Problem Sheet 6

MS121 Semester 2 IT Mathematics

Exercise 1.

Compute the indicated derivatives of the following functions using the definition of derivatives:

(a) $f'(3)$ for $f(x) = x^2 + 2x$,

(b) $f'(0)$ for $f(x) = x^3 + 1$,

(c) $f'(2)$ for $f(x) = \frac{1}{x}$,

(d) $f'(1)$ for $f(x) = \frac{x+1}{x}$,

(e) $f'(0)$ for $f(x) = \frac{1}{2x-1}$,

(f) $f'(1)$ for $f(x) = \frac{1}{x^2}$.

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Solution 1.

These solutions actually compute the derivatives as functions at all points in the domain of f and then substitute the required values:

(a) $f(x) = x^2 + 2x$ has difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 + 2h}{h} = 2x + h + 2$$

and derivative $f'(x) = 2x + 2$, so in particular $f'(3) = 8$,

(b) $f(x) = x^3 + 1$ has difference quotient

$$\frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2$$

and derivative $f'(x) = 3x^2$, so in particular $f'(0) = 0$,

(c) $f(x) = \frac{1}{x}$ has difference quotient

$$\frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \frac{x - (x+h)}{h(x+h)x} = \frac{-1}{(x+h)x}$$

and derivative $f'(x) = \frac{-1}{x^2}$, so in particular $f'(2) = -\frac{1}{4}$,

(d) $f(x) = \frac{x+1}{x}$ means $f(x) = 1 + \frac{1}{x}$, which has derivative $f'(x) = \frac{-1}{x^2}$, as before, so in particular $f'(1) = -1$,

(e) $f(x) = \frac{1}{2x-1}$ has difference quotient

$$\frac{1}{h}\left(\frac{1}{2x+2h-1} - \frac{1}{2x-1}\right) = \frac{2x - 1 - (2x + 2h - 1)}{h(2x+2h-1)(2x-1)} = \frac{-2}{(2x+2h-1)(2x-1)}$$

and derivative $f'(x) = \frac{-2}{(2x-1)^2}$, so in particular $f'(0) = -2$,

(f) $f(x) = \frac{1}{x^2}$ has difference quotient

$$\frac{1}{h}\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right) = \frac{x^2 - (x+h)^2}{h(x+h)^2x^2} = \frac{-2x - h}{(x+h)^2x^2}$$

and derivative $f'(x) = \frac{-2x}{x^4} = \frac{-2}{x^3}$, so in particular $f'(1) = -2$.

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Exercise 2.

Use the rules of differentiation to find the derivatives of the following functions. Indicate which rules of differentiation you used in each part and which known derivatives of basic functions.

(a) $y = x^7 - 3x^5 + 6$,

(b) $y = x^{121} + 45x^2 - 33x + 1$,

(c) $y = x + \frac{1}{x}$,

(d) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$,

(e) $y = 7x^{\frac{1}{20}} + (\sqrt[5]{x})^7$,

(f) $y = (x^3 - 5x^2)(x + 8)$,

(g) $y = x^{\frac{1}{4}}\left(2 + \frac{1}{7}x^21\right)$,

(h) $y = x^{\frac{1}{5}}(x^2 + 3)\frac{1}{\sqrt{x}}$,

(i) $y = (x^3 + 7x)^9$,

(j) $y = \frac{x^2+1}{3+x}$.

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Solution 2.

(a) $y = x^7 - 3x^5 + 6$ has derivative $7x^6 - 15x^4$,

(b) $y = x^{121} + 45x^2 - 33x + 1$ has derivative $1217x^{120} + 90x - 33$,

(c) $y = x + \frac{1}{x}$ has derivative $1 - \frac{1}{x^2}$,

(d) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$ has derivative $\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$,

(e) $y = 7x^{\frac{1}{20}} + (\sqrt[5]{x})^7$ has derivative $\frac{7}{20}x^{\frac{-19}{20}} + \frac{7}{5}x^{-\frac{2}{5}}$,

(f) $y = (x^3 - 5x^2)(x + 8)$ has derivative $(3x^2 - 10x)(x + 8) + (x^3 - 5x^2)$ $(= 4x^3 + 9x^2 - 80x)$,

(g) $y = x^{\frac{1}{4}}\left(2 + \frac{1}{7}x^21\right)$ has derivative $\frac{1}{4}x^{-\frac{3}{4}}\left(2 + \frac{1}{7}x^21\right) + x^{\frac{1}{4}}3x^{20}$,

(h) $y = x^{\frac{1}{5}}(x^2 + 3)\frac{1}{\sqrt{x}}$ has derivative
$\frac{1}{5}x^{-\frac{4}{5}}(x^2 + 3)\frac{1}{\sqrt{x}} + x^{\frac{1}{5}}(2x)\frac{1}{\sqrt{x}} - x^{\frac{1}{5}}(x^2 + 3)\frac{1}{2}x^{-\frac{3}{2}}$,

(i) $y = (x^3 + 7x)^9$ has derivative $9(x^3 + 7x)^8(3x^2 + 7)$,

(j) $y = \frac{x^2+1}{3+x}$ has derivative $\frac{(3+x)2x-(x^2+1)}{(3+x)^2} = \frac{x^2+6x-1}{(3+x)^2}$.

We used that the derivative of x^α is $\alpha x^{\alpha-1}$ for all $\alpha \in \mathbb{R}$. We also used the following rules:

sum rule	all parts
product rule	parts (f), (g) and (h)
chain rule	part (i)
quotient rule	part (j)

Different answers are possible, because sometimes we can choose which rule to use. $\diamondsuit$

Exercise 3.

Find the tangent line to the curve $y = f(x)$ for the following functions at the given values of x:

(a) $f(x) = x^2 + x$, at $x = 2$,

(b) $f(x) = x^3 + 1$, at $x = -1$,

(c) $f(x) = x^2 - 3x$, at $x = -3$,

(d) $f(x) = x^2 + \frac{1}{2x}$, at $x = 1$.

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Solution 3.

(a) $f(x) = x^2 + x$, at $x = 2$, has $f(2) = 6$, so the tangent goes through $(2, 6)$; we have $f'(x) = 2x + 1$ and $f'(2) = 5$, so the tangent has slope 5; its equation is therefore $y = 5x - 4$.

(b) $f(x) = x^3 + 1$, at $x = -1$, has $f(-1) = 0$, so the tangent goes through $(-1, 0)$; we have $f'(x) = 3x^2$ and $f'(-1) = 3$, so the tangent has slope 3; its equation is therefore $y = 3x + 3$.

(c) $f(x) = x^2 - 3x$, at $x = -3$, has $f(-3) = 18$, so the tangent goes through $(-3, 18)$; we have $f'(x) = 2x - 3$ and $f'(-3) = -9$, so the tangent has slope -9; its equation is therefore $y = -9x - 9$.

(d) $f(x) = x^2 + \frac{1}{2x}$, at $x = 1$, has $f(1) = \frac{3}{2}$, so the tangent goes through $(1, \frac{3}{2})$; we have $f'(x) = 2x - \frac{1}{2x^2}$ and $f'(1) = \frac{3}{2}$, so the tangent has slope $\frac{3}{2}$; its equation is therefore $y = \frac{3}{2}x$.

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Exercise 4.

Calculate the following limits by recognizing a derivative:

(a) $\lim_{x\to 0} \dfrac{(x^7 - 1)^8 - 1}{x}$,

(b) $\lim_{x\to 4} \dfrac{\sqrt{x-3} - 1}{x - 4}$,

(c) $\lim_{x\to 0} \dfrac{\sin(x)}{x}$,

(d) $\lim_{x\to 1} \dfrac{1}{x^2(x-1)} - \dfrac{1}{x-1}$.

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Solution 4.

(a) $\lim_{x\to 0} \dfrac{(x^7 - 1)^8 - 1}{x} = f'(0)$ for $f(x) = (x^7 - 1)^8$. Since $f'(x) = 8(x^7 - 1)^7 \cdot 7x^6$ we find $f'(0) = 0$, so the limit vanishes.

(b) $\lim_{x\to 4} \dfrac{\sqrt{x-3} - 1}{x - 4} = f'(4)$ for $f(x) = \sqrt{x-3}$. Since $f'(x) = \frac{1}{2}(x-3)^{-\frac{1}{2}}$ we find that the limit is $f'(4) = \frac{1}{2}$.

(c) $\lim_{x\to 0} \dfrac{\sin(x)}{x} = \sin'(0) = \cos(0) = 1$.

(d) $\lim_{x\to 1} \dfrac{1}{x^2(x-1)} - \dfrac{1}{x-1} = f'(1)$ for $f(x) = x^{-2}$. Since $f'(x) = -2x^{-3}$ we find that the limit is $f'(1) = -2$.

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