If $f : A \to B$ is invertible, what is the composition $f^{-1} \circ f : A \to A$? It takes each point of A to itself.

Definition: If A is a non-empty set we define the identity function

$$\mathrm{id}_A : A \to A : a \mapsto a$$

that is, the function which takes each element to itself.

Proposition: (a) If $f : A \to B$ then

$$\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A.$$

(b) If, furthermore, f is invertible, then

$$f^{-1} \circ f = \mathrm{id}_A \text{ and } f \circ f^{-1} = \mathrm{id}_B.$$

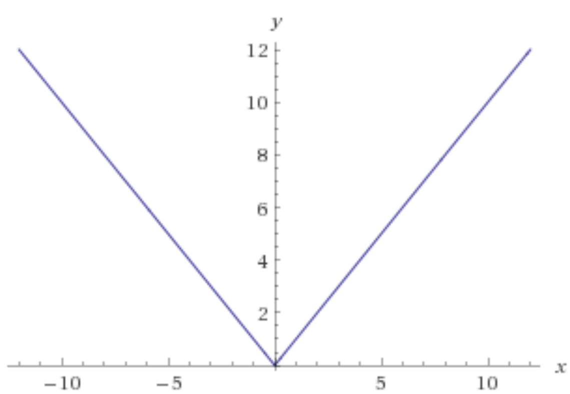
Definition: The absolute value function $|x|$ has domain $\mathbb{R}$ and codomain $\mathbb{R}$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Examples : $|45| = 45$, $|-3| = 3$.

Note : $|x| = \sqrt{x^2}$.

Note : $|x|$ measures the distance on the real line from x to 0. $|x - y|$ measures the distance on the real line from x to y.



$f(x) = |x|$

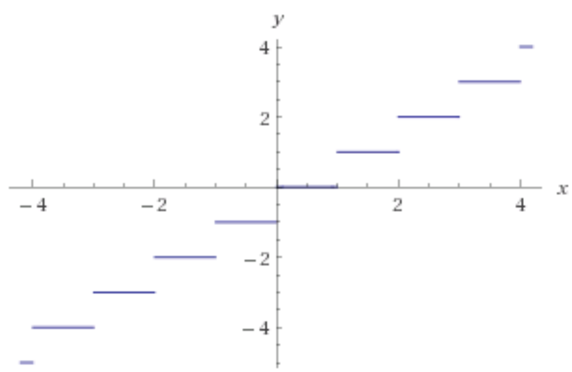
Example : Solve $|x+2| = 3$. Since $|y| = 3$ implies $y = 3$ or $y = -3$, we get $x + 2 = 3$ or $x + 2 = -3$. This is equivalent to $x = 1$ or $x = -5$. This is what we expect: 1 and -5 are the two numbers which are distance 3 from -2. ($|x+2| = |x-(-2)|$.)

Definition: The floor function $\lfloor x \rfloor$ has domain $\mathbb{R}$ and codomain $\mathbb{Z}$ and is defined by

$$\lfloor x \rfloor = \text{largest integer} \leq x$$

Examples : $\lfloor 5.7 \rfloor = 5$, $\lfloor -3.4 \rfloor = -4$.

Note : $\lfloor x \rfloor$ rounds x down to the nearest integer. What does $(1/10)\lfloor 10x \rfloor$ do? (It rounds down to first decimal place.)



$f(x) = \lfloor x \rfloor$

Pigeonhole principle

It is intuitively clear that, when we try to put more than n objects (pigeons) into n containers (pigeonholes) then there will be at least one container with more than one object.

Proposition: Suppose A and B are finite sets and $f : A \to B$ is injective. Then $|B| \geq |A|$.

Proof: Let the elements of A be $a_1, a_2, \ldots, a_n$. Since f is injective the elements $f(a_1), f(a_2), \ldots, f(a_n)$ are distinct in B. So B contains at least $|A|$ elements.

Corollary: (Pigeonhole principle) Suppose A and B are finite sets with $|B| < |A|$ and $f : A \to B$ is a function. Then f cannot be injective.

Example: An extended family of 14 people are gathered for a function. Show that at least two have birthdays in the same month of the year.

Here the set A is the set of family members, the set B is the set of months and the function $f : A \to B$ takes each family member to his/her birth month. Since $|A| = 14 > 12 = |B|$, f cannot be injective and two members have the same birth month.

Example: How big does a crowd have to be to ensure that two people from it share a birthday?

Here the set A is the set of people in the crowd, the set B is the set of days in the year and the function $f : A \to B$ takes each person to his/her birth day. Since we want $|A| > 366 = |B|$ (possible leap year), f will not be injective and two people have the same birthday provided $|A| > 366$.

Example: How many different surnames must be in a telephone directory in order to ensure that at least two surnames have the same first and the same second letter?

Here the set A is the set of names in the directory, L is the set of letters of the alphabet, the set B is $L \times L$, the set of pairs of letters of the alphabet and the function $f : A \to B$ takes each surname to the pair (first letter, second letter). Since we want $|A| > 26^2 = |B|$, f will not be injective and two surnames have the same first and second letter provided $|A| > 676$.