Homework #0

04654: Intro to Probabilistic Graphical Model
Prof. Assane GUEYE
Due: Monday, Jan 20, 2025 at 11:59 PM CAT

Please remember to show your work for all problems and to write down the names of any students that you collaborate with. The full collaboration and grading policies are available on the course website: https://labayifa.github.io/04654. You are strongly encouraged (but not required) to use Latex to typeset your solutions.

Your solutions should be uploaded to Gradescope (https://www.gradescope.com/) in PDF format by the deadline. We will not accept hardcopies. If you choose to hand-write your solutions, please make sure the uploaded copies are legible. Gradescope will ask you to identify which page(s) contain your solutions to which problems, so make sure you leave enough time to finish this before the deadline. We will give you a 30-minute grace period to upload your solutions in case of technical problems.

1 Probability *[45 points]*

1. (10 points) A fair coin is tossed 10 times. The sample space for each trial is Head, Tail and the trials are independent. What is the probability of having:

   1 Zero Tail?

   2 6 Heads?

   3 At least three Heads?

   4 At least three Heads given the first trail was a Head!

2. (10 points) Assuming the probability that it rains on Monday is 0.45; the probability that it rains on Wednesday is 0.4; and the probability that it rains on Wednesday given that rained on Monday is 0.6. What is the probability that:

   1 It rains on both days?

   2 Rain will come next Monday given that it has just finished raining today (Wednesday)?

3. (10 points) Let X denote the outcome of a random experiment with possible values $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ according to the following probability law:

$$p(X = k) = \begin{cases} ck^2 & \text{if } k \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of c?

   (b) Compute the expectation and the variance of X?

4. (10 points) You just built a new COVID test with the following properties:

   - If a person has COVID, the test is positive with 0.95 probability.

- If a person does not have COVID, the test can still be positive with 0.05 probability.

You are told that a random person has COVID with probability 0.001.
You just use your test on a random person and it turns out to be positive.
What is the probability that the person really has COVID?

5. (5 points) You will write a python program to simulate the Monthy Hall problem. It's a famous problem, and you can read more about it online.

   Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door (but don't open it)s, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

   In this simulation, the doors will be represented by an array of three elements. Each element of the array is either a "0" (for the goats) or a "1" (for the car). A python file is provided with four functions to fill.

2 MLE-MAP (Warmup) *[30 points]*

Suppose that $\mathbf{x} \in \mathbb{R}^d$ is fixed and given. Moreover, assume that $\boldsymbol{\beta} \in \mathbb{R}^d$ is a parameter vector and

$$y = \langle \mathbf{x}, \boldsymbol{\beta} \rangle + \epsilon, \text{ where } \epsilon \sim \mathcal{N}(0, \sigma^2), \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner (i.e., dot) product of two vectors, that is, if $\boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \dots & \boldsymbol{\beta}_d \end{bmatrix}^T$ and $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_d \end{bmatrix}^T$, then $\langle \mathbf{x}, \boldsymbol{\beta} \rangle = \sum_{i=1}^{d} \boldsymbol{x}_i \boldsymbol{\beta}_i$.

Therefore, y is a linear function of x with i.i.d. Gaussian noise.

1. (20 points) **Maximum likelihood estimation:**

   (a) Write down the probability density function (PDF) of the conditional distribution $f_{y|\boldsymbol{\beta}}$. Your answer can be in terms of the fixed $\boldsymbol{x} \in \mathbb{R}^d$.

   (b) Assume that we (independently) draw N pairs $(\mathbf{x}_n, y_n) \in \mathbb{R}^d \times \mathbb{R}$ from the above model, where $\mathbf{x}_n$'s are fixed and then y_n is defined according to (1):

   $$y_n = \langle \mathbf{x}_n, \boldsymbol{\beta} \rangle + \epsilon_n, \text{ where } \epsilon_n \sim \mathcal{N}(0, \sigma^2)\,. \tag{2}$$

   What is the PDF of $(y_1, \dots, y_N|\boldsymbol{\beta})$?

   (c) Write down the associated log-likelihood function for the PDF you found in part (b).

   (d) Find $\boldsymbol{\beta}$ that maximizes the log-likelihood function from part (b).

2. (10 points) **Maximum-a-posteriori estimation:** Assume $\boldsymbol{\beta} \sim \mathcal{N}(0, \lambda^2 \mathbf{I}_d)$ where $\mathbf{I}_d$ is $d \times d$ identity matrix.

   (a) After drawing N pairs as above, from Bayes' rule we know the distribution of $\boldsymbol{\beta}|(y_1, \dots, y_N)$ is

   $$P(\boldsymbol{\beta}|y_1, \dots, y_N) = \frac{P(y_1, \dots, y_N, \boldsymbol{\beta})}{P(y_1, \dots, y_N)}.$$

   Find the distribution $P(y_1, \dots, y_N, \boldsymbol{\beta})$.

   (b) Note that from the Bayes rule, finding the MAP estimator of $\boldsymbol{\beta}$ is equivalent to maximizing the numerator $P(y_1, \dots, y_N, \boldsymbol{\beta})$ with respect to $\boldsymbol{\beta}$ since the denominator does not depend on $\boldsymbol{\beta}$. Use the expression from part (a) above to formulate a minimization problem whose solution will give the MAP estimator $\hat{\boldsymbol{\beta}}$.

3 Graph Theory *[25 points]*

1. (5 points) Degree of any vertex of a graph is

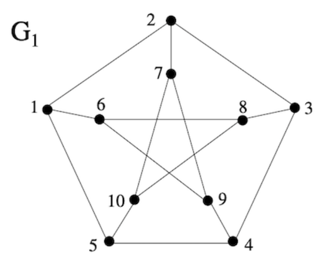
   (a) Number of vertices in a graph

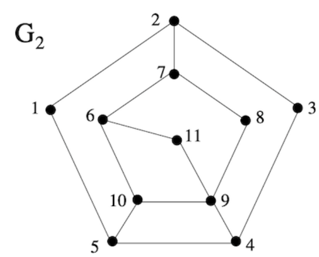
   (b) The number of edges incident with the vertex

   (c) Number of edges in a graph

   (d) Number of vertices adjacent to thatvertices

2. (5 points) In each of the following graphs, find paths of length 9 and 11, and cycles of length 5, 6, 8 and 9, if possible.





3. (5 points) (a) Find the adjacency matrix and the incidence matrix of the graph $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{ab, ac, bc, bd, cd, ce, de\}$.

   (b) Give the adjacency list and a drawing of the graph $G = ([5], E)$ whose adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

4. (5 points) How many of the following statements are correct?

   (a) All cyclic graphs are complete graphs.

   (b) All complete graphs are cyclic graphs.

   (c) All paths are bipartite.

   (d) All cyclic graphs are bipartite.

   (e) There are cyclic graphs which are complete.

5. (5 points) Which of the following statements for a simple graph is correct.

   (a) Every trail is a path

(b) Every path is a trail

(c) path

(d) Path and trail have no relation