1. Consider the system of linear equations below:

$$\begin{aligned} 3x + y &= 6 \\ -2x + 2y + 8z &= -8 \\ 4x + 4y + 8z &= 4 \end{aligned}$$

(a) Express the system in a compact matrix form, Ax = b

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 2 & 8 \\ 4 & 4 & 8 \end{bmatrix} \; x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; b = \begin{bmatrix} 6 \\ -8 \\ 4 \end{bmatrix}$$

To represent the system in a compact matrix form:

$$\begin{bmatrix} 3 & 1 & 0 \\ -2 & 2 & 8 \\ 4 & 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 4 \end{bmatrix}$$

(b) **Use Gaussian elimination to determine if the system has no solution, one unique solution or infinitely many solutions and justify your answer**

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 6 \\ -2 & 2 & 8 & -8 \\ 4 & 4 & 8 & 4 \end{array}\right] R_1 \leftrightarrow R_2 \left[\begin{array}{ccc|c} -2 & 2 & 8 & -8 \\ 3 & 1 & 0 & 6 \\ 4 & 4 & 8 & 4 \end{array}\right] R_1 = R_1 + R_2 \left[\begin{array}{ccc|c} 1 & 3 & 8 & -2 \\ 3 & 1 & 0 & 6 \\ 4 & 4 & 8 & 4 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \left[\begin{array}{ccc|c} 1 & 3 & 8 & -2 \\ 0 & -8 & -24 & 12 \\ 4 & 4 & 8 & 4 \end{array}\right] R_3 = R_3 - 4R_1 \left[\begin{array}{ccc|c} 1 & 3 & 8 & -2 \\ 0 & -8 & -24 & 12 \\ 0 & -8 & -24 & 12 \end{array}\right]$$

$$R_3 = R_3 - R_2 \left[\begin{array}{ccc|c} 1 & 3 & 8 & -2 \\ 0 & -8 & -24 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 = -\frac{1}{8} R_2 \left[\begin{array}{ccc|c} 1 & 3 & 8 & -2 \\ 0 & 1 & 3 & -1.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system has infinitely many solutions since the last row of the matrix is all zeros. The system is consistent.

(c) **If the system has a solution, what is the solution?**

$$\begin{aligned} x + 3y + 8z &= -2 \\ y + 3z &= -1.5 \end{aligned}$$

$$x = -2 - 3y - 8z$$
$$y = -1.5 - 3z$$
$$x = -2 - 3(-1.5 - 3z) - 8z$$
$$x = -2 + 4.5 + 9z - 8z$$
$$x = 2.5 + z$$

The solution to the system is:

$$x = 2.5 + z$$
$$y = -1.5 - 3z$$
$$z = z$$

where Z is a free variable and can take any value.

2. **Determine whether the following systems of equations (or matrix equations) described below have no solution, one unique solution, or infinitely many solutions, and justify your answer.**

(a)

$$ax + by = c$$
$$dx + ey = f$$

where $a, b, c, d, e, f$ are scalars satisfying $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

$$\frac{a}{d} = \frac{b}{c} = \frac{c}{f}$$
$$e = \frac{bd}{a}$$

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ d & \frac{bd}{a} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

in matrix argumented form:

$$\begin{bmatrix} a & b & | & c \\ d & \frac{bd}{a} & | & f \end{bmatrix} R_1 = \frac{1}{a} R_1 \begin{bmatrix} 1 & \frac{b}{a} & | & \frac{c}{a} \\ d & \frac{bd}{a} & | & f \end{bmatrix} R_2 = R_2 - dR_1 \begin{bmatrix} 1 & \frac{b}{a} & | & \frac{c}{a} \\ 0 & 0 & | & f - d\frac{c}{a} \end{bmatrix}$$

The system has no solution if $f - d\frac{c}{a} \neq 0$, or infinitely many solutions if $f - d\frac{c}{a} = 0$

(b) A homogeneous system of 3 equations in 4 unknowns

$$\begin{aligned} ax + by + cz + dw &= 0 \\ ex + fy + gz + hw &= 0 \\ ix + jy + kz + lw &= 0 \end{aligned}$$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has infinitely many solutions since there will be a lot of free variables.

(c) **Ax = b, where the row-reduced echelon form of the augmented matrix [A — b] looks as follows:**

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The system has no solution since the last row of the matrix is all zeros with the last element being 1, this means that the system is inconsistent.

3. The following represents the average life span of smokers according to the average number of cigars they smoke in a day:

| No. of cigars | 5 | 10 | 15 | 20 |
|---|---|---|---|---|
| Life span | 65 | 56 | 48 | 40 |

Let $x$ represent the number of cigars and $y$ the life span.

(a) Find the least squares line for the data set (use two decimal places).[20]

(b) Use this model to predict the average life span of a person who smokes 25 cigars daily (round-off to the nearest year). [5]

4. The following data gives the number of bacteria in a culture (in billions):

| Time (in minutes) | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Population | 7.4 | 8.2 | 30.1 | 50.8 |

Let $x$ be the number of minutes and $y$ the bacteria population:

(a) Construct a scatter plot for the data set. [12]

(b) Does the data follow a linear trend?

(c) Linearize the data set

(d) Find the least squares line for the linearized data set [8]

(e) Predict the bacteria population after 10 minutes. [5]

Note: Answers should be correct to four decimal places for this item.

$$m = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$b = \frac{\sum x^2 \sum y - \sum x \sum xy}{n\sum x^2 - (\sum x)^2}$$