**Definition:** A partial order on a set $A$ is a relation $R$ which is reflexive, antisymmetric and transitive.

**Note:** Recall that
(i) $R$ is reflexive if $(a, a) \in R$ for all $a \in A$,
(ii) $R$ is antisymmetric if whenever we have $(a, b) \in R$ and $(b, a) \in R$ then $a = b$, and
(iii) $R$ is transitive if whenever we have $(a, b) \in R$ and $(b, c) \in R$ we also have $(a, c) \in R$

**Example:** For $A = \mathbb{R}$, the 'less than or equal to' relation,

$$R = \{(x, y) \mid x \leq y\}$$

is a partial order. Think of $x \leq y$ if and only if $y - x \geq 0$. $x \leq x$ for each $x \in \mathbb{R}$ since $x = x$. Thus $\leq$ is reflexive. If $x \leq y$ and $y \leq x$ then $x = y$ so that $\leq$ is antisymmetric. If $x \leq y$ and $y \leq z$ then $x \leq z$ so that $\leq$ is transitive.

**Example:** For $A = P(B)$ (the power set or set of all subsets of a set $B$) the relation

$$R = \{(B_1, B_2) \mid B_1 \subseteq B_2\}$$

is a partial order. Certainly $B_1 \subseteq B_1$ so that $\subseteq$ is reflexive. If $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$ then $B_1$ and $B_2$ have the same elements and $B_1 = B_2$, making $\subseteq$ antisymmetric. If $B_1 \subseteq B_2$ and $B_2 \subseteq B_3$ then each element of $B_1$ is an element of $B_3$ so that $B_1 \subseteq B_3$ and $\subseteq$ is transitive.

**Example:** For $A = \mathbb{N}$ the relation

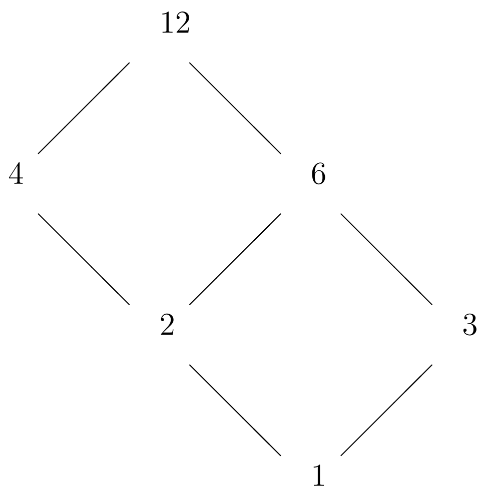
$$R = \{(a, b) \mid a \text{ is a divisor of } b\}$$

is a partial order. For $a \in \mathbb{N}$, $a = a(1)$ so that $a$ is a divisor of $a$ and $R$ is reflexive. If $a, b \in \mathbb{N}$ satisfy $aRb$ and $bRa$, then $b = ca$ and $a = db$. This gives $a = db = dca$ so that $dc = 1$. Since $d$ and $c$ are in $\mathbb{N}$ we have $c = d = 1$ and $b = a$. This makes $R$ antisymmetric. Finally, suppose $a, b, c \in \mathbb{N}$ satisfy $aRb$ and $bRc$. Then $b = pa$ and $c = qb$ so that $c = qb = qpa$. But this gives $aRc$ and $R$ is transitive.

**Terminology:** A set with a partial order on it is called a partially ordered set or poset. If $R$ is a partial order on $A$ and $(a, b) \in R$ with $a \neq b$ we say $a$

is a predecessor of $b$ and $b$ is a successor of $a$. If $a$ is a predecessor of $b$ and there is no successor of $a$ which is a predecessor of $b$ the we say that $a$ is an immediate predecessor of $b$ and write $a \prec b$. Instead of drawing a diagraph of $R$ we often draw a subgraph called a Hasse diagram with an edge connecting $a$ to $b$ whenever $a \prec b$.

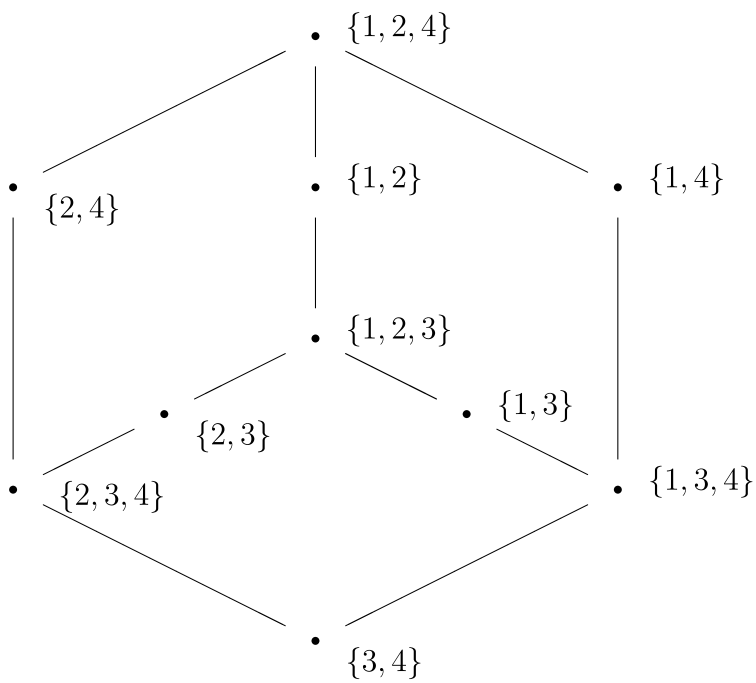
**Example:** $A$ is the set of factors of the integer 12 with partial order given by 'is a divisor of'. Draw the Hasse diagram.



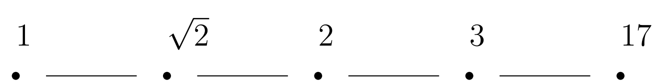

**Example:** $A$ is the set of subsets of $\{1, 2, 3, 4\}$ given by

$$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,3,4\}$$

with partial order given by inclusion. Draw the Hasse diagram.




**Example:** $A$ is the set of real numbers $\{1, 2, \sqrt{2}, 3, 17\}$ with partial order given by $x \leq y$. Draw the Hasse diagram.




**Definition:** A partial order $R$ on a set $A$ is called a total order if whenever $a$ and $b$ belong to $A$ and $a \neq b$ then exactly one of $(a, b) \in R$ or $(b, a) \in R$ holds.

**Example:** $A = \mathbb{R}$ with the partial order $\leq$ is totally ordered. (If $x, y \in \mathbb{R}$ and $x \neq y$, then we have either $y - x > 0$ so that $x \leq y$ or we have $y - x < 0$ so that $y \leq x$.)

**Example:** If $A$ is a totally ordered set then $A \times A$ can be ordered lexicographically by

$$(a_1, a_2) \leq (a_3, a_4) \text{ if } (a_1 < a_3) \textbf{ or } (a_1 = a_3 \textbf{ and } a_2 \leq a_4).$$

This can be extended to several copies of $A$.

**Example:** If $A = \{a, b, \ldots, z\}$ and words are elements of $A \times A \times A \times \ldots$ then the lexicographic order is the one found in the dictionary where we have an end of word character which precedes $a$. Put these words in lexicographic order:
apple, aardvark, anthill, ant, antacid, antithesis

The correct order is

aardvark
ant
antacid
anthill
antithesis
apple

The words all begin with 'a' but the 'aa' precedes the 'an' which in turn precedes the 'ap'. The 'an' words all continue to 'ant' so are ordered by the next character, in this case an end of word character, an 'a', a 'h' and an 'i'.