

Homework 3 - Maths Foundation for Machine Learning

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Question 1: Eigen Values and Eigen Vectors

Consider the matrix $A = \begin{bmatrix} 9 & 1 & -1 \\ -1 & 11 & 1 \\ -2 & 2 & 10 \end{bmatrix}$.

1. Obtain the characteristic polynomial of A . (Hint: find the determinant of $A - \lambda I$)

$$A - \lambda I = \begin{bmatrix} 9 - \lambda & 1 & -1 \\ -1 & 11 - \lambda & 1 \\ -2 & 2 & 10 - \lambda \end{bmatrix}$$

2. From the characteristic polynomial, obtain the 3 eigenvalues of $\lambda_1, \lambda_2, \lambda_3$, such that $\lambda_1 < \lambda_2 < \lambda_3$.

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 9 - \lambda & 1 & -1 \\ -1 & 11 - \lambda & 1 \\ -2 & 2 & 10 - \lambda \end{bmatrix} = 0$$

$$(9 - \lambda)((11 - \lambda)(10 - \lambda) - (2 \times 1)) - 1((-1)(10 - \lambda) - (-2 \times 1)) - 1((-1)(2) - (-2)(11 - \lambda)) = 0$$

$$(9 - \lambda)((11 - \lambda)(10 - \lambda) - 2) - 1((-10 + \lambda) + 2) - 1(-2 + 22 - 2\lambda) = 0$$

$$(9 - \lambda)(110 - 21\lambda + \lambda^2 - 2) - 1(-8 + \lambda) - 1(20 - 2\lambda) = 0$$

$$(9 - \lambda)(108 - 21\lambda + \lambda^2) - 1(-8 + \lambda) - 1(20 - 2\lambda) = 0$$

$$9(108 - 21\lambda + \lambda^2) - \lambda(108 - 21\lambda + \lambda^2) + 8 - \lambda - 20 + 2\lambda = 0$$

$$972 - 189\lambda + 9\lambda^2 - 108\lambda + 21\lambda^2 - \lambda^3 + 8 - \lambda - 20 + 2\lambda = 0$$

$$-\lambda^3 + 30\lambda^2 - 296\lambda + 960 = 0$$

$$\lambda^3 - 30\lambda^2 + 296\lambda - 960 = 0$$

$$(\lambda - 10)(\lambda^2 - 20\lambda + 96) = 0$$

$$(\lambda - 10)(\lambda - 8)(\lambda - 12) = 0$$

$$\lambda_1 = 8, \lambda_2 = 10, \lambda_3 = 12$$

3. Using Gaussian elimination, find the eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$.
for eigen value $\lambda_1 = 8$

$$A - 8I = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & 1 \\ -2 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & 1 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augumented matrix:

$$\begin{aligned} & \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ -1 & 3 & 1 & | & 0 \\ -2 & 2 & 2 & | & 0 \end{bmatrix} \xrightarrow{R_2=R_2+R_1} \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 4 & 0 & | & 0 \\ -2 & 2 & 2 & | & 0 \end{bmatrix} \\ & \xrightarrow{R_3=R_3+2R_1} \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 4 & 0 & | & 0 \\ 0 & 4 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_3=R_3-R_2} \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 4 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \\ & \xrightarrow{R_2=\frac{1}{4}R_2} \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_1=R_1-R_2} \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \end{aligned}$$

$$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$$

$$x_2 = 0$$

Let $x_3 = 1$, then $x_1 = 1$ and $x_2 = 0$. Therefore, the eigenvector corresponding to $\lambda_1 = 8$ is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.
for eigen value $\lambda_2 = 10$

$$A - 10I = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augumented matrix:

$$\begin{bmatrix} -1 & 1 & -1 & | & 0 \\ -1 & 1 & 1 & | & 0 \\ -2 & 2 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_1 = -R_1} \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ -1 & 1 & 1 & | & 0 \\ -2 & 2 & 0 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ -2 & 2 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_3 = R_3 + 2R_1} \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_2 = \frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 + R_2} \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 - x_2 = 0, x_3 = 0 \Rightarrow x_1 = x_2$$

Let $x_2 = 1$, then $x_1 = 1$ and $x_3 = 0$. Therefore, the eigenvector corresponding to $\lambda_2 = 10$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.
for eigen value $\lambda_3 = 12$

$$A - 12I = \begin{bmatrix} -3 & 1 & -1 \\ -1 & -1 & 1 \\ -2 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & -1 \\ -1 & -1 & 1 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & -1 & | & 0 \\ -1 & -1 & 1 & | & 0 \\ -2 & 2 & -2 & | & 0 \end{bmatrix} \xrightarrow{R_1 = -\frac{1}{3}R_1} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & | & 0 \\ -1 & -1 & 1 & | & 0 \\ -2 & 2 & -2 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & | & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} & | & 0 \\ -2 & 2 & -2 & | & 0 \end{bmatrix} \xrightarrow{R_3 = R_3 + 2R_1} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & | & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} & | & 0 \\ 0 & \frac{4}{3} & -\frac{4}{3} & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & | & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_2 = -\frac{3}{4}R_2} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 + \frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 = 0, x_2 = x_3$$

Let $x_2 = 1$, then $x_1 = 0$ and $x_3 = 1$. Therefore, the eigenvector corresponding to $\lambda_3 = 12$ is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

4. (in the notebook)
5. Compare the eigen vectors computed using `np.linalg.eig` and that obtained using Gaussian elimination in (3) above. Are they the same? If they are the same, explain why this is the case. Otherwise, explain why they are different eigen vectors using `np.linalg.eig`:

$$\begin{pmatrix} 9.00258517e-16 & -7.07106781e-01 & 7.07106781e-01 \\ -7.07106781e-01 & -3.36518470e-16 & -7.07106781e-01 \\ -7.07106781e-01 & -7.07106781e-01 & -5.62025848e-15 \end{pmatrix}$$

eigen vectors using Gaussian elimination:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The eigen vectors are different. This is because the eigen vectors obtained using `np.linalg.eig` normalizes the eigen vectors to have a norm of 1. The eigen vectors obtained using Gaussian elimination are not normalized.

6. Now, given a diagonal matrix D , whose diagonal entries are the eigenvalues of A , and a matrix P whose columns are the eigenvectors of A , use the eigen values you obtained in (2) and the eigenvectors you obtained in (3) to show that A as PDP^{-1} .

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 12 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 9 & 1 & -1 \\ -1 & 11 & 1 \\ -2 & 2 & 10 \end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

$$P^{-1} = \begin{pmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \end{pmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$DP^{-1} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 4 \\ 5 & 5 & -5 \\ -6 & 6 & 6 \end{bmatrix}$$

$$P(DP^{-1}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -4 & 4 \\ 5 & 5 & -5 \\ -6 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 1 & -1 \\ -1 & 11 & 1 \\ -2 & 2 & 10 \end{bmatrix}$$

Therefore, $A = PDP^{-1}$.

7. Calculate the determinant of A, $\det(A)$:

$$\begin{aligned} \det(A) &= 9((11 \times 10) - (2 \times 1)) - 1((-1 \times 10) - (-2 \times 1)) - 1((-1 \times 2) - (11 \times -2)) \\ &= 9(110 - 2) - 1(-10 + 2) - 1(-2 + 22) = 9(108) - 1(-8) - 1(20) = 972 + 8 - 20 = \mathbf{960} \end{aligned}$$

8. Calculate the trace of A, $\text{tr}(A)$:

$$\text{tr}(A) = 9 + 11 + 10 = \mathbf{30}$$

9. confirm the following holds:

$$\det(A) = \lambda_1 \lambda_2 \lambda_3$$

$$960 = 8 \times 10 \times 12 = 960$$

$$\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$$

$$30 = 8 + 10 + 12 = 30$$