Conditional Probability: Conditional probability tries to answer the following question: how should we modify the probability associated to an event A if we happen to know that some other event B has occured? To motivate the definition we consider an example where we know all the probabilities involved.

Example: Suppose we are dealt two cards from a standard pack. Let A be the event that the second card is black and let B be the event that the first card is black. Since we are concerned with the order of the cards the sample space is the set of ordered samples of size 2 from 52 without replacement. We would agree that $\mathbb{P}[B] = 1/2$ (26 black cards among the 52) while the probability that A occurs given that B has already occured should be 25/51 (25 black cards left among the 51). Note that $A \cap B$ is the event that both cards are black and hence

$$\mathbb{P}[A \cap B] = \frac{{}^{26}P_2}{{}^{52}P_2} = \frac{26 \times 25}{52 \times 51} = \mathbb{P}[B] \times \frac{25}{51}.$$

It seems that for this example, the probability that A occurs given that B has already occured is $\mathbb{P}[A \cap B]/\mathbb{P}[B]$.

Definition: For any fixed event B, such that $\mathbb{P}[B] > 0$, we define the conditional probability of A given B, which we denote $\mathbb{P}[A|B]$, to be

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Note: It makes sense that $\mathbb{P}[A|B]$ should be proportional to $\mathbb{P}[A \cap B]$ since the outcomes ω contributing to this probability must lie in B since we assume B has occured. Also the denominator is needed for normalisation so that $\mathbb{P}[B|B] = 1$.

Example: Two dice are tossed and the upwards numbers are added. What is the probability that the sum is less than 6, given that the sum is even?

The sample space is $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and the probabilities are $P(2) = 1/36$, $P(3) = 2/36$, $P(4) = 3/36$, $P(5) = 4/36$, $P(6) = 5/36$, $P(7) = 6/36$, $P(8) = 5/36$, $P(9) = 4/36$, $P(10) = 3/36$, $P(11) = 2/36$, $P(12) = 1/36$. If A is the event the sum is even and B is the event that the sum is less than 6 then

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = (1/9)/(1/2) = 2/9.$$