

Homework 2 - Maths Foundation for Machine Learning

kipngeno koech - bkoech

September 30, 2024

1. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of the vector space \mathbb{R}^2 , where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The action of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ on the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is given by

$$T(\mathbf{v}_1) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \quad \text{and} \quad T(\mathbf{v}_2) = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}.$$

Find the formula for $T(\mathbf{x})$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$$A = \begin{pmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{pmatrix}$$

This is because the transformation is linear and the transformation of a vector is a linear combination of the transformation of the basis vectors. Therefore, the transformation of a vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by:

$$Ax = \begin{pmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 4x + 8y \\ 6x + 10y \end{pmatrix}$$

so the formula for $T(\mathbf{x})$ is:

$$T(\mathbf{x}) = \begin{pmatrix} 2x \\ 4x + 8y \\ 6x + 10y \end{pmatrix}$$

2. For an integer $n \geq 0$, let P_n be the vector space of polynomials of degree at most n . The set $B = \{1, x, x^2, \dots, x^n\}$ is a basis of P_n called the standard basis. Let $T : P_n \rightarrow P_{n+1}$ be the map defined by, for $f \in P_n$,

$$T(f)(x) = xf(x).$$

Prove that T is a linear transformation, and find its range and nullspace.

let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, this is a polynomial of degree at most n

Then the transformation of f to $T(f)$ is:

$$T(f)(x) = x(a_0 + a_1x + a_2x^2 + \dots + a_nx^n) = a_0x + a_1x^2 + a_2x^3 + \dots + a_nx^{n+1}$$

This is a polynomial of degree at most $n + 1$, therefore T is a linear transformation. The range of T is the set of all polynomials of degree at most $n + 1$, and the nullspace of T is the set of all polynomials of degree at most n such that $f(x) = 0$. Therefore, the range of T is P_{n+1} and the nullspace of T is P_n .

3. Let $C[0, 3]$ be the vector space of real functions on the interval $[0, 3]$. Let P_3 denote the set of real polynomials of degree 3 or less. Define the map $T : C[0, 3] \rightarrow P_3$ by

$$T(f)(x) = f(0) + f(1)x + f(2)x^2 + f(3)x^3.$$

Determine if T is a linear transformation.

let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, this is a polynomial of degree at most 3

Then the transformation of f to $T(f)$ is:

$$T(f)(x) = a_0 + a_1x + a_2x^2 + a_3x^3 = a_0 + a_1x + a_2x^2 + a_3x^3$$

This is a polynomial of degree at most 3, therefore T is a linear transformation.

4. Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of the vector space \mathbb{R}^2 where

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Let $C = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a basis of the vector space \mathbb{R}^3 where

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Consider the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ on the basis B given by:

$$T(\mathbf{b}_1) = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} \quad \text{and} \quad T(\mathbf{b}_2) = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix}.$$

(a) Find the transformation matrix A_T of the linear transformation T

$$A_T = \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix}$$

(b) Consider a new basis $\tilde{B} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ of \mathbb{R}^2 and $\tilde{C} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ of \mathbb{R}^3 . Under the basis change, what is the transformation matrix \tilde{A}_T ?

The transformation matrix \tilde{A}_T is given by $\tilde{A}_T = P^{-1}A_TS$

let us get the transformation matrix S from the basis \tilde{B} to the basis B :

$$B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so:

$$S = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$$

we have the transformation matrix A_T as:

$$A_T = \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix}$$

we need to find the transformation matrix P from the basis C to the basis \tilde{C} :

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -\frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(c) What is the image and kernel of T

The image of T is the span of the columns of A_T , and the kernel of T is the nullspace of A_T

$$\text{The image of } T = \text{span} \left\{ \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right\}$$

$$\text{The kernel of } T = \text{nullspace} \left\{ \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix} \right\}$$

$$\text{The kernel of } T = \text{nullspace} \left\{ \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix} \right\} = \text{nullspace} \left\{ \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix} \right\} = \text{nullspace} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{\mathbf{0}\}$$

$$\text{The kernel of } T = \{\mathbf{0}\}$$

$$\text{The image of } T = \text{span} \left\{ \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right\}$$

$$\text{The image of } T = \text{span} \left\{ \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right\}$$

5. A rotation in 3-D space (whose Cartesian coordinates we will call x , y , and z as usual) is characterized by three angles. We will characterize them as a rotation around the x -axis, a rotation around the y -axis, and a rotation around the z -axis.

- (a) Derive the rotation matrix R_1 that transforms a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to a new vector $\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$ by rotating it counterclockwise by angle θ around the x -axis, then an angle δ around the y -axis, and finally an angle ϕ around the z -axis.

$$R_1 = R_z(\phi)R_y(\delta)R_x(\theta)$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R_y(\delta) = \begin{pmatrix} \cos(\delta) & 0 & \sin(\delta) \\ 0 & 1 & 0 \\ -\sin(\delta) & 0 & \cos(\delta) \end{pmatrix}$$

$$R_z(\phi) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\delta) & 0 & \sin(\delta) \\ 0 & 1 & 0 \\ -\sin(\delta) & 0 & \cos(\delta) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R_1 = \begin{pmatrix} \cos(\phi)\cos(\delta) & \cos(\phi)\sin(\delta)\sin(\theta) - \sin(\phi)\cos(\theta) & \cos(\phi)\sin(\delta)\cos(\theta) + \sin(\phi)\sin(\theta) \\ \sin(\phi)\cos(\delta) & \sin(\phi)\sin(\delta)\sin(\theta) + \cos(\phi)\cos(\theta) & \sin(\phi)\sin(\delta)\cos(\theta) - \cos(\phi)\sin(\theta) \\ -\sin(\delta) & \cos(\delta)\sin(\theta) & \cos(\delta)\cos(\theta) \end{pmatrix}$$

- (b) Derive the rotation matrix R_2 that transforms a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to a new vector $\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$ by rotating it counterclockwise by an angle δ around the y -axis, then an angle θ around the x -axis, and finally an angle ϕ around the z -axis.

$$R_2 = R_z(\phi)R_x(\theta)R_y(\delta)$$

$$R_2 = \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\delta) + \cos(\phi)\sin(\theta)\sin(\delta) & \sin(\phi)\sin(\delta) + \cos(\phi)\sin(\theta)\cos(\delta) \\ \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\delta) + \sin(\phi)\sin(\theta)\sin(\delta) & -\cos(\phi)\sin(\delta) + \sin(\phi)\sin(\theta)\cos(\delta) \\ -\sin(\theta) & \cos(\theta)\sin(\delta) & \cos(\theta)\cos(\delta) \end{pmatrix}$$

- (c) Confirm that $R_1R_1^\top = R_2R_2^\top = I$ (Hint: Do not directly multiply the matrices from part 1 above but write down the matrices you need to multiply and multiply them in pairs).

$$R_1R_1^\top = R_z(\phi)R_y(\delta)R_x(\theta)R_x^\top(\theta)R_y^\top(\delta)R_z^\top(\phi)$$

$$R_1R_1^\top = R_z(\phi)R_y(\delta)R_x(\theta)R_x(\theta)R_y(\delta)R_z(\phi)$$

$$R_1R_1^\top = R_z(\phi)R_y(\delta)R_y(\delta)R_z(\phi)$$

$$R_1R_1^\top = R_z(\phi)R_z(\phi)$$

$$R_1R_1^\top = R_z(\phi)R_z(\phi) = I$$

$$R_2R_2^\top = R_z(\phi)R_x(\theta)R_y(\delta)R_y^\top(\delta)R_x^\top(\theta)R_z^\top(\phi)$$

$$R_2R_2^\top = R_z(\phi)R_x(\theta)R_y(\delta)R_y(\delta)R_x(\theta)R_z(\phi)$$

$$R_2R_2^\top = R_z(\phi)R_x(\theta)R_x(\theta)R_z(\phi)$$

$$R_2R_2^\top = R_z(\phi)R_z(\phi) = I$$

$$R_2R_2^\top = I$$

$$R_1R_1^\top = R_2R_2^\top = I$$

$$\text{Therefore, } R_1R_1^\top = R_2R_2^\top = I$$

6. Let

$$A = \begin{pmatrix} a & -1 \\ 1 & 4 \end{pmatrix}$$

be a 2×2 matrix, where a is some real number. Suppose that the matrix A has an eigenvalue 3.

- (a) Determine the value of a .

The eigenvalues of A are the roots of the characteristic polynomial $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} a & -1 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} a - \lambda & -1 \\ 1 & 4 - \lambda \end{pmatrix}\right) = (a - \lambda)(4 - \lambda) - (-1)(1)$$

$$(a - \lambda)(4 - \lambda) - (-1)(1) = \lambda^2 - 4\lambda - a\lambda + 4a + 1$$

$$\lambda^2 - 4\lambda - a\lambda + 4a + 1 = \lambda^2 - (4 + a)\lambda + 4a + 1$$

The eigenvalues of A are the roots of $\lambda^2 - (4 + a)\lambda + 4a + 1 = 0$

Given that the matrix A has an eigenvalue of 3, then $3^2 - (4 + a)3 + 4a + 1 = 0$

$$9 - 3(4 + a) + 4a + 1 = 0$$

$$9 - 12 - 3a + 4a + 1 = 0$$

$$-3a + 4a - 2 = 0 = a = 2$$

Therefore, the value of a is $a = 2$

(b) Does the matrix A have eigenvalues other than 3?

$$\text{our matrix } A \text{ is } A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

The eigenvalues of A are the roots of $\lambda^2 - (4 + 2)\lambda + 4(2) + 1 = 0$

$$\lambda^2 - 6\lambda + 9 = 0$$

we expand the equation $(\lambda - 3)(\lambda - 3) = 0$

Therefore, the matrix A has only one eigenvalue of **3**

7. Suppose G_{k+2} is the average of the two previous numbers G_{k+1} and G_k . That is:

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k.$$

Let:

$$\begin{pmatrix} G_{k+2} \\ G_{k+1} \end{pmatrix} = A \begin{pmatrix} G_{k+1} \\ G_k \end{pmatrix},$$

(a) Find the eigenvalues and eigenvectors of A .

The eigenvalues of A are the roots of the characteristic polynomial $\det(A - \lambda I) = 0$

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

$$\det(A - \lambda I) = \det \left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = \det \left(\begin{pmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ 1 & -\lambda \end{pmatrix} \right) = \left(\frac{1}{2} - \lambda \right) (-\lambda) - \left(\frac{1}{2} \right)$$

$$\left(\frac{1}{2} - \lambda \right) (-\lambda) - \left(\frac{1}{2} \right) = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}$$

$$\text{The eigenvalues of } A \text{ are the roots of } \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$$

multiplying by 2 to clear the fractions:

$$2\lambda^2 - \lambda - 1 = 0$$

we expand the equation $(2\lambda + 1)(\lambda - 1) = 0$

Therefore, the eigenvalues of A are $\lambda = -\frac{1}{2}$ and $\lambda = 1$

The matrix A has two eigenvalues $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = 1$

The eigenvectors of A are the solutions to the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$

For $\lambda = -\frac{1}{2}$, the eigenvector is :

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}v_1 \\ -\frac{1}{2}v_2 \end{pmatrix}$$

$$\frac{1}{2}v_1 + \frac{1}{2}v_2 = -\frac{1}{2}v_1$$

multiplying by 2 to clear the fractions:

$$v_1 + v_2 = -v_1$$

$$v_1 + v_2 + v_1 = 0$$

$$2v_1 + v_2 = 0$$

$$v_2 = -2v_1$$

lets represent V_1 in terms of v_2 :

$$v_1 = -\frac{1}{2}v_2$$

so, the eigenvector for $\lambda = -\frac{1}{2}$ is:

$$\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

For $\lambda = 1$, the eigenvector is :

$$\begin{aligned}
\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\frac{1}{2}v_1 + \frac{1}{2}v_2 &= v_1 \\
v_1 + v_2 &= 2v_1 \\
v_1 + v_2 - 2v_1 &= 0 \\
-v_1 + v_2 &= 0 \\
v_1 &= v_2
\end{aligned}$$

so, the eigenvector for $\lambda = 1$ is:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) Find the limit as $n \rightarrow \infty$ of the matrices A^n .

The matrix A has two eigenvalues $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = 1$

The eigenvectors of A are $\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The matrix A can be diagonalized as $A = PDP^{-1}$

where $P = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

The limit as $n \rightarrow \infty$ of the matrices A^n is $\lim_{n \rightarrow \infty} A^n = P \lim_{n \rightarrow \infty} D^n P^{-1}$

The limit as $n \rightarrow \infty$ of the matrix D^n is $\lim_{n \rightarrow \infty} D^n = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} (-\frac{1}{2})^n & 0 \\ 0 & 1^n \end{pmatrix}$

The limit as $n \rightarrow \infty$ of the matrix D^n is $\lim_{n \rightarrow \infty} D^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

The limit as $n \rightarrow \infty$ of the matrices A^n is $\lim_{n \rightarrow \infty} A^n = P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P^{-1}$

The limit as $n \rightarrow \infty$ of the matrices A^n is $\lim_{n \rightarrow \infty} A^n = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

8. You are given bases in \mathbb{R}^3 . Apply the Gram-Schmidt process on each of them to obtain the orthogonal bases. Transform the orthogonal bases to orthonormal bases.

(a) $\left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$

Let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}{\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{0}{12} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}{\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{8}{12} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - (-)\frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 - \frac{4}{3} + \frac{1}{2} \\ 3 - \frac{4}{3} + 0 \\ 1 - \frac{4}{3} - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{3} \\ -\frac{5}{6} \end{pmatrix}$$

The orthogonal basis is $\left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{3} \\ -\frac{5}{6} \end{pmatrix} \right\}$

To make them orthonormal, we divide each vector by its norm:

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad \text{norm} = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$$

$$\mathbf{u}_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{norm} = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{3} \\ -\frac{5}{6} \end{pmatrix} \quad \text{norm} = \sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2} = \sqrt{\frac{25}{36} + \frac{25}{9} + \frac{25}{36}} = \sqrt{\frac{25}{36} + \frac{100}{36} + \frac{25}{36}} = \sqrt{\frac{150}{36}} = \sqrt{\frac{25}{6}} = \frac{5}{\sqrt{6}}$$

$$\mathbf{u}_3 = \frac{1}{5\sqrt{6}} \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{3} \\ -\frac{5}{6} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$$

The orthonormal basis is $\left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}$

$$\text{Let } \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - (-)\frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 + \frac{1}{2} \\ 1 - \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}}{\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - (-)\frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 5 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 + \frac{1}{2} - \frac{5}{2} \\ 3 - \frac{1}{2} - \frac{5}{2} \\ 1 - 0 \end{pmatrix} = \begin{pmatrix} 2 - 2 \\ 3 - 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The orthogonal basis is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

To make them orthonormal, we divide each vector by its norm:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{norm} = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad \text{norm} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 0^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$$

$$\mathbf{u}_2 = \frac{1}{\frac{\sqrt{2}}{2}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{norm} = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$$

$$\mathbf{u}_3 = \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{The orthonormal basis is } \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$