MS121: IT Mathematics

Functions

Domain & Range

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Outline

Outline

Sets & Numbers

Set

- A set is a collection of objects and the objects in the set are called elements.
- Sets are usually denoted by upper-case letters; if x belongs to some set S, this is written as $x \in S$, and if x does not belong to S, this is written as $x \notin S$, e.g.

$$S = \{1, 2, 3, 4, 5\}, \qquad 5 \in S, \qquad 7 \notin S.$$

- Set A is a subset of set B if every element in set A is also in set B, written $A \subset B$. For example,

$$\{1, 3, 17\} \subset \{-4, 0, 1, 2, 3, 13, 17\}$$

Special Number Sets

N	Set of natural (positive whole) numbers: $1, 2, 3, \ldots$
Z	Set of integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
Q	Set of rational numbers (fractions): $\frac{a}{b}$, $a, b \in Z$, $b \neq 0$.
R	Set of all real numbers, consisting of all rational numbers and irrational numbers.
C	Set of all complex numbers.
$\emptyset$	Empty (or null) set (the set which has no elements).

It is easy to see that $N \subset Z$ and, more generally, that

$$N \subset Z \subset Q \subset R$$

Set Operations

Intersection

The intersection of sets A and B is the set of elements which belong to both A and B and is denoted by $A \cap B$. For example:

$$\{1, 2, 10\} \cap \{-3, -1, 0, 2, 6\} = \{2\}$$

whereas

$$\{1, 2, \} \cap \{3, 4\} = \emptyset,$$

the empty (or null) set.

Set Operations (Cont'd)

Union

The union of sets A and B is the set of elements which belong to either A or B and is denoted by $A \cup B$. For example:

$$\{0, 2, 4, 5\} \cup \{1, 3, 5\} = \{0, 1, 2, 3, 4, 5\}$$

and

$$\{1, 2\} \cup \{1, 2, 3\} = \{1, 2, 3\}$$

Set Operations (Cont'd)

Disjoint Sets

The sets A and B are disjoint is they have no elements in common, i.e. if

$$A \cap B = \emptyset$$

The set $A \setminus B$ is "set A less set B" and denotes the set of elements in A that are not in set B. If

$$A = \{1, 3, 10, 19\} \qquad B = \{-1, 1, 14\}$$

then

$$A \setminus B = \{3, 10, 19\}$$

Example

Question

What is the set defined by:

$$\{x \in Z \mid 3 < x \leq 6\}$$

Answer

The expression $x \in Z$ means that x must be an integer whereas $3 < x \leq 6$ means that x must be greater than 3 and less than or equal to 6. Putting these together, we obtain:

$$\{x \in Z \mid 3 < x \leq 6\} \; = \; \{4.5, 6\}$$

Example

Question

What is the set defined by:

$$\{x \in R \mid \sqrt{x} \in N \text{ and } x < 30\}$$

Answer

- We are looking for real numbers (since $x \in R$) less than 30 (since $x < 30$) whose squares must be positive whole numbers (since $\sqrt{x} \in N$).
- The set of numbers less than 30 having whole number square roots are $\sqrt{1} = 1$, $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{25} = 5$.
- Therefore

$$\{x \in R \mid \sqrt{x} \in N \text{ and } x < 30\} \; = \; \{1, 4, 9, 16, 25\}$$

Intervals

What is an "Interval"

Intervals are subsets of the real line without gaps, i.e.

$$\{x \in R \mid -2 < x < 7\}$$

is an interval while

$$\{1, 4, 9\}$$

is not.

Intervals

Open Interval

An interval is <u>open</u> if it does NOT include the end-points. The interval

$$\{x \in R \mid -2 < x < 7\}$$

is an open interval and is written as $(-2, 7)$ (round brackets).

Intervals (Cont'd)

Closed Interval

An interval is <u>closed</u> if it does include the end-points. The interval

$$\{x \in R \mid -1 \leq x \leq 0\}$$

is a closed interval and is written as $[-1, 0]$ (square brackets).

Intervals (Cont'd)

Half-open (or Half-closed) Interval

An interval is called <u>half-open</u> (or half-closed) if it includes only one end-point. The interval

$$\{x \in R \mid -10 \leq x < -2\}$$

is a half-open interval and is written as $[-10, -2)$.

Intervals (Cont'd)

Infinite Intervals

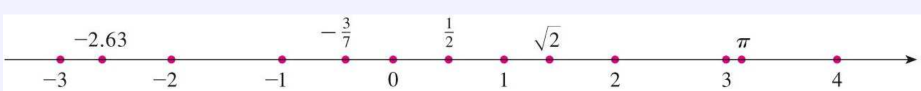
Intervals can extend out to plus and minus infinity (i.e. to $+\infty$ and $-\infty$). The interval

$$\{x \in R \mid x > 2\}$$

really means

$$\{x \in R \mid 2 < x < \infty\}$$

and is written $(2, \infty)$. The entire real line, R, is written as $(-\infty, \infty)$.



Example

Question

Express the following as intervals:

1. $\{x \in R \mid -10 < x \leq 10\}$
2. $\{x \in R \mid x \geq -4\}$
3. $\{x \in R \mid x \leq 2\}$

Answer

1. $(-10, 10]$
2. $[-4, \infty)$
3. $(-\infty, 2]$

Example

Question

Re-write the following sets:

1. $\{x \in Z \mid x^2 < 25\}$
2. $\{x \in R \mid \frac{x}{3} \in N \text{ and } x < 10\}$

Answer

1. $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
2. $\{3, 6, 9\}$

Outline

Functions and their Graphs

Rationale

Functions are the key to describing the real world in mathematical terms.

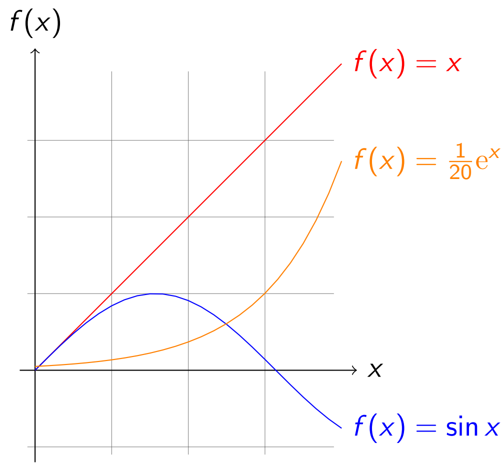
- The interest paid on an investment depends on the length of time the money is invested.
- The distance an object travels at a fixed speed depends on the elapsed time.
- The area of the circle depends on the radius of the circle.
- The run time of an algorithm depends on the length of the input.

Function

Definition

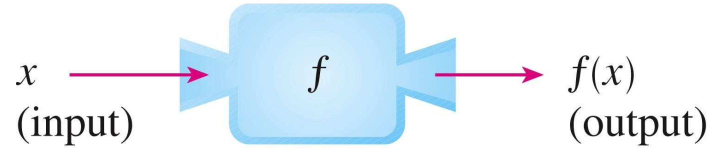
- A function/map from a set D (domain) to a set R (range) is a rule which assigns to each element in D a unique (single) element in R.
- The value of one (dependent) variable, say y, depends on the value of another (independent) variable, say x: we say that y is a function of x and write
$$y = f(x).$$
- The set D of all possible input values is called the domain of f.
- The set of all values of $f(x)$ as x varies throughout D is called the range of f.
- We write $f : D \to \mathbb{R}$.
- The graph of f is the set $\{(x, f(x)) \mid x \in D\}$.

Different Types of Functions



The graph of f is the set $\{(x, f(x)) \mid x \in D\}$.

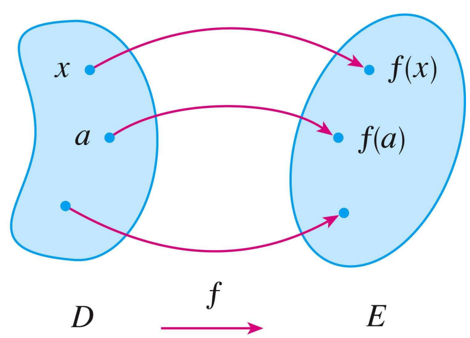
It may be helpful to think of a function as a machine:



If x is in the domain of the function f, then when x enters the machine, it is accepted as an input and the machine produces an output $f(x)$ according to the rule of the function.

Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

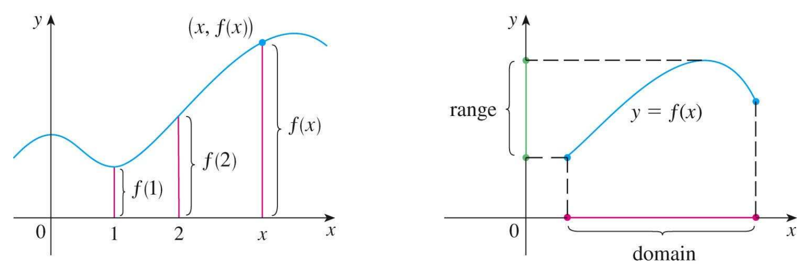
Another way to picture a function is by an arrow diagram



Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, f (a) is associated with a, and so on.

The most common method for visualizing a function is its graph. If f is a function with domain D, then its graph is the set of ordered pairs

$$\{(x,\ f(x)) \mid x \in D\}$$



- In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.
- The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis

Outline

Domain & Range

- We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the domain of the function.
- The number $f(x)$ is the value of f at x and is read "f of x".
- The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.
- A symbol that represents an arbitrary number in the domain of a function f is called an independent variable.
- A symbol that represents a number in the range of f is called a dependent variable.
- In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.
- The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis.

Domain & Range Example

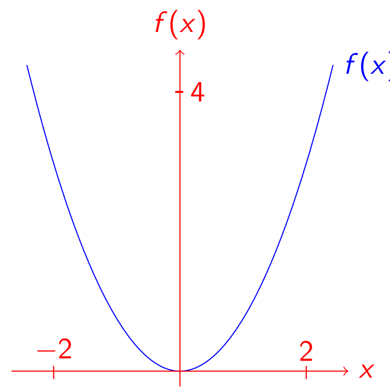
Question

Consider the function $f(x) = x^2$.

Analysis

- The natural domain is simply R, since any real number can be squared.
- However, the range of $f(x)$ is $[0, \infty)$ since the square of any number (i.e. x^2) cannot be negative.

Domain & Range of $f(x) = x^2$



Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $[0, \infty)$

Note that in this graph, we have restricted our domain to $[-2, 2]$ so that the range is $[0, 4]$.

Domain & Range Example

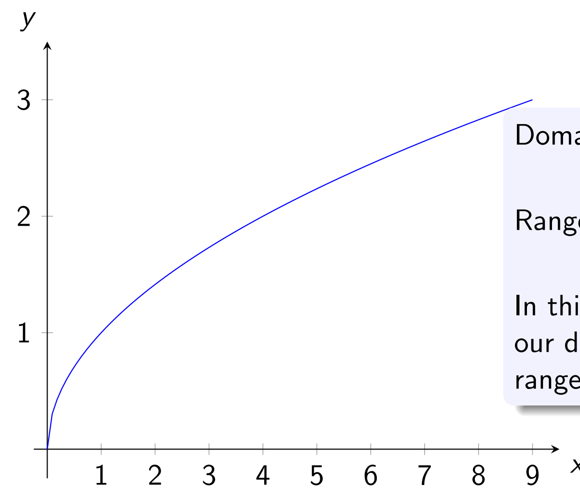
Question

Consider the positive square-root function $f(x) = +\sqrt{x}$.

Analysis

- We cannot take the square root of a negative number so the domain must be restricted to $[0, \infty)$ since the function is the positive square-root.
- Note that, although every positive number has two square-roots, $+\sqrt{}$ and $-\sqrt{}$, the operation $f(x)$ would not be a function unless we defined it uniquely.

Domain & Range of $f(x) = \sqrt{x}$

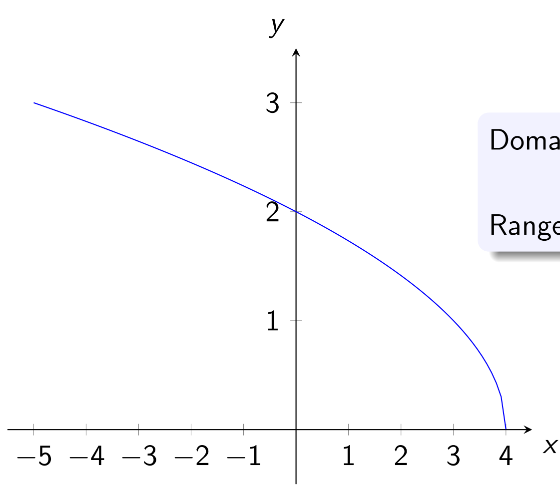


Domain: $[0, \infty)$

Range: $[0, \infty)$

In this picture, we have restricted our domain to $[0, 9]$ and so the range is $[0, 3]$.

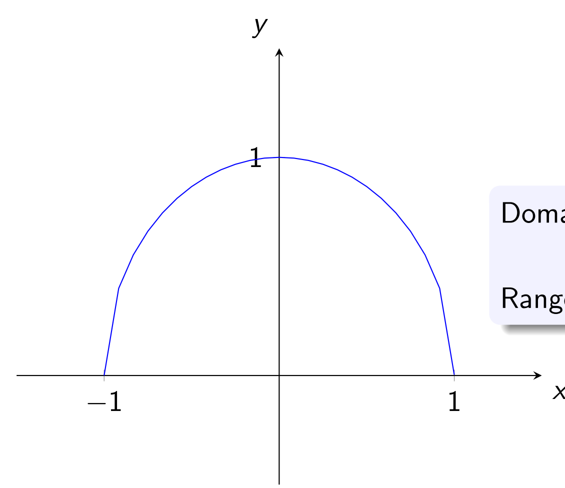
Domain & Range of $f(x) = \sqrt{4 - x}$



Domain: $(-\infty, 4]$

Range: $[0, \infty)$

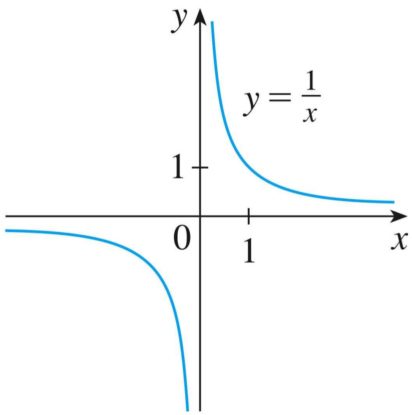
Domain & Range of $f(x) = \sqrt{1 - x^2}$



Domain: $D = (-1, 1)$

Range: $R = [0, 1]$

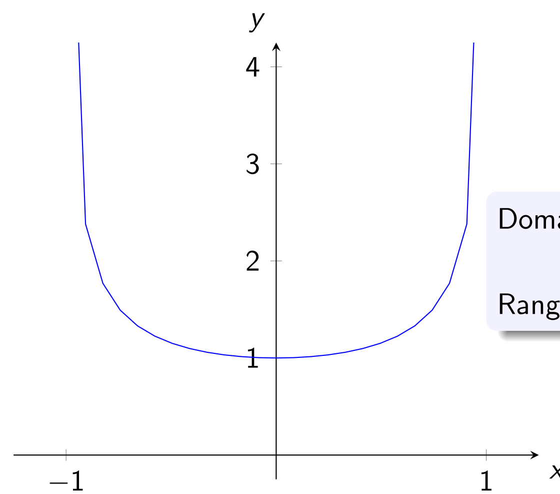
Domain & Range of $f(x) = \frac{1}{x}$



Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

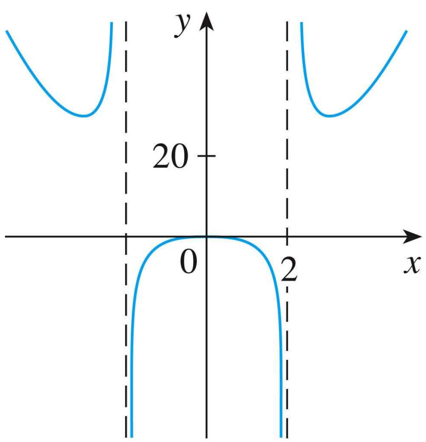
Domain & Range of $f(x) = \dfrac{1}{\sqrt{1-x^2}}$



Domain: $D = (-1, 1)$

Range: $R = [1, \infty)$

Domain & Range of a Rational Function

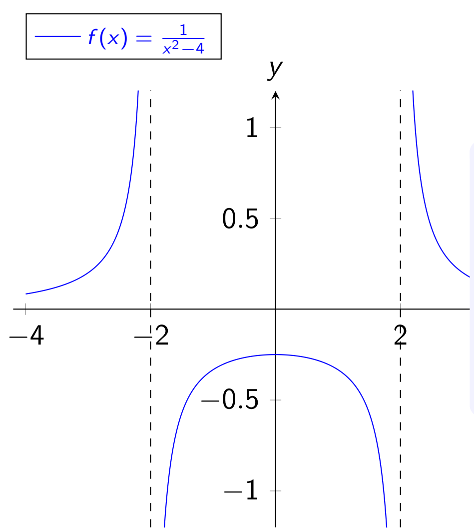


$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

Domain: $\{x \mid x \neq \pm 2\}$

Range: $(-\infty, \infty)$

Domain & Range of a Rational Function



$$f(x) = \frac{1}{x^2 - 4}$$

Domain: $\{x \mid x \neq \pm 2\}$

Range: $(-\infty, -\frac{1}{4}] \cup (0, \infty)$

Domain & Range Example

Question

Consider the function $f(x) = \dfrac{1}{x-3}$.

Analysis

- It is not defined for $x = 3$ since we would be attempting to divide by zero at that point. So the domain is $R \setminus \{3\}$, i.e. all the real numbers except 3.
- The range is also restricted as follows.
- The function $\frac{1}{x-3}$ can produce any real number except 0 since no real value for x will make $\frac{1}{x-3} = 0$.
- Therefore, the range is $R \setminus \{0\}$.

Domain & Range Example

Question

Consider the function $f(x) = \dfrac{1}{\sqrt{x-19}}$.

Analysis

- There are two potential problems in finding the natural domain here.
- We cannot divide by zero and we cannot take the square root of a negative number.
- Thus, we need
$$x - 19 > 0 \quad \Rightarrow \quad x > 19$$
and so the natural domain is $(19, \infty)$.
- To determine the range, note that the square root is positive so the range must be positive, i.e. $(0, \infty)$.

Domain & Range Example

Question

Determine the natural domain of $f(x) = \dfrac{1}{x^2 - x - 2}$.

Answer

We need to avoid dividing by zero and so the natural domain will be all real numbers except those which make $x^2 - x - 2$ equal to zero. In order to eliminate these points, we need to solve

$$x^2 - x - 2 = 0$$

giving the two roots $x = 2$ and $x = -1$. Hence, the natural domain of $f(x)$ is $R \setminus \{-1, 2\}$.

Domain & Range Example

Solving $x^2 - x - 2 = 0$.

Roots of a Quadratic: By factorisation

$$x^2 - x - 2 = (x-2)(x+1) = 0$$

giving the two roots $x = 2$ and $x = -1$.

Domain & Range Example

Solving $x^2 - x - 2 = 0$.

Roots of a Quadratic: Using the formula

We will use

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a, b and c are the coefficients of the quadratic equation

$$ax^2 + bx + c = 0.$$

With $a = 1$, $b = -1$ and $c = -2$, we have

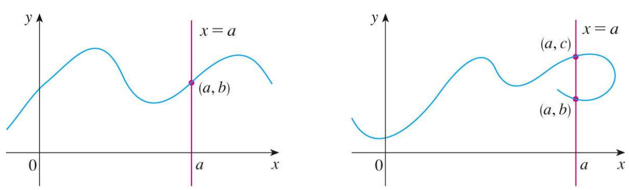
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

giving the two values $x = 2$ and $x = -1$ (as before).

Outline

Which Curves are Graphs of Functions?

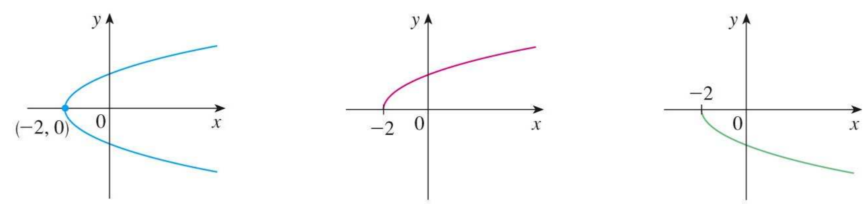
The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions?



This is answered by the following test.

Vertical Line Test

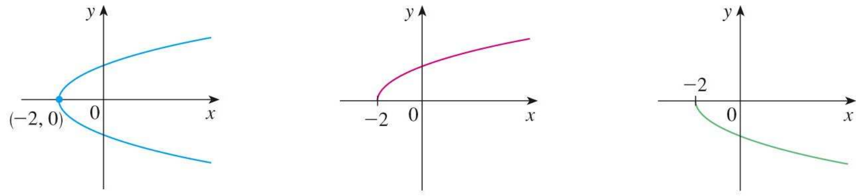
If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But, if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve cannot represent a function because a function cannot assign two different values to a.



(a) $x = y^2 - 2$ (b) $y = \sqrt{x+2}$ (c) $y = -\sqrt{x+2}$

The parabola $x = y^2 - 2$ is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice.

The parabola, however, does contain the graphs of two functions of x.



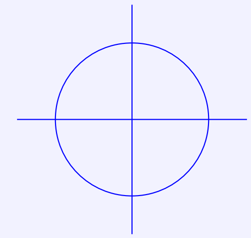
(a) $x = y^2 - 2$ (b) $y = \sqrt{x+2}$ (c) $y = -\sqrt{x+2}$

Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x+2}$.

Thus the upper and lower halves of the parabola are the graphs of the functions $y = \sqrt{x+2}$ and $y = -\sqrt{x+2}$.

Vertical Line Test

A function f can have only one value $f(x)$, for each x in its domain, so no vertical line can intersect the graph of a function more than once.

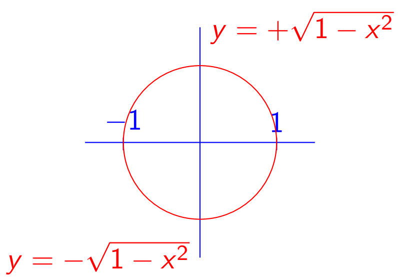


A circle is not the graph of a function as the vertical line $x = 0$ (y-axis) cuts it twice.

Vertical Line Test

However, the unit circle centred on $(0, 0)$ has equation

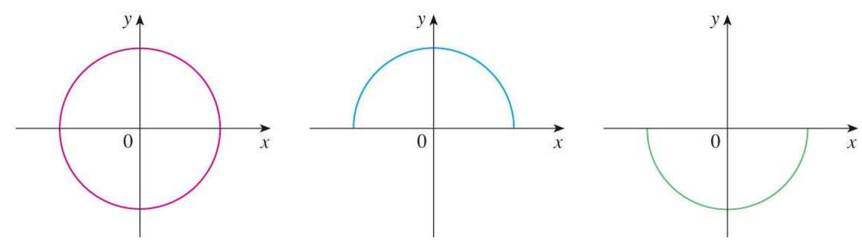
$$x^2 + y^2 = 1 \iff y^2 = 1 - x^2 \iff y = \pm\sqrt{1 - x^2}.$$



The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$.

The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$

The Circle and 2 Functions



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

MS121: IT Mathematics

Functions

Catalogue of Essential Functions

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Outline

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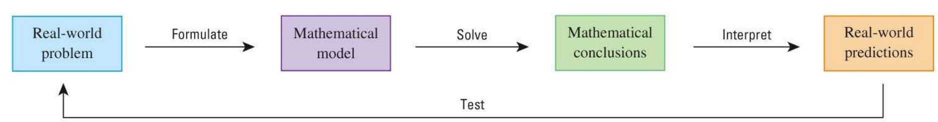
Mathematical Model

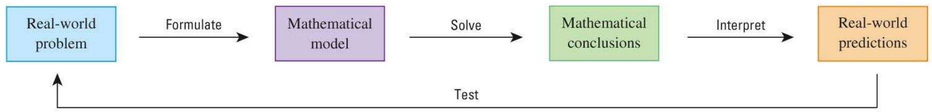


A mathematical model is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as

- the size of a population,
- the demand for a product,
- the speed of a falling object,
- the concentration of a product in a chemical reaction,
- the life expectancy of a person at birth, or
- the cost of emission reductions.

The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

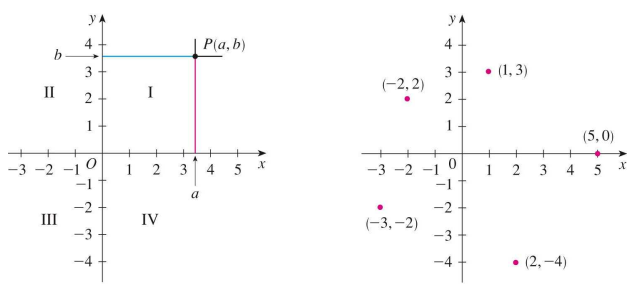
Mathematical Model



- A mathematical model is never a completely accurate representation of a physical situation – it is an idealization.
- A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.
- It is important to realize the limitations of the model.
- In the end, Mother Nature has the final say

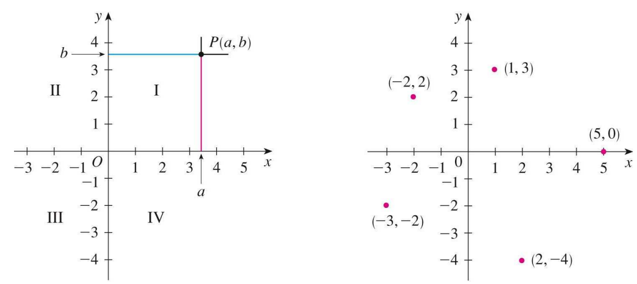
Outline

Coordinate Geometry & Lines



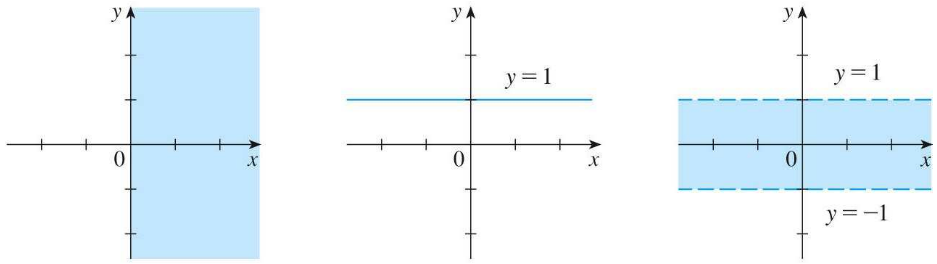
- Any point in the plane can be located by a unique ordered pair of numbers as follows:
- Draw lines through perpendicular to the x- and y-axes.
- These lines intersect the axes in points with coordinates a and b as shown above.

Coordinate Geometry & Lines (Cont'd)



- Then the point P is assigned the ordered pair (a, b).
- The first number a is called the x-coordinate of P; the second number b is called the y-coordinate of P.
- We say that P is the point with coordinates (a, b), and we denote the point by the symbol $P(a, b)$. Several points are labeled with their coordinates in the second plot above.

Regions in the Plane

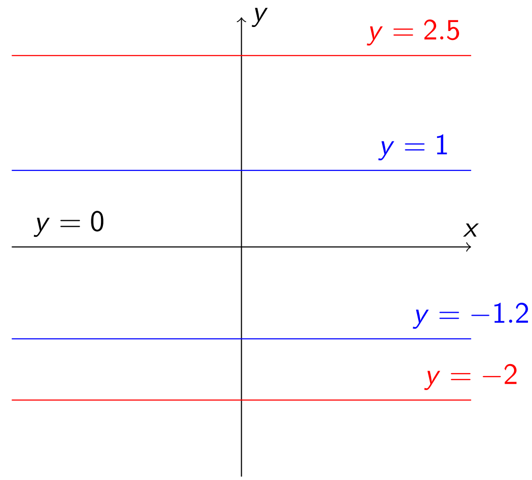


(a) $x \geqslant 0$ (b) $y = 1$ (c) $|y| < 1$

Formally

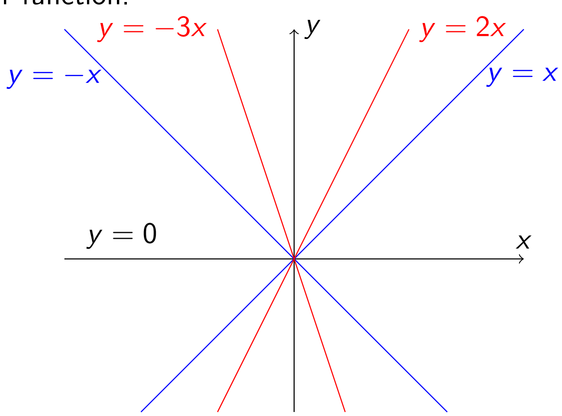
- $\{(x, y) \mid x \geq 0\}$
- $\{(x, y) \mid y = 1\}$
- $\{(x, y) \mid |y| < 1\}$

Constant Functions: Slope $m = 0$



Linear Functions

A function of the form $y = f(x) = mx + c$, where m and c are constants, is called a linear function.



Case 1: $c = 0$

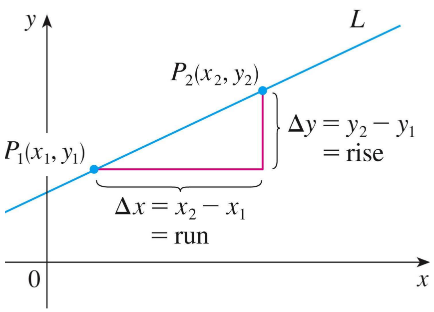
Linear Functions

- When we say that y is a linear function of x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + c$$

  where m is the slope of the line and c is the y-intercept.
- A characteristic feature of linear functions is that they grow at a constant rate.

Slope of a Straight Line

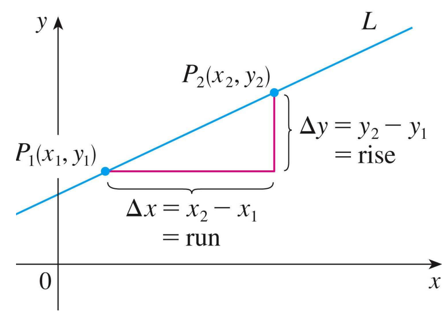


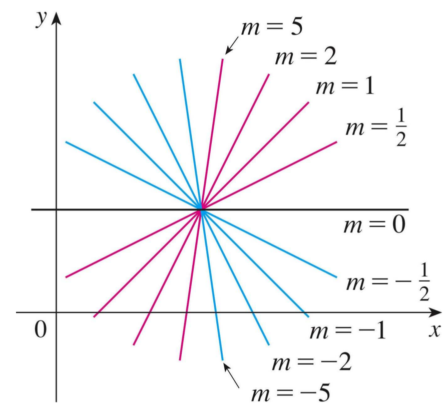
The slope of a nonvertical line which passes through the points $P_1\,(x_1, y_1)$ and $P_2\,(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

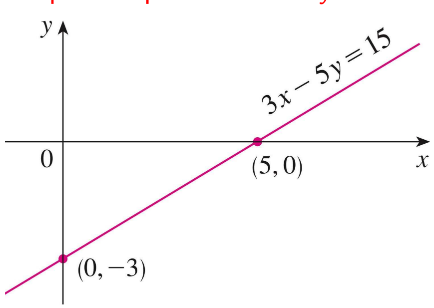
Slope of a Straight Line (Cont'd)



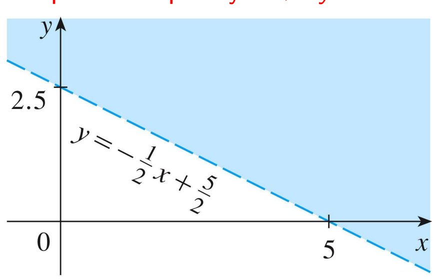


Straight Line: Equation & Inequality

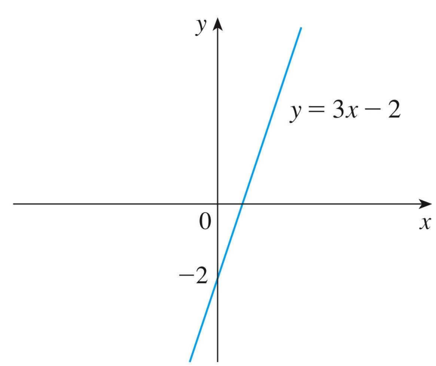
Graph of equation $3x - 5y = 15$



Graph of inequality $x + 2y > 5$



Slope as the Rate of Change



- Consider the graph of the linear function

$$y = f(x) = 3x - 2$$

- You will notice that whenever x increases by 1, the value of y increases by 3.
- So y increases three times as fast as x.
- Thus the slope of the graph, namely 3, can be interpreted as the rate of change of y with respect to x.

Polynomials

Definition

- A function is called a polynomial if

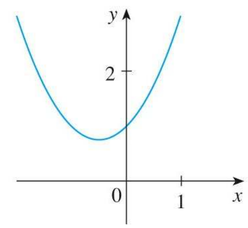
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers a_0, a_1, $\ldots$, a_n are constants called the coefficients of the polynomial.
- The domain of any polynomial $R = (-\infty, +\infty)$.
- If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.
- For example, the function

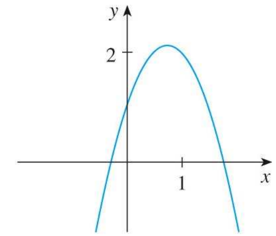
$$P(x) = 2x^6 - x^4 + \frac{2}{5}x + \sqrt{2}$$

is a polynomial of degree 6.

Polynomial of Degree 2: Quadratic Function



(a) $y = x^2 + x + 1$



(b) $y = -2x^2 + 3x + 1$

- A polynomial of degree 2 is of the form

$$P(x) = ax^2 + bx + c$$

and is called a quadratic function.
- Its graph is always a parabola obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$.

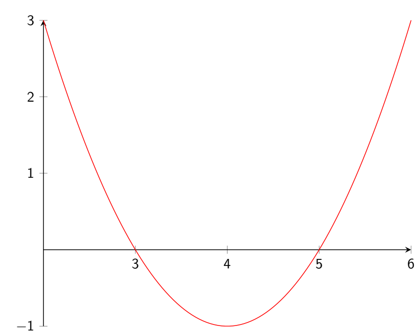
Quadratic functions

$$f(x) = ax^2 + bx + c$$

Questions to ask

- Where is $f(x) = (x-3)(x-5)$ positive?
- Where is $f(x) = (x-a)(x-b)$, when $a < b$, positive?
- Where is $g(x) = x^2 - b^2 = (x-b)(x+b)$, when $0 < b$, positive?
- Where is $h(x) = b^2 - x^2 = (b-x)(b+x) = -g(x)$, when $0 < b$, positive?

Where is $(x-3)(x-5) = x^2 - 8x + 15 > 0$?



- The roots of this function are the solutions to the quadratic equation

$$(x-3)(x-5) = 0$$

i.e. $x = 3$ and $x = 5$.
- So

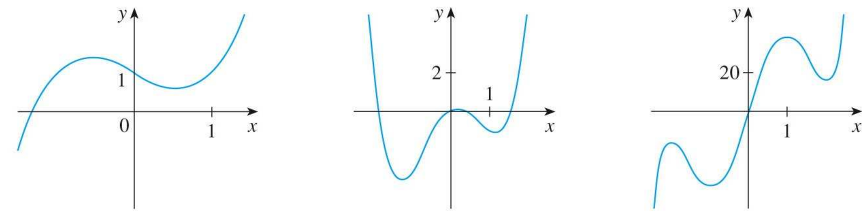
$$f(x) > 0 \quad \Leftrightarrow x < 3 \text{ or } x > 5,$$
$$f(x) < 0 \quad \Leftrightarrow 3 < x < 5.$$

Quadratic functions

Some Answers

- The quadratic function $f(x) = (x-a)(x-b)$, where $a < b$, is
  - zero when $x = a$ and $x = b$;
  - positive if $x < a$ or $x > b$;
  - negative when $a < x < b$.
- The quadratic function $g(x) = x^2 - b^2 = (x-b)(x+b)$, where $0 < b$, is
  - zero when $x = \pm b$;
  - positive if $x < -b$ or $x > b$;
  - negative when $-b < x < b$.
- The quadratic function $h(x) = b^2 - x^2 = (b-x)(b+x) = -g(x)$, where $0 < b$, is
  - zero when $x = \pm b$;
  - positive if $-b < x < b$.
  - negative when $x < -b \text{ or } x > b$;

Polynomial Functions of Degree n



(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d, \qquad a \neq 0$$

and is called a cubic function.

Polynomials: Recap

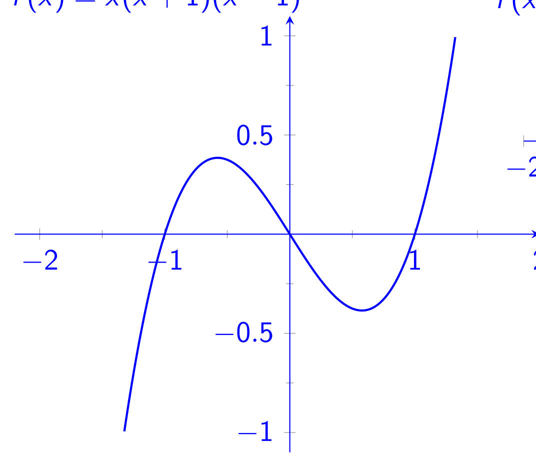
A function f is a polynomial if

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_o$$

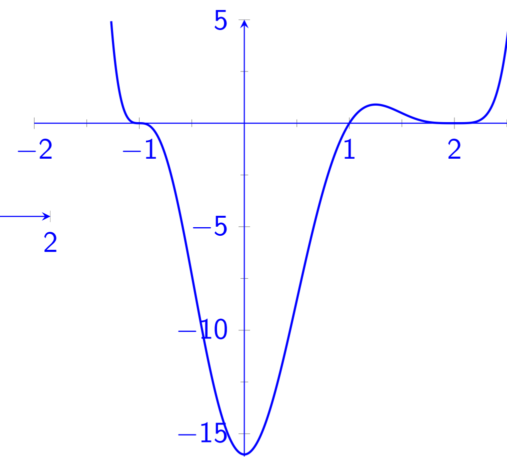
where n is a non negative integer, called the degree of the polynomial, and $a_0, a_1, \ldots, a_n$ are real constants, called coefficients, with $a_n \neq 0$

Examples of Polynomials

$f(x) = x(x+1)(x-1)$



$f(x) = (x-2)^4(x+1)^3(x-1)$



Polynomials

Properties

- All polynomials have domain $\mathbb{R} = (-\infty, \infty)$.
- Linear functions- $y = mx + b$ (with $m \neq 0$) are polynomials of degree 1.
- Quadratic functions $y = ax^2 + bx + c$ (with $a \neq 0$) are polynomials of degree 2.
- Cubic functions $y = ax^3 + bx^2 + cx + d$ (with $a \neq 0$) are polynomials of degree 3.
- If n is even, then f has even degree and if n is odd, then f has odd degree.

Examples of Polynomials

- A ball is thrown into the air with an initial velocity of $160 ft/sec$. It reaches a height, h, after t seconds:

$$h(t) = 160t - 16t^2 \text{ ft.}$$

- The supply and demand functions for a commodity are

$$P = 2Q_S^2 + 9Q_S + 150$$

$$P = -2Q_D^2 - 12Q_D + 275$$

where P is the price per unit.

Evaluating Polynomials

Let $p(x) = 7\,x^3 - 12\,x^2 - 8\,x + 8$. p is a polynomial of degree 3 (a cubic).

We can, for example, evaluate at $x = 0$ to find

$$p(0) = 7\,(0)^3 - 12\,(0)^2 - 8\,(0) + 8 \;=\; 8$$

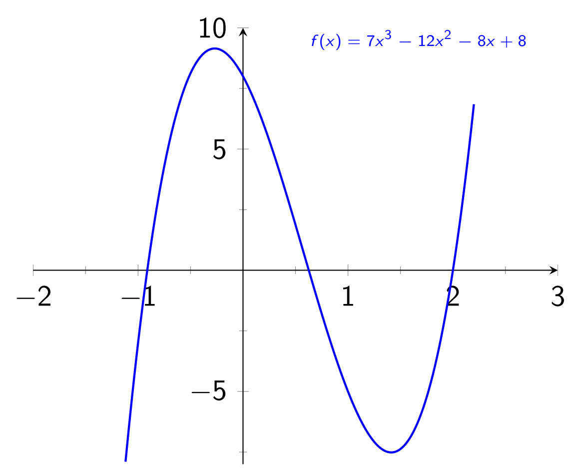
While evaluating at $x = 2$ gives

$$p(2) = 7\,(2)^3 - 12\,(2)^2 - 8\,(2) + 8 \;=\; 0$$

In this case, $x = 2$ is called a root of the polynomial p. Note that

$$p(x) = (x - 2)(7x^2 + 2x - 4).$$

Graph of $p(x) = 7\,x^3 - 12\,x^2 - 8\,x + 8$



Roots of a Polynomial

- In general, given any function f, we call a solution of the equation

$$f(x) = 0$$

a root or zero of the function f.

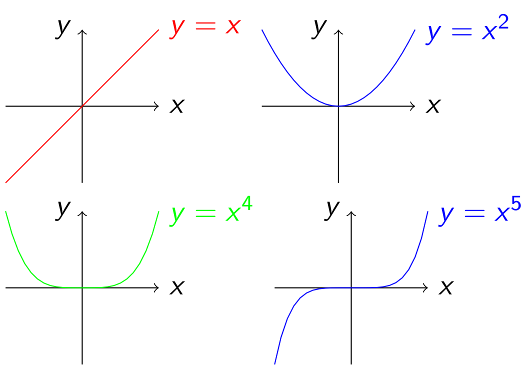
- If $x = a$ is a root of a polynomial p, then $p(x) = (x - a)q(x)$, for some polynomial q.
- We say that $(x - a)$ is a factor of p.
- A root $x = a$ is called a repeated root of the polynomial p, if $p(x) = (x - a)^m s(x)$, for some $m \geq 2$ and some polynomial $s(x)$.
- If p is a polynomial of degree n, then p has at most n real roots.

Outline

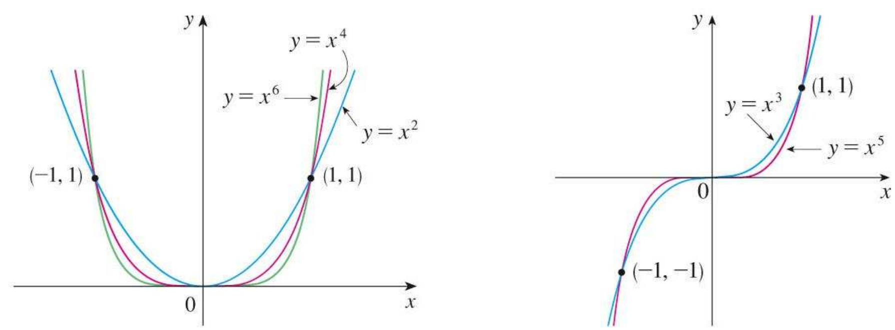
Power Functions

A function of the form $y = f(x) = x^n$, where n is a constant, is called a power function.

The graphs for $n = 1, 2, 3, 4, 5$ are displayed below:

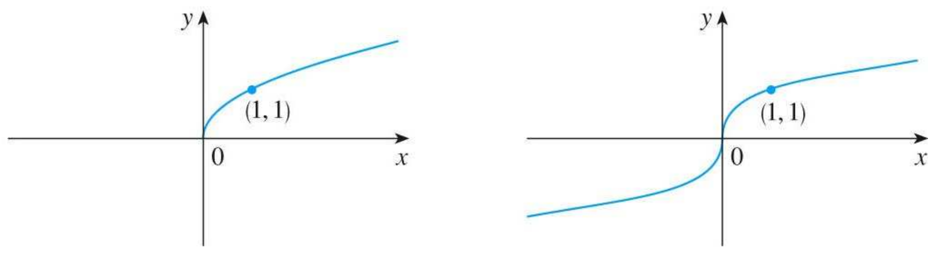


Families of Power Functions



- The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.
- Note that, as n increases, the graph of $y = x^n$ becomes flatter near $x = 0$ and steeper when $x \geq 1$.

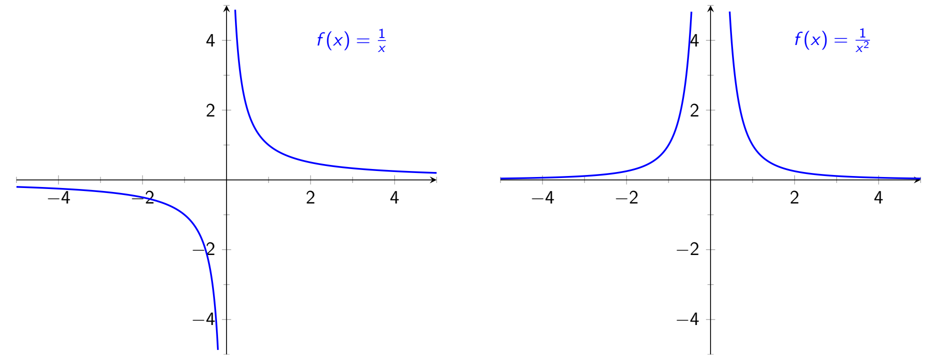
Power Functions: $x^{\frac{1}{n}}$, n a positive integer



(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

- Note that the domain of $\sqrt{x}$ is $[0, \infty)$ whose graph is the upper half of the parabola $x = y^2$.
- The domain of $\sqrt[3]{x}$ is R (every real number has a cube root).

Power Functions: Two Further Examples



Outline

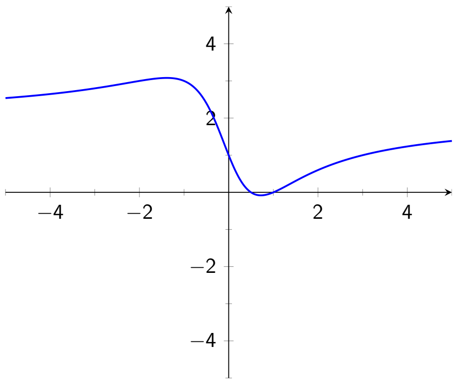
Rational Functions

A rational function is a quotient or ratio of two polynomial functions:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. The domain of a rational function is the set of all x for which $q(x) \neq 0$.

Example of a Rational Function

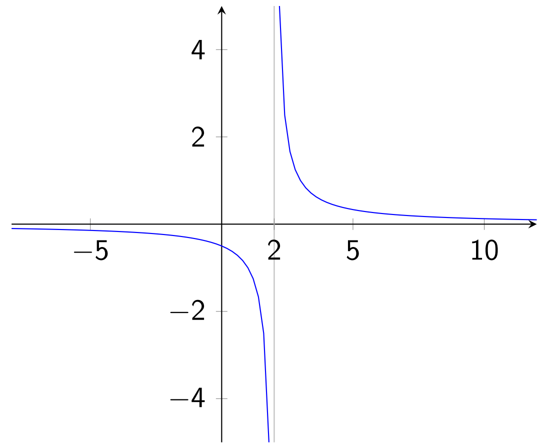


The function

$$f(x) = \frac{2x^2 - 3x + 1}{x^2 + 1}$$

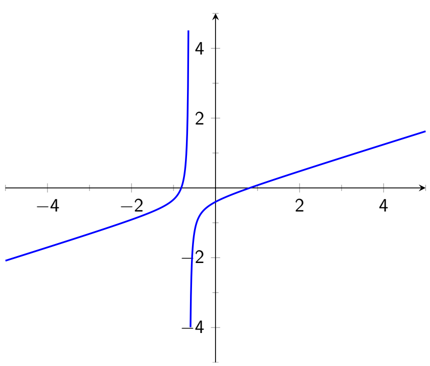
is a rational function with domain all real numbers.

Rational Function with a Discontinuity



$$f(x) = \frac{1}{x - 2}$$

Rational Function with a Discontinuity



The function

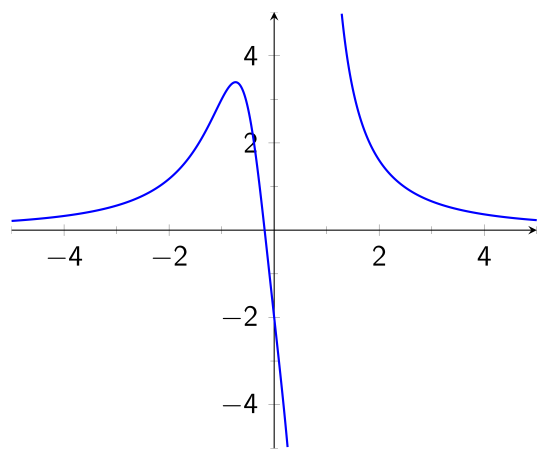
$$f(x) = \frac{3x^2 - 2}{8x + 5}$$

is a rational function with domain all real numbers except when the denominator is zero:

$$8x + 5 = 0 \iff x = -\frac{5}{8}.$$

So the domain is $\mathbb{R} \setminus \left\{-\frac{5}{8}\right\} = \left(-\infty, -\frac{5}{8}\right) \cup \left(-\frac{5}{8}, \infty\right)$.

Rational Function with a Discontinuity



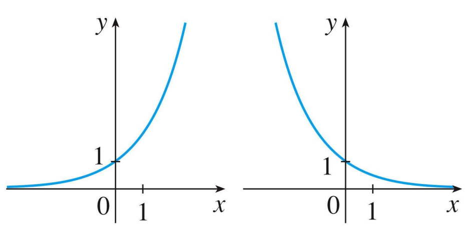
$$f(x) = \frac{11x + 2}{2x^3 - 1}$$

Outline

Exponential Functions

- Functions of the form
$$y = a^x,$$
where $a > 0$ are called exponential functions, a is called the base.
- All exponential functions have domain $\mathbb{R}$.
- If $a \neq 1$, then the range of $y = a^x$ is $(0, \infty)$.

Exponential Function: $f(x) = a^x$

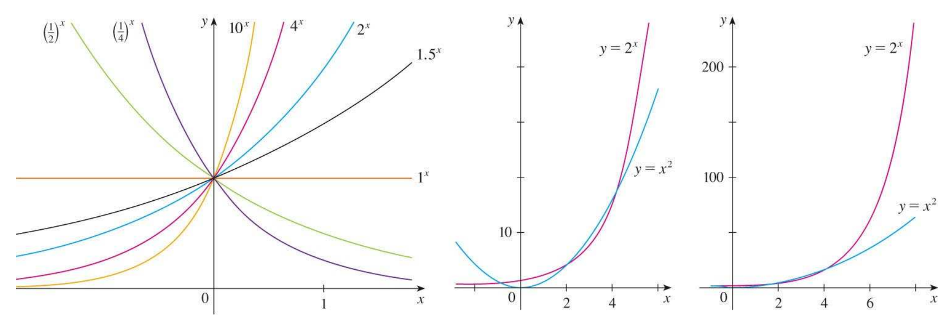


(a) $y = 2^x$ (b) $y = (0.5)^x$

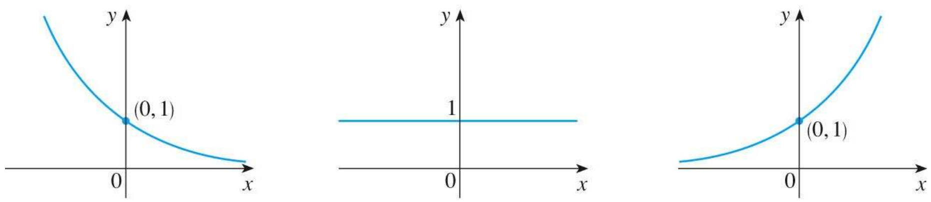
In both cases,

- the domain is $(-\infty, \infty)$, and
- the range is $(0, \infty)$

Exponential Functions: Family Members



The 3 kinds of Exponential Functions $y = a^x$



(a) $y = a^x$, $0 < a < 1$ (b) $y = 1^x$ (c) $y = a^x$, $a > 1$

Exponential Functions

Application

- Suppose that €10,000 euro is invested in an account with an interest rate of 2% per annum.
- The amount of money in the account after t years is

$$P(t) = 10,000(1.02)^t.$$

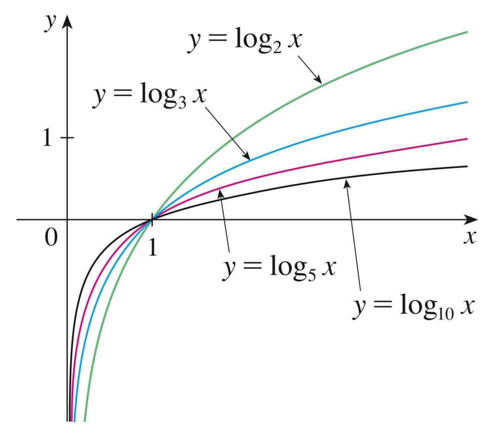
Logarithmic Functions

- Logarithmic functions are functions of the form

$$y = \log_a x$$

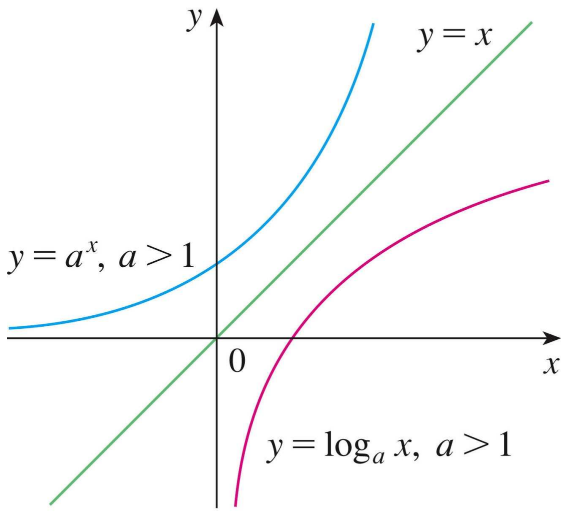
where the base $a \neq 1$ is a positive constant, are the inverse functions of the exponential functions.
- All logarithmic functions have domain $(0, \infty)$ and range $\mathbb{R}$.

Logarithmic Function: $f(x) = \log_a x$

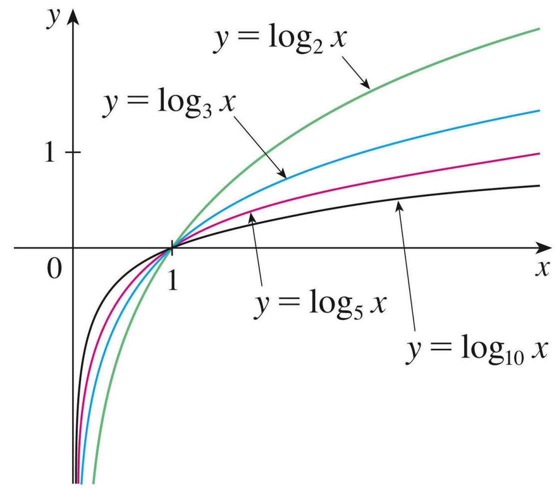


- The domain is $(0, \infty)$, and
- the range is $(-\infty, \infty)$

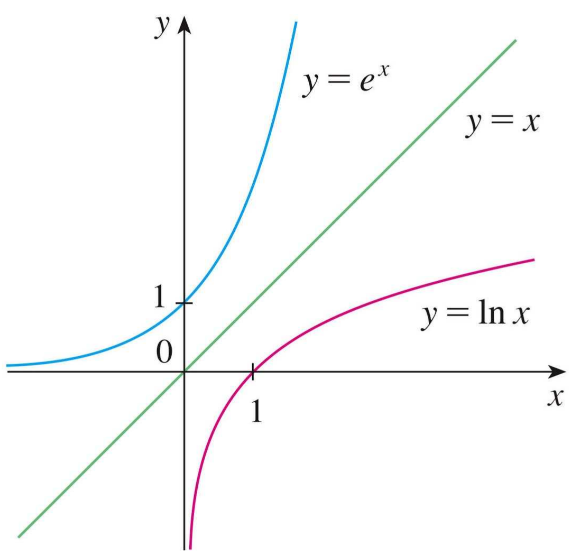
Exponential & Logarithmic Functions $a > 1$



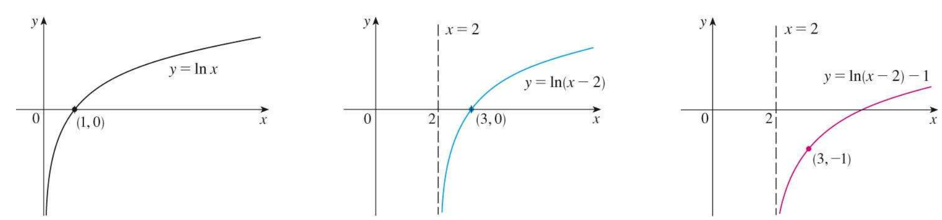
Logarithmic Functions Base a



Natural Logarithmic Function $y = \ln x$



Logarithmic Function $y = \ln(x - 2) - 1$

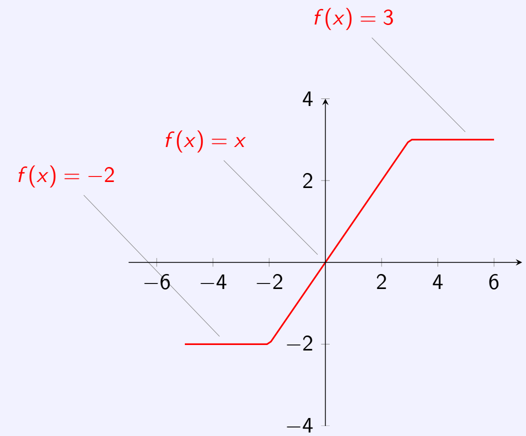


Outline

Piecewise-Defined Functions

Piecewise-defined functions are functions which are defined by different formulae in different parts of their domains.

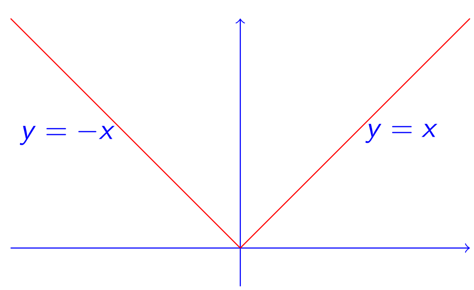
Simple Example



Piecewise-Defined Functions

Another example of a function which is described by using different formulae on different parts of its domain is the absolute value function:

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

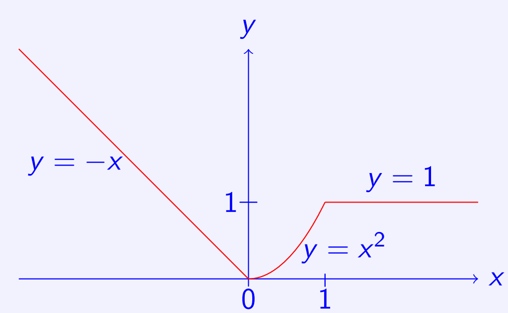


Graphing Piecewise-Defined Functions

The function

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x^2, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } x > 1. \end{cases}$$

is defined on the real line but has values given by different formulas depending on the position of x.



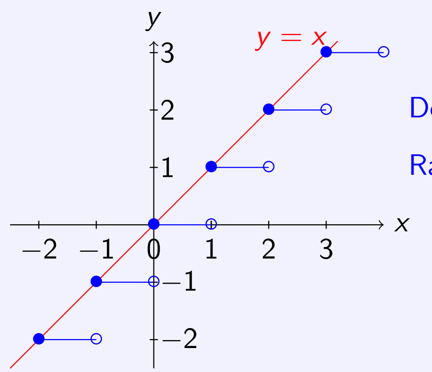
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Piecewise-Defined Functions

The Greatest Integer function is the function whose value at x is the *greatest integer less than or equal to x* it is also called the integer floor function and is denoted $\lfloor x \rfloor$. For example

$$\lfloor 3.2 \rfloor = 3 = \lfloor 3 \rfloor = \lfloor 3.99 \rfloor, \quad \lfloor -3.7 \rfloor = -4 = \lfloor -3.0001 \rfloor.$$



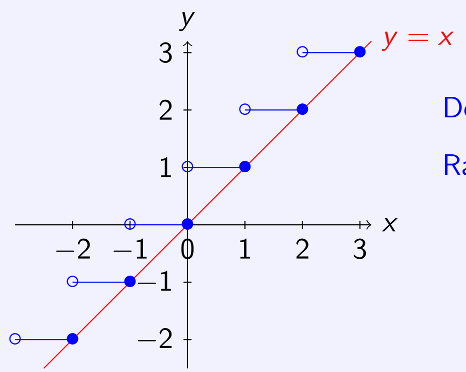
Domain: $(-\infty, \infty)$

Range: $\mathbb{Z}$

Piecewise-Defined Functions

The Least Integer function is the function whose value at x is the *smallest integer greater than or equal to x* it is also called the integer ceiling function and is denoted $\lceil x \rceil$. For example

$$\lceil 3.2 \rceil = 4 = \lceil 4 \rceil = \lceil 3.0001 \rceil, \quad \lceil -3.7 \rceil = -3 = \lceil -3.999999 \rceil.$$



Domain: $(-\infty, \infty)$

Range: $\mathbb{Z}$

MS121: IT Mathematics

Functions

New Functions from Old Functions

John Carroll
School of Mathematical Sciences

Dublin City University

Outline

Outline

Even & Odd Functions

Properties

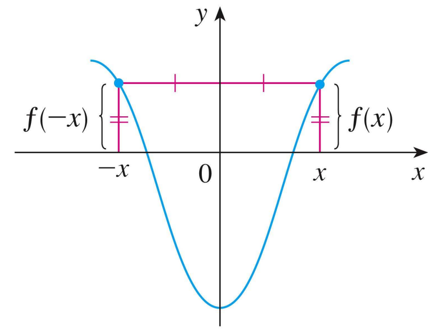
- A function f is an *even function* of x if
$$f(-x) = f(x),$$
for every x in the function's domain.
- A function f is an *odd function* of x if
$$f(-x) = -f(x),$$
for every x in the function's domain.

Even & Odd Functions

Properties (Cont'd)

- The names even and odd come from powers of x.
- If y is an even power of x, e.g. $y = x^6$, then it is an even function as $(-x)^6 = x^6$.
- If y is an odd power of x, e.g. $y = x^7$, then it is an odd function as $(-x)^7 = -x^7$.
- Some functions are even, some are odd, and some are neither.

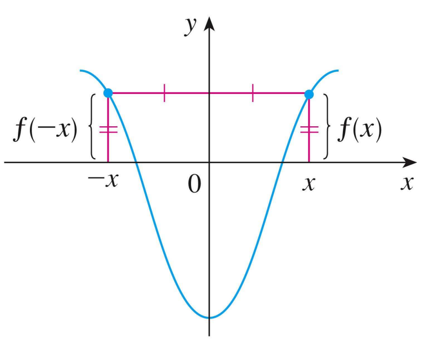
Even Functions



- If a function satisfies $f(-x) = f(x)$ for every number x in its domain, then it is called an even function.
- For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

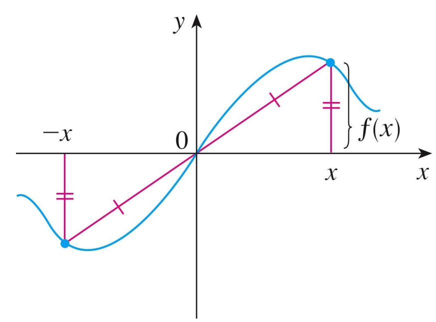
Even Functions



Significance

- The geometric significance of an even function is that its graph is symmetric with respect to the y-axis.
- This means that if we have plotted the graph of $f(x)$ for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

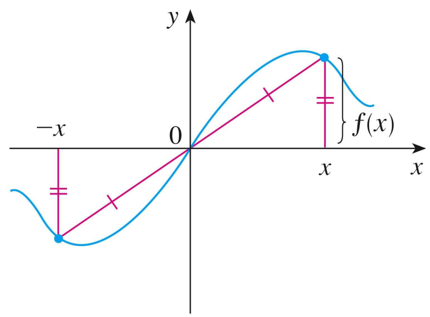
Odd Functions



- If $f(x)$ satisfies $f(-x) = -f(x)$ for every number x in its domain, then $f(x)$ is called an odd function.
- For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Odd Functions



Significance

- The graph of an odd function is symmetric about the origin.
- If we already have the graph of $f(x)$ for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

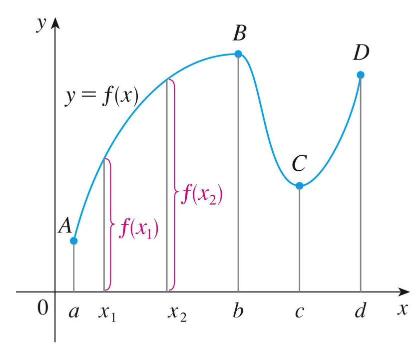
Outline

Increasing/Decreasing Functions

General Situation

- If the graph of a function *climbs* or *rises*, as you move from left to right, then the function is increasing.
- If the graph of a function *descends* or *falls*, as you move from left to right, then the function is decreasing.
- We will look at the more formal definition during this Semester and see how to find the intervals over which a function is increasing and the intervals over which it is decreasing.

Increasing & Decreasing Functions



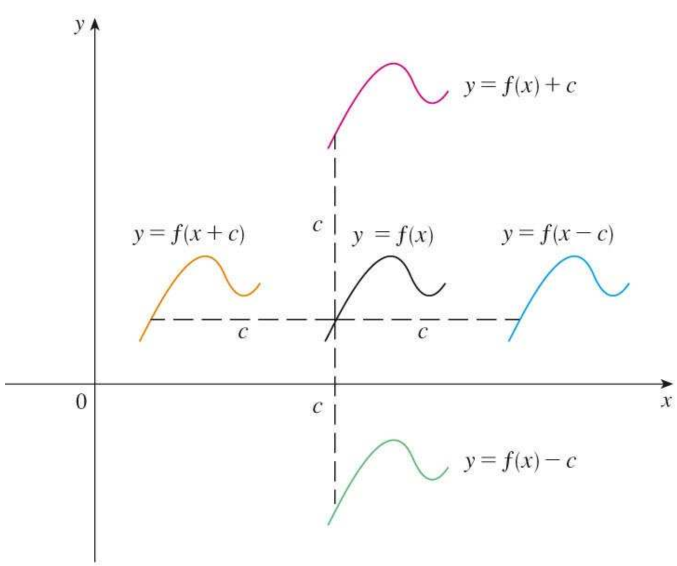
- The graph rises from A to B, falls from B to C, and rises again from to C to D.
- The function is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$.
- Notice that if x_1 and x_2 and are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- We use this as the defining property of an increasing function.

Sample Increasing & Decreasing Functions

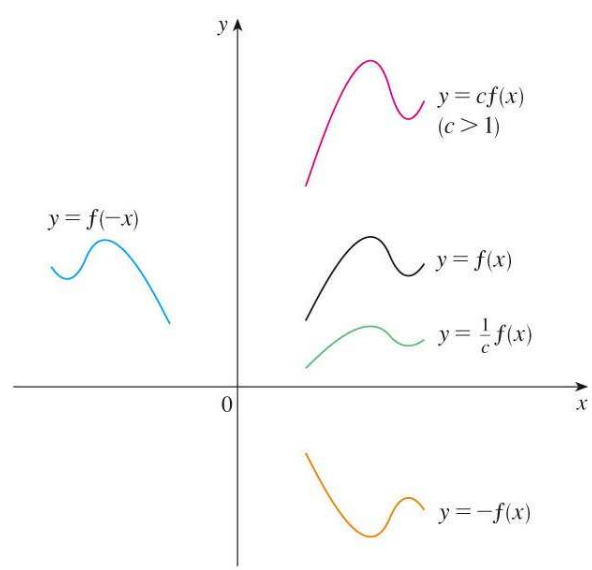
$f(x)$	Where Increasing	Where Decreasing
x^2	$[0, \infty)$	$(-\infty, 0]$
x^3	$(-\infty, \infty)$	Nowhere
$\frac{1}{x}$	Nowhere	$(-\infty, 0) \cup (0, \infty)$
$\frac{1}{x^2}$	$(-\infty, 0)$	$(0, \infty)$
$\sqrt{x}$	$[0, \infty)$	Nowhere
$x^{2/3}$	$[0, \infty)$	$(-\infty, 0]$

Outline

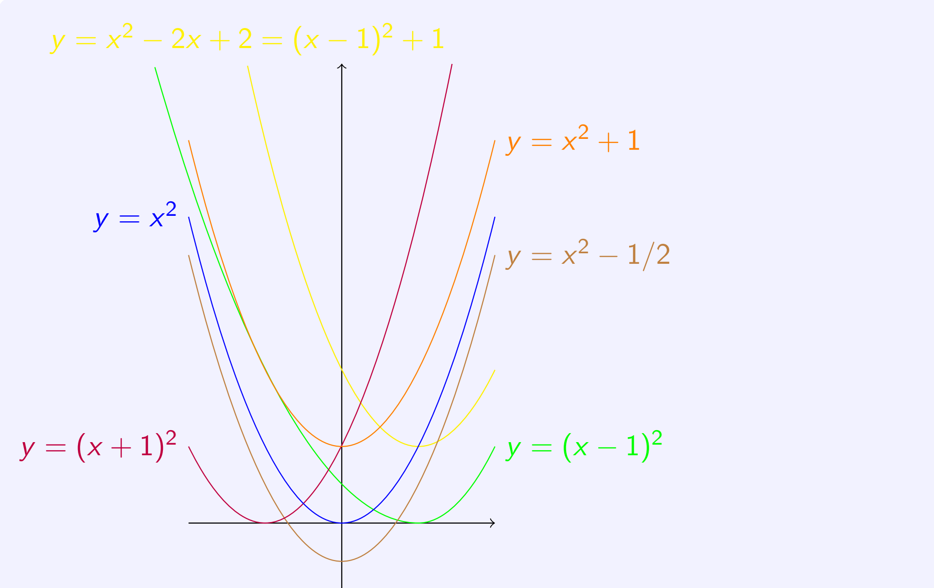
Vertical & Horizontal Shifts



Vertical & Horizontal Shifts



Shifting Graphs



Shifting Graphs

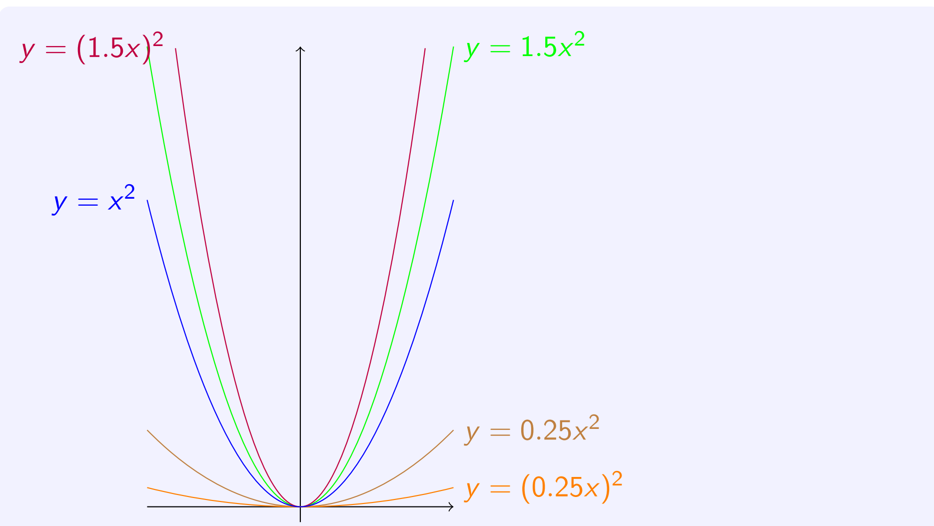
Vertical shifts: $y = f(x) + k$ shifts the graph of f

- *up* k units if $k > 0$;
- *down* $|k|$ units if $k < 0$.

Horizontal shifts: $y = f(x + h)$ shifts the graph of f

- *left* h units if $h > 0$;
- *right* $|h|$ units if $h < 0$.

Scaling a Graph



Scaling a Graph

Vertically: $y = kf(x)$, for $k > 0$,

- *stretches* the graph of f vertically by a factor of k if $k > 1$;
- *compresses* the graph of f vertically by a factor of k if $k < 1$;

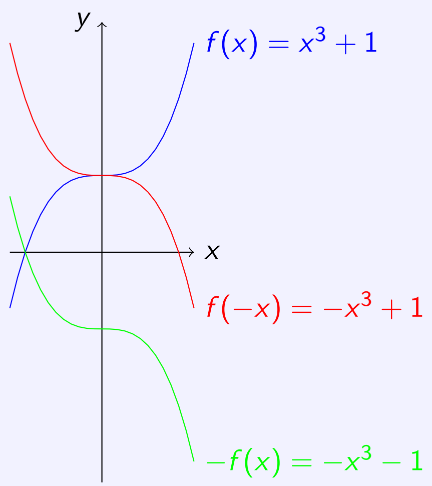
Horizontally: $y = f(kx)$, for $k > 0$,

- *stretches* the graph of f horizontally by a factor of k if $k < 1$;
- *compresses* the graph of f horizontally by a factor of k if $k > 1$;

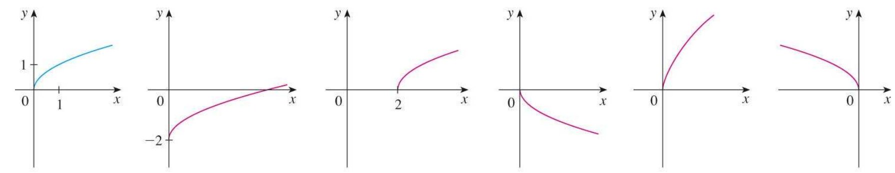
Reflections

$y = -f(x)$ is a reflection of $y = f(x)$ across the x-axis;

$y = f(-x)$ is a reflection of $y = f(x)$ across the y-axis;



Example: Transformation of Function $\sqrt{x}$



(a) $y = \sqrt{x}$ (b) $y = \sqrt{x} - 2$ (c) $y = \sqrt{x-2}$ (d) $y = -\sqrt{x}$ (e) $y = 2\sqrt{x}$ (f) $y = \sqrt{-x}$

We sketch:

- $y = \sqrt{x} - 2$ by shifting 2 units downward
- $y = \sqrt{x - 2}$ by shifting 2 units to the right
- $y = -\sqrt{x}$ by reflecting about the x-axis
- $y = 2\sqrt{x}$ by stretching vertically by a factor of 2
- $y = \sqrt{-x}$ by reflecting about the y-axis

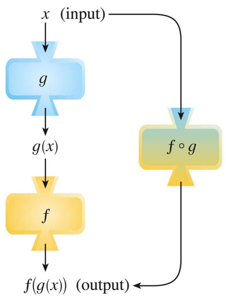
Combining Functions

In general, given any two functions f and g with domains $D(f)$, $D(g)$ respectively and constant c:

- $(cf)(x) = c(f(x))$ with domain $D(f)$.
- $(f + g)(x) = f(x) + g(x)$, domain $D(f) \cap D(g)$.
- $(f - g)(x) = f(x) - g(x)$, domain $D(f) \cap D(g)$.
- $(f \star g)(x) = f(x)g(x)$, domain $D(f) \cap D(g)$.
- $(f/g)(x) = \dfrac{f(x)}{g(x)}$ with domain $D(f) \cap D(g)$ less any points where g is zero.

Outline

Composite Functions



If f and g are functions, the *composite* function $f \circ g$ (f composed with g or f after g)is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of all x in the domain of g for which $g(x)$ lies in the domain of f.

$$\xrightarrow{\quad x \quad} \boxed{g} \xrightarrow{\quad g(x) \quad} \boxed{f} \longrightarrow f(g(x))$$

Viewing a Function as a Composite

To evaluate the function

$$y = \sqrt{1 - x^2}$$

we first find $1 - x^2$ and then take the square root:

$$\boxed{x} \xrightarrow{\quad 1 - x^2 \quad} \boxed{1 - x^2} \xrightarrow{\quad \sqrt{x} \quad} \boxed{y = \sqrt{1 - x^2}}$$

So that y is the composite $f \circ g$ where

$$f(x) = \sqrt{x} \ \text{ and } \ g(x) = 1 - x^2$$

Composition of Functions Example

The Functions

$f(x) = 2x + 3$ and $g(x) = x^2$

Composition

We obtain

$$g \circ f(x) = g\,(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9$$

while

$$f \circ g(x) = f\,(x^2) = 2x^2 + 3.$$

Note: In this example, $f \circ g(x) \neq g \circ f(x)$. This is also true in general as the composition of functions is not commutative.

Composition of Functions Example

The Functions

$f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$

Composition

We have

$$g \circ f(x) = g\left(x^2 + 1\right) = \sqrt{x^2 + 1}$$

while

$$f \circ g(x) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 + 1 = |x| + 1$$

Note: For $f \circ g(x)$, the domain of f can only include points from the range of g.

Composition of Functions Example

The Functions

$f(x) = x^2$ and $g(x) = \sqrt{x}$

Composition

We obtain

$$g \circ f(x) = g\left(x^2\right) = \sqrt{x^2} = |x|$$

and

$$f \circ g(x) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = |x|$$

Although $f \circ g(x)$ and $g \circ f(x)$ appear to be identical functions, their natural domains and ranges could be different. $g \circ f$ has natural domain $(-\infty, \infty)$ and range $[0, \infty)$ whereas $f \circ g$ has natural domain $[0, \infty)$ and range $[0, \infty)$.

Outline

Inverting Functions

Context

The functions $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$ "cancel each other out" in the sense that

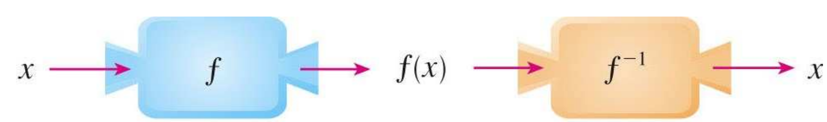
$$f \circ g(x) = f\left(x^{\frac{1}{3}}\right) = \left(x^{\frac{1}{3}}\right)^3 = x$$

and

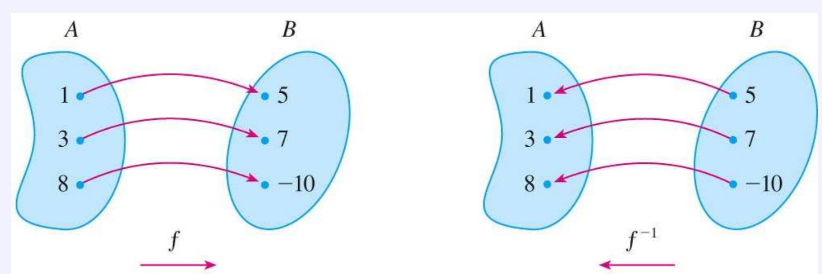
$$g \circ f(x) = g\left(x^3\right) = \left(x^3\right)^{\frac{1}{3}} = x$$

i.e. $f \circ g$ and $g \circ f$ leave x unchanged — each behave as the identity function. We say that the functions f and g are the inverses of each other.

Inverting Functions



f^{-1} reverses the effect of f



Inverting Functions

Definition: Inverse of a Function

f is said to be an invertible function, with inverse f^{-1}, if the function f^{-1} exists and obeys the following property:

$$f^{-1} \circ f(x) = x$$

for all x in the domain of f.

- Note that f^{-1} does not mean $\frac{1}{f}$ but is simply notation to mean "the inverse function".
- Note also that the inverse f^{-1} must itself be a function, i.e. it must produce unique values.

Finding the Inverse of $f(x)$

The Function

Where it exists, find $f^{-1}(x)$ where $f(x) = \dfrac{x+1}{x-2}$.

Answer

We begin by writing $y = \dfrac{x+1}{x-2}$, from which we see that y is given explicitly in terms of x. To invert the function, we need to write x explicitly in terms of y. Cross-multiply and transpose as follows:

$$\begin{aligned} y(x-2) &= x+1 \\ \Rightarrow xy - 2y &= x+1 \\ \Rightarrow xy - x &= 2y+1 \\ \Rightarrow x(y-1) &= 2y+1 \;\Rightarrow\; x = \frac{2y+1}{y-1} \end{aligned}$$

Finding the Inverse of $f(x)$

Function & Inverse

$$y = \frac{x+1}{x-2} \qquad\qquad x = \frac{2y+1}{y-1}$$

Answer (Cont'd)

Since $y = f(x)$ then $x = f^{-1}(y)$ and so $f^{-1}(y) = \dfrac{2y+1}{y-1}$. For convenience, we can replace y by x (because it is conventional to express a function in terms of x not y) and write

$$f^{-1}(x) = \frac{2x+1}{x-1}$$

Note that the inverse function has domain $R \setminus \{1\}$.

Finding the Inverse of $f(x)$

The Function

$f(x) = (x-4)^2$ on the domain $[4, \infty)$.

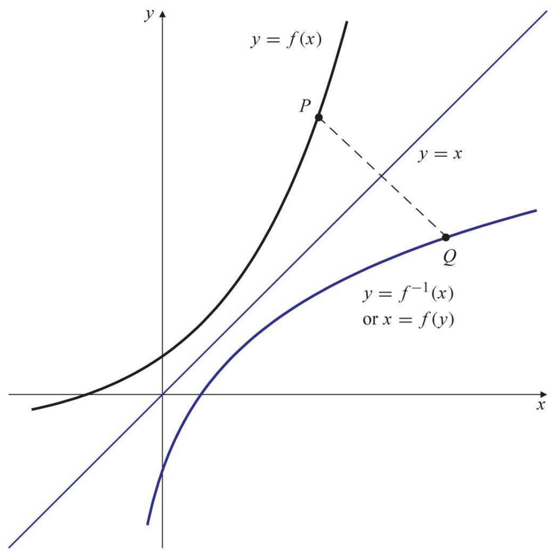
Answer

We write

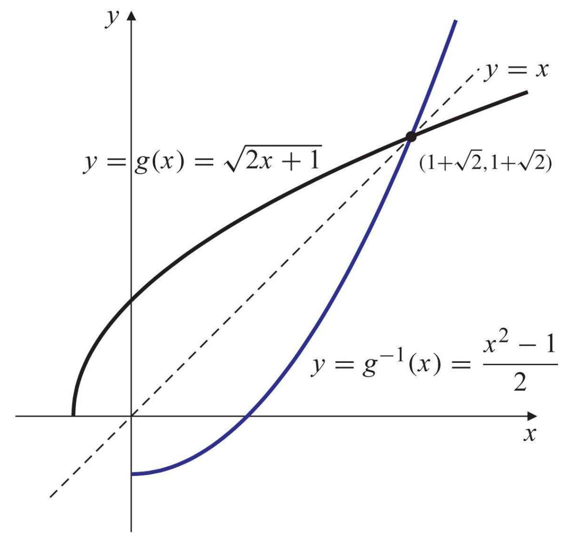
$$\begin{aligned} y &= (x-4)^2 \\ \Rightarrow +\sqrt{y} &= x-4 \qquad (+\sqrt{\ } \text{ since } x \geq 4) \\ \Rightarrow x &= 4+\sqrt{y} \\ \Rightarrow f^{-1}(x) &= 4+\sqrt{x} \end{aligned}$$

Note that f^{-1} has natural domain $[0, \infty)$ and range $[4, \infty)$ (which is the domain of f).

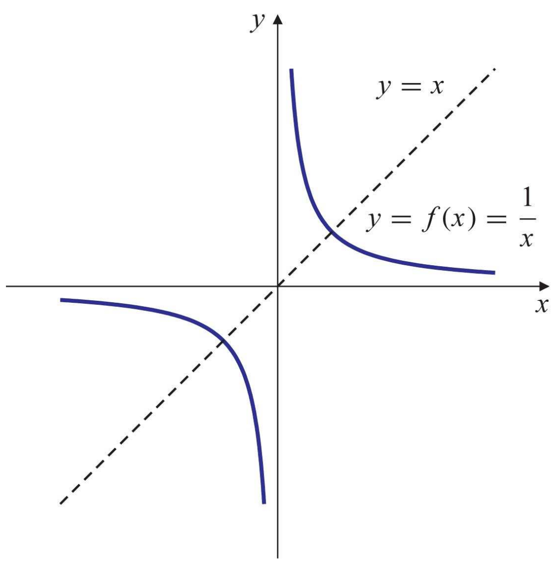
The Inverse Function: Conclusion (1/3)



The Inverse Function: Conclusion (2/3)



The Inverse Function: Conclusion (3/3)



MS121: IT Mathematics

Functions

Absolute Value & Inequalities

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Outline

Outline

Absolute Value & Inequalities

Illustration 1

The equation

$$|2x+1| = 3$$

has two solutions, obtained from solving

$$2x+1 = +3 \qquad \text{and} \qquad 2x+1 = -3$$

or $x = 1$ and $x = -2$. Therefore, the solution to the absolute value equation $|2x+1| = 3$ is the set:

$$\{-2, 1\}$$

Absolute Value & Inequalities

Illustration 2

Consider the equation

$$|x| \geq 2$$

Either $x \geq 2$ or $x \leq -2$ and the solution may be expressed in interval form as

$$(-\infty, -2] \cup [2, \infty)$$

Absolute Value & Inequalities

Solve the inequality $|x^2 - 4| > 3$.

The solutions are found from either:

$$\begin{array}{ccc} x^2 - 4 > 3 & \quad or \quad & x^2 - 4 < -3 \\ \downarrow & & \downarrow \\ x^2 > 7 & & x^2 < 1 \\ |x| > \sqrt{7} & & |x| < 1 \end{array}$$

giving 4 cases:

$$x > \sqrt{7}, \qquad x < -\sqrt{7}, \qquad x < 1, \qquad x > -1.$$

which can be written in interval form as follows:

$$x \in (-\infty, -\sqrt{7}) \cup (-1, 1) \cup (\sqrt{7}, \infty)$$

Absolute Value & Inequalities

Find all the real roots of the equation

$$|x + 1| + |2x - 3| = 4.$$

Since either $|x + 1| = x + 1$ or $-(x + 1)$ and $|2x - 3| = 2x - 3$ or $-(2x - 3)$, there are 4 cases to consider:

$$\begin{array}{lll} +(x+1) + (2x-3) = 4 & \Rightarrow \; 3x - 2 = 4 & \Rightarrow x = 2 \\ -(x+1) + (2x-3) = 4 & \Rightarrow \; x - 4 = 4 & \Rightarrow x = 8 \\ +(x+1) - (2x-3) = 4 & \Rightarrow \; -x + 4 = 5 & \Rightarrow x = 0 \\ -(x+1) - (2x-3) = 4 & \Rightarrow \; -3x + 2 = 4 & \Rightarrow x = -\dfrac{2}{3} \end{array}$$

The possible solutions are therefore:

$$\left\{2, 8, 0, -\frac{2}{3}\right\}$$

Absolute Value & Inequalities

Possible solutions: $\{2, 8, 0, -\frac{2}{3}\}$.

Answer (cont'd)

We now test each solution in the original inequality:

$$\begin{array}{rcll} |x+1| + |2x-3| & = & 4 & \\ |3| + |1| & = & 4 & \text{True} \quad x = 2 \\ |9| + |13| & \neq & 4 & \text{False} \quad x = 8 \\ |1| + |-4| & = & 4 & \text{True} \quad x = 0 \\ \left|\dfrac{2}{3}\right| + \left|-\dfrac{13}{3}\right| & \neq & 4 & \text{False} \quad x = -\dfrac{2}{3} \end{array}$$

Therefore, the solutions are $x = 0$ and $x = 2$.

Outline

1. Inequalities involving the Absolute Value
2. Solving More General Inequalities

Absolute Value & Inequalities

Question

Solve the inequality $x^2 - 5x + 6 \leq 0$.

Answer

First we factor the left side:

$$x^2 - 5x + 6 = (x-2)(x-3)$$

We know that the corresponding equation has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \qquad (2,3) \qquad (3, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Absolute Value & Inequalities

$x^2 - 5x + 6 \leq 0$ (Cont'd)

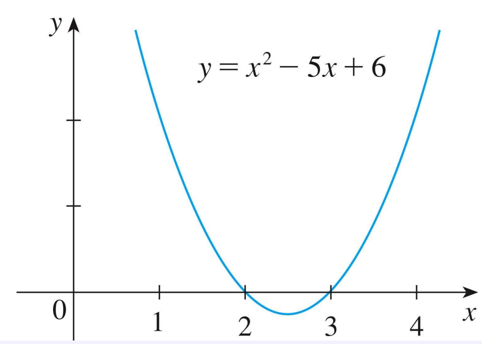
Then we record these signs in the following chart:

	$x-2$	$x-3$	$(x-2)(x-3)$
$(-\infty, 2)$	$-$	$-$	$+$
$(2,3)$	$+$	$-$	$-$
$(3,\infty)$	$+$	$+$	$+$

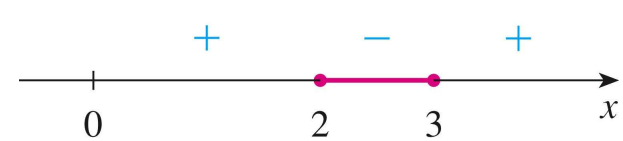
Therefore, $x^2 - 5x - 6 \leq 0$ has solution

$$\{x \mid 2 \leq x \leq 3\} = [2,3]$$

Absolute Value & Inequalities



$$x^2 - 5x - 6 \leq 0 \qquad \Rightarrow \qquad \{x \mid 2 \leq x \leq 3\} = [2,3]$$



Absolute Value & Inequalities

Question

Solve the inequality $\dfrac{(x+1)(x-3)}{(x+4)} > 0$.

Answer

Rather than attempt to sketch the function, we will first determine the points where it changes sign, i.e. those x-values which make the numerator or denominator zero:

$$\begin{aligned} x+1=0 &\Rightarrow x=-1 \\ x-3=0 &\Rightarrow x=3 \\ x+4=0 &\Rightarrow x=-4 \end{aligned}$$

We now divide the x-axis into intervals according to these points, namely $(-\infty,-4)$, $(-4,-1)$, $(-1,3)$ and $(3,\infty)$.

$$\frac{(x+1)(x-3)}{(x+4)} > 0$$

Answer (Cont'd)

We construct a table with these intervals and with the factors of the function as follows:

	$x+1$	$x-3$	$x+4$	$f(x)$
$(-\infty,-4)$	$-$	$-$	$-$	$-$
$(-4,-1)$	$-$	$-$	$+$	$+$
$(-1,3)$	$+$	$-$	$+$	$-$
$(3,\infty)$	$+$	$+$	$+$	$+$

We see that the function is positive on the two intervals $(-4,-1)$ and $(3,\infty)$. We therefore write

$$\frac{(x+1)(x-3)}{x+4} > 0 \quad \text{on} \quad (-4,-1) \cup (3,\infty)$$

MS121: IT Mathematics

Limits & Continuity

Rates of Change & Tangents

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Outline

1. Limits: Athletics Perspective
2. Rates of Change
3. Slope & Tangent

Outline

1. Limits: Athletics Perspective
2. Rates of Change
3. Slope & Tangent

Is there a limit to how fast a man can run?

3 Recent World Records

- Jamaica's Asafa Powell clocked 9.74 seconds for the Men's 100m sprint on September 9, 2007 in Rieti, Italy.
- This was followed by Usain Bolt's showboating Olympic run of 9.69 seconds in August 2008.
- His more recent assault on the track and field record book running 9.58 on 16 August 2009, when he reached a top speed of nearly 28 mph, has again raised the question if humans are fast approaching their physical limit?

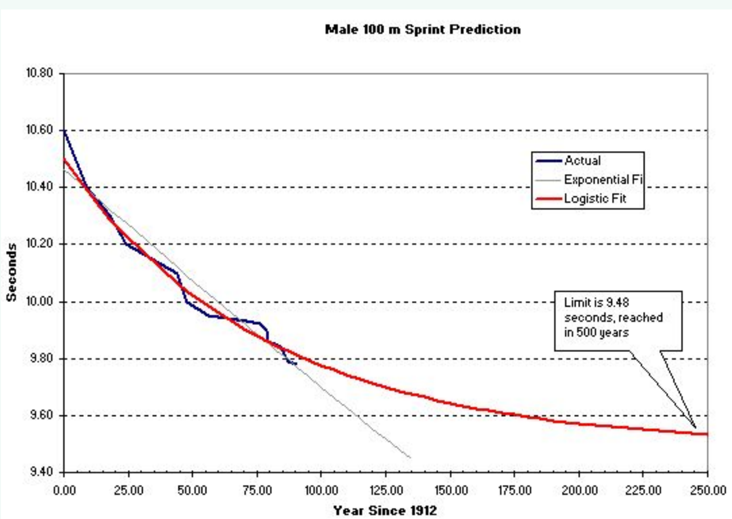
The Men's 100m World Record in 2008 & 2009

Is there a limit to how fast a man can run?

Some Observations

- A linear regression least-square error best-fit line over 90 years shows a slope of 7.7 millisecond fall per year.
- The correlation is 0.97, which is considered good for some applications (a best fit exponential gives virtually identical results).
- Even if this straight line trend were correct, it would imply that, in about 1,300 years, the 100m record would be run literally in no time flat (the record would be 0 seconds).
- More sensibly, a logistic equation reaches a limit of 9.48 seconds in 2412.

Men's 100m World Record in 2412?



Is there a limit to human performance?

Empirical Studies

- A 2008 study by the French Institute of Sport concluded that athletics will finally hit the ceiling in 2060. After that, no more world records.
- The institute analysed all 3,260 world records set since the first modern Olympics in 1896, and says that athletes are nudging their physiological limits.
- It estimates that athletes were operating at 75% of their potential in 1896, while in 2008, they were at 99%. By 2027, the athletes in about half of the events will have reached 100%, and by 2060 they all will.

Outline

Rates of Change

Calculus, to a great extent, is the study of the rate at which quantities change. We would like to answer questions like:

- how fast is a population growing?
- how fast is a car travelling?
- how fast will a ball travel if dropped from a height?

Rates of Change

Example

A rock breaks free from the top of the cliff. What is the average speed during the first two seconds of the fall?

- Experiments have shown that the rock will fall

$$s = f(t) = 16t^2$$

  feet in the first t seconds.

- The average speed in the first 2 seconds is the distance travelled divided by 2 *(length of time interval)*:

$$u = \frac{16 \star 2^2}{2} = \frac{64}{2} = 32\text{ft/sec}$$

Instantaneous Speed

What is the instantaneous speed at time $t = 2$ (secs)?

We can estimate it by working out the average speed over a shorter time period (including $t = 2$):

- from $t = 1$ to $t = 2$ is

$$\frac{f(2) - f(1)}{2 - 1} = \frac{16(2^2) - 16(1^2)}{1} = 48ft/sec;$$

- from $t = 1.9$ to $t = 2$ is $\frac{f(2) - f(1.9)}{2 - 1.9} = 62.4ft/sec$;
- from $t = 1.99$ to $t = 2$ is $\frac{f(2) - f(1.99)}{2 - 1.99} = 63.84ft/sec$.

In fact, the average speed is tending towards the instantaneous speed of $64ft/sec$ at $t = 2$

Instantaneous Speed

MS121 Perspective

This example encapsulates what we will do in calculus:

- The limit of the estimates is $64 ft/sec$.
- The speed is obtained from the derivative of f.
- The function

$$f(t) = 16t^2$$

is obtained using integration, using the fact that gravity causes the rock to accelerate at a rate of $32 ft/sec$

To Calculate the Instantaneous Speed

Using Average Spped

- We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$ having length $\Delta t = h$, as

$$\frac{\Delta s}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$$

- We cannot use this formula to calculate the *instantaneous* speed at t_0 by substituting $h = 0$ as we cannot divide by zero.
- As before, we calculate average speeds over increasingly short time intervals.

Instantaneous Speed

Increasingly shorter time intervals

h	1	0.1	0.01	0.001	0.0001
Av.speed	80	65.6	64.16	64.016	64.0016

$$\frac{\Delta s}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h}$$

$$= \frac{64h + 16h^2}{h} = 64 + 16h$$

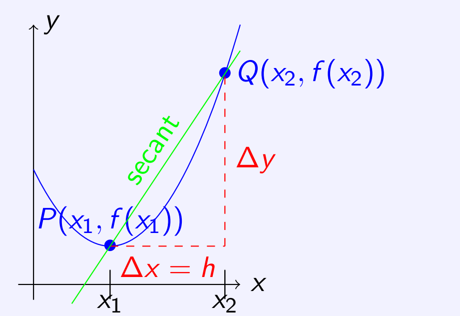
Observation

The average speed has the limiting value 64 as h approaches zero, at $t_0 = 2$ secs.

Average Rate of Change

The *average rate of change* of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0$$



A line joining two points of a curve is a *secant* to the curve — the average rate of change of f from x_1 to x_2 is the slope of secant PQ.

Average Rate of Change

Examples

(i) Find the average rate of change of $f(x) = \sqrt{4x+1}$ on the interval $[0,2]$.

$$\frac{\Delta y}{\Delta x} = \frac{f(2)-f(0)}{2-0} = \frac{3-1}{2} = 1$$

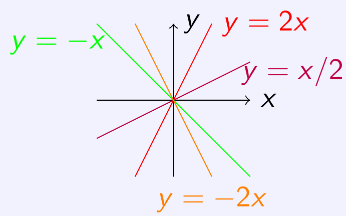
(ii) Find the average rate of change of $f(x) = x^3 - 4x^2 + 5x$ on the interval $[1,2]$.

$$\frac{\Delta y}{\Delta x} = \frac{f(2)-f(1)}{2-1} = 2-2 = 0$$

Outline

Slope

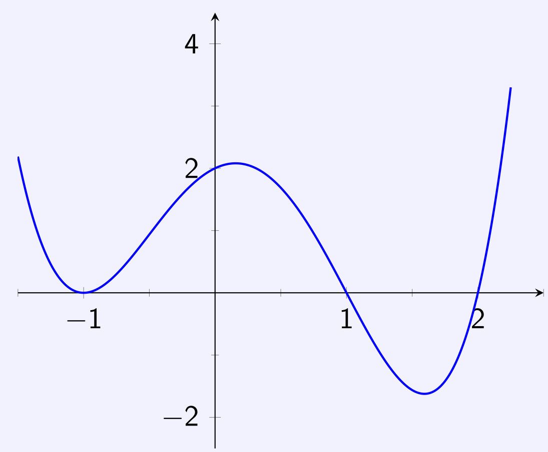
The graph of the function $y = mx + c$ is a straight line. The number m is called the slope and it tells us which way and by how much the line is tilted, i.e. it tells us whether y is *increasing or decreasing* as x increases



and how rapidly this occurs.

- $m = \frac{1}{2}$: y increases as x increases;
- $m = 2$: y increases more rapidly as x increases;
- $m = -1$: y decreases as x increases;
- $m = -2$: y decreases more rapidly as x increases.

Example: $f(x) = (x+1)^2(x-1)(x-2)$



Slope

Behaviour of $f(x) = (x+1)^2(x-1)(x-2)$

- When $x = -0.5,\ y$ is increasing
- When $x = 2.2,\ y$ is increasing more rapidly;
- When $x = 1,\ y$ is decreasing;
- When $x = -1.5,\ y$ is decreasing more rapidly.

The major difference here is that the "slope" may change from point to point.

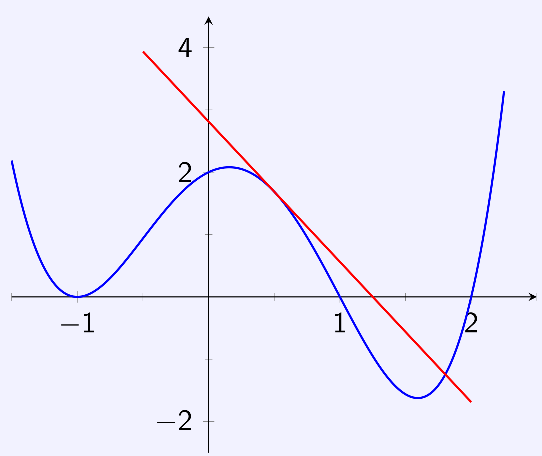
Tangents

Definition

- The *tangent* to a curve $\mathcal{C}$ at the point P on the curve is a straight line which touches $\mathcal{C}$ at P and may cross the curve at P but otherwise does not cross the curve near P.
- The *slope* of the curve at $x = c$ is equal to the slope of the tangent line T at $x = c$.

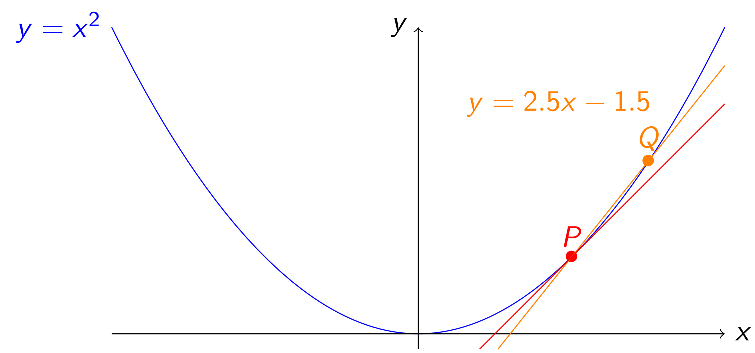
Tangent to $f(x) = (x+1)^2(x-1)(x-2)$ at $x = 0.5$

Slope of the Tangent $= -\frac{9}{4}$



How to find Tangents

- We want to find the tangent to $y = x^2$ at $P = (1, 1)$
- Take a point Q on the curve near P: i.e. $Q = (1.5, 2.25)$
- The line through P and Q is $y = 2.5x - 1.5$
- The line segment PQ is called a secant.

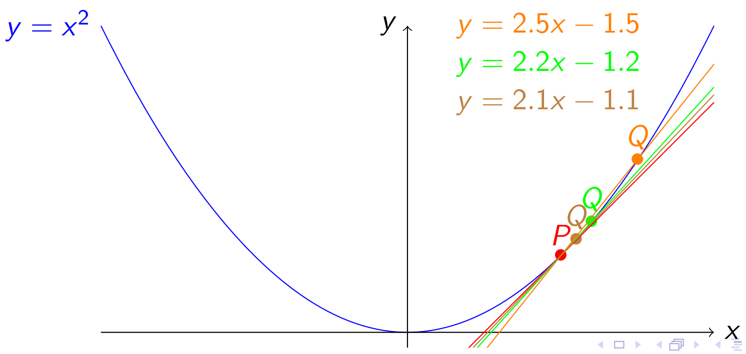


Find the Tangent: Let Q get closer to P

- Suppose that $Q = (1+h, (1+h)^2)$, where h is small and positive.
- The slope of the secant PQ is

$$\frac{\Delta y}{\Delta x} = \frac{(1+h)^2 - 1}{1+h-1} = \frac{2h+h^2}{h} = 2 + h.$$

So as h gets smaller the slope approaches 2.

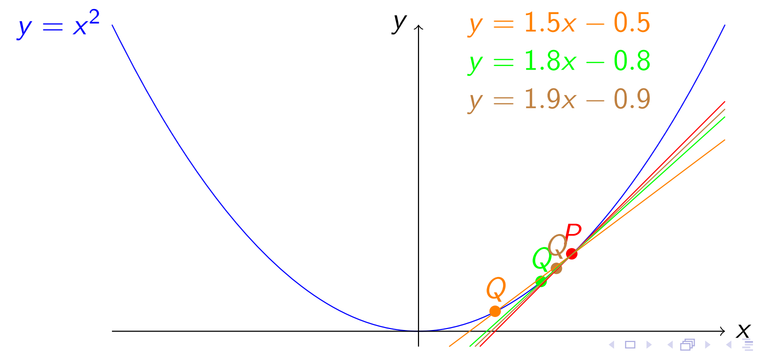


Let Q approaching P from the other direction

Assume that $Q = (1+h, (1+h)^2)$, where h is small and negative. The slope of the secant PQ is

$$\frac{\Delta y}{\Delta x} = \frac{(1+h)^2 - 1}{1+h-1} = \frac{2h+h^2}{h} = 2 + h.$$

So as h gets smaller the slope approaches 2.



Tangent to $y = x^2$: Conclusion

- If Q is close to P, then $Q = (1+h, (1+h)^2) = (1+h, 1+2h+h^2)$ where h is small.
- If $h > 0$, then Q lies above and to the right of P.
- If $h < 0$, then Q lies below and to the left of P.
- As Q approaches P, $h \to 0$.
- The slope of the secant PQ is

$$\frac{1+2h+h^2-1}{1+h-1} = \frac{2h+h^2}{h} = 2 + h.$$

- The slope of the tangent to $y = x^2$ at $x = 1$ is 2.
- The tangent passes through the point $P = (1, 1)$.
- The point-slope equation of the tangent is

$$y = 1 + 2(x-1) = 2x - 1.$$

MS121: IT Mathematics

Limits & Continuity

Introduction to Limits

John Carroll
School of Mathematical Sciences

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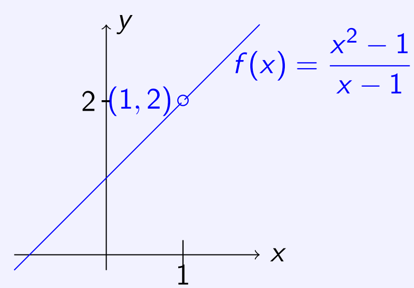
Outline

Outline

Limits of Function Values

Example: Rational Function

- Let $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x-1)(x+1)}{x-1}$
- Then f is not defined when $x = 1$.
- However for $x \neq 1$, $f(x) = x + 1$ and as x gets closer to 1, $f(x)$ gets closer to 2.
- So $\lim\limits_{x \to 1} f(x) = 2$.



Limits of Function Values

Table of Values of $f(x) = \frac{x^2-1}{x-1}$

x	$f(x)$
0.9	1.9
0.99	1.99
0.999	1.999
1.1	2.1
1.01	2.01
1.001	2.001

So we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1 or as

$$x \to 1 \qquad f(x) \to 2.$$

Outline

Limits of Function Values

Definition

- Let f be defined on an open interval about x_0, *except possibly at x_0 itself.*
- We say that f approaches the limit L (L a real number) as x approaches x_0 if, however small a distance we choose, $f(x)$ gets closer than this distance to L for x sufficiently close to (but not equal to) x_0. We write

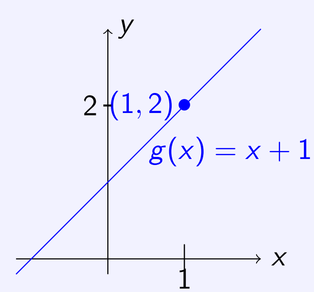
$$\lim_{x \to x_0} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to x_0.$$

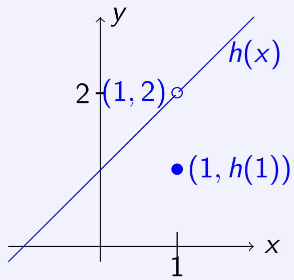
Example

- Let $g(x) = x + 1$.
- As x gets closer to 1, $g(x)$ gets closer to $2 = g(1)$.
- So $\lim_{x \to 1} g(x) = 2 = g(1)$.



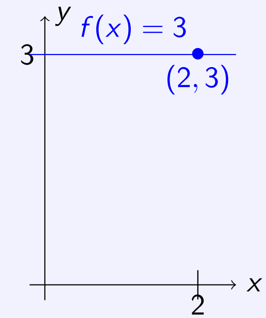
Example

- Let
$$h(x) = \begin{cases} x+1, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$$
- For $x \neq 1$, $h(x) = x + 1$ and as x gets closer to 1, $h(x)$ gets closer to 2.
- So $\lim_{x \to 1} h(x) = 2 \neq h(1)$.



Trivial Example

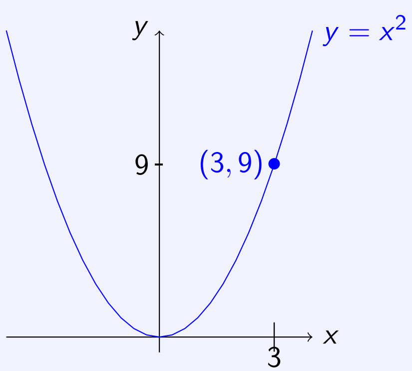
Let $f(x) = 3$, for all x. $\lim_{x \to 2} f(x) = 3$



Fact: If $f(x) = k$, for some constant k, then $\lim_{x \to x_0} f(x) = k$, for any x_0.

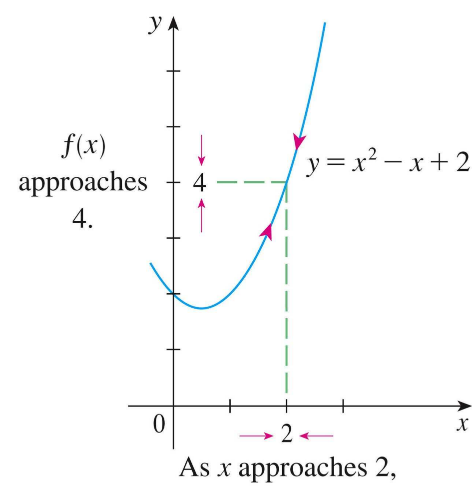
Example $f(x) = x^2$

$\lim_{x \to 3} x^2 = 9$

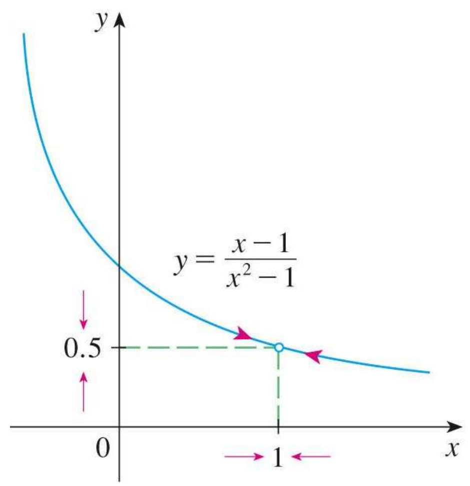


Fact: If f is any polynomial, then $\lim_{x \to x_0} f(x) = f(x_0)$ for any x_0.

$\lim_{x \to 2} (x^2 - x - 2) = 4$

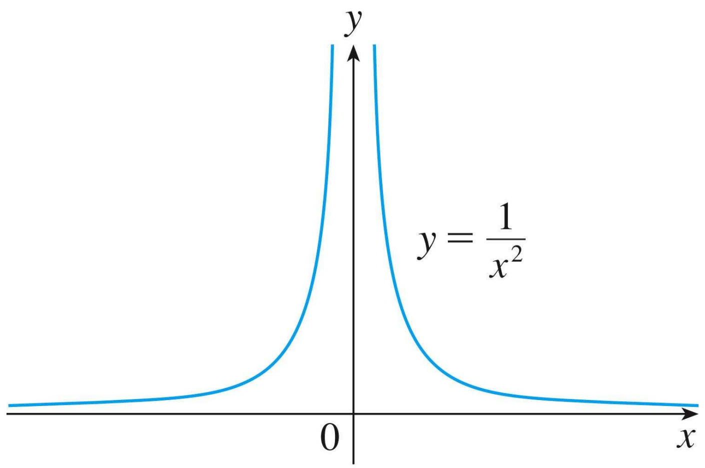


$$\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\frac{1}{2}$$

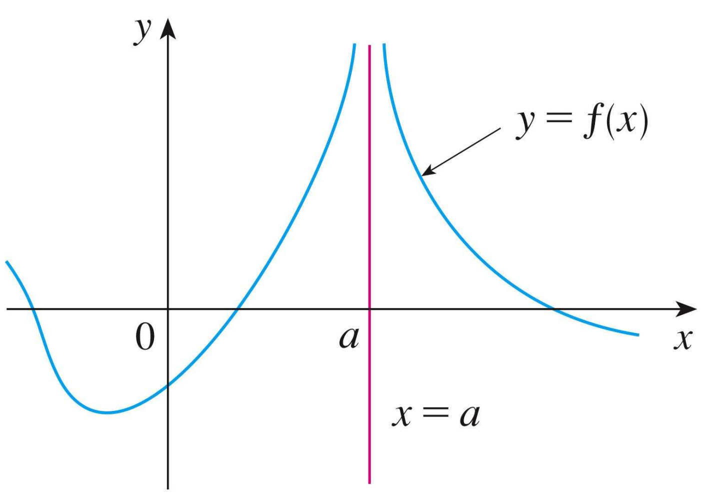


$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

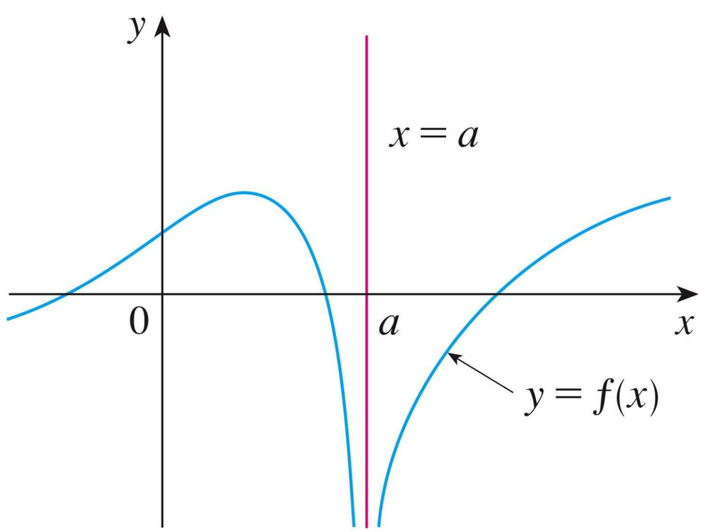
Limits do not always Exist!



$$\lim_{x\to a} f(x)=\infty$$



$$\lim_{x\to a} f(x)=-\infty$$



Outline

Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x\to c} f(x) = L, \qquad \lim_{x\to c} g(x) = M,$$

then

(i) Sum Rule: $\lim_{x\to c} f(x) + g(x) = L + M$.

(ii) Difference Rule: $\lim_{x\to c} f(x) - g(x) = L - M$.

(iii) Product Rule: $\lim_{x\to c} f(x)g(x) = LM$.

(iv) Constant Multiple Rule: $\lim_{x\to c} kf(x) = kL$.

(v) Quotient Rule: If $M \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Applying the Limit Laws

- If f is a polynomial, then $\lim_{x\to c} f(x) = f(c) = L$.
- $\lim_{x\to c} x^3 - 4x^2 - 3 = c^3 - 4c^2 - 3$.
- If f and g are polynomials and $g(c) \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$$

- Rational function:

$$\lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

- Other functions, e.g.

$$\lim_{x\to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x\to -2} 4x^2 - 3} = \sqrt{13}$$

Sandwich Theorem

Definition

Suppose that

$$g(x) \leq f(x) \leq h(x)$$

for all x in some open interval containing c, except possibly at c itself.

Furthermore, suppose that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then

$$\lim_{x\to c} f(x) = L$$

One-sided Limits

- Ordinary limits are called *two-sided*. If f fails to have a two sided limit at c it may still have a *one-sided* limit — that is, as x approaches c from one side.
- If f is defined on an interval (c, b), where $c < b$, and $f(x)$ gets arbitrarily close to L as x approaches c from the right, then f has a right-hand limit L at c and we write

$$\lim_{x \to c^+} f(x) = L$$

- If f is defined on an interval (b, c), where $b < c$, and $f(x)$ gets arbitrarily close to L as x approaches c from the left, then f has a left-hand limit L at c and we write

$$\lim_{x \to c^-} f(x) = L$$

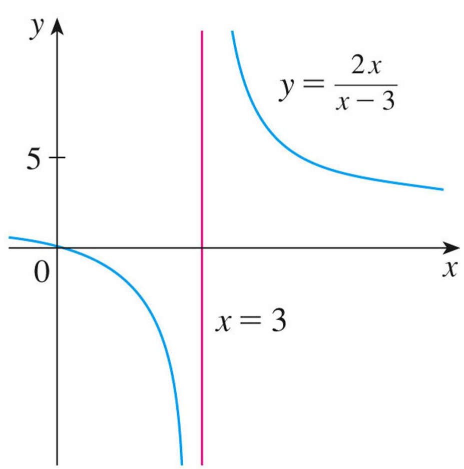
One-sided Limits (Cont'd)

Important Property

$\lim_{x \to c} f(x)$ exists if and only if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty \qquad \lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$



Outline

Eliminating Zero Denominators Algebraically

$$\lim_{x\to1}\frac{x^2+x-2}{x^2-x}$$

- We cannot substitute $x=1$ as it makes the denominator zero.
- We test the numerator to see if it is also zero at $x=1$ (if so, it has a factor of $(x-1)$ in common with the denominator):

$$1^2+1-2=0$$

- We can cancel the $(x-1)$ terms to get:

$$\lim_{x\to1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to1}\frac{(x-1)(x+2)}{x(x-1)}=\lim_{x\to1}\frac{x+2}{x}=3$$

Eliminating Zero Denominators Algebraically

$$\lim_{x\to-2}\frac{x^2+5x+6}{x^2-4}$$

- We cannot substitute $x=-2$ as it makes the denominator zero.
- We test the numerator to see if it is zero at $x=-2$:
- $(-2)^2+5(-2)+6=4-10+6=0$.
- We can cancel the $(x+2)$ terms to get:

$$\begin{aligned}\lim_{x\to-2}\frac{x^2+5x+6}{x^2-4}&=\lim_{x\to-2}\frac{(x+3)(x+2)}{(x-2)(x+2)}\\&=\lim_{x\to-2}\frac{x+3}{x-2}=-\frac{1}{4}\end{aligned}$$

Eliminating Zero Denominators Algebraically

$$\lim_{x\to2}\frac{x^3-8}{x^4-16}$$

- We cannot substitute $x=2$ as it makes the denominator zero.
- We test the numerator to see if it is zero at $x=2$:
- $2^3-8=8-8=0$.
- We can cancel the $(x-2)$ terms to get:

$$\begin{aligned}\lim_{x\to2}\frac{x^3-8}{x^4-16}&=\lim_{x\to2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x^3+2x^2+4x+8)}\\&=\lim_{x\to2}\frac{x^2+2x+4}{x^3+2x^2+4x+8}=\frac{12}{32}=\frac{3}{8}\end{aligned}$$

An Aside: Long Division Illustrations

Illustration 1

$$\begin{array}{r|rrrrrrr}
 & x^2 & + & x & - & 12 & & \\ \hline
x-3 & x^3 & - & 2x^2 & - & 15x & + & 36 \\
 & x^3 & - & 3x^2 & & & & \\ \cline{2-4}
 & & & x^2 & - & 15x & & \\
 & & & x^2 & - & 3x & & \\ \cline{4-6}
 & & & & - & 12x & + & 36 \\
 & & & & - & 12x & + & 36 \\ \cline{5-8}
 & & & & & & & 0
\end{array}$$

Long Division Cont'd

Illustration 2

$$
\begin{array}{r|rrrrrrr}
 & x^2 & - & 2x & - & 8 & & \\
\cline{2-8}
x+2 & x^3 & & & - & 12x & - & 16 \\
 & x^3 & + & 2x^2 & & & & \\
\cline{2-4}
 & & - & 2x^2 & - & 12x & & \\
 & & - & 2x^2 & - & 4x & & \\
\cline{4-6}
 & & & & - & 8x & - & 16 \\
 & & & & - & 8x & - & 16 \\
\cline{6-8}
 & & & & & & & 0
\end{array}
$$

Eliminating Zero Denominators Algebraically

Example: $\lim\limits_{x\to 2} \dfrac{x^3 - 2x^2 + x - 2}{2x^2 - x - 6}$

We factorise the numerator and denominator and simplify as follows:

$$\frac{x^3 - 2x^2 + x - 2}{2x^2 - x - 6} = \frac{(x-2)(x^2+1)}{(x-2)(2x+3)} = \frac{x^2+1}{2x+3}$$

The limit is then found from

$$\lim_{x\to 2} \frac{x^3 - 2x^2 + x - 2}{2x^2 - x - 6} = \lim_{x\to 2} \frac{x^2+1}{2x+3} = \frac{2^2+1}{2\cdot 2+3} = \frac{5}{7}.$$

Eliminating Zero Denominators Algebraically

Example: $\lim\limits_{x\to -1} \dfrac{x^2 + 2x + 1}{x^3 + 3x^2 + 3x + 1}$

We factorise the numerator and denominator and simplify as follows:

$$\frac{x^2 + 2x + 1}{x^3 + 3x^2 + 3x + 1} = \frac{(x+1)^2}{(x+1)^3} = \frac{1}{x+1}.$$

The limit is then found from

$$\lim_{x\to -1} \frac{x^2 + 2x + 1}{x^3 + 3x^2 + 3x + 1} = \lim_{x\to -1} \frac{1}{x+1} = \frac{1}{0} = \infty.$$

Outline

Special Solution Method: Rationalize the Numerator

Question

Evaluate $\lim\limits_{x \to 1} \dfrac{\sqrt{x^2+x+23}-5}{x-1}$

Formula Required

$$A - B = (A-B) \star \left\{ \frac{A+B}{A+B} \right\}$$

Special Solution Method: Rationalize the Numerator

Solution

Consider

$$\begin{aligned}
\frac{\sqrt{x^2+x+23}-5}{x-1} &= \frac{\sqrt{x^2+x+23}-5}{x-1} \cdot \left\{ \frac{\sqrt{x^2+x+23}+5}{\sqrt{x^2+x+23}+5} \right\} \\
&= \frac{x^2+x-2}{(x-1)\sqrt{x^2+x+23}+5} \\
&= \frac{x+2}{\sqrt{x^2+x+23}+5}
\end{aligned}$$

and so

$$\lim_{x \to 1} \frac{\sqrt{x^2+x+23}-5}{x-1} = \frac{1+2}{\sqrt{1^2+1+23}+5} = \frac{3}{10}$$

Outline

Piecewise-Defined Functions

Let

$$f(x) = \begin{cases} \dfrac{1}{2-3x}, & \text{if } x < -3 \\[2ex] x+2, & \text{if } x \geq -3 \end{cases}$$

Find the left and righthand limits at -3. Does $\lim\limits_{x \to -3} f(x)$ exist?

$$\begin{aligned}
\lim_{x \to -3^-} f(x) &= \lim_{x \to -3^-} \frac{1}{2-3x} = \frac{1}{2-3(-3)} &= \frac{1}{11} \\
\lim_{x \to -3^+} f(x) &= \lim_{x \to 3^+} x+2 = -3+2 &= -1
\end{aligned}$$

As $\lim\limits_{x \to -3^-} f(x) \neq \lim\limits_{x \to -3^+} f(x)$, $\lim_{x \to -3} f(x)$ does not exist.

Piecewise-defined Functions

Let

$$f(x) = \begin{cases} 5-3x & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ \sqrt{2x+2} & \text{if } x > 1 \end{cases}$$

Find the left and righthand limits at 1. Does $\lim\limits_{x \to 1} f(x)$ exist?

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 5-3x \quad = \quad 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{2x+2} = \sqrt{4} = 2$$

As $\lim\limits_{x \to 1^-} f(x) = 2 = \lim\limits_{x \to 1^+} f(x)$, $\lim_{x \to 1} f(x) = 2$.

MS121: IT Mathematics

Limits & Continuity

Limits at Infinity

John Carroll
School of Mathematical Sciences

Dublin City University

Outline

1. Overview
2. Rules for Limits
3. Rational Functions: Illustration
4. Limits at Infinity: Worked Examples
5. Limits at Infinity: A Special Case

Outline

1. Overview
2. Rules for Limits
3. Rational Functions: Illustration
4. Limits at Infinity: Worked Examples
5. Limits at Infinity: A Special Case

Limits as $x \to \infty$

Introduction

- So far, we have only considered limits as $x \to c$ where c is some finite value.
- We now examine what happens to functions as x becomes infinitely large, i.e. as $x \to \infty$.

Limits as $x \to \infty$

Simple Illustration

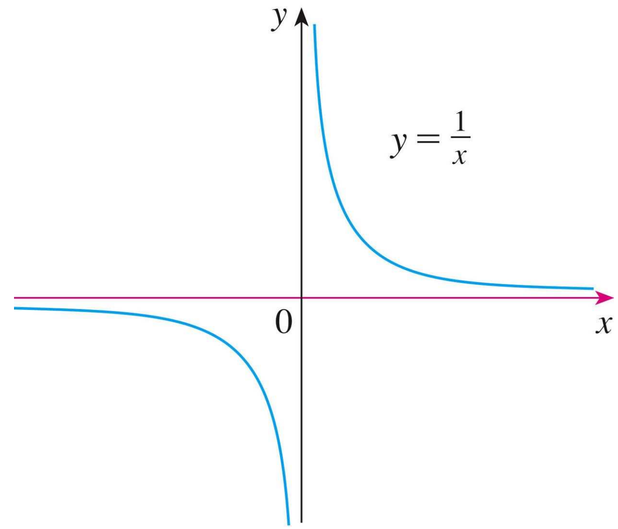
One example to consider first is $f(x) = \frac{1}{x}$.

x-value	$f(x)$-value
10	0.1
100	0.01
1000	0.001
1000000	0.000001

As x becomes infinitely large, then $\frac{1}{x}$ becomes smaller and smaller, and we write:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0 \qquad \lim_{x\to\infty} \frac{1}{x} = 0$$



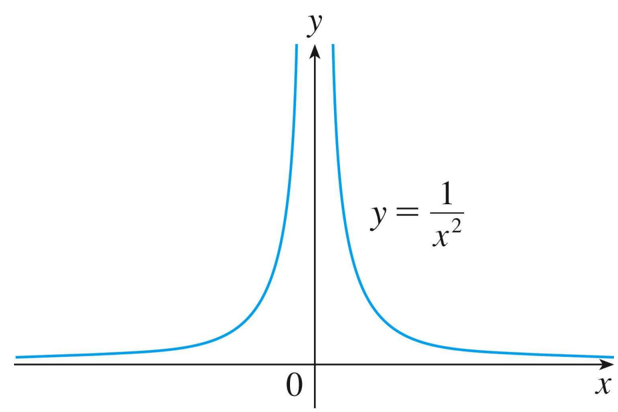
Limits as $x \to \infty$

A Second (Comparative) Illustration

- Consider the evaluation of $\lim_{x\to\infty} \frac{1}{x^2}$.
- As x becomes large, then $\frac{1}{x}$ becomes small but x^2 is even larger than x and so $\frac{1}{x^2}$ is even smaller than $\frac{1}{x}$.
- Hence, we conclude that

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x^2} = 0 \qquad \lim_{x\to\infty} \frac{1}{x^2} = 0$$



Limits as $x \to \infty$

More Generally

- The more general result
$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$
holds for any $n > 0$.
- Note that n can be less than 1, for example $n = \frac{1}{2}$ when we may write
$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Limits as $x \to \infty$

Making Comparisons

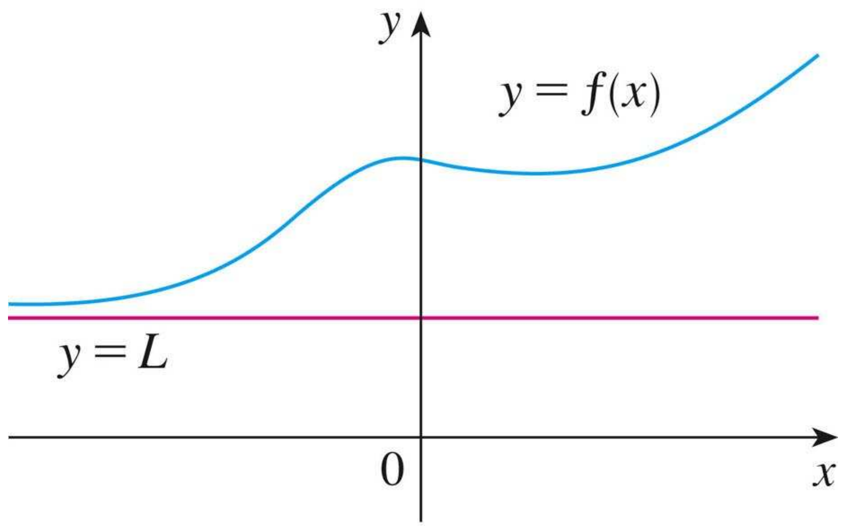
- If, instead, we require $\lim_{x\to\infty} \frac{1}{x-1}$, we can reason as follows:
- If x is infinitely large, then $x-1$ is also infinitely large, and dividing by an infinitely large number produces an infinitely small number, and so that the limit must be zero.
- We could also proceed as follows: divide above and below by x:
$$\frac{1}{x-1} = \frac{\frac{1}{x}}{1-\frac{1}{x}}$$
- Now, take the limit as $x \to \infty$:
$$\lim_{x\to\infty} \frac{1}{x-1} = \frac{\lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x}} = \frac{0}{1-0} = 0$$

Outline

Finite Limits as $x \to \pm\infty$

- The function f has a real limit L as x tends to ∞ if, however small a distance we choose, $f(x)$ gets closer than this distance to L and stays closer, no matter how large x becomes and we write
$$\lim_{x\to\infty} f(x) = L \text{ or } f(x) \to L, \text{ as } x \to \infty$$
- The function f has a real limit L as x tends to $-\infty$ if, however small a distance we choose, $f(x)$ gets closer than this distance to L and stays closer, no matter how large x and negative becomes and we write
$$\lim_{x\to-\infty} f(x) = L \text{ or } f(x) \to L, \text{ as } x \to -\infty$$

$\lim_{x\to-\infty} f(x) = L$



Rules for Limits as $x \to \pm\infty$

If L, M and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L, \qquad \lim_{x\to\pm\infty} g(x) = M \qquad \text{then}$$

(i) Sum Rule: $\lim_{x\to\pm\infty} f(x) + g(x) = L + M$.

(ii) Difference Rule: $\lim_{x\to\pm\infty} f(x) - g(x) = L - M$.

(iii) Product Rule: $\lim_{x\to\pm\infty} f(x)g(x) = LM$.

(iv) Constant Multiple Rule: $\lim_{x\to\pm\infty} kf(x) = kL$.

(v) Quotient Rule: If $M \neq 0$, then $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$.

(vi) Power Rule: If r and s are integers with no common factors and $s \neq 0$, then $\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$, provided that $L^{r/s}$ is a real number.

Outline

Rational Functions:Limits as $x \to \pm\infty$

Simple Rational Function

- Let $f(x) = x^2 - 1$ and $g(x) = x^2 + 1$.
- Therefore,

$$\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$$

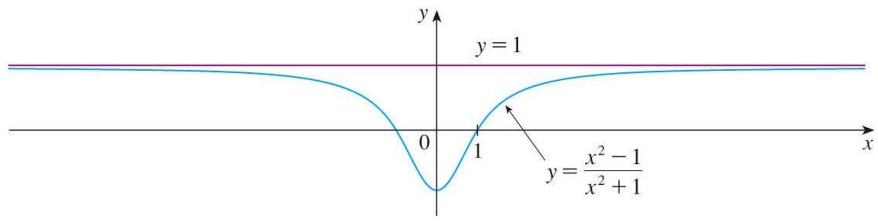
- Set

$$h(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 1}{x^2 + 1}$$

- What is

$$\lim_{x\to\infty} h(x)?$$

Plot of $\dfrac{x^2-1}{x^2+1}$



$\lim_{x\to\infty} \dfrac{x^2-1}{x^2+1}$

Rational Functions

- As we are assuming x is large (and hence non-zero), we can divide through by x^2 (the highest power of x occurring in the denominator) to get:

$$h(x) = \frac{x^2-1}{x^2+1} = \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}$$

- As $x \to \infty$, $\dfrac{1}{x^2} \to 0$ and $1 \to 1$.
- Therefore

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1}{1} = 1$$

Outline

Problem Solving

General Approach

We will evaluate some limits at infinity in the following examples by dividing above and below by the highest power of x in the original expression.

Example 1

To evaluate

$$\lim_{x\to\infty} \frac{1}{x^2-4}$$

we divide above and below by x^2 and take limits as follows:

$$\lim_{x\to\infty} \frac{1}{x^2-4} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{1-\frac{4}{x^2}} = \frac{0}{1-0} = 0$$

Example 2

To evaluate

$$\lim_{x\to\infty} \frac{x}{x^2-4}$$

we again divide above and below by x^2 and take limits as follows:

$$\lim_{x\to\infty} \frac{x}{x^2-4} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{4}{x^2}} = \frac{0}{1-0} = 0$$

Note

Although the numerator x becomes infinitely large, the denominator x^2-4 was still infinitely larger than the numerator and so the overall ratio was zero in the limit.

Example 3

Consider

$$\lim_{x\to\infty} \frac{x^3+27x^2+1}{x^4+6}$$

Solution

We divide above and below by the highest power which, in this case, is x^4:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3+27x^2+1}{x^4+6} &= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{27}{x^2}+\frac{1}{x^4}}{1+\frac{6}{x^4}} \\ &= \frac{0+0+0}{1+0} \\ &= 0\end{aligned}$$

Example 4

Consider

$$\lim_{x\to\infty} \frac{2x^3+1}{3x^3+4x^2+2}$$

Solution

We divide above and below by the highest power which, in this case, is x^3, to obtain:

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^3+1}{3x^3+4x^2+2} &= \lim_{x\to\infty} \frac{2+\frac{1}{x^3}}{3+\frac{4}{x}+\frac{2}{x^3}} \\ &= \frac{2+0}{3+0+0} \\ &= \frac{2}{3}\end{aligned}$$

Example 5

To evaluate

$$\lim_{x\to\infty} \frac{2x+4x^2+x^5}{1+x^4}$$

note that the highest power is x^5 and so we obtain:

$$\begin{aligned}\lim_{x\to\infty} \frac{2x+4x^2+x^5}{1+x^4} &= \lim_{x\to\infty} \frac{\frac{2}{x^4}+\frac{4}{x^3}+1}{\frac{1}{x^5}+\frac{1}{x}} \\ &= \frac{0+0+1}{0+0} \\ &= \frac{1}{0} = \infty\end{aligned}$$

Example 6

Consider

$$\lim_{x\to\infty} \frac{\sqrt{x}+1}{x}$$

Solution

We divide above and below by the highest power, namely x, to obtain:

$$\lim_{x\to\infty} \frac{\sqrt{x}+1}{x} = \lim_{x\to\infty} \frac{\frac{1}{x^{\frac{1}{2}}}+\frac{1}{x}}{1} = \frac{0+0}{1} = 0$$

Note that

$$\frac{\sqrt{x}}{x} = \frac{x^{\frac{1}{2}}}{x} = \frac{1}{x^{\frac{1}{2}}}$$

Example 7

Consider

$$\lim_{x\to\infty} \frac{x^{\frac{5}{2}}+x}{3x^{\frac{5}{2}}+2x^2+1}$$

Solution

We divide above and below by the highest power, $x^{\frac{5}{2}}$, to obtain:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^{\frac{5}{2}}+x}{3x^{\frac{5}{2}}+2x^2+1} &= \lim_{x\to\infty} \frac{1+\frac{1}{x^{\frac{3}{2}}}}{3+\frac{2}{x^{\frac{1}{2}}}+\frac{1}{x^{\frac{5}{2}}}} \\ &= \frac{1+0}{3+0+0} = \frac{1}{3}\end{aligned}$$

Example 8

Consider

$$\lim_{x\to\infty} \frac{|x|}{x}$$

Solution

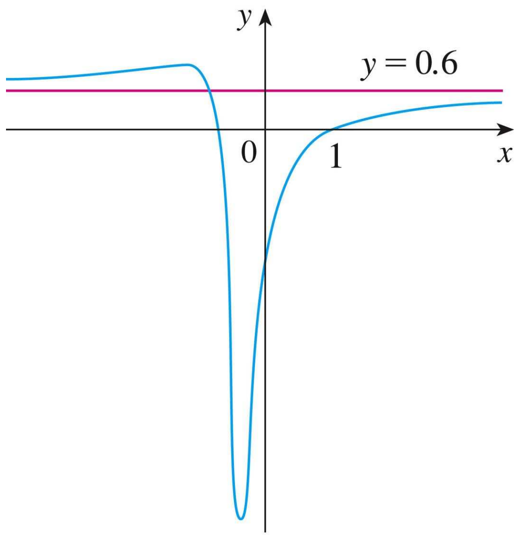
As $x \to +\infty$, then certainly $x > 0$ and, when $x > 0$, we have

$$\frac{|x|}{x} = 1$$

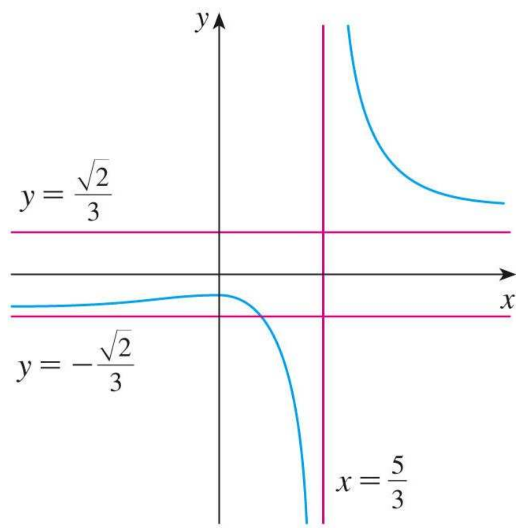
Hence, the limit which we require must be the limit of the constant value 1, i.e.

$$\lim_{x\to\infty} \frac{|x|}{x} = \lim_{x\to\infty} 1 = 1$$

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}=\frac{3}{5}$$

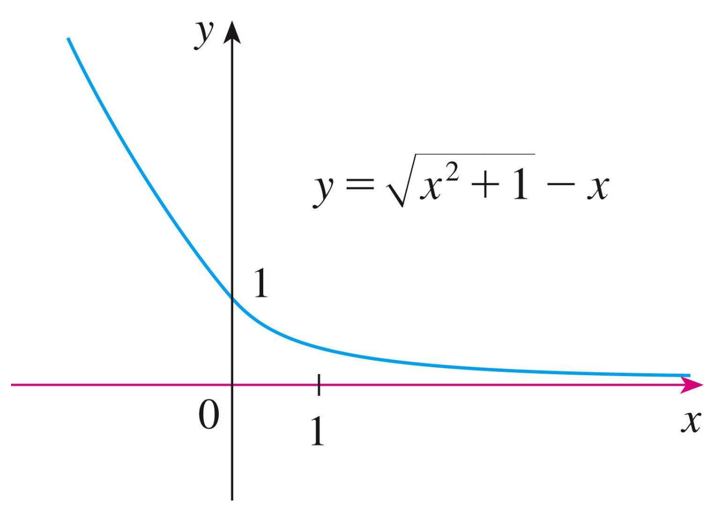


$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\frac{\sqrt{2}}{3}$$



Outline

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=0$$



Show that $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=0$

Rationalize the Numerator

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\star\left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right]\\
&= \lim_{x\to\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}\\
&= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}\\
&= 0
\end{aligned}$$

MS121: IT Mathematics

Limits & Continuity

Continuity

John Carroll
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Outline

1. Overview
2. Using Limits
3. Continuity Test
4. Examples

Outline

1. Overview
2. Using Limits
3. Continuity Test
4. Examples

Continuity

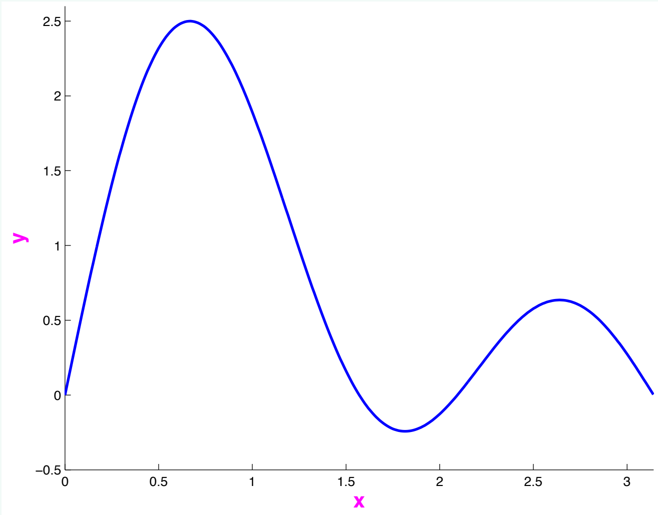
Informal Definition

An informal definition of a continuous function is that its graph contains no gaps.

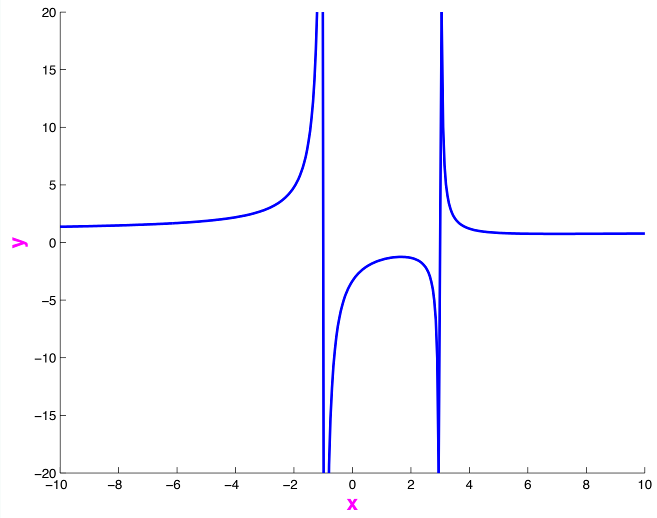
Graphical Interpretation

If the domain of $f(x)$ contains a neighbourhood of a fixed real number c, then the graph of f can be drawn through the point $(c, f(c))$ without lifting the pen from the paper.

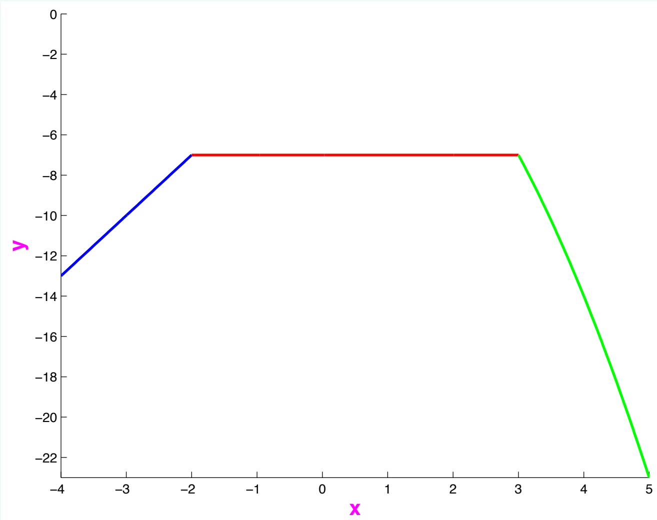
A Continuous Function



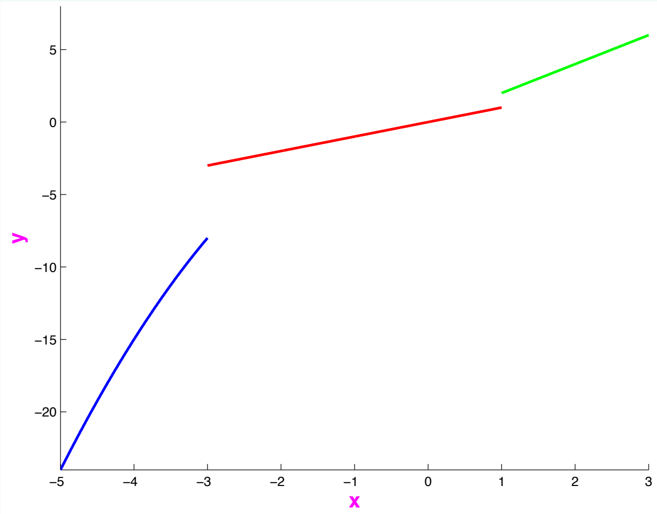
A Discontinuous Function



A “Piecewise” Continuous Function



A Discontinuous Function



Continuity

Use of Limits

We use limits to decide if a function is continuous or not. If we wish to determine whether or not a function is continuous at a given point, then we approach the point in question . . .

from the right of the point on the x-axis (from above)

and . . .

from the left of the point on the x-axis (from below)

to see if there are any jumps in the y-values.

Outline

Continuity

Use of Limits

Suppose we are interested in finding the limit of a function $f(x)$ at a point c. The expression

$$\lim_{x \to c^+} f(x)$$

means letting x approach the value c from the right (or from above) while the expression

$$\lim_{x \to c^-} f(x)$$

means letting x approach the value c from the left (or from below). If the direction of approach is not important, then we simply write

$$\lim_{x \to c} f(x)$$

Example 1

The function $\sqrt{4-x^2}$ has domain $[-2, 2]$. Its graph is a semicircle centred on the origin. Also

$$\lim_{x \to -2^+} \sqrt{4-x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4-x^2} = 0,$$

but the function does not have a left-sided limit at $x = -2$ nor a right-sided limit at $x = +2$.

The function $\sqrt{4-x^2}$ with domain $[-2, 2]$

One-sided & Two-sided Limits

Definition

A function $f(x)$ has a limit as x approaches c if and only if it has left-sided and right-sided limits there, and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x).$$

Outline

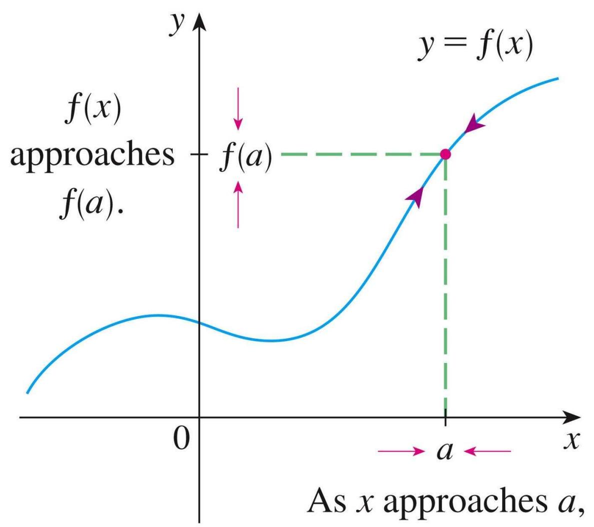
Continuous Function

Definition

A function f is continuous at an interior point $x = c$ of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

f is continuous at a



Continuity Test

Definition

A function f is continuous at $x = c$ if and only if it meets the following three conditions:

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value)

Outline

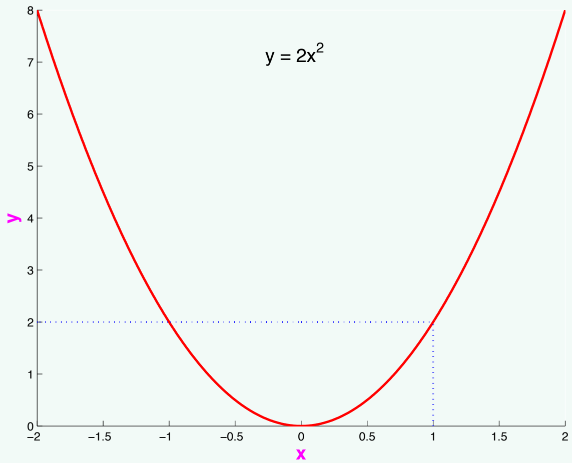
Example 2

We ask if the function $f(x) = 2x^2$ is continuous at the point $x = 1$?

Solution

- First, evaluate the function at $x = 1$, namely $f(1) = 2 \times (1)^2 = 2$.
- Next, investigate what happens to the $f(x)$-values as we approach x from either side of $x = 1$.

The function $2x^2$ on the domain $[-2, 4]$



$$f(x) = 2x^2$$

Example 2 (Cont'd)

Approaching from the right (from above), we find:

x-value	y-value
1.5	4.5
1.1	2.42
1.05	2.205
1.01	2.0402
1.001	2.004002
1.0001	2.000400
1.00001	2.000004 ...

$$f(x) = 2x^2$$

Example 2 (Cont'd)

Next, approaching $x = 1$ from the left (from below):

x-value	y-value
0.5	0.5
0.9	1.62
0.99	1.9602
0.999	1.996002
0.9999	1.999600
0.99999	1.999960

$$f(x) = 2x^2$$

Example 2 — Conclusion

- The pattern is clear. As the x-values approach $x = 1$ from both the right and the left of $x = 1$, the y-values approach the y-value which corresponds to $x = 1$, namely $y = 2$.
- We therefore say that the function $f(x) = 2x^2$ is continuous at $x = 1$.

Footnote

You could examine the function of the last example, namely $f(x) = 2x^2$ at any x-value and achieve the same result. The function is continuous everywhere.

Example 3

Is the function

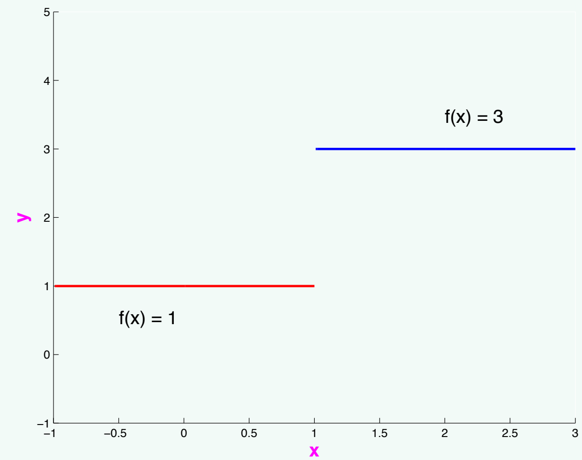
$$f(x) = \begin{cases} 3 & \text{if } x \geq 1 \\ 1 & \text{if } x < 1 \end{cases}$$

continuous at $x = 1$?

Solution

- First, evaluate the function at $x = 1$ to find $f(x) = 3$.
- Then, examine the function from the right and from the left.

A function with a discontinuity



$$f(x) = \begin{cases} 3 & \text{if } x \geq 1 \\ 1 & \text{if } x < 1 \end{cases}$$

Example 3 (Cont'd)

Approach $x = 1$ from the right:

x-value	y-value
1.5	3
1.1	3
1.001	3
1.00001	3

$$f(x) = \begin{cases} 3 & \text{if } x \geq 1 \\ 1 & \text{if } x < 1 \end{cases}$$

Example 3 (Cont'd)

And now approach $x = 1$ from the left:

x-value	y-value
0.5	1
0.9	1
0.999	1
0.99999	1

$$f(x) = \begin{cases} 3 & \text{if } x \geq 1 \\ 1 & \text{if } x < 1 \end{cases}$$

Example 3 — Conclusion

- It is clear that, as we approach $x = 1$ from the left, we are not approaching the desired y-value given by $f(1) = 3$.
- We conclude that the function is not continuous at $x = 1$.
- This is a point of discontinuity.
- Note, however, that the function is continuous everywhere else.

Continuous & Discontinuous Functions

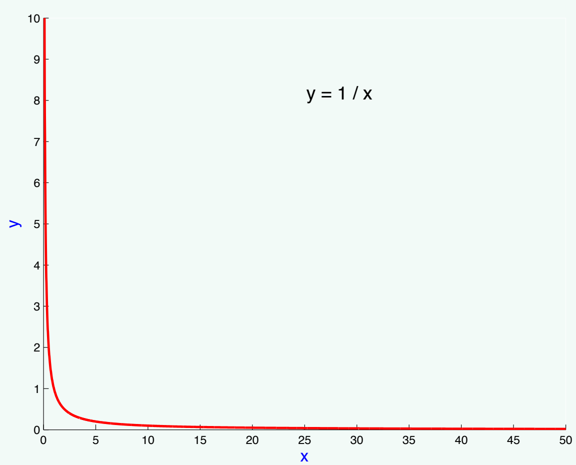
Context

- Some apparently simple functions are discontinuous.
- Consider, for example, the function $f : R \to R$, defined by

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  Note that $\frac{1}{x}$ is not defined at $x = 0$.
- A rough sketch of $f(x)$ will show that there is a gap in the graph at $x = 0$ and hence the function is discontinuous at $x = 0$.
- Furthermore, in this case, there is no value which we could assign to $f(x)$ at $x = 0$ to make the function continuous.

$f(x) = \dfrac{1}{x}$ is undefined at $x = 0$



Continuous & Discontinuous Functions

"Piecewise" Continuous Functions

We can examine more complicated functions which are constructed piecewise from continuous functions and determine whether these functions are continuous or discontinuous.

Example 4

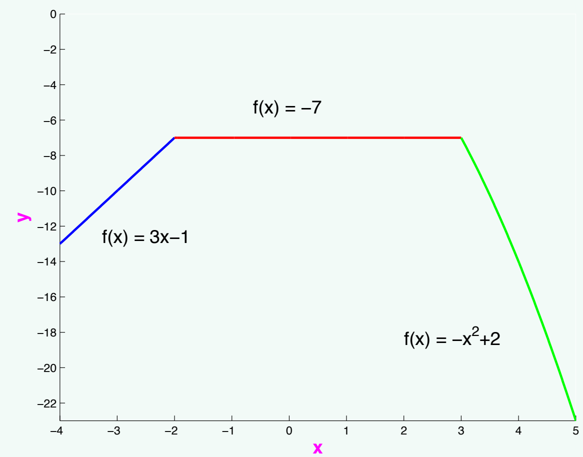
Consider the function $f : R \to R$, defined by

$$f(x) = \begin{cases} 3x - 1 & x \le -2 \\ -7 & -2 < x \le 3 \\ -x^2 + 2 & x > 3 \end{cases}$$

Method of Solution

We must examine the function at each of the “breakpoints”, that is, at $x = -2$ and $x = 3$.

A “piecewise” continuous function



$$f(x) = \begin{cases} 3x - 1 & x \le -2 \\ -7 & -2 < x \le 3 \\ -x^2 + 2 & x > 3 \end{cases}$$

Example 4 (Cont’d)

At $x = -2$:

$$\begin{aligned} \lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (3x - 1) &= -7 \\ \lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (-7) &= -7 \end{aligned}$$

Because the two limits are equal, the function is continuous at $x = -2$.

$$f(x) = \begin{cases} 3x - 1 & x \le -2 \\ -7 & -2 < x \le 3 \\ -x^2 + 2 & x > 3 \end{cases}$$

Example 4 (Cont’d)

At $x = 3$, we find

$$\begin{aligned} \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-7) &= -7 \\ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-x^2 + 2) &= -7 \end{aligned}$$

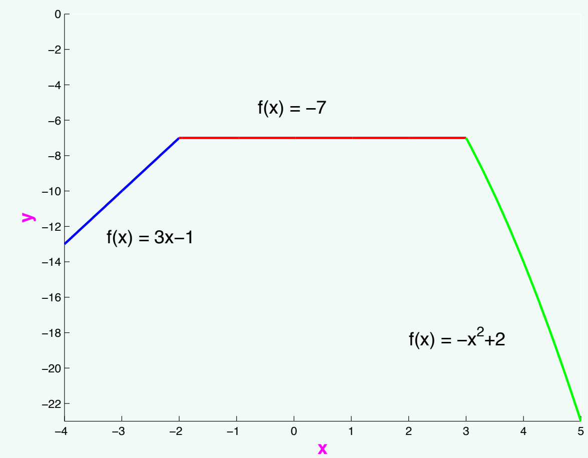
and so the function is also continuous at $x = 3$.

$$f(x) = \begin{cases} 3x - 1 & x \leq -2 \\ -7 & -2 < x \leq 3 \\ -x^2 + 2 & x > 3. \end{cases}$$

Example 4 — Conclusion

- The function is also continuous at every other value of x and hence we can say that $f(x)$ is continuous everywhere or simply "continuous".
- A sketch of $f(x)$ will show that, although there is a "sharp corner" at $x = -2$, there is no gap in the graph — confirming that the function is continuous.

A "piecewise" continuous function



Example 5

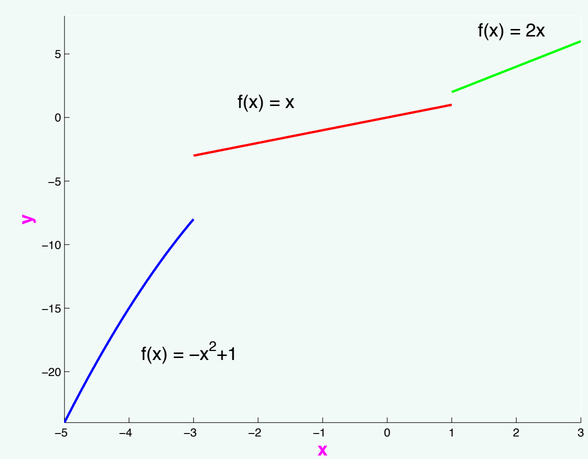
Consider the function $f : R \to R$,
defined by:

$$f(x) = \begin{cases} -x^2 + 1 & x < -3 \\ x & -3 \leq x < 1 \\ 2x & x \geq 1 \end{cases}$$

Method of Solution

We again examine the function at each of the "breakpoints", that is, at $x = -3$ and $x = 1$.

A "piecewise" discontinuous function



$$f(x) = \begin{cases} -x^2+1 & x < -3 \\ x & -3 \le x < 1 \\ 2x & x \ge 1 \end{cases}$$

Example 5 (Cont'd)

At $x = -3$, we find

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-x^2+1) = -8$$
$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} x = -3$$

Because the two limits are not equal, the function is discontinuous at $x = -3$.

$$f(x) = \begin{cases} -x^2+1 & x < -3 \\ x & -3 \le x < 1 \\ 2x & x \ge 1 \end{cases}$$

Example 5 (Cont'd)

At $x = 1$, we find

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x = 2$$

and again, as the two limits are not equal, we conclude that the function is also discontinuous at $x = 3$.

$$f(x) = \begin{cases} -x^2+1 & x < -3 \\ x & -3 \le x < 1 \\ 2x & x \ge 1 \end{cases}$$

Example 5 — Conclusion

- Even though the function is continuous at every other value of x, we say that $f(x)$ is discontinuous.
- A sketch of $f(x)$ will show clearly the two points of discontinuity, namely at $x = -3$ and at $x = 1$.

Example 6

Given

$$f(x) = \begin{cases} 2x - 2 & \text{if } x < -1 \\ Ax + B & \text{if } -1 \le x < 1 \\ 5x + 7 & \text{if } x > 1 \end{cases}$$

determine the constants A and B so that the function $f(x)$ is continuous for all real values of x.

Method of Solution

- The function $f(x)$ is clearly continuous for $x < -1$, for $-1 < x < 1$ and for $x > 1$.
- Examine the (left/right-sided) limits at the intersection points of the three intervals

$$f(x) = \begin{cases} 2x - 2 & \text{if } x < -1 \\ Ax + B & \text{if } -1 \leq x < 1 \\ 5x + 7 & \text{if } x > 1 \end{cases}$$

Example 6 (Cont'd)

At $x = -1$:

$$\lim_{x \to (-1)^-} 2x - 2 = -4$$

$$\lim_{x \to (-1)^+} Ax + B = -A + B$$

Continuity at $x = -1$ therefore requires that

$$-A + B = -4$$

$$f(x) = \begin{cases} 2x - 2 & \text{if } x < -1 \\ Ax + B & \text{if } -1 \leq x < 1 \\ 5x + 7 & \text{if } x > 1 \end{cases}$$

Example 6 (Cont'd)

At $x = 1$:

$$\lim_{x \to 1^-} Ax + B = A + B$$

$$\lim_{x \to 1^+} 5x + 7 = 12$$

Continuity at $x = 1$ therefore requires that

$$A + B = 12$$

$$f(x) = \begin{cases} 2x - 2 & \text{if } x < -1 \\ Ax + B & \text{if } -1 \leq x < 1 \\ 5x + 7 & \text{if } x > 1 \end{cases}$$

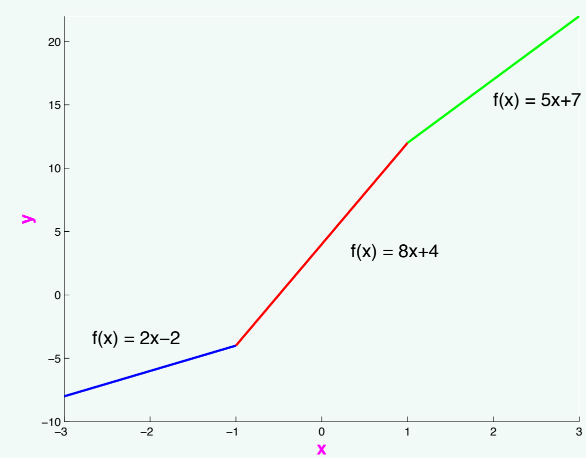
Example 6 (Cont'd)

$f(x)$ is continuous for all real x if and only if the left- and right-sided limits are equal in each case, i.e. if and only if

$$-A + B = -4 \qquad\qquad A + B = 12$$

Solving these two equations gives $A = 8$ and $B = 4$.

The resulting "piecewise" continuous function



MS121: IT Mathematics

Differentiation

Introduction

John Carroll
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Outline

Outline

Differentiation

What is Differentiation

- Differentiation is a mathematical technique for analysing the way in which functions change.
- In particular, it determines how rapidly a function is changing at any specific point.
- Differentiation also allows us to find maximum and minimum values of a function which can be useful in determining optimum values of key variables.

Differentiation from First Principles

Slope of a Line

Recall how we find the slope of a straight line connecting two points (x_1, y_1) and (x_2, y_2), given by

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

We could also write this as

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{dy}{dx}$$

Differentiation from First Principles

Slope of a Curve?

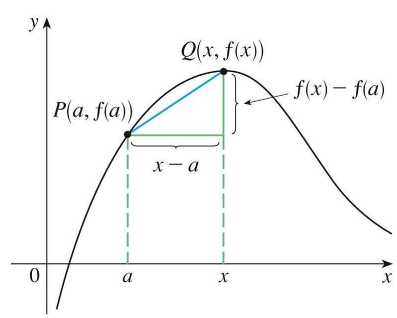
How do we define the slope of a curve? We define it in terms of the slope of a straight line which we already know.

Finding the Slope of a Curve

- Take any curve and the point at which the slope is required.
- At this point, draw the tangent to the curve.
- The tangent is a straight line which, at the point in question, appears to be parallel to the curve.
- The slope of the curve at this point is simply the slope of the tangent.

Outline

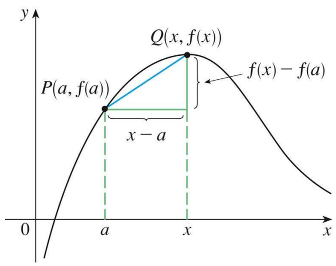
Differentiation: An Introduction — Tangents

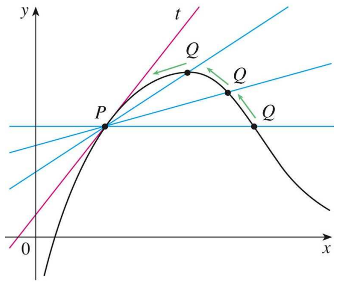


If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Differentiation: An Introduction — Tangents

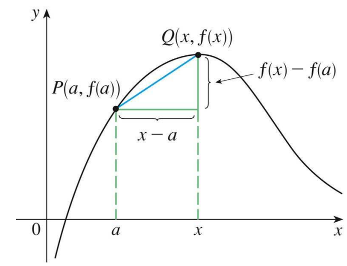


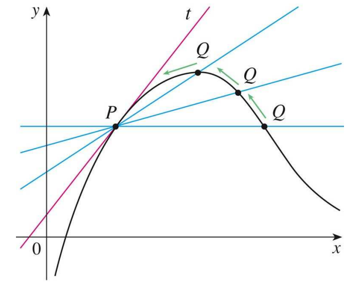


$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

- Then we let Q approach P along the curve C by letting x approach a.
- If m_{PQ} approaches a number m, then we define the tangent t to be the line through P with slope m.
- The tangent line is the limiting position of the secant line PQ as Q approaches P.

Differentiation: An Introduction — Tangents





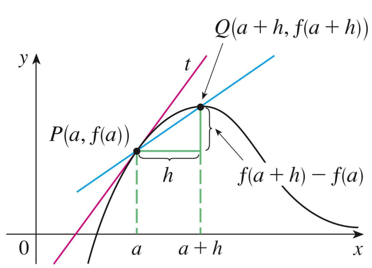
Definition

The tangent line to the curve $y = f(x)$ at the point at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Differentiation: An Introduction — Tangents



Easier Expression

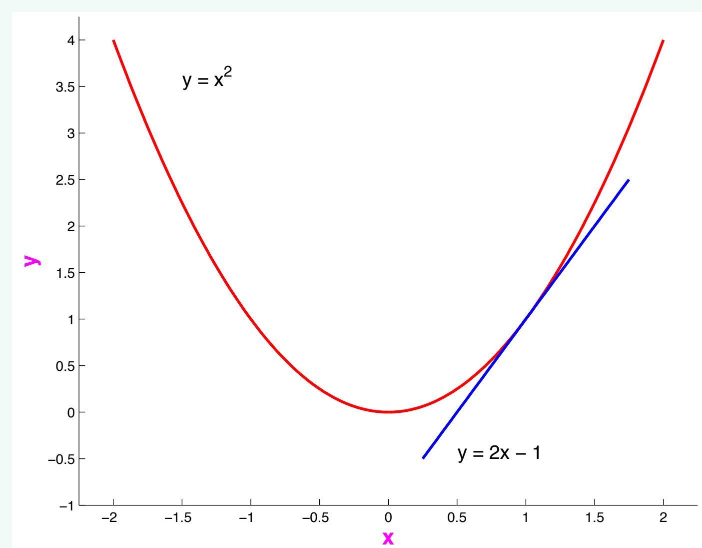
- If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

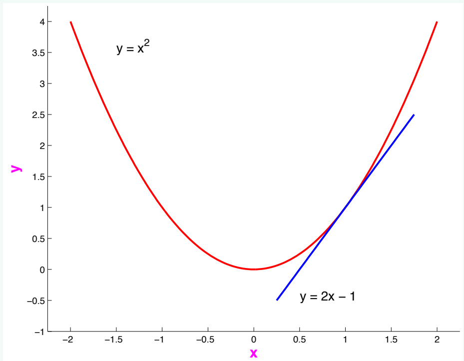
- As x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line becomes

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Tangent to $y = x^2$ at the point $x = 1$



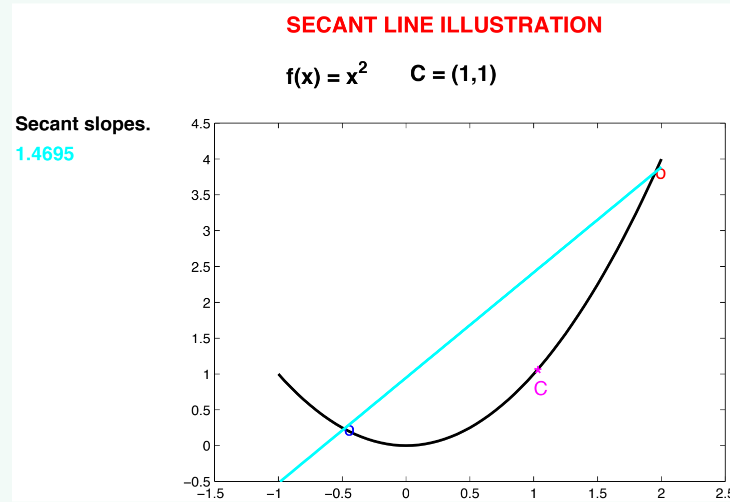
Differentiation from First Principles



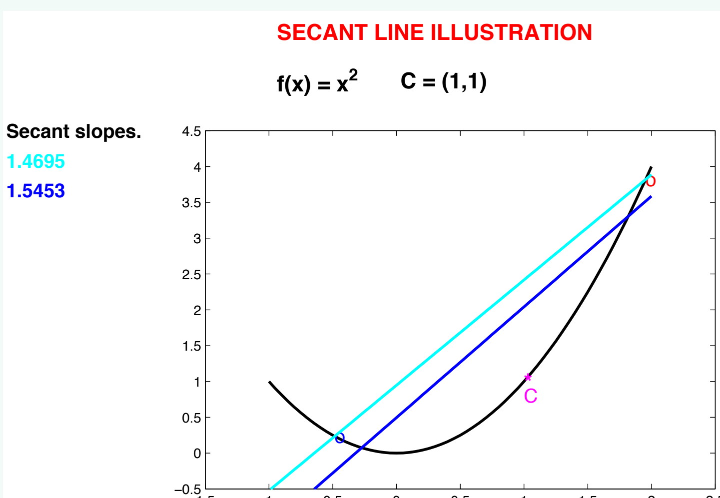
Slope of a Curve

As you will see from the graph, the slopes at different points on the curve will be different (unlike the straight line where the slope is always the same, no matter where it is evaluated).

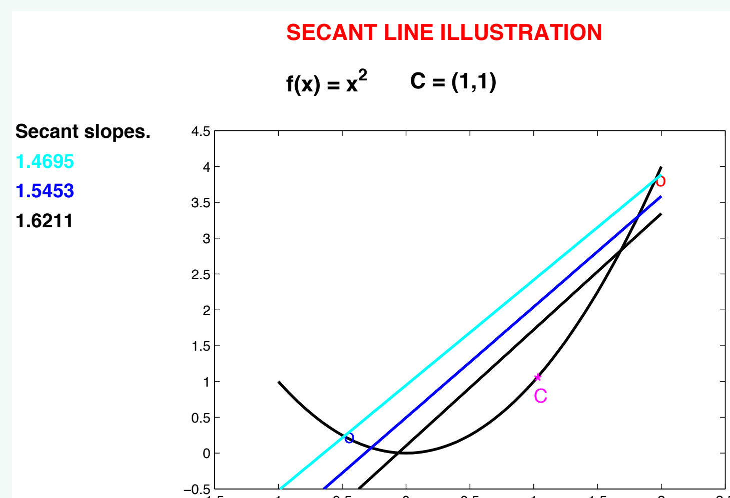
Limit of Secant Lines: $y = x^2$ at the point $x = 1$



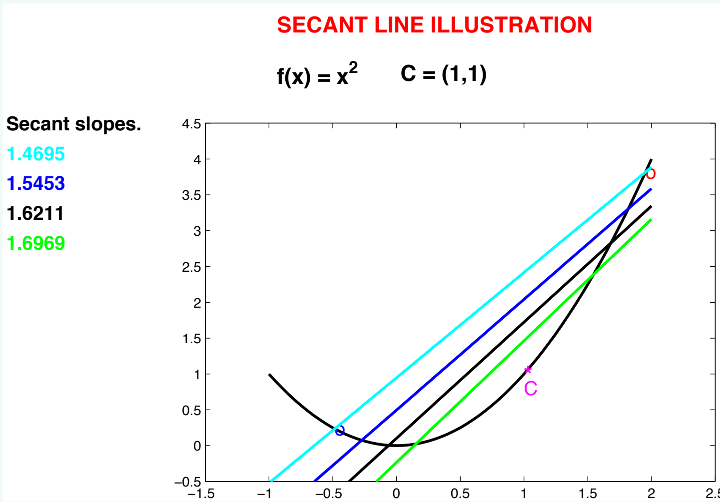
Limit of Secant Lines: $y = x^2$ at the point $x = 1$



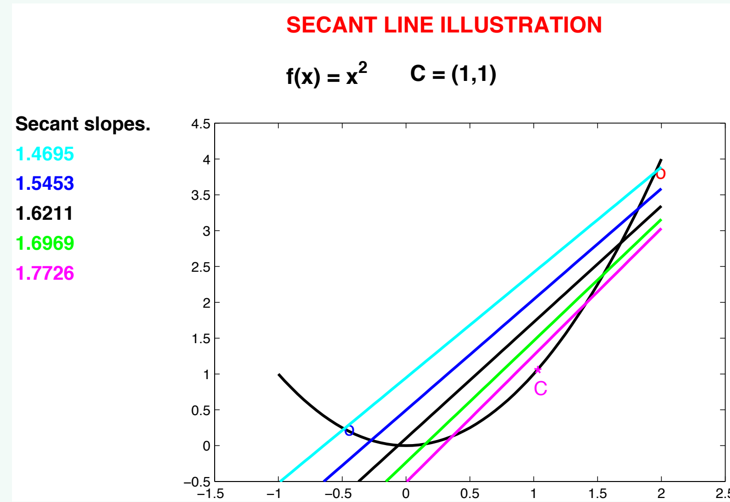
Limit of Secant Lines: $y = x^2$ at the point $x = 1$



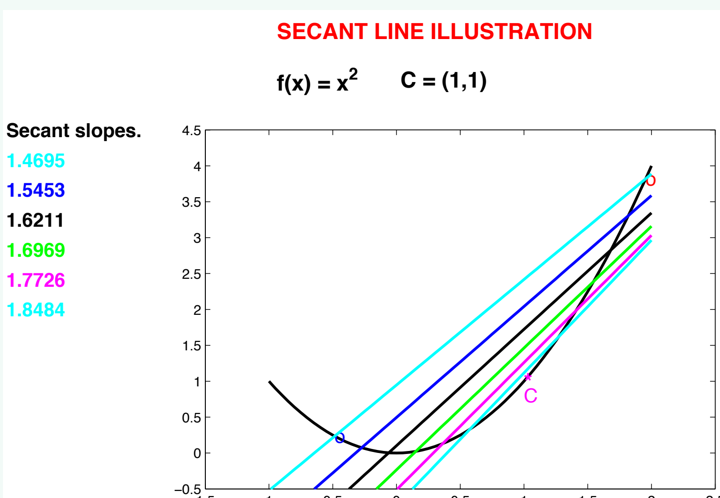
Limit of Secant Lines: $y = x^2$ at the point $x = 1$



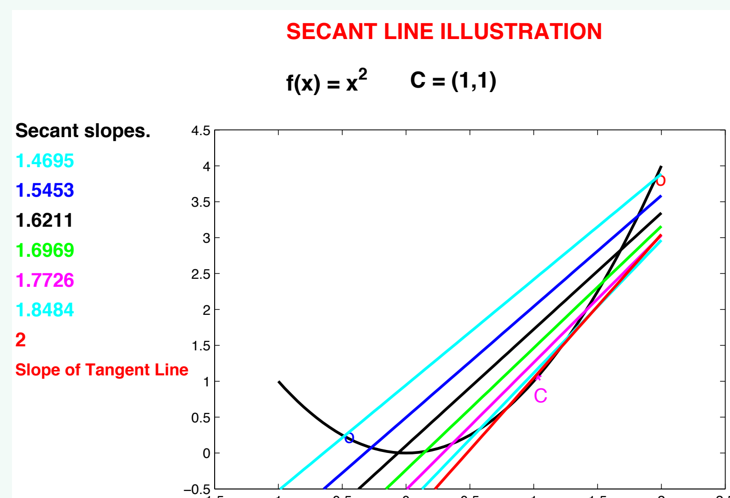
Limit of Secant Lines: $y = x^2$ at the point $x = 1$

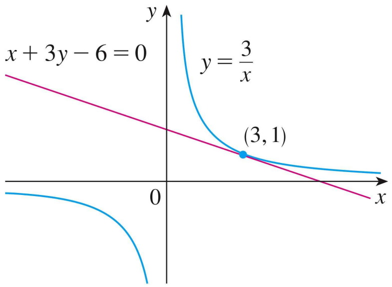


Limit of Secant Lines: $y = x^2$ at the point $x = 1$



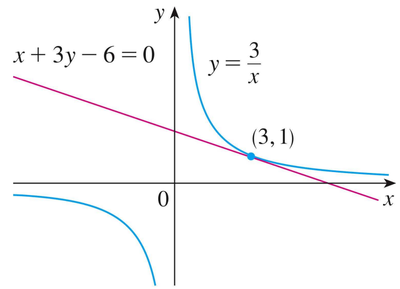
Limit of Secant Lines: $y = x^2$ at the point $x = 1$





Example

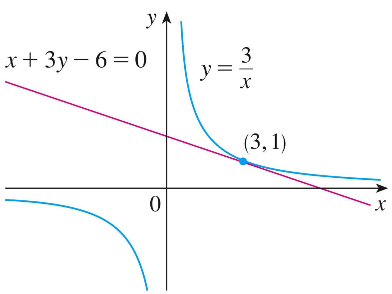
Find an equation of the tangent line to the hyperbola $y = \frac{3}{x}$ at the point $(3, 1)$.



Solution (1/2): Slope of Tangent

Let $f(x) = \frac{3}{x}$. Then the slope of the tangent at $(3, 1)$ is:

$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{3+h} - 1}{h} \\ &= \lim_{h\to 0} \frac{-1}{3+h} \\ &= -\frac{1}{3} \end{aligned}$$



Solution (2/2): Equation of Tangent

Therefore, an equation of the tangent at the point $(3, 1)$ is

$$y - 1 = -\frac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$

Outline

Differentiation from First Principles

Method of Differentiation

- We do not need to keep drawing tangents to find the slopes — we can use differentiation.
- The derivative (obtained by differentiation) of a curve at a point is the slope of the curve at that point.
- We shall first differentiate functions from first principles.

The Derivative of a Function

Definition

The derivative of the function $f : R \to R$ at a value $x = a$, if it exists, is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Differentiation from First Principles

Basic Approach

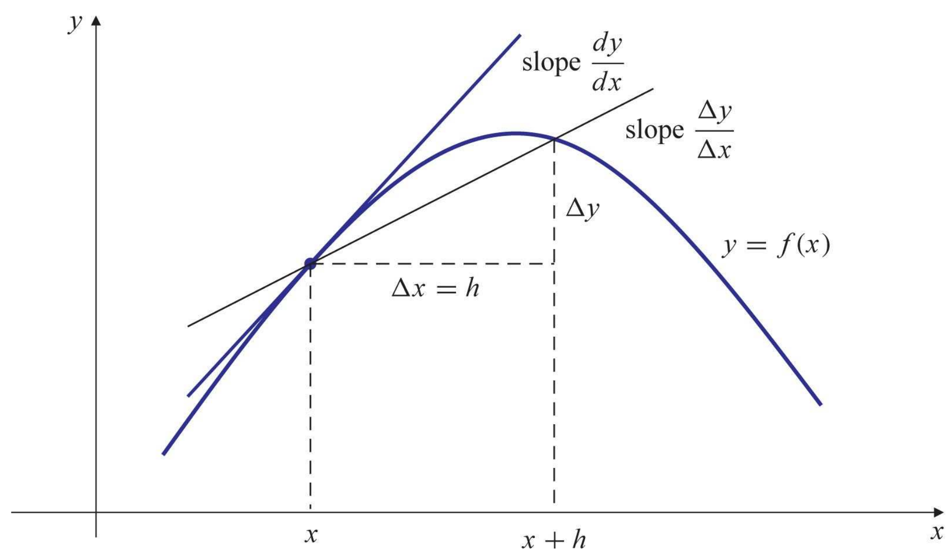
- The idea is to consider the line joining the two points $(a, f(a))$ and $(a+h, f(a+h))$ which, for h small, will approximate the tangent to the curve at the point $x = a$.
- Compute the slope of this line:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

and then take the limit as h tends to zero, giving

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Differentiation from First Principles



Outline

Example 1

We will find the derivative from first principles of $f(x) = x^2$ at $x = a$.

Solution

We will find the derivative from first principles of $f(x) = x^2$ at $x = a$. First compute

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a+h)^2 - a^2}{h} \\ &= \frac{(a^2 + 2ah + h^2) - a^2}{h} \\ &= \frac{2ah + h^2}{h} \\ &= 2a + h \end{aligned}$$

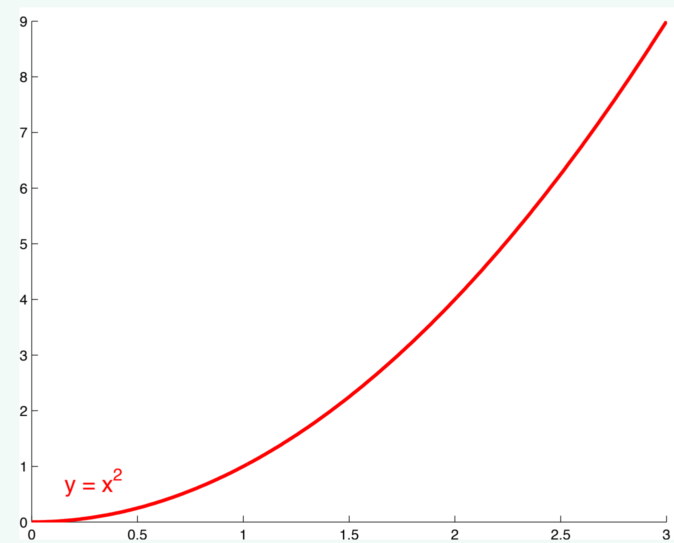
$$f(x) = x^2$$

Example 1 (Cont'd)

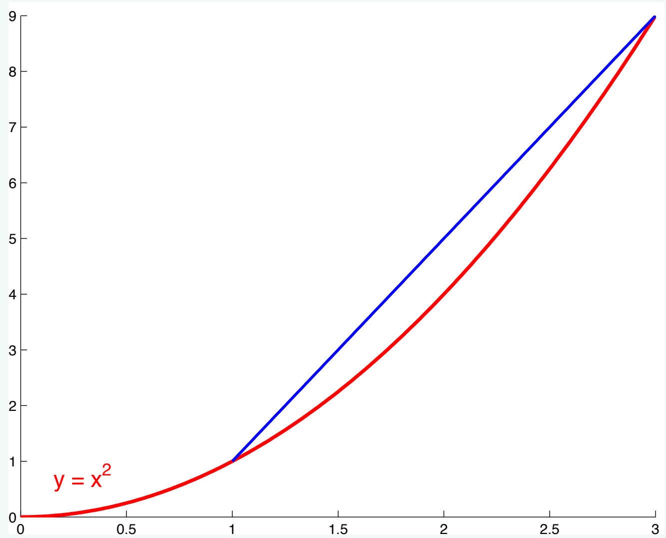
Then take limits to obtain

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2a + h) = 2a$$

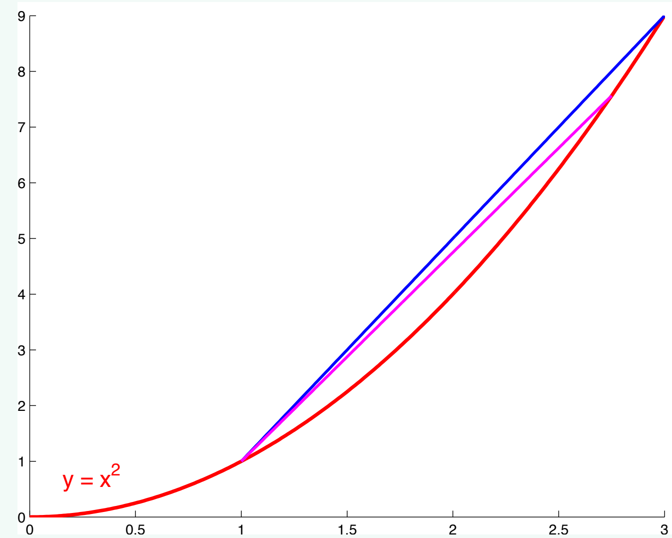
The function $y = x^2$ on the interval $[0, 3]$



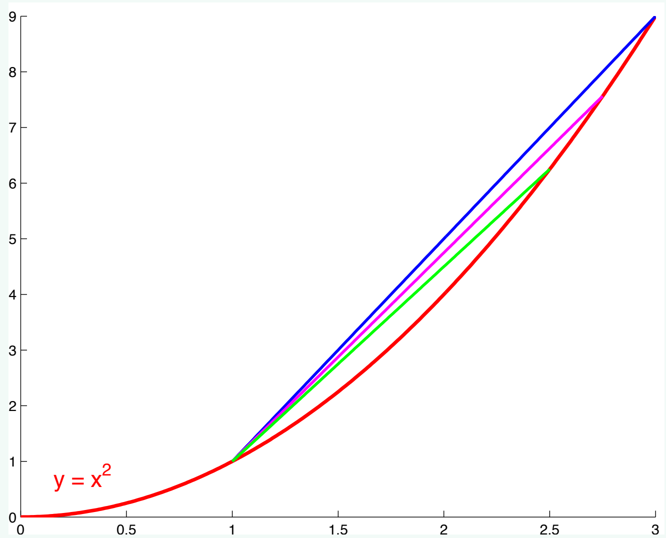
Function $y = x^2$ 1st Secant Line with $h = 2$



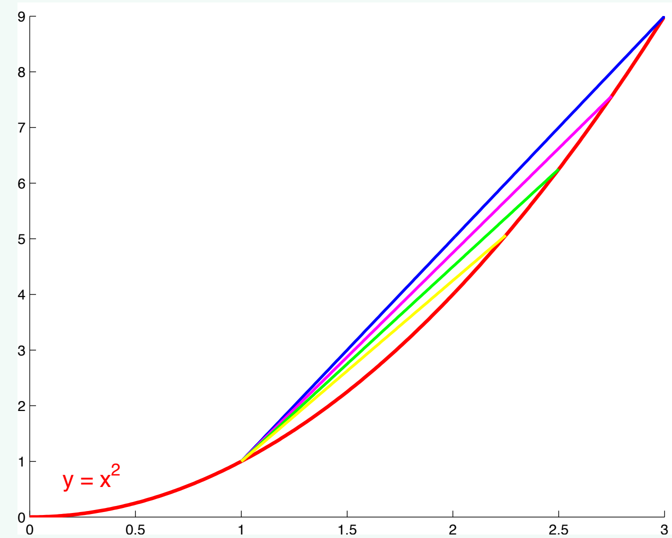
Function $y = x^2$ 2nd Secant Line with $h = 1.75$



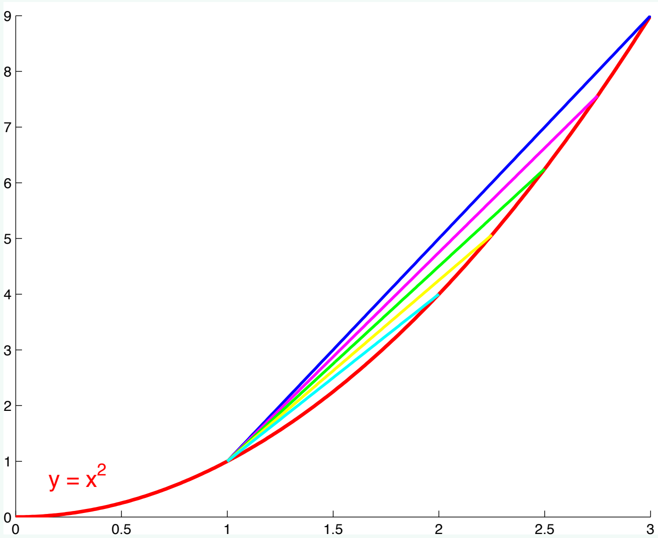
Function $y = x^2$ 3rd Secant Line with $h = 1.5$



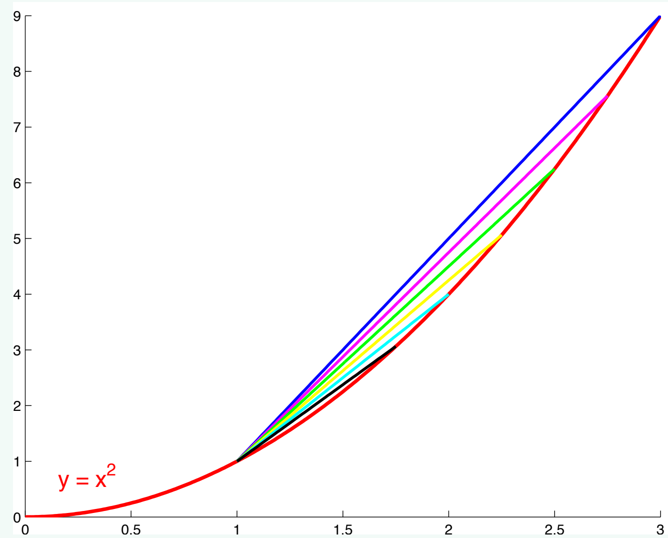
Function $y = x^2$ 4th Secant Line with $h = 1.25$



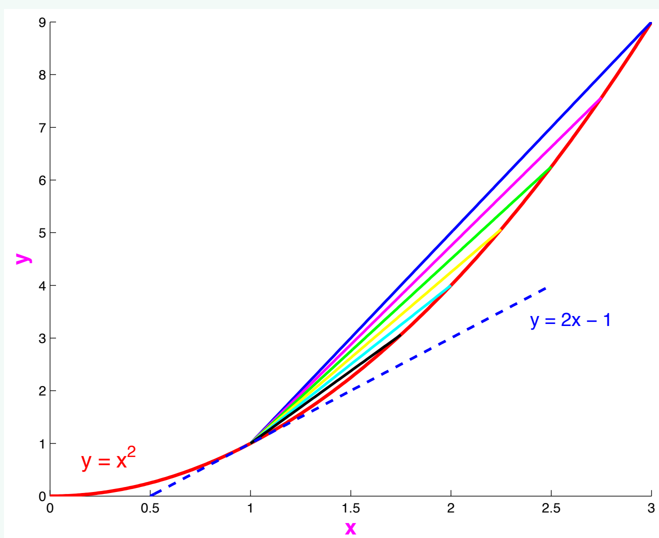
Function $y = x^2$ 5th Secant Line with $h = 1$



Function $y = x^2$ 6th Secant Line with $h = 0.75$



Function $y = x^2$ Limit as $h \to 0$



Differentiation

Notation

The notation we use is as follows:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \left.\frac{df(x)}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

where the vertical line (above) denotes "evaluated at".

Differentiation

Other Notation

It is important to note at this point that an alternative ("Δx") notation is also widely used, namely:

$$\lim_{\Delta x \to 0} \frac{(y+\Delta y)-y}{(x+\Delta x)-x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

We illustrate the use of the "Δ" in the following example.

Example 2

Differentiate the function $y = x^3 + 1$ from first principles at $x = a$.

Solution

First compute the quotient $\frac{\Delta y}{\Delta x}$:

$$\begin{aligned} y &= x^3 + 1 \\ y + \Delta y &= (x+\Delta x)^3 + 1 \\ \Delta y &= (x+\Delta x)^3 - x^3 \\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 \\ &= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \\ \frac{\Delta y}{\Delta x} &= 3x^2 + 3x\Delta x + (\Delta x)^2 \end{aligned}$$

$$y = x^3 + 1$$

Example 2 (Cont'd)

We then take the limit as Δx tends to zero:

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= 3x^2 + 3x\Delta x + (\Delta x)^2 \\ \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= 3x^2 \\ \Rightarrow \left.\frac{dy}{dx}\right|_{x=a} &= 3a^2 \end{aligned}$$

Example 3

As a simpler example, we will find the derivative from first principles of $f(x) = x$.

Solution

We compute

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)-a}{h} = \frac{h}{h} = 1$$

so that, irrespective of h, we obtain

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} 1 = 1$$

This is as expected since $y = x$ is just the straight line with constant slope $(= 1)$.

Example 4

Differentiate from first principles the function $y = \sqrt{x}$ for $x > 0$.

Solution

Since

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left[\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right] \\ &= \frac{1}{\sqrt{x+h} + \sqrt{x}} \end{aligned}$$

we obtain

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

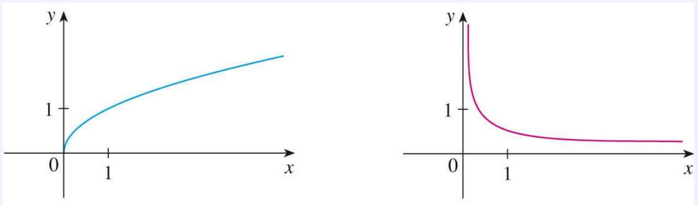
$y = \sqrt{x}$ $\qquad x > 0$

Solution (Cont'd)

We then obtain

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$y = \sqrt{x}$ $\qquad x > 0$



(a) $f(x) = \sqrt{x}$ (b) $f'(x) = \frac{1}{2\sqrt{x}}$

Footnote to Example 4

- Note that, in order to simplify the expression $\sqrt{x+h} - \sqrt{x}$, we multiplied above and below by the "conjugate" expression $\sqrt{x+h} + \sqrt{x}$.
- We are making use of the identity

$$A^2 - B^2 = (A - B)(A + B)$$

and on the basis that it may sometimes be easier to deal with $A^2 - B^2$ than $A - B$ (as in this case).

Example 5

What is the slope of the curve $y = \frac{1}{x}$ at the point $x = 2$?

Solution

As before, we compute:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{1}{x+h}-\frac{1}{x}}{h}\\ &= \frac{x-(x+h)}{x(x+h)h}\\ &= \frac{-h}{x(x+h)h}\\ &= \frac{-1}{x(x+h)}\end{aligned}$$

$$y = \frac{1}{x}$$

Example 5 (Cont'd)

Then take limits to obtain

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{-1}{x(x+0)} = -\frac{1}{x^2}.$$

Finally, evaluate at $x = 2$ to obtain

$$\left.\frac{dy}{dx}\right|_{x=2} = -\left.\frac{1}{x^2}\right|_{x=2} = -\frac{1}{4}.$$

Outline

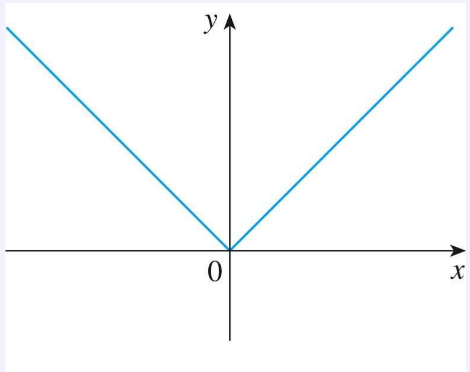
When is a Function Differentiable?

Definition

- A function is differentiable at a if $f'(a)$ exists.
- It is differentiable on an open interval (a, b) if it is differentiable at every number in the interval.

When is a Function Differentiable?

Example: $f(x) = |x|$



(a) $y = f(x) = |x|$

When is a Function Differentiable?

Example: $f(x) = |x|$: Case $x > 0$

If $x > 0$, then $|x| = x$ and we can choose h small enough so that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \to 0} \frac{x + h - x}{h} \\ &= \lim_{h \to 0} \frac{h}{h} \\ &= \lim_{h \to 0} 1 = 1 \end{aligned}$$

and so f is differentiable for any $x > 0$.

When is a Function Differentiable?

Example: $f(x) = |x|$: Case $x < 0$

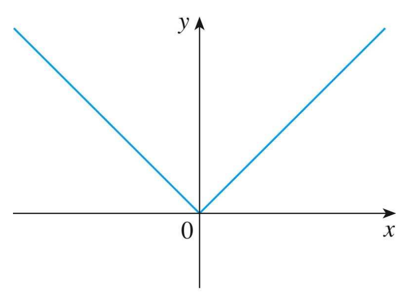
If $x < 0$, then $|x| = -x$ and we can choose h small enough so that $x + h < 0$ and hence $|x + h| = -(x + h)$. Therefore, for $x < 0$, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h \to 0} \frac{-x - h + x}{h} \\ &= \lim_{h \to 0} \frac{-h}{h} \\ &= \lim_{h \to 0} -1 = -1 \end{aligned}$$

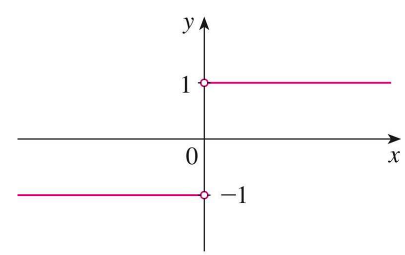
and so f is differentiable for any $x < 0$.

When is a Function Differentiable?

$f'(0)$ does not exist.

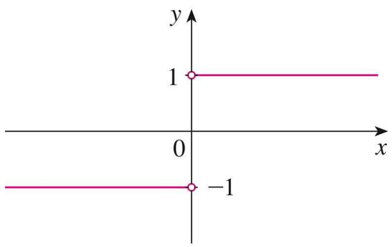


(a) $y = f(x) = |x|$



(b) $y = f'(x)$

When is a Function Differentiable?



(b) $y = f'(x)$

A formula for f' is given by

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases}$$

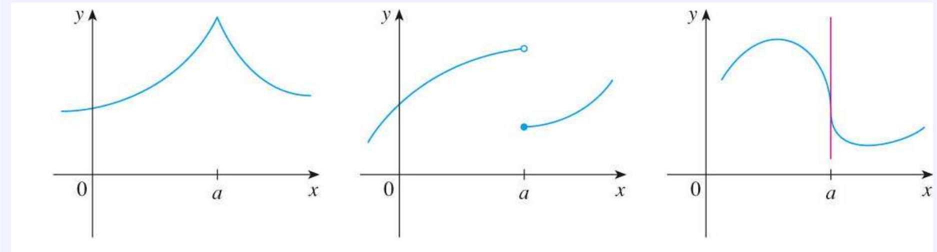
Note

$f(x) = |x|$ is differentiable at all x except 0.

How Can a Function Fail to Be Differentiable?

If the graph of a function f has a "corner" or "kink" in it, then the graph of has no tangent at this point and is not differentiable there.

The Three Cases



(a) A corner (b) A discontinuity (c) A vertical tangent

MS121: IT Mathematics

Differentiation

Rules for Differentiation: Part 1

John Carroll
School of Mathematical Sciences

Dublin City University

Outline

Outline

Differentiation by Formula

Pattern Observed

You may have noticed the following pattern when we were differentiating from first principles:

$$\begin{array}{ccc} x & \to & 1 \\ x^2 & \to & 2x \\ x^3 & \to & 3x^2 \\ \vdots & \vdots & \vdots \\ x^{-1} & \to & -1 \cdot x^{-2} \\ x^{-2} & \to & -2 \cdot x^{-3} \\ \vdots & \vdots & \vdots \end{array}$$

Differentiation by Formula

General Rule

$$\boxed{\frac{d}{dx}x^n = n \cdot x^{n-1}}$$

and this rule is true for all values of n.

Example 1

To find the derivative of $\sqrt{x}$, note that $\sqrt{x} = x^{\frac{1}{2}}$ so that the general rule can be applied with $n = \frac{1}{2}$ to obtain

$$\frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}.$$

Note

This rule is found on page 25 of the “formulae and tables” booklet, along with a number of other useful derivatives which you may use without proof unless you have been explicitly asked to differentiate from first principles.

Differentiation by Formula

Other Entries in the Mathematical Tables

The derivatives of the trigonometric functions are also available, for example

$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x \\ \frac{d}{dx}\cos x &= -\sin x \\ \frac{d}{dx}\tan x &= \sec^2 x \end{aligned}$$

Note that trigonometric definitions are on pages 13–16 .

The Exponential and Log Functions

Rate of Growth

- Another important derivative found in the log tables is for the exponential function.
- We know that the derivative of a function is equal to its slope which we also think of as being its rate of growth.

The Exponential and Log Functions

Question

What function has a derivative equal to the function itself?

Answer

- The answer is the unique function

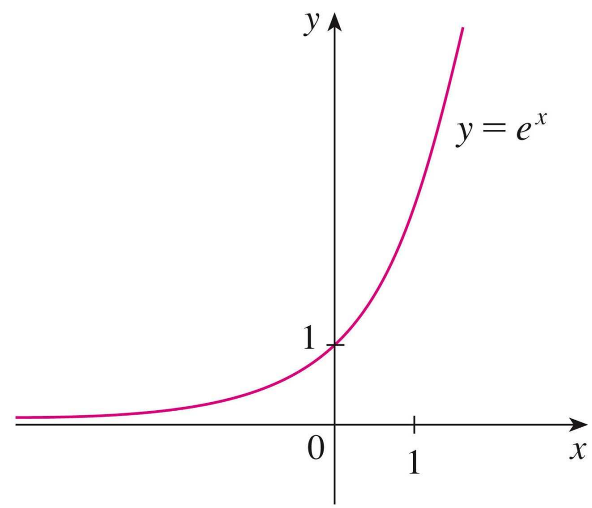
$$y = e^x = \exp(x)$$

the exponential function.

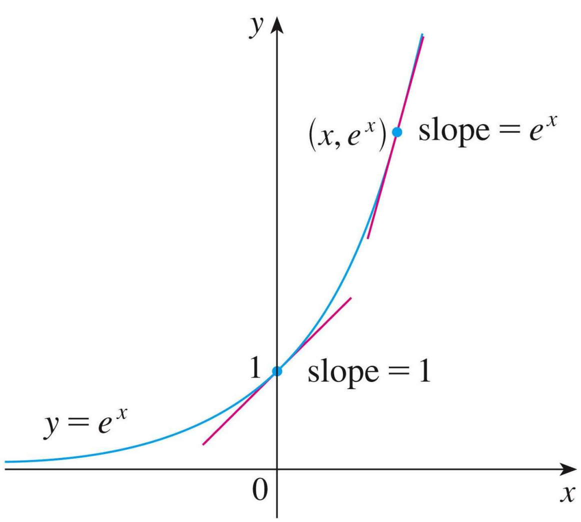
- For this function only,

$$\frac{dy}{dx} = y, \qquad \text{i.e.} \qquad \frac{d}{dx} e^x = e^x.$$

The function $y = e^x$



$y = e^x$ and $\frac{dy}{dx} = e^x$



The Exponential and Log Functions

The Exponential Function (Cont'd)

- Note that e^x is simply the number which we call "e" raised to the power of x ($e = e^1 = 2.718281828\ldots$).
- This number, like the number π, is an irrational number, i.e. a non-repeating decimal.
- Since $2 < e < 3$, the function e^x satisfies

$$2^x < e^x < 3^x,$$

and the limit of e^x as $x \to \infty$ must be infinite, i.e.

$$\lim_{x\to\infty} e^x = \infty,$$

The Exponential and Log Functions

The Exponential Function (Cont'd)

- The limit of e^x as $x \to -\infty$ is

$$\lim_{x \to -\infty} e^x = \lim_{z \to \infty} e^{-z} = \lim_{z \to \infty} \frac{1}{e^z} = \frac{1}{\infty} = 0.$$

- In the foregoing, we simply made the substitution $z = -x$.

The Exponential and Log Functions

The Logarithmic Function

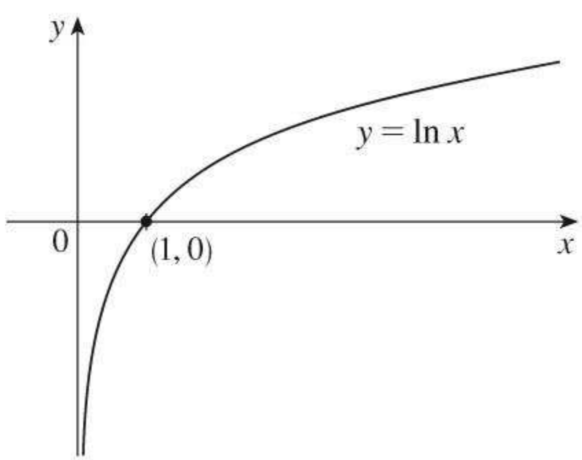
- The function $y = e^x$ has an inverse, namely

$$y = \ln x,$$

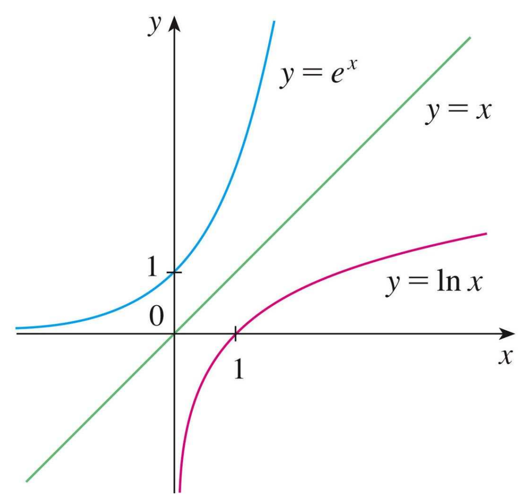
  the natural logarithm of x.
- A graph of $y = \ln x$ will show that the function is only defined on $(0, \infty)$ which is the range of the exponential function $y = e^x$ and hence the domain of $y = \ln x$.
- The derivative of $\ln x$ is also in the tables:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

The function $y = \ln x$



$y = e^x$ is a reflection of $y = \ln x$ in $y = x$



Outline

Sums & Differences of Functions

Work Plan

- We now introduce some rules for differentiation which will allow us to take the derivative of sums, products, quotients and compositions of functions.
- In the remainder of this section, we deal with sums of functions while the products, quotients and compositions of functions are dealt with in separate sections to follow.

Sums & Differences of Functions

Derivative of the Sum

The derivative of the sum is simply the sum of the derivatives:

$$\boxed{\frac{d}{dx}\left[u(x)+v(x)\right]=\frac{du}{dx}+\frac{dv}{dx}}$$

for any 2 functions of x.

Example 1

For $y=x^2+7$, we obtain

$$\frac{d}{dx}\left[x^2+7\right]=\frac{d}{dx}x^2+\frac{d}{dx}7=2x+0=2x.$$

Note how the derivative of any constant term is zero. You can prove this from first principles or simply apply the general for x^n with $n=0$, i.e.

$$\frac{d}{dx}7=\frac{d}{dx}7x^0=7\frac{d}{dx}x^0=7\cdot 0\cdot x^{-1}=0.$$

Example 2

Using the same rule, we find

$$\frac{d}{dx}\left[e^x + \ln x\right] = \frac{d}{dx}e^x + \frac{d}{dx}\ln x = e^x + \frac{1}{x}.$$

Outline

The Product Rule

Formula

To differentiate the product of two functions, $u(x)$ and $v(x)$, we must use the product rule, which is given in the Math Tables:

$$\boxed{\frac{d}{dx}\left[u(x) \star v(x)\right] = v\frac{du}{dx} + u\frac{dv}{dx}}$$

Example 3

Consider the function $y = xe^x$. To use the product rule, let $u = x$ and $v = e^x$ so that

$$\frac{du}{dx} = 1, \qquad \frac{dv}{dx} = e^x.$$

The product rule then gives:

$$\begin{aligned} v\frac{du}{dx} + u\frac{dv}{dx} &= e^x \cdot 1 + x \cdot e^x \\ &= e^x(1+x) \end{aligned}$$

Example 4

For the function $y = \ln x \tan x$, we let

$$u = \ln x, \qquad v = \tan x,$$

so that

$$\frac{du}{dx} = \frac{1}{x}, \qquad \frac{dv}{dx} = \sec^2 x.$$

The product rule then gives:

$$\begin{aligned} v\frac{du}{dx} + u\frac{dv}{dx} &= \tan x \cdot \frac{1}{x} + \ln x \cdot \sec^2 x \\ &= \frac{1}{x}\tan x + \ln x \sec^2 x \end{aligned}$$

Example 5

Consider $y = x^{\frac{1}{4}}\left(2 + 3x + x^2\right)$.

Solution

With

$$u = x^{\frac{1}{4}}, \qquad v = 2 + 3x + x^2,$$

we obtain

$$\frac{du}{dx} = \frac{1}{4}x^{-\frac{3}{4}}, \qquad \frac{dv}{dx} = 3 + 2x.$$

The product rule then gives:

$$\begin{aligned} v\frac{du}{dx} + u\frac{dv}{dx} &= \left(2 + 3x + x^2\right) \cdot \frac{1}{4}x^{-\frac{3}{4}} + x^{\frac{1}{4}} \cdot (3 + 2x) \\ &= \frac{1}{4}x^{-\frac{3}{4}}\left(2 + 3x + x^2\right) + x^{\frac{1}{4}}(3 + 2x) \end{aligned}$$

Extension of the Formula

If you need to find the derivative of 3 functions, say $u(x)$, $v(x)$ and $w(x)$ multiplied together, then the formula to use is an extension of the product rule, namely

$$\boxed{\tfrac{d}{dx}\left[u(x) \star v(x) \star w(x)\right] = \tfrac{du}{dx}vw + u\tfrac{dv}{dx}w + uv\tfrac{dw}{dx}}$$

This rule is not found in the Math Tables because it is simply the product rule applied twice.

Example 6

Differentiate $y = e^x \sin x \tan x$.

Solution

We let

$$u = e^x, \qquad v = \sin x, \qquad w = \tan x,$$

so that

$$\frac{du}{dx} = e^x, \qquad \frac{dv}{dx} = \cos x, \qquad \frac{dw}{dx} = \sec^2 x.$$

$$y = e^x \sin x \tan x$$

Example 6 (Cont'd)

We then obtain

$$\begin{aligned}
\frac{d}{dx}[uvw] &= \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx} \\
&= e^x \sin x \tan x + e^x \cos x \tan x + e^x \sin x \sec^2 x \\
&= e^x \left\{\sin x \tan x + \cos x \tan x + \sin x \sec^2 x\right\} \\
&= e^x \sin x \left\{\tan x + 1 + \sec^2 x\right\}.
\end{aligned}$$

Outline

The Quotient Rule

Formula

To differentiate the quotient of two functions, $u(x)$ and $v(x)$, namely $y = \frac{u(x)}{v(x)}$, we must use the quotient rule, which is given in the log tables:

$$\boxed{\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Example 7

Find $\frac{dy}{dx}$ where $y = \dfrac{x^{\frac{1}{4}}}{\cos x}$

Solution

With

$$\begin{aligned}
u &= x^{\frac{1}{4}} & v &= \cos x \\
\frac{du}{dx} &= \frac{1}{4}x^{-\frac{3}{4}} & \frac{dv}{dx} &= -\sin x
\end{aligned}$$

$$\begin{aligned}
\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} &= \frac{\cos x \cdot \frac{1}{4}x^{-\frac{3}{4}} - x^{\frac{1}{4}} \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\frac{1}{4}x^{-\frac{3}{4}}\cos x + x^{\frac{1}{4}}\sin x}{\cos^2 x}.
\end{aligned}$$

Example 8

Find $\frac{dy}{dx}$ where $y = \dfrac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{\ln x}$

Solution

We let

$$u = x^{\frac{1}{2}} + x^{-\frac{1}{2}}, \qquad v = \ln x,$$

so that

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}, \qquad \frac{dv}{dx} = \frac{1}{x}.$$

The quotient rule then gives:

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\ln x \cdot \frac{1}{2}\left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right) - \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) \cdot \frac{1}{x}}{\ln^2 x}$$

$$y = \frac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{\ln x}$$

Example 8 Cont'd

The quotient rule:

$$\begin{aligned}\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} &= \frac{\ln x \cdot \frac{1}{2}\left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right) - \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) \cdot \frac{1}{x}}{\ln^2 x} \\ &= \frac{\frac{1}{2}\ln x\left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right) - \frac{1}{x}\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)}{\ln^2 x}\end{aligned}$$

Example 9

Consider the function $y = \frac{\sin x}{\cos x}$.

Solution

In this case, we have

$$u = \sin x, \qquad v = \cos x,$$

so that

$$\frac{du}{dx} = \cos x, \qquad \frac{dv}{dx} = -\sin x.$$

The quotient rule then gives

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$y = \frac{\sin x}{\cos x}$$

Example 9 Cont'd

The quotient rule:

$$\begin{aligned}\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$

Note that this is just the result $\frac{d}{dx}\tan x = \sec^2 x$.

MS121: IT Mathematics

Differentiation

Rules for Differentiation: Part 2

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Outline

1. The Chain Rule
2. Examples on Composition of 2 Functions
3. Examples on Composition of 3 Functions
4. Examples involving Product & Quotient Rules

Outline

1. The Chain Rule
2. Examples on Composition of 2 Functions
3. Examples on Composition of 3 Functions
4. Examples involving Product & Quotient Rules

The Chain Rule

Context

The Chain Rule is used to differentiate the composition of functions.

Composition of 2 Functions

The function

$$y = \ln(\sin x)$$

is the composition $f \circ g$ where

$$f(\cdot) = \ln(\cdot) \quad \text{and} \quad g(\cdot) = \sin(\cdot).$$

The Chain Rule

Composition of 2 Functions

The function

$$y = \ln(\sin x)$$

is the composition $f \circ g$ where $f(\cdot) = \ln(\cdot)$ and $g(\cdot) = \sin(\cdot)$.

The Chain Rule

We can differentiate $y = f \circ g$ as

$$\frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

where f is written in terms of g and we only differentiate f with respect to its argument, namely g. Since g is written in terms of x, this differentiation is simply with respect to x.

The Chain Rule

Composition of 2 Functions

$$y = \ln(\sin x)$$

Using the Chain Rule

We obtain:

$$\begin{aligned} g &= \sin x \Rightarrow \frac{dg}{dx} = \frac{d}{dx}\sin x = \cos x \\ f &= \ln(g) \Rightarrow \frac{df}{dg} = \frac{d}{dg}\ln g = \frac{1}{g} = \frac{1}{\sin x} \\ \Rightarrow \frac{dy}{dx} &= \frac{df}{dg}\cdot\frac{dg}{dx} = \frac{1}{\sin x}\cdot\cos x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} = \cot x \end{aligned}$$

Outline

Example 1: Composition of 2 Functions

$$y = \cos\left(2x^2\right)$$

Solution

$$u = 2x^2 \quad \Rightarrow \quad \frac{du}{dx} = 4x$$

$$y = \cos(u) \quad \Rightarrow \quad \frac{dy}{du} = -\sin(u)$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\sin(u)\cdot 4x = -4x\sin\left(2x^2\right)$$

Note

Note that you must always express your answer wholly in terms of x (just as the question was expressed in the first place).

Example 2: Composition of 2 Functions

$$y = \tan\left(e^x\right)$$

Solution

$$u = e^x \quad \Rightarrow \quad \frac{du}{dx} = e^x$$

$$y = \tan(u) \quad \Rightarrow \quad \frac{dy}{du} = \sec^2(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2(u) \cdot e^x = e^x \sec^2\left(e^x\right)$$

Example 3: Composition of 2 Functions

$$y = \sin\left(\ln(x)\right)$$

Solution

$$u = \ln(x) \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$y = \sin(u) \quad \Rightarrow \quad \frac{dy}{du} = \cos(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \cdot \frac{1}{x} = \frac{1}{x} \cos\left(\ln(x)\right)$$

Outline

Example 4: Composition of 3 Functions

$$y = \ln\left(\sin\left(e^x\right)\right)$$

Solution

$$u = e^x \quad \Rightarrow \quad \frac{du}{dx} = e^x$$

$$v = \sin(u) \quad \Rightarrow \quad \frac{dv}{du} = \cos(u)$$

$$y = \ln(v) \quad \Rightarrow \quad \frac{dy}{dv} = \frac{1}{v}$$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = \frac{1}{v} \cdot \cos(u) \cdot e^x = \frac{1}{\sin\left(e^x\right)} \cos\left(e^x\right) e^x = e^x \cot(x)$$

Example 5: Composition of 3 Functions

$$y = \ln(\ln(\ln(x)))$$

Solution

$$u = \ln(x) \quad \Rightarrow \quad \frac{\mathrm{du}}{\mathrm{dx}} = \frac{1}{x}$$

$$v = \ln(u) \quad \Rightarrow \quad \frac{\mathrm{dv}}{\mathrm{du}} = \frac{1}{u}$$

$$y = \ln(v) \quad \Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dv}} = \frac{1}{v}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dv}} \cdot \frac{\mathrm{dv}}{\mathrm{du}} \cdot \frac{\mathrm{du}}{\mathrm{dx}} = \frac{1}{v} \cdot \frac{1}{u} \cdot \frac{1}{x} = \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

Outline

Example 6: Composition of Functions & Product Rule

$$y = e^{x \sin x}$$

Solution

$$u = x \sin x \quad \Rightarrow \quad \frac{\mathrm{du}}{\mathrm{dx}} = \sin x + x \cos x$$

$$y = e^u \quad \Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{du}} = e^u$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{du}} \cdot \frac{\mathrm{du}}{\mathrm{dx}} = e^u \cdot (\sin x + x \cos x) = e^{x \sin x}(\sin x + x \cos x)$$

Example 7: Composition of Functions & Quotient Rule

$$y = \frac{\sqrt{x}}{e^{\tan x}}$$

Observation

At first sight, this appears only to be an example involving the quotient rule with

$$u = \sqrt{x}, \qquad v = e^{\tan x}.$$

However, we must first use the chain rule to determine $\frac{dv}{dx}$.

Example 7: Step 1 — Evaluate $\frac{dv}{dx}$

$$v = e^{\tan x}$$

Solution

$$w = \tan x \quad \Rightarrow \quad \frac{\mathrm{dw}}{\mathrm{dx}} = \sec^2 x$$

$$v = e^w \quad \Rightarrow \quad \frac{\mathrm{dv}}{\mathrm{dw}} = e^w$$

$$\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{dv}}{\mathrm{dw}} \cdot \frac{\mathrm{dw}}{\mathrm{dx}} = e^w \cdot \sec^2(x) = e^{\tan x} \sec^2 x$$

Example 7: Step 2 — Evaluate $\frac{dy}{dx}$

$$y = \frac{\sqrt{x}}{e^{\tan x}}$$

Solution

$$u = \sqrt{x} \quad \Rightarrow \quad \frac{\mathrm{du}}{\mathrm{dx}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$v = e^{\tan x} \quad \Rightarrow \quad \frac{\mathrm{dv}}{\mathrm{dx}} = e^{\tan x} \sec^2 x$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{e^{\tan x} \cdot \frac{1}{2\sqrt{x}} - \sqrt{x} \cdot e^{\tan x} \sec^2 x}{\left(e^{\tan x}\right)^2} = \frac{\frac{1}{2\sqrt{x}} - \sqrt{x} \sec^2 x}{e^{\tan x}}$$

MS121: IT Mathematics

Applications of Differentiation 1

Curve Sketching

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Outline

Outline

Optimization Problems

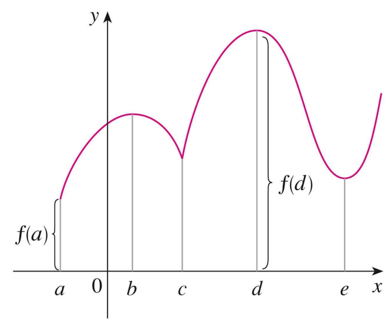
Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function.

Outline

Maximum & Minimum Values



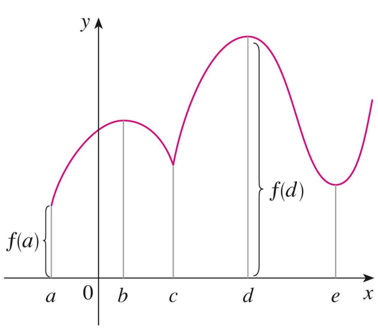
Definition

Let c be a number in the domain D of a function f. Then f is the

- absolute maximum value of f on D if $f(c) \geq f(x)$ for all $x \in D$.
- absolute minimum value of f on D if $f(c) \leq f(x)$ for all $x \in D$.

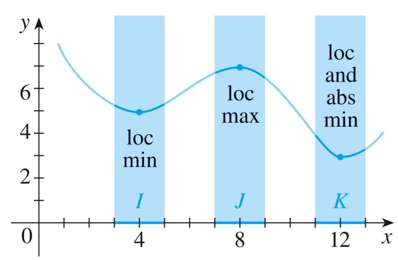
An absolute maximum or minimum is sometimes called a global maximum or minimum. The maximum and minimum values of f are called extreme values of f.

Maximum & Minimum Values



- $f(a)$ is the absolute minimum on $[a, e]$.
- $f(d)$ is the absolute maximum on $[a, e]$.
- $f(c)$ is the local minimum on $[b, d]$.
- $f(b)$ is the absolute maximum on $[a, c]$.

Local Maximum & Minimum Values

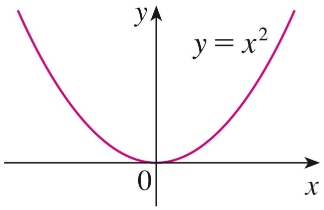


Definition

The number $f(c)$ is a

- local maximum value of f if $f(c) \geq f(x)$ when x is near c.
- local minimum value of f if $f(c) \leq f(x)$ when x is near c.

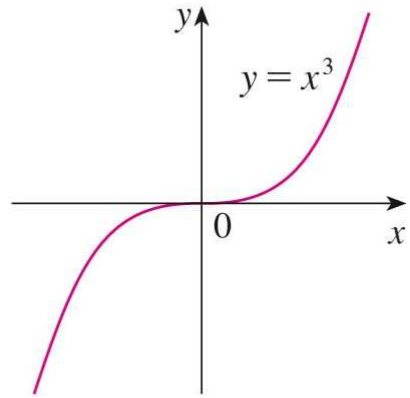
Global & Local Minimum Value



Absolute & Local Mimiumum

- If $f(x) = x^2$, then $f(x) \geq f(0)$ because $x^2 \geq 0$ for all x.
- Therefore $f(0) = 0$ is the absolute (and local) minimum value of f. This corresponds to the fact that the origin is the lowest point on the parabola.
- However, there is no highest point on the parabola and so this function has no maximum value.

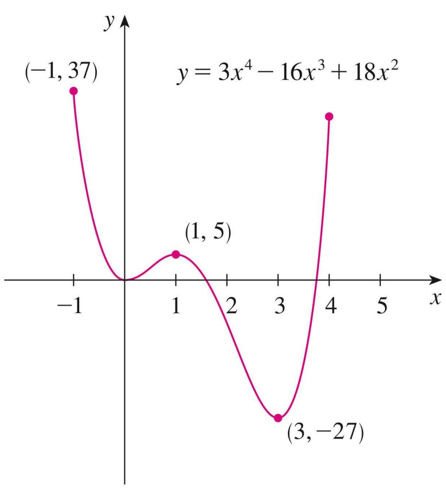
No Local/Global Maximum or Minimum Values



No "Extreme" Values

- From the graph of the function, we see that this function has neither an absolute maximum value nor an absolute minimum value.
- In fact, it has no local extreme values either.

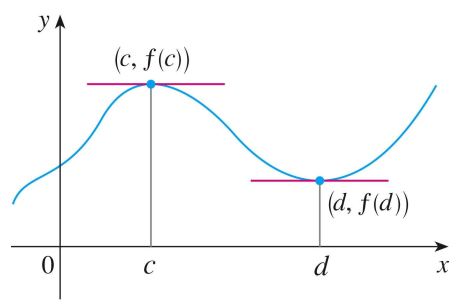
Local v Global Maximum Values



Both "Extreme" Values

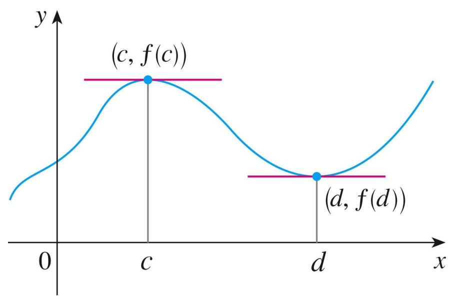
- $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1) = 37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.)
- Also, $f(0) = 0$ is a local minimum and $f(3) = -27$ is both a local and an absolute minimum.
- Note f that has neither a local nor an absolute maximum at $x = 4$.

When does a Max/Min Occur?



- The graph of a function f shows a local maximum at c and a local minimum at d.
- It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0.
- We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$.

When does a Max/Min Occur?

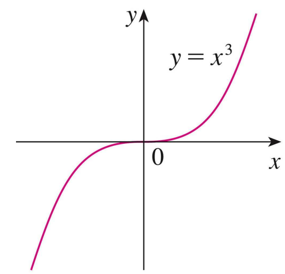


Fermat's Theorem

If f has a local maximum or minimum at $f'(c) = 0$, and if $f'(c)$ exists, then

$$f'(c) = 0$$

A Word of Caution on Fermat's Theorem



The Converse is False!

- If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$.
- But f has no maximum or minimum at 0, as you can see from its graph.
- The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at $(0, 0)$.
- Instead of having a maximum or minimum at $(0, 0)$, the curve crosses its horizontal tangent there.

Outline

Finding the Critical Points

Definition: A Critical Number (Point)

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's Theorem: Re-Stated

If f has a local maximum or minimum at c, then c is a critical number of f.

The Closed Interval Method: Absolute Max/Min Values

3-Step Procedure

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example

Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \le x \le 4$$

Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method:

$$\begin{aligned} f(x) &= x^3 - 3x^2 + 1 \\ f'(x) &= 3x^2 - 6x = 3x(x-2) \end{aligned}$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $\left(-\frac{1}{2}, 4\right)$.

Example (Cont'd)

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \le x \le 4$$

The critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. The values of at these critical numbers are

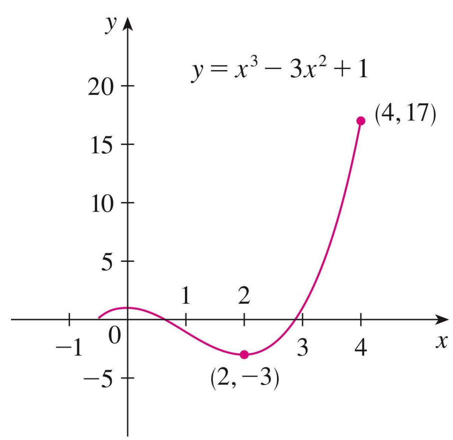
$$f(0) = 1 \qquad f(2) = -3$$

The values of at the endpoints of the interval are

$$f\left(-\frac{1}{2}\right) = \frac{1}{8} \qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Example (Cont'd)

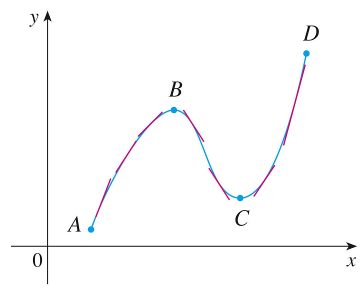


Note

In this example, the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number.

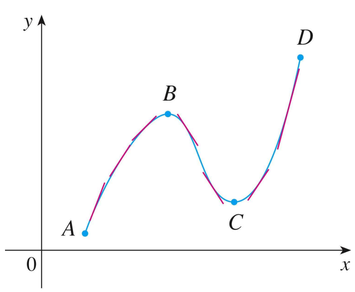
Outline

Increasing/Decreasing Test



- Between A and B and between C and D, the tangent lines have positive slope and so $f'(x) > 0$.
- Between B and C, the tangent lines have negative slope and so $f'(x) < 0$.
- Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative.

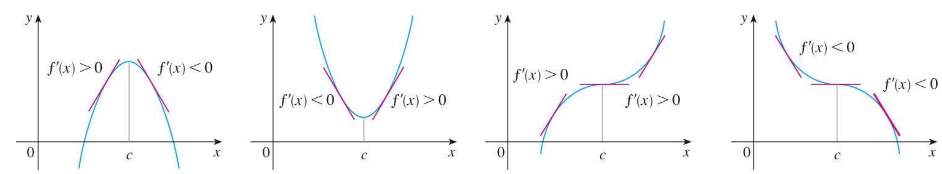
Increasing/Decreasing Test



Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test



(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maximum or minimum

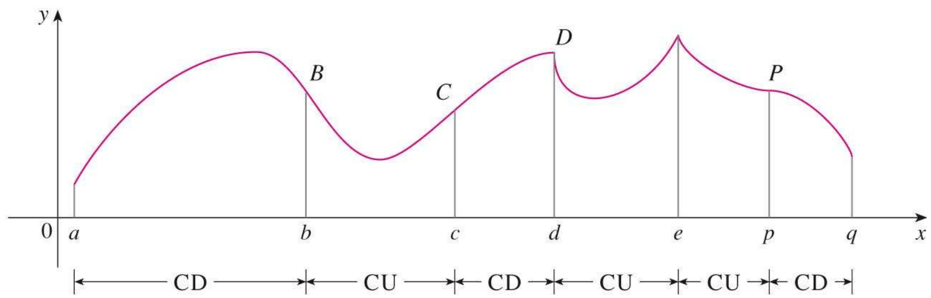
Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

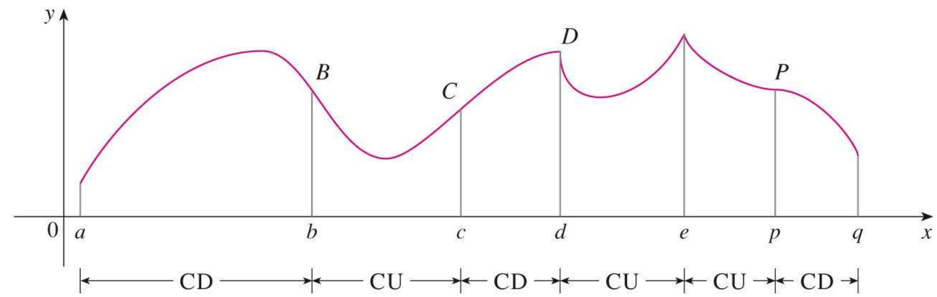
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Using the 2nd Derivative



- The slope of the tangent decreases from a to b, so f' decreases and therefore f'' is negative.
- The slope of the tangent increases from b to c, and this means that the derivative is an increasing function and therefore its derivative (i.e. f'') is positive.

Concavity Test



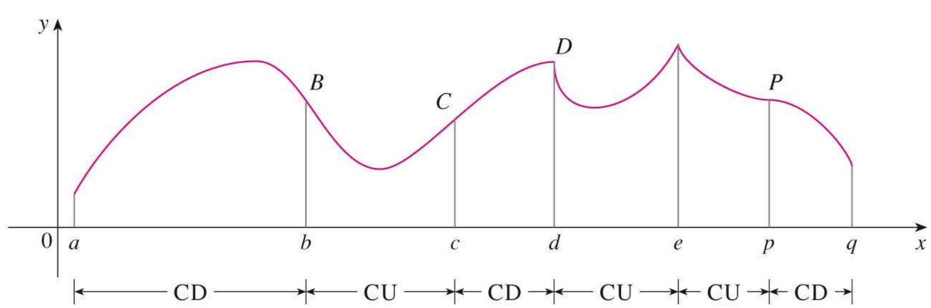
(a) If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

Point of Inflection

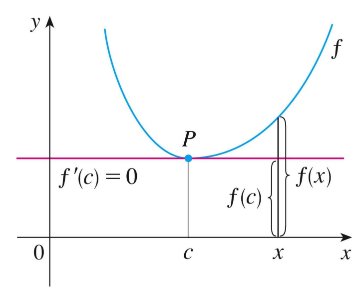
Definition

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.



B, C, D and P are the points of inflection.

Identifying the Maximum/Minumum Points



Second Derivative Test

Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

A Practical Example

- Discuss the curve
$$y = x^4 - 4x^3$$
with respect to concavity, points of inflection, and local maxima and minima.
- Use this information to sketch the curve.

Note

$$\begin{aligned} f(x) &= x^4 - 4x^3 \\ f'(x) &= 4x^3 - 12x^2 = 4x^2(x-3) \\ f''(x) &= 12x^2 - 24x = 12x(x-2) \end{aligned}$$

A Practical Example (Cont'd)

$$\begin{aligned} f(x) &= x^4 - 4x^3 \\ f'(x) &= 4x^3 - 12x^2 = 4x^2(x-3) \\ f''(x) &= 12x^2 - 24x = 12x(x-2) \end{aligned}$$

Solution

$$\begin{aligned} f'(x) = 0 &\quad\Rightarrow\quad x = 0 \text{ and } x = 3 \\ f''(x) = 0 &\quad\Rightarrow\quad x = 0 \text{ and } x = 2 \end{aligned}$$

Furthernore, note that $f''(0) = 0$ and $f''(3) = 36 > 0$. Since $f'(3) = 0$ and $f''(3) > 0$, we have that $f(3) = -27$ is a local minimum.

Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0.

A Practical Example (Cont'd)

$$\begin{aligned} f(x) &= x^4 - 4x^3 \\ f'(x) &= 4x^3 - 12x^2 = 4x^2(x-3) \\ f''(x) &= 12x^2 - 24x = 12x(x-2) \end{aligned}$$

$f'(0) = 0$ and $f''(0) = 0$

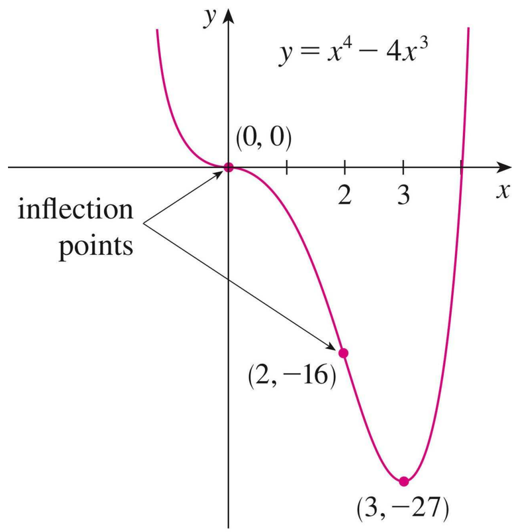
- But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that does not have a local maximum or minimum at 0.
- Since $f'(0) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

A Practical Example (Cont'd)

Interval	$f''(x) = 12x(x-2)$	Concavity
$(-\infty, 0)$	$+$	Upward
$(0, 2)$	-	Downward
$(2, \infty)$	$+$	Upward

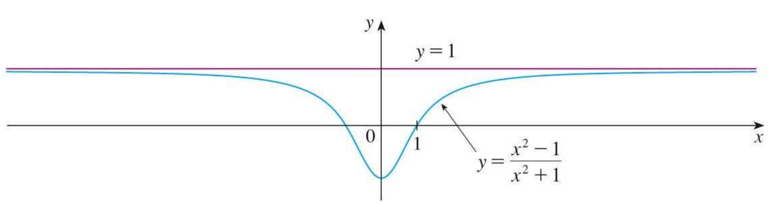
- The point $(0, 0)$ is an inflection point since the curve changes from concave upward to concave downward there.
- Also $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

A Practical Example (Concluded)



Outline

A Reminder about Limits



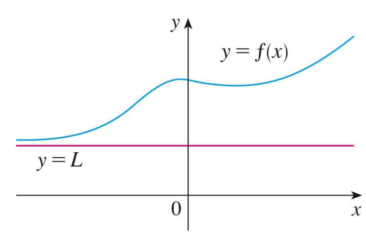
Definition

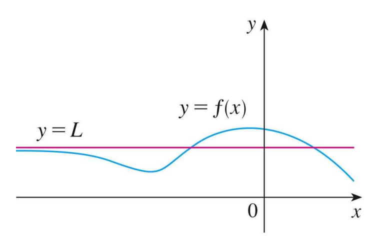
Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of f can be made arbitrarily close to L by taking x sufficiently large.

Horizontal Asymptote





Definition

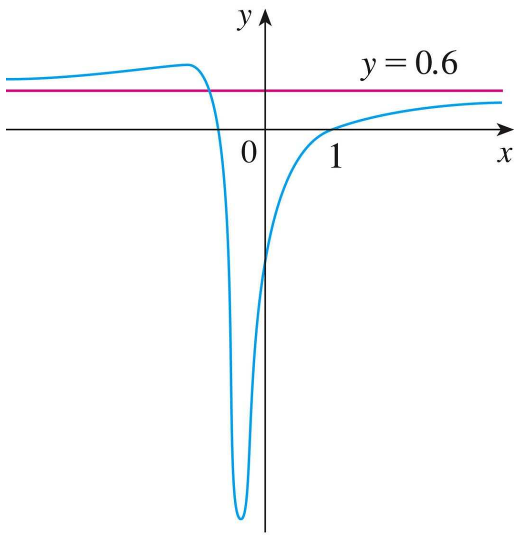
The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L$$

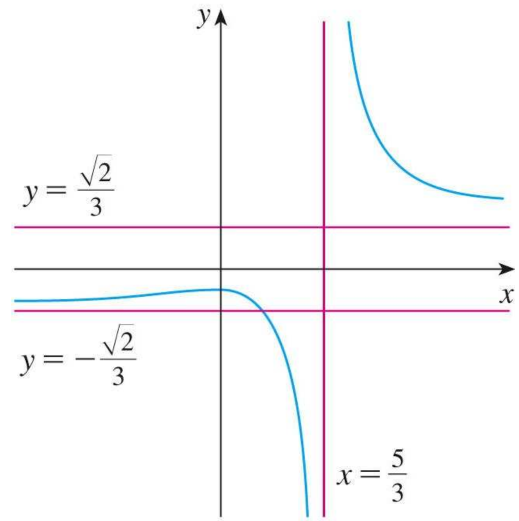
or

$$\lim_{x \to -\infty} f(x) = L$$

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$



$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \frac{\sqrt{2}}{3}$$



Outline

8 Steps for Example 1

Requirement

Identify the critical points and asymptotes of

$$y = \frac{2x^2}{x^2 - 1}$$

Sketch the curve.

Steps 1 & 2

1 Domain:
The domain is

$$\{x \mid x^2 - 1 \neq 0\} \quad = \quad \{x \mid x \neq \pm 1\}$$

2 Intercepts:
The x- and y- intercepts are both 0.

8 Steps for Example 1 Cont'd

$$y = \frac{2x^2}{x^2 - 1}$$

Steps 3 & 4

3 Symmetry:
Since $f(-x) = f(x)$, the function is even. The curve is symmetric about the y-axis

4 Asymptotes:

$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - \frac{1}{x^2}} = 2$$

Therefore, the line $y = 2$ is a horizintal asymptote.

8 Steps for Example 1 Cont'd

$$y = \frac{2x^2}{x^2 - 1}$$

Step 4 Cont'd

4 Asymptotes:
Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = +\infty$$

Therefore the lines $x = -1$ and $x = +1$ are vertical asymptotes.

8 Steps for Example 1 Cont'd

$$y = \frac{2x^2}{x^2 - 1}$$

Steps 5 & 6

5 Intervals of Increase or Decrease:

$$f'(x) = \frac{(x^2 - 1)(4x) - (2x^2)(2x)}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

6 Local Maximum or Minimum Values:
The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

8 Steps for Example 1 Cont'd

$$y = \frac{2x^2}{x^2 - 1}$$

Step 7

7 Concavity & Points of Inflection:

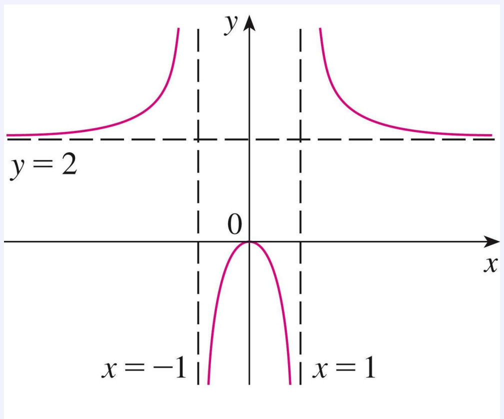
$$f''(x) = \frac{(x^2 - 1)^2(-4) + 8x(x^2 - 1)(2x)}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have
$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$ and
$f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and and $(1, \infty)$and concave downward on $(-1, 1)$.
It has no point of inflection since 1 and -1 are not in the domain of f.

8 Steps for Example 1 Concluded

$$y = \frac{2x^2}{x^2 - 1}$$

Step 8: Sketch



Curve Sketching: A Final Example

Problem Statement

For the function

$$y = \frac{x^2 + 2x + 4}{2x}$$

find and classify all the critical points, determine the asymptotes and hence sketch the curve.

Curve Sketching — Final Example: $y = \frac{x^2 + 2x + 4}{2x}$

Identify the Critical Points

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$:

$$\begin{aligned}
y &= \frac{x^2 + 2x + 4}{2x} \\
\frac{dy}{dx} &= \frac{2x(2x + 2) - (x^2 + 2x + 4)2}{4x^2} \\
&= \frac{2x^2 - 8}{4x^2} = \frac{x^2 - 4}{2x^2} \\
\frac{d^2y}{dx^2} &= \frac{2x^2(2x) - (x^2 - 4)4x}{4x^4} \\
&= \frac{16x}{4x^4} = \frac{4}{x^3}
\end{aligned}$$

Curve Sketching — Final Example: $y = \frac{x^2 + 2x + 4}{2x}$

Identify the Critical Points (Cont'd)

Since

$$\frac{dy}{dx} = \frac{x^2 - 4}{2x^2} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{4}{x^3}$$

we see that

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x^2 - 4 = 0 \quad \Rightarrow \quad x = \pm 2$$

and since

$$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = -\frac{1}{2} < 0 \qquad \left.\frac{d^2y}{dx^2}\right|_{x=+2} = \frac{1}{2} > 0$$

there is a local maximum at $(-2, -1)$ and a local minimum at $(2, 3)$.

Curve Sketching — Final Example: $y = \dfrac{x^2+2x+4}{2x}$

Identify the Asymptotes

Note also that the denominator of y is zero when $x = 0$ and,

$$\lim_{x\to 0^-} \frac{x^2+2x+4}{2x} = -\infty \qquad \lim_{x\to 0^+} \frac{x^2+2x+4}{2x} = +\infty$$

there is a vertical asymptote at $x = 0$ as $x \to 0^-$ and as $x \to 0^+$.
Furthermore, by division, we find that

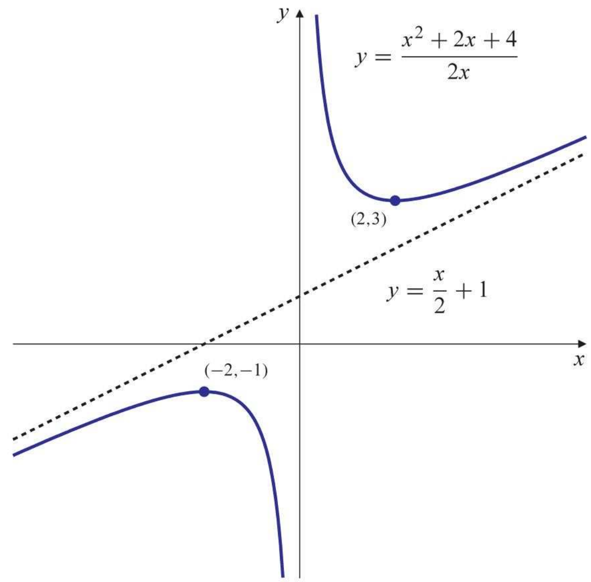
$$\frac{x^2+2x+4}{2x} = \frac{x}{2} + 1 + \frac{2}{x}$$

and, since

$$\lim_{x\to-\infty} \frac{2}{x} = \lim_{x\to+\infty} \frac{2}{x} = 0$$

There is an asymptote in the line $y = \frac{x}{2} + 1$ as $x \to \pm\infty$.

Curve Sketching — Final Example — The Graph



MS121: IT Mathematics

Applications of Differentiation 2

Optimization Problems

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Outline

Outline

Optimization Problems

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function.

How to Solve Optimization Problems

Convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized:

Basic Steps

1. Understand the Problem:
   Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. Draw a Diagram
3. Introduce Notation:
   Assign a symbol to the quantity that is to be maximized or minimized (lets call it Q for now). Assign other symbols as required.
4. Express Q in terms of some of the other symbols from Step 3.
5. Use the methods developed earlier to find the absolute maximum or minimum value of Q.

Outline

Example 1: Finding The Maximum Area

Problem Statement

- A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river.
- He does not require a fence along the river.
- What are the dimensions of the field that has the largest area?

Example 1: Finding The Maximum Area

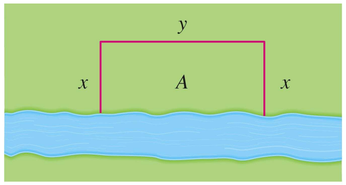


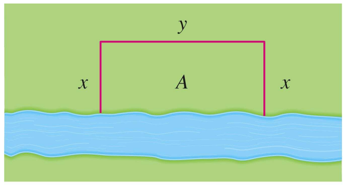
Diagram & Notation

- Let x and y be the depth and width of the rectangle (in meters).
- Then we express A in terms of x and y:

$$A = xy$$

- We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x.

Example 1: Finding The Maximum Area



Formulate A in terms of x

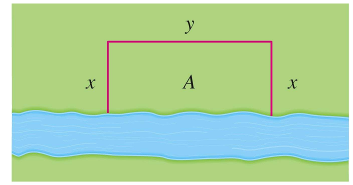
- We use the given information that the total length of the fencing is 1200 m. Thus

$$2x + y = 1200 \Rightarrow y = 1200 - 2x$$

- Since $A = xy$, we therefore have

$$A = x(1200 - 2x) = 1200x - 2x^2$$

Example 1: Finding The Maximum Area



Maximize $A = x(1200 - 2x)$

- Note that $x \geq 0$ and $x \leq 600$ (otherwise $A < 0$). So the function that we wish to maximize is

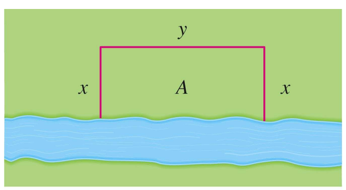
$$A = 1200x - 2x^2 \qquad 0 < x \leq 600$$

- The derivative is $A'(x) = 1200 - 4x$), so to find the critical numbers we solve the equation

$$1200 - 4x = 0$$

which gives $x = 300$).

Example 1: Finding The Maximum Area



Maximum of $A = x(1200 - 2x)$ occurs when $x = 300$

- Observe that

$$A''(x) = -4 < 0$$

for all x, so A is always concave downward and the local maximum at $x = 300$ must be an absolute maximum.

- Thus the rectangular field should be 300 m deep and 600 m wide (an area of $180,000\ m^2$).

Outline

Example 2: Minimize Total Surface Area

Problem Statement

- A cylindrical can is to be made to hold 1 L of oil.
- Find the dimensions that will minimize the cost of the metal to manufacture the can.

Example 2: Minimize Total Surface Area

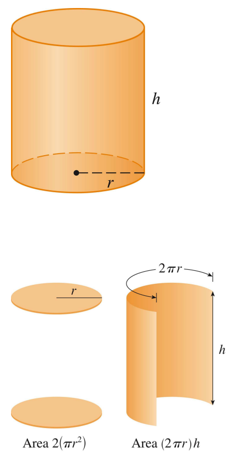
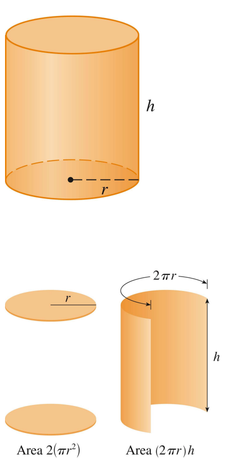


Diagram & Notation

- Let r be the radius and h the height (both in centimeters).
- In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides).
- The sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

Example 2: Minimize Total Surface Area



Eliminate h

To eliminate h, we use the fact that the volume is given as $1L$ which we take to be 1000 cm^3. Thus

$$\pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$$

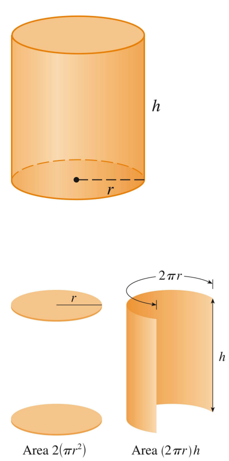
Substituting into the expression for 1000 gives

$$A = 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

Therefore, the function that we want to minimize is

$$A = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

Example 2: Minimize Total Surface Area



Find the Mimiumum Value

To find the critical numbers, we differentiate:

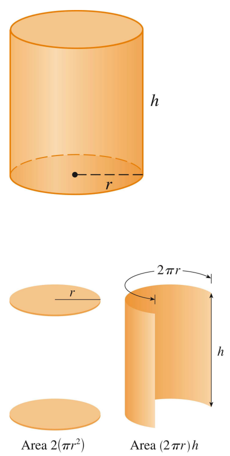
$$A = 2\pi r^2 + \frac{2000}{r}$$

to obtain

$$A' = 4\pi r - \frac{2000}{r^2} = \frac{4\left(\pi r^3 - 500\right)}{r^2}$$

Thus, $A' = 0$ when $\pi r^3 = 500$, so the only critical number is $\sqrt[3]{\frac{500}{\pi}}$.

Example 2: Minimize Total Surface Area



Mimiumum Value: $r^* = \sqrt[3]{\dfrac{500}{\pi}}$

Note that $A'(r) < 0$ for $r < r^*$ and $A'(r) > 0$ for $r > r^*$, so A is decreasing for all to the left of the critical number and increasing for all to the right. Thus r^* must give rise to an absolute minimum. Note also that

$$A'' = 4\pi + \frac{4000}{r^3} > 0$$

for all $r > 0$. Finally, the value of h to achive this minimum is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

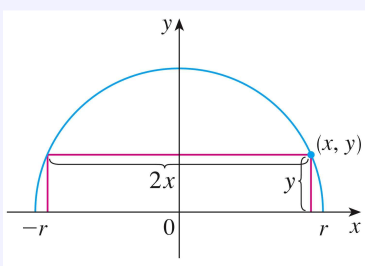
Outline

Example 3: Rectangle Inscribed in a Semicircle

Problem Statement

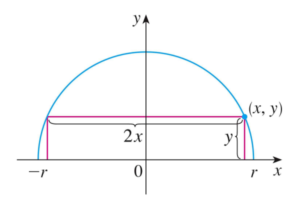
Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Structure



The word inscribed means that the rectangle has two vertices on the semicircle and two vertices on the x-axis.

Example 3: Rectangle Inscribed in a Semicircle



Establish the Variables

The rectangle has sides of lengths $2x$ and y, so its area is

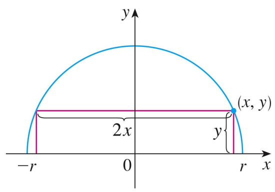
$$A = 2xy$$

To eliminate y, we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$) and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \le x \le r$.

Example 3: Rectangle Inscribed in a Semicircle



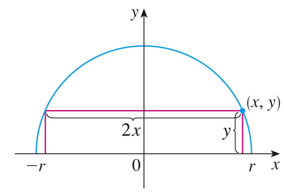
Identify the Critical Point(s)

$$A = 2x\sqrt{r^2 - x^2}$$

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2\left(r^2 - 2x^2\right)}{\sqrt{r^2 - x^2}}$$

Note that $A' = 0$ when $2x^2 = r^2$, that is, $x = \frac{r}{\sqrt{2}}$ (since $x \geq 0$).

Example 3: Rectangle Inscribed in a Semicircle



Maximum at $x = \frac{r}{\sqrt{2}}$

This value of x gives a maximum value of A since $A(0) = 0$ and $A(r) = 0$. Therefore, the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2 \star \frac{r}{\sqrt{2}} \star \sqrt{r^2 - \frac{r^2}{2}} = r^2$$

MS121: IT Mathematics

Integration

Introduction — The Area Problem

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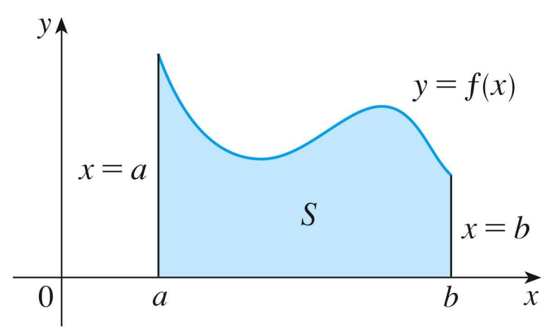
Outline

Outline

The Area Problem

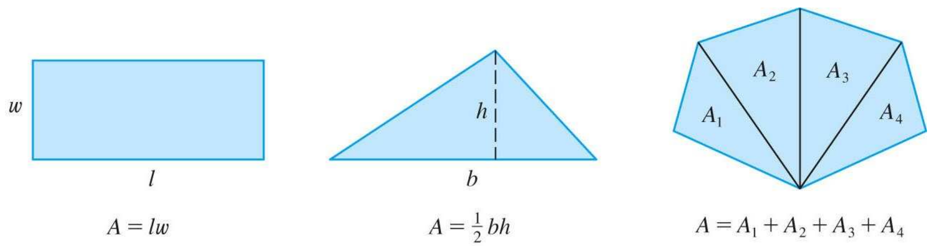
We begin by attempting to solve the area problem: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.



This means that S is bounded by the graph of a continuous function f, where $f(x) \geq 0$, the vertical lines $x = a$ and $x = b$, and the x-axis.

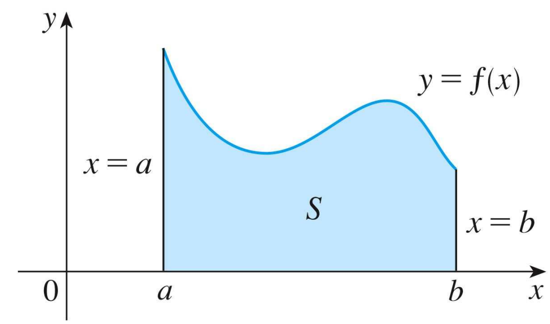
Area Problems that can be Solved

For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height.



The area of a polygon is found by dividing it into triangles and adding the areas of the triangles.

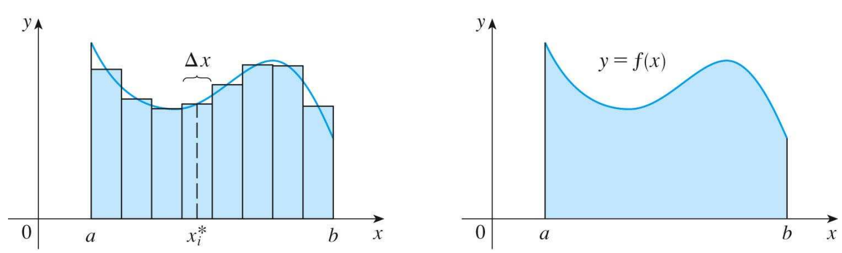
The Area Problem



- It is not so easy to find the area of a region with curved sides.
- We all have an intuitive idea of what the area of a region is.
- Part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

The Area Problem: Strategy

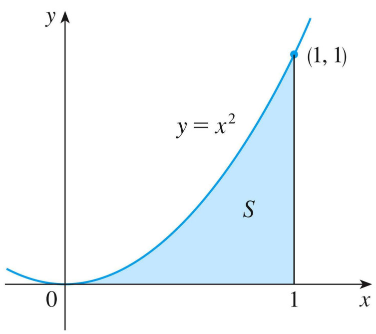
Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations.



We pursue a similar idea for areas. We first approximate the shaded region by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

Outline

The Area Problem: An Illustration

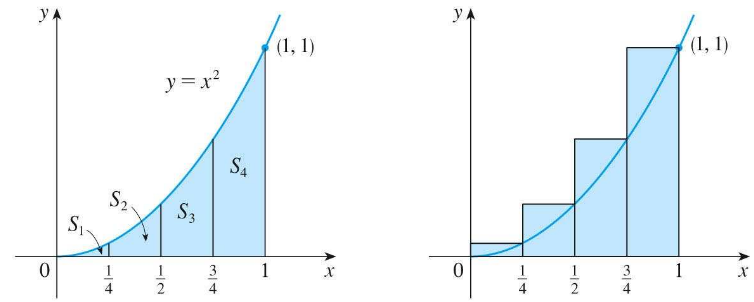


The Challenge

Use rectangles to estimate the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$ (the parabolic region S illustrated above).

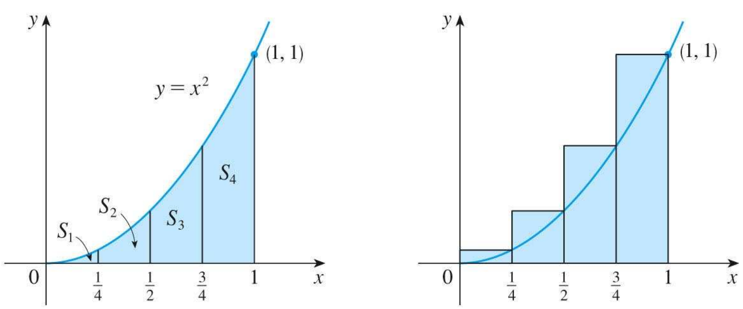
The Area Problem: An Illustration

Suppose we divide S into four strips $S1$, $S2$, $S3$ and $S4$ by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$ and $x = \frac{3}{4}$.



We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip.

The Area Problem: An Illustration

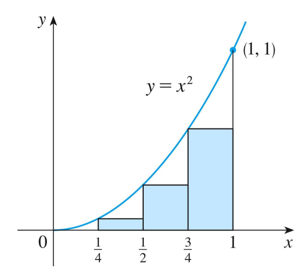


If we let R_4 be the sum of the areas of these approximating rectangles, we obtain

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot(1)^2 = 0.46875$$

From the diagram, we can see that the area A of S is less than R_4, so $A < 0.46875$.

The Area Problem: An Illustration



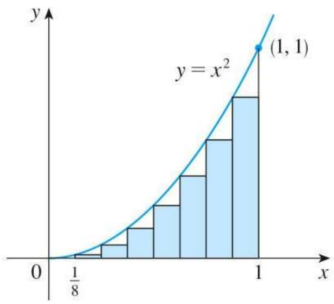
Instead of using the larger rectangles in the previous slide, we could use the smaller rectangles:

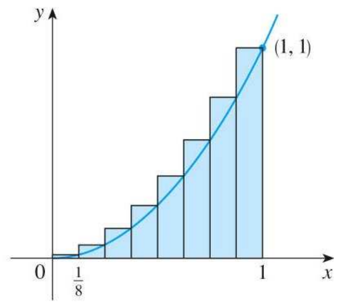
$$L_4 = \frac{1}{4}\cdot(0)^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A: $0.21875 < A < 0.46875$.

The Area Problem: An Illustration

We can repeat this procedure with a larger number of strips (8).

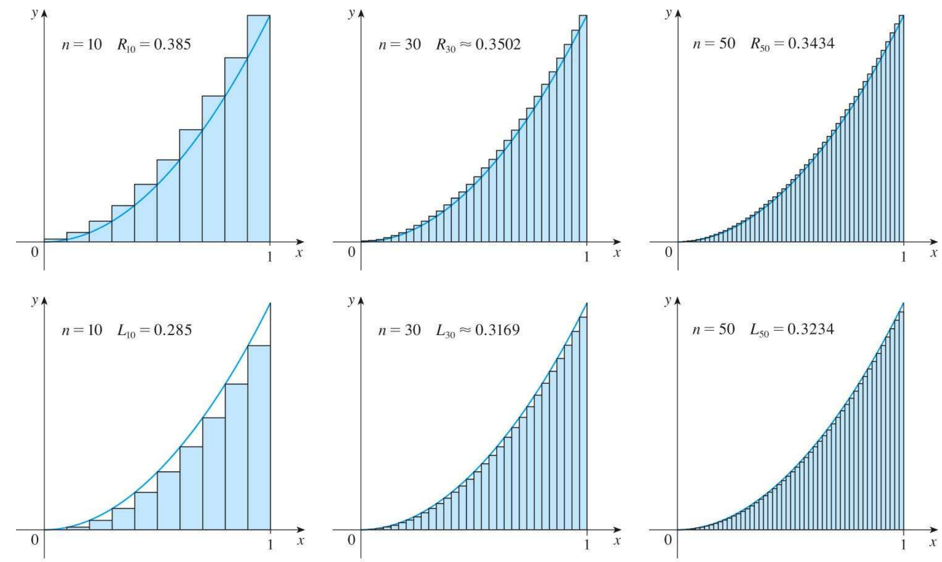




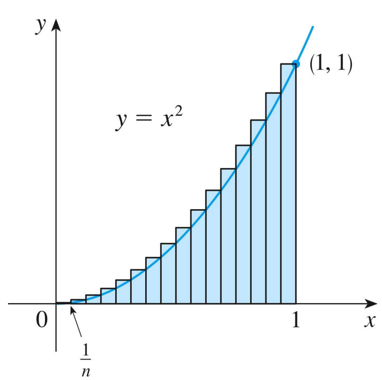
By computing the sum of the areas of the smaller rectangles L_8 and the sum of the areas of the larger rectangles R_8, we obtain better lower and upper estimates for A: $0.2734375 < A < 0.3984375$.

The Area Problem: An Illustration

Repeating this procedure with a even larger number of strips:

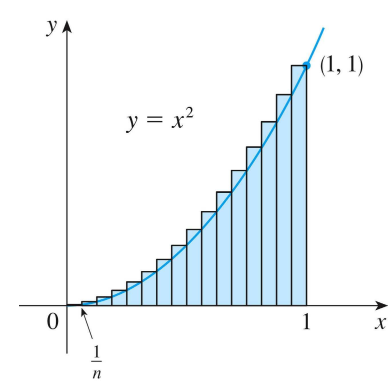


The Area Problem: An Illustration (Exact Calculation)



$$
\begin{aligned}
R_n &= \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{3}{n}\right)^2 + \cdots + + \frac{1}{n}\cdot\left(\frac{n}{n}\right)^2 \\
&= \frac{(n+1)(2n+1)}{6n^2}
\end{aligned}
$$

The Area Problem: An Illustration (Exact Calculation)

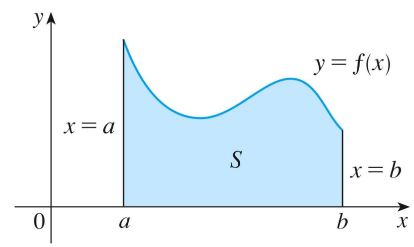


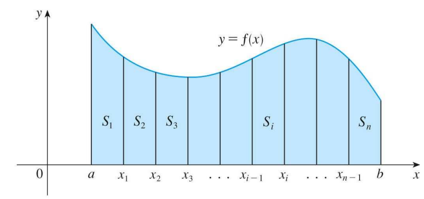
$$
\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{3}
\end{aligned}
$$

Outline

The Area Problem: General Formulation

We will apply the idea of the last example ($y = x^2$) to the more general region S:

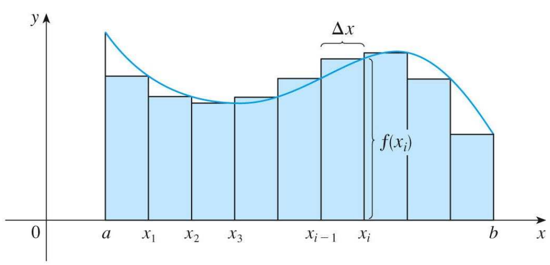




We subdivide S into n strips S_1, S_2, ..., S_n of equal width.

The Area Problem: General Formulation

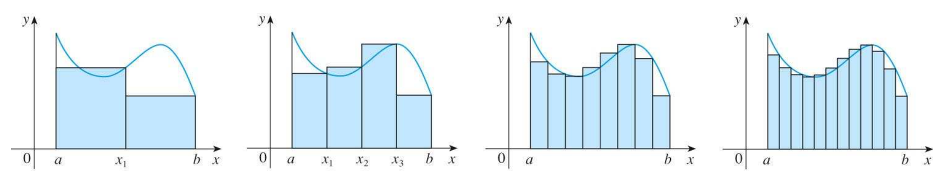
We approximate the i-th strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint.



Then the area of the i-th rectangle is $f(x_i)\,\Delta x$. The area of S is approximated by the sum of the areas of these rectangles:

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

The Area Problem: General Formulation



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

Notice that the approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

Definition

The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

Outline

Graphical Explanation

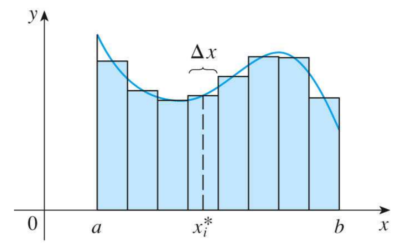


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

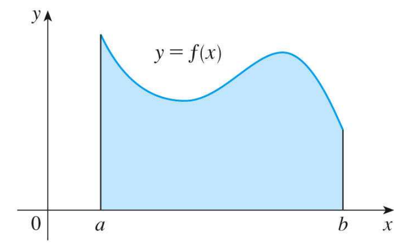
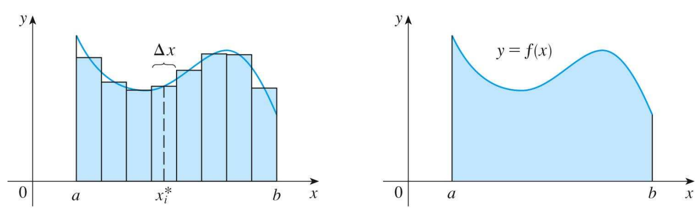


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

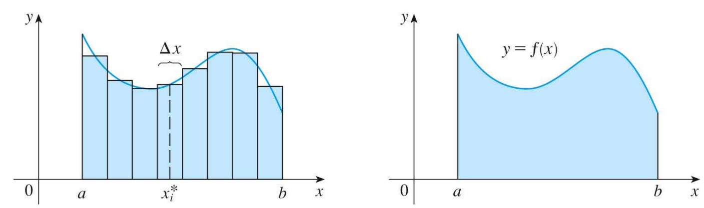
A More Mathematical Explanation



Definition (Slide 1/2)

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a$, x_1, . . ., $x_n = b$ be the endpoints of these subintervals and we let x_1^*, x_2^*, . . ., x_n^* be any sample points in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$.

A More Mathematical Explanation



Definition (Slide 2/2)

Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

A More Mathematical Explanation

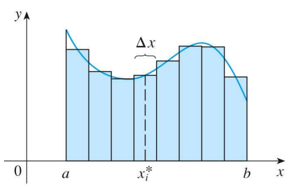


FIGURE 1
If $f(x) \geq 0$, the Riemann sum $\sum f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

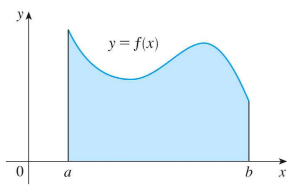


FIGURE 2
If $f(x) \geq 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

- The symbol $\int$ was introduced by Leibniz and is called an integral sign.
- It is an elongated S and was chosen because an integral is a limit of sums.
- $f(x)$ is called the integrand and a and b are called the limits of integration; a is the lower limit and b is the upper limit.

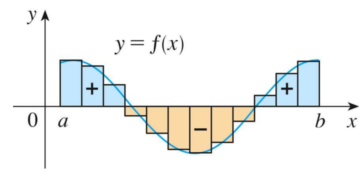


FIGURE 3
$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.

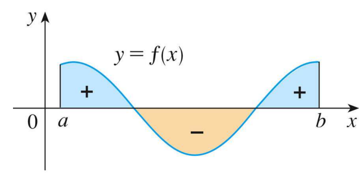


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

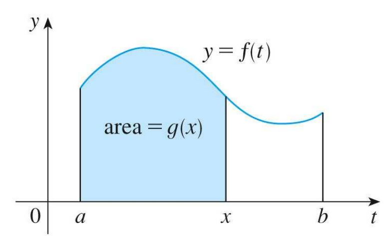
- If $f(x)$ takes on both positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis.
- A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

Outline

Differentiation & Integration as Inverse Processes



The Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. If $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is $F' = f$.

MS121: IT Mathematics

Integration

Some Rules of Integration

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Outline

Outline

Inverse Process of Differentiation

Overview

- The previous section of the syllabus dealt with the process of differentiation which involved finding the rate of change of a function with respect to a given variable, for example

$$\frac{d}{dx}x^3 = 3x^2 \qquad\qquad y = \ln x \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{x}$$

- We now consider the inverse process of differentiation, that of finding a function whose derivative is known, and this is called the process of integration.

Inverse Process of Differentiation

Notation

The symbol $\int$ is used to denote the operation of integration, i.e.

$$\int_a^b f(x)\,dx = \text{ the } \textit{integral of } f \text{ from } x = a \text{ to } x = b$$

where a and b are called the limits of integration (a is the lower limit and b is the upper limit).

The key to calculating integrals is to recognize that

Integration reverses the action of Differentiation

Inverse Process of Differentiation

Integration reverses the action of Differentiation

The statement that

differentiating x^3 gives $3x^2$

can be inverted to one which states that

integrating $3x^2$ gives x^3.

Integration reverses the action of Differentiation

$f(x)$	$\int f(x)\,dx$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
e^x	e^x
$\frac{1}{x}$	$\log x$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$

Note that differentiating the entry in the right-hand column (integral) gives the corresponding entry in the left-hand column.

Integration reverses the action of Differentiation

Examples

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x\,dx = \sin x$$

$$\frac{d}{dx}(e^x) = e^x \quad \Rightarrow \quad \int e^x\,dx = e^x$$

$$\frac{d}{dx}\left(\frac{x^5}{5}\right) = \frac{1}{5}\cdot 5x^4 = x^4 \quad \Rightarrow \quad \int x^4\,dx = \frac{x^5}{5}$$

Outline

The Constant of Integration

Illustration

Since

$$\begin{aligned} \frac{d}{dx}x^2 &= 2x \\ \frac{d}{dx}(x^2+5) &= 2x \\ \frac{d}{dx}(x^2-100) &= 2x \end{aligned}$$

it is quite common to write

$$\int 2x\,dx = x^2 + C,$$

where C is a constant called the constant of integration.

The Constant of Integration

$$\int 2x\,dx = x^2 + C,$$

Rationale

- If we differentiate $x^2 + C$, we obtain $2x$.
- This is because the derivative of any constant C is always 0.
- In this way, for example, the statement

$$\frac{d}{dx}(\sin x + C) = \cos x$$

is equivalent to

$$\int \cos x\,dx = \sin x + C.$$

Outline

Definite & Indefinite Integrals

When we are not specific, or definite, about where to start and stop integrating, we call the integral the indefinite integral:

$$\int f(x)\,dx, \quad \text{which is a function of } x$$

When we define the lower and upper limits of integration (a and b), we call the integral the definite integral:

$$\int_a^b f(x)\,dx \quad \text{which is a number}$$

Definite & Indefinite Integrals

Evaluating Definite Integrals

Given $F(x) = \int f(x)\,dx$, we can evaluate the definite integral

$$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{b} := F(b) - F(a).$$

Note that the area between the curve $y = f(x)$ and the x-axis between the vertical lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$

Outline

Examples: Evaluating Definite Integrals

Calculate $\int_1^2 x\,dx$

We can find

$$\int_1^2 x\,dx$$

by first noting that

$$\int x\,dx = \frac{x^2}{2} \qquad \left(\text{since } \frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2}\cdot 2x = x\right)$$

So

$$\int_1^2 x\,dx = \frac{x^2}{2}\Big|_{x=1}^{2} = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$$

Examples: Evaluating Definite Integrals

Calculate $\int_2^4 e^x\,dx$

The integral

$$\int_2^4 e^x\,dx$$

is found by observing that

$$\int e^x\,dx = e^x$$

and so

$$\int_2^4 e^x\,dx = e^x\Big|_{x=2}^{4} = e^4 - e^2$$

Examples: Evaluating Definite Integrals

Calculate $\int_1^2 \frac{1}{x}\,dx$

We find

$$\int_1^2 \frac{1}{x}\,dx$$

by referring to the formulae and tables:

$$\int_1^2 \frac{1}{x}\,dx = \log x\Big|_{x=1}^{2} = \log 2 - \log 1$$

Examples: Evaluating Definite Integrals

Calculate $\int_1^2 \frac{1}{x^2}\,dx$

We must first write

$$\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx$$

and use the general formula for $n \neq -1$:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

so that, in this instance, with $n = -2$, must have

$$\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

Examples: Evaluating Definite Integrals

Calculate $\int_1^2 \frac{1}{x^2}\,dx$ (Cont'd)

Since

$$\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

we therefore obtain

$$\int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{x=1}^{2} = -\frac{1}{2} - \left(-\frac{1}{1}\right) = -\frac{1}{2} + 1 = \frac{1}{2}$$

Outline

Two Basic Rules of Integration

(P1) The integral of the sum is the sum of the integrals:

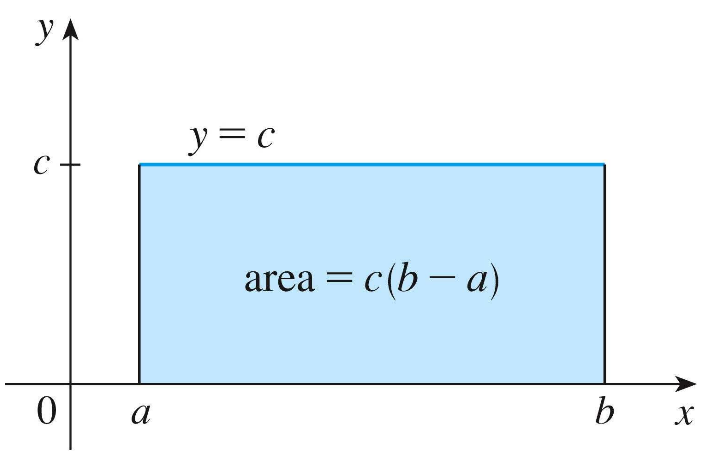
$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

(P2) A constant can be factored outside the integral:

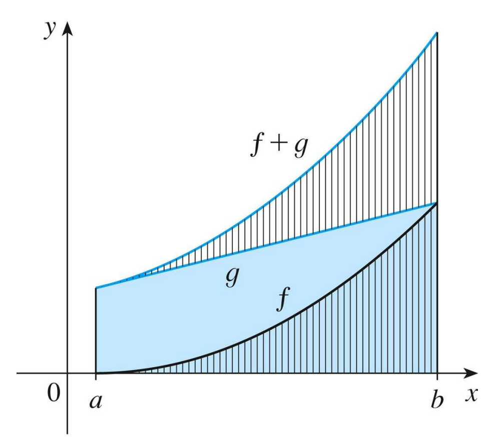
$$\int c\, f(x)\, dx = c \int f(x)\, dx, \quad \text{where } c \text{ is any } \textit{constant}$$

It is then straightforward to calculate the integrals of a wide class of functions.

$\int_a^b c\, dx = c(b-a)$, c a constant



$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$



Examples: Evaluating Definite Integrals

Calculate $\int \left(3x^{\frac{1}{2}} - \frac{2}{x} + 4\sin x\right)\,dx$

Using the two rules:

$$\begin{aligned}
\int \left(3x^{\frac{1}{2}} - \frac{2}{x} + 4\sin x\right)\,dx &= \int 3x^{\frac{1}{2}}\,dx + \int \left(-\frac{2}{x}\right)\,dx + \int 4\sin x\,dx \\
&= 3\int x^{\frac{1}{2}}\,dx - 2\int \frac{1}{x}\,dx + 4\int \sin x\,dx \\
&= 3\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) - 2\log x + 4\left(-\cos x\right) \\
&= 2x^{\frac{3}{2}} - 2\log x - 4\cos x
\end{aligned}$$

MS121: IT Mathematics

Integration

Two Techniques of Integration

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Outline

Integration by Substitution

Learn from Differentiation

To differentiate

$$y = e^{x^3}$$

we substitute $u = x^3$ and differentiate using the Chain Rule. We have

$$\begin{aligned} y &= e^u \qquad u = x^3 \\ \Rightarrow \frac{dy}{du} &= e^u \qquad \frac{du}{dx} = 3x^2 \\ \Rightarrow \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot \left(3x^2\right) = 3x^2\, e^{x^3} \end{aligned}$$

Integration by Substitution

$$y = e^{x^3} \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2\, e^{x^3}$$

Applying this to Integration

From the above, we therefore know that

$$\int 3x^2 e^{x^3}\, dx = e^{x^3}$$

How could we calculate the integral directly? The reason that it is difficult to calculate

$$\int 3x^2 e^{x^3}\, dx$$

is similar to that which makes it difficult to calculate the derivative of e^{x^3}.

Integration by Substitution

Find $\int 3x^2\, e^{x^3}\, dx$

We try the same substitution again, so

$$u = x^3 \qquad \Rightarrow \qquad \frac{du}{dx} = 3x^2 \qquad \Rightarrow \qquad du = 3x^2\, dx$$

Now replace the x's under the integral (including dx) by the u's (and du):

$$\begin{aligned} \int 3x^2 e^{x^3}\, dx &= \int e^{x^3} \star 3x^2\, dx \\ &= \int e^u\, du \\ &= e^u = e^{x^3} \end{aligned}$$

which gives the same answer as before.

Integration by Substitution

Technique: What is the best choice for u?

The following are some useful tips:

- Let $u =$ the "most complicated expression", particularly if that expression is the argument of another function. This is the same choice of u that one would use if differentiating using the chain rule (e.g., choosing $u = x^3$ above).
- Let $u = f(x)$ if $f'(x)\, dx$ is also present under the integral.
- If the integrand (function to be differentiated) is a quotient, let $u =$ the denominator.

These hints may be contradictory and they are not foolproof. However, they are sufficient to tackle a large class of integrals and, with enough practice, one can develop an intuition for the correct strategy.

Outline

Integration by Substitution: Examples

Calculate $\displaystyle\int \frac{x^4}{x^5+2}\,dx$

We let

$$u = x^5 + 2 \quad \Rightarrow \quad \frac{du}{dx} = 5x^4 \quad \Rightarrow \quad du = 5x^4\,dx$$

Replacing the x's by u's:

$$\int \frac{x^4}{x^5+2}\,dx = \int \frac{x^4\,dx}{x^5+2} = \int \frac{\frac{1}{5}\,du}{u} = \frac{1}{5}\int u^{-1}\,du = \frac{1}{5}\log u = \frac{1}{5}\log(x^5+2)$$

Note: Always write the final answer in terms of the original variable (in this case x).

Integration by Substitution: Examples

Calculate $\displaystyle\int x^3\sqrt{16+x^4}\,dx$

Note that the expression which is the argument of the square root is a good candidate for substitution. Accordingly, Let

$$u = 16 + x^4 \quad \Rightarrow \quad \frac{du}{dx} = 4x^3 \quad \Rightarrow \quad du = 4x^3\,dx$$

Look to arrange $4x^3\,dx (= du)$ in one group in the integral:

$$\int x^3\sqrt{16+x^4}\,dx = \int \sqrt{16+x^4} \star x^3\,dx = \int \sqrt{u} \star \frac{1}{4}\,du = \frac{1}{4}\int u^{1/2}\,du = \frac{1}{4}\frac{u^{1+1/2}}{1+1/2} = \frac{u^{3/2}}{6} = \frac{(16+x^4)^{3/2}}{6}$$

Integration by Substitution: Examples

Calculate $\displaystyle\int \frac{\log x}{x}\,dx$

If $f(x) = \log x$, then $f'(x)\,dx = \frac{1}{x}\,dx$ is present in the integrand. Therefore, put

$$u = \log x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \quad \Rightarrow \quad du = \frac{dx}{x}$$

$$\int \frac{\log x}{x}\,dx = \int \log x \star \frac{dx}{x} = \int u\,du = \frac{u^2}{2} = \frac{1}{2}(\log x)^2$$

Integration by Substitution: Examples

Calculate $\displaystyle\int x^3(x^4+4)^5\,dx$

Let $u = x^4 + 4$, the argument of the fifth power. Then

$$\frac{du}{dx} = 4x^3 \quad \Rightarrow \quad du = 4x^3\,dx$$

Group $4x^3\,dx (= du)$ together and substitute:

$$\int x^3(x^4+4)^5\,dx = \int (x^4+4)^5.x^3\,dx = \int u^5 \star \frac{1}{4}\,du = \frac{1}{4}\int u^5\,du = \frac{1}{4}\frac{u^6}{6} = \frac{(x^4+4)^6}{24}$$

Integration by Substitution: Examples

Calculate $\int \frac{x}{\sqrt{2x+2}}\,dx$

Let $u = 2x+2$, the argument of the square root. Thus

$$\frac{du}{dx} = 2 \quad \Rightarrow \quad du = 2\,dx$$

Now, we replace x (and dx) by u (and du):

$$\int \frac{x}{\sqrt{2x+2}}\,dx = \int \frac{x}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{2}\int \frac{x}{\sqrt{u}}\,du$$

Integration by Substitution: Examples

$$\int \frac{x}{\sqrt{2x+2}}\,dx = \int \frac{x}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{2}\int \frac{x}{\sqrt{u}}\,du$$

Solution (Cont'd)

We now replace x by expressing it in terms of u: $x = \frac{1}{2}(u-2)$, so

$$\begin{aligned}\int \frac{x}{\sqrt{2x+2}}\,dx &= \frac{1}{2}\int \frac{\frac{1}{2}(u-2)}{\sqrt{u}}\,du \\ &= \frac{1}{4}\int\left(\sqrt{u} - \frac{2}{\sqrt{u}}\right)du = \frac{1}{4}\left(\frac{u^{3/2}}{3/2} - 2\frac{u^{1/2}}{1/2}\right) \\ &= \frac{1}{6}u^{3/2} - u^{1/2} = \frac{1}{6}(2x+2)^{3/2} - (2x+2)^{1/2}\end{aligned}$$

Integration by Substitution: Trogonometry Examples

Calculate $\int \sin 6x\,dx$

We let $u = 6x$, the argument of the "sin" function:

$$\frac{du}{dx} = 6 \quad \Rightarrow \quad du = 6\,dx$$

Therefore:

$$\begin{aligned}\Rightarrow \int \sin 6x\,dx &= \int \sin u \star \frac{1}{6}\,du = \frac{1}{6}\int \sin u\,du \\ &= \frac{1}{6}\cdot(-\cos u) = -\frac{1}{6}\cos 6x\end{aligned}$$

Integration by Substitution: Trogonometry Examples

Calculate $\int \frac{\sin x}{2+\cos x}\,dx$

Since the integrand is a quotient, we put $u = 2+\cos x$, so $du = -\sin x\,dx$. Therefore, we have

$$\begin{aligned}\int \frac{\sin x}{2+\cos x}\,dx &= \int \frac{\sin x\,dx}{2+\cos x} = \int \frac{-du}{u} \\ &= -\log u = -\log(2+\cos x)\end{aligned}$$

Integration by Substitution: Trogonometry Examples

Calculate $\int \sin x \cos^3 x \, dx$

Let $u = \cos x$, the argument of the cube. Then

$$\frac{du}{dx} = -\sin x \quad \Rightarrow \quad du = -\sin x \, dx$$

Group $-\sin x \, dx (= du)$ together and substitute:

$$\begin{aligned} \int \sin x \cos^3 x \, dx &= \int \cos^3 x \star \sin x \, dx = \int u^3 \star (-du) \\ &= -\int u^3 \, du = -\frac{u^4}{4} = -\frac{\cos^4 x}{4} \end{aligned}$$

Integration by Substitution: Trogonometry Examples

Calculate $\int \frac{\tan^5 x}{\cos^2 x} \, dx$

We solve as follows:

$$\frac{d}{dx} \tan x = \sec^2 x, \quad \text{so put } u = \tan x \text{ to obtain}$$

$$du = \sec^2 x \, dx = \frac{dx}{\cos^2 x}$$

$$\Rightarrow \int \frac{\tan^5 x}{\cos^2 x} \, dx = \int u^5 \, du = \frac{u^6}{6} = \frac{\tan^6 x}{6}$$

Outline

Integration by Parts

The Link to Differentiaton

- We have seen how integration by substitution corresponds to the chain rule for differentiation.
- It is reasonable to ask if there is a rule of integration corresponding to the product rule?
- The answer to this question is "yes" and the rule is that of integration by parts.
- This is a rule which is useful for calculating the integrals of products:

$$\boxed{\int u \, dv = uv - \int v \, du}$$

Outline

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x\,e^{2x}\,dx$

We will let $u = x$ and $dv = e^{2x}\,dx$ so that

$$\int xe^{2x}\,dx = \int u\,dv$$

We also need du and v:

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$$

i.e. we differentiate u to get du. Also

$$dv = e^{2x}\,dx \quad \Rightarrow \quad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$$

i.e. we integrate dv to get v.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x\,e^{2x}\,dx$ (Cont'd)

Let

$$u = x \qquad \text{and} \qquad dv = e^{2x}\,dx$$

so that

$$du = dx \qquad \text{and} \qquad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$$

Now, substitute u, v, du and dv into the integration by parts formula to obtain:

$$\int x\,e^{2x}\,dx = \int u\,dv = uv - \int v\,du \;=\; x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \frac{1}{2}x\,e^{2x} - \frac{1}{4}e^{2x}$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x\cos x\,dx$

Choose $u = x$ and $dv = \cos x\,dx$, so that Inspecting the right-hand side of the integration by parts formula, we also need du and v:

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad \Rightarrow du = dx$$

$$dv = \cos x\,dx \quad \Rightarrow \quad v = \int \cos x\,dx = \sin x$$

Substitute u, v, du and dv into the formula to obtain:

$$\int x\cos x\,dx = \int u\,dv = uv - \int v\,du$$

$$= \quad x\sin x - \int \sin x\,dx = x\sin x + \cos x$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Important Question

- How do we know in practice what to choose for u and dv?
- To answer the question, we first consider another example where we make our choice by trial and error.
- We then present a set of guidelines to assist in the decision-making process.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x^2 \log x\,dx$

We could choose either

$$u = x^2 \qquad \text{or} \qquad u = \log x$$

Which should one try?

(a) $u = x^2$: Then $dv = \log x\,dx$, so

$$v = \int \log x\,dx.$$

This is not listed in the formulae and tables. We will therefore abandon this approach.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x^2 \log x\,dx$ (Cont'd)

(b) $u = \log x$: Then $dv = x^2\,dx$, so

$$v = \int x^2\,dx = \frac{1}{3}x^3 \qquad \text{and} \qquad \frac{du}{dx} = \frac{1}{x} \qquad \Rightarrow \qquad du = \frac{1}{x}\,dx$$

Substitute these expressions into the integration by parts formula:

$$\begin{aligned}\int x^2 \log x\,dx &= \int u\,dv = uv - \int v\,du \\ &= \log x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx \\ &= \frac{1}{3}x^3 \log x - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3 \log x - \frac{1}{9}x^3\end{aligned}$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Answer to the Important Question

To decide how u should be chosen in general, we advocate that the choice should be made in the following order:

L – **L**ogarithm
A – **A**lgebraic: powers of x, e.g. x, x^3
T – **T**rigonometric
E – **E**xponential

Integration by Parts: $\int u\,dv = uv - \int v\,du$

"LATE" Guidelines

The guideline has been used in all the examples:

1. $\int xe^{2x}\,dx$: $x \to \mathbf{A}$, $e^{2x} \to \mathbf{E}$
   LATE $\Rightarrow u = x$ (**A**), so $dv = e^{2x}\,dx$.
2. $\int x\cos x\,dx$: $x \to \mathbf{A}$, $\cos x \to \mathbf{T}$
   LATE $\Rightarrow u = x$ (**A**), so $dv = \cos x\,dx$.
3. $\int x^2 \log x\,dx$ $x^2 \to \mathbf{A}$, $\log x \to \mathbf{L}$
   LATE $\Rightarrow u = \log x$ (**L**), so $dv = x^2\,dx$.

This rule gives very reliable guidance, but occasionally it does not work. Sometimes, we must integrate by parts successively to get an answer as the following example will illustrate.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Calculate $\int x^2 \cos x\,dx$

According to our notation, this is of type LATE $\Rightarrow u = x^2$ (**A**), so $dv = \cos x\,dx$ (**T**). Therefore, we differentiate u and integrate dv to obtain:

$$\frac{du}{dx} = 2x, \qquad v = \int \cos x\,dx = \sin x$$

Substitute u, v, du, dv into the formula:

$$\int x^2 \cos x\,dx = \int u\,dv = uv - \int v\,du$$

$$= x^2 \sin x - \int \sin x \star 2x\,dx = x^2 \sin x - 2\int x \sin x\,dx$$

We see that we cannot calculate the last integral directly.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

$\int x^2 \cos x\,dx = x^2 \sin x - 2\int x \sin x\,dx$

We must use the integration by parts formula again for

$$\int x \sin x\,dx$$

which is again of type LATE $\Rightarrow u = x$ (**A**), so $dv = \sin x\,dx$ (**T**). We differentiate u and integrate dv to obtain:

$$\frac{du}{dx} = 1, \qquad v = \int \sin x\,dx = -\cos x$$

Substituting into the formula yields:

$$\int x \sin x\,dx = -x\cos x - \int (-\cos x)\,dx = -x\cos x + \sin x$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\int x \sin x\,dx$$

$$\int x \sin x\,dx = -x\cos x + \sin x$$

$\int x^2 \cos x\,dx$ (Cont'd)

Substituting the second result into the first yields:

$$\begin{aligned}\int x^2 \cos x\,dx &= x^2 \sin x - 2\int x \sin x\,dx \\ &= x^2 \sin x - 2(-x\cos x + \sin x) \\ &= x^2 \sin x + 2x\cos x - 2\sin x\end{aligned}$$

Outline

Definite Integrals: Integration by Substitution Example

Calculate $\displaystyle\int_1^4 \frac{x}{3x^2+1}\,dx$

Substitution

$$u = 3x^2 + 1 \quad\Rightarrow\quad \frac{du}{dx} = 6x \quad\Rightarrow\quad du = 6x\,dx \quad\Rightarrow\quad \frac{1}{6}du = x\,dx$$

This means that $\displaystyle\int \frac{x}{3x^2+1}\,dx = \frac{1}{6}\int \frac{1}{u}\,du$

Definite Integrals: Integration by Substitution Example

Substitution $u = 3x^2 + 1$ leads to $\displaystyle\int \frac{x}{3x^2+1}\,dx = \frac{1}{6}\int \frac{1}{u}\,du$

Change the limits of integration

$x = 1 \quad\Rightarrow\quad u = 3x^2 + 1 = 4 \qquad\qquad x = 4 \quad\Rightarrow\quad u = 3x^2 + 1 = 49$

Therefore

$$\begin{aligned}\int_1^4 \frac{x}{3x^2+1}\,dx &= \int_1^4 \frac{1}{3x^2+1}\,(x\,dx) = \int_4^{49} \frac{1}{u}\left(\frac{1}{6}\,du\right)\\ &= \frac{1}{6}\int_4^{49} \frac{1}{u}\,du\\ &= \frac{1}{6}\left[\ln u\right]_4^{49} = \frac{1}{6}(\ln 49 - \ln 4)\end{aligned}$$

Definite Integrals: Integration by Parts Example

Calculate $\displaystyle\int_1^3 r^3 \ln r\,dr$

Integration by Parts

$$u = \ln r \quad\Rightarrow\quad du = \frac{1}{r}\,dr$$

$$dv = r^3\,dr \quad\Rightarrow\quad v = \frac{1}{4}r^4$$

Definite Integrals: Integration by Parts Example

$\int_1^3 r^3 \ln r \, dr$ with $u = \ln r$ and $dv = r^3 \, dr \;\Rightarrow\; v = \frac{1}{4}r^4$

Using the integration by parts formula

$$\begin{aligned}
\int u \, dv &= \int_1^3 r^3 \ln r \, dr = \left[uv - \int v \, du\right]_{r=1}^{3} \\
&= \left[\frac{1}{4}r^4 \ln r\right]_1^3 - \int_1^3 \frac{1}{4}r^3 \, dr \\
&= \frac{81}{4}\ln 3 - 0 - \frac{1}{4}\left[\frac{1}{4}r^4\right]_1^3 \\
&= \frac{81}{4}\ln 3 - \frac{1}{16}(81-1) = \frac{81}{4}\ln 3 - 5
\end{aligned}$$

MS121: IT Mathematics

Integration

An Important Application of Integration

John Carroll
School of Mathematical Sciences

Dublin City University

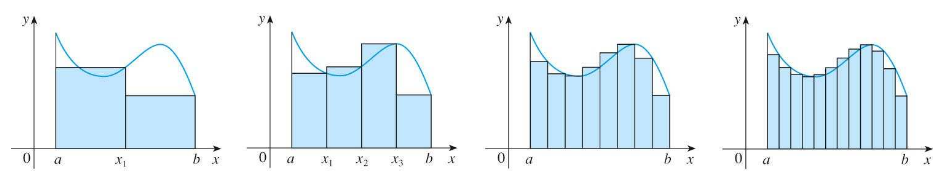
Outline

1. Area Under the Curve: A Brief Reminder
2. Area Between Curves: The Formula
3. Area Between Curves: Worked Examples
4. Concluding Special-Case Examples

Outline

1. Area Under the Curve: A Brief Reminder
2. Area Between Curves: The Formula
3. Area Between Curves: Worked Examples
4. Concluding Special-Case Examples

The Area Problem: General Formulation



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

We saw that the approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

Definition

The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x\right]$$

The Area Problem: The Definite Integral

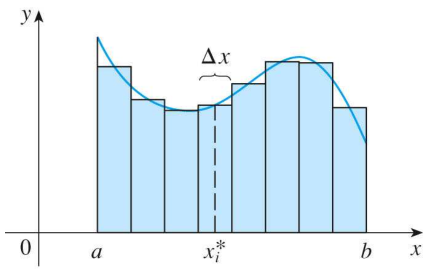


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

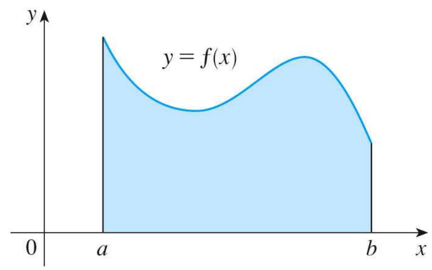


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

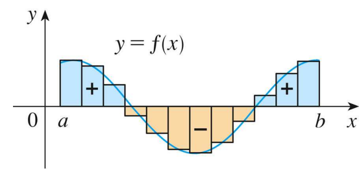


FIGURE 3
$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area.

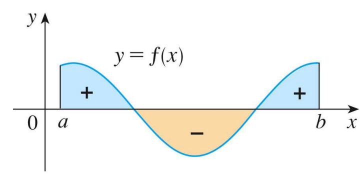


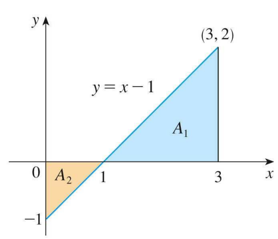
FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

- If $f(x)$ takes on both positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis.
- A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

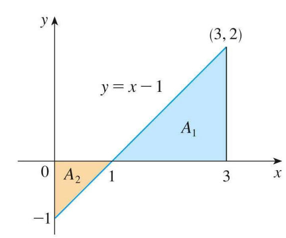
Positive & Negative Areas: Example 1



No need to perform integration. Simple calculations yield:

$$\begin{aligned}\int_0^3 (x-1)\,dx &= A_1 - A_2 \\ &= \frac{1}{2}(2 \star 2) - \frac{1}{2}(1 \star 1) \\ &= \frac{3}{2}\end{aligned}$$

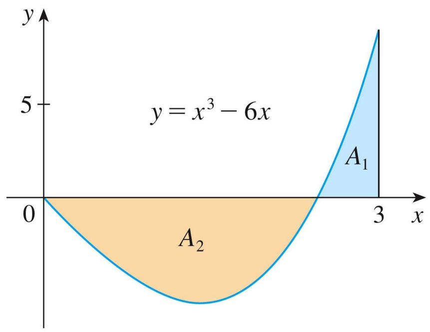
Positive & Negative Areas: Example 1 Re-Visited



Alternatively, integration yields:

$$\begin{aligned}\int_0^3 (x-1)\,dx &= \left[\frac{x^2}{2} - x\right]_0^3 \\ &= \left(\frac{3^2}{2} - 3\right) - \left(\frac{0^2}{2} - 0\right) \\ &= \frac{3}{2}\end{aligned}$$

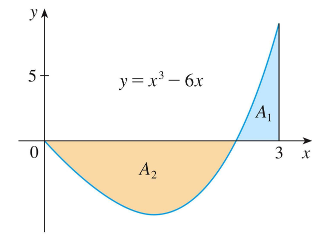
Positive & Negative Areas: Example 2



$A1$ is the area of the region above the x-axis and below the graph of f

$A2$ is the area of the region below the x-axis and above the graph of f.

Positive & Negative Areas: Example 2



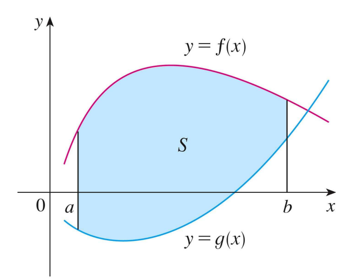
Calculations yield:

$$\begin{aligned}\int_0^3 \left(x^3 - 6x\right) dx &= \left[\frac{x^4}{4} - 3x^2\right]_0^3 \\ &= \left(\frac{3^4}{4} - 3(3^2)\right) - \left(\frac{0}{4} - 3(0^2)\right) \\ &= -\frac{27}{4}\end{aligned}$$

Outline

Area Between Curves: Developing The Formula

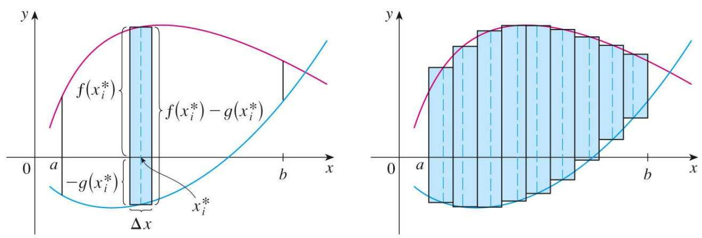
Earlier, we defined and calculated areas of regions that lie under the graphs of functions. Here we use integrals to find areas of regions that lie between the graphs of two functions.



Consider the region that lies between two curves $f(x)$ and $f(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all $x \in [a, b]$.

Area Between Curves: Developing The Formula

We divide S into n strips of equal width and then we approximate the i-th strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$:



(a) Typical rectangle (b) Approximating rectangles

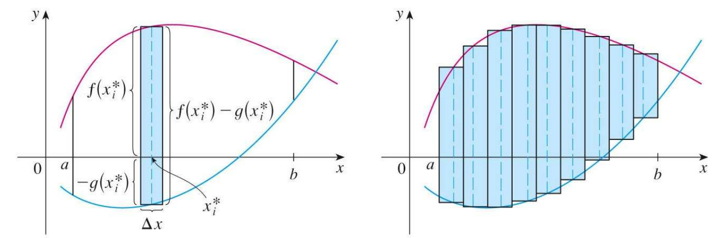
The Riemann sum

$$\sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*)\right] \Delta x$$

is an approximation to what we intuitively think of as the area of S.

Area Between Curves: Developing The Formula

This approximation appears to become better and better as $n \to \infty$.



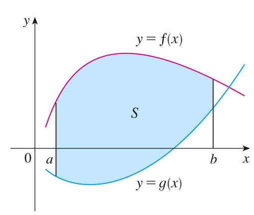
(a) Typical rectangle (b) Approximating rectangles

Therefore we define the area A of the region S as the limiting value of the sum of the areas of these approximating rectangles:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*)\right] \Delta x$$

Area Between Curves: Definition

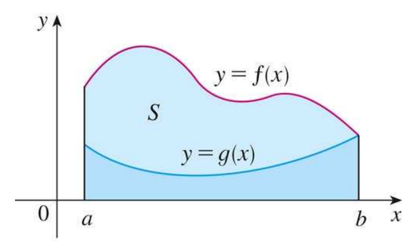
We recognize this limit as the definite integral of $f - g$. Therefore, we have the following formula for area.



The area A of the region bounded by the curves $f(x)$ and $f(x)$, and the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all $x \in [a, b]$ is

$$A = \int_a^b \left[f(x) - g(x)\right]\, dx$$

Area Between Curves: The Formula



In the case where both f and g are positive, you can see from the diagram why this is true.

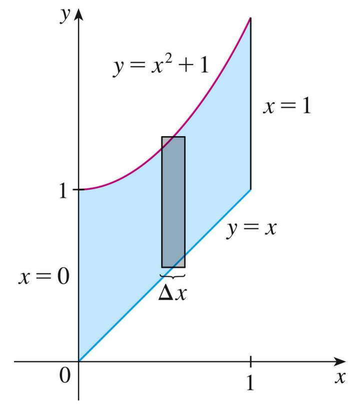
$$\begin{aligned} A &= [\text{area under } y = f(x)] - [\text{area under } y = g(x)] \\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \left[f(x) - g(x)\right]\,dx \end{aligned}$$

Outline

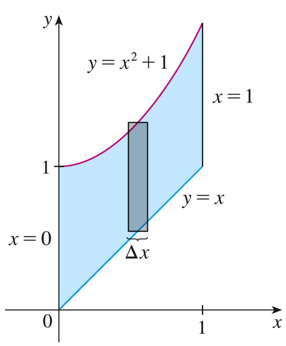
Area Between Curves: Worked Examples

Problem Statement

Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



Area Between Curves: Worked Examples



The area of the shaded region is

$$
\begin{aligned}
A &= \int_0^1 \left[\left(x^2 + 1\right) - x\right]\, dx \\
&= \int_0^1 \left(x^2 - x + 1\right)\, dx \\
&= \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^1 \\
&= \frac{5}{6}
\end{aligned}
$$

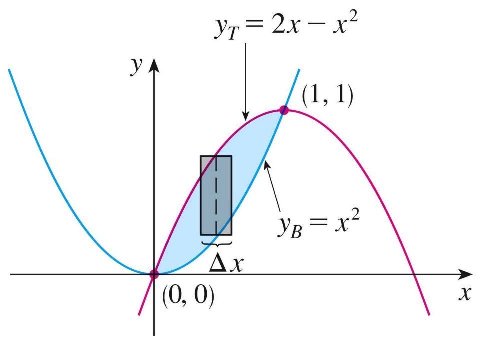
Outline

Concluding Special-Case Examples

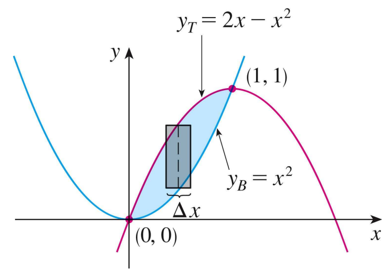
Problem Statement

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

We first have to find the points of intersection of the parabolas by solving their equations simultaneously.



Concluding Special-Case Examples



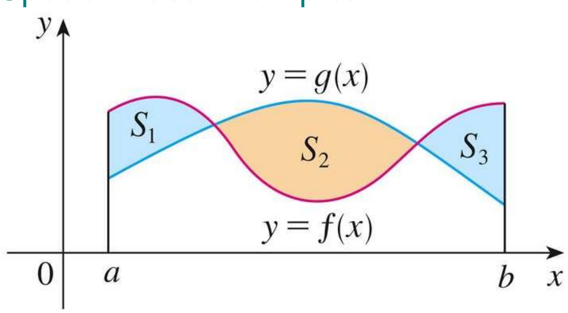
Solving the equations simultaneously:

$$x^2 = 2x - x^2 \Rightarrow 2x(x-1) \Rightarrow x = 0,\ 1$$

so the region lies between $x = 0$ and $x = 0$ and the total area is

$$\begin{aligned} A &= \int_0^1 \left(2x - 2x^2\right)\, dx \\ &= \left[2\left(\frac{x^2}{2}\right) - 2\left(\frac{x^3}{3}\right)\right]_0^1 \\ &= \frac{1}{3} \end{aligned}$$

Concluding Special-Case Examples

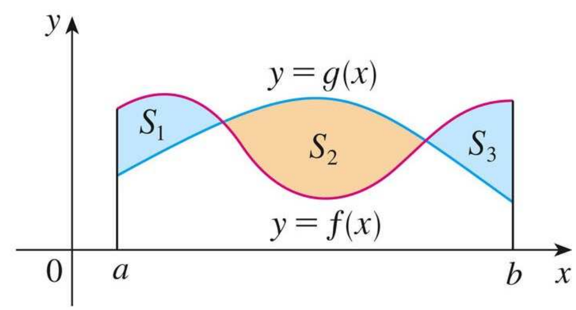


Issue

If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x. Note that

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases}$$

Concluding Special-Case Examples



Definition

The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

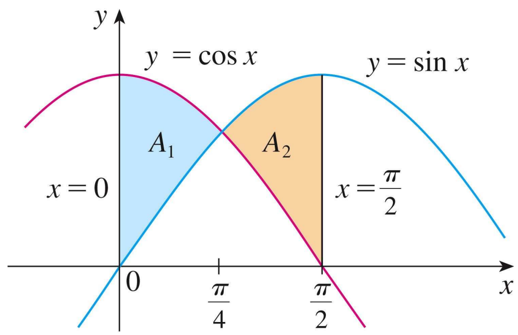
$$A = \int_a^b |f(x) - g(x)|\, dx$$

Concluding Special-Case Examples

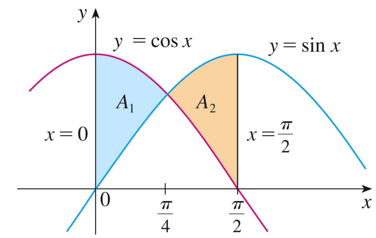
Problem Statement

Find the area of the region bounded by the curves $y = \sin(x)$, $y = \cos(x)$, $x = 0$ and $x = \pi$.

We first have to find the point of intersection of the curves:



Concluding Special-Case Examples

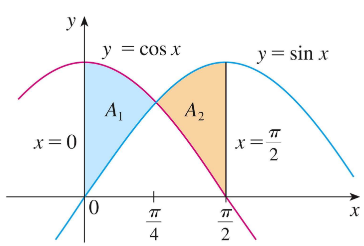


Solution

The points of intersection of the curve is when $\sin(x) = \cos(x)$ which is equivalent to $\tan(x) = 1$ and occurs when $x = \frac{\pi}{4}$. The required area is

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} |\cos(x) - \sin(x)|\ dx \\ &= \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\ dx + \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\ dx \end{aligned}$$

Concluding Special-Case Examples

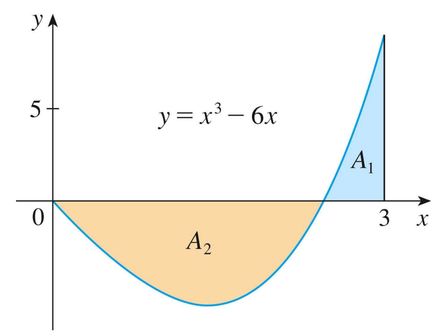


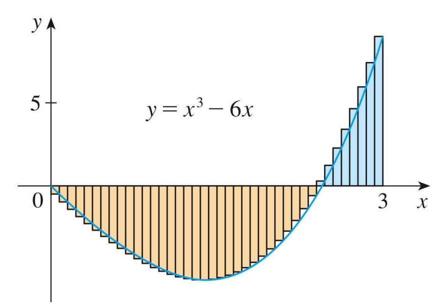
Solution (Cont'd)

The required area is

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\ dx + \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\ dx \\ &= [\sin(x) + \cos(x)]_0^{\frac{\pi}{4}} + [-\cos(x) - \sin(x)]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= 2\sqrt{2} - 2 \end{aligned}$$

Using Absolute Values: Summary Example

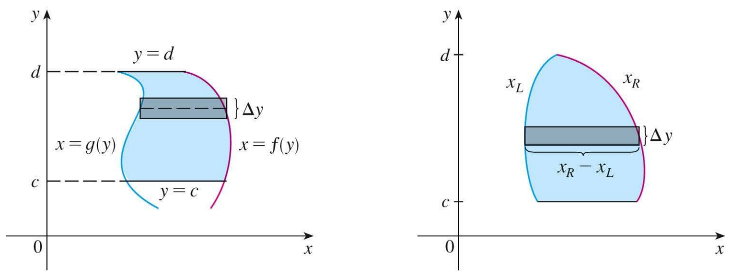




- $\int_0^{\sqrt{6}} (x^3 - 6x)\ dx = -9$ $\qquad \int_{\sqrt{6}}^{3} (x^3 - 6x)\ dx = \frac{9}{4}$
- $\int_0^{3} (x^3 - 6x)\ dx = -\frac{27}{4}$ $\qquad$ Area: $|A_2| + A_1 = 9 + \frac{9}{4} = \frac{45}{4}$

Area Between Curves: Writing x as a function of y

Some regions are best treated by regarding x as a function of y.

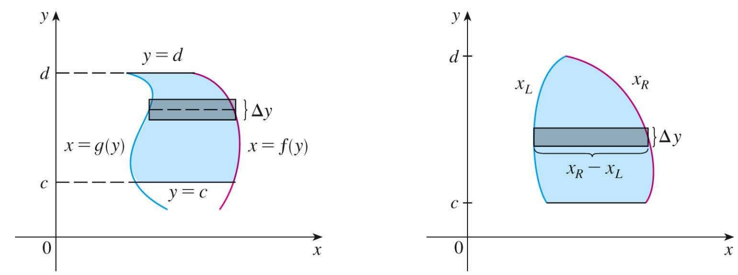


If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then its area is

$$A = \int_c^d [f(y) - g(y)]\, dy$$

Area Between Curves: Writing x as a function of y

$$A = \int_c^d [f(y) - g(y)]\, dy$$



If we write x_R for the right boundary and x_L for the left boundary, then we have

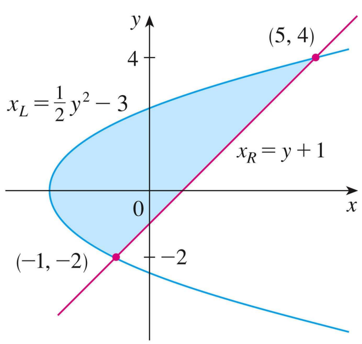
$$A = \int_c^d (x_R - x_L)\, dy$$

Area Between Curves: Writing x as a function of y

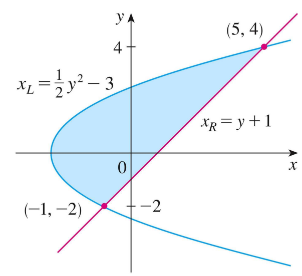
Problem Statement

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

By solving the two equations, we find that the points of intersection are $(-1, -2)$ and $(5, 4)$.



Area Between Curves: Writing x as a function of y



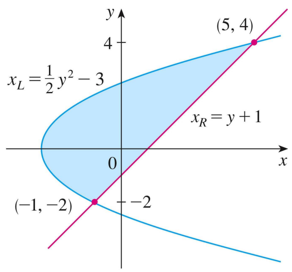
Solution

We solve the equation of the parabola for x and notice the left- and right-boundary curves are

$$x_L = \frac{1}{2}y^2 - 3 \quad \text{and} \quad x_R = y + 1$$

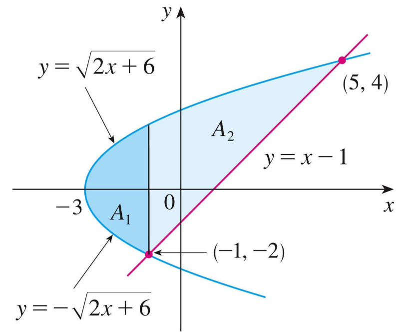
We must integrate between the appropriate y-values, $y = -2$ and $y = 4$.

Area Between Curves: Writing x as a function of y



$$
\begin{aligned}
A &= \int_{-2}^{4} (x_R - x_L)\, dy = \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3 \right) \right] dy \\
&= \int_{-2}^{4} \left(\frac{1}{2}y^2 + y + 4 \right) dy \\
&= \left[\frac{1}{2}\left(\frac{y^3}{3} \right) + \frac{y^2}{2} + y \right]_{-2}^{4} = 18
\end{aligned}
$$

Area Between Curves: Writing x as a function of y



Footnote to the Example

We could have found the area by integrating with respect to x instead of y, but the calculation is much more involved. It would have meant splitting the region in two and computing the areas labelled A_1 and A_2 (as shown). The method that we have just used is much easier.