An old MS121 final Exam question:

> (c) (i) **State** the inclusion-exclusion principle for three finite sets A, B and C.
> (ii) Suppose that A, B and C are finite sets with the following properties:
>
> B has two more elements than A; C is twice as big as B;
>
> $A \cap B$ is the same size as $A \cap C$; B and C have no elements in common.
>
> Prove that $|A \cup B \cup C|$ is divisible by 2 (i.e. has an even number of elements).

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| \\
&\quad -|A \cap B| - |A \cap C| - |B \cap C| \\
&\quad +|A \cap B \cap C| \\
&= |A| + (|A| + 2) + 2(|A| + 2) \\
&\quad -|A \cap B| - (|A \cap B|) - (0) \\
&\quad +(0) \\
&= 2(2|A| + 3 - |A \cap B|)
\end{aligned}$$

Here we are not told the sizes of the sets, just some relationships. We translate
B has two more elements than A: $|B| = |A| + 2$
C is twice as big as B: $|C| = 2|B|$
$A \cap B$ is the same size as $A \cap C$: $|A \cap B| = |A \cap C|$
B and C have no elements in common: $|B \cap C| = 0$.
There is still the problem of knowing the size of $A \cap B \cap C$. However, set theory facts help us here. Since B and C have no elements in common, $B \cap C = \emptyset$. Furthermore, since $X \cap Y \subseteq X$ for any two sets X and Y we get

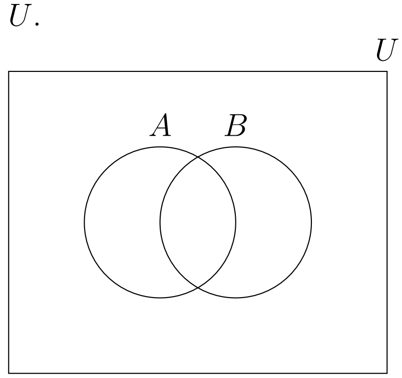
$$A \cap B \cap C = A \cap (B \cap C) \subseteq B \cap C = \emptyset$$

and $A \cap B \cap C = \emptyset$ which gives $|A \cap B \cap C| = 0$.

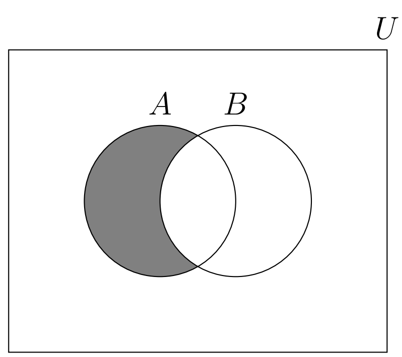
Set theory and methods of proof.

Suppose we want to prove $P \Rightarrow Q$. Let U be the universal set of all things under discussion (Integers, real numbers, triangles, pairs of integers, etc.,) Set $A = \{x \in U \mid P(x)\}$, $B = \{x \in U \mid Q(x)\}$. Proving $P \Rightarrow Q$ is equivalent to showing $A \subseteq B$. Showing $x \in A \Rightarrow x \in B$ is the direct proof.

Now look at a Venn diagram of two general sets A and B in a universal set U.



To say $A \subseteq B$ is equivalent to saying $\sim B \subseteq \sim A$. So we could approach proving $P \Rightarrow Q$ by proving **not** $Q \Rightarrow$ **not** P. This is the contrapositive approach.



Finally, again looking at the Venn diagram we see that $A \subseteq B$ is equivalent to saying $\sim B \cap A = \emptyset$. This gives the proof by contradiction approach.

Product sets

Definition: If A and B are sets their Cartesian product, denoted $A \times B$, is the set whose elements are all possible ordered pairs (a, b), where $a \in A$ and $b \in B$. That is,

$$A \times B = \{(a, b) \mid a \in A \textbf{ and } b \in B\}$$

Example: If $A = \{1, 2\}$ and $B = \{p, q, r\}$ then

$$A \times B = \{(1, p), (1, q), (1, r), (2, p), (2, q), (2, r)\}$$

If A and B are finite we can picture the Cartesian product as a grid. In the case of the last example this would be

$$\begin{array}{cc} .\ (1, r) & .\ (2, r) \\ \\ .\ (1, q) & .\ (2, q) \\ \\ .\ (1, p) & .\ (2, p) \end{array}$$

Note: If A and B are finite, then $|A \times B| = |A||B|$. (That is the product of the two cardinalities.)

Note: We already view the plane $\mathbb{R}^2$ as $\mathbb{R} \times \mathbb{R}$.

Note: If $A = B$ then we write $A \times A$ as A^2.

Note: We can extend this definition to more than two sets and to a Cartesian product of a set with itself several times.

Example: If $A = \{0, 1\}$, then

$$A^3 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$

This is the same as the set of binary strings of length 3. We usually write these as

$$000, 001, 010, 011, 100, 101, 110, 111$$

and binary strings of length 3 can be used to represent subsets of a three element set.

Example: The number of elements in A^3 is $|A|^3$. In the above example, $|A| = 2$ and A^3 has $8 = 2^3$ elements. In general, the number of binary strings of length n is 2^n.