Problem Sheet 2

MS121 Semester 2 IT Mathematics

Exercise 1.
Find the equations that determine the following lines:

(a) The line through $(-1, 2)$ with slope -2.

(b) The line through $(-1, -2)$ and $(1, 3)$.

(c) The line through $(0, 5)$ which is parallel to the line $y = 7 - 3x$.

(d) The line through $(1, 1)$ which is perpendicular to the line $2x - y = 5$.

$\oslash$

Solution 1.

(a) For the line through $(-1, 2)$ with slope -2 we start with the general formula $y = mx + b$. We insert the slope $m = -2$ and then determine b using the point $(-1, 2)$: $2 = -2(-1) + b$, so $b = 0$. Hence, $y = -2x$.

(b) For the line through $(-1, -2)$ and $(1, 3)$ we first determine the slope $m = \frac{3-(-2)}{1-(-1)} = \frac{5}{2}$. Inserting this into $y = mx + b$ we may use either of the points to determine $b = \frac{1}{2}$ and hence $y = \frac{1}{2}(5x + 1)$.

(c) The line through $(0, 5)$ which is parallel to the line $y = 7 - 3x$ must have the same slope, $m = -3$. Using $(0, 5)$ to determine b in $y = -3x + b$ yields $b = 5$ and hence $y = -3x + 5$.

(d) For the line through $(1, 1)$ which is perpendicular to the line $2x - y = 5$ we first note that the given line can be written as $y = 2x - 5$, so it has a slope 2. Any line perpendicular to it must have a slope $m = -\frac{1}{2}$ (the products of the slopes must be -1). Determining b in $y = -\frac{1}{2}x + b$ with the given point $(1, 1)$ yields $b = \frac{3}{2}$ and hence $y = -\frac{1}{2}x + \frac{3}{2}$.

$\diamondsuit$

Exercise 2.
Which of the following functions are even, which ones are odd, and which ones are neither?

(a) $f(x) = 4x^3 - 2x$,

(b) $f(x) = 5x^6$,

(c) $f(x) = x - 3$,

(d) $f(x) = \sin(x)$,

(e) $f(x) = x\sin(x) + \cos(x)$,

(f) $f(x) = 0$.

$\oslash$

Solution 2.

(a) $f(x) = 4x^3 - 2x$ is odd (polynomial with only odd powers),

(b) $f(x) = 5x^6$ is even,

(c) $f(x) = x - 3$ is neither even nor odd,

(d) $f(x) = \sin(x)$ is odd,

(e) $f(x) = x\sin(x) + \cos(x)$ is even,

(f) $f(x) = 0$ is the only function which is both even and odd.

$\diamondsuit$

Exercise 3.
Find the roots of the following functions and determine where the functions are positive and where they are negative:

(a) $f(x) = 49 - x^2$,

(b) $f(x) = x^2 - 5x - 6$,

(c) $g(y) = y^2 - 4y + 4$,

(d) $h(t) = t^2 + 2t + 3$,

(e) $y = -x^2 + 3x - 2$,

(f) $h(x) = x^4 + 4x^2 + 3$.

$\oslash$

Solution 3.

(a) $f(x) = 49 - x^2$ has roots $x = -7$, and $x = 7$, and it is positive on $(-7, 7)$ and negative on $(-\infty, -7) \cup (7, \infty)$,

(b) $f(x) = x^2 - 5x - 6$ has roots $x = -1$, and $x = 6$, and it is positive on $(-\infty, -1) \cup (6, \infty)$ and negative on $(-1, 6)$,

(c) $g(y) = y^2 + 4y + 4$ has one (double) root $y = 2$, and it is positive on $(-\infty, 2) \cup (2, \infty)$ and nowhere negative,

(d) $h(t) = t^2 + t + 1$ has no roots (discriminant is negative) and it is positive on all of $\mathbb{R}$,

(e) $y = -x^2 + 3x - 2$ has roots $x = 1$, and $x = 2$, and it is positive on $(1, 2)$ and negative on $(-\infty, 1) \cup (2, \infty)$,

(f) for $h(x) = x^4 + 4x^2 + 3$ we first note $h(x) = g(x^2)$ with $g(y) = y^2 + 4y + 3$; $g(y)$ has roots at $y = -3$ and $y = -1$, so that $g(y) = (y + 3)(y + 1)$; it follows that $h(x) = (x^2 + 3)(x^2 + 1)$ and neither factor has roots, because $x^2 = -3$ and $x^2 = -1$ have no solutions in $\mathbb{R}$; hence h has no roots and it is positive on the entire $\mathbb{R}$.

$\diamondsuit$

Exercise 4.
Solve the following inequalities for $x \in \mathbb{R}$. Write your answer in terms of intervals.

(a) $1 - 3x \leq -2$,

(b) $1 \leq 7 - 2x < 3$,

(c) $|3x - 4| < 5$,

(d) $2x^2 - 4x < 16$,

(e) $\frac{x+3}{2x+7} > 0$, (Hint: multiply both sides by the positive number $(2x + 7)^2$)

(f) $\frac{3}{\sqrt{-2x^2+7x-5}} > 0$,

(g) $\frac{x+2}{3x+4} > 5x + 6$.

$\oslash$

Solution 4.

(a) $1 - 3x \leq -2$ means $3x \geq 1 + 2 = 3$ and $x \geq 1$, i.e. $x \in [1, \infty)$,

(b) $1 \leq 7 - 2x < 3$ means $2x \leq 7 - 1 = 6$ and $2x > 7 - 3 = 4$, i.e. $x \in (2, 3]$,

(c) to solve $2x^2 - 4x < 16$ we first find the solutions to $2x^2 - 4x = 16$, which means $x^2 - 2x - 8 = 0$ and therefore $x = -2$ or $x = 4$. From the shape of the graph $y = 2x^2 - 4x - 16$ we see that the solutions are $x \in (-2, 4)$,

(d) $|3x - 4| < 5$ means $3x - 4 < 5$ *and* $4 - 3x < 5$ (because $|y| = \max\{y, -y\}$), and therefore $x \in (-\frac{1}{3}, 3)$,

(e) $\frac{x+3}{2x+7} > 0$ means either $x + 3 > 0$ and $2x + 7 > 0$, or $x + 3 < 0$ and $2x + 7 < 0$, i.e. $x \in (-\infty, -3\frac{1}{2}) \cup (-3, \infty)$,

(f) $\frac{3}{\sqrt{-2x^2+7x-5}} > 0$ means $\sqrt{-2x^2 + 7x - 5} > 0$ and therefore $-2x^2 + 7x - 5 > 0$; first solving $-2x^2 + 7x - 5 = 0$ we have $x = 1$ or $x = \frac{5}{2}$ and from the shape of the graph of $y = -2x^2 + 7x - 5$ we see that the solutions are $x \in (1, \frac{5}{2})$,

(g) $\frac{x+2}{3x+4} > 5x + 6$ means either $3x + 4 > 0$ and $x + 2 > (5x + 6)(3x + 4)$ (multiplying both sides by $3x + 4$), or $3x + 4 < 0$ and $x + 2 < (5x + 6)(3x + 4)$; the polynomial $(5x + 6)(3x + 4) - (x + 2) = 15x^2 + 37x + 22$ has roots at $x = -\frac{22}{15}$ and $x = -1$ and it is negative in between these roots; using the ordering $-\frac{22}{15} < -\frac{4}{3} < -1$ we then find that $x \in (-\infty, -\frac{22}{15}) \cup (-\frac{4}{3}, -1)$.

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