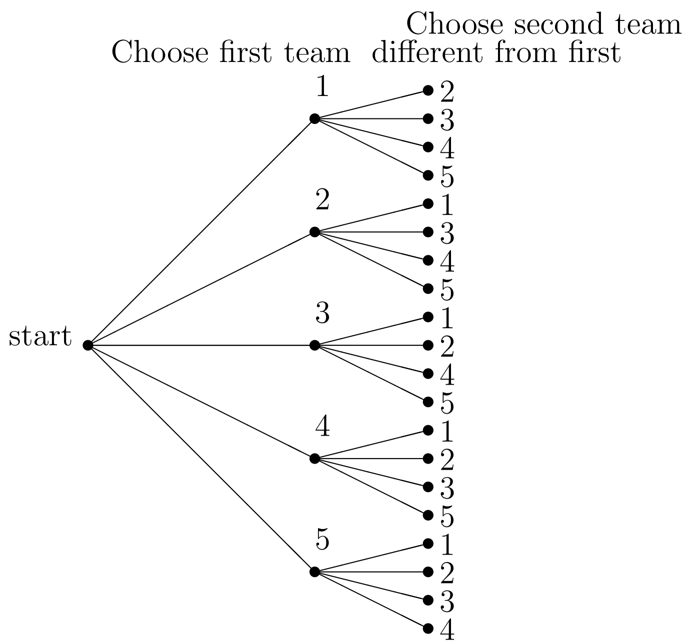
Example: How many ways can the first three places be filled in a league table if the league has 20 teams? Answer: (20)(19)(18). E_1 is choice of first team, E_2 is choice of second team different from first, E_3 is choice of third team different from first and second.

Note: It may help to view each element as the result of applying a number of choices and visualising the process as a tree.

Example: If the league has 5 teams and we are concerned with the first two places then the tree is



Note: Just as there is a subtraction rule, we sometimes count using division.

Example: If the top three teams in a league of twenty go into a superleague, in how many ways can this happen? Answer: (20)(19)(18)/(3)(2)(1). We have 20 choices for first team, 19 for second and 18 for third. However each set of three teams $\{a, b, c\}$ has been counted 6 times as

$$abc, acb, bac, bca, cab, cba$$

with 3 choices for the first of the three, 2 choices for the second and 1 choice for the last of the three.

Note: We are seeing products of consecutive integers in decreasing order such as (20)(19)(18) arising in counting problems.

Definition: A permutation of a finite set of objects is an ordering of the objects in a row.

Example: Above we saw the 6 permutations of three objects

$$abc, acb, bac, bca, cab, cba.$$

Example: There are 24 permutations of four objects

$$abcd, abdc, acbd, acdb, adbc, adcb,$$

$$bacd, badc, bcad, bcda, bdac, bdca,$$

$$cabd, cadb, cbad, cbda, cdab, cdba,$$

$$dabc, dacb, dbac, dbca, dcab, dcba,$$

These are found by picking one of the elements to be first in 4 ways and following it with one of the 6 permutations of the remaining three. Note that the last row is obtained from the previous example by putting a d in front of the 6 permutations of $\{a, b, c\}$.

Definition: If n is a positive integer we define n factorial, written $n!$ to be the product of the first n positive integers.

$$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

Note: By convention, we agree that $0! = 1$.

Note: n factorial can be defined inductively by

$$1! = 1 \text{ and } n! = n[(n-1)!]$$

Example: The number of permutations of 5 objects is $5! = 120$.

Example: In how many ways can a group of five world leaders be arranged in a row for a photoshoot? In how many of these ways will president A be beside president B?

In the first case the answer is $5! = 120$. In the second case, we treat $\{A, B\}$ as a unit, permute the four sets $\{A, B\}, \{C\}, \{D\}, \{E\}$ in $4!$ ways and permute the set $\{A, B\}$ in 2! ways to give a total, by the multiplication rule of $4!2! = 24(2) = 48$.

Example: In how many ways can a group of five world leaders be arranged around a circular table where two arrangements are the same if one is obtained from the other by rotation?

Here we take any circular arrangement and rotate it so that president A is in a fixed place, say the north end, on the table. Now arrange the others to their right in $4! = 24$ ways.

Example: The number of permutations of 3 objects out of 20 is (20)(19)(18), which we can write as (20)!/(17)!.

Definition: If n and r are positive integers with $r \leq n$ then we define the quantity ${}^nP_r$ by

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

This is the number of r-permutations from a set of size n.

Note: In the case $r = n$ we get $n!$ if we agree $0! = 1$.

Example: How many 4-digit PINs are there without repeated digits?

Here the number is ${}^{10}P_4 = 5040$.

Example: In how many ways can we arrange 3 people from a group of 7 in a row? In how many of these arrangements will a particular person from the seven be the first person in the row?

In the first case we have just ${}^7P_3 = (7)(6)(5)$. In the second case, the first person is fixed and we just have a 2-permutation from 6 people to fill the second and third places, giving ${}^6P_2 = (6)(5)$. Note that this is 1/7 of the answer for the first part.

Example: The number of 3 element subsets of 20 objects is (20)(19)(18)/(3)(2)(1), which we can write as (20)!/(17)!(3)!.

Definition: If n and r are positive integers with $r \leq n$ then we define the binomial coefficient ${}^nC_r$ or $\left(\begin{array}{c} n \\ r \end{array} \right)$ by

$$ \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!} $$

Example: For $n = 6$ these numbers are

$$
\begin{aligned}
\begin{pmatrix} 6 \\ 0 \end{pmatrix} &= \frac{6!}{0!(6-0)!} = \frac{6!}{(1)6!} = 1 \\
\begin{pmatrix} 6 \\ 1 \end{pmatrix} &= \frac{6!}{1!(6-1)!} = \frac{6!}{1!5!} = 6 \\
\begin{pmatrix} 6 \\ 2 \end{pmatrix} &= \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{6 \times 5}{2 \times 1} = 15 \\
\begin{pmatrix} 6 \\ 3 \end{pmatrix} &= \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20 \\
\begin{pmatrix} 6 \\ 4 \end{pmatrix} &= \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = \frac{6 \times 5}{2 \times 1} = 15 \\
\begin{pmatrix} 6 \\ 5 \end{pmatrix} &= \frac{6!}{5!(6-5)!} = \frac{6!}{5!1!} = 6 \\
\begin{pmatrix} 6 \\ 6 \end{pmatrix} &= \frac{6!}{6!(6-6)!} = \frac{6!}{6!(1)} = 1
\end{aligned}
$$

In each case we are cancelling the largest factorial on the bottom with the corresponding part of $6!$.