Problem Sheet 8

MS121 Semester 2 IT Mathematics

Exercise 1.

Find and classify all critical points of the following functions, and sketch the curve $y = f(x)$:

(a) $f(x) = x^3 - 3x^2 + 5$,

(b) $f(x) = \cosh(x)$,

(c) $f(x) = x^4 - 4x^3 + 7$,

(d) $f(x) = xe^{-x^2}$,

(e) $f(x) = \frac{x+1}{x^2+3}$,

(f) $f(x) = \sqrt{2 - \cos(x)}$.

$\oslash$

Solution 1.

(a) $f(x) = x^3 - 3x^2 + 5$ has $f'(x) = 3x^2 - 6x = 3x(x-2)$, so the critical points are at $x = 0$ and $x = 2$. We have $f''(x) = 6x - 6 = 6(x-1)$, so $f''(0) = -6 < 0$ and $f''(2) = 6 > 0$, with a local maximum $f(0) = 5$ and local minimum $f(2) = 1$.

(b) $f(x) = \cosh(x)$ has $f'(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$, so the only critical point is at $x = 0$ (because e^x is invertible). Since $f''(x) = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$ we have $f''(0) = 1 > 0$ and we have a (global) minimum at $f(0) = 1$.

(c) $f(x) = x^4 - 4x^3 + 7$ has $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$, so the critical points are at $x = 0$ and $x = 3$. We have $f''(x) = 12x^2 - 24x = 12x(x-2)$, so $f''(0) = 0$ and $f''(3) = 36 > 0$. $f(3) = -20$ is a (global) minimum. Because $f'(x)$ does not change sign at $x = 0$ we have an inflection point there with $f(0) = 7$.

(d) $f(x) = xe^{-x^2}$ has $f'(x) = (1-2x^2)e^{-x^2}$, so the critical points are at $x = -\frac{1}{2}\sqrt{2}$ and $x = \frac{1}{2}\sqrt{2}$. We have $f''(x) = (-4x - 2x + 4x^3)e^{-x^2} = 2x(2x^2-3)e^{-x^2}$ and $f''(\pm\frac{1}{2}\sqrt{2}) = \mp 2\sqrt{\frac{2}{e}}$, so we find a (global) maximum $f(\frac{1}{2}\sqrt{2}) = \frac{1}{2}\sqrt{2/e}$ and a (global) minimum $f(-\frac{1}{2}\sqrt{2}) = -\frac{1}{2}\sqrt{2/e}$.

(e) $f(x) = \frac{x+1}{x^2+3}$ has $f'(x) = \frac{(x^2+3)-2x(x+1)}{(x^2+3)^2} = \frac{-x^2-2x+3}{(x^2+3)^2}$, so the critical points are at $x = -3$ and $x = 1$. We have $f''(x) = \frac{(x^2+3)(-2x-2)+(x^2+2x-3)4x}{(x^2+3)^3} = \frac{2x^3+6x^2-18x-6}{(x^2+3)^3}$, so $f''(-3) = \frac{48}{12^3} = \frac{1}{36} > 0$ and $f''(1) = \frac{-16}{4^3} = -\frac{1}{4} < 0$ and we find a (global) minimum $f(-3) = -\frac{1}{6}$ and a (global) maximum $f(1) = \frac{1}{2}$.

(f) $f(x) = \sqrt{2 - \cos(x)}$ has $f'(x) = \frac{1}{2}(2 - \cos(x))^{-\frac{1}{2}} \sin(x)$, so critical points are at $x = k\pi$ for all $k \in \mathbb{Z}$. We have

$$\begin{aligned} f''(x) &= -\frac{1}{4}(2-\cos(x))^{-\frac{3}{2}}(\sin(x)^2 - 2(2-\cos(x))\cos(x)) \\ &= -\frac{1}{4}(2-\cos(x))^{-\frac{3}{2}}(2 - \sin(x)^2 - 4\cos(x)), \end{aligned}$$

so $f''(k\pi) = \frac{1}{2} > 0$ when $k \in 2\mathbb{Z}$ (because $\cos(k\pi) = 1$) and $f''(k\pi) = -\frac{1}{2\sqrt{3}} < 0$ else (because $\cos(k\pi) = -1$). We therefore have (global) minima $f(k\pi) = 1$ when $x = k\pi$ with k even (i.e. $k \in 2\mathbb{Z}$) and (global) maxima $f(k\pi) = \sqrt{3}$ when k is odd. (This oscillating behaviour is similar as for $2 - \cos(x)$, but the square root changes the amplitudes above and below in different ways.)

$\diamondsuit$

Exercise 2.
Consider the function $f(x) = 2x^3 - 15x^2 + 24x + 20$.

(a) Determine the absolute maximum and minimum on the closed interval $[0, 6]$.

(b) Does $f(x)$ also have an absolute maximum and/or minimum on the open interval $(0, 6)$?

$\oslash$

Solution 2.

(a) We have $f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x-4)(x-1)$, so the critical points are $x = 1$ and $x = 4$, which do lie in the interval $[0, 6]$. At the boundaries and the critical points we have: $f(0) = 20$, $f(1) = 31$, $f(4) = 4$ and $f(6) = 56$. The absolute minimum on $[0, 6]$ is therefore $f(4) = 4$ and the absolute maximum is $f(6) = 56$.

(b) On the open interval $(0, 6)$ there is an absolute minimum, $f(4) = 4$, but no absolute maximum: the function approaches $f(6) = 56$ as closely as we like, but it never actually attains this value in the open interval $(0, 6)$.

$\diamondsuit$

Exercise 3.
We take a rectangular piece of paper of 12 cm by 12 cm and we remove squares of x cm by x cm from each of the four corners. We fold the ends up to a box of height x. For what x does the box have a maximal volume? How big is this volume? $\oslash$

Solution 3.
The height of the box is x and the length and width are both $12 - 2x$ (in cm). The volume is therefore $V(x) = x(12 - 2x)^2 = 4x^3 - 48x^2 + 144x$ (in cm^3). Note that $0 \leq x \leq 6$. To find the maximum we compute $V'(x) = 12x^2 - 96x + 144 = 12(x^2 - 8x + 12)$ and we set $V'(x) = 0$ to find the critical points of V. These are $x_\pm = 4 \pm \frac{1}{2}\sqrt{16} = 4 \pm 2$, i.e. $x = 2$ and $x = 6$. (The larger critical point is on the boundary.) We have $V(0) = 0$, $V(6) = 0$ and $V(2) = 2(12 - 4)^2 = 128$. We therefore find the maximum volume 128 cm^3 at $x = 2$ cm. $\diamondsuit$

Exercise 4.
Determine whether the following functions are continuous and/or differentiable at the indicated points:

(a) $f(x) = |x|$ at $x = 0$,

(b) $f(x) = |x|^3$ at $x = 0$,

(c)
$$g(t) = \begin{cases} 2t - 1 & \text{if } -1 < t < 1 \\ t^2 & \text{if } |t| \geq 1 \end{cases}$$

at $t = 1$ and at $t = -1$.

$\oslash$

Solution 4.

(a) Note that

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} .$$

This is continuous at $x = 0$, but not differentiable, because

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} .$$

has unequal left and right hand limits as $x \to 0$.

(b) $f(x) = |x|^3 = |x|x^2$ is continuous at $x = 0$, because it is a product of continuous functions (see part (a)). It is even differentiable there, because

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x > 0 \end{cases} .$$

tends to 0 as $x \to 0$.

(c)

$$g(t) = \begin{cases} 2t - 1 & \text{if } -1 < t < 1 \\ t^2 & \text{if } |t| \geq 1 \end{cases}$$

is not continuous at $t = -1$, because the left and right hand limits differ, and it is therefore not differentiable either. $g(t)$ is continuous at $t = 1$ and it is even differentiable there, because

$$\begin{aligned} \frac{g(t) - g(1)}{t - 1} &= \begin{cases} \frac{2t-2}{t-1} & \text{if } -1 < t < 1 \\ \frac{t^2-1}{t-1} & \text{if } |t| \geq 1 \end{cases} \\ &= \begin{cases} 2 & \text{if } -1 < t < 1 \\ t+1 & \text{if } |t| \geq 1 \end{cases} \end{aligned}$$

has the limit 2 as $t \to 1$.

$\diamondsuit$