Homework 2 - FALL 2024 MATH 2250

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1. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of the vector space $\mathbb{R}^2$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The action of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ on the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is given by

$$T(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad \text{and} \quad T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}.$$

Find the formula for $T(\mathbf{x})$, where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Solution:

Given $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, we can express $\mathbf{x}$ as a linear combination of the basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2$$

where a and b are scalars. To find a and b, we solve the system:

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This gives us the equations:

$$x = a + b \quad \text{and} \quad y = a - b$$

Solving for a and b:

$$a = \frac{x + y}{2} \quad \text{and} \quad b = \frac{x - y}{2}$$

Now, we can find $T(\mathbf{x})$:

$$T(\mathbf{x}) = T(a\mathbf{v}_1 + b\mathbf{v}_2) = aT(\mathbf{v}_1) + bT(\mathbf{v}_2)$$

Substitute a and b into the equation:

$$T(\mathbf{x}) = \frac{x+y}{2}\begin{bmatrix}2\\4\\6\end{bmatrix} + \frac{x-y}{2}\begin{bmatrix}0\\8\\10\end{bmatrix}$$

Simplify the expression:

$$T(\mathbf{x}) = \begin{bmatrix}(x+y)\\2(x+y)+4(x-y)\\3(x+y)+5(x-y)\end{bmatrix}$$

Therefore, the formula for $T(\mathbf{x})$ is:

$$T(\mathbf{x}) = \begin{bmatrix}x+y\\6x-2y\\8x-2y\end{bmatrix}$$