Chapter 2: Sets

A set is a collection of objects. In fact, we do not define the term 'set'. All that matters about the set is what its elements are, and that we can decide if a given object is or is not an element of a given set. There is much similarity between the theory of sets and that of logic. The two subjects should complement each other.

Notation: If an object x belongs to a set A we write $x \in A$, if not we write $x \notin A$. Two sets are equal if they have exactly the same elements. Write $X = Y$ if for every z, $z \in X \Rightarrow z \in Y$ and $z \in Y \Rightarrow z \in X$. Otherwise write $X \neq Y$.

Note: So for a set A and an element x the statement $x \in A$ should be a proposition, either true or false.

Notation: We write our sets by enclosing a description of the elements between braces '{' on left and '}' on right. We can give a finite or infinite list or a predicate description using properties.

Example: $P = \{\text{Norwich City}, \text{Sheffield United}, \text{Aston Villa}\}$ or

$$P = \{T \text{ a Premiership team} \mid T \text{ was promoted this year}\}.$$

Example: $E = \{2, 4, 6, 8, \ldots\}$ or

$$E = \{x \text{ a positive integer } \mid x = 2k \text{ for some integer } k\}$$

Note: You have to be a little careful with the predicate notation in order to avoid things like Russell's Paradox but we will not worry about such things.

Example: There are some standard sets of numbers

$$\begin{aligned}
\mathbb{N} &= \text{the set of all natural numbers} = \{1, 2, 3, \ldots\} \\
\mathbb{Z} &= \text{the set of all integers} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbb{Q} &= \text{the set of all rational numbers or fractions} \\
\mathbb{R} &= \text{the set of all real numbers} \\
\mathbb{C} &= \text{the set of all complex numbers}
\end{aligned}$$

Definition: There is a special set with no elements called the empty set. There are two notations $\emptyset$ and $\{\}$. The definition is

$$\emptyset = \{x \mid x \neq x\}.$$

Definition: We say that a set A is a subset of a set B or A is contained in B and write $A \subseteq B$ if

$$x \in A \Rightarrow x \in B.$$

Other notation for this situation is $B \supseteq A$, and we say B is a superset of A or B contains A.

Example: The even integers form a subset of $\mathbb{Z}$. We also have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

Proposition: Notice the following three properties

$$A \subseteq A$$

$$A \subseteq B \text{ and } B \subseteq A \Leftrightarrow A = B$$

$$A \subseteq B \text{ and } B \subseteq C \Rightarrow A \subseteq C$$

Definition: If A is a set then the set of all subsets of A is called the power set of A. We will write $P(A)$ for this power set, although some people write 2^A.

Example: If $A = \{1, 2, 3\}$ then

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Note that A has 3 elements while $P(A)$ has $8 = 2^3$. This is where the notation 2^A comes from. In general a set with n elements will have 2^n subsets, one subset B for each combination of truth values of the n propositions

object one $\in B$, object two $\in B$, $\ldots$, object $n \in B$.

For this example, we get

$1 \in B$	$2 \in B$	$3 \in B$	B
T	T	T	$\{1, 2, 3\}$
T	T	F	$\{1, 2\}$
T	F	T	$\{1, 3\}$
T	F	F	$\{1\}$
F	T	T	$\{2, 3\}$
F	T	F	$\{2\}$
F	F	T	$\{3\}$
F	F	F	$\emptyset$

Definition: If A and B are sets, we define their union by

$$A \cup B = \{x \mid x \in A \textbf{ or } x \in B\}$$

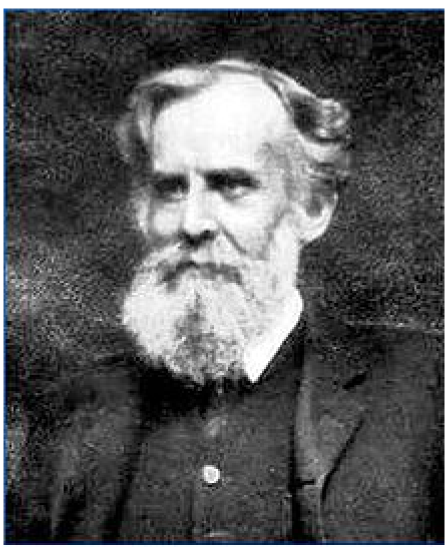
and their intersection by

$$A \cap B = \{x \mid x \in A \textbf{ and } x \in B\}.$$

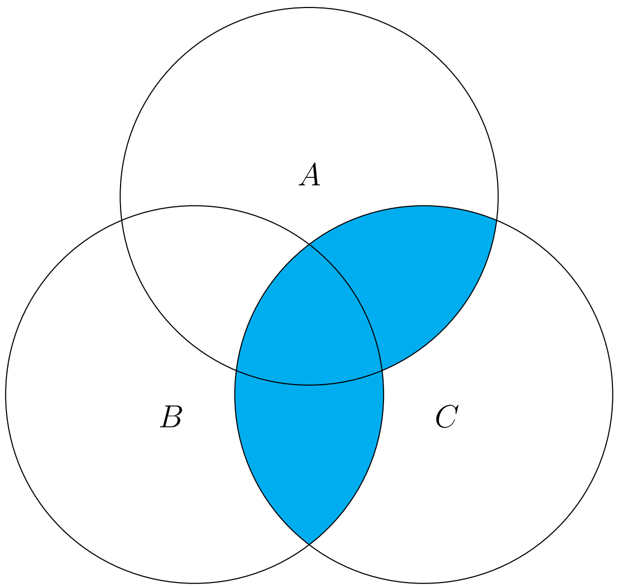
Example: If $A = \{1, 2, 3\}$ and $B = \{0, 2, 5\}$ then

$$A \cap B = \{2\} \quad A \cup B = \{0, 1, 2, 3, 5\}.$$

Venn Diagrams
A Venn diagram is a device for pictorially representing relationships between sets. It is named after John Venn (1834-1923) who popularised its use.



Elliptic regions are drawn to represent sets. The overlapping ellipses define regions corresponding to intersections and several regions will combine to represent unions.



The blue region is $(A \cup B) \cap C$. However it can also be described as $(A \cap C) \cup (B \cap C)$. The equality of these two sets is one of the identities of set theory.

Proposition: The following properties hold

$$A \cap B \subseteq A, \;\; A \cap B \subseteq B, \;\; A \subseteq A \cup B, \;\; B \subseteq A \cup B$$

$$A \cup A = A, \;\; A \cap A = A, \;\; A \cup B = B \cup A, \;\; A \cap B = B \cap A$$

$$A \cup B = B \Leftrightarrow A \subseteq B, \;\; A \cap B = A \Leftrightarrow A \subseteq B$$

$$A \cup (B \cup C) = (A \cup B) \cup C, \;\; A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \;\; A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Each of these can be translated into a logical equivalence. We consider the one illustrated by the Venn diagram above, namely

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

Let x be an element and consider the three propositions

$$P : x \in A, \quad Q : x \in B, \quad R : x \in C.$$

Then $x \in (A \cup B) \cap C$ is the compound proposition (P **or** Q) **and** R, while $x \in (A \cap C) \cup (B \cap C)$ is the compound proposition (P **and** R) **or** (Q **and** R). So establishing the equality is the same as establishing the logical equivalence

$$(\text{P } \textbf{or} \text{ Q}) \textbf{ and } \text{R} \equiv (\text{P } \textbf{and} \text{ R}) \textbf{ or } (\text{Q } \textbf{and} \text{ R})$$

We do this in the usual way with a truth table. To save space let X be P **or** Q, Y be X **and** R, U be P **and** R, V be Q **and** R and W be U **or** V.

P	Q	R	X	Y	U	V	W
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

Since the Y and W columns are the same the logical equivalence holds. We note that the eight rows of the table correspond to the eight regions of our Venn diagram. For example, the TFT row corresponds to P and R true but Q false which translates to $x \in A$ and $x \in C$ but $x \notin B$ and corresponds to the region in the intersection of A with C which is not in B.