Problem Sheet 9

MS121 Semester 2 IT Mathematics

Exercise 1.
For the following functions, determine the Taylor polynomial of degree 2 at the indicated supporting point:

(a) $f(x) = \sin(x)$ at $x_0 = \frac{\pi}{2}$,

(b) $g(x) = (1+x)^3$ at $x_0 = 0$,

(c) $h(x) = \ln(x)$ at $x_0 = 1$.

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Solution 1.
(a) $f(x) = \sin(x)$ has $f'(x) = \cos(x)$ and $f''(x) = -\sin(x)$, so $f(\frac{\pi}{2}) = 1$, $f'(\frac{\pi}{2}) = 0$ and $f''(\frac{\pi}{2}) = -1$. This leads to the Taylor polynomial

$$\begin{aligned} P(x) &= f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x - \frac{\pi}{2}) + \frac{1}{2}f''(\frac{\pi}{2})(x - \frac{\pi}{2})^2 \\ &= 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 = -\frac{1}{2}x^2 + \frac{\pi}{2}x + 1 - \frac{\pi^2}{8}. \end{aligned}$$

(b) $g(x) = (1+x)^3$ has $g'(x) = 3(1+x)^2$ and $g''(x) = 6(1+x)$, so $g(0) = 1$, $g'(0) = 3$ and $g''(0) = 6$. This leads to the Taylor polynomial

$$\begin{aligned} P(x) &= g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 \\ &= 1 + 3x + \frac{6}{2}x^2 = 1 + 3x + 3x^2. \end{aligned}$$

The Taylor polynomial of degree 3 would be $1 + 3x + 3x^2 + x^3$ which equals $g(x)$.

(c) $h(x) = \ln(x)$ has $h'(x) = x^{-1}$ and $h''(x) = -x^{-2}$, so $f(1) = 0$, $f'(1) = 1$ and $f''(1) = -1$. This leads to the Taylor polynomial

$$\begin{aligned} P(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 \\ &= (x-1) - \frac{1}{2}(x-1)^2 = -\frac{1}{2}x^2 + 2x - \frac{3}{2}. \end{aligned}$$

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Exercise 2.
For the following functions $f(x)$ find a primitive function $F(x)$ such that F takes the given value at the specified point:

(a) $f(x) = 3x^2 + 2$, with $F(1) = 2$,

(b) $f(x) = 2x^7 + 6x^3 + 2x$, with $F(0) = 1$,

(c) $f(x) = (2x+3)^2$, with $F(-2) = -\frac{1}{6}$,

(d) $f(x) = 2\sin(x)$, with $F(0) = 0$,

(e) $f(x) = \cos(x) + \sin(3x)$, with $F\left(\frac{\pi}{2}\right) = 0$,

(f) $f(x) = e^x$, with $F(0) = 3$,

(g) $f(x) = \frac{6}{x}$ (on $x > 0$), with $F(1) = 4$,

(h) $f(x) = \frac{2x}{(x^2+3)^2}$, with $F(0) = -\frac{1}{3}$.

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Solution 2.
In all cases we can first find a ny primitive function (or anti-derivative) and add an appropriate constant to get the given value. The results are:

(a) $F(x) = x^3 + 2x - 1$,

(b) $F(x) = \frac{1}{4}x^8 + \frac{3}{2}x^4 + x^2 + 1$,

(c) $F(x) = \frac{1}{6}(2x+3)^3$,

(d) $F(x) = -2\cos(x) + 2$,

(e) $F(x) = \sin(x) - \frac{1}{3}\cos(3x) - 1$,

(f) $F(x) = e^x + 2$,

(g) $F(x) = 6\ln(x) + 4$,

(h) $F(x) = \frac{-1}{x^2+3}$.

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Exercise 3.
Evaluate the following definite integrals:

(a) $\int_0^2 x^3 \, \mathrm{d}x$,

(b) $\int_{-2}^1 4t^2(1-t)\mathrm{d}t$,

(c) $\int_1^2 t^4 + 8t^{-9}\mathrm{d}t$,

(d) $\int_1^5 \frac{x^{\frac{3}{2}}-7x}{x}\mathrm{d}x$.

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Solution 3.

(a) $\int_0^2 x^3 \, \mathrm{d}x = \left[\frac{1}{4}x^4\right]_0^2 = \frac{2^4}{4} - \frac{0}{4} = 4$,

(b) $\int_{-2}^1 4t^2(1-t)\mathrm{d}t = \left[\frac{4}{3}t^3 - t^4\right]_{-2}^1 = (\frac{4}{3} - 1) - (-8 \cdot \frac{4}{3} - 16) = 27$,

(c) $\int_1^2 t^4 + 8t^{-9}\mathrm{d}t = \left[\frac{1}{5}t^5 - t^{-8}\right]_1^2 = \left(\frac{32}{5} - \frac{1}{256}\right) - (\frac{1}{5} - 1) = 7\frac{251}{1280}$,

(d) $\int_1^5 \frac{x^{\frac{3}{2}}-7x}{x}\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}} - 7x\right]_1^5 = (3\frac{1}{3}\sqrt{5} - 35) - (\frac{2}{3} - 7) = 3\frac{1}{3}\sqrt{5} - 28\frac{2}{3}$.

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Exercise 4.
We consider the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad 0 \leq x \leq 3 \\ 1 & \text{if} \quad x > 3 \end{cases} .$$

(a) Find a function $F(x)$ which is continuous and such that $F'(x) = f(x)$ for all $x \in \mathbb{R} \setminus \{0, 3\}$.

(b) Compute the area A under the curve $y = f(x)$ between $x = 1$ and $x = 5$.

(c) Verify that $F(5) - F(1) = A$.

(d) Show that $F'(0) = 0 = f(0)$, but $F'(3)$ does not exist. (Hint: use one-sided limits.)

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Solution 4.

(a) We find

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{1}{2}x^2 & \text{if} \quad 0 \le x \le 3 \\ \frac{3}{2} + x & \text{if} \quad x > 3 \end{cases}$$

up to an arbitrary additive constant.

(b) The total area is $A = 4 + 2 = 6$:
Between $x = 3$ and $x = 5$ we have a block of area $2 \cdot 1 = 2$.
Between $x = 1$ and $x = 3$ we have a shape with area $2 \cdot \frac{f(1)+f(3)}{2} = 2 \cdot \frac{4}{2} = 4$. (We use the formula: width times average height, which can be justified geometrically. One can also view this as a block of area 2 plus a triangle of area $\frac{1}{2} \cdot 2 \cdot (3-1) = 2$.)

(c) For our choice of $F(x)$ we have

$$F(5) - F(1) = \left(\frac{3}{2} + 5\right) - \frac{1}{2} = 6 = A.$$

(d) This part is a bit harder. Using the definitions we compute:

$$\begin{aligned}
\lim_{x\to 0^-} \frac{F(x) - F(0)}{x} &= \lim_{x\to 0^-} 0 = 0, \\
\lim_{x\to 0^+} \frac{F(x) - F(0)}{x} &= \lim_{x\to 0^+} \frac{1}{2}x = 0, \\
\lim_{x\to 3^-} \frac{F(x) - F(3)}{x-3} &= \lim_{x\to 3^-} \frac{\frac{1}{2}(x+3)(x-3)}{x-3} = \lim_{x\to 3^-} \frac{1}{2}(x+3) = 3, \\
\lim_{x\to 3^+} \frac{F(x) - F(3)}{x-3} &= \lim_{x\to 3^+} \frac{\frac{3}{2} + x - \frac{9}{2}}{x-3} = \lim_{x\to 3^+} 1 = 1.
\end{aligned}$$

From the first lines we see that $F'(0) = \lim_{x\to 0} \frac{F(x)-F(0)}{x} = 0$ exists, but from the last two one-sided limits we see that $F'(3)$ does not exist.

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