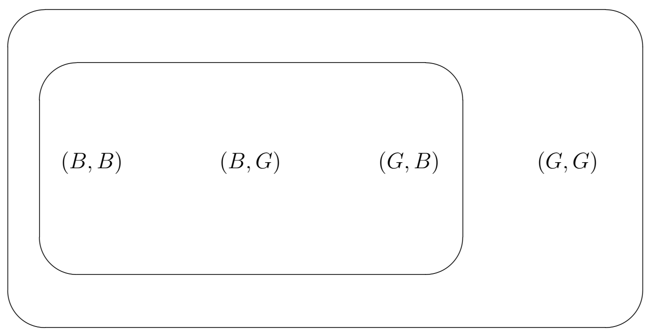
Example: Imagine a couple with two children, each of whom is equally likely to be a boy or a girl. Now suppose you are given the information that one is a boy. What is the probability that the other child is a boy?

Here the sample space is the set of pairs $\{(B,B),(B,G),(G,B),(G,G)\}$ with each pair equally likely. The event A consists of the first 3 outcomes and has $\mathbb{P}[A] = 3/4$. The event B consists of $\{(B,B)\}$ and has $\mathbb{P}[B] = 1/4$. The event $A \cap B$ is the same event as B. Thus

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = (1/4)/(3/4) = 1/3.$$



Definition: Recall that we say that a collection of subsets

$$B_1, B_2, \ldots, B_k$$

forms a partition of the sample space Ω if

$$B_1 \cup B_2 \cup \ldots \cup B_k = \Omega \quad \text{and} \quad B_i \cap B_j = \emptyset \quad \text{for} \quad i \neq j$$

Example: If A is any event then the collection $\{A, \overline{A}\}$ always gives a partition.

Proposition: (The law of total probability) Let $B_1, B_2, \ldots, B_k$ be a partition of Ω into events of positive probability. Then, for every $A \subset \Omega$

$$\mathbb{P}[A] = \mathbb{P}[A|B_1]\mathbb{P}[B_1] + \mathbb{P}[A|B_2]\mathbb{P}[B_2] + \ldots + \mathbb{P}[A|B_k]\mathbb{P}[B_k]$$

Proof: Since the B_j form a partition of Ω we can write A as a disjoint union

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k)$$

so that

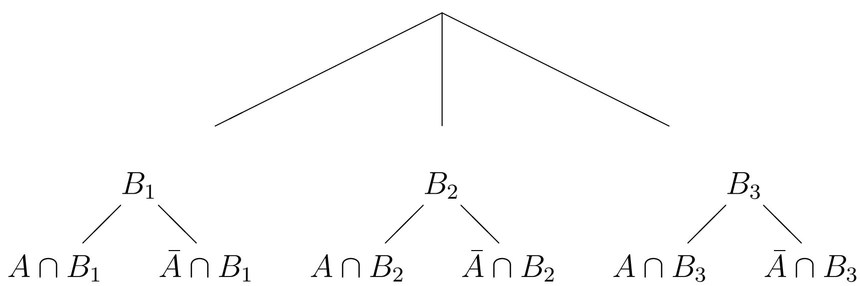
$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}[A \cap B_1] + \mathbb{P}[A \cap B_2] + \ldots + \mathbb{P}[A \cap B_k] \\ &= \mathbb{P}[A|B_1]\mathbb{P}[B_1] + \mathbb{P}[A|B_2]\mathbb{P}[B_2] + \ldots + \mathbb{P}[A|B_k]\mathbb{P}[B_k]\end{aligned}$$

where we use the fact

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \quad \Rightarrow \quad \mathbb{P}[A \cap B] = \mathbb{P}[A|B]\mathbb{P}[B]$$

Note: The law of total probability is useful for breaking up a complicated problem into manageable pieces.

Note: Sometimes it helps to view the law in terms of a tree diagram:



Example: A bag contains two coins, a fair one and a biased one. The biased one falls heads with probability 2/3. A coin is selected at random and tossed. What is the probability that it falls heads?

Let F be the event that the fair coin is selected, B the event that the biased coin is selected and H the event that heads show. Since there are only two coins, $\{B, F\}$ form a partition of the sample space. Hence

$$\mathbb{P}[H] = \mathbb{P}[H|F]\mathbb{P}[F] + \mathbb{P}[H|B]\mathbb{P}[B] = (1/2)(1/2) + (2/3)(1/2) = 7/12$$

Note: It might be helpful to think of this example as follows. Suppose we do this experiment 12 times. We expect the fair coin to be selected half of the time, so we expect the fair coin to occur 6 times. When we toss the fair coin we expect to get heads half of the time. So three of the outcomes should be the fair coin showing heads. Similarly, three of the outcomes should be the fair coin showing tails. Likewise we expect the biased coin to be selected half of the time, so we expect the biased coin to occur 6 times. When we toss the biased coin we expect to get heads two thirds of the time. So four of the outcomes should be the biased coin showing heads. Similarly, two of the outcomes should be the biased coin showing tails. Our 12 outcomes should be

$$\{(F,H),(F,H),(F,H),(F,T),(F,T),(F,T),$$
$$(B,H),(B,H),(B,H),(B,H),(B,T),(B,T)\}.$$

Sure enough, heads show 7 times out of the 12.

Note: When considering $\mathbb{P}[A|B]$ it is natural to think of B as the 'cause' and of A as the 'effect'. With this interpretation the following result allows us to derive the probability that a given effect is due to a certain cause.

Proposition: (Bayes' formula) Let $B_1, B_2, \ldots, B_k$ be a partition of Ω into events of positive probability. Then, for every $A \subset \Omega$

$$\mathbb{P}[B_j|A] = \frac{\mathbb{P}[A|B_j]\mathbb{P}[B_j]}{\mathbb{P}[A|B_1]\mathbb{P}[B_1] + \mathbb{P}[A|B_2]\mathbb{P}[B_2] + \ldots + \mathbb{P}[A|B_k]\mathbb{P}[B_k]}$$

Proof: Using the definition of conditional probability and the law of total probability we get

$$\begin{aligned}\mathbb{P}[B_j|A] &= \frac{\mathbb{P}[B_j \cap A]}{\mathbb{P}[A]} \\ &= \frac{\mathbb{P}[A|B_j]\mathbb{P}[B_j]}{\mathbb{P}[A]} \\ &= \frac{\mathbb{P}[A|B_j]\mathbb{P}[B_j]}{\mathbb{P}[A|B_1]\mathbb{P}[B_1] + \mathbb{P}[A|B_2]\mathbb{P}[B_2] + \ldots + \mathbb{P}[A|B_k]\mathbb{P}[B_k]}\end{aligned}$$

Example: A bag contains two coins, a fair one and a biased one. The biased one falls heads with probability 2/3. A coin is selected at random and tossed. Suppose now the selected coin falls H. What is the probability that the fair coin was selected?

Using the same notation as before, we know that

$$\mathbb{P}[H|F] = 1/2, \quad \mathbb{P}[H|B] = 2/3, \quad \mathbb{P}[F] = \mathbb{P}[B] = 1/2.$$

So we can use Bayes' formula to give

$$\mathbb{P}[F|H] \quad = \quad \frac{\mathbb{P}[H|F]\mathbb{P}[F]}{\mathbb{P}[H|F]\mathbb{P}[F] + \mathbb{P}[H|B]\mathbb{P}[B]}$$

$$
\begin{aligned}
&= \frac{(1/2)(1/2)}{(1/2)(1/2) + (2/3)(1/2)} \\
&= \frac{3}{7}.
\end{aligned}
$$

Note: When we last looked at this example we argued that doing the experiment 12 times should yield

$$\{(F,H),(F,H),(F,H),(F,T),(F,T),(F,T),$$

$$(B,H),(B,H),(B,H),(B,H),(B,T),(B,T)\}.$$

We see that of the 7 times heads show, 3 of them occur when we have selected the fair coin.

Example: A company produces components in two factories. It is know that a component from Factory A has a probability of $1/100$ of being faulty while a component from Factory B has a probability of $1/200$ of being faulty. It is also known that a finished component has a probability of $1/3$ of being from Factory A and a probability of $2/3$ of being from Factory B. Use Baye's Theorem to compute the probability that a finished component which is faulty was produced in Factory A.

Using the obvious names for the events we have

$$\mathbb{P}[F|A] = 1/100, \quad \mathbb{P}[F|B] = 1/200, \quad \mathbb{P}[A] = 1/3, \quad \mathbb{P}[B] = 2/3.$$

So we can use Bayes' formula to give

$$
\begin{aligned}
\mathbb{P}[A|F] &= \frac{\mathbb{P}[F|A]\mathbb{P}[A]}{\mathbb{P}[F|A]\mathbb{P}[A] + \mathbb{P}[F|B]\mathbb{P}[B]} \\
&= \frac{(1/100)(1/3)}{(1/100)(1/3) + (1/200)(2/3)} \\
&= \frac{1}{2}.
\end{aligned}
$$