Example: (MS121 May 2017) In a referendum 55% of the voters vote 'yes' and 45% vote 'no'. A TV station conducts an exit poll. All of those who respond say truthfully how they voted. Of those who vote 'yes' 70% respond and of those who vote 'no' 90% respond. Use Bayes' formula to compute the percentage of 'yes' votes the poll will predict.

For a given voter let R be the event that they respond, Y be the event that they vote 'yes' and N be the event that they vote 'no'. We know that $\mathbb{P}[R \mid Y] = 70/100$, $\mathbb{P}[R \mid N] = 90/100$, $\mathbb{P}[Y] = 55/100$ and $\mathbb{P}[N] = 45/100$, while $\{Y, N\}$ partitions the sample space. We seek $\mathbb{P}[Y \mid R]$. Apply Bayes' Theorem.

$$\begin{aligned}
\mathbb{P}[Y \mid R] &= \frac{\mathbb{P}[Y \cap R]}{\mathbb{P}[R]} \\
&= \frac{\mathbb{P}[R \mid Y]\mathbb{P}[Y]}{\mathbb{P}[R \mid Y]\mathbb{P}[Y] + \mathbb{P}[R \mid N]\mathbb{P}[N]} \\
&= \frac{(70/100)(55/100)}{(70/100)(55/100) + (90/100)(45/100)} \\
&= 77/158.
\end{aligned}$$

Example: (Monty Hall Problem) In a game show a contestant is told that a prize is behind one of three doors. After the contestant picks one door the host (Monty) opens another revealing no prize. The contestant is invited to switch his/her choice to the remaining door or not. What should he/she do?

Solution: Suppose the choice of door has occurred. So we now name the doors with the contestant choosing door 1. Let P_i be the event that the prize is behind door i, and H_i the event that the host opens door i. Conditional probabilities are computed after the contestant has chosen door 1. So $\mathbb{P}[H_1] = 0$ (host will not open the prize door) and $\mathbb{P}[H_i|P_i] = 0$ (host will not open the chosen door). Also $\mathbb{P}[H_i|P_j] = 1/2$ for $j = 1$ (host is equally likely to open either of the other doors if contestant has chosen the prize door) and $\mathbb{P}[H_i|P_j] = 1$ for $j \neq 1$ (host can only open the one other door if contestant has not chosen the prize door).

$$\begin{aligned}
\mathbb{P}[P_1|H_2] &= \frac{\mathbb{P}[P_1 \cap H_2]}{\mathbb{P}[H_2]} \\
&= \frac{\mathbb{P}[H_2|P_1]\mathbb{P}[P_1]}{\mathbb{P}[H_2|P_1]\mathbb{P}[P_1] + \mathbb{P}[H_2|P_2]\mathbb{P}[P_2] + \mathbb{P}[H_2|P_3]\mathbb{P}[P_3]}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1/2)(1/3)}{(1/2)(1/3)+(0)(1/3)+(1)(1/3)} \\
&= 1/3.
\end{aligned}
$$

Just to be sure let's check the other possibility.

$$
\begin{aligned}
\mathbb{P}[P_3|H_2] &= \frac{\mathbb{P}[P_3 \cap H_2]}{\mathbb{P}[H_2]} \\
&= \frac{\mathbb{P}[H_2|P_3]\mathbb{P}[P_3]}{\mathbb{P}[H_2|P_1]\mathbb{P}[P_1]+\mathbb{P}[H_2|P_2]\mathbb{P}[P_2]+\mathbb{P}[H_2|P_3]\mathbb{P}[P_3]} \\
&= \frac{(1)(1/3)}{(1/2)(1/3)+(0)(1/3)+(1)(1/3)} \\
&= 2/3.
\end{aligned}
$$

So be sure to change your choice if you are a contestant in such a show!

Note: It can happen that knowing the event B has occured does not change our view on the probability of A happening. This is the case if $\mathbb{P}[A|B] = \mathbb{P}[A]$ or

$$\mathbb{P}[A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

Definition: Two events A and B are said to be independent for the probability measure $\mathbb{P}$ if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$$

Example: A card is selected at random from a standard deck. If A is the event that the card is an ace and S is the event that the card is a spade then A and S are independent.

Clearly, if all the cards are equally likely to be picked

$$\mathbb{P}[A] = 1/13, \quad \mathbb{P}[S] = 1/4, \quad \mathbb{P}[A \cap S] = 1/52$$

Example: A coin is tossed three times. If A is the event that H comes up at least twice and B is the event that the first time H occurs is on the second toss then A and B are independent.

The sample space has eight equally likely outcomes and the events of interest are

$$A = \{HHT, HTH, THH, HHH\}, B = \{THH, THT\}, A \cap B = \{THH\}$$

The probabilities are

$$\mathbb{P}[A] = 1/2, \quad \mathbb{P}[B] = 1/4, \quad \mathbb{P}[A \cap B] = 1/8$$

So A and B are independent.

Example: Two cards are chosen from a deck of 52 without replacement. If A is the event that the first card is black and B is the event that the second card is black then A and B are not independent.

$$\mathbb{P}[A] = 1/2, \mathbb{P}[B] = 1/2, \mathbb{P}[A \cap B] = \binom{26}{2} / \binom{52}{2} = \frac{25}{102} \neq \frac{1}{4}$$

Note: Do not confuse independent events ($\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$) with disjoint events ($A \cap B = \emptyset$). The two are very different:
Being disjoint is an absolute property, not relative to any probability measure.
If two events are disjoint, knowing that one event has occurred gives you a lot of information about the other event (namely that it cannot occur).
If $A \cap B = \emptyset$ then $\mathbb{P}[A \cap B] = 0$ and independence can only occur if one of A or B has probability zero.

Note: The idea of independence can be extended to collections of n events. A collection of events is said to be mutually independent for the probability measure $\mathbb{P}$ if, for any subcollection of the events, the probability of their intersection is equal to the product of their probabilities.

Definition: A Bernoulli trial is an experiment which can have only two outcomes, often described as 'success' and 'failure'; the standard notation for the probability of success is p so that the probability of failure is $1-p$. A sequence of n Bernoulli trials is an experiment in which n trials take place in succession. The trials are conducted under identical conditions (so that the probability of success in any trial remains p throughout) and independently of each other.

Example: The prototype of a sequence of Bernoulli trials is the experiment 'toss a coin n times '; whether you call H a success or T a success is up to you.

Example: Four coins are tossed and the number of heads showing is noted. The sample space is $\{0, 1, 2, 3, 4\}$ and the probabilities are computed using the sample space of all strings of length 4 in H and T. The event that no heads show is the subset consisting of the single string TTTT with probability (1/2)(1/2)(1/2)(1/2) since the probability of T is 1/2 in each toss. The event that one head shows is the subset $\{HTTT, THTT, TTHT, TTTH\}$ with probability 4(1/2)(1/2)(1/2)(1/2) since the probability of T is 1/2 in each toss and the probability of H on each toss is 1/2. The event that two heads show is the subset

$$\{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}$$

with probability 6(1/2)(1/2)(1/2)(1/2). The number in this subset is $\binom{4}{2}$ corresponding to the number of ways of choosing the two positions for the H's in the string of length 4. We complete the other probabilities similarly.

Sample point	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Example: A die is rolled 6 times and the number of 1's showing is noted. The sample space is $\{0, 1, 2, 3, 4, 5, 6\}$ and the probabilities are computed using the sample space of all strings of length 6 in S and F, where S (success) is an occurence of 1 while F (failure) is an occurence of any of $\{2, 3, 4, 5, 6\}$. Thus $p = 1/6$ and $1 - p = 5/6$. The event that no 1's show is the subset consisting of the single string FFFFFF with probability $(5/6)^6$ since the probability of F is 5/6 for each roll. The event that one 1 shows is the subset $\{SFFFFF, FSFFFF, \ldots, FFFFFS\}$ with probability $6(1/6)(5/6)^5$ since the probability of S is 1/6 in each roll and the probability of F on each roll is 5/6. The event that two 1's show is the subset

$$\{SSFFFF, SFSFFF, \ldots, FFFFSS\}$$

with probability $(15)(1/6)^2(5/6)^4$. The number in this subset is $\binom{6}{2}$ corresponding to the number of ways of choosing the two positions for the S's in the string of length 6. We complete the other probabilities similarly.

Proposition: The probability of exactly j successes in n Bernoulli trials is

$$\binom{n}{j} p^j (1-p)^{n-j} \quad \text{for} \quad 0 \le j \le n$$

where p is the probability of success in any individual trial.