Problem Sheet 1

MS121 Semester 2 IT Mathematics

Exercise 1.
Express the following sets as intervals:

(a) $\{x \in \mathbb{R} | \ -5 \leq x, \ x < 20\}$,

(b) $\{x \in \mathbb{R} | \ x > 7\}$,

(c) $\{x \in \mathbb{R} | \ x \leq 27\}$.

$\oslash$

Solution 1.
(a) $[-5, 20)$,

(b) $(7, \infty)$, (∞ is not a real number, so it is never included)

(c) $(-\infty, 27]$.

$\diamondsuit$

Exercise 2.
Draw or sketch the graphs of the following functions. Sometimes it helps to first make a short table with well chosen values of x and $f(x)$.

(a) $f(x) = (x-2)^2 - 1$,

(b) g, the straight line through the points $(0, 2)$ and $(3, -2)$,

(c) $\sin(x)$.

$\oslash$

Solution 2.
(a) $f(x) = (x-2)^2 + 1$ is a parabola with minimum at $(2, -1)$ and zeroes at $(1, 0)$ and $(3, 0)$.

(b) g is given by $g(x) = 2 - \frac{4}{3}x$, so the graph is $y = 2 - \frac{4}{3}x$.

(c) $\sin(x)$ is a standard oscillating function, with $\sin(0) = 0$.

$\diamondsuit$

Exercise 3.
The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given by the formulae

$$f(x) = 2 - x, \qquad g(x) = 4x - x^3.$$

Express each of the following functions as a formula in terms of x:

(a) $f + g$,

(b) $g - f$,

(c) $f \cdot g$,

(d) $f \circ g$,

(e) $g \circ f$,

(f) g/f.

$\oslash$

Solution 3.

(a) $(f + g)(x) = f(x) + g(x) = (2 - x) + (4x - x^3) = 2 + 3x - x^3$,

(b) $(g - f)(x) = g(x) - f(x) = (4x - x^3) - (2 - x) = -2 + 5x - x^3$,

(c) $f \cdot g(x) = f(x) \cdot g(x) = (2 - x)(4x - x^3) = 8x - 4x^2 - 2x^3 + x^4$,

(d) $f \circ g(x) = f(g(x)) = 2 - g(x) = 2 - 4x + x^3$,

(e) $g \circ f(x) = g(f(x)) = 4f(x) - f(x)^3 = 4(2 - x) - (2 - x)^3 = 8x - 6x^2 + x^3$,

(f) $g/f(x) = \frac{g(x)}{f(x)} = \frac{4x-x^3}{2-x}$. This may be simplified $\frac{x(2-x)(2+x)}{2-x} = x(2 + x) = 2x + x^2$ (on $x \neq 2$).

$\diamondsuit$

Exercise 4.
Find the natural domain of the following functions. Express your answers in terms of intervals.

(a) $f(x) = 2x^2$,

(b) $f(x) = x^8$,

(c) $f(x) = x^9$,

(d) $f(x) = \frac{1}{x-7}$,

(e) $f(x) = |x - 4|$,

(f) $f(x) = \sqrt{x^2 - 4}$.

(g) $f(x) = \frac{\sqrt{2-x}}{x^2-1}$,

(h) $f(x) = \frac{\sqrt{4-\sqrt{x}}}{\sqrt{x^2+1}}$.

For the functions in parts $(a) - (f)$, can you also give the range?

$\oslash$

Solution 4.

(a) Domain$(f) = \mathbb{R}$ and Range$(f) = [0, \infty)$,

(b) Domain$(f) = \mathbb{R}$ and Range$(f) = [0, \infty)$,

(c) Domain$(f) = \mathbb{R}$ and Range$(f) = \mathbb{R}$,

(d) Domain$(f) = (-\infty, 7) \cup (7, \infty)$ and Range$(f) = (-\infty, 0) \cup (0, \infty)$,

(e) Domain$(f) = \mathbb{R}$ and Range$(f) = [0, \infty)$,

(f) Domain$(f) = (-\infty, -2] \cup [2, \infty)$ and Range$(f) = [0, \infty)$.

(g) Domain$(f) = (-\infty, -1) \cup (-1, 1) \cup (1, 2]$, because the numerator only makes sense when $x \leq 2$ and the quotient makes no sense when $x = -1$ or $x = 1$.

(h) Domain$(f) = [0, 16]$, because the numerator only makes sense when $x \geq 0$ and $\sqrt{x} \leq 4$, i.e. $0 \leq x \leq 16$. The quotient is no problem, because $\sqrt{x^2 + 1} \geq 1 > 0$ for all $x \in \mathbb{R}$.

$\diamondsuit$

Problem Sheet 2

MS121 Semester 2 IT Mathematics

Exercise 1.

Find the equations that determine the following lines:

(a) The line through $(-1, 2)$ with slope -2.

(b) The line through $(-1, -2)$ and $(1, 3)$.

(c) The line through $(0, 5)$ which is parallel to the line $y = 7 - 3x$.

(d) The line through $(1, 1)$ which is perpendicular to the line $2x - y = 5$.

$\oslash$

Solution 1.

(a) For the line through $(-1, 2)$ with slope -2 we start with the general formula $y = mx + b$. We insert the slope $m = -2$ and then determine b using the point $(-1, 2)$: $2 = -2(-1) + b$, so $b = 0$. Hence, $y = -2x$.

(b) For the line through $(-1, -2)$ and $(1, 3)$ we first determine the slope $m = \frac{3-(-2)}{1-(-1)} = \frac{5}{2}$. Inserting this into $y = mx + b$ we may use either of the points to determine $b = \frac{1}{2}$ and hence $y = \frac{1}{2}(5x + 1)$.

(c) The line through $(0, 5)$ which is parallel to the line $y = 7 - 3x$ must have the same slope, $m = -3$. Using $(0, 5)$ to determine b in $y = -3x + b$ yields $b = 5$ and hence $y = -3x + 5$.

(d) For the line through $(1, 1)$ which is perpendicular to the line $2x - y = 5$ we first note that the given line can be written as $y = 2x - 5$, so it has a slope 2. Any line perpendicular to it must have a slope $m = -\frac{1}{2}$ (the products of the slopes must be -1). Determining b in $y = -\frac{1}{2}x + b$ with the given point $(1, 1)$ yields $b = \frac{3}{2}$ and hence $y = -\frac{1}{2}x + \frac{3}{2}$.

$\diamondsuit$

Exercise 2.

Which of the following functions are even, which ones are odd, and which ones are neither?

(a) $f(x) = 4x^3 - 2x$,

(b) $f(x) = 5x^6$,

(c) $f(x) = x - 3$,

(d) $f(x) = \sin(x)$,

(e) $f(x) = x\sin(x) + \cos(x)$,

(f) $f(x) = 0$.

$\oslash$

Solution 2.

(a) $f(x) = 4x^3 - 2x$ is odd (polynomial with only odd powers),

(b) $f(x) = 5x^6$ is even,

(c) $f(x) = x - 3$ is neither even nor odd,

(d) $f(x) = \sin(x)$ is odd,

(e) $f(x) = x\sin(x) + \cos(x)$ is even,

(f) $f(x) = 0$ is the only function which is both even and odd.

$\diamondsuit$

Exercise 3.
Find the roots of the following functions and determine where the functions are positive and where they are negative:

(a) $f(x) = 49 - x^2$,

(b) $f(x) = x^2 - 5x - 6$,

(c) $g(y) = y^2 - 4y + 4$,

(d) $h(t) = t^2 + 2t + 3$,

(e) $y = -x^2 + 3x - 2$,

(f) $h(x) = x^4 + 4x^2 + 3$.

$\oslash$

Solution 3.

(a) $f(x) = 49 - x^2$ has roots $x = -7$, and $x = 7$, and it is positive on $(-7, 7)$ and negative on $(-\infty, -7) \cup (7, \infty)$,

(b) $f(x) = x^2 - 5x - 6$ has roots $x = -1$, and $x = 6$, and it is positive on $(-\infty, -1) \cup (6, \infty)$ and negative on $(-1, 6)$,

(c) $g(y) = y^2 + 4y + 4$ has one (double) root $y = 2$, and it is positive on $(-\infty, 2) \cup (2, \infty)$ and nowhere negative,

(d) $h(t) = t^2 + t + 1$ has no roots (discriminant is negative) and it is positive on all of $\mathbb{R}$,

(e) $y = -x^2 + 3x - 2$ has roots $x = 1$, and $x = 2$, and it is positive on $(1, 2)$ and negative on $(-\infty, 1) \cup (2, \infty)$,

(f) for $h(x) = x^4 + 4x^2 + 3$ we first note $h(x) = g(x^2)$ with $g(y) = y^2 + 4y + 3$; $g(y)$ has roots at $y = -3$ and $y = -1$, so that $g(y) = (y + 3)(y + 1)$; it follows that $h(x) = (x^2 + 3)(x^2 + 1)$ and neither factor has roots, because $x^2 = -3$ and $x^2 = -1$ have no solutions in $\mathbb{R}$; hence h has no roots and it is positive on the entire $\mathbb{R}$.

$\diamondsuit$

Exercise 4.
Solve the following inequalities for $x \in \mathbb{R}$. Write your answer in terms of intervals.

(a) $1 - 3x \leq -2$,

(b) $1 \leq 7 - 2x < 3$,

(c) $|3x - 4| < 5$,

(d) $2x^2 - 4x < 16$,

(e) $\frac{x+3}{2x+7} > 0$, (Hint: multiply both sides by the positive number $(2x + 7)^2$)

(f) $\frac{3}{\sqrt{-2x^2+7x-5}} > 0$,

(g) $\frac{x+2}{3x+4} > 5x + 6$.

$\oslash$

Solution 4.

(a) $1 - 3x \leq -2$ means $3x \geq 1 + 2 = 3$ and $x \geq 1$, i.e. $x \in [1, \infty)$,

(b) $1 \leq 7 - 2x < 3$ means $2x \leq 7 - 1 = 6$ and $2x > 7 - 3 = 4$, i.e. $x \in (2, 3]$,

(c) to solve $2x^2 - 4x < 16$ we first find the solutions to $2x^2 - 4x = 16$, which means $x^2 - 2x - 8 = 0$ and therefore $x = -2$ or $x = 4$. From the shape of the graph $y = 2x^2 - 4x - 16$ we see that the solutions are $x \in (-2, 4)$,

(d) $|3x - 4| < 5$ means $3x - 4 < 5$ *and* $4 - 3x < 5$ (because $|y| = \max\{y, -y\}$), and therefore $x \in (-\frac{1}{3}, 3)$,

(e) $\frac{x+3}{2x+7} > 0$ means either $x + 3 > 0$ and $2x + 7 > 0$, or $x + 3 < 0$ and $2x + 7 < 0$, i.e. $x \in (-\infty, -3\frac{1}{2}) \cup (-3, \infty)$,

(f) $\frac{3}{\sqrt{-2x^2+7x-5}} > 0$ means $\sqrt{-2x^2 + 7x - 5} > 0$ and therefore $-2x^2 + 7x - 5 > 0$; first solving $-2x^2 + 7x - 5 = 0$ we have $x = 1$ or $x = \frac{5}{2}$ and from the shape of the graph of $y = -2x^2 + 7x - 5$ we see that the solutions are $x \in (1, \frac{5}{2})$,

(g) $\frac{x+2}{3x+4} > 5x + 6$ means either $3x + 4 > 0$ and $x + 2 > (5x + 6)(3x + 4)$ (multiplying both sides by $3x + 4$), or $3x + 4 < 0$ and $x + 2 < (5x + 6)(3x + 4)$; the polynomial $(5x + 6)(3x + 4) - (x + 2) = 15x^2 + 37x + 22$ has roots at $x = -\frac{22}{15}$ and $x = -1$ and it is negative in between these roots; using the ordering $-\frac{22}{15} < -\frac{4}{3} < -1$ we then find that $x \in (-\infty, -\frac{22}{15}) \cup (-\frac{4}{3}, -1)$.

$\diamondsuit$

Problem Sheet 3

MS121 Semester 2 IT Mathematics

Exercise 1.
Write the following functions as a rational function.

(a) $f(x) = \frac{2x}{x^2+1} - \frac{3x}{x^2+2}$,

(b) $g(x) = \frac{x^2-4}{x+2} + \frac{x^2-4}{x-2}$,

(c) $h(x) = \frac{x^3}{x^4+3} - \frac{x}{x^2-1}$.

$\oslash$

Solution 1.
(a) $f(x) = \frac{2x(x^2+2)-3x(x^2+1)}{(x^2+1)(x^2+2)} = \frac{x-x^3}{x^4+3x^2+2}$.

(b) $g(x) = \frac{(x^2-4)(x-2)+(x^2-4)(x+2)}{(x+2)(x-2)} = \frac{(x^2-4)2x}{x^2-4} = 2x$. (Notes: This is a monomial, which is a special case of a polynomial and of a rational function, $g(x) = \frac{2x}{1}$; in principle we should take $g(x)$ with domain $\mathbb{R} \setminus \{-2, 2\}$, because the original sum is not defined at these points.)

(c) $h(x) = \frac{x^3(x^2-1)-x(x^4+3)}{(x^4+3)(x^2-1)} = \frac{-x^3-3x}{x^6-x^4+3x^2-3}$.

$\diamondsuit$

Exercise 2.
Explain in your own words and in detail why the following equalities are true:

$$3^5 \cdot 4^5 = 12^5 \ ,$$
$$5^3 \cdot 5^4 = 5^7 \ .$$

Then Compute the following numbers by hand and write your answers in a form without powers.

(a) 5^2,

(b) 3^3,

(c) $16^{\frac{1}{2}}$,

(d) $8^{\frac{2}{3}}$,

$\oslash$

Solution 2.
We can reorder the products as

$$\begin{aligned}
3^5 \cdot 4^5 &= (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (4 \cdot 4 \cdot 4 \cdot 4 \cdot 4) \\
&= (3 \cdot 4) \cdot (3 \cdot 4) \cdot (3 \cdot 4) \cdot (3 \cdot 4) \cdot (3 \cdot 4) \\
&= 12^5 \ , \\
5^3 \cdot 5^4 &= (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5) \\
&= 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \\
&= 5^7 \ .
\end{aligned}$$

Of course these equalities derive from general rules, $a^x b^x = (ab)^x$ and $a^x a^y = a^{x+y}$ for all $a > 0$, $b > 0$ and $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is not the intention that the students compute numbers, e.g. $3^5 = 243$, $4^5 = 1024$ and $12^5 = 1024 * 243 = 248832$, or $5^3 = 125$, $5^4 = 625$ and $5^7 = 78125$.

(a) $5^2 = 25$,

(b) $3^3 = 27$,

(c) $16^{\frac{1}{2}} = 4$,

(d) $8^{\frac{2}{3}} = 2^2 = 4$.

$\diamondsuit$

Exercise 3.
Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

(a) $f(x)$ is a linear function with exactly one root,

(b) $f(x)$ is an even monomial,

(c) $f(x)$ is an odd polynomial, but not a monomial,

(d) $f(x)$ is a rational function (with domain $\mathbb{R}$),

(e) $f(x)$ is an odd rational function, but not a polynomial.

$\oslash$

Solution 3.

(a) E.g. $f(x) = 2x - 7$. Any linear function $f(x) = mx + b$ which is not constant ($m \neq 0$) will do.

(b) E.g. $f(x) = 2x^4$. Any monomial $f(x) = cx^n$ with even power n will do.

(c) E.g. $f(x) = x^3 - 4x$. This requires a sum of at least two odd monomials with different odd exponents and non-zero coefficients.

(d) E.g. $f(x) = \frac{2x}{x^2+1}$. The denominator must be a polynomial without roots. It must always be even and for degree 2 we can easily check whether it has roots. Degree 0 would also work, e.g. $f(x) = \frac{2x}{3}$, but this is really just a polynomial.

(e) E.g. $f(x) = \frac{2x}{x^2+1}$. If the numerator is odd and the denominator even, or the other way around, the quotient is always odd. (If the domain is still $\mathbb{R}$, we must choose the denominator even and hence the numerator odd.) The denominator should not be a constant function, otherwise the quotient is a polynomial.

$\diamondsuit$

Exercise 4.

Evaluate the following limits and indicate which rules for limits you used, if any.

(a) $\lim_{x\to 5} x - 19$,

(b) $\lim_{x\to 3}(x+7)(x-7)$,

(c) $\lim_{x\to 0}(1-x^3)^{125}$,

(d) $\lim_{x\to 0}\dfrac{x^2}{x}$.

$\oslash$

Solution 4.

Using the facts that $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = a$ for any $a, c \in \mathbb{R}$ we find:

(a) $\lim_{x\to 5} x - 19 = 5 - 19 = -14$ from the sum rule for limits,

(b) $\lim_{x\to 3}(x+7)(x-7) = 10 \cdot (-4) = -40$ from the product and sum rules for limits,

(c) $\lim_{x\to 0}(1-x^3)^{125} = \left(\lim_{x\to 0}(1-x^3)\right)^{125} = (1-0^3)^{125} = 1$ using the continuity of $f(y) = y^{125}$ and the rule for limits under compositions with continuous functions, as well as the rules for sums and products of limits; it is possible to expand $(1-x^3)^{125}$, in which case one doesn't need the rule for limits under compositions with continuous functions

(d) $\lim_{x\to 0}\dfrac{x^2}{x} = \lim_{x\to 0} x = 0$, because $\frac{x^2}{x} = x$ on $x \neq 0$ and the limit as $x \to 0$ does not depend on the value at $x = 0$.

$\diamondsuit$

Problem Sheet 4

MS121 Semester 2 IT Mathematics

Exercise 1.

Evaluate the following limits:

(a) $\lim_{y\to 2} y^3 - \dfrac{21}{y}$,

(b) $\lim_{x\to 2} \dfrac{x^3+x^2+2x+1}{x-1}$,

(c) $\lim_{x\to 3\pi} \sin(x)$,

(d) $\lim_{x\to \pi} \cos(2x-\pi)$,

(e) $\lim_{x\to 2} \sqrt{x^2+3x-7}$,

(f) $\lim_{x\to \frac{1}{2}\pi} \sqrt{\sin(x)}$.

You may use the fact that the functions cos, sin and $\sqrt{\ }$ are continuous.

$\oslash$

Solution 1.

(a) $\lim_{y\to 2} y^3 - \dfrac{21}{y} = 2^3 - \dfrac{21}{2} = \dfrac{-5}{2}$ from the continuity of rational functions on their domain and the rules for sums and multiples,

(b) $\lim_{x\to 2} \dfrac{x^3+x^2+2x+1}{x-1} = 17$, from the continuity of rational functions on their domain

(c) $\lim_{x\to 3\pi} \sin(x) = \sin(3\pi) = 0$, using the continuity of sin,

(d) $\lim_{x\to \pi} \cos(2x-\pi) = \cos(2\pi-\pi) = -1$, using the continuity of the composition of the continuous functions cos and $2x-\pi$,

(e) $\lim_{x\to 2} \sqrt{x^2+3x-7} = \sqrt{2^2+3\cdot 2-7} = \sqrt{3}$, using the continuity of the composition of the continuous functions $\sqrt{\ }$ and x^2+3x-7,

(f) $\lim_{x\to \frac{1}{2}\pi} \sqrt{\sin(x)} = \sqrt{\sin(\frac{1}{2}\pi)} = \sqrt{1} = 1$, using the continuity of the composition of the continuous functions $\sqrt{\ }$ and sin.

$\diamondsuit$

Exercise 2.

The following functions have domain $(0,\infty)$. Find their inverses and the domains of these inverses.

(a) $f(x) = \frac{1}{x}$,

(b) $f(x) = x^2$,

(c) $g(y) = \frac{1}{1+y^2}$,

(d) $f(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ (x+1)^2 & \text{if } x > 1 \end{cases}$,

(e) $h(t) = \begin{cases} t+1 & \text{if } 0 < t \leq 1 \\ \frac{1}{t} & \text{if } t > 1 \end{cases}$.

$\oslash$

Solution 2.

(a) $f(x) = \frac{1}{x}$ is its own inverse,

(b) $f(x) = x^2$ has inverse $f^{-1}(y) = \sqrt{y}$ with domain $(0, \infty)$,

(c) $g(y) = \frac{1}{1+y^2}$ has inverse $g^{-1}(x) = \sqrt{\frac{1}{x} - 1}$ with domain $(0, 1)$,

(d) $f(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ (x+1)^2 & \text{if } x > 1 \end{cases}$
has inverse

$$f^{-1}(y) = \begin{cases} y & \text{if } 0 < y \leq 1 \\ \sqrt{y} - 1 & \text{if } y > 4 \end{cases}$$

with domain $(0, 1] \cup (4, \infty)$.

(e) $h(t) = \begin{cases} t+1 & \text{if } 0 < t \leq 1 \\ \frac{1}{t} & \text{if } t > 1 \end{cases}$
is not increasing or decreasing on its entire domain, but it does have an inverse

$$h^{-1}(s) = \begin{cases} s - 1 & \text{if } 1 < s \leq 2 \\ \frac{1}{s} & \text{if } 0 < s < 1 \end{cases}$$

with domain $(0, 1) \cup (1, 2]$.

$\diamondsuit$

Exercise 3.

For the following functions, indicate how we can restrict their natural domain to make them invertible, and find the inverse. (Sketch the graph first, if this is helpful.)

(a) $f(x) = (x-3)^2 + 5$,

(b) $g(y) = \frac{1}{(y-5)^2}$,

(c) $f(x) = |x - \pi|$,

(d) $h(t) = \sin(t)$.

$\oslash$

Solution 3.

(a) $f(x) = (x-3)^2 + 5$ can be restricted e.g. to $x \geq 3$ with inverse
$f^{-1}(y) = \sqrt{y-5} + 3$ on $y \geq 5$,

(b) $g(y) = \frac{1}{(y-5)^2}$ can be restricted e.g. to $y > 5$ with inverse
$g^{-1}(x) = \frac{1}{\sqrt{x}} + 5$ on $x > 0$,

(c) $f(x) = |x - \pi|$ can be restricted e.g. to $x \geq \pi$ with inverse $f^{-1}(y) = y + \pi$ on $y \geq 0$.

(d) $h(t) = \sin(t)$ can be restricted e.g. to $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$. Its inverse is then called $\arcsin(y)$ and has domain $[-1, 1]$.

$\diamondsuit$

Exercise 4.
Use left and right hand limits to determine which of the following functions are continuous.

(a) $f(x) = \begin{cases} 3-5x & \text{if} \quad x \leq 0 \\ \frac{6}{4x+2} & \text{if} \quad x > 0 \end{cases},$

(b) $g(x) = \begin{cases} -2x & \text{if} \quad x < -2 \\ x^2 & \text{if} \quad -2 \leq x \leq 1 \\ 2x & \text{if} \quad x > 1 \end{cases}.$

$\oslash$

Solution 4.
(a)

$$f(x) = \begin{cases} 3-5x & \text{if} \quad x \leq 0 \\ \frac{6}{4x+2} & \text{if} \quad x > 0 \end{cases}$$

is continuous, because the one-sided limits

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3-5x = 3-0 = 3$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{6}{4x+2} = \frac{6}{0+2} = 3$$

are equal to the function value $f(0) = 3 - 0 = 3$.

(b)

$$g(x) = \begin{cases} -2x & \text{if} \quad x < -2 \\ x^2 & \text{if} \quad -2 \leq x \leq 1 \\ 2x & \text{if} \quad x > 1 \end{cases}$$

is not continuous, because the one-sided limits

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1$$
$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} 2x = 2$$

differ. (g is continuous at $x = -2$.)

$\diamondsuit$

Problem Sheet 1

MS121 Semester 2 IT Mathematics

Exercise 1.
Express the following sets as intervals:

(a) $\{x \in \mathbb{R}| \ -5 \leq x, \ x < 20\}$,

(b) $\{x \in \mathbb{R}| \ x > 7\}$,

(c) $\{x \in \mathbb{R}| \ x \leq 27\}$.

$\oslash$

Solution 1.
(a) $[-5, 20)$,

(b) $(7, \infty)$, (∞ is not a real number, so it is never included)

(c) $(-\infty, 27]$.

$\diamondsuit$

Exercise 2.
Draw or sketch the graphs of the following functions. Sometimes it helps to first make a short table with well chosen values of x and $f(x)$.

(a) $f(x) = (x-2)^2 - 1$,

(b) g, the straight line through the points $(0, 2)$ and $(3, -2)$,

(c) $\sin(x)$.

$\oslash$

Solution 2.
(a) $f(x) = (x-2)^2 + 1$ is a parabola with minimum at $(2, -1)$ and zeroes at $(1, 0)$ and $(3, 0)$.

(b) g is given by $g(x) = 2 - \frac{4}{3}x$, so the graph is $y = 2 - \frac{4}{3}x$.

(c) $\sin(x)$ is a standard oscillating function, with $\sin(0) = 0$.

$\diamondsuit$

Exercise 3.
The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given by the formulae

$$f(x) = 2 - x, \qquad g(x) = 4x - x^3.$$

Express each of the following functions as a formula in terms of x:

(a) $f + g$,

(b) $g - f$,

(c) $f \cdot g$,

(d) $f \circ g$,

(e) $g \circ f$,

(f) g/f.

$\oslash$

Solution 3.

(a) $(f + g)(x) = f(x) + g(x) = (2 - x) + (4x - x^3) = 2 + 3x - x^3$,

(b) $(g - f)(x) = g(x) - f(x) = (4x - x^3) - (2 - x) = -2 + 5x - x^3$,

(c) $f \cdot g(x) = f(x) \cdot g(x) = (2 - x)(4x - x^3) = 8x - 4x^2 - 2x^3 + x^4$,

(d) $f \circ g(x) = f(g(x)) = 2 - g(x) = 2 - 4x + x^3$,

(e) $g \circ f(x) = g(f(x)) = 4f(x) - f(x)^3 = 4(2 - x) - (2 - x)^3 = 8x - 6x^2 + x^3$,

(f) $g/f(x) = \frac{g(x)}{f(x)} = \frac{4x-x^3}{2-x}$. This may be simplified $\frac{x(2-x)(2+x)}{2-x} = x(2 + x) = 2x + x^2$ (on $x \neq 2$).

$\diamondsuit$

Exercise 4.
Find the natural domain of the following functions. Express your answers in terms of intervals.

(a) $f(x) = 2x^2$,

(b) $f(x) = x^8$,

(c) $f(x) = x^9$,

(d) $f(x) = \frac{1}{x-7}$,

(e) $f(x) = |x - 4|$,

(f) $f(x) = \sqrt{x^2 - 4}$.

(g) $f(x) = \frac{\sqrt{2-x}}{x^2-1}$,

(h) $f(x) = \frac{\sqrt{4-\sqrt{x}}}{\sqrt{x^2+1}}$.

For the functions in parts $(a) - (f)$, can you also give the range?

$\oslash$

Solution 4.

(a) Domain$(f) = \mathbb{R}$ and Range$(f) = [0, \infty)$,

(b) Domain$(f) = \mathbb{R}$ and Range$(f) = [0, \infty)$,

(c) Domain$(f) = \mathbb{R}$ and Range$(f) = \mathbb{R}$,

(d) Domain$(f) = (-\infty, 7) \cup (7, \infty)$ and Range$(f) = (-\infty, 0) \cup (0, \infty)$,

(e) Domain$(f) = \mathbb{R}$ and Range$(f) = [0, \infty)$,

(f) Domain$(f) = (-\infty, -2] \cup [2, \infty)$ and Range$(f) = [0, \infty)$.

(g) Domain$(f) = (-\infty, -1) \cup (-1, 1) \cup (1, 2]$, because the numerator only makes sense when $x \leq 2$ and the quotient makes no sense when $x = -1$ or $x = 1$.

(h) Domain$(f) = [0, 16]$, because the numerator only makes sense when $x \geq 0$ and $\sqrt{x} \leq 4$, i.e. $0 \leq x \leq 16$. The quotient is no problem, because $\sqrt{x^2+1} \geq 1 > 0$ for all $x \in \mathbb{R}$.

$\diamondsuit$

Problem Sheet 2

MS121 Semester 2 IT Mathematics

Exercise 1.
Find the equations that determine the following lines:

(a) The line through $(-1, 2)$ with slope -2.

(b) The line through $(-1, -2)$ and $(1, 3)$.

(c) The line through $(0, 5)$ which is parallel to the line $y = 7 - 3x$.

(d) The line through $(1, 1)$ which is perpendicular to the line $2x - y = 5$.

$\oslash$

Solution 1.

(a) For the line through $(-1, 2)$ with slope -2 we start with the general formula $y = mx + b$. We insert the slope $m = -2$ and then determine b using the point $(-1, 2)$: $2 = -2(-1) + b$, so $b = 0$. Hence, $y = -2x$.

(b) For the line through $(-1, -2)$ and $(1, 3)$ we first determine the slope $m = \frac{3-(-2)}{1-(-1)} = \frac{5}{2}$. Inserting this into $y = mx + b$ we may use either of the points to determine $b = \frac{1}{2}$ and hence $y = \frac{1}{2}(5x + 1)$.

(c) The line through $(0, 5)$ which is parallel to the line $y = 7 - 3x$ must have the same slope, $m = -3$. Using $(0, 5)$ to determine b in $y = -3x + b$ yields $b = 5$ and hence $y = -3x + 5$.

(d) For the line through $(1, 1)$ which is perpendicular to the line $2x - y = 5$ we first note that the given line can be written as $y = 2x - 5$, so it has a slope 2. Any line perpendicular to it must have a slope $m = -\frac{1}{2}$ (the products of the slopes must be -1). Determining b in $y = -\frac{1}{2}x + b$ with the given point $(1, 1)$ yields $b = \frac{3}{2}$ and hence $y = -\frac{1}{2}x + \frac{3}{2}$.

$\diamond$

Exercise 2.
Which of the following functions are even, which ones are odd, and which ones are neither?

(a) $f(x) = 4x^3 - 2x$,

(b) $f(x) = 5x^6$,

(c) $f(x) = x - 3$,

(d) $f(x) = \sin(x)$,

(e) $f(x) = x\sin(x) + \cos(x)$,

(f) $f(x) = 0$.

$\oslash$

Solution 2.

(a) $f(x) = 4x^3 - 2x$ is odd (polynomial with only odd powers),

(b) $f(x) = 5x^6$ is even,

(c) $f(x) = x - 3$ is neither even nor odd,

(d) $f(x) = \sin(x)$ is odd,

(e) $f(x) = x\sin(x) + \cos(x)$ is even,

(f) $f(x) = 0$ is the only function which is both even and odd.

$\diamondsuit$

Exercise 3.
Find the roots of the following functions and determine where the functions are positive and where they are negative:

(a) $f(x) = 49 - x^2$,

(b) $f(x) = x^2 - 5x - 6$,

(c) $g(y) = y^2 - 4y + 4$,

(d) $h(t) = t^2 + 2t + 3$,

(e) $y = -x^2 + 3x - 2$,

(f) $h(x) = x^4 + 4x^2 + 3$.

$\oslash$

Solution 3.

(a) $f(x) = 49 - x^2$ has roots $x = -7$, and $x = 7$, and it is positive on $(-7, 7)$ and negative on $(-\infty, -7) \cup (7, \infty)$,

(b) $f(x) = x^2 - 5x - 6$ has roots $x = -1$, and $x = 6$, and it is positive on $(-\infty, -1) \cup (6, \infty)$ and negative on $(-1, 6)$,

(c) $g(y) = y^2 + 4y + 4$ has one (double) root $y = 2$, and it is positive on $(-\infty, 2) \cup (2, \infty)$ and nowhere negative,

(d) $h(t) = t^2 + t + 1$ has no roots (discriminant is negative) and it is positive on all of $\mathbb{R}$,

(e) $y = -x^2 + 3x - 2$ has roots $x = 1$, and $x = 2$, and it is positive on $(1, 2)$ and negative on $(-\infty, 1) \cup (2, \infty)$,

(f) for $h(x) = x^4 + 4x^2 + 3$ we first note $h(x) = g(x^2)$ with $g(y) = y^2 + 4y + 3$; $g(y)$ has roots at $y = -3$ and $y = -1$, so that $g(y) = (y + 3)(y + 1)$; it follows that $h(x) = (x^2 + 3)(x^2 + 1)$ and neither factor has roots, because $x^2 = -3$ and $x^2 = -1$ have no solutions in $\mathbb{R}$; hence h has no roots and it is positive on the entire $\mathbb{R}$.

$\diamondsuit$

Exercise 4.
Solve the following inequalities for $x \in \mathbb{R}$. Write your answer in terms of intervals.

(a) $1 - 3x \leq -2$,

(b) $1 \leq 7 - 2x < 3$,

(c) $|3x - 4| < 5$,

(d) $2x^2 - 4x < 16$,

(e) $\frac{x+3}{2x+7} > 0$, (Hint: multiply both sides by the positive number $(2x + 7)^2$)

(f) $\frac{3}{\sqrt{-2x^2+7x-5}} > 0$,

(g) $\frac{x+2}{3x+4} > 5x + 6$.

$\oslash$

Solution 4.

(a) $1-3x \leq -2$ means $3x \geq 1+2=3$ and $x \geq 1$, i.e. $x \in [1,\infty)$,

(b) $1 \leq 7-2x < 3$ means $2x \leq 7-1=6$ and $2x > 7-3=4$, i.e. $x \in (2,3]$,

(c) to solve $2x^2-4x < 16$ we first find the solutions to $2x^2-4x=16$, which means $x^2-2x-8=0$ and therefore $x=-2$ or $x=4$. From the shape of the graph $y=2x^2-4x-16$ we see that the solutions are $x \in (-2,4)$,

(d) $|3x-4| < 5$ means $3x-4<5$ *and* $4-3x<5$ (because $|y| = \max\{y,-y\}$), and therefore $x \in (-\frac{1}{3},3)$,

(e) $\frac{x+3}{2x+7} > 0$ means either $x+3>0$ and $2x+7>0$, or $x+3<0$ and $2x+7<0$, i.e. $x \in (-\infty,-3\frac{1}{2}) \cup (-3,\infty)$,

(f) $\frac{3}{\sqrt{-2x^2+7x-5}} > 0$ means $\sqrt{-2x^2+7x-5} > 0$ and therefore $-2x^2+7x-5>0$; first solving $-2x^2+7x-5=0$ we have $x=1$ or $x=\frac{5}{2}$ and from the shape of the graph of $y=-2x^2+7x-5$ we see that the solutions are $x \in (1,\frac{5}{2})$,

(g) $\frac{x+2}{3x+4} > 5x+6$ means either $3x+4>0$ and $x+2>(5x+6)(3x+4)$ (multiplying both sides by $3x+4$), or $3x+4<0$ and $x+2<(5x+6)(3x+4)$; the polynomial $(5x+6)(3x+4)-(x+2) = 15x^2+37x+22$ has roots at $x=-\frac{22}{15}$ and $x=-1$ and it is negative in between these roots; using the ordering $-\frac{22}{15} < -\frac{4}{3} < -1$ we then find that $x \in (-\infty,-\frac{22}{15}) \cup (-\frac{4}{3},-1)$.

$\diamondsuit$

Problem Sheet 3

MS121 Semester 2 IT Mathematics

Exercise 1.
Write the following functions as a rational function.

(a) $f(x) = \frac{2x}{x^2+1} - \frac{3x}{x^2+2}$,

(b) $g(x) = \frac{x^2-4}{x+2} + \frac{x^2-4}{x-2}$,

(c) $h(x) = \frac{x^3}{x^4+3} - \frac{x}{x^2-1}$.

$\oslash$

Solution 1.

(a) $f(x) = \frac{2x(x^2+2)-3x(x^2+1)}{(x^2+1)(x^2+2)} = \frac{x-x^3}{x^4+3x^2+2}$.

(b) $g(x) = \frac{(x^2-4)(x-2)+(x^2-4)(x+2)}{(x+2)(x-2)} = \frac{(x^2-4)2x}{x^2-4} = 2x$. (Notes: This is a monomial, which is a special case of a polynomial and of a rational function, $g(x) = \frac{2x}{1}$; in principle we should take $g(x)$ with domain $\mathbb{R} \setminus \{-2, 2\}$, because the original sum is not defined at these points.)

(c) $h(x) = \frac{x^3(x^2-1)-x(x^4+3)}{(x^4+3)(x^2-1)} = \frac{-x^3-3x}{x^6-x^4+3x^2-3}$.

$\diamondsuit$

Exercise 2.
Explain in your own words and in detail why the following equalities are true:

$$3^5 \cdot 4^5 = 12^5 \ ,$$
$$5^3 \cdot 5^4 = 5^7 \ .$$

Then Compute the following numbers by hand and write your answers in a form without powers.

(a) 5^2,

(b) 3^3,

(c) $16^{\frac{1}{2}}$,

(d) $8^{\frac{2}{3}}$,

$\oslash$

Solution 2.
We can reorder the products as

$$\begin{aligned}
3^5 \cdot 4^5 &= (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (4 \cdot 4 \cdot 4 \cdot 4 \cdot 4) \\
&= (3 \cdot 4) \cdot (3 \cdot 4) \cdot (3 \cdot 4) \cdot (3 \cdot 4) \cdot (3 \cdot 4) \\
&= 12^5 \ , \\
5^3 \cdot 5^4 &= (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5) \\
&= 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \\
&= 5^7 \ .
\end{aligned}$$

Of course these equalities derive from general rules, $a^x b^x = (ab)^x$ and $a^x a^y = a^{x+y}$ for all $a > 0$, $b > 0$ and $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is not the intention that the students compute numbers, e.g. $3^5 = 243$, $4^5 = 1024$ and $12^5 = 1024 * 243 = 248832$, or $5^3 = 125$, $5^4 = 625$ and $5^7 = 78125$.

(a) $5^2 = 25$,

(b) $3^3 = 27$,

(c) $16^{\frac{1}{2}} = 4$,

(d) $8^{\frac{2}{3}} = 2^2 = 4$.

$\diamondsuit$

Exercise 3.
Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

(a) $f(x)$ is a linear function with exactly one root,

(b) $f(x)$ is an even monomial,

(c) $f(x)$ is an odd polynomial, but not a monomial,

(d) $f(x)$ is a rational function (with domain $\mathbb{R}$),

(e) $f(x)$ is an odd rational function, but not a polynomial.

$\oslash$

Solution 3.
(a) E.g. $f(x) = 2x - 7$. Any linear function $f(x) = mx + b$ which is not constant ($m \neq 0$) will do.

(b) E.g. $f(x) = 2x^4$. Any monomial $f(x) = cx^n$ with even power n will do.

(c) E.g. $f(x) = x^3 - 4x$. This requires a sum of at least two odd monomials with different odd exponents and non-zero coefficients.

(d) E.g. $f(x) = \frac{2x}{x^2+1}$. The denominator must be a polynomial without roots. It must always be even and for degree 2 we can easily check whether it has roots. Degree 0 would also work, e.g. $f(x) = \frac{2x}{3}$, but this is really just a polynomial.

(e) E.g. $f(x) = \frac{2x}{x^2+1}$. If the numerator is odd and the denominator even, or the other way around, the quotient is always odd. (If the domain is still $\mathbb{R}$, we must choose the denominator even and hence the numerator odd.) The denominator should not be a constant function, otherwise the quotient is a polynomial.

$\diamondsuit$

Exercise 4.

Evaluate the following limits and indicate which rules for limits you used, if any.

(a) $\lim_{x\to 5} x - 19$,

(b) $\lim_{x\to 3} (x+7)(x-7)$,

(c) $\lim_{x\to 0} (1-x^3)^{125}$,

(d) $\lim_{x\to 0} \dfrac{x^2}{x}$.

⊘

Solution 4.

Using the facts that $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = a$ for any $a, c \in \mathbb{R}$ we find:

(a) $\lim_{x\to 5} x - 19 = 5 - 19 = -14$ from the sum rule for limits,

(b) $\lim_{x\to 3} (x+7)(x-7) = 10 \cdot (-4) = -40$ from the product and sum rules for limits,

(c) $\lim_{x\to 0} (1-x^3)^{125} = \left(\lim_{x\to 0} (1-x^3)\right)^{125} = (1-0^3)^{125} = 1$ using the continuity of $f(y) = y^{125}$ and the rule for limits under compositions with continuous functions, as well as the rules for sums and products of limits; it is possible to expand $(1-x^3)^{125}$, in which case one doesn't need the rule for limits under compositions with continuous functions

(d) $\lim_{x\to 0} \dfrac{x^2}{x} = \lim_{x\to 0} x = 0$, because $\frac{x^2}{x} = x$ on $x \neq 0$ and the limit as $x \to 0$ does not depend on the value at $x = 0$.

♢

Problem Sheet 4

MS121 Semester 2 IT Mathematics

Exercise 1.

Evaluate the following limits:

(a) $\lim_{y\to 2} y^3 - \frac{21}{y}$,

(b) $\lim_{x\to 2} \frac{x^3+x^2+2x+1}{x-1}$,

(c) $\lim_{x\to 3\pi} \sin(x)$,

(d) $\lim_{x\to \pi} \cos(2x-\pi)$,

(e) $\lim_{x\to 2} \sqrt{x^2+3x-7}$,

(f) $\lim_{x\to \frac{1}{2}\pi} \sqrt{\sin(x)}$.

You may use the fact that the functions cos, sin and $\sqrt{\ }$ are continuous.

$\oslash$

Solution 1.

(a) $\lim_{y\to 2} y^3 - \frac{21}{y} = 2^3 - \frac{21}{2} = \frac{-5}{2}$ from the continuity of rational functions on their domain and the rules for sums and multiples,

(b) $\lim_{x\to 2} \frac{x^3+x^2+2x+1}{x-1} = 17$, from the continuity of rational functions on their domain

(c) $\lim_{x\to 3\pi} \sin(x) = \sin(3\pi) = 0$, using the continuity of sin,

(d) $\lim_{x\to \pi} \cos(2x-\pi) = \cos(2\pi-\pi) = -1$, using the continuity of the composition of the continuous functions cos and $2x-\pi$,

(e) $\lim_{x\to 2} \sqrt{x^2+3x-7} = \sqrt{2^2+3\cdot 2-7} = \sqrt{3}$, using the continuity of the composition of the continuous functions $\sqrt{\ }$ and x^2+3x-7,

(f) $\lim_{x\to \frac{1}{2}\pi} \sqrt{\sin(x)} = \sqrt{\sin(\frac{1}{2}\pi)} = \sqrt{1} = 1$, using the continuity of the composition of the continuous functions $\sqrt{\ }$ and sin.

$\diamondsuit$

Exercise 2.

The following functions have domain $(0,\infty)$. Find their inverses and the domains of these inverses.

(a) $f(x) = \frac{1}{x}$,

(b) $f(x) = x^2$,

(c) $g(y) = \frac{1}{1+y^2}$,

(d) $f(x) = \begin{cases} x & \text{if } 0 < x \le 1 \\ (x+1)^2 & \text{if } x > 1 \end{cases}$,

(e) $h(t) = \begin{cases} t+1 & \text{if } 0 < t \le 1 \\ \frac{1}{t} & \text{if } t > 1 \end{cases}$.

$\oslash$

Solution 2.

(a) $f(x) = \frac{1}{x}$ is its own inverse,

(b) $f(x) = x^2$ has inverse $f^{-1}(y) = \sqrt{y}$ with domain $(0, \infty)$,

(c) $g(y) = \frac{1}{1+y^2}$ has inverse $g^{-1}(x) = \sqrt{\frac{1}{x} - 1}$ with domain $(0, 1)$,

(d) $f(x) = \begin{cases} x & \text{if} \quad 0 < x \leq 1 \\ (x+1)^2 & \text{if} \quad x > 1 \end{cases}$
has inverse
$$f^{-1}(y) = \begin{cases} y & \text{if} \quad 0 < y \leq 1 \\ \sqrt{y} - 1 & \text{if} \quad y > 4 \end{cases}$$
with domain $(0, 1] \cup (4, \infty)$.

(e) $h(t) = \begin{cases} t+1 & \text{if} \quad 0 < t \leq 1 \\ \frac{1}{t} & \text{if} \quad t > 1 \end{cases}$
is not increasing or decreasing on its entire domain, but it does have an inverse
$$h^{-1}(s) = \begin{cases} s-1 & \text{if} \quad 1 < s \leq 2 \\ \frac{1}{s} & \text{if} \quad 0 < s < 1 \end{cases}$$
with domain $(0, 1) \cup (1, 2]$.

$\diamondsuit$

Exercise 3.

For the following functions, indicate how we can restrict their natural domain to make them invertible, and find the inverse. (Sketch the graph first, if this is helpful.)

(a) $f(x) = (x-3)^2 + 5$,

(b) $g(y) = \frac{1}{(y-5)^2}$,

(c) $f(x) = |x - \pi|$,

(d) $h(t) = \sin(t)$.

$\oslash$

Solution 3.

(a) $f(x) = (x-3)^2 + 5$ can be restricted e.g. to $x \geq 3$ with inverse
$f^{-1}(y) = \sqrt{y-5} + 3$ on $y \geq 5$,

(b) $g(y) = \frac{1}{(y-5)^2}$ can be restricted e.g. to $y > 5$ with inverse
$g^{-1}(x) = \frac{1}{\sqrt{x}} + 5$ on $x > 0$,

(c) $f(x) = |x - \pi|$ can be restricted e.g. to $x \geq \pi$ with inverse $f^{-1}(y) = y + \pi$ on $y \geq 0$.

(d) $h(t) = \sin(t)$ can be restricted e.g. to $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$. Its inverse is then called $\arcsin(y)$ and has domain $[-1, 1]$.

$\diamondsuit$

Exercise 4.
Use left and right hand limits to determine which of the following functions are continuous.

(a) $f(x) = \begin{cases} 3-5x & \text{if} \quad x \le 0 \\ \frac{6}{4x+2} & \text{if} \quad x > 0 \end{cases}$,

(b) $g(x) = \begin{cases} -2x & \text{if} \quad x < -2 \\ x^2 & \text{if} \quad -2 \le x \le 1 \\ 2x & \text{if} \quad x > 1 \end{cases}$.

$\oslash$

Solution 4.
(a)

$$f(x) = \begin{cases} 3-5x & \text{if} \quad x \le 0 \\ \frac{6}{4x+2} & \text{if} \quad x > 0 \end{cases}$$

is continuous, because the one-sided limits

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} 3-5x = 3-0 = 3 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{6}{4x+2} = \frac{6}{0+2} = 3 \end{aligned}$$

are equal to the function value $f(0) = 3 - 0 = 3$.

(b)

$$g(x) = \begin{cases} -2x & \text{if} \quad x < -2 \\ x^2 & \text{if} \quad -2 \le x \le 1 \\ 2x & \text{if} \quad x > 1 \end{cases}$$

is not continuous, because the one-sided limits

$$\begin{aligned} \lim_{x\to 1^-} g(x) &= \lim_{x\to 1^-} x^2 = 1^2 = 1 \\ \lim_{x\to 1^+} g(x) &= \lim_{x\to 1^+} 2x = 2 \end{aligned}$$

differ. (g is continuous at $x = -2$.)

$\diamondsuit$

Problem Sheet 5

MS121 Semester 2 IT Mathematics

Exercise 1.

Evaluate the following limits:

(a) $\lim_{x\to\infty} \dfrac{7x+8}{x^2-5x+3}$,

(b) $\lim_{x\to-\infty} \dfrac{2x+5}{8x-3}$,

(c) $\lim_{x\to\infty} \dfrac{(1-x)^3}{x^3}$,

(d) $\lim_{x\to\infty} \dfrac{x^2-2x+1}{7x}$,

(e) $\lim_{x\to-\infty} \dfrac{(x-1)^2}{x+1}$.

Hint: Substitute $x = \frac{1}{y}$ and consider $\lim_{y\to 0^+}$ or $\lim_{y\to 0^-}$ as appropriate. ⊘

Solution 1.

(a) $\lim_{x\to\infty} \dfrac{7x+8}{x^2-5x+3} = \lim_{x\to\infty} \dfrac{7x^{-1}+8x^{-2}}{1-5x^{-1}+3x^{-2}} = \lim_{y\to 0^+} \dfrac{7y+8y^2}{1-5y+3y^2} = \dfrac{0}{1} = 0$, using the continuity of rational functions on their domain, the rule for compositions with $y(x) = \frac{1}{x}$ and $\lim_{x\to\infty} \dfrac{1}{x} = 0$,

(b) $\lim_{x\to-\infty} \dfrac{2x+5}{8x-3} = \lim_{x\to-\infty} \dfrac{2+5x^{-1}}{8-3x^{-1}} = \lim_{y\to 0^-} \dfrac{2+5y}{8-3y} = \dfrac{2}{8} = \dfrac{1}{4}$,

(c) $\lim_{x\to\infty} \dfrac{(1-x)^3}{x^3} = \lim_{x\to\infty} (x^{-1}-1)^3 = \lim_{y\to 0^+} (y-1)^3 = (-1)^3 = -1$,

(d) $\lim_{x\to\infty} \dfrac{x^2-2x+1}{7x} = \lim_{x\to\infty} \dfrac{x-2+x^{-1}}{7} = \infty$ (from the definition, i.e. we can make the RHS as large as we like by choosing x large enough),

(e) $\lim_{x\to-\infty} \dfrac{(x-1)^2}{x+1} = \lim_{x\to-\infty} \dfrac{x-2+x^{-1}}{1+x^{-1}} = -\infty$ (again from the definition).

♢

Exercise 2.

The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$

(a) Show for all $x \neq 0$ that $|f(x)| \leq |x|$.

(b) Use the Squeeze Theorem to show that f is continuous at $x = 0$.

⊘

Solution 2.

(a) For all $y \in \mathbb{R}$ we have $\sin(y) \in [-1, 1]$, which means that for $x \neq 0$ we have $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$. It follows that $|f(x)| = |x| \cdot \left|\sin\left(\frac{1}{x}\right)\right| \leq |x|$.

(b) From part (a) we see that $-|x| \leq f(x) \leq |x|$ for all $x \neq 0$. (Note that $f(x)$ has no absolute value here.) Because $|x|$ is continuous we have $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$. We may therefore apply the Squeeze Theorem and conclude that $\lim_{x\to 0} f(x) = 0$ (and in particular that it exists). Because $f(0) = 0$ as well we conclude that f is continuous at 0.

$\diamondsuit$

Exercise 3.

Determine the numbers $A \in \mathbb{R}$ and $B \in \mathbb{R}$ such that the following function $f : \mathbb{R} \to \mathbb{R}$ is continuous:

$$f(x) = \begin{cases} 3^x & \text{if } \; x < -1 \\ \frac{Ax+B}{4-x^2} & \text{if } \; -1 \leq x \leq 1 \\ \sqrt{x-1} & \text{if } \; x > 1 \end{cases} .$$

(You may use the fact that exponential functions like $x \mapsto 3^x$, rational functions and power functions like $x^{\frac{1}{2}}$ are continuous on their domains.) $\oslash$

Solution 3.

The function is continuous for $x \notin \{-1, 1\}$. Note in particular that $\frac{Ax+B}{4-x^2}$ is continuous for $x \notin \{-2, 2\}$ and therefore on $(-1, 1)$. Also $\sqrt{x-1}$ is continuous for $x - 1 > 0$.

At $x = -1$ we have

$$\begin{aligned} \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} 3^x = 3^{-1} = \frac{1}{3} \\ \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} \frac{Ax+B}{4-x^2} = \frac{A\cdot(-1)+B}{4-(-1)^2} = \frac{B-A}{3} \\ f(-1) &= \frac{B-A}{3} \end{aligned}$$

and at $x = 1$ we have

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{Ax+B}{4-x^2} = \frac{A+B}{4-1^2} = \frac{A+B}{3} \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \sqrt{x-1} = \sqrt{1-1} = 0 \\ f(1) &= \frac{A+B}{3} . \end{aligned}$$

To have continuity at both $x = -1$ and $x = 1$ we need to require the equalities

$$\frac{B-A}{3} = \frac{1}{3}, \qquad \frac{A+B}{3} = 0.$$

The second equation means that $A = -B$ and substituting into the first equation gives $B = \frac{1}{2}$ and therefore $A = -\frac{1}{2}$. $\diamondsuit$

Exercise 4.

Evaluate the following limits. (Hint: you can factorise the numerators and denominators and divide out the common factors.)

(a) $\displaystyle\lim_{y\to 1} \frac{y^2+2y-3}{y-1}$,

(b) $\displaystyle\lim_{x\to 1} \frac{x^2+2x-3}{x^2-1}$,

(c) $\displaystyle\lim_{y\to -1} \left(\frac{1}{y+1} + \frac{2}{y^2-1}\right)$.

$\oslash$

Solution 4.

(a) $\displaystyle\lim_{y\to 1} \frac{y^2+2y-3}{y-1} = \lim_{y\to 1} \frac{(y-1)(y+3)}{y-1} = \lim_{y\to 1} y+3 = 1+3 = 4,$

(b) $\displaystyle\lim_{x\to 1} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x+3)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x+3}{x+1} = \frac{4}{2} = 2,$

(c)

$$\begin{aligned}
\lim_{y\to -1} \left(\frac{1}{y+1} + \frac{2}{y^2-1}\right) &= \lim_{y\to -1} \left(\frac{1}{y+1} + \frac{2}{(y+1)(y-1)}\right) \\
&= \lim_{y\to -1} \frac{y-1+2}{(y+1)(y-1)} \\
&= \lim_{y\to -1} \frac{1}{y-1} = -\frac{1}{2}.
\end{aligned}$$

$\diamondsuit$

Problem Sheet 6

MS121 Semester 2 IT Mathematics

Exercise 1.

Compute the indicated derivatives of the following functions using the definition of derivatives:

(a) $f'(3)$ for $f(x) = x^2 + 2x$,

(b) $f'(0)$ for $f(x) = x^3 + 1$,

(c) $f'(2)$ for $f(x) = \frac{1}{x}$,

(d) $f'(1)$ for $f(x) = \frac{x+1}{x}$,

(e) $f'(0)$ for $f(x) = \frac{1}{2x-1}$,

(f) $f'(1)$ for $f(x) = \frac{1}{x^2}$.

$\oslash$

Solution 1.

These solutions actually compute the derivatives as functions at all points in the domain of f and then substitute the required values:

(a) $f(x) = x^2 + 2x$ has difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 + 2h}{h} = 2x + h + 2$$

and derivative $f'(x) = 2x + 2$, so in particular $f'(3) = 8$,

(b) $f(x) = x^3 + 1$ has difference quotient

$$\frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2$$

and derivative $f'(x) = 3x^2$, so in particular $f'(0) = 0$,

(c) $f(x) = \frac{1}{x}$ has difference quotient

$$\frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \frac{x - (x+h)}{h(x+h)x} = \frac{-1}{(x+h)x}$$

and derivative $f'(x) = \frac{-1}{x^2}$, so in particular $f'(2) = -\frac{1}{4}$,

(d) $f(x) = \frac{x+1}{x}$ means $f(x) = 1 + \frac{1}{x}$, which has derivative $f'(x) = \frac{-1}{x^2}$, as before, so in particular $f'(1) = -1$,

(e) $f(x) = \frac{1}{2x-1}$ has difference quotient

$$\frac{1}{h}\left(\frac{1}{2x+2h-1} - \frac{1}{2x-1}\right) = \frac{2x - 1 - (2x + 2h - 1)}{h(2x+2h-1)(2x-1)} = \frac{-2}{(2x+2h-1)(2x-1)}$$

and derivative $f'(x) = \frac{-2}{(2x-1)^2}$, so in particular $f'(0) = -2$,

(f) $f(x) = \frac{1}{x^2}$ has difference quotient

$$\frac{1}{h}\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right) = \frac{x^2 - (x+h)^2}{h(x+h)^2x^2} = \frac{-2x - h}{(x+h)^2x^2}$$

and derivative $f'(x) = \frac{-2x}{x^4} = \frac{-2}{x^3}$, so in particular $f'(1) = -2$.

$\diamondsuit$

Exercise 2.

Use the rules of differentiation to find the derivatives of the following functions. Indicate which rules of differentiation you used in each part and which known derivatives of basic functions.

(a) $y = x^7 - 3x^5 + 6$,

(b) $y = x^{121} + 45x^2 - 33x + 1$,

(c) $y = x + \frac{1}{x}$,

(d) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$,

(e) $y = 7x^{\frac{1}{20}} + (\sqrt[5]{x})^7$,

(f) $y = (x^3 - 5x^2)(x + 8)$,

(g) $y = x^{\frac{1}{4}}\left(2 + \frac{1}{7}x^21\right)$,

(h) $y = x^{\frac{1}{5}}(x^2 + 3)\frac{1}{\sqrt{x}}$,

(i) $y = (x^3 + 7x)^9$,

(j) $y = \frac{x^2+1}{3+x}$.

$\oslash$

Solution 2.

(a) $y = x^7 - 3x^5 + 6$ has derivative $7x^6 - 15x^4$,

(b) $y = x^{121} + 45x^2 - 33x + 1$ has derivative $1217x^{120} + 90x - 33$,

(c) $y = x + \frac{1}{x}$ has derivative $1 - \frac{1}{x^2}$,

(d) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$ has derivative $\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$,

(e) $y = 7x^{\frac{1}{20}} + (\sqrt[5]{x})^7$ has derivative $\frac{7}{20}x^{\frac{-19}{20}} + \frac{7}{5}x^{-\frac{2}{5}}$,

(f) $y = (x^3 - 5x^2)(x + 8)$ has derivative $(3x^2 - 10x)(x + 8) + (x^3 - 5x^2)$ $(= 4x^3 + 9x^2 - 80x)$,

(g) $y = x^{\frac{1}{4}}\left(2 + \frac{1}{7}x^21\right)$ has derivative $\frac{1}{4}x^{-\frac{3}{4}}\left(2 + \frac{1}{7}x^21\right) + x^{\frac{1}{4}}3x^{20}$,

(h) $y = x^{\frac{1}{5}}(x^2 + 3)\frac{1}{\sqrt{x}}$ has derivative
$\frac{1}{5}x^{-\frac{4}{5}}(x^2 + 3)\frac{1}{\sqrt{x}} + x^{\frac{1}{5}}(2x)\frac{1}{\sqrt{x}} - x^{\frac{1}{5}}(x^2 + 3)\frac{1}{2}x^{-\frac{3}{2}}$,

(i) $y = (x^3 + 7x)^9$ has derivative $9(x^3 + 7x)^8(3x^2 + 7)$,

(j) $y = \frac{x^2+1}{3+x}$ has derivative $\frac{(3+x)2x-(x^2+1)}{(3+x)^2} = \frac{x^2+6x-1}{(3+x)^2}$.

We used that the derivative of x^α is $\alpha x^{\alpha-1}$ for all $\alpha \in \mathbb{R}$. We also used the following rules:

sum rule	all parts
product rule	parts (f), (g) and (h)
chain rule	part (i)
quotient rule	part (j)

Different answers are possible, because sometimes we can choose which rule to use. $\diamondsuit$

Exercise 3.

Find the tangent line to the curve $y = f(x)$ for the following functions at the given values of x:

(a) $f(x) = x^2 + x$, at $x = 2$,

(b) $f(x) = x^3 + 1$, at $x = -1$,

(c) $f(x) = x^2 - 3x$, at $x = -3$,

(d) $f(x) = x^2 + \frac{1}{2x}$, at $x = 1$.

$\oslash$

Solution 3.

(a) $f(x) = x^2 + x$, at $x = 2$, has $f(2) = 6$, so the tangent goes through $(2, 6)$; we have $f'(x) = 2x + 1$ and $f'(2) = 5$, so the tangent has slope 5; its equation is therefore $y = 5x - 4$.

(b) $f(x) = x^3 + 1$, at $x = -1$, has $f(-1) = 0$, so the tangent goes through $(-1, 0)$; we have $f'(x) = 3x^2$ and $f'(-1) = 3$, so the tangent has slope 3; its equation is therefore $y = 3x + 3$.

(c) $f(x) = x^2 - 3x$, at $x = -3$, has $f(-3) = 18$, so the tangent goes through $(-3, 18)$; we have $f'(x) = 2x - 3$ and $f'(-3) = -9$, so the tangent has slope -9; its equation is therefore $y = -9x - 9$.

(d) $f(x) = x^2 + \frac{1}{2x}$, at $x = 1$, has $f(1) = \frac{3}{2}$, so the tangent goes through $(1, \frac{3}{2})$; we have $f'(x) = 2x - \frac{1}{2x^2}$ and $f'(1) = \frac{3}{2}$, so the tangent has slope $\frac{3}{2}$; its equation is therefore $y = \frac{3}{2}x$.

$\diamondsuit$

Exercise 4.

Calculate the following limits by recognizing a derivative:

(a) $\displaystyle\lim_{x \to 0} \frac{(x^7 - 1)^8 - 1}{x}$,

(b) $\displaystyle\lim_{x \to 4} \frac{\sqrt{x - 3} - 1}{x - 4}$,

(c) $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x}$,

(d) $\displaystyle\lim_{x \to 1} \frac{1}{x^2(x - 1)} - \frac{1}{x - 1}$.

$\oslash$

Solution 4.

(a) $\displaystyle\lim_{x \to 0} \frac{(x^7 - 1)^8 - 1}{x} = f'(0)$ for $f(x) = (x^7 - 1)^8$. Since $f'(x) = 8(x^7 - 1)^7 \cdot 7x^6$ we find $f'(0) = 0$, so the limit vanishes.

(b) $\displaystyle\lim_{x \to 4} \frac{\sqrt{x - 3} - 1}{x - 4} = f'(4)$ for $f(x) = \sqrt{x - 3}$. Since $f'(x) = \frac{1}{2}(x - 3)^{-\frac{1}{2}}$ we find that the limit is $f'(4) = \frac{1}{2}$.

(c) $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x} = \sin'(0) = \cos(0) = 1$.

(d) $\displaystyle\lim_{x \to 1} \frac{1}{x^2(x - 1)} - \frac{1}{x - 1} = f'(1)$ for $f(x) = x^{-2}$. Since $f'(x) = -2x^{-3}$ we find that the limit is $f'(1) = -2$.

$\diamondsuit$

Problem Sheet 7

MS121 Semester 2 IT Mathematics

Exercise 1.

Find the tangent lines to the following curves at the indicated points:

(a) $y = \sin(t)$ at $t = 0$,

(b) $y = \sqrt{x}$ at $x = 1$,

(c) $y = \frac{1}{x+1}$ at $x = 2$,

(d) $y = 2^t$ at $t = 3$.

$\oslash$

Solution 1.

(a) $y = \sin(t)$ at $t = 0$ has $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}t}(0) = \cos(0) = 1$, so the tangent line is $y = t$.

(b) $y = \sqrt{x}$ at $x = 1$ has $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}(1) = \frac{1}{2} \cdot 1^{-\frac{1}{2}} = \frac{1}{2}$, so the tangent line is $y - 1 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}(x + 1)$.

(c) $y = \frac{1}{x+1}$ at $x = 2$ has $y = \frac{1}{3}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}(2) = \frac{-1}{(2+1)^2} = -\frac{1}{9}$, so the tangent line is $y - \frac{1}{3} = -\frac{1}{9}(x-2)$ or $y = -\frac{1}{9}x + \frac{5}{9}$.

(d) $y = 2^t$ at $t = 3$ has $y = 2^3 = 8$ and $\frac{\mathrm{d}y}{\mathrm{d}t}(3) = \ln(2) \cdot 2^3 = 8\ln(2)$, so the tangent line is $y - 8 = 8\ln(2)(x - 3)$ or $y = 8\ln(2)x + 8 - 24\ln(2)$.

$\diamondsuit$

Exercise 2.

Differentiate the following functions with respect to t:

(a) $f(t) = t^\pi$,

(b) $f(t) = t^{\frac{5}{2}} \cos(t)$,

(c) $f(t) = t^2 \ln(t)$,

(d) $f(t) = e^t + \frac{5t}{(t+1)^3}$,

(e) $f(t) = \frac{\sqrt{t^2+1}}{3+\sqrt{t}}$,

(f) $f(t) = \sqrt{\frac{t-9}{t^2+7}}$,

(g) $f(t) = \left(t^2 - t^{-2}\right)^{-5}$.

$\oslash$

Solution 2.

(a) $f(t) = t^\pi$ has $f'(t) = \pi t^{\pi-1}$,

(b) $f(t) = t^{\frac{5}{2}} \cos(t)$ has $f'(t) = \frac{5}{2}t^{\frac{3}{2}} \cos(t) - t^{\frac{5}{2}} \sin(t)$,

(c) $f(t) = t^2 \ln(t)$ has $f'(t) = 2t\ln(t) + t$ (on $t > 0$),

(d) $f(t) = e^t + \frac{5t}{(t+1)^3}$ has $f'(t) = e^t + \frac{5(t+1)-15t}{(t+1)^4}$,

(e) $f(t) = \frac{\sqrt{t^2+1}}{3+\sqrt{t}}$ has

$$f'(t) = \frac{(3 + \sqrt{t})t(t^2 + 1)^{-\frac{1}{2}} - \frac{1}{2}\sqrt{t^2 + 1}t^{-\frac{1}{2}}}{(3 + \sqrt{t})^2},$$

(f) $f(t) = \sqrt{\frac{t-9}{t^2+7}}$ has

$$f'(t) = \frac{1}{2}\sqrt{\frac{t^2+7}{t-9}}\,\frac{(t^2+7)-2(t-9)t}{(t^2+7)^2},$$

(g) $f(t) = \left(t^2 - t^{-2}\right)^{-5}$ has

$$f'(t) = -5\left(t^2 - t^{-2}\right)^{-6}(2t + 2t^{-3}).$$

$\diamondsuit$

Exercise 3.
For the following functions, find all critical points:

(a) $f(x) = x^3 - 3x^2 - 9x$,

(b) $f(x) = x^4 - 2x^3 + 5$,

(c) $f(x) = (x^3 - 27)^8$,

(d) $f(x) = \frac{7x}{x^2+1}$.

$\oslash$

Solution 3.

(a) $f(x) = x^3 - 3x^2 - 9x$ has $f'(x) = 3x^2 - 6x - 9$ and $f''(x) = 6x - 6$. The critical points $f'(x) = 0$ are at $x = -1$ and $x = 3$.

(b) $f(x) = x^4 - 2x^3 + 5$ has $f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$ and $f''(x) = 12x^2 - 12x$. The critical points $f'(x) = 0$ are at $x = 0$ (double) and $x = \frac{3}{2}$.

(c) $f(x) = (x^3 - 27)^8$ has $f'(x) = 8(x^3 - 27)^7 \cdot 3x^2$. The critical points $f'(x) = 0$ are at $x = 0$ (double) and $x = 3$ (7-fold).

(d) $f(x) = \frac{7x}{x^2+1}$ has $f'(x) = \frac{7(x^2+1)-14x^2}{(x^2+1)^2}$. The critical points $f'(x) = 0$ are at $x = -1$ and $x = +1$.

$\diamondsuit$

Exercise 4.
Assume that the function f is differentiable at $a \in \mathbb{R}$. Argue that f must be continuous at $a \in \mathbb{R}$. Hint: First write down the assumption in terms of limits. Then show that $\lim_{x\to a} f(x) - f(a) = 0$.

$\oslash$

Solution 4.
From

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

we conclude that

$$\begin{aligned}
\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}(x-a)\frac{f(x)-f(a)}{x-a} \\
&= \left(\lim_{x\to a}(x-a)\right)\cdot\left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right) \\
&= 0 \cdot f'(a) = 0 \ .
\end{aligned}$$

Because $f(a)$ is constant (independent of x) it follows that $\lim_{x \to a} f(x) = f(a)$, which means that f is continuous at $a \in \mathbb{R}$. $\diamondsuit$

Problem Sheet 8

MS121 Semester 2 IT Mathematics

Exercise 1.

Find and classify all critical points of the following functions, and sketch the curve $y = f(x)$:

(a) $f(x) = x^3 - 3x^2 + 5$,

(b) $f(x) = \cosh(x)$,

(c) $f(x) = x^4 - 4x^3 + 7$,

(d) $f(x) = xe^{-x^2}$,

(e) $f(x) = \frac{x+1}{x^2+3}$,

(f) $f(x) = \sqrt{2 - \cos(x)}$.

$\oslash$

Solution 1.

(a) $f(x) = x^3 - 3x^2 + 5$ has $f'(x) = 3x^2 - 6x = 3x(x-2)$, so the critical points are at $x = 0$ and $x = 2$. We have $f''(x) = 6x - 6 = 6(x-1)$, so $f''(0) = -6 < 0$ and $f''(2) = 6 > 0$, with a local maximum $f(0) = 5$ and local minimum $f(2) = 1$.

(b) $f(x) = \cosh(x)$ has $f'(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$, so the only critical point is at $x = 0$ (because e^x is invertible). Since $f''(x) = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$ we have $f''(0) = 1 > 0$ and we have a (global) minimum at $f(0) = 1$.

(c) $f(x) = x^4 - 4x^3 + 7$ has $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$, so the critical points are at $x = 0$ and $x = 3$. We have $f''(x) = 12x^2 - 24x = 12x(x-2)$, so $f''(0) = 0$ and $f''(3) = 36 > 0$. $f(3) = -20$ is a (global) minimum. Because $f'(x)$ does not change sign at $x = 0$ we have an inflection point there with $f(0) = 7$.

(d) $f(x) = xe^{-x^2}$ has $f'(x) = (1-2x^2)e^{-x^2}$, so the critical points are at $x = -\frac{1}{2}\sqrt{2}$ and $x = \frac{1}{2}\sqrt{2}$. We have $f''(x) = (-4x - 2x + 4x^3)e^{-x^2} = 2x(2x^2 - 3)e^{-x^2}$ and $f''(\pm\frac{1}{2}\sqrt{2}) = \mp 2\sqrt{\frac{2}{e}}$, so we find a (global) maximum $f(\frac{1}{2}\sqrt{2}) = \frac{1}{2}\sqrt{2/e}$ and a (global) minimum $f(-\frac{1}{2}\sqrt{2}) = -\frac{1}{2}\sqrt{2/e}$.

(e) $f(x) = \frac{x+1}{x^2+3}$ has $f'(x) = \frac{(x^2+3)-2x(x+1)}{(x^2+3)^2} = \frac{-x^2-2x+3}{(x^2+3)^2}$, so the critical points are at $x = -3$ and $x = 1$. We have $f''(x) = \frac{(x^2+3)(-2x-2)+(x^2+2x-3)4x}{(x^2+3)^3} = \frac{2x^3+6x^2-18x-6}{(x^2+3)^3}$, so $f''(-3) = \frac{48}{12^3} = \frac{1}{36} > 0$ and $f''(1) = \frac{-16}{4^3} = -\frac{1}{4} < 0$ and we find a (global) minimum $f(-3) = -\frac{1}{6}$ and a (global) maximum $f(1) = \frac{1}{2}$.

(f) $f(x) = \sqrt{2 - \cos(x)}$ has $f'(x) = \frac{1}{2}(2 - \cos(x))^{-\frac{1}{2}} \sin(x)$, so critical points are at $x = k\pi$ for all $k \in \mathbb{Z}$. We have

$$\begin{aligned} f''(x) &= -\frac{1}{4}(2 - \cos(x))^{-\frac{3}{2}}(\sin(x)^2 - 2(2 - \cos(x))\cos(x)) \\ &= -\frac{1}{4}(2 - \cos(x))^{-\frac{3}{2}}(2 - \sin(x)^2 - 4\cos(x)), \end{aligned}$$

so $f''(k\pi) = \frac{1}{2} > 0$ when $k \in 2\mathbb{Z}$ (because $\cos(k\pi) = 1$) and $f''(k\pi) = -\frac{1}{2\sqrt{3}} < 0$ else (because $\cos(k\pi) = -1$). We therefore have (global) minima $f(k\pi) = 1$ when $x = k\pi$ with k even (i.e. $k \in 2\mathbb{Z}$) and (global) maxima $f(k\pi) = \sqrt{3}$ when k is odd. (This oscillating behaviour is similar as for $2 - \cos(x)$, but the square root changes the amplitudes above and below in different ways.)

$\diamondsuit$

Exercise 2.
Consider the function $f(x) = 2x^3 - 15x^2 + 24x + 20$.

(a) Determine the absolute maximum and minimum on the closed interval $[0, 6]$.

(b) Does $f(x)$ also have an absolute maximum and/or minimum on the open interval $(0, 6)$?

$\oslash$

Solution 2.

(a) We have $f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x - 4)(x - 1)$, so the critical points are $x = 1$ and $x = 4$, which do lie in the interval $[0, 6]$. At the boundaries and the critical points we have: $f(0) = 20$, $f(1) = 31$, $f(4) = 4$ and $f(6) = 56$. The absolute minimum on $[0, 6]$ is therefore $f(4) = 4$ and the absolute maximum is $f(6) = 56$.

(b) On the open interval $(0, 6)$ there is an absolute minimum, $f(4) = 4$, but no absolute maximum: the function approaches $f(6) = 56$ as closely as we like, but it never actually attains this value in the open interval $(0, 6)$.

$\diamondsuit$

Exercise 3.
We take a rectangular piece of paper of 12 cm by 12 cm and we remove squares of x cm by x cm from each of the four corners. We fold the ends up to a box of height x. For what x does the box have a maximal volume? How big is this volume? $\oslash$

Solution 3.
The height of the box is x and the length and width are both $12 - 2x$ (in cm). The volume is therefore $V(x) = x(12 - 2x)^2 = 4x^3 - 48x^2 + 144x$ (in cm^3). Note that $0 \leq x \leq 6$. To find the maximum we compute $V'(x) = 12x^2 - 96x + 144 = 12(x^2 - 8x + 12)$ and we set $V'(x) = 0$ to find the critical points of V. These are $x_\pm = 4 \pm \frac{1}{2}\sqrt{16} = 4 \pm 2$, i.e. $x = 2$ and $x = 6$. (The larger critical point is on the boundary.) We have $V(0) = 0$, $V(6) = 0$ and $V(2) = 2(12 - 4)^2 = 128$. We therefore find the maximum volume 128 cm^3 at $x = 2$ cm. $\diamondsuit$

Exercise 4.
Determine whether the following functions are continuous and/or differentiable at the indicated points:

(a) $f(x) = |x|$ at $x = 0$,

(b) $f(x) = |x|^3$ at $x = 0$,

(c)
$$g(t) = \begin{cases} 2t - 1 & \text{if } -1 < t < 1 \\ t^2 & \text{if } |t| \geq 1 \end{cases}$$

at $t = 1$ and at $t = -1$.

$\oslash$

Solution 4.

(a) Note that

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} .$$

This is continuous at $x = 0$, but not differentiable, because

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} .$$

has unequal left and right hand limits as $x \to 0$.

(b) $f(x) = |x|^3 = |x|x^2$ is continuous at $x = 0$, because it is a product of continuous functions (see part (a)). It is even differentiable there, because

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x > 0 \end{cases} .$$

tends to 0 as $x \to 0$.

(c)

$$g(t) = \begin{cases} 2t - 1 & \text{if } -1 < t < 1 \\ t^2 & \text{if } |t| \geq 1 \end{cases}$$

is not continuous at $t = -1$, because the left and right hand limits differ, and it is therefore not differentiable either. $g(t)$ is continuous at $t = 1$ and it is even differentiable there, because

$$\begin{aligned} \frac{g(t) - g(1)}{t - 1} &= \begin{cases} \frac{2t-2}{t-1} & \text{if } -1 < t < 1 \\ \frac{t^2-1}{t-1} & \text{if } |t| \geq 1 \end{cases} \\ &= \begin{cases} 2 & \text{if } -1 < t < 1 \\ t + 1 & \text{if } |t| \geq 1 \end{cases} \end{aligned}$$

has the limit 2 as $t \to 1$.

$\diamondsuit$

Problem Sheet 9

MS121 Semester 2 IT Mathematics

Exercise 1.
For the following functions, determine the Taylor polynomial of degree 2 at the indicated supporting point:

(a) $f(x) = \sin(x)$ at $x_0 = \frac{\pi}{2}$,

(b) $g(x) = (1+x)^3$ at $x_0 = 0$,

(c) $h(x) = \ln(x)$ at $x_0 = 1$.

$\oslash$

Solution 1.
(a) $f(x) = \sin(x)$ has $f'(x) = \cos(x)$ and $f''(x) = -\sin(x)$, so $f(\frac{\pi}{2}) = 1$, $f'(\frac{\pi}{2}) = 0$ and $f''(\frac{\pi}{2}) = -1$. This leads to the Taylor polynomial

$$\begin{aligned}P(x) &= f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x - \frac{\pi}{2}) + \frac{1}{2}f''(\frac{\pi}{2})(x - \frac{\pi}{2})^2 \\ &= 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 = -\frac{1}{2}x^2 + \frac{\pi}{2}x + 1 - \frac{\pi^2}{8}.\end{aligned}$$

(b) $g(x) = (1+x)^3$ has $g'(x) = 3(1+x)^2$ and $g''(x) = 6(1+x)$, so $g(0) = 1$, $g'(0) = 3$ and $g''(0) = 6$. This leads to the Taylor polynomial

$$\begin{aligned}P(x) &= g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 \\ &= 1 + 3x + \frac{6}{2}x^2 = 1 + 3x + 3x^2.\end{aligned}$$

The Taylor polynomial of degree 3 would be $1 + 3x + 3x^2 + x^3$ which equals $g(x)$.

(c) $h(x) = \ln(x)$ has $h'(x) = x^{-1}$ and $h''(x) = -x^{-2}$, so $f(1) = 0$, $f'(1) = 1$ and $f''(1) = -1$. This leads to the Taylor polynomial

$$\begin{aligned}P(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 \\ &= (x-1) - \frac{1}{2}(x-1)^2 = -\frac{1}{2}x^2 + 2x - \frac{3}{2}.\end{aligned}$$

$\diamondsuit$

Exercise 2.
For the following functions $f(x)$ find a primitive function $F(x)$ such that F takes the given value at the specified point:

(a) $f(x) = 3x^2 + 2$, with $F(1) = 2$,

(b) $f(x) = 2x^7 + 6x^3 + 2x$, with $F(0) = 1$,

(c) $f(x) = (2x+3)^2$, with $F(-2) = -\frac{1}{6}$,

(d) $f(x) = 2\sin(x)$, with $F(0) = 0$,

(e) $f(x) = \cos(x) + \sin(3x)$, with $F\left(\frac{\pi}{2}\right) = 0$,

(f) $f(x) = e^x$, with $F(0) = 3$,

(g) $f(x) = \frac{6}{x}$ (on $x > 0$), with $F(1) = 4$,

(h) $f(x) = \frac{2x}{(x^2+3)^2}$, with $F(0) = -\frac{1}{3}$.

$\oslash$

Solution 2.
In all cases we can first find a ny primitive function (or anti-derivative) and add an appropriate constant to get the given value. The results are:

(a) $F(x) = x^3 + 2x - 1$,

(b) $F(x) = \frac{1}{4}x^8 + \frac{3}{2}x^4 + x^2 + 1$,

(c) $F(x) = \frac{1}{6}(2x+3)^3$,

(d) $F(x) = -2\cos(x) + 2$,

(e) $F(x) = \sin(x) - \frac{1}{3}\cos(3x) - 1$,

(f) $F(x) = e^x + 2$,

(g) $F(x) = 6\ln(x) + 4$,

(h) $F(x) = \frac{-1}{x^2+3}$.

$\diamondsuit$

Exercise 3.
Evaluate the following definite integrals:

(a) $\int_0^2 \; x^3 \; \mathrm{d}x$,

(b) $\int_{-2}^1 \; 4t^2(1-t)\mathrm{d}t$,

(c) $\int_1^2 \; t^4 + 8t^{-9}\mathrm{d}t$,

(d) $\int_1^5 \; \frac{x^{\frac{3}{2}} - 7x}{x}\mathrm{d}x$.

$\oslash$

Solution 3.

(a) $\int_0^2 \; x^3 \; \mathrm{d}x = \left[\frac{1}{4}x^4\right]_0^2 = \frac{2^4}{4} - \frac{0}{4} = 4$,

(b) $\int_{-2}^1 \; 4t^2(1-t)\mathrm{d}t = \left[\frac{4}{3}t^3 - t^4\right]_{-2}^1 = (\frac{4}{3} - 1) - (-8 \cdot \frac{4}{3} - 16) = 27$,

(c) $\int_1^2 \; t^4 + 8t^{-9}\mathrm{d}t = \left[\frac{1}{5}t^5 - t^{-8}\right]_1^2 = \left(\frac{32}{5} - \frac{1}{256}\right) - (\frac{1}{5} - 1) = 7\frac{251}{1280}$,

(d) $\int_1^5 \; \frac{x^{\frac{3}{2}} - 7x}{x}\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}} - 7x\right]_1^5 = (3\frac{1}{3}\sqrt{5} - 35) - (\frac{2}{3} - 7) = 3\frac{1}{3}\sqrt{5} - 28\frac{2}{3}$.

$\diamondsuit$

Exercise 4.
We consider the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad 0 \le x \le 3 \\ 1 & \text{if} \quad x > 3 \end{cases} .$$

(a) Find a function $F(x)$ which is continuous and such that $F'(x) = f(x)$ for all $x \in \mathbb{R} \setminus \{0, 3\}$.

(b) Compute the area A under the curve $y = f(x)$ between $x = 1$ and $x = 5$.

(c) Verify that $F(5) - F(1) = A$.

(d) Show that $F'(0) = 0 = f(0)$, but $F'(3)$ does not exist. (Hint: use one-sided limits.)

$\oslash$

Solution 4.

(a) We find

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{1}{2}x^2 & \text{if} \quad 0 \leq x \leq 3 \\ \frac{3}{2} + x & \text{if} \quad x > 3 \end{cases}$$

up to an arbitrary additive constant.

(b) The total area is $A = 4 + 2 = 6$:
Between $x = 3$ and $x = 5$ we have a block of area $2 \cdot 1 = 2$.
Between $x = 1$ and $x = 3$ we have a shape with area $2 \cdot \frac{f(1)+f(3)}{2} = 2 \cdot \frac{4}{2} = 4$. (We use the formula: width times average height, which can be justified geometrically. One can also view this as a block of area 2 plus a triangle of area $\frac{1}{2} \cdot 2 \cdot (3-1) = 2$.)

(c) For our choice of $F(x)$ we have

$$F(5) - F(1) = \left(\frac{3}{2} + 5\right) - \frac{1}{2} = 6 = A.$$

(d) This part is a bit harder. Using the definitions we compute:

$$\begin{aligned}
\lim_{x \to 0^-} \frac{F(x) - F(0)}{x} &= \lim_{x \to 0^-} 0 = 0, \\
\lim_{x \to 0^+} \frac{F(x) - F(0)}{x} &= \lim_{x \to 0^+} \frac{1}{2}x = 0, \\
\lim_{x \to 3^-} \frac{F(x) - F(3)}{x - 3} &= \lim_{x \to 3^-} \frac{\frac{1}{2}(x+3)(x-3)}{x-3} = \lim_{x \to 3^-} \frac{1}{2}(x+3) = 3, \\
\lim_{x \to 3^+} \frac{F(x) - F(3)}{x - 3} &= \lim_{x \to 3^+} \frac{\frac{3}{2} + x - \frac{9}{2}}{x-3} = \lim_{x \to 3^+} 1 = 1.
\end{aligned}$$

From the first lines we see that $F'(0) = \lim_{x \to 0} \frac{F(x)-F(0)}{x} = 0$ exists, but from the last two one-sided limits we see that $F'(3)$ does not exist.

$\diamondsuit$

Problem Sheet 10

MS121 Semester 2 IT Mathematics

Exercise 1.

Use an integration by parts to evaluate the following integrals:

(a) $\int\ xe^{-x}\ \mathrm{d}x$,

(b) $\int\ t\sin(3t)\ \mathrm{d}t$,

(c) $\int\ x\ln(x)\ \mathrm{d}x$,

(d) $\int_0^1\ (s^2-1)e^s\ \mathrm{d}s$,

(e) $\int_1^2\ (x^3-2x)\ln(x)\ \mathrm{d}x$,

$\oslash$

Solution 1.

(a) $\int\ xe^{-x}\ \mathrm{d}x = x(-e^{-x}) - \int -e^{-x}\mathrm{d}x = -xe^{-x} - e^{-x}$,

(b) $\int\ t\sin(3t)\ \mathrm{d}t = t(-\frac{1}{3}\cos(3t)) - \int -\frac{1}{3}\cos(3t)\mathrm{d}t = -\frac{1}{3}t\cos(3t) + \frac{1}{9}\sin(3t)$,

(c) $\int\ x\ln(x)\ \mathrm{d}x = (\frac{1}{2}x^2)\ln(x) - \int \frac{1}{2}x^2 \cdot x^{-1}\ \mathrm{d}x = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2$,

(d) We integrate by parts twice to find:

$$\begin{aligned}
\int_0^1\ (s^2-1)e^s\ \mathrm{d}s &= \left[(s^2-1)e^s\right]_0^1 - \int_0^1\ 2se^s\ \mathrm{d}s \\
&= 1 - [2se^s]_0^1 + \int_0^1\ 2e^s\ \mathrm{d}s \\
&= 1 - 2e + [2e^s]_0^1 = 1 - 2e + 2e - 2 = -1,
\end{aligned}$$

(e) $\int_1^2\ (x^3-2x)\ln(x)\ \mathrm{d}x = \left[(\frac{1}{4}x^4 - x^2)\ln(x)\right]_1^2 - \int_1^2\ (\frac{1}{4}x^4 - x^2)x^{-1}\ \mathrm{d}x = 0 - \left[\frac{1}{16}x^4 - \frac{1}{2}x^2\right]_1^2 = -(1-2) + (\frac{1}{16} - \frac{1}{2}) = \frac{9}{16}$,

$\diamondsuit$

Exercise 2.

Use a substitution to evaluate the following integrals:

(a) $\int\ \frac{\ln(x)}{x}\ \mathrm{d}x$,

(b) $\int\ t\sqrt{t-5}\ \mathrm{d}t$,

(c) $\int_0^{\pi^2}\ \frac{\cos(\sqrt{x})}{\sqrt{x}}\ \mathrm{d}x$,

(d) $\int_0^1\ \frac{x^3}{x^4+5}\ \mathrm{d}x$,

(e) $\int_0^1\ te^{-t^2}\mathrm{d}t$,

(f) $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\ e^{\sin(t)}\cos(t)\mathrm{d}t$.

$\oslash$

Solution 2.

(a) $\int\ \frac{\ln(x)}{x}\ \mathrm{d}x$ can be computed with

$$y = \ln(x), \quad x = e^y, \quad \frac{\mathrm{d}x}{\mathrm{d}y} = e^y$$

as

$$\int\ \frac{y}{e^y}\ e^y\mathrm{d}y = \int\ y\mathrm{d}y = \frac{1}{2}y^2 = \frac{1}{2}(\ln(x))^2.$$

(b) $\int\ t\sqrt{t-5}\ \mathrm{d}t$ can be computed with

$$s = t-5, \quad t = s+5, \quad \frac{\mathrm{d}t}{\mathrm{d}s} = 1$$

as

$$\int\ (s+5)\sqrt{s}\ \mathrm{d}s = \int\ s^{\frac{3}{2}} + 5s^{\frac{1}{2}}\ \mathrm{d}s = \frac{2}{5}s^{\frac{5}{2}} + 5\cdot\frac{2}{3}s^{\frac{3}{2}} = \frac{2}{5}(t-5)^{\frac{5}{2}} + 3\frac{1}{3}(t-5)^{\frac{3}{2}}.$$

(c) $\int_0^{\pi^2}\ \frac{\cos(\sqrt{x})}{\sqrt{x}}\ \mathrm{d}x$ can be computed with

$$y = \sqrt{x}, \quad x = y^2, \quad \frac{\mathrm{d}x}{\mathrm{d}y} = 2y$$

as

$$\int_0^{\pi}\ \frac{\cos(y)}{y}\ 2y\mathrm{d}y = \int_0^{\pi}\ 2\cos(y)\ \mathrm{d}y = [2\sin(y)]_0^{\pi} = 0.$$

(d) $\int_0^1\ \frac{x^3}{x^4+5}\ \mathrm{d}x$ can be computed with

$$y = x^4+5, \quad x = \sqrt[4]{y-5}, \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{4}(y-5)^{-\frac{3}{4}} = \frac{1}{4}x^{-3}$$

as

$$\int_5^6\ \frac{(y-5)^{\frac{3}{4}}}{y}\ \frac{1}{4}(y-5)^{-\frac{3}{4}}\mathrm{d}y = \frac{1}{4}\int_5^6\ y^{-1}\ \mathrm{d}y = \frac{1}{4}\left[\ln(y)\right]_5^6 = \frac{1}{4}\ln\left(\frac{6}{5}\right).$$

(e) $\int_0^1\ te^{-t^2}\mathrm{d}t$ can be computed with

$$s = -t^2, \quad t = \sqrt{-s}, \quad \frac{\mathrm{d}t}{\mathrm{d}s} = -\frac{1}{2}(-s)^{-\frac{1}{2}}$$

as

$$\int_0^{-1}\ \sqrt{-s}e^s\ \left(-\frac{1}{2}\right)(-s)^{-\frac{1}{2}}\mathrm{d}s = \frac{1}{2}\int_{-1}^0\ e^s\ \mathrm{d}s = \frac{1}{2}\left[e^s\right]_{-1}^0 = \frac{1}{2}(1-e^{-1}).$$

(f) $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\ e^{\sin(t)}\cos(t)\mathrm{d}t$ can be computed with

$$x = \sin(t), \quad t = \sin^{-1}(x), \quad \frac{\mathrm{d}t}{\mathrm{d}x} = (1-x^2)^{-\frac{1}{2}}$$

as

$$\int_{-1}^1\ e^x\sqrt{1-x^2}\ (1-x^2)^{-\frac{1}{2}}\mathrm{d}x = [e^x]_{-1}^1 = e - e^{-1}.$$

$\diamondsuit$

Exercise 3.
Find the following areas:

(a) the area above the x-axis and below the curve $y = 9 - x^2$,

(b) the area between the curves $y = x^2$ and $y = x$,

(c) the area above the curve $y = (x-3)^2$ and below the curve $y = 9 + 2x - x^2$,

(d) the area above the line $y = 1$ and below the curve $y = \frac{6x}{x^2+5}$.

$\oslash$

Solution 3.

(a) the parabola intersects the x-axis at $x = -3$ and $x = 3$, so the desired area is

$$\int_{-3}^{3} 9 - x^2 \, \mathrm{d}x = \left[9x - \frac{1}{3}x^3\right]_{-3}^{3} = (27-9) - (-27+9) = 36,$$

(b) noting that the curves $y = x^2$ and $y = x$ intersect at $x = 0$ and at $x = 1$, and that $x > x^2$ in between, we find the area

$$\int_{0}^{1} x - x^2 \, \mathrm{d}x = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{0}^{1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

(c) the curves intersect where $0 = (9+2x-x^2) - (x-3)^2 = -2x^2 + 8x$, i.e. at $x = 0$ and $x = 4$, and in between we have $9 + 2x - x^2 > (x-3)^2$, so the desired area is

$$\int_{0}^{4} (9+2x-x^2) - (x-3)^2 \, \mathrm{d}x = \int_{0}^{4} 8x - 2x^2 \, \mathrm{d}x = \left[4x^2 - \frac{2}{3}x^3\right]_{0}^{4} = 64 - 64 \cdot \frac{2}{3} = 21\frac{1}{3},$$

(d) noting that the curves intersect when $x^2 + 5 - 6x = 0$, i.e. when $x = 1$ and $x = 5$, we find the area

$$\int_{1}^{5} \frac{6x}{x^2+5} - 1 \, \mathrm{d}x = \int_{6}^{30} \frac{3}{y} \, \mathrm{d}y - [x]_1^5 = [3\ln(y)]_6^{30} - 4 = 3\ln\left(\frac{30}{6}\right) - 4 = 3\ln(5) - 4,$$

where we substituted $y = x^2 + 5$, $x = \sqrt{y-5}$, $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{2}(y-5)^{-\frac{1}{2}}$ in the first term.

$\diamondsuit$

Exercise 4.
For a continuous function $f : \mathbb{R} \to \mathbb{R}$ and any number $a > 0$ we consider the integral

$$\int_{-a}^{a} f(x) \, \mathrm{d}x.$$

Show that this integral is 0 when f is an odd function, i.e. when $f(x) = f(-x)$. (Hint: use the substitution $y = -x$.)

$\oslash$

Solution 4.
Using $f(x) = -f(-x)$ and the substitution $y = -x$ with $\frac{\mathrm{d}x}{\mathrm{d}y} = -1$ we find

$$\begin{aligned}
\int_{-a}^{a} f(x)\ \mathrm{d}x &= \int_{-a}^{a} -f(-x)\ \mathrm{d}x \\
&= \int_{a}^{-a} f(y)\ \mathrm{d}y \\
&= -\int_{-a}^{a} f(y)\ \mathrm{d}y \\
&= -\int_{-a}^{a} f(x)\ \mathrm{d}x.
\end{aligned}$$

where the reversal of the boundaries incurs a sign in the third line and we just renamed the (dummy) variable in the last line. It now follows that the integral is a number $I \in \mathbb{R}$ such that $I = -I$, which means that $2I = 0$ and hence $I = 0$. (Geometrically, any area below (resp. above) the curve in the region $x > 0$ is compensated for by an area above (resp. below) the curve in the region $x < 0$.) $\diamondsuit$