) For $a_2 \in A$ and $c_2 \in C$ we see $(a_2, b_1), (a_2, b_3) \in R$ but $(b_1, c_2) \notin S$ and $(b_3, c_2) \notin S$, so $(a_2, c_2) \notin S \circ R$. For $a_3 \in A$ and $c_1 \in C$ we see $(a_3, b_2), (a_3, b_4) \in R$ but $(b_2, c_1) \notin S$ and $(b_4, c_1) \notin S$, so $(a_3, c_1) \notin S \circ R$. For $a_3 \in A$ and $c_2 \in C$ we see $(a_3, b_4) \in R$ and $(b_4, c_2) \in S$, so $(a_3, c_2) \in S \circ R$. Thus

$$S \circ R = \{(a_2, c_1), (a_3, c_2)\}$$

Note: The matrix of the composition of two relations can be deduced from the matrices of the relations using a logical matrix product. (We will not pursue this.)