Note: We recognise the graph of a function by the feature that a vertical line $x = a$ crosses the graph exactly once for each point a in the domain of f.

Example: The relation from $\mathbb{R}$ to $\mathbb{R}$ given by

$$R = \{(x, y) \mid x^2 = y^2\}$$

is not a function. The set of points in R is the pair of lines $y = x$ and $y = -x$ since

$$x^2 = y^2 \Leftrightarrow x^2 - y^2 = 0 \Leftrightarrow (x - y)(x + y) = 0 \Leftrightarrow (y = x) \textbf{ or } (y = -x)$$

A vertical line $x = a$ crosses this set at (a, a) and $(a, -a)$ which are different if $a \neq 0$.

Definition: We say two functions f and g are equal and write $f = g$ if f and g have the same domain and the same codomain and $f(x) = g(x)$ for each x in the domain.

Note: When f and g are obtained by complicated processes knowing that they are equal will be important.

Example: Sometimes we use equality to define new functions, as in

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

Example: Sometimes equality follows from some arithmetical fact, as in $f = g$ for $f(x) = (-2)(x - 2)$ and $g(x) = -2x + 4$.

Note: When looking at the digraph of a function we see a lack of symmetry between the domain and the codomain. We require exactly one arrow leaving each point in the domain but can have several or none ending at a point in the codomain.

Definition: A function f from a set A to a set B is called surjective or onto if $\text{Range}(f) = B$, that is, if $b \in B$ then $b = f(a)$ for at least one $a \in A$.

Note: The digraph of a surjective function will have at least one arrow ending at each element of the codomain.

Definition: A function f from a set A to a set B is called injective or one-to-one if no two elements in A have the same image in B, that is

$$[f(a_1) = f(a_2)] \Rightarrow [a_1 = a_2].$$

Note: The digraph of an injective function will have at most one arrow ending at each element of the codomain.

Note: Can think of one-to-one as 'not two-to-one'.

Definition: A function f from a set A to a set B which is both injective and surjective is called a bijective function or a one-to-one correspondence.

Example: $A = \{a, b, c\}$, $B = \{p, q, r, s\}$ with

$$R_3 = \{(a, q), (b, q), (c, r)\}$$

The corresponding function is neither surjective nor injective. There is no element of A mapping onto p, so not surjective. There are two elements a and b both mapping to q, so not injective.

Example: Suppose $A = \mathbb{Z}$, $B = \{0, 1, 2\}$ and $f : A \to B$ is defined by

$$f(n) = \text{the remainder when } n^2 \text{ is divided by } 3.$$

Is f injective? Is f surjective?

Compute some values and see if there is a pattern:
$n = 0$: Here $n^2 = 0 = 0(3) + 0$ so the remainder after dividing n^2 by 3 is 0.
$n = 1$: Here $n^2 = 1 = 0(3) + 1$ so the remainder after dividing n^2 by 3 is 1.
$n = 2$: Here $n^2 = 4 = 1(3) + 1$ so the remainder after dividing n^2 by 3 is 1.
$n = 3$: Here $n^2 = 9 = 3(3) + 0$ so the remainder after dividing n^2 by 3 is 0.

$n = -1$: Here $n^2 = 1 = 0(3) + 1$ so the remainder after dividing n^2 by 3 is 1.
$n = -2$: Here $n^2 = 4 = 1(3) + 1$ so the remainder after dividing n^2 by 3 is 1.
$n = -3$: Here $n^2 = 9 = 3(3) + 0$ so the remainder after dividing n^2 by 3 is 0.

It looks like $f(n)$ is always 0 or 1.

f is not injective since $f(3) = 0 = f(0)$. Two different integers are taken by f to the same point.

f is not surjective since $f(n) \neq 2$ for any integer n. To see this look at the remainder when n itself is divided by 3. Suppose $n = 3k + r$ where $r \in \{0, 1, 2\}$. Then

$$n^2 = (3k + r)^2 = 9k^2 + 6kr + r^2.$$

Since the first two terms are multiples of 3, the remainder when n^2 is divided by 3 is the same as the remainder when r^2 is divided by 3. But $r \in \{0, 1, 2\}$ so that $r^2 \in \{0, 1, 4\}$ and the remainder is either 0 or 1.