# ARMA(p,q) Process

Kushal Kharel

9/19/2021

Definition:

$\{X_t\}$ is an $ARMA(p,q)$ process if $\{X_t\}$ is stationary and if for every $t$,

$X_t - \phi X_{t-1} - ... - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q}, \ldots$ (i)

where,

$\{Z_t\} \sim WN(0, \sigma^2)$ and the polynomials $(1 - \phi_1 z - ... - \phi_p z^p)$ and $1 + \theta_1 z + ... + \theta_q z^q$ have no common factors.

The process $\{X_t\}$ is said to be an $ARMA(p,q)$ process with mean $\mu$ if $\{X_t - \mu\}$ is an $ARMA(p,q)$ process.

The more concise form of equation (i) above is,

$\phi(B)X_t = \theta(B)Z_t, \ldots$ (ii)

where $\phi(.)$ and $\theta(.)$ are the $p$th and $q$th - degree polynomials,

$\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p$ and

$\theta(z) = 1 + \theta_1 z + ... + \theta_q z^q$,

and B is the backward shift operator $(B^j X_t = X_{t-j}, B^j Z_t = Z_{t-j}, j = 0, \pm 1, ...)$.

The time series $\{X_t\}$ is said to be an Autoregressive process of order $p$ (or $AR(p)$) if $\theta(z) \equiv 1$, and a moving average process of order $q$ (or $MA(q)$ if) $\phi(z) \equiv 1$.

For the $ARMA(1,1)$ equations, a stationary solution exists (and is unique) if and only if $\phi_1 \neq \pm 1$. The latter is equivalent to the condition that autoregressive polynomial $\phi(z) = 1 - \phi_1 z \neq 0$ for $z = \pm 1$. The analogous condition for the general $ARMA(p,q)$ process is $\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p \neq 0$ for all complex $z$ with $|z| = 1$. The region defined by the set of complex $z$ such that $|z| = 1$ is our unit circle.

If $\phi(z) \neq 0$ for all $z$ on the unit circle, then there exists $\delta > 0$ such that

$\frac{1}{\phi(z)} = \sum_{j=-\infty}^{\infty} \chi_j z^j$ for $1 - \delta < |z| < 1 + \delta$,

and $\sum_{j=-\infty}^{\infty} |\chi_j| < \infty$. We can now define $\frac{1}{\phi(B)}$ as the linear filter with absolutely summable coefficients $\frac{1}{\phi(B)} = \sum_{j=-\infty}^{\infty} \chi_j B^j$.

Applying the operator $\chi(B) := \frac{1}{\phi(B)}$ to both sides of equation (ii), we get

$X_t = \chi(B)\phi(B)X_t = \chi(B)\theta(B)Z_t = \psi(B)Z_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \ldots$ (iii)

where,

$\psi(z) = \chi(z)\theta(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j$

Existence and Uniqueness:

A stationary solution $\{X_t\}$ of equation (i) exists (and is also the unique stationary solution)

if and only if $\phi(z) = 1 - \phi_1(z) - ... - \phi_p z^p \neq 0$ for all $|z| = 1, \ldots$ (iv)

Causality:

An $ARMA(p,q)$ process $\{X_t\}$ is causal, or a causal function of $\{Z_t\}$, if there exist constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ for all $t$ ... (v)

Causality is equivalent to the condition $\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p \neq 0$ for all $|z| \leq 1$, ...(vi)

Invertibility:

An $ARMA(p,q)$ process $\{X_t\}$ is invertible if there exist constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$ for all $t$ ... (vi)

Invertibility is equivalent to the condition $\theta(z) = 1 + \theta_1 z + ... + \theta_q z^q \neq 0$ for all $|z| \leq 1$, ...(vii).

In simple terms,

Stationarity: $\phi(.)$ roots are not on the unit circle.

Causality: $\phi(.)$ roots are outside the unit circle.

Invertibility: $\theta(.)$ roots are outside the unit circle.

If we have a real number then we just need to check whether it is greater than 1 or less than -1. If we have a complex number then we are going to check the values in between the major axis.

Example from Brockwell and Davis:

Consider the $ARMA(1,1)$ process $\{X_t\}$ satisfying the equations,

$X_t - 0.5X_{t-1} = Z_t + 0.4Z_{t-1}$, $\{Z_t \sim WN(o, \sigma^2)\}$, ...(viii).

Since the autoregressive polynomial $\phi(z) = 1 - 0.5z$ has a zero at $z = 2$, which is located outside the unit circle, we conclude from (iv) and (vi) that there exists a unique ARMA process satisfyinh (viii) that is also causal. The coefficients $\{\psi_j\}$ in the $MA(\infty)$ representation of $X_t$ can be found from the equation below:

$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j$, $j = 0, 1, ...,$ ...(ix)

where $\theta_0 := 1$, $\theta_j := 0$ for $j > q$, and $\psi_j := 0$ for $j < 0$

Using (ix), we get,

$\psi_0 = 1,$

$\psi_1 = 0.4 + 0.5,$

$\psi_2 = 0.5(0.4 + 0.5),$

$\psi_j = 0.5^{j-1}(0.4 + 0.5)$, $j = 1, 2, ....$

The MA polynomial $\theta(z) = 1 + 0.4z$ has a zero at $z = \frac{-1}{.4} = -2.5$, which is also located outside the unit circle which implies that $\{X_t\}$ is invertible with coefficients $\{\pi_j\}$ given by,

$\pi_0 = 1,$

$\pi_1 = -(0.4 + 0.5),$

$\pi_2 = -(0.4 + 0.5)(-0.4),$

$\pi_j = -(0.4 + 0.5)(-0.4)^{j-1}$, $j = 1, 2, ...$

Another Example:

Consider an $AR(1)$ model which is equivalent to $ARMA(1,0)$ and the stochastic process is given by,

$X_t = 0.9X_{t-1} + Z_t.$

The first step is to write the above stochastic process in general format given by,

$\phi(B)X_t = \theta(B)Z_t$. So,

$X_t - 0.9X_{t-1} = Z_t$. Now we want to write $X_t$ times a function of backward shift operators.

$(1 - 0.9B)X_t = Z_t$.

Next step is to plug in $Z$ for the backward shift operators. So, we are going to write out our characteristic functions for $\phi$ and $\theta$,

$\phi(z) = 1 - 0.9z$

$\theta(z) = 1$

Next step is to check for invertibility, stationarity and causality baed on determining the roots of Z.

First, let us check for inveritibility and the rule for it is that $\theta$ roots are outside the unit circle. In our example, we don't have any roots to check because $\theta(z) = 1$ and there is no Z. So, the given stochastic process is invertible.

Next, let us check for stationarity and causality by solving for Z. We need to find the roots for Z using $\phi$ and setting it equal to 0. So,

$\phi(z) = 1 - 0.9z = 0$

$z = \frac{1}{0.9} > 1$ and also $\neq 1$. So since it is greater than 1, we have causality and since it is not equal to 1, we also have stationarity.

Final Example from Davis & Brockwell:

Consider $ARMA(2,1)$ process defined by the equations,

$X_t - 0.75X_{t-1} + 0.5625X_{t-2} = Z_t + 1.25Z_{t-1}$, $\{Z_t\} \sim WN(o, \sigma^2)$.

First get the equations in terms of backward shift operator.

$(1 - 0.75B - 0.5625B^2)X_t = (1 + 1.25B)Z_t$

Next step is to get characteristic polynomial,

$\phi(z) = 1 - 0.75z - 0.5625z^2$

$\theta(z) = 1 + 1.25z$

Now checking the roots,

$1 + 1.25z = 0$

$z = -0.8 > -1$ meaning it is greater than -1 and inside the unit circle implying that it is not invertible.

$1 - 0.75z - 0.5625z^2 = 0$ which is our quadratic equation. Using quadratic formula we get,

$z = \frac{-.75 \pm \sqrt{0.5625 - (4)(1)(0.5625)}}{2(0.5625)}$

$z = \frac{2}{3} \pm \frac{2\sqrt{3}i}{3}$ which suggests that this is a complex number with real part $\frac{2}{3}$ and imaginary part $\frac{2\sqrt{3}i}{3}$ which is not on major axes and is somewhere between so we have to determine the length or radius of Z to determine whether it is greater than the unit circle or within the unit circle. So,

$|z| = \sqrt{x^2 + y^2} = \sqrt{(2/3)^2 + ((2\sqrt{3})/3)^2} = \frac{4}{3} \neq 1$ suggests that we have stationarity and $\frac{4}{3} > 1$ is outside the unit circle suggesting causality.