Problem Sheet 4

MS121 Semester 2 IT Mathematics

Exercise 1.

Evaluate the following limits:

(a) $\lim_{y\to 2} y^3 - \dfrac{21}{y}$,

(b) $\lim_{x\to 2} \dfrac{x^3+x^2+2x+1}{x-1}$,

(c) $\lim_{x\to 3\pi} \sin(x)$,

(d) $\lim_{x\to\pi} \cos(2x-\pi)$,

(e) $\lim_{x\to 2} \sqrt{x^2+3x-7}$,

(f) $\lim_{x\to \frac{1}{2}\pi} \sqrt{\sin(x)}$.

You may use the fact that the functions cos, sin and $\sqrt{\ }$ are continuous.

$\oslash$

Solution 1.

(a) $\lim_{y\to 2} y^3 - \dfrac{21}{y} = 2^3 - \dfrac{21}{2} = \dfrac{-5}{2}$ from the continuity of rational functions on their domain and the rules for sums and multiples,

(b) $\lim_{x\to 2} \dfrac{x^3+x^2+2x+1}{x-1} = 17$, from the continuity of rational functions on their domain

(c) $\lim_{x\to 3\pi} \sin(x) = \sin(3\pi) = 0$, using the continuity of sin,

(d) $\lim_{x\to\pi} \cos(2x-\pi) = \cos(2\pi-\pi) = -1$, using the continuity of the composition of the continuous functions cos and $2x-\pi$,

(e) $\lim_{x\to 2} \sqrt{x^2+3x-7} = \sqrt{2^2+3\cdot 2-7} = \sqrt{3}$, using the continuity of the composition of the continuous functions $\sqrt{\ }$ and x^2+3x-7,

(f) $\lim_{x\to \frac{1}{2}\pi} \sqrt{\sin(x)} = \sqrt{\sin(\frac{1}{2}\pi)} = \sqrt{1} = 1$, using the continuity of the composition of the continuous functions $\sqrt{\ }$ and sin.

$\diamondsuit$

Exercise 2.

The following functions have domain $(0,\infty)$. Find their inverses and the domains of these inverses.

(a) $f(x) = \frac{1}{x}$,

(b) $f(x) = x^2$,

(c) $g(y) = \frac{1}{1+y^2}$,

(d) $f(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ (x+1)^2 & \text{if } x > 1 \end{cases}$,

(e) $h(t) = \begin{cases} t+1 & \text{if } 0 < t \leq 1 \\ \frac{1}{t} & \text{if } t > 1 \end{cases}$.

$\oslash$

Solution 2.

(a) $f(x) = \frac{1}{x}$ is its own inverse,

(b) $f(x) = x^2$ has inverse $f^{-1}(y) = \sqrt{y}$ with domain $(0, \infty)$,

(c) $g(y) = \frac{1}{1+y^2}$ has inverse $g^{-1}(x) = \sqrt{\frac{1}{x} - 1}$ with domain $(0, 1)$,

(d) $f(x) = \begin{cases} x & \text{if} \quad 0 < x \leq 1 \\ (x+1)^2 & \text{if} \quad x > 1 \end{cases}$
has inverse

$$f^{-1}(y) = \begin{cases} y & \text{if} \quad 0 < y \leq 1 \\ \sqrt{y} - 1 & \text{if} \quad y > 4 \end{cases}$$

with domain $(0, 1] \cup (4, \infty)$.

(e) $h(t) = \begin{cases} t+1 & \text{if} \quad 0 < t \leq 1 \\ \frac{1}{t} & \text{if} \quad t > 1 \end{cases}$
is not increasing or decreasing on its entire domain, but it does have an inverse

$$h^{-1}(s) = \begin{cases} s-1 & \text{if} \quad 1 < s \leq 2 \\ \frac{1}{s} & \text{if} \quad 0 < s < 1 \end{cases}$$

with domain $(0, 1) \cup (1, 2]$.

$\diamondsuit$

Exercise 3.

For the following functions, indicate how we can restrict their natural domain to make them invertible, and find the inverse. (Sketch the graph first, if this is helpful.)

(a) $f(x) = (x-3)^2 + 5$,

(b) $g(y) = \frac{1}{(y-5)^2}$,

(c) $f(x) = |x - \pi|$,

(d) $h(t) = \sin(t)$.

$\oslash$

Solution 3.

(a) $f(x) = (x-3)^2 + 5$ can be restricted e.g. to $x \geq 3$ with inverse
$f^{-1}(y) = \sqrt{y-5} + 3$ on $y \geq 5$,

(b) $g(y) = \frac{1}{(y-5)^2}$ can be restricted e.g. to $y > 5$ with inverse
$g^{-1}(x) = \frac{1}{\sqrt{x}} + 5$ on $x > 0$,

(c) $f(x) = |x - \pi|$ can be restricted e.g. to $x \geq \pi$ with inverse $f^{-1}(y) = y + \pi$ on $y \geq 0$.

(d) $h(t) = \sin(t)$ can be restricted e.g. to $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$. Its inverse is then called $\arcsin(y)$ and has domain $[-1, 1]$.

$\diamondsuit$

Exercise 4.
Use left and right hand limits to determine which of the following functions are continuous.

(a) $f(x) = \begin{cases} 3-5x & \text{if} \quad x \leq 0 \\ \frac{6}{4x+2} & \text{if} \quad x > 0 \end{cases},$

(b) $g(x) = \begin{cases} -2x & \text{if} \quad x < -2 \\ x^2 & \text{if} \quad -2 \leq x \leq 1 \\ 2x & \text{if} \quad x > 1 \end{cases}.$

$\oslash$

Solution 4.
(a)

$$f(x) = \begin{cases} 3-5x & \text{if} \quad x \leq 0 \\ \frac{6}{4x+2} & \text{if} \quad x > 0 \end{cases}$$

is continuous, because the one-sided limits

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} 3-5x = 3-0 = 3 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{6}{4x+2} = \frac{6}{0+2} = 3 \end{aligned}$$

are equal to the function value $f(0) = 3 - 0 = 3$.

(b)

$$g(x) = \begin{cases} -2x & \text{if} \quad x < -2 \\ x^2 & \text{if} \quad -2 \leq x \leq 1 \\ 2x & \text{if} \quad x > 1 \end{cases}$$

is not continuous, because the one-sided limits

$$\begin{aligned} \lim_{x\to 1^-} g(x) &= \lim_{x\to 1^-} x^2 = 1^2 = 1 \\ \lim_{x\to 1^+} g(x) &= \lim_{x\to 1^+} 2x = 2 \end{aligned}$$

differ. (g is continuous at $x = -2$.)

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