Similarly, the predicate can be turned into the proposition
There is an integer x, satisfying $x^2 > 3$

This new proposition is true, since $x = 2$ is an integer with $x^2 > 3$.

Since the expressions 'for all x ' and 'there exists an x' come up so regularly we have shorthand notation for them:

$\forall x$ 'for all x ' $\exists x$ 'there exists an x'

More complicated predicates
When several quantifiers are involved care needs to be taken in determining the truth values.

Example: If x and y are integers and P(x,y) is the statement $xy = 1$, express the following propositions in words and determine their truth value:
$\exists\ x(\exists\ y\ (\mathrm{P}(x,\ y)))$ $\forall\ x(\exists\ y\ (\mathrm{P}(x,\ y)))$

Solution The first says 'there exists an x with the property that there exists a y satisfying $xy = 1$'.

This proposition is true. We could use the values $x = -1$ and $y = -1$.

The second says 'for all x there is a y with $xy = 1$'.

This proposition is false, since we can take $x = 2$ so that $xy = 1$ gives $y = 1/2$ which is not an integer.

Negating Propositions involving Quantifiers
If we construct a proposition from a predicate using a quantifier we can apply the logical operation **not** to it.

The negation of the proposition
For all integers x, $x^2 > 3$
is the proposition
There is an integer x with $x^2 \leq 3$.

The negation of the proposition
There is an integer x satisfying $x^2 > 3$
is the proposition
For all integers x, $x^2 \leq 3$.

In general, we use the following logical equivalences for their negations

$\textbf{not } (\exists\, x\, (\mathrm{P}(x))) \equiv \forall\, x\ (\textbf{not } (\mathrm{P}(x)))$

$\textbf{not } (\forall\, x\, (\mathrm{P}(x))) \equiv \exists\, x\ (\textbf{not } (\mathrm{P}(x)))$

Note: It may help you remember these negations if you consider the case where x belongs to a finite set $X = \{x_1, x_2, \ldots, x_n\}$. Now the proposition $\exists\, x$ in X satisfying $\mathrm{P}(x)$ is equivalent to $\mathrm{P}(x_1)$ **or** $\mathrm{P}(x_2)$ **or** $\ldots$ **or** $\mathrm{P}(x_n)$ whose negation should be

not$(\mathrm{P}(x_1))$ **and not**$(\mathrm{P}(x_2))$ **and** $\ldots$ **and not**$(\ \mathrm{P}(x_n))$.

To spell this out further, suppose $X = \{x_1, x_2, x_3\}$. Now $\exists x(P(x))$ is equivalent to $\mathrm{P}(x_1)$ **or** $\mathrm{P}(x_2)$ **or** $\mathrm{P}(x_3)$. Using the rule

not(P **or** Q) $\equiv$ (**not**P) **and** (**not**Q)

we deduce

not$[\mathrm{P}(x_1)$ **or** $\mathrm{P}(x_2)$ **or** $\mathrm{P}(x_3)]$ $\equiv$ **not**$(\mathrm{P}(x_1))$ **and not**$[\ \mathrm{P}(x_2)$ **or** $\mathrm{P}(x_3)]$

$\equiv$ **not**$(\mathrm{P}(x_1))$ **and not**$(\mathrm{P}(x_2))$ **and not**$(\mathrm{P}(x_3))$

Methods of proof

We use logical arguments to prove theorems. In computing, these proofs are used in the verification of algorithms.

We will consider three methods of proof of a proposition of the form ($P \Rightarrow Q$):

1. Direct argument: Here we simply assume that P is true and show that Q must be true. Since ($P \Rightarrow Q$) is only false if P is true and Q is false this will complete the proof.

2. Contrapositive argument: Here we assume Q is false and prove P is also false. This uses the equivalence
$(P \Rightarrow Q) \equiv ((\textbf{not } Q) \Rightarrow (\textbf{not } P))$.

3. Proof by contradiction: To show ($P \Rightarrow Q$) we assume P is true and Q is false and derive a contradiction. That is, we show (P **and** (**not** Q)) is false.

Example: Use a direct argument to show that the sum of two even integers has to be even.

Solution: Recall that an integer is even if it is a multiple of 2, that is, an integer x is even if $x = 2y$ for some integer y.

Now suppose a and b are even integers. So $a = 2c$ and $b = 2d$ for some integers c and d.

Now form their sum $a + b$:
$a + b = 2c + 2d = 2(c + d)$

But $c + d$ is an integer so that $a + b$ is an even integer.

Here P was the statement
  a is even **and** b is even
while Q was the statement
  $a + b$ is even.