MS121: IT Mathematics

Integration

Two Techniques of Integration

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Outline

1. Integration by Substitution: The Technique
2. Integration by Substitution: Worked Examples
3. Integration by Parts: The Technique
4. Integration by Parts: Worked Examples
5. Two Definite Integral Examples

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Integration by Substitution

Learn from Differentiation

To differentiate

$$y = e^{x^3}$$

we substitute $u = x^3$ and differentiate using the Chain Rule. We have

$$\begin{aligned} y &= e^u \qquad u = x^3 \\ \Rightarrow \frac{dy}{du} &= e^u \qquad \frac{du}{dx} = 3x^2 \\ \Rightarrow \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot \left(3x^2\right) = 3x^2\, e^{x^3} \end{aligned}$$

Integration by Substitution

$$y = e^{x^3} \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2\, e^{x^3}$$

Applying this to Integration

From the above, we therefore know that

$$\int 3x^2 e^{x^3}\, dx = e^{x^3}$$

How could we calculate the integral directly? The reason that it is difficult to calculate

$$\int 3x^2 e^{x^3}\, dx$$

is similar to that which makes it difficult to calculate the derivative of e^{x^3}.

Integration by Substitution

Find $\int 3x^2\, e^{x^3}\, dx$

We try the same substitution again, so

$$u = x^3 \qquad \Rightarrow \qquad \frac{du}{dx} = 3x^2 \qquad \Rightarrow \qquad du = 3x^2\, dx$$

Now replace the x's under the integral (including dx) by the u's (and du):

$$\begin{aligned} \int 3x^2 e^{x^3}\, dx &= \int e^{x^3} \star 3x^2\, dx \\ &= \int e^u\, du \\ &= e^u = e^{x^3} \end{aligned}$$

which gives the same answer as before.

Integration by Substitution

Technique: What is the best choice for u?

The following are some useful tips:

- Let $u =$ the "most complicated expression", particularly if that expression is the argument of another function. This is the same choice of u that one would use if differentiating using the chain rule (e.g., choosing $u = x^3$ above).
- Let $u = f(x)$ if $f'(x)\, dx$ is also present under the integral.
- If the integrand (function to be differentiated) is a quotient, let $u =$ the denominator.

These hints may be contradictory and they are not foolproof. However, they are sufficient to tackle a large class of integrals and, with enough practice, one can develop an intuition for the correct strategy.

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Integration by Substitution: Examples

Calculate $\int \frac{x^4}{x^5+2}\,dx$

We let

$$u = x^5 + 2 \quad \Rightarrow \quad \frac{du}{dx} = 5x^4 \quad \Rightarrow \quad du = 5x^4\,dx$$

Replacing the x's by u's:

$$\int \frac{x^4}{x^5+2}\,dx = \int \frac{x^4\,dx}{x^5+2} = \int \frac{\frac{1}{5}\,du}{u} = \frac{1}{5}\int u^{-1}\,du$$
$$= \frac{1}{5}\log u = \frac{1}{5}\log(x^5+2)$$

Note: Always write the final answer in terms of the original variable (in this case x).

Integration by Substitution: Worked Examples | Finding the "Right" Substutution

Integration by Substitution: Examples

Calculate $\int x^3\sqrt{16+x^4}\,dx$

Note that the expression which is the argument of the square root is a good candidate for substitution. Accordingly, Let

$$u = 16 + x^4 \quad \Rightarrow \quad \frac{du}{dx} = 4x^3 \quad \Rightarrow \quad du = 4x^3\,dx$$

Look to arrange $4x^3\,dx (= du)$ in one group in the integral:

$$\int x^3\sqrt{16+x^4}\,dx = \int \sqrt{16+x^4} \star x^3\,dx$$
$$= \int \sqrt{u} \star \frac{1}{4}\,du = \frac{1}{4}\int u^{1/2}\,du$$
$$= \frac{1}{4}\frac{u^{1+1/2}}{1+1/2} = \frac{u^{3/2}}{6} = \frac{(16+x^4)^{3/2}}{6}$$

Integration by Substitution: Worked Examples | Finding the "Right" Substutution

Integration by Substitution: Examples

Calculate $\int \frac{\log x}{x}\,dx$

If $f(x) = \log x$, then $f'(x)\,dx = \frac{1}{x}\,dx$ is present in the integrand. Therefore, put

$$u = \log x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \quad \Rightarrow \quad du = \frac{dx}{x}$$

$$\int \frac{\log x}{x}\,dx = \int \log x \star \frac{dx}{x} = \int u\,du = \frac{u^2}{2} = \frac{1}{2}(\log x)^2$$

Integration by Substitution: Worked Examples | Finding the "Right" Substutution

Integration by Substitution: Examples

Calculate $\int x^3(x^4+4)^5\,dx$

Let $u = x^4 + 4$, the argument of the fifth power. Then

$$\frac{du}{dx} = 4x^3 \quad \Rightarrow \quad du = 4x^3\,dx$$

Group $4x^3\,dx (= du)$ together and substitute:

$$\int x^3(x^4+4)^5\,dx = \int (x^4+4)^5 . x^3\,dx$$
$$= \int u^5 \star \frac{1}{4}\,du = \frac{1}{4}\int u^5\,du = \frac{1}{4}\frac{u^6}{6} = \frac{(x^4+4)^6}{24}$$

Integration by Substitution: Examples

Calculate $\displaystyle\int \frac{x}{\sqrt{2x+2}}\,dx$

Let $u = 2x + 2$, the argument of the square root. Thus

$$\frac{du}{dx} = 2 \quad \Rightarrow \quad du = 2\,dx$$

Now, we replace x (and dx) by u (and du):

$$\int \frac{x}{\sqrt{2x+2}}\,dx = \int \frac{x}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{2}\int \frac{x}{\sqrt{u}}\,du$$

Integration by Substitution: Examples

$$\int \frac{x}{\sqrt{2x+2}}\,dx = \int \frac{x}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{2}\int \frac{x}{\sqrt{u}}\,du$$

Solution (Cont'd)

We now replace x by expressing it in terms of u: $x = \frac{1}{2}(u-2)$, so

$$\begin{aligned}\int \frac{x}{\sqrt{2x+2}}\,dx &= \frac{1}{2}\int \frac{\frac{1}{2}(u-2)}{\sqrt{u}}\,du \\ &= \frac{1}{4}\int\left(\sqrt{u} - \frac{2}{\sqrt{u}}\right)du = \frac{1}{4}\left(\frac{u^{3/2}}{3/2} - 2\frac{u^{1/2}}{1/2}\right) \\ &= \frac{1}{6}u^{3/2} - u^{1/2} = \frac{1}{6}(2x+2)^{3/2} - (2x+2)^{1/2}\end{aligned}$$

Integration by Substitution: Trogonometry Examples

Calculate $\displaystyle\int \sin 6x\,dx$

We let $u = 6x$, the argument of the "sin" function:

$$\frac{du}{dx} = 6 \quad \Rightarrow \quad du = 6\,dx$$

Therefore:

$$\begin{aligned}\Rightarrow \int \sin 6x\,dx &= \int \sin u \star \frac{1}{6}\,du = \frac{1}{6}\int \sin u\,du \\ &= \frac{1}{6}\cdot(-\cos u) = -\frac{1}{6}\cos 6x\end{aligned}$$

Integration by Substitution: Trogonometry Examples

Calculate $\displaystyle\int \frac{\sin x}{2+\cos x}\,dx$

Since the integrand is a quotient, we put $u = 2 + \cos x$, so $du = -\sin x\,dx$. Therefore, we have

$$\begin{aligned}\int \frac{\sin x}{2+\cos x}\,dx &= \int \frac{\sin x\,dx}{2+\cos x} = \int \frac{-du}{u} \\ &= -\log u = -\log(2+\cos x)\end{aligned}$$

Integration by Substitution: Trogonometry Examples

Calculate $\int \sin x \cos^3 x \, dx$

Let $u = \cos x$, the argument of the cube. Then

$$\frac{du}{dx} = -\sin x \quad \Rightarrow \quad du = -\sin x \, dx$$

Group $-\sin x \, dx (= du)$ together and substitute:

$$\begin{aligned} \int \sin x \cos^3 x \, dx &= \int \cos^3 x \star \sin x \, dx = \int u^3 \star (-du) \\ &= -\int u^3 \, du = -\frac{u^4}{4} = -\frac{\cos^4 x}{4} \end{aligned}$$

Integration by Substitution: Trogonometry Examples

Calculate $\int \frac{\tan^5 x}{\cos^2 x} dx$

We solve as follows:

$$\frac{d}{dx} \tan x = \sec^2 x, \quad \text{so put } u = \tan x \text{ to obtain}$$

$$du = \sec^2 x \, dx = \frac{dx}{\cos^2 x}$$

$$\Rightarrow \int \frac{\tan^5 x}{\cos^2 x} dx = \int u^5 \, du = \frac{u^6}{6} = \frac{\tan^6 x}{6}$$

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Integration by Parts

The Link to Differentiaton

- We have seen how integration by substitution corresponds to the chain rule for differentiation.
- It is reasonable to ask if there is a rule of integration corresponding to the product rule?
- The answer to this question is "yes" and the rule is that of integration by parts.
- This is a rule which is useful for calculating the integrals of products:

$$\boxed{\int u \, dv = uv - \int v \, du}$$

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Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x\,e^{2x}\,dx$

We will let $u = x$ and $dv = e^{2x}\,dx$ so that

$$\int xe^{2x}\,dx = \int u\,dv$$

We also need du and v:

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$$

i.e. we differentiate u to get du. Also

$$dv = e^{2x}\,dx \quad \Rightarrow \quad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$$

i.e. we integrate dv to get v.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x\,e^{2x}\,dx$ (Cont'd)

Let

$$u = x \qquad \text{and} \qquad dv = e^{2x}\,dx$$

so that

$$du = dx \qquad \text{and} \qquad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$$

Now, substitute u, v, du and dv into the integration by parts formula to obtain:

$$\begin{aligned}\int x\,e^{2x}\,dx = \int u\,dv = uv - \int v\,du &= x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx \\ &= \frac{1}{2}x\,e^{2x} - \frac{1}{4}e^{2x}\end{aligned}$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x\cos x\,dx$

Choose $u = x$ and $dv = \cos x\,dx$, so that Inspecting the right-hand side of the integration by parts formula, we also need du and v:

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad \Rightarrow du = dx$$

$$dv = \cos x\,dx \quad \Rightarrow \quad v = \int \cos x\,dx = \sin x$$

Substitute u, v, du and dv into the formula to obtain:

$$\begin{aligned}\int x\cos x\,dx &= \int u\,dv = uv - \int v\,du \\ &= x\sin x - \int \sin x\,dx = x\sin x + \cos x\end{aligned}$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Important Question

- How do we know in practice what to choose for u and dv?
- To answer the question, we first consider another example where we make our choice by trial and error.
- We then present a set of guidelines to assist in the decision-making process.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x^2 \log x\,dx$

We could choose either

$$u = x^2 \qquad \text{or} \qquad u = \log x$$

Which should one try?

(a) $u = x^2$: Then $dv = \log x\,dx$, so

$$v = \int \log x\,dx.$$

This is not listed in the formulae and tables. We will therefore abandon this approach.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x^2 \log x\,dx$ (Cont'd)

(b) $u = \log x$: Then $dv = x^2\,dx$, so

$$v = \int x^2\,dx = \frac{1}{3}x^3 \qquad \text{and} \qquad \frac{du}{dx} = \frac{1}{x} \qquad \Rightarrow \qquad du = \frac{1}{x}\,dx$$

Substitute these expressions into the integration by parts formula:

$$\begin{aligned}\int x^2 \log x\,dx &= \int u\,dv = uv - \int v\,du \\ &= \log x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx \\ &= \frac{1}{3}x^3 \log x - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3 \log x - \frac{1}{9}x^3\end{aligned}$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Answer to the Important Question

To decide how u should be chosen in general, we advocate that the choice should be made in the following order:

L – **L**ogarithm
A – **A**lgebraic: powers of x, e.g. x, x^3
T – **T**rigonometric
E – **E**xponential

Integration by Parts: $\int u\,dv = uv - \int v\,du$

"LATE" Guidelines

The guideline has been used in all the examples:

1. $\int xe^{2x}\,dx$: $x \to$ **A**, $e^{2x} \to$ **E**
   LATE $\Rightarrow u = x$ (**A**), so $dv = e^{2x}\,dx$.
2. $\int x\cos x\,dx$: $x \to$ **A**, $\cos x \to$ **T**
   LATE $\Rightarrow u = x$ (**A**), so $dv = \cos x\,dx$.
3. $\int x^2\log x\,dx$ $x^2 \to$ **A**, $\log x \to$ **L**
   LATE $\Rightarrow u = \log x$ (**L**), so $dv = x^2\,dx$.

This rule gives very reliable guidance, but occasionally it does not work. Sometimes, we must integrate by parts successively to get an answer as the following example will illustrate.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x^2\cos x\,dx$

According to our notation, this is of type LATE $\Rightarrow u = x^2$ (**A**), so $dv = \cos x\,dx$ (**T**). Therefore, we differentiate u and integrate dv to obtain:

$$\frac{du}{dx} = 2x, \qquad v = \int \cos x\,dx = \sin x$$

Substitute u, v, du, dv into the formula:

$$\int x^2\cos x\,dx = \int u\,dv = uv - \int v\,du$$
$$= x^2\sin x - \int \sin x \star 2x\,dx = x^2\sin x - 2\int x\sin x\,dx$$

We see that we cannot calculate the last integral directly.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$

We must use the integration by parts formula again for

$$\int x\sin x\,dx$$

which is again of type LATE $\Rightarrow u = x$ (**A**), so $dv = \sin x\,dx$ (**T**). We differentiate u and integrate dv to obtain:

$$\frac{du}{dx} = 1, \qquad v = \int \sin x\,dx = -\cos x$$

Substituting into the formula yields:

$$\int x\sin x\,dx = -x\cos x - \int(-\cos x)\,dx = -x\cos x + \sin x$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$$
$$\int x\sin x\,dx = -x\cos x + \sin x$$

$\int x^2\cos x\,dx$ (Cont'd)

Substituting the second result into the first yields:

$$\begin{aligned}\int x^2\cos x\,dx &= x^2\sin x - 2\int x\sin x\,dx\\ &= x^2\sin x - 2(-x\cos x + \sin x)\\ &= x^2\sin x + 2x\cos x - 2\sin x\end{aligned}$$

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Definite Integrals: Integration by Substitution Example

Calculate $\displaystyle\int_1^4 \frac{x}{3x^2+1}\,dx$

Substitution

$$u = 3x^2+1 \quad\Rightarrow\quad \frac{du}{dx} = 6x \quad\Rightarrow\quad du = 6x\,dx \quad\Rightarrow\quad \frac{1}{6}du = x\,dx$$

This means that $\displaystyle\int \frac{x}{3x^2+1}\,dx = \frac{1}{6}\int \frac{1}{u}\,du$

Definite Integrals: Integration by Substitution Example

Substitution $u = 3x^2+1$ leads to $\displaystyle\int \frac{x}{3x^2+1}\,dx = \frac{1}{6}\int \frac{1}{u}\,du$

Change the limits of integration

$x = 1 \quad\Rightarrow\quad u = 3x^2+1 = 4 \qquad\qquad x = 4 \quad\Rightarrow\quad u = 3x^2+1 = 49$

Therefore

$$\begin{aligned}
\int_1^4 \frac{x}{3x^2+1}\,dx &= \int_1^4 \frac{1}{3x^2+1}\,(x\,dx) = \int_4^{49} \frac{1}{u}\left(\frac{1}{6}\,du\right) \\
&= \frac{1}{6}\int_4^{49} \frac{1}{u}\,du \\
&= \frac{1}{6}\left[\ln u\right]_4^{49} = \frac{1}{6}(\ln 49 - \ln 4)
\end{aligned}$$

Definite Integrals: Integration by Parts Example

Calculate $\displaystyle\int_1^3 r^3 \ln r\,dr$

Integration by Parts

$$\begin{aligned}
u &= \ln r &\quad\Rightarrow\quad& du = \frac{1}{r}\,dr \\
dv &= r^3\,dr &\quad\Rightarrow\quad& v = \frac{1}{4}r^4
\end{aligned}$$

Definite Integrals: Integration by Parts Example

$\int_1^3 r^3 \ln r \, dr$ with $u = \ln r$ and $dv = r^3 \, dr \;\Rightarrow\; v = \frac{1}{4} r^4$

Using the integration by parts formula

$$\begin{aligned}
\int u \, dv &= \int_1^3 r^3 \ln r \, dr = \left[uv - \int v \, du \right]_{r=1}^{3} \\
&= \left[\frac{1}{4} r^4 \ln r \right]_1^3 - \int_1^3 \frac{1}{4} r^3 \, dr \\
&= \frac{81}{4} \ln 3 - 0 - \frac{1}{4} \left[\frac{1}{4} r^4 \right]_1^3 \\
&= \frac{81}{4} \ln 3 - \frac{1}{16}(81 - 1) = \frac{81}{4} \ln 3 - 5
\end{aligned}$$