Example: A program consists of three nested loops of form

while $1 \leq i \leq 4$
    while $1 \leq j \leq i$
        while $1 \leq k \leq j$
            $\boxed{\text{Some commands}}$
        $k \to k+1$
    $j \to j+1$
$i \to i+1$

How many times is the inner loop iterated? During each iteration the variables i, j and k take values in $\{1,2,3,4\}$ but can coincide. The triple (k,j,i) is a selection of 3 from $\{1,2,3,4\}$ with repetition allowed. The number is

$$\binom{3+4-1}{4-1} = \binom{6}{3} = 20.$$

Note: We summarise the formulae for the number of ways of choosing r objects from n, when order is important or not and repetition is allowed or not.

	ordered	unordered
repeated	n^r r-samples	$\binom{r+n-1}{n-1}$ r-selections
unrepeated	$\frac{n!}{(n-r)!}$ r-permutations	$\binom{n}{r}$ r-combinations

Multinomial coefficients: Our final concept in counting will be multinomial coefficients. We will introduce this idea using three examples, which all compute the same number.

Example: How many five-letter words can be formed using the letters of NAVAN?

NNAAV, NNAVA, NNVAA, NANAV, NANVA, NAANV, NAAVN, NAVNA, NAVAN, NVNAA, NVANA, NVAAN, ANNAV, ANNVA, ANANV, ANAVN, ANVNA, ANVAN, AANNV, AANVN, AAVNN, AVNNA, AVNAN, AVANN, VNNAA, VNANA, VNAAN, VANNA, VANAN, VAANN. (30 in total)

These can be counted by looking at possibilities for first letter, then second, etc., However it is easier to first distinguish the two N's and the two A's to give 5! words and then note that these words are in groups of four which all read the same if the two N's and the two A's are undistinguished.

Theorem: The number of arrangements of $n = n_1 + \ldots + n_r$ objects of which n_1 are identical, n_2 are identical, ... , n_r are identical is

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

Example: In how many ways can a group of 5 people be partitioned into 3 ordered sets of sizes 2, 2 and 1?

Break the problem into 3 stages and use the product principle. First choose 2 from 5 for the first subset, then 2 from 3 for the second and finally 1 from 1 for the last. The total is

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{5!}{2!3!}\frac{3!}{2!1!}\frac{1!}{1!0!} = \frac{5!}{2!2!1!}$$

Note the cancelling of the right hand term below the line with the term above the line of the next factor.

Note: The connection between this an the first problem is that each five-letter word formed using the letters of NAVAN is determined by a choice of 2 places from 5 for the N's, a choice of 2 places from the remaining 3 places for the A's leaving 1 place for the single V.

Theorem: The number of ways of partitioning a set of $n = n_1 + \ldots + n_r$ objects into ordered subsets of sizes $n_1, n_2, \ldots, n_r$ is

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

Example: What is the coefficient of x^2y^2z in $(x + y + z)^5$?

Expanding while keeping track of the order of letters will yield 3^5 terms. The ones contributing to x^2y^2z will have 2 x's, 2 y's and 1 z. The number will be the number of ways of partitioning 5 into ordered subsets of sizes 2, 2 and 1 and is thus

$$\frac{5!}{2!2!1!}$$

Theorem: If $n = n_1 + n_2 + \ldots + n_r$ then the coefficient of $x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$ in $(x_1 + x_2 + \ldots + x_r)^n$ is

$$\frac{n!}{n_1! n_2! \ldots n_r!}$$

Definition: Given positive integers $n_1, n_2, \ldots, n_r$ and $n = n_1 + n_2 + \ldots + n_r$, we define the corresponding multinomial coefficient to be

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

Note: When $r = 2$ we have the usual binomial coefficients.
If $1 \leq s < n$ then set $t = n - s$ and write

$$\binom{n}{s} = \frac{n!}{s!(n-s)!} = \frac{(s+t)!}{s!t!} = \binom{s+t}{s, t}$$

Note: A multinomial coefficient can be interpreted as a coefficient in an expansion, the number of ordered partitions of a finite set or the number of rearrangements of a string of letters with repetitions.