MS121 Discrete Mathematics, Tutorial 5

2. Consider the relations '$\subseteq$' (contained in), '$\subsetneq$' (contained in but not equal to equal to) on the set $P(A)$ of subsets of a set A. Are these relations reflexive, symmetric, antisymmetric, transitive?

First be clear about the meaning of the symbols $\subseteq$ and $\subsetneq$. If B and C are subsets of A then $B \subseteq C$ if and only every element of B is also an element of C. On the other hand $B \subsetneq C$ means every element of B is also an element of C and there is at least one element of C which is not an element of B.

Next recall the definitions of properties of relations:
A relation R on a set S is called reflexive if $(s, s) \in R$ for all $s \in S$.
A relation R on a set S is called symmetric if whenever we have $(s, t) \in R$ we also have $(t, s) \in R$.
A relation R on a set S is called antisymmetric if whenever we have $(s, t) \in R$ and $(t, s) \in R$, then $s = t$.
A relation R on a set S is called transitive if whenever we have $(s, t) \in R$ and $(t, u) \in R$ we also have $(s, u) \in R$.
(We have rewritten the definitions in terms of an underlying set S since there is already an A in the question with a different meaning.)

Part of the difficulty with this question is that the underlying set is a set of subsets. So the S is $P(A)$.

(It may help to consider a particular set. This will make the problem more concrete. For example, consider the subsets of $\{1, 2, 3\}$. Here the set $S = P(A)$ consists of the 8 subsets $\emptyset$, $\{1\}$, $\{2\}$,$\{3\}$,$\{1, 2\}$,$\{1, 3\}$,$\{2, 3\}$,$\{1, 2, 3\}$.)

The relation $\subseteq$ is reflexive: $a \in A \Rightarrow a \in A$ so $A \subseteq A$.
The relation $\subseteq$ is not symmetric if $A \neq \emptyset$: If $a \in A$ then $S = P(A)$ contains the one-element set $\{a\}$ and $\emptyset \subseteq \{a\}$ but $\{a\} \not\subseteq \emptyset$.
The relation $\subseteq$ is antisymmetric, If $B \subseteq C$ and $C \subseteq B$ then every element of B is an element of C and vice versa so that $B = C$.
The relation $\subseteq$ is transitive: $B \subseteq C$ and $C \subseteq D$ means every element of B is an element of C and every element of C is an element of D so every element of B is an element of D and $B \subseteq D$.

The relation $\subsetneq$ is not reflexive: We cannot have $B \subsetneq B$ since B has no elements which do not belong to B.
The relation $\subsetneq$ is not symmetric: $B \subsetneq C$ means C has an element which is not in B, so we cannot have $C \subsetneq B$
The relation $\subsetneq$ is antisymmetric: We cannot have $B \subsetneq C$ and $C \subsetneq B$.
The relation $\subsetneq$ is transitive: $B \subsetneq C$ and $C \subsetneq D$ means $B \subseteq D$. But D contains an element not in C and hence not in B so that $B \subsetneq D$.

3. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and define the relation R on A by $(a, b) \in R$ if and only if $b - a$ is a multiple of 5. R is an equivalence relation. (The proof is similar to the case where $b - a$ is a multiple of 2.) What is the equivaence class of 1? Of 2? Of 6? What is the partition of A defined by R? Draw the digraph of R.

Recall the definition of equivalence class: If R is an equivalence relation on a set A and $a \in A$ we define the equivalence class of a to be the subset of A given by

$$E_a = \{b \in A \mid (a, b) \in R\}.$$

In words, that is the set of elements b in A that a is related to.

Here $(a, b) \in R$ if and only if $b-a$ is a multiple of 5. Rewrite this as $b = a+5k$ and we see that we find the possible b's by going up or down from a in steps of size 5 but not overshooting and leaving A. Faor $a = 1$, $a - 5 = -4 \notin A$ so just go up from 1 in steps of size 5. $1 + 5 = 6$ and $1 + 2(5) = 11$. After that we leave A. Thus the equivalence class of 1 is $\{1, 6, 11\}$. In exactly the same way the equivalence class of 2 is $\{2, 7, 12\}$. The equivalence class of 6 is also $\{1, 6, 11\}$, but here we can go back one step and forward one step. We can also use the fact that when $(a, b) \in R$ for an equivalence relation R the $E_a = E_b$. The partition will be

$$\{1, 6, 11\}, \{2, 7, 12\}, \{3, 8\}, \{, 4, 9\}, \{3, 10\}.$$

The digraph will be a disjoint union of the complete graphs on the 5 blocks of the partition.