Chapter 4: Functions.

Recall the notion of function from school mathematics. Let X and Y be sets. A function from X to Y is a rule that assigns to each element of X exactly one element of Y. Usually we think of X and Y as being subsets of $\mathbb{R}$ and the function as being given by a formula.

Example : Consider the square root function. Even though we have a square root function on the calculator you have to careful with what it means. What is the square root of -1? The calculator gives an error message. If the square root of x is the number whose square is x why is the square root of 4 not -2?

Example : Let f be the function which assigns to the ID of each student registered for MS121 their score in test 1. There is no formula for f. We don't have anything like

$$\text{score} = \sqrt{\sin(\text{ID number}) - 5}.$$

Instead we have a table

ID number	Score
19237961	2
19247961	1
$\vdots$	$\vdots$
19949494	0

which is really just a set of pairs of the form (a, b) where A is the set of IDs of students registered for MS121 and B is the set $\{0, 1, 2, 3, 4\}$. So the function is a relation between A and B. We notice that for each $a \in A$, or each student ID, there has to be one and only one element of B, or score. It is precisely those relations with this further property which we call functions.

Definition: A function f from a set A to a set B is a relation between A and B which satisfies two properties:
(1) every element in A is related to some element in B, and
(2) no element in A is related to more than one element in B.

In other words, given any element $a \in A$, there is a unique element $b \in B$ with $(a, b) \in f$.

Note: The digraph of a relation which is a function has exactly one arrow leaving each point in the set A. The matrix of a relation which is a function has exactly one T in each row.

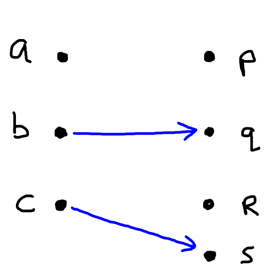
Example: $A = \{a, b, c\}$, $B = \{p, q, r, s\}$ with 3 relations

$$R_1 = \{(b, q), (c, s)\}$$

$$R_2 = \{(a, r), (b, q), (b, s), (c, p)\}$$

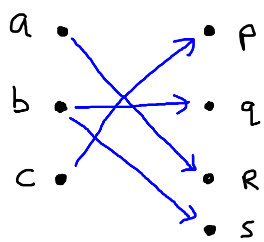
$$R_3 = \{(a, q), (b, q), (c, r)\}$$

Only R_3 is a function.



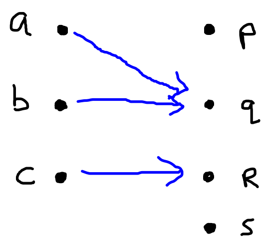
	p	q	R	s
a	F	F	F	F
b	F	T	F	F
c	F	F	F	T

R_1



	p	q	R	s
a	F	F	T	F
b	F	T	F	T
c	T	F	F	F

R_2



	p	q	r	s
a	F	T	F	F
b	F	T	F	F
c	F	F	T	F

R_3

Notation: If a relation f is a function we usually think of it as a rule which assigns to each $a \in A$ the unique element $b \in B$ with $(a, b) \in f$. We often write $b = f(a)$ and call b the image of a under f. The set A is called the domain of the function. The set B is called the codomain of the function. We will use the notation $f : A \to B : a \mapsto f(a)$ as shorthand for:
'f is a function with domain A and codomain B which takes a typical element a in A to the element in B given by $f(a)$.'

Example: If $A = \mathbb{R}$ and $B = \mathbb{R}$, the relation

$$R = \{(x, y) \mid y = \sin(x)\}$$

defines the function $f(x) = \sin(x)$.

Example: If $A = \{0, 1\}^4$ and $B = \{0, 1, 2, 3, 4\}$, the relation

$$S = \{(abcd, n) \mid n = \text{the number of 1's among } a, b, c, d\}$$

defines the function which measures the number of 1's in a binary string of length 4.

Example: If $A = \mathbb{Z}$ and $B = \{0, 1, 2\}$ we can define a function $f : A \to B$ with $f(n)$ equal to the remainder when n is divided by 3. There are similar functions where 3 is replaced by some other number. These are used to construct hashing functions. Sometimes this function is denoted by the % symbol as in $5\%3 = 2$ meaning that when 5 is divided by 3 the remainder is 2.

Example: The set of all functions from the finite set $A = \{a, b, c\}$ to $B = \{0, 1\}$ can be identified with the subsets of A by $f \leftrightarrow C$ where C is the subset of A consisting of those x with $f(x) = 1$. So writing a function $f : A \to B$ as a binary string $f(a)f(b)f(c)$ we get

$$000 \leftrightarrow \emptyset, \quad 001 \leftrightarrow \{c\}, \quad 010 \leftrightarrow \{b\}, \quad 011 \leftrightarrow \{b, c\},$$

$$100 \leftrightarrow \{a\}, \quad 101 \leftrightarrow \{a, c\}, \quad 110 \leftrightarrow \{a, b\}, \quad 111 \leftrightarrow \{a, b, c\}.$$

Notation: The range of a function f is the set of all images of elements of A under f. That is,

$$\text{Range}(f) = \{b \in B \mid (a, b) \in f \text{ for some } a \in A\}.$$

Note: In terms of the digraph Range(f) is the subset of the codomain which are the endpoints of arrows. In terms of the matrix of the relation Range(f) is the subset of the codomain for which the corresponding columns have at least one T in them.

Examples: $f : \{a, b, c\} \to \{p, q, r, s\}$ given by $f(a) = q$, $f(b) = q$ and $f(c) = r$ has range $\{q, r\}$.
$g : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ has range $\{x \in \mathbb{R} \mid x \geq 0\}$.
$h : \mathbb{Z} \to \mathbb{Z} : n \to r$ where r the remainder when n is divided by 3, has range $\{0, 1, 2\}$.

Note: When A or B or both are infinite it is not possible to draw a digraph of the relation but a variation on the matrix of the relation called the graph of f is useful when A and B are subsets of $\mathbb{R}$. The graph is given by

$$\text{graph}(f) = \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$$

Example: $A = \{-2, -1, 0, 1, 2\}$, $B = \{-1, 0, 1, 2, 3, 4\}$, $f : A \to B : x \mapsto x^2$. Matrix is

f	-1	0	1	2	3	4
-2	F	F	F	F	F	T
-1	F	F	T	F	F	F
0	F	T	F	F	F	F
1	F	F	T	F	F	F
2	F	F	F	F	F	T

Replace F by blank space and T by $*$:

f	-1	0	1	2	3	4
-2						$*$
-1			$*$			
0		$*$				
1			$*$			
2						$*$

Usually we interchange the axes and get

f	-2	-1	0	1	1
4	$*$				$*$
3					
2					
1		$*$		$*$	
0			$*$		
-1					

When we enlarge A to $\{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$ and B to $\{y \in \mathbb{R} \mid -1 \leq y \leq 4\}$ we get