Example: For the function $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ given by

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n-1 & \text{if } n < 0 \end{cases}$$

we can write a formula for the inverse function $f^{-1} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$

$$f^{-1}(n) = \begin{cases} n/2 & \text{if } n = 2k \\ -(n+1)/2 & \text{if } n = 2k+1 \end{cases}$$

Example: The function $f : \mathbb{R} \to \mathbb{R} : x \mapsto 3x - 5$ is both injective and surjective and is hence a bijection. By the theorem it is invertible. We can, in this case, construct the inverse function.

$$y = 3x - 5 \Leftrightarrow y + 5 = 3x \Leftrightarrow (y+5)/3 = x$$

so that the inverse function is $g : \mathbb{R} \to \mathbb{R} : y \mapsto (y+5)/3$.

Example: Let $A = B = \{0, 1, 2, 3, 4, 5, 6\}$ and define the function $f : A \to B$ by

$$f(n) = r \text{ where } 3n = 7(q) + r \text{ with } 0 \leq r < 7.$$

So r is the remainder we get when we divide $3n$ by 7. Thus $f(n) = 3n \text{mod} 7$ or $f(n) = 3n\%7$.
This f is both injective and surjective and is hence a bijection.

$$f(0) = 0, f(1) = 3, f(2) = 6, f(3) = 2, f(4) = 5, f(5) = 1, f(6) = 4.$$

By the theorem it is invertible. The inverse is $g : B \to A$ given by

$$g(0) = 0, g(3) = 1, g(6) = 2, g(2) = 3, g(5) = 4, g(1) = 5, g(4) = 6.$$

We can, in this case, check that the inverse has a similar formula

$$g(n) = s \text{ where } 5n = 7(t) + s \text{ with } 0 \leq s < 7.$$

Note: We have seen that a function is invertible if and only if it is bijective. How do we reconcile this with the fact that the function $\sin(x)$ is not injective but $\sin^{-1}(x)$ is defined? For that matter, $f(x) = x^2$ is not injective but its inverse $g(x) = \sqrt{x}$ exists.

The answer to these questions is in the definition of a function, which specifies a domain and a codomain. The function $f : \mathbb{R} \to \mathbb{R} : x \to x^2$ is neither injective (since $(-2)^2 = 4 = 2^2$) nor surjective (since $x^2 \neq -1$) but the function $h : [0, \infty) \to [0, \infty) : x \mapsto x^2$ is bijective and has inverse $k : [0, \infty) \to [0, \infty) : x \mapsto \sqrt{x}$. Here $[0, \infty)$ is the set of non-negative real numbers.

Example : Suppose $f(x) = (2x - 1)/(x + 2)$. What is the natural domain of f? What is the range of f? Show that $f(x)$ is bijective as a function from its natural domain to its range and compute the inverse function.

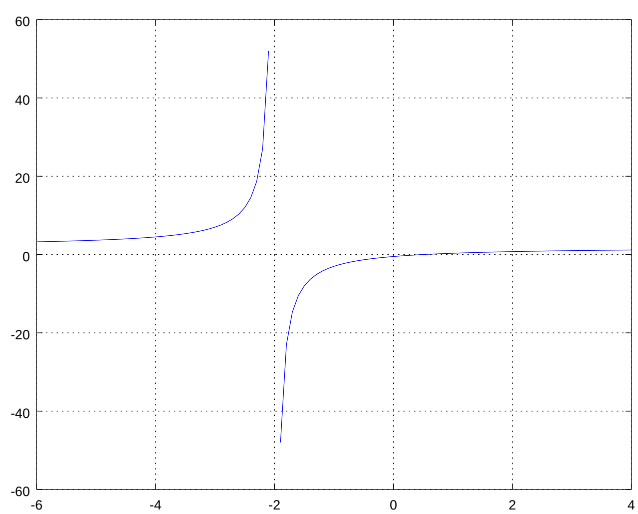
We cannot divide by zero so $x = -2$ is not in the domain. For every other real number x, $f(x)$ is real so the domain of f is $\mathbb{R} \setminus \{-2\}$. To work out the range suppose

$$y = \frac{2x - 1}{x + 2}.$$

$$\text{So } \; yx + 2y = 2x - 1 \; \text{ and } \; x(y - 2) = -1 - 2y \; \text{ which gives}$$

$$x = \frac{-1 - 2y}{y - 2}$$

Thus the range is $\mathbb{R} \setminus \{2\}$. For any $y \neq 2$ we can find an x with $f(x) = y$, namely, $x = (-1 - 2y)/y - 2)$ and for $y = 2$ we cannot find any x with $f(x) = y$. Furthermore, f is bijective from $\mathbb{R} \setminus \{-2\}$ to $\mathbb{R} \setminus \{2\}$ and the inverse function is $g(y) = (-1 - 2y)/(y - 2)$.



$$f(x) = \frac{2x - 1}{x + 2}$$

Note: Recall that, if R is a relation between a set A and a set B and S is a relation between B and a set C then the composition of S with R, written $S \circ R$, is the relation between A and C given by

$$S \circ R = \{(a, c) \in A \times C \mid \text{ for some } b \in B,\ [((a, b) \in R) \textbf{ and } ((b, c) \in S)]\}.$$

This reduces to something much simpler in the case where R and S are functions.

Definition: If $f : A \to B$ and $g : B \to C$ are functions, then the composition of g with f, written $g \circ f$, is the function

$$g \circ f : A \to C : a \mapsto g(f(a)).$$

Proposition : The composition of functions is a function.

Proof : Suppose $f : A \to B$ and $g : B \to C$ are functions. Check the two properties for the relation $g \circ f$:
(1) If $a \in A$, then, since f is a function $(a, b) \in f$ for some unique $b \in B$. This b is denoted $f(a)$. Since g is a function $(b, c) \in g$ for some unique $c \in C$. This c is denoted $g(b)$. However, $c = g(b) = g(f(a)) = (g \circ f)(a)$ is an element of C with $(a, c) \in g \circ f$.
(2) Since b is uniquely determined by a and c is uniquely determined by b, c is uniquely determined by a.

Example : Suppose $A = \{a, b, c\}, B = \{p, q, r, s\}$ and $C = \{x, y\}$, with the two functions $f : A \to B$ and $g : B \to C$ defined by

$$f(a) = q, f(b) = r, f(c) = q \ \text{ and } \ g(p) = y, g(q) = x, g(r) = y, g(s) = x$$

Then $g \circ f : A \to C$ is given by

$$(g \circ f)(a) = g(f(a))) = g(q) = x,$$

$$(g \circ f)(b) = g(f(b)) = g(r) = y,$$
$$(g \circ f)(c) = g(f(c)) = g(q) = x.$$

Example : Suppose $f : \mathbb{R} \to \mathbb{R} : x \mapsto x+1$ and $g : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$. Then

$$(g \circ f) : \mathbb{R} \to \mathbb{R}; x \mapsto g(f(x)) = g(x+1) = (x+1)^2 = x^2 + 2x + 1.$$

However the composition $f \circ g$ is also defined and

$$(f \circ g) : \mathbb{R} \to \mathbb{R}; x \mapsto f(g(x)) = f(x^2) = x^2 + 1.$$

Thus, in general, $g \circ f \neq f \circ g$.

Proposition: If $f : A \to B$, $g : B \to C$ and $h : C \to D$ are any mappings then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

That is, composition of mappings is always associative.

Proof: We simply calculate the value of each composition on a typical element of the domain A. Let $a \in A$ and set $b = f(a)$, $c = g(b)$ and $d = h(c)$. So $c = (g \circ f)(a)$ and $d = (h \circ g)(b)$. This gives

$$\begin{aligned}
(h \circ (g \circ f))(a) &= h((g \circ f)(a)) \\
&= h(c) \\
&= d \\
&= (h \circ g)(b) \\
&= (h \circ g)(f(a)) \\
&= ((h \circ g) \circ f)(a)
\end{aligned}$$