MS121: IT Mathematics

Differentiation

Rules for Differentiation: Part 1

John Carroll
School of Mathematical Sciences

Dublin City University

Outline

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Differentiation by Formula

Pattern Observed

You may have noticed the following pattern when we were differentiating from first principles:

$$\begin{array}{ccc} x & \to & 1 \\ x^2 & \to & 2x \\ x^3 & \to & 3x^2 \\ \vdots & \vdots & \vdots \\ x^{-1} & \to & -1 \cdot x^{-2} \\ x^{-2} & \to & -2 \cdot x^{-3} \\ \vdots & \vdots & \vdots \end{array}$$

Differentiation by Formula

General Rule

$$\boxed{\frac{d}{dx}x^n = n \cdot x^{n-1}}$$

and this rule is true for all values of n.

Example 1

To find the derivative of $\sqrt{x}$, note that $\sqrt{x} = x^{\frac{1}{2}}$ so that the general rule can be applied with $n = \frac{1}{2}$ to obtain

$$\frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}.$$

Note

This rule is found on page 25 of the "formulae and tables" booklet, along with a number of other useful derivatives which you may use without proof unless you have been explicitly asked to differentiate from first principles.

Differentiation by Formula

Other Entries in the Mathematical Tables

The derivatives of the trigonometric functions are also available, for example

$$\begin{aligned}\frac{d}{dx}\sin x &= \cos x\\ \frac{d}{dx}\cos x &= -\sin x\\ \frac{d}{dx}\tan x &= \sec^2 x\end{aligned}$$

Note that trigonometric definitions are on pages 13–16 .

The Exponential and Log Functions

Rate of Growth

- Another important derivative found in the log tables is for the exponential function.
- We know that the derivative of a function is equal to its slope which we also think of as being its rate of growth.

The Exponential and Log Functions

Question

What function has a derivative equal to the function itself?

Answer

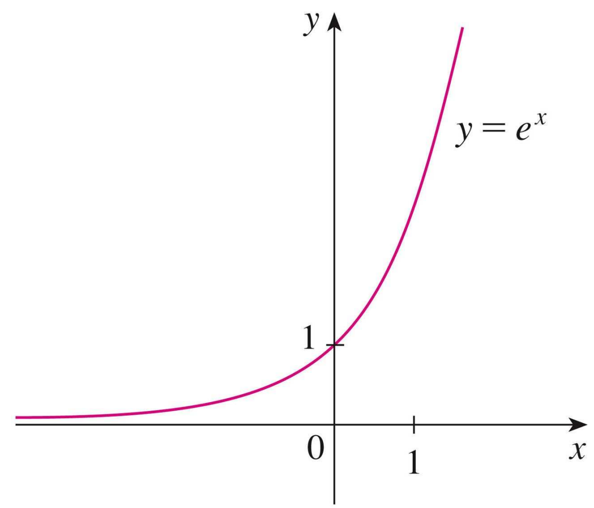
- The answer is the unique function

$$y = e^x = \exp(x)$$

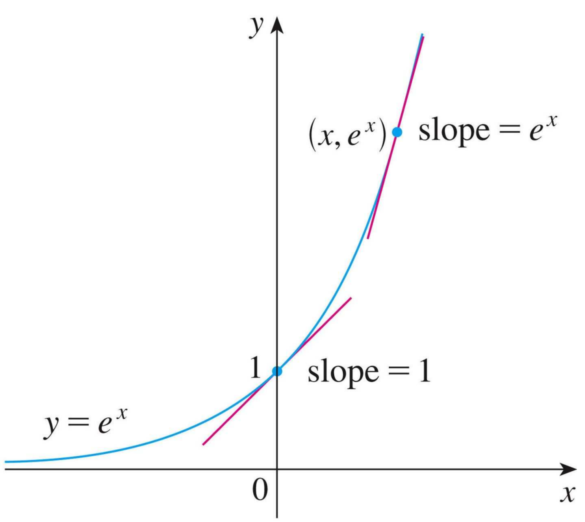
  the exponential function.
- For this function only,

$$\frac{dy}{dx} = y, \qquad \text{i.e.} \qquad \frac{d}{dx} e^x = e^x.$$

The function $y = e^x$



$y = e^x$ and $\frac{dy}{dx} = e^x$



The Exponential and Log Functions

The Exponential Function (Cont'd)

- Note that e^x is simply the number which we call "e" raised to the power of x ($e = e^1 = 2.718281828\ldots$).
- This number, like the number π, is an irrational number, i.e. a non-repeating decimal.
- Since $2 < e < 3$, the function e^x satisfies

$$2^x < e^x < 3^x,$$

  and the limit of e^x as $x \to \infty$ must be infinite, i.e.

$$\lim_{x \to \infty} e^x = \infty,$$

The Exponential and Log Functions

The Exponential Function (Cont'd)

- The limit of e^x as $x \to -\infty$ is

$$\lim_{x \to -\infty} e^x = \lim_{z \to \infty} e^{-z} = \lim_{z \to \infty} \frac{1}{e^z} = \frac{1}{\infty} = 0.$$

- In the foregoing, we simply made the substitution $z = -x$.

The Exponential and Log Functions

The Logarithmic Function

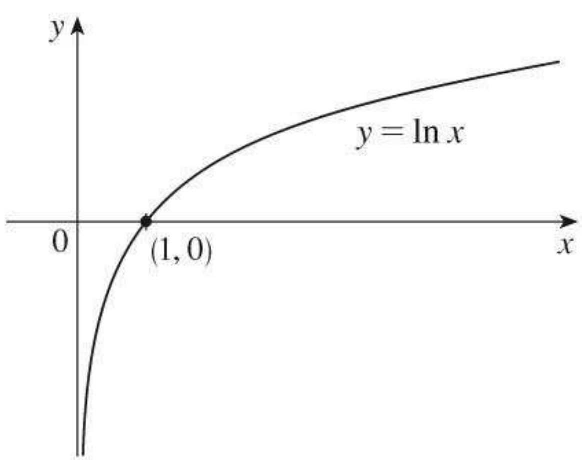
- The function $y = e^x$ has an inverse, namely

$$y = \ln x,$$

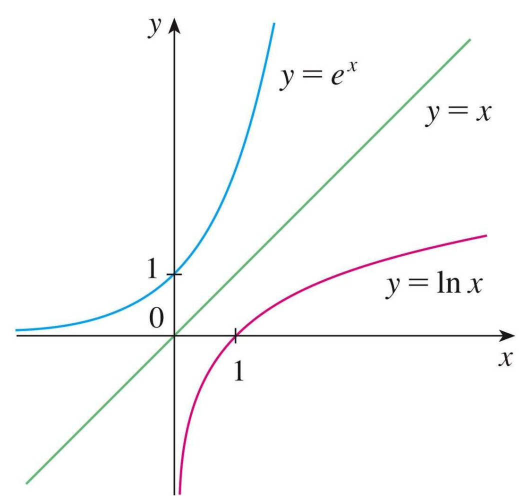
  the natural logarithm of x.
- A graph of $y = \ln x$ will show that the function is only defined on $(0, \infty)$ which is the range of the exponential function $y = e^x$ and hence the domain of $y = \ln x$.
- The derivative of $\ln x$ is also in the tables:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

The function $y = \ln x$



$y = e^x$ is a reflection of $y = \ln x$ in $y = x$



Outline

Sums & Differences of Functions

Work Plan

- We now introduce some rules for differentiation which will allow us to take the derivative of sums, products, quotients and compositions of functions.
- In the remainder of this section, we deal with sums of functions while the products, quotients and compositions of functions are dealt with in separate sections to follow.

Sums & Differences of Functions

Derivative of the Sum

The derivative of the sum is simply the sum of the derivatives:

$$\boxed{\frac{d}{dx}\left[u(x)+v(x)\right]=\frac{du}{dx}+\frac{dv}{dx}}$$

for any 2 functions of x.

Example 1

For $y=x^2+7$, we obtain

$$\frac{d}{dx}\left[x^2+7\right]=\frac{d}{dx}x^2+\frac{d}{dx}7=2x+0=2x.$$

Note how the derivative of any constant term is zero. You can prove this from first principles or simply apply the general for x^n with $n=0$, i.e.

$$\frac{d}{dx}7=\frac{d}{dx}7x^0=7\frac{d}{dx}x^0=7\cdot 0\cdot x^{-1}=0.$$

Example 2

Using the same rule, we find

$$\frac{d}{dx}\left[e^x + \ln x\right] = \frac{d}{dx}e^x + \frac{d}{dx}\ln x = e^x + \frac{1}{x}.$$

Outline

The Product Rule

Formula

To differentiate the product of two functions, $u(x)$ and $v(x)$, we must use the product rule, which is given in the Math Tables:

$$\boxed{\frac{d}{dx}\left[u(x) \star v(x)\right] = v\frac{du}{dx} + u\frac{dv}{dx}}$$

Example 3

Consider the function $y = xe^x$. To use the product rule, let $u = x$ and $v = e^x$ so that

$$\frac{du}{dx} = 1, \qquad \frac{dv}{dx} = e^x.$$

The product rule then gives:

$$\begin{aligned} v\frac{du}{dx} + u\frac{dv}{dx} &= e^x \cdot 1 + x \cdot e^x \\ &= e^x(1 + x) \end{aligned}$$

Example 4

For the function $y = \ln x \tan x$, we let

$$u = \ln x, \qquad v = \tan x,$$

so that

$$\frac{du}{dx} = \frac{1}{x}, \qquad \frac{dv}{dx} = \sec^2 x.$$

The product rule then gives:

$$\begin{aligned} v\frac{du}{dx} + u\frac{dv}{dx} &= \tan x \cdot \frac{1}{x} + \ln x \cdot \sec^2 x \\ &= \frac{1}{x}\tan x + \ln x \sec^2 x \end{aligned}$$

Example 5

Consider $y = x^{\frac{1}{4}}\left(2 + 3x + x^2\right)$.

Solution

With

$$u = x^{\frac{1}{4}}, \qquad v = 2 + 3x + x^2,$$

we obtain

$$\frac{du}{dx} = \frac{1}{4}x^{-\frac{3}{4}}, \qquad \frac{dv}{dx} = 3 + 2x.$$

The product rule then gives:

$$\begin{aligned} v\frac{du}{dx} + u\frac{dv}{dx} &= \left(2 + 3x + x^2\right) \cdot \frac{1}{4}x^{-\frac{3}{4}} + x^{\frac{1}{4}} \cdot (3 + 2x) \\ &= \frac{1}{4}x^{-\frac{3}{4}}\left(2 + 3x + x^2\right) + x^{\frac{1}{4}}(3 + 2x) \end{aligned}$$

Extension of the Formula

If you need to find the derivative of 3 functions, say $u(x)$, $v(x)$ and $w(x)$ multiplied together, then the formula to use is an extension of the product rule, namely

$$\boxed{\frac{d}{dx}\left[u(x) \star v(x) \star w(x)\right] = \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx}}$$

This rule is not found in the Math Tables because it is simply the product rule applied twice.

Example 6

Differentiate $y = e^x \sin x \tan x$.

Solution

We let

$$u = e^x, \qquad v = \sin x, \qquad w = \tan x,$$

so that

$$\frac{du}{dx} = e^x, \qquad \frac{dv}{dx} = \cos x, \qquad \frac{dw}{dx} = \sec^2 x.$$

$$y = e^x \sin x \tan x$$

Example 6 (Cont'd)

We then obtain

$$\begin{aligned}
\frac{d}{dx}[uvw] &= \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx} \\
&= e^x \sin x \tan x + e^x \cos x \tan x + e^x \sin x \sec^2 x \\
&= e^x \left\{\sin x \tan x + \cos x \tan x + \sin x \sec^2 x\right\} \\
&= e^x \sin x \left\{\tan x + 1 + \sec^2 x\right\}.
\end{aligned}$$

Outline

The Quotient Rule

Formula

To differentiate the quotient of two functions, $u(x)$ and $v(x)$, namely $y = \frac{u(x)}{v(x)}$, we must use the quotient rule, which is given in the log tables:

$$\boxed{\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Example 7

Find $\frac{dy}{dx}$ where $y = \dfrac{x^{\frac{1}{4}}}{\cos x}$

Solution

With

$$\begin{aligned}
u &= x^{\frac{1}{4}} & v &= \cos x \\
\frac{du}{dx} &= \frac{1}{4}x^{-\frac{3}{4}} & \frac{dv}{dx} &= -\sin x
\end{aligned}$$

$$\begin{aligned}
\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} &= \frac{\cos x \cdot \frac{1}{4}x^{-\frac{3}{4}} - x^{\frac{1}{4}} \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\frac{1}{4}x^{-\frac{3}{4}}\cos x + x^{\frac{1}{4}}\sin x}{\cos^2 x}.
\end{aligned}$$

Example 8

Find $\frac{dy}{dx}$ where $y = \dfrac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{\ln x}$

Solution

We let

$$u = x^{\frac{1}{2}} + x^{-\frac{1}{2}}, \qquad v = \ln x,$$

so that

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}, \qquad \frac{dv}{dx} = \frac{1}{x}.$$

The quotient rule then gives:

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\ln x \cdot \frac{1}{2}\left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right) - \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\cdot\frac{1}{x}}{\ln^2 x}$$

$$y = \frac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{\ln x}$$

Example 8 Cont'd

The quotient rule:

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\ln x \cdot \frac{1}{2}\left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right) - \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\cdot\frac{1}{x}}{\ln^2 x}$$

$$= \frac{\frac{1}{2}\ln x\left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right) - \frac{1}{x}\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)}{\ln^2 x}$$

Example 9

Consider the function $y = \frac{\sin x}{\cos x}$.

Solution

In this case, we have

$$u = \sin x, \qquad v = \cos x,$$

so that

$$\frac{du}{dx} = \cos x, \qquad \frac{dv}{dx} = -\sin x.$$

The quotient rule then gives

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$y = \frac{\sin x}{\cos x}$$

Example 9 Cont'd

The quotient rule:

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

Note that this is just the result $\frac{d}{dx}\tan x = \sec^2 x$.