MS121: IT Mathematics

Applications of Differentiation 2

Optimization Problems

John Carroll
School of Mathematical Sciences

Dublin City University

Outline

Outline

Optimization Problems

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function.

How to Solve Optimization Problems

Convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized:

Basic Steps

1. Understand the Problem:
   Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. Draw a Diagram
3. Introduce Notation:
   Assign a symbol to the quantity that is to be maximized or minimized (lets call it Q for now). Assign other symbols as required.
4. Express Q in terms of some of the other symbols from Step 3.
5. Use the methods developed earlier to find the absolute maximum or minimum value of Q.

Outline

Example 1: Finding The Maximum Area

Problem Statement

- A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river.
- He does not require a fence along the river.
- What are the dimensions of the field that has the largest area?

Example 1: Finding The Maximum Area

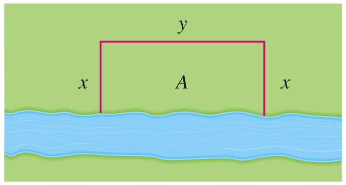


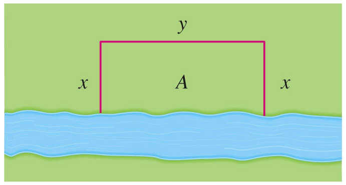
Diagram & Notation

- Let x and y be the depth and width of the rectangle (in meters).
- Then we express A in terms of x and y:

$$A = xy$$

- We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x.

Example 1: Finding The Maximum Area



Formulate A in terms of x

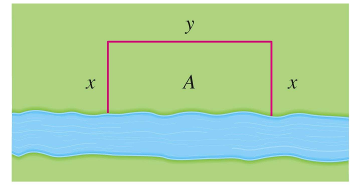
- We use the given information that the total length of the fencing is 1200 m. Thus

$$2x + y = 1200 \Rightarrow y = 1200 - 2x$$

- Since $A = xy$, we therefore have

$$A = x(1200 - 2x) = 1200x - 2x^2$$

Example 1: Finding The Maximum Area



Maximize $A = x(1200 - 2x)$

- Note that $x \geq 0$ and $x \leq 600$ (otherwise $A < 0$). So the function that we wish to maximize is

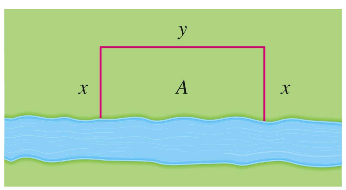
$$A = 1200x - 2x^2 \qquad 0 < x \leq 600$$

- The derivative is $A'(x) = 1200 - 4x$), so to find the critical numbers we solve the equation

$$1200 - 4x = 0$$

which gives $x = 300$).

Example 1: Finding The Maximum Area



Maximum of $A = x(1200 - 2x)$ occurs when $x = 300$

- Observe that

$$A''(x) = -4 < 0$$

for all x, so A is always concave downward and the local maximum at $x = 300$ must be an absolute maximum.

- Thus the rectangular field should be 300 m deep and 600 m wide (an area of $180,000\ m^2$).

Outline

Example 2: Minimize Total Surface Area

Problem Statement

- A cylindrical can is to be made to hold 1 L of oil.
- Find the dimensions that will minimize the cost of the metal to manufacture the can.

Example 2: Minimize Total Surface Area

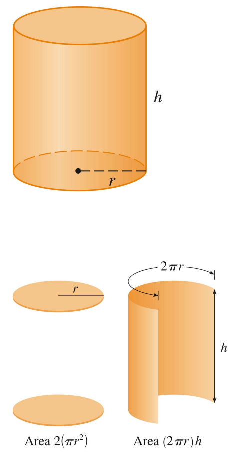
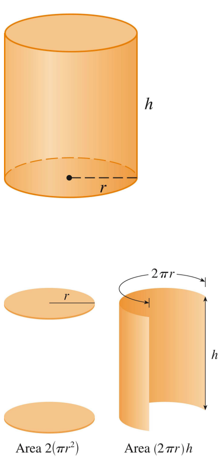


Diagram & Notation

- Let r be the radius and h the height (both in centimeters).
- In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides).
- The sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

Example 2: Minimize Total Surface Area



Eliminate h

To eliminate h, we use the fact that the volume is given as $1L$ which we take to be 1000 cm^3. Thus

$$\pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$$

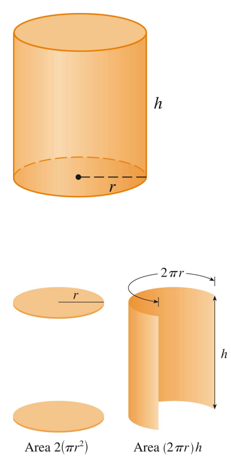
Substituting into the expression for 1000 gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

Therefore, the function that we want to minimize is

$$A = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

Example 2: Minimize Total Surface Area



Find the Mimiumum Value

To find the critical numbers, we differentiate:

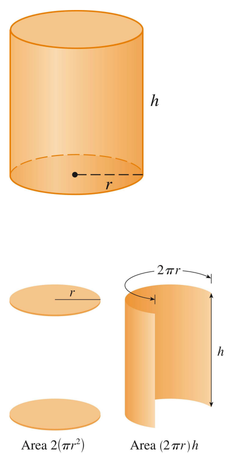
$$A = 2\pi r^2 + \frac{2000}{r}$$

to obtain

$$A' = 4\pi r - \frac{2000}{r^2} = \frac{4\left(\pi r^3 - 500\right)}{r^2}$$

Thus, $A' = 0$ when $\pi r^3 = 500$, so the only critical number is $\sqrt[3]{\frac{500}{\pi}}$.

Example 2: Minimize Total Surface Area



Mimiumum Value: $r^* = \sqrt[3]{\dfrac{500}{\pi}}$

Note that $A'(r) < 0$ for $r < r^*$ and $A'(r) > 0$ for $r > r^*$, so A is decreasing for all to the left of the critical number and increasing for all to the right. Thus r^* must give rise to an absolute minimum. Note also that

$$A'' = 4\pi + \frac{4000}{r^3} > 0$$

for all $r > 0$. Finally, the value of h to achive this minimum is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

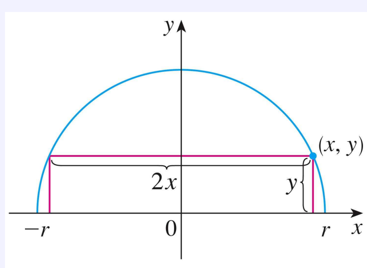
Outline

Example 3: Rectangle Inscribed in a Semicircle

Problem Statement

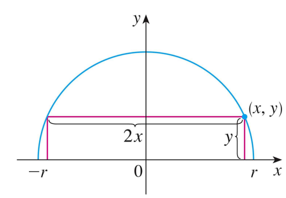
Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Structure



The word inscribed means that the rectangle has two vertices on the semicircle and two vertices on the x-axis.

Example 3: Rectangle Inscribed in a Semicircle



Establish the Variables

The rectangle has sides of lengths $2x$ and y, so its area is

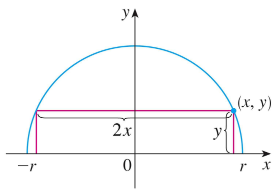
$$A = 2xy$$

To eliminate y, we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$) and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \leq x \leq r$.

Example 3: Rectangle Inscribed in a Semicircle

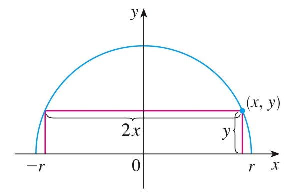


Identify the Critical Point(s)

$$\begin{aligned} A &= 2x\sqrt{r^2 - x^2} \\ A' &= 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2\left(r^2 - 2x^2\right)}{\sqrt{r^2 - x^2}} \end{aligned}$$

Note that $A' = 0$ when $2x^2 = r^2$, that is, $x = \frac{r}{\sqrt{2}}$ (since $x \geq 0$).

Example 3: Rectangle Inscribed in a Semicircle



Maximum at $x = \frac{r}{\sqrt{2}}$

This value of x gives a maximum value of A since $A(0) = 0$ and $A(r) = 0$. Therefore, the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2 \star \frac{r}{\sqrt{2}} \star \sqrt{r^2 - \frac{r^2}{2}} = r^2$$