Problem Sheet 5

MS121 Semester 2 IT Mathematics

Exercise 1.
Evaluate the following limits:

(a) $\lim\limits_{x\to\infty} \dfrac{7x+8}{x^2-5x+3}$,

(b) $\lim\limits_{x\to-\infty} \dfrac{2x+5}{8x-3}$,

(c) $\lim\limits_{x\to\infty} \dfrac{(1-x)^3}{x^3}$,

(d) $\lim\limits_{x\to\infty} \dfrac{x^2-2x+1}{7x}$,

(e) $\lim\limits_{x\to-\infty} \dfrac{(x-1)^2}{x+1}$.

Hint: Substitute $x = \frac{1}{y}$ and consider $\lim\limits_{y\to 0^+}$ or $\lim\limits_{y\to 0^-}$ as appropriate. ⊘

Solution 1.

(a) $\lim\limits_{x\to\infty} \dfrac{7x+8}{x^2-5x+3} = \lim\limits_{x\to\infty} \dfrac{7x^{-1}+8x^{-2}}{1-5x^{-1}+3x^{-2}} = \lim\limits_{y\to 0^+} \dfrac{7y+8y^2}{1-5y+3y^2} = \dfrac{0}{1} = 0$, using the continuity of rational functions on their domain, the rule for compositions with $y(x) = \frac{1}{x}$ and $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$,

(b) $\lim\limits_{x\to-\infty} \dfrac{2x+5}{8x-3} = \lim\limits_{x\to-\infty} \dfrac{2+5x^{-1}}{8-3x^{-1}} = \lim\limits_{y\to 0^-} \dfrac{2+5y}{8-3y} = \dfrac{2}{8} = \dfrac{1}{4}$,

(c) $\lim\limits_{x\to\infty} \dfrac{(1-x)^3}{x^3} = \lim\limits_{x\to\infty} (x^{-1}-1)^3 = \lim\limits_{y\to 0^+} (y-1)^3 = (-1)^3 = -1$,

(d) $\lim\limits_{x\to\infty} \dfrac{x^2-2x+1}{7x} = \lim\limits_{x\to\infty} \dfrac{x-2+x^{-1}}{7} = \infty$ (from the definition, i.e. we can make the RHS as large as we like by choosing x large enough),

(e) $\lim\limits_{x\to-\infty} \dfrac{(x-1)^2}{x+1} = \lim\limits_{x\to-\infty} \dfrac{x-2+x^{-1}}{1+x^{-1}} = -\infty$ (again from the definition).

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Exercise 2.
The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(a) Show for all $x \neq 0$ that $|f(x)| \leq |x|$.

(b) Use the Squeeze Theorem to show that f is continuous at $x = 0$.

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Solution 2.

(a) For all $y \in \mathbb{R}$ we have $\sin(y) \in [-1,1]$, which means that for $x \neq 0$ we have $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$. It follows that $|f(x)| = |x| \cdot \left|\sin\left(\frac{1}{x}\right)\right| \leq |x|$.

(b) From part (a) we see that $-|x| \leq f(x) \leq |x|$ for all $x \neq 0$. (Note that $f(x)$ has no absolute value here.) Because $|x|$ is continuous we have $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$. We may therefore apply the Squeeze Theorem and conclude that $\lim_{x\to 0} f(x) = 0$ (and in particular that it exists). Because $f(0) = 0$ as well we conclude that f is continuous at 0.

$\diamondsuit$

Exercise 3.

Determine the numbers $A \in \mathbb{R}$ and $B \in \mathbb{R}$ such that the following function $f : \mathbb{R} \to \mathbb{R}$ is continuous:

$$f(x) = \begin{cases} 3^x & \text{if} \quad x < -1 \\ \frac{Ax+B}{4-x^2} & \text{if} \quad -1 \leq x \leq 1 \\ \sqrt{x-1} & \text{if} \quad x > 1 \end{cases} .$$

(You may use the fact that exponential functions like $x \mapsto 3^x$, rational functions and power functions like $x^{\frac{1}{2}}$ are continuous on their domains.) $\oslash$

Solution 3.

The function is continuous for $x \notin \{-1,1\}$. Note in particular that $\frac{Ax+B}{4-x^2}$ is continuous for $x \notin \{-2,2\}$ and therefore on $(-1,1)$. Also $\sqrt{x-1}$ is continuous for $x - 1 > 0$.

At $x = -1$ we have

$$\begin{aligned} \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} 3^x = 3^{-1} = \frac{1}{3} \\ \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} \frac{Ax+B}{4-x^2} = \frac{A\cdot(-1)+B}{4-(-1)^2} = \frac{B-A}{3} \\ f(-1) &= \frac{B-A}{3} \end{aligned}$$

and at $x = 1$ we have

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{Ax+B}{4-x^2} = \frac{A+B}{4-1^2} = \frac{A+B}{3} \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \sqrt{x-1} = \sqrt{1-1} = 0 \\ f(1) &= \frac{A+B}{3} . \end{aligned}$$

To have continuity at both $x = -1$ and $x = 1$ we need to require the equalities

$$\frac{B-A}{3} = \frac{1}{3}, \qquad \frac{A+B}{3} = 0.$$

The second equation means that $A = -B$ and substituting into the first equation gives $B = \frac{1}{2}$ and therefore $A = -\frac{1}{2}$. $\diamondsuit$

Exercise 4.

Evaluate the following limits. (Hint: you can factorise the numerators and denominators and divide out the common factors.)

(a) $\lim_{y\to 1} \dfrac{y^2+2y-3}{y-1}$,

(b) $\lim_{x\to 1} \dfrac{x^2+2x-3}{x^2-1}$,

(c) $\lim_{y\to -1} \left(\dfrac{1}{y+1}+\dfrac{2}{y^2-1}\right)$. ⊘

Solution 4.

(a) $\lim_{y\to 1} \dfrac{y^2+2y-3}{y-1} = \lim_{y\to 1} \dfrac{(y-1)(y+3)}{y-1} = \lim_{y\to 1} y+3 = 1+3 = 4$,

(b) $\lim_{x\to 1} \dfrac{x^2+2x-3}{x^2-1} = \lim_{x\to 1} \dfrac{(x-1)(x+3)}{(x-1)(x+1)} = \lim_{x\to 1} \dfrac{x+3}{x+1} = \dfrac{4}{2} = 2$,

(c)

$$\begin{aligned}\lim_{y\to -1} \left(\frac{1}{y+1}+\frac{2}{y^2-1}\right) &= \lim_{y\to -1} \left(\frac{1}{y+1}+\frac{2}{(y+1)(y-1)}\right)\\ &= \lim_{y\to -1} \frac{y-1+2}{(y+1)(y-1)}\\ &= \lim_{y\to -1} \frac{1}{y-1} = -\frac{1}{2}.\end{aligned}$$

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