MS121: IT Mathematics

Limits & Continuity

Continuity

John Carroll
School of Mathematical Sciences

Dublin City University

Outline

1. Overview
2. Using Limits
3. Continuity Test
4. Examples

Outline

1. Overview
2. Using Limits
3. Continuity Test
4. Examples

Continuity

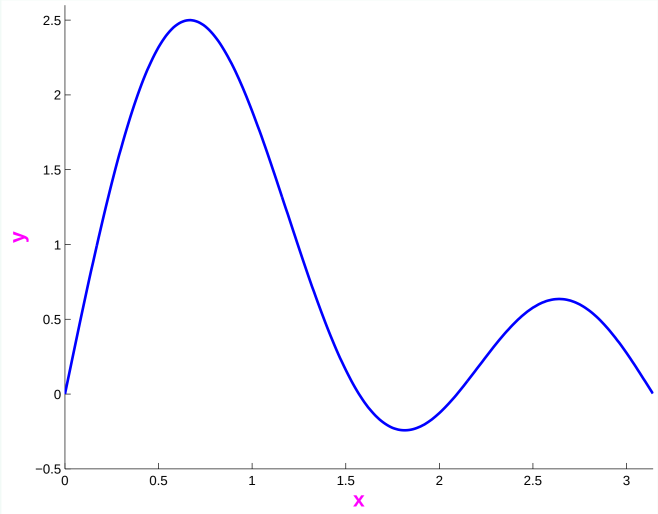
Informal Definition

An informal definition of a continuous function is that its graph contains no gaps.

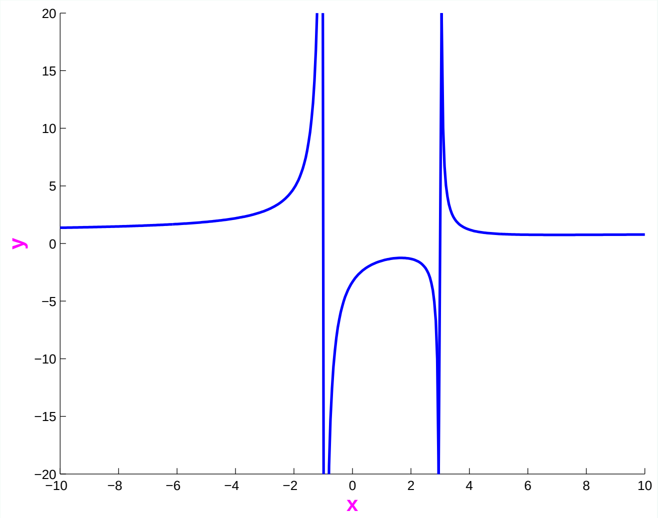
Graphical Interpretation

If the domain of $f(x)$ contains a neighbourhood of a fixed real number c, then the graph of f can be drawn through the point $(c, f(c))$ without lifting the pen from the paper.

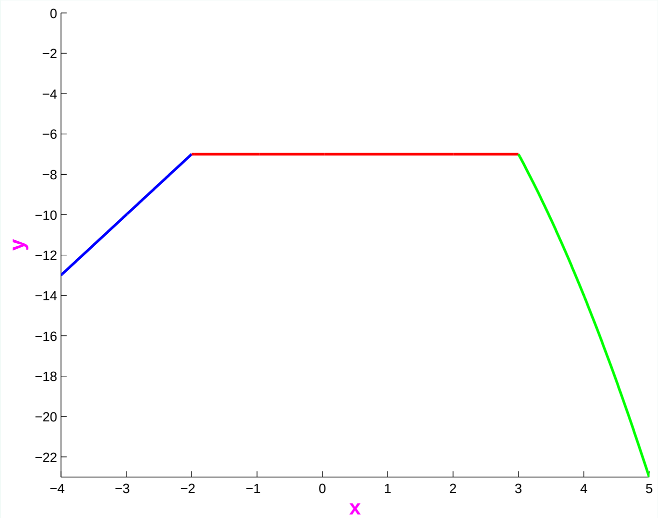
A Continuous Function



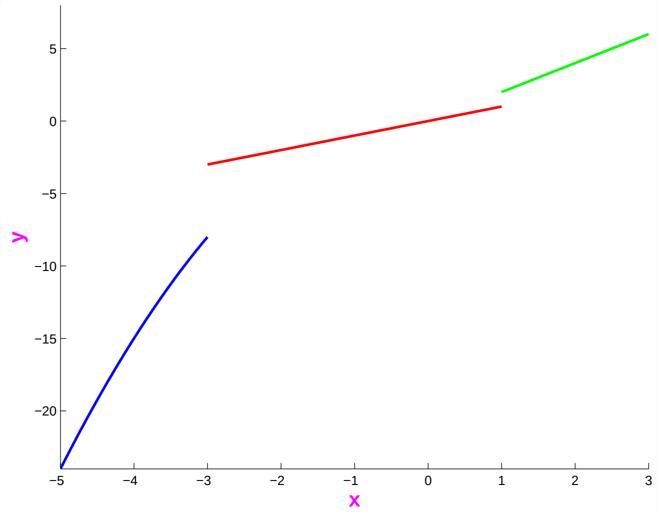
A Discontinuous Function



A "Piecewise" Continuous Function



A Discontinuous Function



Continuity

Use of Limits

We use limits to decide if a function is continuous or not. If we wish to determine whether or not a function is continuous at a given point, then we approach the point in question ...

from the right of the point on the x-axis (from above)

and ...

from the left of the point on the x-axis (from below)

to see if there are any jumps in the y-values.

Outline

Continuity

Use of Limits

Suppose we are interested in finding the limit of a function $f(x)$ at a point c. The expression

$$\lim_{x \to c^+} f(x)$$

means letting x approach the value c from the right (or from above) while the expression

$$\lim_{x \to c^-} f(x)$$

means letting x approach the value c from the left (or from below). If the direction of approach is not important, then we simply write

$$\lim_{x \to c} f(x)$$

Example 1

The function $\sqrt{4 - x^2}$ has domain $[-2, 2]$. Its graph is a semicircle centred on the origin. Also

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0,$$

but the function does not have a left-sided limit at $x = -2$ nor a right-sided limit at $x = +2$.

The function $\sqrt{4 - x^2}$ with domain $[-2, 2]$

One-sided & Two-sided Limits

Definition

A function $f(x)$ has a limit as x approaches c if and only if it has left-sided and right-sided limits there, and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x).$$

Outline

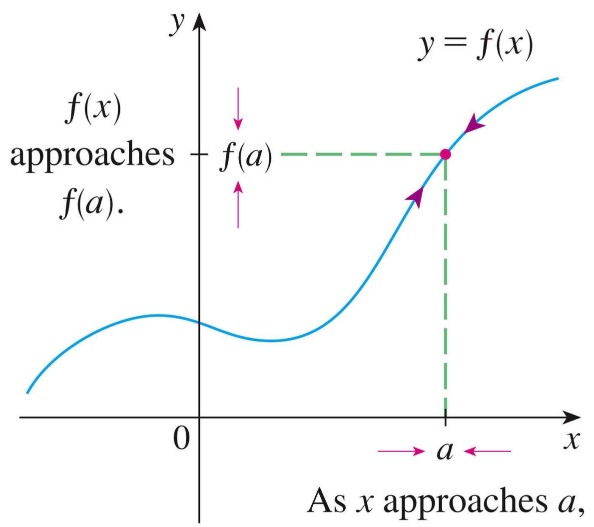
Continuous Function

Definition

A function f is continuous at an interior point $x = c$ of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

f is continuous at a



Continuity Test

Definition

A function f is continuous at $x = c$ if and only if it meets the following three conditions:

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value)

Outline

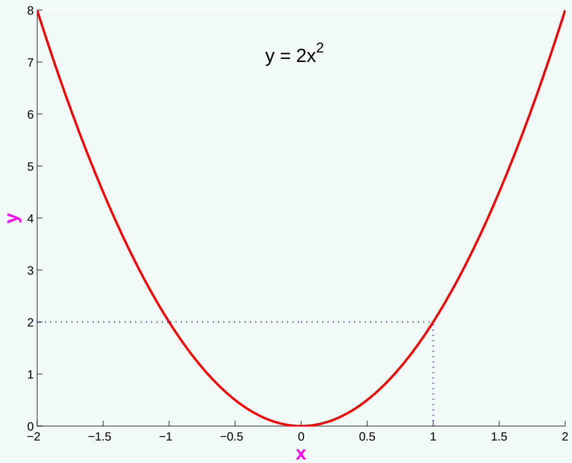
Example 2

We ask if the function $f(x) = 2x^2$ is continuous at the point $x = 1$?

Solution

- First, evaluate the function at $x = 1$, namely $f(1) = 2 \times (1)^2 = 2$.
- Next, investigate what happens to the $f(x)$-values as we approach x from either side of $x = 1$.

The function $2x^2$ on the domain $[-2, 4]$



$$f(x) = 2x^2$$

Example 2 (Cont'd)

Approaching from the right (from above), we find:

x-value	y-value
1.5	4.5
1.1	2.42
1.05	2.205
1.01	2.0402
1.001	2.004002
1.0001	2.000400
1.00001	2.000004 ...

$$f(x) = 2x^2$$

Example 2 (Cont'd)

Next, approaching $x = 1$ from the left (from below):

x-value	y-value
0.5	0.5
0.9	1.62
0.99	1.9602
0.999	1.996002
0.9999	1.999600
0.99999	1.999960

$$f(x) = 2x^2$$

Example 2 — Conclusion

- The pattern is clear. As the x-values approach $x = 1$ from both the right and the left of $x = 1$, the y-values approach the y-value which corresponds to $x = 1$, namely $y = 2$.
- We therefore say that the function $f(x) = 2x^2$ is continuous at $x = 1$.

Footnote

You could examine the function of the last example, namely $f(x) = 2x^2$ at any x-value and achieve the same result. The function is continuous everywhere.

Example 3

Is the function

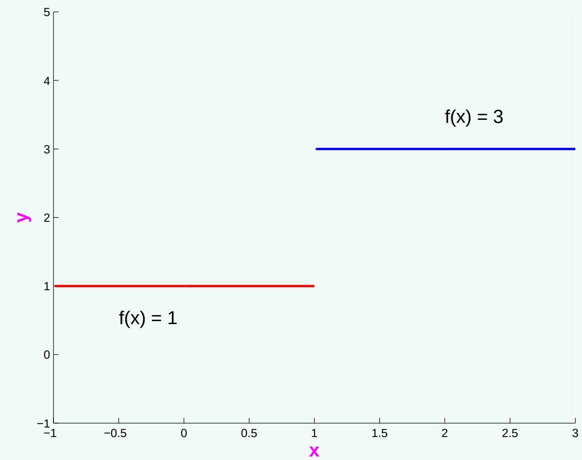
$$f(x) = \begin{cases} 3 & \text{if} \quad x \geq 1 \\ 1 & \text{if} \quad x < 1 \end{cases}$$

continuous at $x = 1$?

Solution

- First, evaluate the function at $x = 1$ to find $f(x) = 3$.
- Then, examine the function from the right and from the left.

A function with a discontinuity



$$f(x) = \begin{cases} 3 & \text{if} \quad x \geq 1 \\ 1 & \text{if} \quad x < 1 \end{cases}$$

Example 3 (Cont'd)

Approach $x = 1$ from the right:

x-value	y-value
1.5	3
1.1	3
1.001	3
1.00001	3

$$f(x) = \begin{cases} 3 & \text{if} \quad x \geq 1 \\ 1 & \text{if} \quad x < 1 \end{cases}$$

Example 3 (Cont'd)

And now approach $x = 1$ from the left:

x-value	y-value
0.5	1
0.9	1
0.999	1
0.99999	1

$$f(x) = \begin{cases} 3 & \text{if} \quad x \geq 1 \\ 1 & \text{if} \quad x < 1 \end{cases}$$

Example 3 — Conclusion

- It is clear that, as we approach $x = 1$ from the left, we are not approaching the desired y-value given by $f(1) = 3$.
- We conclude that the function is not continuous at $x = 1$.
- This is a point of discontinuity.
- Note, however, that the function is continuous everywhere else.

Continuous & Discontinuous Functions

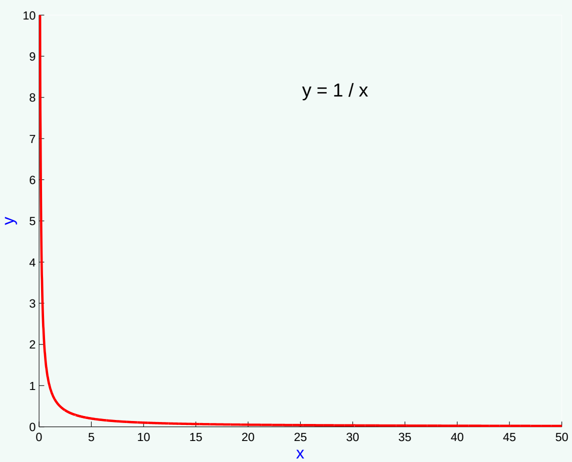
Context

- Some apparently simple functions are discontinuous.
- Consider, for example, the function $f : R \to R$, defined by

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  Note that $\frac{1}{x}$ is not defined at $x = 0$.
- A rough sketch of $f(x)$ will show that there is a gap in the graph at $x = 0$ and hence the function is discontinuous at $x = 0$.
- Furthermore, in this case, there is no value which we could assign to $f(x)$ at $x = 0$ to make the function continuous.

$f(x) = \dfrac{1}{x}$ is undefined at $x = 0$



Continuous & Discontinuous Functions

"Piecewise" Continuous Functions

We can examine more complicated functions which are constructed piecewise from continuous functions and determine whether these functions are continuous or discontinuous.

Example 4

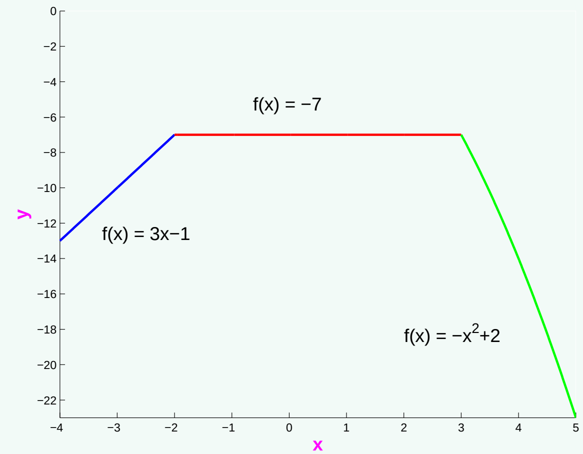
Consider the function $f : R \to R$, defined by

$$f(x) = \begin{cases} 3x - 1 & x \leq -2 \\ -7 & -2 < x \leq 3 \\ -x^2 + 2 & x > 3 \end{cases}$$

Method of Solution

We must examine the function at each of the "breakpoints", that is, at $x = -2$ and $x = 3$.

A "piecewise" continuous function



$$f(x) = \begin{cases} 3x - 1 & x \leq -2 \\ -7 & -2 < x \leq 3 \\ -x^2 + 2 & x > 3 \end{cases}$$

Example 4 (Cont'd)

At $x = -2$:

$$\begin{aligned} \lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (3x - 1) &= -7 \\ \lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (-7) &= -7 \end{aligned}$$

Because the two limits are equal, the function is continuous at $x = -2$.

$$f(x) = \begin{cases} 3x - 1 & x \leq -2 \\ -7 & -2 < x \leq 3 \\ -x^2 + 2 & x > 3 \end{cases}$$

Example 4 (Cont'd)

At $x = 3$, we find

$$\begin{aligned} \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-7) &= -7 \\ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-x^2 + 2) &= -7 \end{aligned}$$

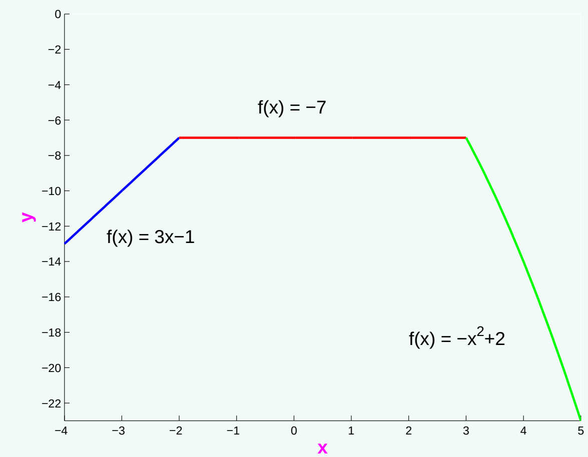
and so the function is also continuous at $x = 3$.

$$f(x) = \begin{cases} 3x - 1 & x \leq -2 \\ -7 & -2 < x \leq 3 \\ -x^2 + 2 & x > 3. \end{cases}$$

Example 4 — Conclusion

- The function is also continuous at every other value of x and hence we can say that $f(x)$ is continuous everywhere or simply "continuous".
- A sketch of $f(x)$ will show that, although there is a "sharp corner" at $x = -2$, there is no gap in the graph — confirming that the function is continuous.

A "piecewise" continuous function



Example 5

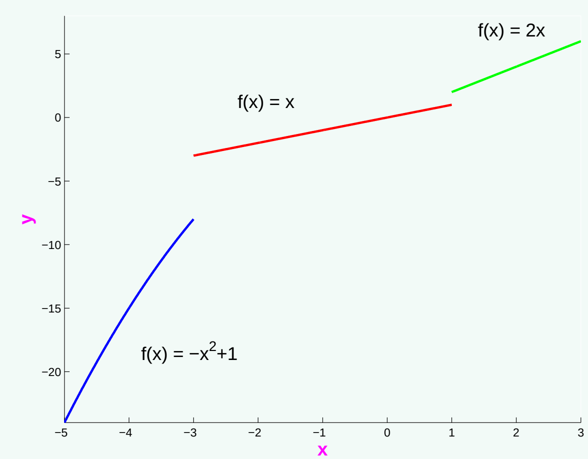
Consider the function $f : R \to R$,
defined by:

$$f(x) = \begin{cases} -x^2 + 1 & x < -3 \\ x & -3 \leq x < 1 \\ 2x & x \geq 1 \end{cases}$$

Method of Solution

We again examine the function at each of the "breakpoints", that is, at $x = -3$ and $x = 1$.

A "piecewise" discontinuous function



$$f(x) = \begin{cases} -x^2+1 & x < -3 \\ x & -3 \le x < 1 \\ 2x & x \ge 1 \end{cases}$$

Example 5 (Cont'd)

At $x = -3$, we find

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-x^2+1) = -8$$
$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} x = -3$$

Because the two limits are not equal, the function is discontinuous at $x = -3$.

$$f(x) = \begin{cases} -x^2+1 & x < -3 \\ x & -3 \le x < 1 \\ 2x & x \ge 1 \end{cases}$$

Example 5 (Cont'd)

At $x = 1$, we find

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x = 2$$

and again, as the two limits are not equal, we conclude that the function is also discontinuous at $x = 3$.

$$f(x) = \begin{cases} -x^2+1 & x < -3 \\ x & -3 \le x < 1 \\ 2x & x \ge 1 \end{cases}$$

Example 5 — Conclusion

- Even though the function is continuous at every other value of x, we say that $f(x)$ is discontinuous.
- A sketch of $f(x)$ will show clearly the two points of discontinuity, namely at $x = -3$ and at $x = 1$.

Example 6

Given

$$f(x) = \begin{cases} 2x - 2 & \text{if } x < -1 \\ Ax + B & \text{if } -1 \le x < 1 \\ 5x + 7 & \text{if } x > 1 \end{cases}$$

determine the constants A and B so that the function $f(x)$ is continuous for all real values of x.

Method of Solution

- The function $f(x)$ is clearly continuous for $x < -1$, for $-1 < x < 1$ and for $x > 1$.
- Examine the (left/right-sided) limits at the intersection points of the three intervals

$$f(x) = \begin{cases} 2x-2 & \text{if } x < -1 \\ Ax+B & \text{if } -1 \le x < 1 \\ 5x+7 & \text{if } x > 1 \end{cases}$$

Example 6 (Cont'd)

At $x = -1$:

$$\lim_{x \to (-1)^-} 2x - 2 = -4$$

$$\lim_{x \to (-1)^+} Ax + B = -A + B$$

Continuity at $x = -1$ therefore requires that

$$-A + B = -4$$

$$f(x) = \begin{cases} 2x-2 & \text{if } x < -1 \\ Ax+B & \text{if } -1 \le x < 1 \\ 5x+7 & \text{if } x > 1 \end{cases}$$

Example 6 (Cont'd)

At $x = 1$:

$$\lim_{x \to 1^-} Ax + B = A + B$$

$$\lim_{x \to 1^+} 5x + 7 = 12$$

Continuity at $x = 1$ therefore requires that

$$A + B = 12$$

$$f(x) = \begin{cases} 2x-2 & \text{if } x < -1 \\ Ax+B & \text{if } -1 \le x < 1 \\ 5x+7 & \text{if } x > 1 \end{cases}$$

Example 6 (Cont'd)

$f(x)$ is continuous for all real x if and only if the left- and right-sided limits are equal in each case, i.e. if and only if

$$-A + B = -4 \qquad A + B = 12$$

Solving these two equations gives $A = 8$ and $B = 4$.

The resulting "piecewise" continuous function