MS121: IT Mathematics

Limits & Continuity

Introduction to Limits

John Carroll
School of Mathematical Sciences

Dublin City University

Outline

1. Introduction to Limits
2. Definition of a Limit
3. Limit Laws
4. Limits of Rational Functions
5. Special Solution Method: Rationalize the Numerator
6. Limits of Piecewise-Defined Functions

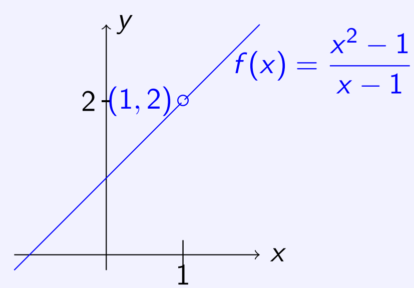
Outline

1. Introduction to Limits
2. Definition of a Limit
3. Limit Laws
4. Limits of Rational Functions
5. Special Solution Method: Rationalize the Numerator
6. Limits of Piecewise-Defined Functions

Limits of Function Values

Example: Rational Function

- Let $f(x) = \dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{x-1}$
- Then f is not defined when $x = 1$.
- However for $x \neq 1$, $f(x) = x + 1$ and as x gets closer to 1, $f(x)$ gets closer to 2.
- So $\lim_{x \to 1} f(x) = 2$.



Limits of Function Values

Table of Values of $f(x) = \frac{x^2-1}{x-1}$

x	$f(x)$
0.9	1.9
0.99	1.99
0.999	1.999
1.1	2.1
1.01	2.01
1.001	2.001

So we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1 or as

$$x \to 1 \qquad f(x) \to 2.$$

Outline

1 Introduction to Limits
2 Definition of a Limit
3 Limit Laws
4 Limits of Rational Functions
5 Special Solution Method: Rationalize the Numerator
6 Limits of Piecewise-Defined Functions

Limits of Function Values

Definition

- Let f be defined on an open interval about x_0, *except possibly at x_0 itself.*
- We say that f approaches the limit L (L a real number) as x approaches x_0 if, however small a distance we choose, $f(x)$ gets closer than this distance to L for x sufficiently close to (but not equal to) x_0. We write

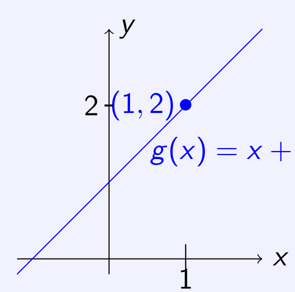
$$\lim_{x \to x_0} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to x_0.$$

Example

- Let $g(x) = x + 1$.
- As x gets closer to 1, $g(x)$ gets closer to $2 = g(1)$.
- So $\lim_{x \to 1} g(x) = 2 = g(1)$.

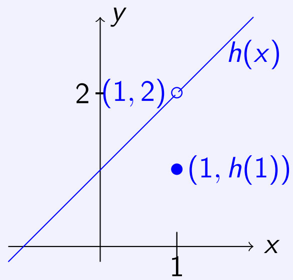


Example

- Let

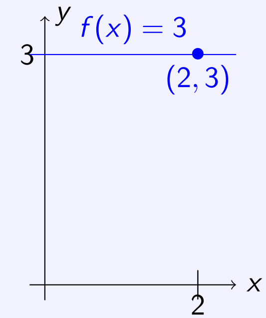
$$h(x) = \begin{cases} x+1, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$$

- For $x \neq 1$, $h(x) = x + 1$ and as x gets closer to 1, $h(x)$ gets closer to 2.
- So $\lim\limits_{x \to 1} h(x) = 2 \neq h(1)$.



Trivial Example

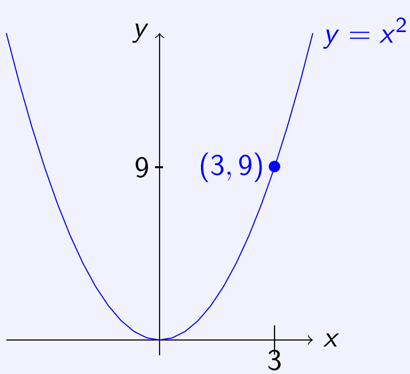
Let $f(x) = 3$, for all x. $\lim\limits_{x \to 2} f(x) = 3$



Fact: If $f(x) = k$, for some constant k, then $\lim\limits_{x \to x_0} f(x) = k$, for any x_0.

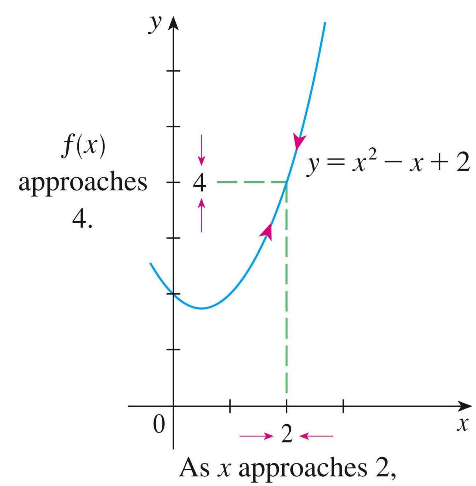
Example $f(x) = x^2$

$\lim\limits_{x \to 3} x^2 = 9$

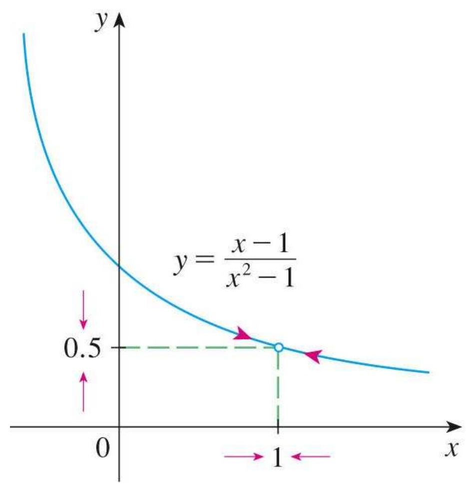


Fact: If f is any polynomial, then $\lim\limits_{x \to x_0} f(x) = f(x_0)$ for any x_0.

$\lim\limits_{x \to 2} (x^2 - x - 2) = 4$

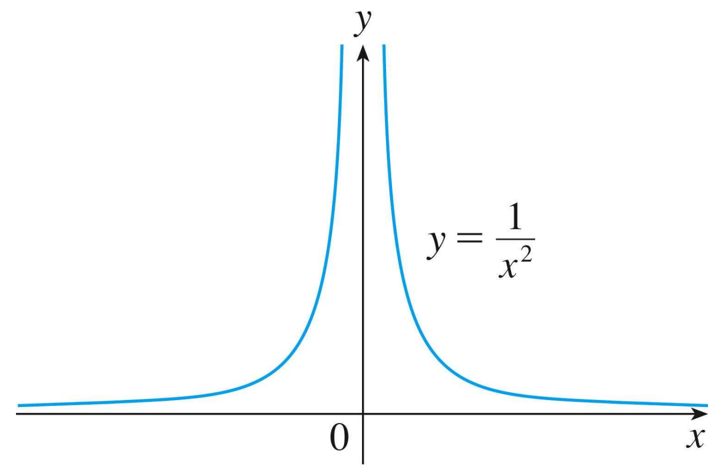


$$\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\frac{1}{2}$$

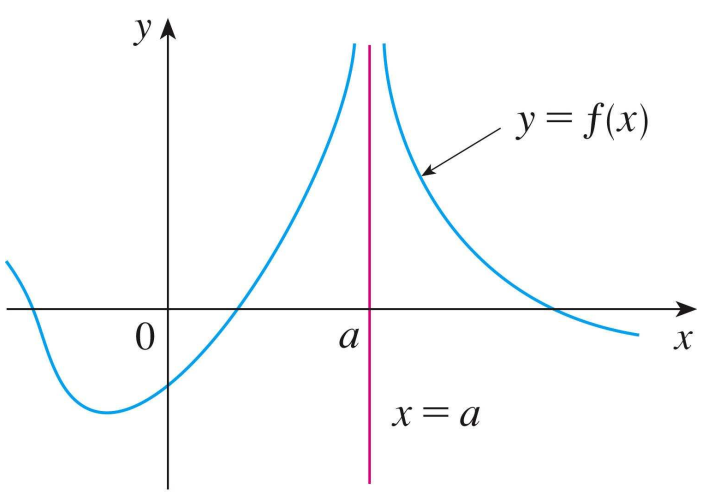


$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

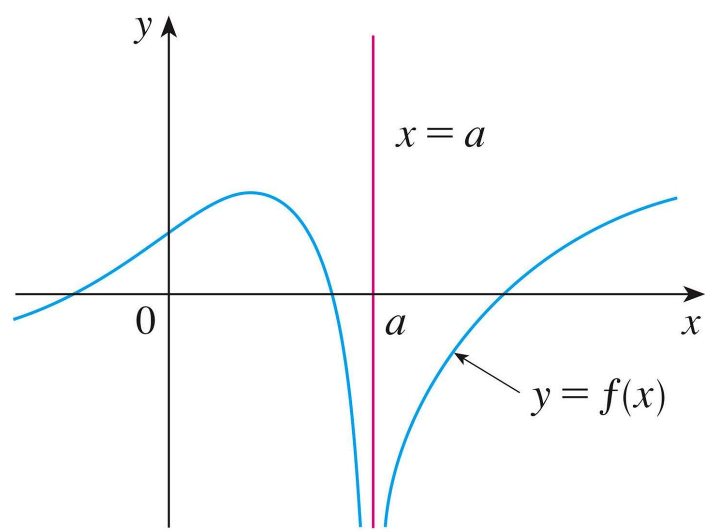
Limits do not always Exist!



$$\lim_{x\to a} f(x)=\infty$$



$$\lim_{x\to a} f(x)=-\infty$$



Outline

1. Introduction to Limits
2. Definition of a Limit
3. Limit Laws
4. Limits of Rational Functions
5. Special Solution Method: Rationalize the Numerator
6. Limits of Piecewise-Defined Functions

Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L, \qquad \lim_{x \to c} g(x) = M,$$

then

(i) Sum Rule: $\lim_{x \to c} f(x) + g(x) = L + M$.

(ii) Difference Rule: $\lim_{x \to c} f(x) - g(x) = L - M$.

(iii) Product Rule: $\lim_{x \to c} f(x)g(x) = LM$.

(iv) Constant Multiple Rule: $\lim_{x \to c} kf(x) = kL$.

(v) Quotient Rule: If $M \neq 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Applying the Limit Laws

- If f is a polynomial, then $\lim_{x \to c} f(x) = f(c) = L$.
- $\lim_{x \to c} x^3 - 4x^2 - 3 = c^3 - 4c^2 - 3$.
- If f and g are polynomials and $g(c) \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$$

- Rational function:

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

- Other functions, e.g.

$$\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2} 4x^2 - 3} = \sqrt{13}$$

Sandwich Theorem

Definition

Suppose that

$$g(x) \leq f(x) \leq h(x)$$

for all x in some open interval containing c, except possibly at c itself.

Furthermore, suppose that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then

$$\lim_{x \to c} f(x) = L$$

One-sided Limits

- Ordinary limits are called *two-sided*. If f fails to have a two sided limit at c it may still have a *one-sided* limit — that is, as x approaches c from one side.
- If f is defined on an interval (c, b), where $c < b$, and $f(x)$ gets arbitrarily close to L as x approaches c from the right, then f has a right-hand limit L at c and we write

$$\lim_{x \to c^+} f(x) = L$$

- If f is defined on an interval (b, c), where $b < c$, and $f(x)$ gets arbitrarily close to L as x approaches c from the left, then f has a left-hand limit L at c and we write

$$\lim_{x \to c^-} f(x) = L$$

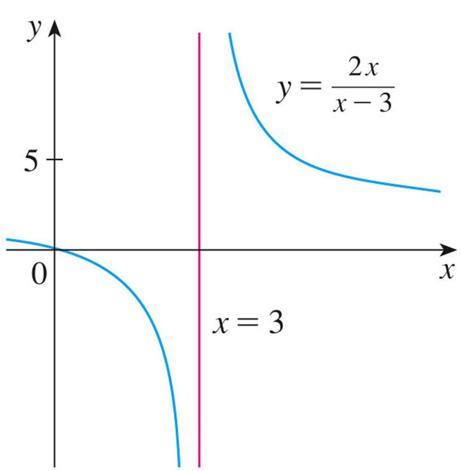
One-sided Limits (Cont'd)

Important Property

$\lim_{x \to c} f(x)$ exists if and only if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty \qquad \lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$



Outline

1 Introduction to Limits
2 Definition of a Limit
3 Limit Laws
4 Limits of Rational Functions
5 Special Solution Method: Rationalize the Numerator
6 Limits of Piecewise-Defined Functions

Eliminating Zero Denominators Algebraically

$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$

- We cannot substitute $x = 1$ as it makes the denominator zero.
- We test the numerator to see if it is also zero at $x = 1$ (if so, it has a factor of $(x - 1)$ in common with the denominator):

$$1^2 + 1 - 2 = 0$$

- We can cancel the $(x - 1)$ terms to get:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x \to 1} \frac{x+2}{x} = 3$$

Eliminating Zero Denominators Algebraically

$\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 - 4}$

- We cannot substitute $x = -2$ as it makes the denominator zero.
- We test the numerator to see if it is zero at $x = -2$:
- $(-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0.$
- We can cancel the $(x + 2)$ terms to get:

$$\begin{aligned} \lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 - 4} &= \lim_{x \to -2} \frac{(x+3)(x+2)}{(x-2)(x+2)} \\ &= \lim_{x \to -2} \frac{x+3}{x-2} = -\frac{1}{4} \end{aligned}$$

Eliminating Zero Denominators Algebraically

$\lim_{x \to 2} \frac{x^3 - 8}{x^4 - 16}$

- We cannot substitute $x = 2$ as it makes the denominator zero.
- We test the numerator to see if it is zero at $x = 2$:
- $2^3 - 8 = 8 - 8 = 0.$
- We can cancel the $(x - 2)$ terms to get:

$$\begin{aligned} \lim_{x \to 2} \frac{x^3 - 8}{x^4 - 16} &= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)}{(x-2)(x^3 + 2x^2 + 4x + 8)} \\ &= \lim_{x \to 2} \frac{x^2 + 2x + 4}{x^3 + 2x^2 + 4x + 8} = \frac{12}{32} = \frac{3}{8} \end{aligned}$$

An Aside: Long Division Illustrations

Illustration 1

$$\begin{array}{r|rrrrrrr}
 & x^2 & + & x & - & 12 & & \\ \hline
x - 3 & x^3 & - & 2x^2 & - & 15x & + & 36 \\
 & x^3 & - & 3x^2 & & & & \\ \cline{2-4}
 & & & x^2 & - & 15x & & \\
 & & & x^2 & - & 3x & & \\ \cline{4-6}
 & & & & - & 12x & + & 36 \\
 & & & & - & 12x & + & 36 \\ \cline{5-8}
 & & & & & & & 0
\end{array}$$

Long Division Cont'd

Illustration 2

$$\begin{array}{rrrrrrrr}
 & x^2 & - & 2x & - & 8 & & \\ \cline{2-8}
x+2 \big) & x^3 & & & - & 12x & - & 16 \\
 & x^3 & + & 2x^2 & & & & \\ \cline{2-4}
 & & - & 2x^2 & - & 12x & & \\
 & & - & 2x^2 & - & 4x & & \\ \cline{4-6}
 & & & & - & 8x & - & 16 \\
 & & & & - & 8x & - & 16 \\ \cline{6-8}
 & & & & & & & 0
\end{array}$$

Eliminating Zero Denominators Algebraically

Example: $\lim\limits_{x \to 2} \dfrac{x^3 - 2x^2 + x - 2}{2x^2 - x - 6}$

We factorise the numerator and denominator and simplify as follows:

$$\frac{x^3 - 2x^2 + x - 2}{2x^2 - x - 6} = \frac{(x-2)(x^2+1)}{(x-2)(2x+3)} = \frac{x^2+1}{2x+3}$$

The limit is then found from

$$\lim_{x \to 2} \frac{x^3 - 2x^2 + x - 2}{2x^2 - x - 6} = \lim_{x \to 2} \frac{x^2+1}{2x+3} = \frac{2^2+1}{2 \cdot 2 + 3} = \frac{5}{7}.$$

Eliminating Zero Denominators Algebraically

Example: $\lim\limits_{x \to -1} \dfrac{x^2 + 2x + 1}{x^3 + 3x^2 + 3x + 1}$

We factorise the numerator and denominator and simplify as follows:

$$\frac{x^2 + 2x + 1}{x^3 + 3x^2 + 3x + 1} = \frac{(x+1)^2}{(x+1)^3} = \frac{1}{x+1}.$$

The limit is then found from

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x^3 + 3x^2 + 3x + 1} = \lim_{x \to -1} \frac{1}{x+1} = \frac{1}{0} = \infty.$$

Outline

1. Introduction to Limits
2. Definition of a Limit
3. Limit Laws
4. Limits of Rational Functions
5. Special Solution Method: Rationalize the Numerator
6. Limits of Piecewise-Defined Functions

Special Solution Method: Rationalize the Numerator

Question

Evaluate $\lim\limits_{x\to 1} \dfrac{\sqrt{x^2+x+23}-5}{x-1}$

Formula Required

$$A-B=(A-B)\star\left\{\frac{A+B}{A+B}\right\}$$

Special Solution Method: Rationalize the Numerator

Solution

Consider

$$\begin{aligned}\frac{\sqrt{x^2+x+23}-5}{x-1} &= \frac{\sqrt{x^2+x+23}-5}{x-1}\cdot\left\{\frac{\sqrt{x^2+x+23}+5}{\sqrt{x^2+x+23}+5}\right\}\\ &= \frac{x^2+x-2}{(x-1)\sqrt{x^2+x+23}+5}\\ &= \frac{x+2}{\sqrt{x^2+x+23}+5}\end{aligned}$$

and so

$$\lim_{x\to 1}\frac{\sqrt{x^2+x+23}-5}{x-1}=\frac{1+2}{\sqrt{1^2+1+23}+5}=\frac{3}{10}$$

Outline

1. Introduction to Limits
2. Definition of a Limit
3. Limit Laws
4. Limits of Rational Functions
5. Special Solution Method: Rationalize the Numerator
6. **Limits of Piecewise-Defined Functions**

Piecewise-Defined Functions

Let

$$f(x)=\begin{cases}\dfrac{1}{2-3x}, & \text{if } x<-3\\[2ex] x+2, & \text{if } x\geq -3\end{cases}$$

Find the left and righthand limits at -3. Does $\lim\limits_{x\to -3} f(x)$ exist?

$$\begin{aligned}\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-}\frac{1}{2-3x} = \frac{1}{2-3(-3)} &= \frac{1}{11}\\ \lim_{x\to -3^+} f(x) = \lim_{x\to 3^+} x+2 = -3+2 &= -1\end{aligned}$$

As $\lim\limits_{x\to -3^-} f(x)\neq \lim\limits_{x\to -3^+} f(x)$, $\lim_{x\to -3} f(x)$ does not exist.

Piecewise-defined Functions

Let

$$f(x) = \begin{cases} 5 - 3x & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ \sqrt{2x+2} & \text{if } x > 1 \end{cases}$$

Find the left and righthand limits at 1. Does $\lim_{x \to 1} f(x)$ exist?

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 5 - 3x = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{2x+2} = \sqrt{4} = 2$$

As $\lim_{x \to 1^-} f(x) = 2 = \lim_{x \to 1^+} f(x)$, $\lim_{x \to 1} f(x) = 2$.