Proposition: The following identities hold

$$\binom{n}{r} = \binom{n}{n-r} \quad , \quad \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

Proof: For the first identity, using the interpretation of $\binom{n}{r}$ as the number of r element subsets of a set A of size n, note that each choice of an r element subset B of A automatically identifies a $(n-r)$-element subset $A - B$.

For the second identity, again using the interpretation of $\binom{n+1}{r}$ as the number of r element subsets of a set A of size $n+1$, we can single out an element a_{n+1} of A and partition A into

$$A_1 = \{a_1, \ldots, a_n\}, \quad A_2 = \{a_{n+1}\}.$$

By the addition rule a subset of size r from A either contains a_{n+1} or not and by the multiplication rule the total number is

$$\binom{n}{r-1}\binom{1}{1} + \binom{n}{r}\binom{1}{0}.$$

(Pick $r-1$ from A_1 **and** 1 from A_2) **or** (Pick r from A_1 **and** 0 from A_2)

Note: We have seen that ${}^nP_r$ is the number of permutations of n objects taken r at a time, while ${}^nC_r$ is the number of subsets of size r from n. The difference between ${}^nP_r$ and ${}^nC_r$ is that order is important in the first case but not in the second. We have also seen that n^r is the number of ways of choosing r objects from n if order is important and each chosen object is replaced before the next choice is made. This leaves one other case, where we select r objects from n with replacement but order is not important.

Example: We have seen that there are $10^4 = 10,000$ four-digit strings, ${}^{10}P_4 = 5040$ of these have no repeated digits and there are ${}^{10}C_4 = 210$ 4-element subsets of a set of size 10. The final computation is the number of distributions of digits occurring among all four-digit strings. We could list the possibilities as

$$0000, 0001, \ldots, 9998, 9999$$

where occurrences of 0 are put on the left followed by occurrences of 1, etc. However, this will take a long time and there is an easier way.

Definition: An r-selection from n is an unordered selection of r objects from n with repetition allowed.

Theorem: The number of r-selections from n is

$$\binom{r+n-1}{n-1}$$

Proof: Start by ordering the types of elements from 1 to n. For each r-selection arrange the elements of the selection so that type 1 elements appear first, type 2 elements appear next, etc. Between each type of element in the selection put a separating marker of the form $\ldots xxx|yy\ldots$ including extra markers for types unrepresented in the selection:

$$\ldots xxx||zz\ldots$$

The result is a string of length $r+n-1$ since there are r elements and we need $n-1$ markers to separate the n types. Therefore an r-selection can be identified with a choice of $n-1$ places for the markers in a string of length $r+n-1$.

Example: Suppose we want to count the number of distributions of digits occurring among all four-digit strings. Here $n = 10$ and $r = 4$ and the number is

$$\binom{4+10-1}{10-1} = \binom{13}{9} = \binom{13}{4} = 715.$$

Example: If 5 cards chosen from a standard deck of 52, the number of different distributions of hearts $\heartsuit$, diamonds $\diamondsuit$, spades $\spadesuit$ and clubs $\clubsuit$ in such a hand is the number of 5-selections from 4 objects and is thus

$$\binom{4+5-1}{4-1} = \binom{8}{3} = \frac{8\times 7\times 6}{3\times 2\times 1} = 56.$$

Example: 3 dice are thrown. How many distributions of the numbers $1,2,3,4,5,6$ are possible?

We are selecting 3 unordered things from 6 with repetition so the number is

$$\binom{3+6-1}{6-1} = \binom{8}{5} = \binom{8}{3} = 56.$$