MS121: IT Mathematics

Functions

Absolute Value & Inequalities

John Carroll
School of Mathematical Sciences

Dublin City University

Outline

1. Inequalities involving the Absolute Value
2. Solving More General Inequalities

Outline

1. Inequalities involving the Absolute Value
2. Solving More General Inequalities

Absolute Value & Inequalities

Illustration 1

The equation

$$|2x + 1| = 3$$

has two solutions, obtained from solving

$$2x + 1 = +3 \quad \text{and} \quad 2x + 1 = -3$$

or $x = 1$ and $x = -2$. Therefore, the solution to the absolute value equation $|2x + 1| = 3$ is the set:

$$\{-2, 1\}$$

Absolute Value & Inequalities

Illustration 2

Consider the equation

$$|x| \geq 2$$

Either $x \geq 2$ or $x \leq -2$ and the solution may be expressed in interval form as

$$(-\infty, -2] \cup [2, \infty)$$

Absolute Value & Inequalities

Solve the inequality $|x^2 - 4| > 3$.

The solutions are found from either:

$$\begin{array}{ccc} x^2 - 4 > 3 & \textit{or} & x^2 - 4 < -3 \\ \downarrow & & \downarrow \\ x^2 > 7 & & x^2 < 1 \\ |x| > \sqrt{7} & & |x| < 1 \end{array}$$

giving 4 cases:

$$x > \sqrt{7}, \quad x < -\sqrt{7}, \quad x < 1, \quad x > -1.$$

which can be written in interval form as follows:

$$x \in (-\infty, -\sqrt{7}) \cup (-1, 1) \cup (\sqrt{7}, \infty)$$

Absolute Value & Inequalities

Find all the real roots of the equation

$$|x + 1| + |2x - 3| = 4.$$

Since either $|x + 1| = x + 1$ or $-(x + 1)$ and $|2x - 3| = 2x - 3$ or $-(2x - 3)$, there are 4 cases to consider:

$$\begin{array}{lll} +(x+1) + (2x-3) = 4 & \Rightarrow \; 3x - 2 = 4 & \Rightarrow x = 2 \\ -(x+1) + (2x-3) = 4 & \Rightarrow \; x - 4 = 4 & \Rightarrow x = 8 \\ +(x+1) - (2x-3) = 4 & \Rightarrow \; -x + 4 = 5 & \Rightarrow x = 0 \\ -(x+1) - (2x-3) = 4 & \Rightarrow \; -3x + 2 = 4 & \Rightarrow x = -\dfrac{2}{3} \end{array}$$

The possible solutions are therefore:

$$\left\{2, 8, 0, -\frac{2}{3}\right\}$$

Absolute Value & Inequalities

Possible solutions: $\{2, 8, 0, -\frac{2}{3}\}$.

Answer (cont'd)

We now test each solution in the original inequality:

$$\begin{array}{rcll} |x+1| + |2x-3| & = & 4 & \\ |3| + |1| & = & 4 & \text{True} \quad x = 2 \\ |9| + |13| & \neq & 4 & \text{False} \quad x = 8 \\ |1| + |-4| & = & 4 & \text{True} \quad x = 0 \\ \left|\dfrac{2}{3}\right| + \left|-\dfrac{13}{3}\right| & \neq & 4 & \text{False} \quad x = -\dfrac{2}{3} \end{array}$$

Therefore, the solutions are $x = 0$ and $x = 2$.

Outline

1 Inequalities involving the Absolute Value

2 Solving More General Inequalities

Absolute Value & Inequalities

Question

Solve the inequality $x^2 - 5x + 6 \leq 0$.

Answer

First we factor the left side:

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

We know that the corresponding equation has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \qquad (2, 3) \qquad (3, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Absolute Value & Inequalities

$x^2 - 5x + 6 \leq 0$ (Cont'd)

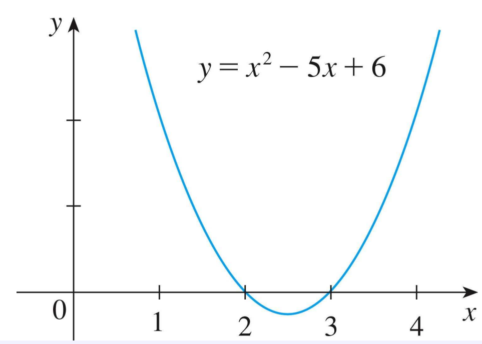
Then we record these signs in the following chart:

	$x-2$	$x-3$	$(x-2)(x-3)$
$(-\infty, 2)$	$-$	$-$	$+$
$(2, 3)$	$+$	$-$	$-$
$(3, \infty)$	$+$	$+$	$+$

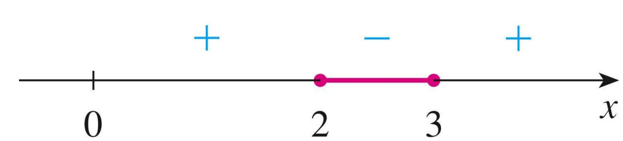
Therefore, $x^2 - 5x - 6 \leq 0$ has solution

$$\{x \mid 2 \leq x \leq 3\} = [2, 3]$$

Absolute Value & Inequalities



$$x^2 - 5x - 6 \leq 0 \qquad \Rightarrow \qquad \{x \mid 2 \leq x \leq 3\} = [2, 3]$$



Absolute Value & Inequalities

Question

Solve the inequality $\dfrac{(x+1)(x-3)}{(x+4)} > 0$.

Answer

Rather than attempt to sketch the function, we will first determine the points where it changes sign, i.e. those x-values which make the numerator or denominator zero:

$$\begin{aligned} x+1=0 &\quad\Rightarrow\quad x=-1 \\ x-3=0 &\quad\Rightarrow\quad x=3 \\ x+4=0 &\quad\Rightarrow\quad x=-4 \end{aligned}$$

We now divide the x-axis into intervals according to these points, namely $(-\infty,-4)$, $(-4,-1)$, $(-1,3)$ and $(3,\infty)$.

$\dfrac{(x+1)(x-3)}{(x+4)} > 0$

Answer (Cont'd)

We construct a table with these intervals and with the factors of the function as follows:

	$x+1$	$x-3$	$x+4$	$f(x)$
$(-\infty,-4)$	$-$	$-$	$-$	$-$
$(-4,-1)$	$-$	$-$	$+$	$+$
$(-1,3)$	$+$	$-$	$+$	$-$
$(3,\infty)$	$+$	$+$	$+$	$+$

We see that the function is positive on the two intervals $(-4,-1)$ and $(3,\infty)$. We therefore write

$$\frac{(x+1)(x-3)}{x+4} > 0 \qquad \text{on} \qquad (-4,-1)\ \cup\ (3,\infty)$$