MS121: IT Mathematics

Functions

New Functions from Old Functions

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Outline

Outline

Even & Odd Functions

Properties

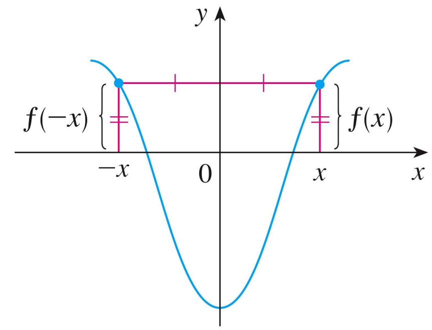
- A function f is an *even function* of x if
$$f(-x) = f(x),$$
for every x in the function's domain.
- A function f is an *odd function* of x if
$$f(-x) = -f(x),$$
for every x in the function's domain.

Even & Odd Functions

Properties (Cont'd)

- The names even and odd come from powers of x.
- If y is an even power of x, e.g. $y = x^6$, then it is an even function as $(-x)^6 = x^6$.
- If y is an odd power of x, e.g. $y = x^7$, then it is an odd function as $(-x)^7 = -x^7$.
- Some functions are even, some are odd, and some are neither.

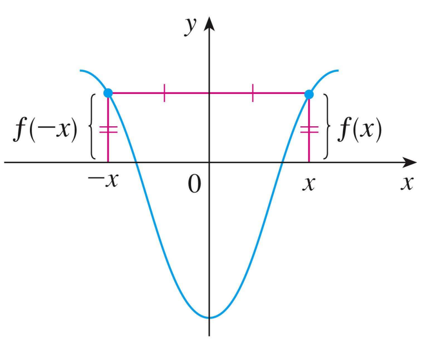
Even Functions



- If a function satisfies $f(-x) = f(x)$ for every number x in its domain, then it is called an even function.
- For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

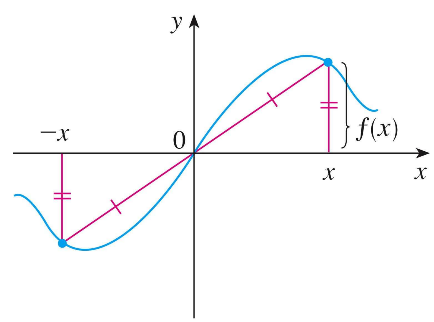
Even Functions



Significance

- The geometric significance of an even function is that its graph is symmetric with respect to the y-axis.
- This means that if we have plotted the graph of $f(x)$ for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

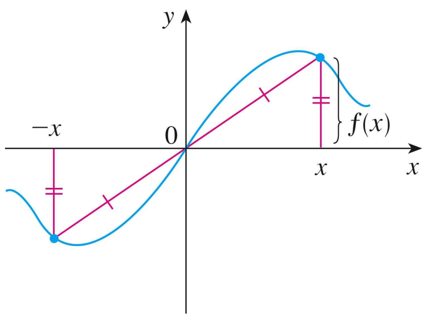
Odd Functions



- If $f(x)$ satisfies $f(-x) = -f(x)$ for every number x in its domain, then $f(x)$ is called an odd function.
- For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Odd Functions



Significance

- The graph of an odd function is symmetric about the origin.
- If we already have the graph of $f(x)$ for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

Outline

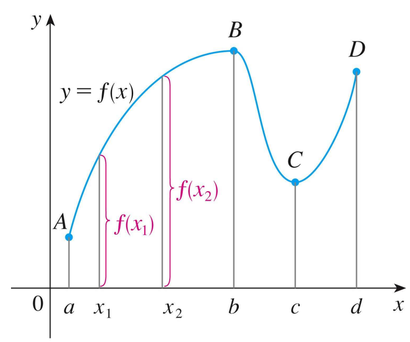
1. Even & Odd Functions
2. Increasing & Decreasing Functions
3. Transformation of Functions
4. Composite Functions
5. Inverse Functions

Increasing/Decreasing Functions

General Situation

- If the graph of a function *climbs* or *rises*, as you move from left to right, then the function is increasing.
- If the graph of a function *descends* or *falls*, as you move from left to right, then the function is decreasing.
- We will look at the more formal definition during this Semester and see how to find the intervals over which a function is increasing and the intervals over which it is decreasing.

Increasing & Decreasing Functions



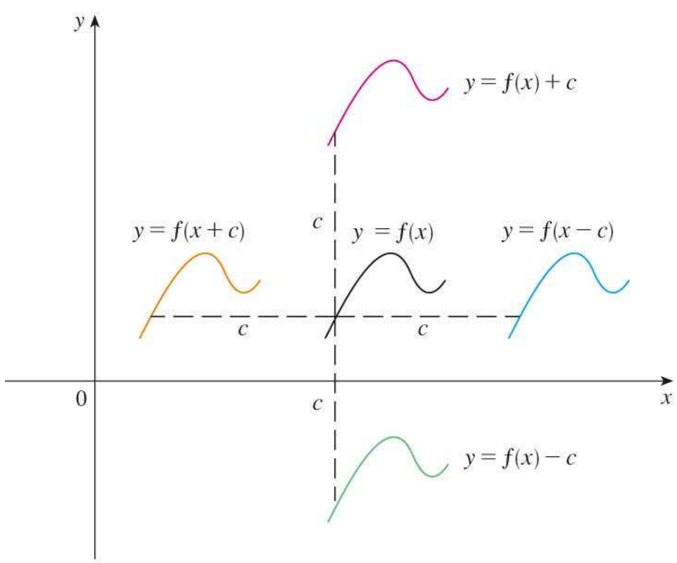
- The graph rises from A to B, falls from B to C, and rises again from to C to D.
- The function is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$.
- Notice that if x_1 and x_2 and are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- We use this as the defining property of an increasing function.

Sample Increasing & Decreasing Functions

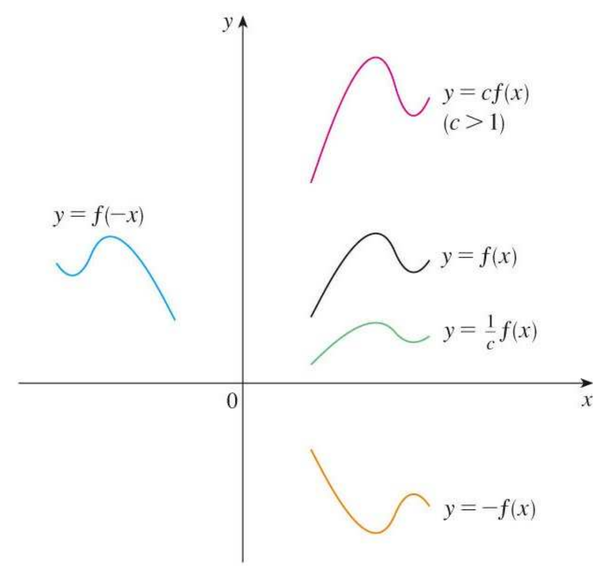
$f(x)$	Where Increasing	Where Decreasing
x^2	$[0, \infty)$	$(-\infty, 0]$
x^3	$(-\infty, \infty)$	Nowhere
$\frac{1}{x}$	Nowhere	$(-\infty, 0) \cup (0, \infty)$
$\frac{1}{x^2}$	$(-\infty, 0)$	$(0, \infty)$
$\sqrt{x}$	$[0, \infty)$	Nowhere
$x^{2/3}$	$[0, \infty)$	$(-\infty, 0]$

Outline

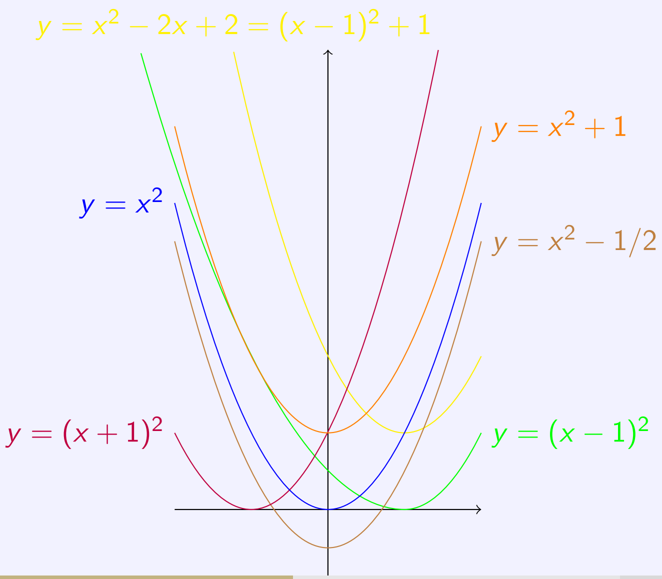
Vertical & Horizontal Shifts



Vertical & Horizontal Shifts



Shifting Graphs



Shifting Graphs

Vertical shifts: $y = f(x) + k$ shifts the graph of f

up k units if $k > 0$;

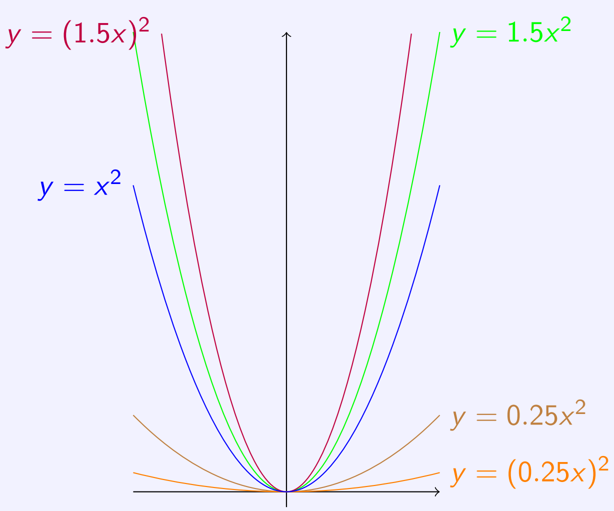
down $|k|$ units if $k < 0$.

Horizontal shifts: $y = f(x+h)$ shifts the graph of f

left h units if $h > 0$;

right $|h|$ units if $h < 0$.

Scaling a Graph



Scaling a Graph

Vertically: $y = kf(x)$, for $k > 0$,

stretches the graph of f vertically by a factor of k if $k > 1$;

compresses the graph of f vertically by a factor of k if $k < 1$;

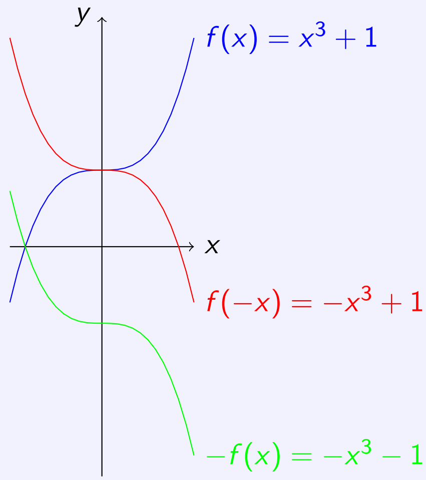
Horizontally: $y = f(kx)$, for $k > 0$,

stretches the graph of f horizontally by a factor of k if $k < 1$;

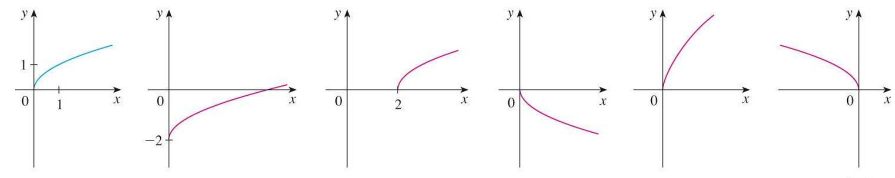
compresses the graph of f horizontally by a factor of k if $k > 1$;

Reflections

$y = -f(x)$ is a reflection of $y = f(x)$ across the x-axis;
$y = f(-x)$ is a reflection of $y = f(x)$ across the y-axis;



Example: Transformation of Function $\sqrt{x}$



(a) $y=\sqrt{x}$ (b) $y=\sqrt{x}-2$ (c) $y=\sqrt{x-2}$ (d) $y=-\sqrt{x}$ (e) $y=2\sqrt{x}$ (f) $y=\sqrt{-x}$

We sketch:

- $y = \sqrt{x} - 2$ by shifting 2 units downward
- $y = \sqrt{x-2}$ by shifting 2 units to the right
- $y = -\sqrt{x}$ by reflecting about the x-axis
- $y = 2\sqrt{x}$ by stretching vertically by a factor of 2
- $y = \sqrt{-x}$ by reflecting about the y-axis

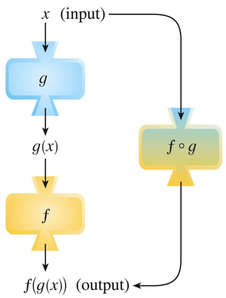
Combining Functions

In general, given any two functions f and g with domains $D(f)$, $D(g)$ respectively and constant c:

- $(cf)(x) = c(f(x))$ with domain $D(f)$.
- $(f+g)(x) = f(x) + g(x)$, domain $D(f) \cap D(g)$.
- $(f-g)(x) = f(x) - g(x)$, domain $D(f) \cap D(g)$.
- $(f \star g)(x) = f(x)g(x)$, domain $D(f) \cap D(g)$.
- $(f/g)(x) = \dfrac{f(x)}{g(x)}$ with domain $D(f) \cap D(g)$ less any points where g is zero.

Outline

Composite Functions



If f and g are functions, the *composite* function $f \circ g$ (f composed with g or f after g)is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of all x in the domain of g for which $g(x)$ lies in the domain of f.

$$\xrightarrow{x} \boxed{g} \xrightarrow{g(x)} \boxed{f} \longrightarrow f(g(x))$$

Viewing a Function as a Composite

To evaluate the function

$$y = \sqrt{1 - x^2}$$

we first find $1 - x^2$ and then take the square root:

$$\boxed{x} \xrightarrow{1 - x^2} \boxed{1 - x^2} \xrightarrow{\sqrt{x}} \boxed{y = \sqrt{1 - x^2}}$$

So that y is the composite $f \circ g$ where

$$f(x) = \sqrt{x} \text{ and } g(x) = 1 - x^2$$

Composition of Functions Example

The Functions

$f(x) = 2x + 3$ and $g(x) = x^2$

Composition

We obtain

$$g \circ f(x) = g\,(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9$$

while

$$f \circ g(x) = f\left(x^2\right) = 2x^2 + 3.$$

Note: In this example, $f \circ g(x) \neq g \circ f(x)$. This is also true in general as the composition of functions is not commutative.

Composition of Functions Example

The Functions

$f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$

Composition

We have

$$g \circ f(x) = g\left(x^2 + 1\right) = \sqrt{x^2 + 1}$$

while

$$f \circ g(x) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 + 1 = |x| + 1$$

Note: For $f \circ g(x)$, the domain of f can only include points from the range of g.

Composition of Functions Example

The Functions

$f(x) = x^2$ and $g(x) = \sqrt{x}$

Composition

We obtain

$$g \circ f(x) = g\left(x^2\right) = \sqrt{x^2} = |x|$$

and

$$f \circ g(x) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = |x|$$

Although $f \circ g(x)$ and $g \circ f(x)$ appear to be identical functions, their natural domains and ranges could be different. $g \circ f$ has natural domain $(-\infty, \infty)$ and range $[0, \infty)$ whereas $f \circ g$ has natural domain $[0, \infty)$ and range $[0, \infty)$.

Outline

Inverting Functions

Context

The functions $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$ "cancel each other out" in the sense that

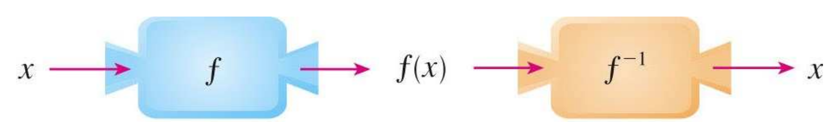
$$f \circ g(x) = f\left(x^{\frac{1}{3}}\right) = \left(x^{\frac{1}{3}}\right)^3 = x$$

and

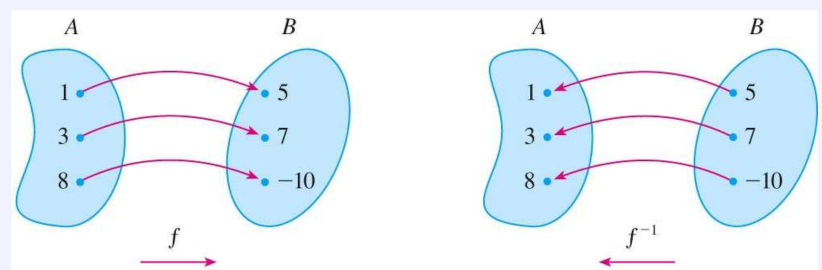
$$g \circ f(x) = g\left(x^3\right) = \left(x^3\right)^{\frac{1}{3}} = x$$

i.e. $f \circ g$ and $g \circ f$ leave x unchanged — each behave as the identity function. We say that the functions f and g are the inverses of each other.

Inverting Functions



f^{-1} reverses the effect of f



Inverting Functions

Definition: Inverse of a Function

f is said to be an invertible function, with inverse f^{-1}, if the function f^{-1} exists and obeys the following property:

$$f^{-1} \circ f(x) = x$$

for all x in the domain of f.

- Note that f^{-1} does not mean $\frac{1}{f}$ but is simply notation to mean "the inverse function".
- Note also that the inverse f^{-1} must itself be a function, i.e. it must produce unique values.

Finding the Inverse of $f(x)$

The Function

Where it exists, find $f^{-1}(x)$ where $f(x) = \dfrac{x+1}{x-2}$.

Answer

We begin by writing $y = \dfrac{x+1}{x-2}$, from which we see that y is given explicitly in terms of x. To invert the function, we need to write x explicitly in terms of y. Cross-multiply and transpose as follows:

$$\begin{aligned} y(x-2) &= x+1 \\ \Rightarrow xy - 2y &= x+1 \\ \Rightarrow xy - x &= 2y+1 \\ \Rightarrow x(y-1) &= 2y+1 \quad \Rightarrow \quad x = \frac{2y+1}{y-1} \end{aligned}$$

Finding the Inverse of $f(x)$

Function & Inverse

$$y = \frac{x+1}{x-2} \qquad\qquad x = \frac{2y+1}{y-1}$$

Answer (Cont'd)

Since $y = f(x)$ then $x = f^{-1}(y)$ and so $f^{-1}(y) = \dfrac{2y+1}{y-1}$. For convenience, we can replace y by x (because it is conventional to express a function in terms of x not y) and write

$$f^{-1}(x) = \frac{2x+1}{x-1}$$

Note that the inverse function has domain $R \setminus \{1\}$.

Finding the Inverse of $f(x)$

The Function

$f(x) = (x-4)^2$ on the domain $[4, \infty)$.

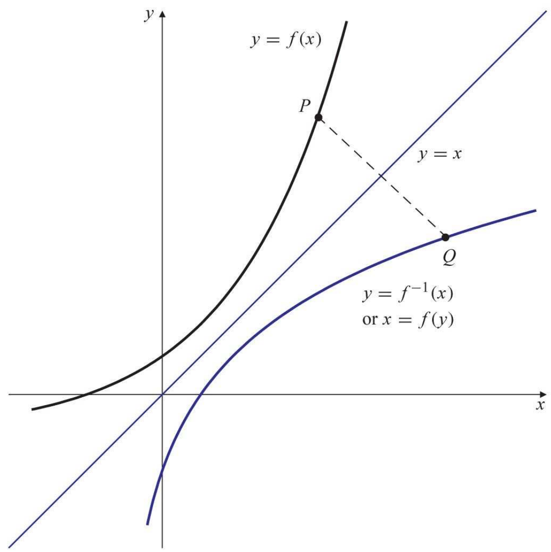
Answer

We write

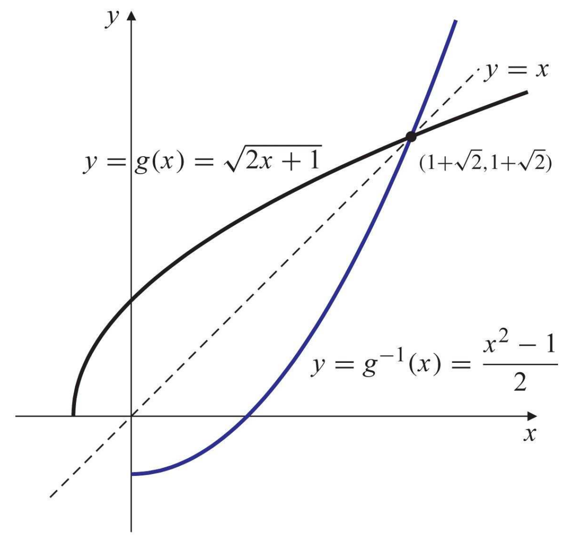
$$\begin{aligned} y &= (x-4)^2 \\ \Rightarrow +\sqrt{y} &= x-4 \qquad (+\sqrt{\ }\text{ since } x \geq 4) \\ \Rightarrow x &= 4+\sqrt{y} \\ \Rightarrow f^{-1}(x) &= 4+\sqrt{x} \end{aligned}$$

Note that f^{-1} has natural domain $[0, \infty)$ and range $[4, \infty)$ (which is the domain of f).

The Inverse Function: Conclusion (1/3)



The Inverse Function: Conclusion (2/3)



The Inverse Function: Conclusion (3/3)