Problem Sheet 3

MS121 Semester 2 IT Mathematics

Exercise 1.

Write the following functions as a rational function.

(a) $f(x) = \frac{2x}{x^2+1} - \frac{3x}{x^2+2}$,

(b) $g(x) = \frac{x^2-4}{x+2} + \frac{x^2-4}{x-2}$,

(c) $h(x) = \frac{x^3}{x^4+3} - \frac{x}{x^2-1}$.

$\oslash$

Solution 1.

(a) $f(x) = \frac{2x(x^2+2)-3x(x^2+1)}{(x^2+1)(x^2+2)} = \frac{x-x^3}{x^4+3x^2+2}$.

(b) $g(x) = \frac{(x^2-4)(x-2)+(x^2-4)(x+2)}{(x+2)(x-2)} = \frac{(x^2-4)2x}{x^2-4} = 2x$. (Notes: This is a monomial, which is a special case of a polynomial and of a rational function, $g(x) = \frac{2x}{1}$; in principle we should take $g(x)$ with domain $\mathbb{R} \setminus \{-2, 2\}$, because the original sum is not defined at these points.)

(c) $h(x) = \frac{x^3(x^2-1)-x(x^4+3)}{(x^4+3)(x^2-1)} = \frac{-x^3-3x}{x^6-x^4+3x^2-3}$.

$\diamondsuit$

Exercise 2.

Explain in your own words and in detail why the following equalities are true:

$$3^5 \cdot 4^5 = 12^5 \ ,$$
$$5^3 \cdot 5^4 = 5^7 \ .$$

Then Compute the following numbers by hand and write your answers in a form without powers.

(a) 5^2,

(b) 3^3,

(c) $16^{\frac{1}{2}}$,

(d) $8^{\frac{2}{3}}$,

$\oslash$

Solution 2.

We can reorder the products as

$$\begin{aligned}
3^5 \cdot 4^5 &= (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (4 \cdot 4 \cdot 4 \cdot 4 \cdot 4) \\
&= (3 \cdot 4) \cdot (3 \cdot 4) \cdot (3 \cdot 4) \cdot (3 \cdot 4) \cdot (3 \cdot 4) \\
&= 12^5 \ , \\
5^3 \cdot 5^4 &= (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5) \\
&= 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \\
&= 5^7 \ .
\end{aligned}$$

Of course these equalities derive from general rules, $a^x b^x = (ab)^x$ and $a^x a^y = a^{x+y}$ for all $a > 0$, $b > 0$ and $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is not the intention that the students compute numbers, e.g. $3^5 = 243$, $4^5 = 1024$ and $12^5 = 1024 * 243 = 248832$, or $5^3 = 125$, $5^4 = 625$ and $5^7 = 78125$.

(a) $5^2 = 25$,

(b) $3^3 = 27$,

(c) $16^{\frac{1}{2}} = 4$,

(d) $8^{\frac{2}{3}} = 2^2 = 4$.

$\diamondsuit$

Exercise 3.
Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

(a) $f(x)$ is a linear function with exactly one root,

(b) $f(x)$ is an even monomial,

(c) $f(x)$ is an odd polynomial, but not a monomial,

(d) $f(x)$ is a rational function (with domain $\mathbb{R}$),

(e) $f(x)$ is an odd rational function, but not a polynomial.

$\oslash$

Solution 3.
(a) E.g. $f(x) = 2x - 7$. Any linear function $f(x) = mx + b$ which is not constant $(m \neq 0)$ will do.

(b) E.g. $f(x) = 2x^4$. Any monomial $f(x) = cx^n$ with even power n will do.

(c) E.g. $f(x) = x^3 - 4x$. This requires a sum of at least two odd monomials with different odd exponents and non-zero coefficients.

(d) E.g. $f(x) = \frac{2x}{x^2+1}$. The denominator must be a polynomial without roots. It must always be even and for degree 2 we can easily check whether it has roots. Degree 0 would also work, e.g. $f(x) = \frac{2x}{3}$, but this is really just a polynomial.

(e) E.g. $f(x) = \frac{2x}{x^2+1}$. If the numerator is odd and the denominator even, or the other way around, the quotient is always odd. (If the domain is still $\mathbb{R}$, we must choose the denominator even and hence the numerator odd.) The denominator should not be a constant function, otherwise the quotient is a polynomial.

$\diamondsuit$

Exercise 4.

Evaluate the following limits and indicate which rules for limits you used, if any.

(a) $\lim\limits_{x\to 5} x - 19$,

(b) $\lim\limits_{x\to 3} (x+7)(x-7)$,

(c) $\lim\limits_{x\to 0} (1-x^3)^{125}$,

(d) $\lim\limits_{x\to 0} \dfrac{x^2}{x}$.

$\oslash$

Solution 4.

Using the facts that $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = a$ for any $a, c \in \mathbb{R}$ we find:

(a) $\lim\limits_{x\to 5} x - 19 = 5 - 19 = -14$ from the sum rule for limits,

(b) $\lim\limits_{x\to 3} (x+7)(x-7) = 10 \cdot (-4) = -40$ from the product and sum rules for limits,

(c) $\lim\limits_{x\to 0} (1-x^3)^{125} = \left(\lim\limits_{x\to 0} (1-x^3)\right)^{125} = (1-0^3)^{125} = 1$ using the continuity of $f(y) = y^{125}$ and the rule for limits under compositions with continuous functions, as well as the rules for sums and products of limits; it is possible to expand $(1-x^3)^{125}$, in which case one doesn't need the rule for limits under compositions with continuous functions

(d) $\lim\limits_{x\to 0} \dfrac{x^2}{x} = \lim\limits_{x\to 0} x = 0$, because $\frac{x^2}{x} = x$ on $x \neq 0$ and the limit as $x \to 0$ does not depend on the value at $x = 0$.

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