MS121: IT Mathematics

DIFFERENTIATION

Rules for Differentiation: Part 2

John Carroll School of Mathematical Sciences

Dublin City University

The Chain Rule

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The Chain Rule

1 The Chain Rule

Outline

- 2 Examples on Composition of 2 Functions
- 3 Examples on Composition of 3 Functions
- 4 Examples involving Product & Quotient Rules

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The Chain Rule

Context

The Chain Rule is used to differentiate the composition of functions.

Composition of 2 Functions

The function

$$y = \ln\left(\sin x\right)$$

is the composition $f \circ g$ where

$$f(\cdot) = \ln(\cdot)$$
 and $g(\cdot) = \sin(\cdot)$.

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The Chain Rule

Composition of 2 Functions

The function

$$y = \ln(\sin x)$$

is the composition $f \circ g$ where $f(\cdot) = \ln(\cdot)$ and $g(\cdot) = \sin(\cdot)$.

The Chain Rule

We can differentiate $y = f \circ g$ as

$$\frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

where f is written in terms of g and we only differentiate f with respect to its argument, namely g. Since g is written in terms of x, this differentiation is simply with respect to x.

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Two Functions

Outline

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The Chain Rule

Composition of 2 Functions

$$y = \ln(\sin x)$$

Using the Chain Rule

We obtain:

$$g = \sin x \Rightarrow \frac{dg}{dx} = \frac{d}{dx} \sin x = \cos x$$

$$f = \ln(g) \Rightarrow \frac{df}{dg} = \frac{d}{dg} \ln g = \frac{1}{g} = \frac{1}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} = \cot x$$

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Two Functions

Example 1: Composition of 2 Functions

$$y = \cos\left(2x^2\right)$$

Solution

$$u = 2x^2$$
 \Rightarrow $\frac{du}{dx} = 4x$

$$y = \cos(u) \Rightarrow \frac{dy}{du} = -\sin(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u) \cdot 4x = -4x \sin(2x^2)$$

Note

Note that you must always express your answer wholly in terms of x (just as the question was expressed in the first place).

Example 2: Composition of 2 Functions

$$y = \tan(e^x)$$

Solution

$$u = e^x$$
 \Rightarrow $\frac{du}{dx} = e^x$

$$y = \tan(u) \quad \Rightarrow \quad \frac{dy}{du} = \sec^2(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2(u) \cdot e^x = e^x \sec^2(e^x)$$

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Example 3: Composition of 2 Functions

$$y = \sin(\ln(x))$$

Solution

$$u = \ln(x)$$
 \Rightarrow $\frac{du}{dx} = \frac{1}{x}$

$$y = \sin(u) \Rightarrow \frac{dy}{du} = \cos(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \cdot \frac{1}{x} = \frac{1}{x} \cos(\ln(x))$$

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Three Functions

Outline

- The Chain Rule
- 2 Examples on Composition of 2 Functions
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Three Functions

Example 4: Composition of 3 Functions

$$y = \ln(\sin(e^x))$$

Solution

$$u = e^x$$
 \Rightarrow $\frac{du}{dx} = e^x$

$$v = \sin(u) \quad \Rightarrow \quad \frac{dv}{du} = \cos(u)$$

$$y = \ln(v)$$
 \Rightarrow $\frac{dy}{dv} = \frac{1}{v}$

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$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = \frac{1}{v} \cdot \cos(u) \cdot e^{x} = \frac{1}{\sin(e^{x})} \cos(e^{x}) e^{x} = e^{x} \cot(x)$$

Example 5: Composition of 3 Functions

$$y = \ln(\ln(\ln(x)))$$

Solution

$$u = \ln(x) \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$v = \ln(u) \quad \Rightarrow \quad \frac{dv}{du} = \frac{1}{u}$$

$$y = \ln(v) \quad \Rightarrow \quad \frac{dy}{dv} = \frac{1}{v}$$

$$\frac{\mathsf{d} \mathsf{y}}{\mathsf{d} \mathsf{x}} = \frac{\mathsf{d} \mathsf{y}}{\mathsf{d} \mathsf{v}} \cdot \frac{\mathsf{d} \mathsf{v}}{\mathsf{d} \mathsf{u}} \cdot \frac{\mathsf{d} \mathsf{u}}{\mathsf{d} \mathsf{x}} = \frac{1}{\mathsf{v}} \cdot \frac{1}{\mathsf{u}} \cdot \frac{1}{\mathsf{x}} = \frac{1}{\mathsf{ln}\left(\mathsf{ln}(\mathsf{x})\right)} \cdot \frac{1}{\mathsf{ln}(\mathsf{x})} \cdot \frac{1}{\mathsf{x}}$$

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Product/Quotient Rule

Example 6: Composition of Functions & Product Rule

$$y = e^{x \sin x}$$

Solution

$$u = x \sin x \quad \Rightarrow \quad \frac{du}{dx} = \sin x + x \cos x$$

$$y = e^u$$
 \Rightarrow $\frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{u} \cdot (\sin x + x \cos x) = e^{x \sin x} (\sin x + x \cos x)$$

Product/Quotient Rule

Example 7: Composition of Functions & Quotient Rule

$$y = \frac{\sqrt{x}}{e^{\tan x}}$$

Observation

At first sight, this appears only to be an example involving the quotient rule with

$$u=\sqrt{x}, \qquad v=e^{\tan x}.$$

However, we must first use the chain rule to determine $\frac{dv}{dx}$.

Example 7: Step 1 — Evaluate $\frac{dv}{dx}$

$$v = e^{\tan x}$$

Solution

$$w = \tan x \quad \Rightarrow \quad \frac{dw}{dx} = \sec^2 x$$

$$v = e^w \qquad \Rightarrow \quad \frac{dv}{dw} = e^w$$

$$\frac{\mathrm{d} \mathsf{v}}{\mathrm{d} \mathsf{x}} = \frac{\mathrm{d} \mathsf{v}}{\mathrm{d} \mathsf{w}} \cdot \frac{\mathrm{d} \mathsf{w}}{\mathrm{d} \mathsf{x}} = e^{\mathsf{w}} \cdot \sec^2(x) = e^{\tan x} \sec^2 x$$



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Solution
$$u = \sqrt{x} \qquad \Rightarrow \quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example 7: Step 2 — Evaluate $\frac{dy}{dx}$

$$v = e^{\tan x} \implies \frac{dv}{dx} = e^{\tan x} \sec^2 x$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} = \frac{e^{\tan x} \cdot \frac{1}{2\sqrt{x}} - \sqrt{x} \cdot e^{\tan x} \sec^2 x}{\left(e^{\tan x}\right)^2} = \frac{\frac{1}{2\sqrt{x}} - \sqrt{x} \sec^2 x}{e^{\tan x}}$$

 $y = \frac{\sqrt{x}}{e^{\tan x}}$



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