Properties of a relation on a single set.

To illustrate important properties that a relation on a single set might have we will consider the following examples of relations.

Example 1: If $A = \{1, 2, 3, 4\}$ and $R = \{(2, 1), (2, 3), (3, 2), (4, 4)\}$.

Example 2: If A is the set of all pages on the web and R is the relation 'web page X links to web page Y'.

Example 3: If $A = \mathbb{N} = \{1, 2, 3, \ldots\}$ and R is the relation given by 'is a divisor of'. The subset R of $A \times A$ is shown in a figure below. $A \times A$ is an infinite grid with lower left corner at $(1, 1)$ and the elements of R are circled. (The parallel lines we see are explained as follows: if $(a, b) \in R$ then $b = ca$ so that $b + kc = ca + kc = c(a + k)$ giving $(a + k, b + kc) \in R$.)

Example 4: If $A = \mathbb{Z}$ and R is the relation given by $(x, y) \in R$ if and only if $y - x$ is even. The subset R of $A \times A$ is shown in a figure below. $A \times A$ is an infinite grid with no corners this time and the elements of R are circled. (The recurring pattern of circles is due to: even - even = even, even - odd = odd, odd - even = odd, odd - odd = even.)

Example 5: Let A be the set of first year students in DCU and define the relation R on A by $(a, b) \in R$ if b is in the same programme as a.

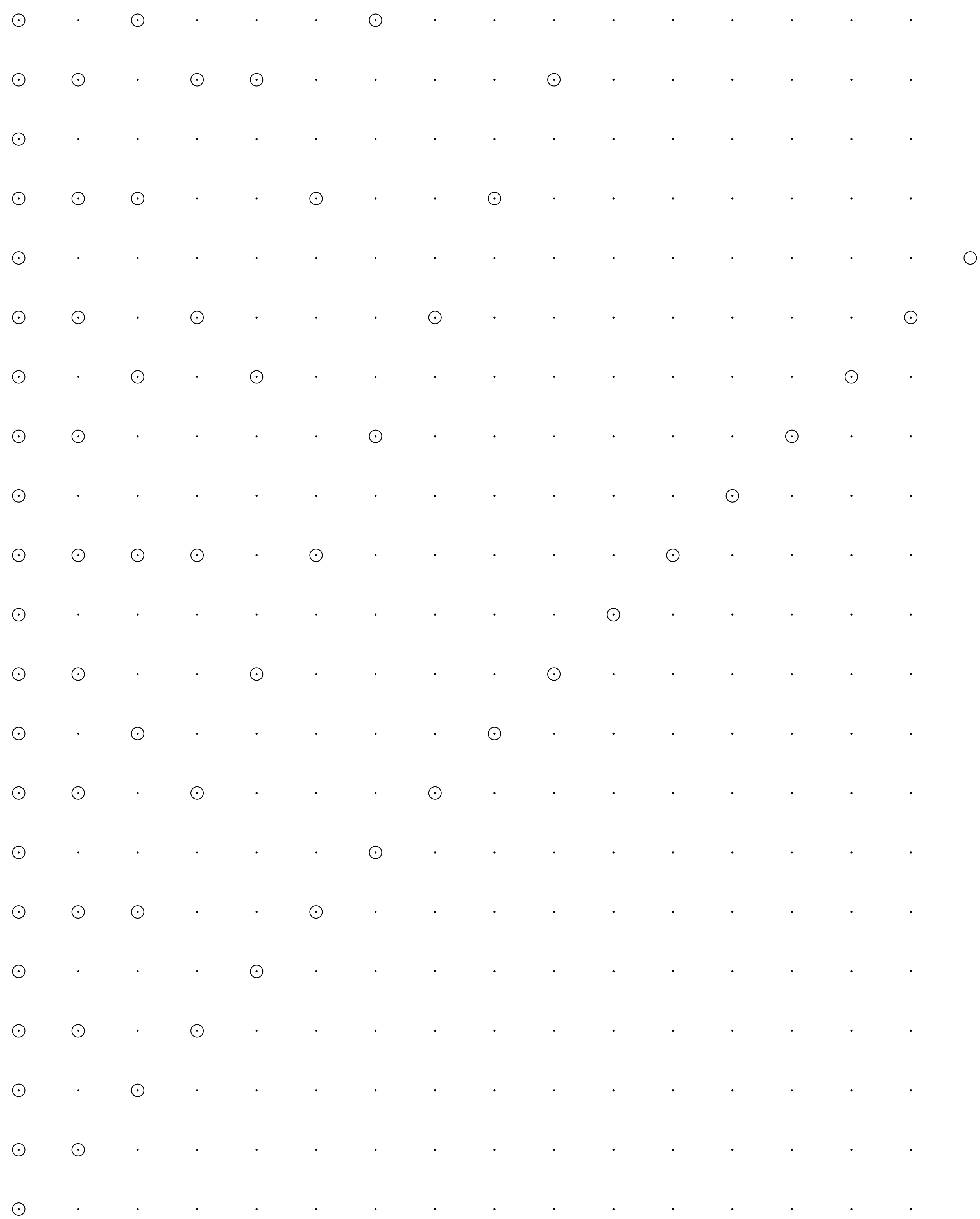
Definition: A relation R on a set A is called reflexive if $(a, a) \in R$ for all $a \in A$.

Note: The digraph of a reflexive relation will have loops at every element of A and the matrix will have T's on the diagonal.

Examples: The R in Example 1 is not reflexive. $(1, 1) \notin R$. The R in Example 2 is not reflexive. Most webpages do not link to themselves. The R in Example 3 is reflexive. If $a \in \{1, 2, 3, \ldots\}$ then a divides a since $a = a(1)$. The R in Example 4 is reflexive. If $x \in \mathbb{Z}$ then $x - x = 0 = 2(0)$ is even so $(x, x) \in R$. The R in Example 5 is reflexive. Every student is in the same programme as himself/herself.

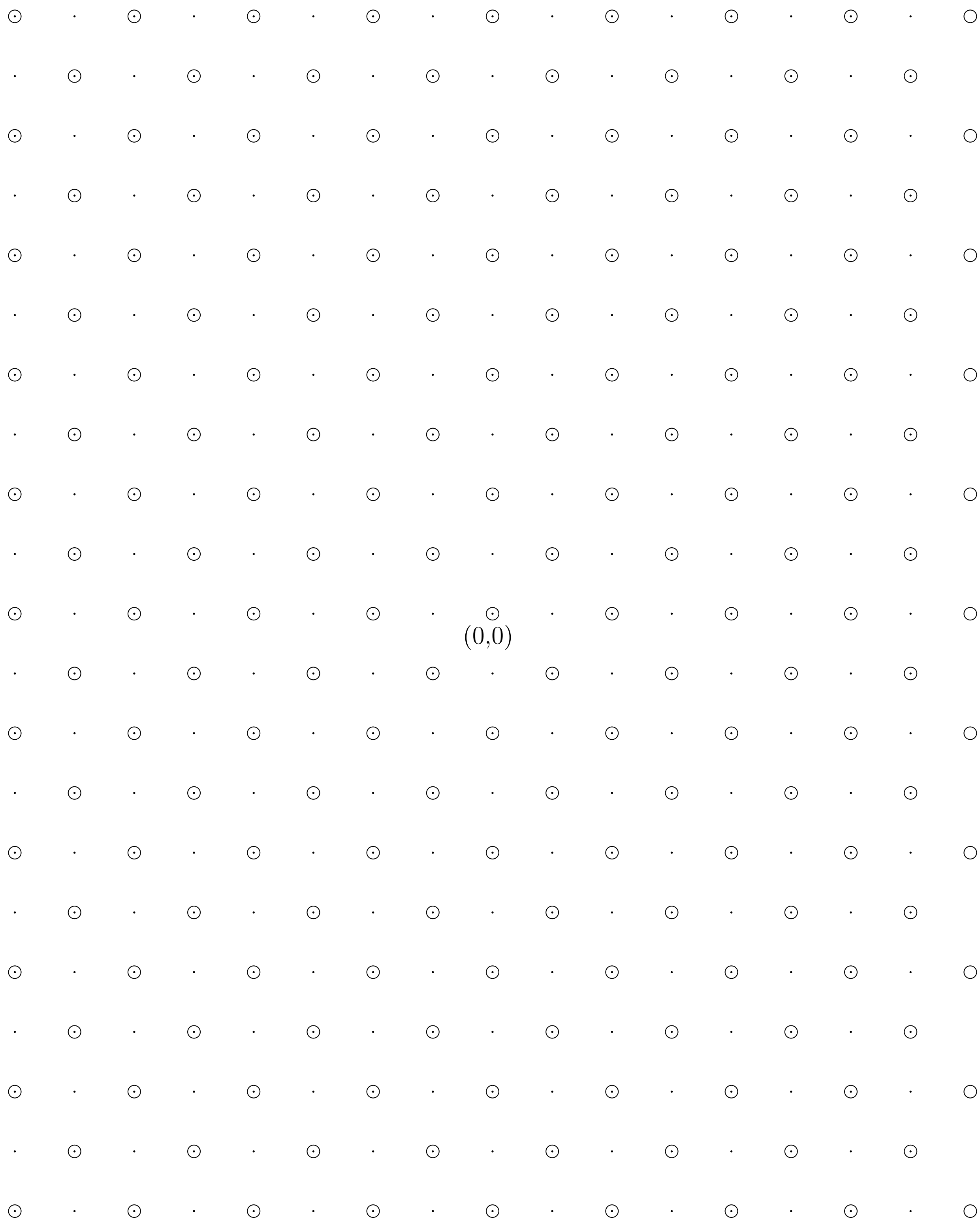
Definition: A relation R on a set A is called symmetric if whenever we have $(a, b) \in R$ we also have $(b, a) \in R$.

Note: The digraph of a symmetric relation will have, for each edge from a to b, an edge going in the opposite direction from b to a, and the matrix will have a T below the diagonal if and only if there is a T above the diagonal in



(1,1)

The relation 'a divides b' inside $\{1, 2, 3, 4, \ldots\} \times \{1, 2, 3, 4, \ldots\}$.



The relation '$y - x$ is even' inside $\mathbb{Z} \times \mathbb{Z}$.

the corresponding position.

Examples: The R in Example 1 is not symmetric. $(2,1) \in R$ but $(1,2) \notin R$. The R in Example 2 is not symmetric. Your page may link to an Irish Times article page but the page will not link back to you. The R in Example 3 is not symmetric. $(2,4) \in R$ since $4 = 2(2)$ but $(4,2) \notin R$ since $2 = (1/2)4$ and $1/2$ is not an integer. The R in Example 4 is symmetric. If $(x,y) \in R$ then $y-x = 2k$ for some integer k. However this gives $x-y = (-1)(y-x) = -2k = 2(-k)$ and $(y,x) \in R$. The R in Example 5 is symmetric. If student A is in the same programme as student B then student B is in the same programme as student A.

Definition: A relation R on a set A is called antisymmetric if whenever we have $(a,b) \in R$ and $(b,a) \in R$, then $a = b$.

Note: The digraph of an antisymmetric relation will have, for each edge from a to b with $b \neq a$, no edge going in the opposite direction from b to a, and the matrix will have for every T off the diagonal an F across the diagonal in the corresponding position.

Examples: The R in Example 1 is not antisymmetric. $(2,3) \in R$ and $(3,2) \in R$ but $2 \neq 3$. The R in Example 2 is not antisymmetric. There are many pairs of friends who link to each others webpages. The R in Example 3 is antisymmetric. If $(a,b) \in R$ and $(b,a) \in R$ the $b = ka$ and $a = lb$ for some positive integers k and l. However this means $a = lb = l(ka) = (lk)a$ so that $lk = 1$ and hence $k = 1$ and $l = 1$. This gives $a = b$. The R in Example 4 is not antisymmetric. $(2,6) \in R$ since $6-2 = 4$ is even and $(6,2) \in R$ since $2-6 = -4$ is even but $2 \neq 6$. The R in Example 5 is not antisymmetric as long as there are two different students in the same programme.