# MS121: IT Mathematics

## Integration

## Some Rules of Integration

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## Outline





## Outline







## Inverse Process of Differentiation

**Overview**

- The previous section of the syllabus dealt with the process of differentiation which involved finding the rate of change of a function with respect to a given variable, for example

$$\frac{d}{dx}x^3 = 3x^2 \qquad\qquad y = \ln x \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{x}$$

- We now consider the inverse process of differentiation, that of finding a function whose derivative is known, and this is called the process of integration.

## Inverse Process of Differentiation

**Notation**

The symbol $\int$ is used to denote the operation of integration, i.e.

$$\int_a^b f(x)\,dx = \text{ the } \textit{integral of } f \text{ from } x = a \text{ to } x = b$$

where $a$ and $b$ are called the limits of integration ($a$ is the lower limit and $b$ is the upper limit).

The key to calculating integrals is to recognize that

> Integration reverses the action of Differentiation





## Inverse Process of Differentiation

> Integration reverses the action of Differentiation

The statement that

differentiating $x^3$ gives $3x^2$

can be inverted to one which states that

integrating $3x^2$ gives $x^3$.





## Integration reverses the action of Differentiation

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $x^n \ (n \neq -1)$ | $\frac{x^{n+1}}{n+1}$ |
| $e^x$ | $e^x$ |
| $\frac{1}{x}$ | $\log x$ |
| $\cos x$ | $\sin x$ |
| $\sin x$ | $-\cos x$ |

Note that differentiating the entry in the right-hand column (integral) gives the corresponding entry in the left-hand column.





## Integration reverses the action of Differentiation

**Examples**

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x\,dx = \sin x$$

$$\frac{d}{dx}(e^x) = e^x \quad \Rightarrow \quad \int e^x\,dx = e^x$$

$$\frac{d}{dx}\left(\frac{x^5}{5}\right) = \frac{1}{5}\cdot 5x^4 = x^4 \quad \Rightarrow \quad \int x^4\,dx = \frac{x^5}{5}$$

## Outline







## The Constant of Integration

**Illustration**

Since

$$\begin{aligned} \frac{d}{dx}x^2 &= 2x \\ \frac{d}{dx}\left(x^2+5\right) &= 2x \\ \frac{d}{dx}\left(x^2-100\right) &= 2x \end{aligned}$$

it is quite common to write

$$\int 2x\,dx = x^2 + C,$$

where $C$ is a constant called the constant of integration.





## The Constant of Integration

$$\int 2x\,dx = x^2 + C,$$

**Rationale**

- If we differentiate $x^2 + C$, we obtain $2x$.
- This is because the derivative of any constant $C$ is always $0$.
- In this way, for example, the statement

$$\frac{d}{dx}\left(\sin x + C\right) = \cos x$$

is equivalent to

$$\int \cos x\,dx = \sin x + C.$$





## Outline

## Definite & Indefinite Integrals

When we are not specific, or definite, about where to start and stop integrating, we call the integral the indefinite integral:

$$\int f(x)\,dx, \quad \text{which is a } \underline{\text{function}} \text{ of } x$$

When we define the lower and upper limits of integration ($a$ and $b$), we call the integral the definite integral:

$$\int_a^b f(x)\,dx \quad \text{which is a } \underline{\text{number}}$$





## Definite & Indefinite Integrals

### Evaluating Definite Integrals

Given $F(x) = \int f(x)\,dx$, we can evaluate the definite integral

$$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{b} := F(b) - F(a).$$

Note that the area between the curve $y = f(x)$ and the $x$-axis between the vertical lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$





## Outline







## Examples: Evaluating Definite Integrals

### Calculate $\int_1^2 x\,dx$

We can find

$$\int_1^2 x\,dx$$

by first noting that

$$\int x\,dx = \frac{x^2}{2} \qquad \left(\text{since } \frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2}\cdot 2x = x\right)$$

So

$$\int_1^2 x\,dx = \frac{x^2}{2}\Big|_{x=1}^{2} = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$$

## Examples: Evaluating Definite Integrals

**Calculate $\displaystyle\int_2^4 e^x\,dx$**

The integral
$$\int_2^4 e^x\,dx$$
is found by observing that
$$\int e^x\,dx = e^x$$
and so
$$\int_2^4 e^x\,dx = e^x\Big|_{x=2}^{4} = e^4 - e^2$$





## Examples: Evaluating Definite Integrals

**Calculate $\displaystyle\int_1^2 \frac{1}{x}\,dx$**

We find
$$\int_1^2 \frac{1}{x}\,dx$$
by referring to the formulae and tables:
$$\int_1^2 \frac{1}{x}\,dx = \log x\Big|_{x=1}^{2} = \log 2 - \log 1$$





## Examples: Evaluating Definite Integrals

**Calculate $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$**

We must first write
$$\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx$$
and use the general formula for $n \neq -1$:
$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$
so that, in this instance, with $n = -2$, must have
$$\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$





## Examples: Evaluating Definite Integrals

**Calculate $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$ (Cont'd)**

Since
$$\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$
we therefore obtain
$$\int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{x=1}^{2} = -\frac{1}{2} - \left(-\frac{1}{1}\right) = -\frac{1}{2} + 1 = \frac{1}{2}$$

## Outline







## Two Basic Rules of Integration

(P1) The integral of the sum is the sum of the integrals:

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

(P2) A constant can be factored outside the integral:

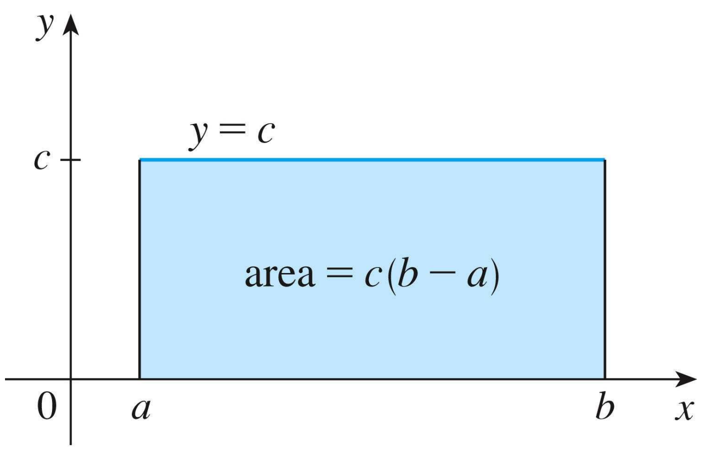
$$\int c\, f(x)\, dx = c \int f(x)\, dx, \quad \text{where } c \text{ is any } \textit{constant}$$

It is then straightforward to calculate the integrals of a wide class of functions.





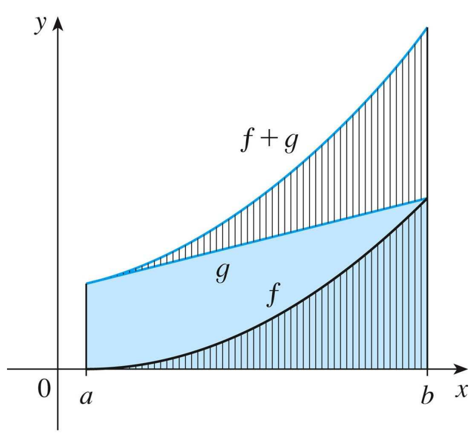
## $\int_a^b c\, dx = c(b-a)$, $c$ a constant








## $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$



# Examples: Evaluating Definite Integrals

Calculate $\int \left(3x^{\frac{1}{2}} - \frac{2}{x} + 4\sin x\right)\, dx$

Using the two rules:

$$\begin{aligned}
\int \left(3x^{\frac{1}{2}} - \frac{2}{x} + 4\sin x\right) dx &= \int 3x^{\frac{1}{2}}\, dx + \int \left(-\frac{2}{x}\right) dx + \int 4\sin x\, dx \\
&= 3\int x^{\frac{1}{2}}\, dx - 2\int \frac{1}{x}\, dx + 4\int \sin x\, dx \\
&= 3\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) - 2\log x + 4\left(-\cos x\right) \\
&= 2x^{\frac{3}{2}} - 2\log x - 4\cos x
\end{aligned}$$