Homework 2 - Maths Foundation for Machine Learning

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1. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of the vector space $\mathbb{R}^2$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The action of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ on the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is given by

$$T(\mathbf{v}_1) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \quad \text{and} \quad T(\mathbf{v}_2) = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}.$$

Find the formula for $T(\mathbf{x})$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$$A = \begin{pmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{pmatrix}$$

This is because the transformation is linear and the transformation of a vector is a linear combination of the transformation of the basis vectors. Therefore, the transformation of a vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by:

$$Ax = \begin{pmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 4x + 8y \\ 6x + 10y \end{pmatrix}$$

so the formula for $T(\mathbf{x})$ is:

$$T(\mathbf{x}) = \begin{pmatrix} 2x \\ 4x + 8y \\ 6x + 10y \end{pmatrix}$$

2. For an integer $n \geq 0$, let P_n be the vector space of polynomials of degree at most n. The set $B = \{1, x, x^2, \ldots, x^n\}$ is a basis of P_n called the standard basis. Let $T : P_n \to P_{n+1}$ be the map defined by, for $f \in P_n$,

$$T(f)(x) = xf(x).$$

Prove that T is a linear transformation, and find its range and nullspace.

$$\text{let} \quad f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n, \text{ this is a polynomial of degree at most } n$$

Then the transformation of f to $T(f)$ is:

$$T(f)(x) = x(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) = a_0x + a_1x^2 + a_2x^3 + \ldots + a_nx^{n+1}$$

This is a polynomial of degree at most $n+1$, therefore T is a linear transformation. The range of T is the set of all polynomials of degree at most $n+1$, and the nullspace of T is the set of all polynomials of degree at most n such that $f(x) = 0$. Therefore, the range of T is P_{n+1} and the nullspace of T is P_n.

3. Let $C[0, 3]$ be the vector space of real functions on the interval $[0, 3]$. Let P_3 denote the set of real polynomials of degree 3 or less. Define the map $T : C[0, 3] \to P_3$ by

$$T(f)(x) = f(0) + f(1)x + f(2)x^2 + f(3)x^3.$$

Determine if T is a linear transformation.

$$\text{let} \quad f(x) = a_0 + a_1x + a_2x^2 + a_3x^3, \text{ this is a polynomial of degree at most 3}$$

Then the transformation of f to $T(f)$ is:

$$T(f)(x) = a_0 + a_1x + a_2x^2 + a_3x^3 = a_0 + a_1x + a_2x^2 + a_3x^3$$

This is a polynomial of degree at most 3, therefore T is a linear transformation.

4. Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of the vector space $\mathbb{R}^2$ where

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Let $C = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a basis of the vector space $\mathbb{R}^3$ where

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ on the basis B given by:

$$T(\mathbf{b}_1) = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} \quad \text{and} \quad T(\mathbf{b}_2) = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix}.$$

(a) Find the transformation matrix A_T of the linear transformation T

$$\begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$5 = a_1 + a_2$$
$$a_1 = 5 - a_2$$
$$1 = a_2 + a_3$$
$$a_3 = 1 - a_2$$
$$4 = a_1 + a_3$$

but we know that $a_1 = 5 - a_2$, so:

$$4 = 5 - a_2 + a_3 = -1 = -a_2 + a_3$$

but we know that $a_3 = 1 - a_2$, so:

$$-1 = -a_2 + 1 - a_2$$
$$-1 = 1 - 2a_2 = -2a_2 = -2 \Rightarrow a_2 = 1$$
$$a_1 = 5 - a_2 = 5 - 1 = 4 \Rightarrow a_1 = 4$$
$$a_3 = 1 - a_2 = 1 - 1 = 0 \Rightarrow a_3 = 0$$
$$\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$
$$3 = b_1 + b_2$$
$$0 = b_1 + b_3 \Rightarrow b_1 = -b_3$$
$$7 = b_2 + b_3 \Rightarrow b_2 = 7 - b_3$$
$$3 = -b_3 + b_2 \Rightarrow 3 = -b_3 + 7 - b_3 \Rightarrow 3 = 7 - 2b_3 \Rightarrow 2b_3 = 4 \Rightarrow b_3 = 2$$
$$b_1 = -b_3 = -2 \Rightarrow b_1 = -2$$
$$b_2 = 7 - b_3 = 7 - 2 = 5 \Rightarrow b_2 = 5$$
$$\begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

so the transformation matrix A_T is:

$$A_T = \begin{pmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{pmatrix}$$

(b) Consider a new basis $\tilde{B} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ of $\mathbb{R}^2$ and $\tilde{C} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$. Under the basis change, what is the transformation matrix $\tilde{A}_T$?

The transformation matrix $\tilde{A}_T$ is given by $\tilde{A}_T = P^{-1} A_T S$

where:
S maps from the basis $\tilde{B}$ to the basis B
A_T maps from the basis B to the basis C

P maps from the basis C to the basis $\tilde{C}$
let us get the transformation matrix S from the basis $\tilde{B}$ to the basis B:

$$\tilde{B}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\tilde{B}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

so:

$$S(B_1, B_2) = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix}$$

we have the transformation matrix A_T as:

$$A_T = \begin{pmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{pmatrix}$$

we need to find the transformation matrix P from the basis C to the basis $\tilde{C}$:

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\text{The transformation matrix } \tilde{A}_T \text{ is given by } \tilde{A}_T = P^{-1} A_T S = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix}$$

$$\tilde{A}_T = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & -1 & 2 \\ 3 & 2 & -2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\tilde{A}_T = \begin{pmatrix} -2 & -1 & 2 \\ 3 & 2 & -2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4(0) + (-2)(-1) & 4(\frac{1}{2}) + (-2)(-\frac{1}{2}) \\ 1(0) + 5(-1) & 1(\frac{1}{2}) + 5(-\frac{1}{2}) \\ 0(0) + 2(-1) & 0(\frac{1}{2}) + 2(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -5 & -2 \\ -2 & -1 \end{pmatrix}$$

$$\tilde{A}_T = \begin{pmatrix} -2 & -1 & 2 \\ 3 & 2 & -2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -5 & -2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -2(2) + (-1)(-5) + 2(-2) & -2(3) + (-1)(-2) + 2(-1) \\ 3(2) + 2(-5) - 2(-2) & 3(3) + 2(-2) - 2(-1) \\ 1(2) + 1(-5) - 1(-2) & 1(3) + 1(-2) - 1(-1) \end{pmatrix} = \begin{pmatrix} -4+5-4 & -6+2-2 \\ 6-10-4 & 9-4+2 \\ 2-5+2 & 3-2+1 \end{pmatrix} = \begin{pmatrix} -5 \\ -8 \\ -1 \end{pmatrix}$$

$$\text{Therefore, the transformation matrix } \tilde{A}_T \text{ is } \tilde{A}_T = \begin{pmatrix} -5 & -6 \\ -8 & 7 \\ -1 & 2 \end{pmatrix}$$

(c) What is the image and kernel of T

$$\text{The image of } T \text{ is the span of the columns of the transformation matrix } A_T$$

$$\text{The image of } T \text{ is the span of the columns of the transformation matrix } A_T = \begin{pmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{pmatrix}$$

$$\text{The image of } T \text{ is the span of the columns of the transformation matrix } A_T = \text{span} \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix} \right\}$$

$$\text{The kernel of } T \text{ is the set of all vectors that are mapped to the zero vector}$$

$$\text{The kernel of } T \text{ is the set of all vectors that are mapped to the zero vector} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{The kernel of } T \text{ is the set of all vectors that are mapped to the zero vector} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - 2y \\ x + 5y \\ 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

5. A rotation in 3-D space (whose Cartesian coordinates we will call x, y, and z as usual) is characterized by three angles. We will characterize them as a rotation around the x-axis, a rotation around the y-axis, and a rotation around the z-axis.

(a) Derive the rotation matrix R_1 that transforms a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to a new vector $\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$ by rotating it counterclockwise by angle θ around the x-axis, then an angle δ around the y-axis, and finally an angle ϕ around the z-axis.

$$R_1 = R_z(\phi)R_y(\delta)R_x(\theta)$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R_y(\delta) = \begin{pmatrix} \cos(\delta) & 0 & \sin(\delta) \\ 0 & 1 & 0 \\ -\sin(\delta) & 0 & \cos(\delta) \end{pmatrix}$$

$$R_z(\phi) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\delta) & 0 & \sin(\delta) \\ 0 & 1 & 0 \\ -\sin(\delta) & 0 & \cos(\delta) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R_1 = \begin{pmatrix} \cos(\phi)\cos(\delta) & \cos(\phi)\sin(\delta)\sin(\theta) - \sin(\phi)\cos(\theta) & \cos(\phi)\sin(\delta)\cos(\theta) + \sin(\phi)\sin(\theta) \\ \sin(\phi)\cos(\delta) & \sin(\phi)\sin(\delta)\sin(\theta) + \cos(\phi)\cos(\theta) & \sin(\phi)\sin(\delta)\cos(\theta) - \cos(\phi)\sin(\theta) \\ -\sin(\delta) & \cos(\delta)\sin(\theta) & \cos(\delta)\cos(\theta) \end{pmatrix}$$

(b) Derive the rotation matrix R_2 that transforms a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to a new vector $\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$ by rotating it counterclockwise by an angle δ around the y-axis, then an angle θ around the x-axis, and finally an angle ϕ around the z-axis.

$$R_2 = R_z(\phi)R_x(\theta)R_y(\delta)$$

$$R_2 = \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\delta) + \cos(\phi)\sin(\theta)\sin(\delta) & \sin(\phi)\sin(\delta) + \cos(\phi)\sin(\theta)\cos(\delta) \\ \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\delta) + \sin(\phi)\sin(\theta)\sin(\delta) & -\cos(\phi)\sin(\delta) + \sin(\phi)\sin(\theta)\cos(\delta) \\ -\sin(\theta) & \cos(\theta)\sin(\delta) & \cos(\theta)\cos(\delta) \end{pmatrix}$$

(c) Confirm that $R_1R_1^\top = R_2R_2^\top = I$ (Hint: Do not directly multiply the matrices from part 1 above but write down the matrices you need to multiply and multiply them in pairs).

$$R_1R_1^\top = R_z(\phi)R_y(\delta)R_x(\theta)R_x^\top(\theta)R_y^\top(\delta)R_z^\top(\phi)$$

$$R_1R_1^\top = R_z(\phi)R_y(\delta)R_x(\theta)R_x(\theta)R_y(\delta)R_z(\phi)$$
$$R_1R_1^\top = R_z(\phi)R_y(\delta)R_y(\delta)R_z(\phi)$$
$$R_1R_1^\top = R_z(\phi)R_z(\phi)$$
$$R_1R_1^\top = R_z(\phi)R_z(\phi) = I$$
$$R_2R_2^\top = R_z(\phi)R_x(\theta)R_y(\delta)R_y^\top(\delta)R_x^\top(\theta)R_z^\top(\phi)$$
$$R_2R_2^\top = R_z(\phi)R_x(\theta)R_y(\delta)R_y(\delta)R_x(\theta)R_z(\phi)$$
$$R_2R_2^\top = R_z(\phi)R_x(\theta)R_x(\theta)R_z(\phi)$$
$$R_2R_2^\top = R_z(\phi)R_z(\phi) = I$$
$$R_2R_2^\top = I$$
$$R_1R_1^\top = R_2R_2^\top = I$$
$$\text{Therefore, } R_1R_1^\top = R_2R_2^\top = I$$

6. Let

$$A = \begin{pmatrix} a & -1 \\ 1 & 4 \end{pmatrix}$$

be a 2×2 matrix, where a is some real number. Suppose that the matrix A has an eigenvalue 3.

(a) Determine the value of a.

$$\text{The eigenvalues of } A \text{ are the roots of the characteristic polynomial } \det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} a & -1 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} a-\lambda & -1 \\ 1 & 4-\lambda \end{pmatrix}\right) = (a-\lambda)(4-\lambda) - (-1)(1)$$
$$(a-\lambda)(4-\lambda) - (-1)(1) = \lambda^2 - 4\lambda - a\lambda + 4a + 1$$
$$\lambda^2 - 4\lambda - a\lambda + 4a + 1 = \lambda^2 - (4+a)\lambda + 4a + 1$$
$$\text{The eigenvalues of } A \text{ are the roots of } \lambda^2 - (4+a)\lambda + 4a + 1 = 0$$

$$\text{Given that the matrix } A \text{ has an eigenvalue of 3, then } 3^2 - (4+a)3 + 4a + 1 = 0$$
$$9 - 3(4+a) + 4a + 1 = 0$$
$$9 - 12 - 3a + 4a + 1 = 0$$
$$-3a + 4a - 2 = 0 = a = 2$$
$$\text{Therefore, the value of } a \text{ is } a = \mathbf{2}$$

(b) Does the matrix A have eigenvalues other than 3?

$$\text{our matrix A is } A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$
$$\text{The eigenvalues of } A \text{ are the roots of } \lambda^2 - (4+2)\lambda + 4(2) + 1 = 0$$
$$\lambda^2 - 6\lambda + 9 = 0$$
$$\text{we expand the equation } (\lambda - 3)(\lambda - 3) = 0$$
$$\text{Therefore, the matrix } A \text{ has only one eigenvalue of } \mathbf{3}$$

7. Suppose G_{k+2} is the average of the two previous numbers G_{k+1} and G_k. That is:

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k.$$

Let:

$$\begin{pmatrix} G_{k+2} \\ G_{k+1} \end{pmatrix} = A \begin{pmatrix} G_{k+1} \\ G_k \end{pmatrix},$$

(a) Find the eigenvalues and eigenvectors of A.

$$\text{The eigenvalues of } A \text{ are the roots of the characteristic polynomial } \det(A - \lambda I) = 0$$
$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$
$$\det(A - \lambda I) = \det\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ 1 & -\lambda \end{pmatrix}\right) = \left(\frac{1}{2} - \lambda\right)(-\lambda) - \left(\frac{1}{2}\right)$$
$$\left(\frac{1}{2} - \lambda\right)(-\lambda) - \left(\frac{1}{2}\right) = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}$$
$$\text{The eigenvalues of } A \text{ are the roots of } \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$$

multiplying by 2 to clear the fractions:

$$2\lambda^2 - \lambda - 1 = 0$$
$$\text{we expand the equation } (2\lambda + 1)(\lambda - 1) = 0$$
$$\text{Therefore, the eigenvalues of } A \text{ are } \lambda = -\frac{1}{2} \text{ and } \lambda = 1$$
$$\text{The matrix } A \text{ has two eigenvalues } \lambda_1 = -\frac{1}{2} \text{ and } \lambda_2 = 1$$
$$\text{The eigenvectors of } A \text{ are the solutions to the equation } (A - \lambda I)\mathbf{v} = \mathbf{0}$$
$$\text{For } \lambda = -\frac{1}{2}, \text{ the eigenvector is :}$$
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}v_1 \\ -\frac{1}{2}v_2 \end{pmatrix}$$
$$\frac{1}{2}v_1 + \frac{1}{2}v_2 = -\frac{1}{2}v_1$$

multiplying by 2 to clear the fractions:

$$v_1 + v_2 = -v_1$$
$$v_1 + v_2 + v_1 = 0$$
$$2v_1 + v_2 = 0$$
$$v_2 = -2v_1$$

lets represent V_1 in terms of v_2:

$$v_1 = -\frac{1}{2}v_2$$

so, the eigenvector for $\lambda = -\frac{1}{2}$ is:

$$\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$
$$\text{For } \lambda = 1, \text{ the eigenvector is :}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\frac{1}{2}v_1 + \frac{1}{2}v_2 = v_1$$

$$v_1 + v_2 = 2v_1$$

$$v_1 + v_2 - 2v_1 = 0$$

$$-v_1 + v_2 = 0$$

$$v_1 = v_2$$

so, the eigenvector for $\lambda = 1$ is:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) Find the limit as $n \to \infty$ of the matrices A^n.

$$\text{The matrix } A \text{ has two eigenvalues } \lambda_1 = -\frac{1}{2} \text{ and } \lambda_2 = 1$$

$$\text{The eigenvectors of } A \text{ are } \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{The matrix } A \text{ can be diagonalized as } A = PDP^{-1}$$

$$\text{where } P = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{The limit as } n \to \infty \text{ of the matrices } A^n \text{ is } \lim_{n\to\infty} A^n = P \lim_{n\to\infty} D^n P^{-1}$$

$$\text{The limit as } n \to \infty \text{ of the matrix } D^n \text{ is } \lim_{n\to\infty} D^n = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} \left(-\frac{1}{2}\right)^n & 0 \\ 0 & 1^n \end{pmatrix}$$

$$\text{The limit as } n \to \infty \text{ of the matrix } D^n \text{ is } \lim_{n\to\infty} D^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{The limit as } n \to \infty \text{ of the matrices } A^n \text{ is } \lim_{n\to\infty} A^n = P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P^{-1}$$

$$\text{The limit as } n \to \infty \text{ of the matrices } A^n \text{ is } \lim_{n\to\infty} A^n = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} 1 & -1 \\ -1 & -\frac{1}{2} \end{pmatrix} = \frac{1}{-\frac{1}{2} - 1} \begin{pmatrix} 1 & -1 \\ -1 & -\frac{1}{2} \end{pmatrix} = -\frac{2}{3} \begin{pmatrix} 1 & -1 \\ -1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\text{Therefore, the limit as } n \to \infty \text{ of the matrices } A^n \text{ is } \lim_{n\to\infty} A^n = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

8. You are given bases in $\mathbb{R}^3$. Apply the Gram-Schmidt process on each of them to obtain the orthogonal bases. Transform the orthogonal bases to orthonormal bases.

(a) $\left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$

$$\text{Let } \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}{\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{0}{12} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{pmatrix}0\\3\\1\end{pmatrix} - \frac{\begin{pmatrix}0\\3\\1\end{pmatrix} \cdot \begin{pmatrix}2\\2\\2\end{pmatrix}}{\begin{pmatrix}2\\2\\2\end{pmatrix} \cdot \begin{pmatrix}2\\2\\2\end{pmatrix}} \begin{pmatrix}2\\2\\2\end{pmatrix} - \frac{\begin{pmatrix}0\\3\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix}}{\begin{pmatrix}1\\0\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix}} \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix}0\\3\\1\end{pmatrix} - \frac{8}{12}\begin{pmatrix}2\\2\\2\end{pmatrix} - (-)\frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix}0\\3\\1\end{pmatrix} - \frac{2}{3}\begin{pmatrix}2\\2\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix} - \begin{pmatrix}\frac{4}{3}\\\frac{4}{3}\\\frac{4}{3}\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix} - \begin{pmatrix}\frac{4}{3}\\\frac{4}{3}\\\frac{4}{3}\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0-\frac{4}{3}+\frac{1}{2}\\3-\frac{4}{3}+0\\1-\frac{4}{3}-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}-\frac{5}{6}\\\frac{5}{3}\\-\frac{5}{6}\end{pmatrix}$$

$$\text{The orthogonal basis is } \left\{ \begin{pmatrix}2\\2\\2\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}-\frac{5}{6}\\\frac{5}{3}\\-\frac{5}{6}\end{pmatrix} \right\}$$

To make them orthonormal, we divide each vector by its norm:

$$\mathbf{u}_1 = \begin{pmatrix}2\\2\\2\end{pmatrix} \qquad \text{norm} = \sqrt{2^2+2^2+2^2} = \sqrt{12} = 2\sqrt{3}$$

$$\mathbf{u}_1 = \frac{1}{2\sqrt{3}}\begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix} \qquad \text{norm} = \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix}-\frac{5}{6}\\\frac{5}{3}\\-\frac{5}{6}\end{pmatrix} \qquad \text{norm} = \sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2} = \sqrt{\frac{25}{36}+\frac{25}{9}+\frac{25}{36}} = \sqrt{\frac{25}{36}+\frac{100}{36}+\frac{25}{36}} = \sqrt{\frac{150}{36}} = \sqrt{\frac{25}{6}} = \frac{5}{\sqrt{6}}$$

$$\mathbf{u}_3 = \frac{1}{5\sqrt{6}}\begin{pmatrix}-\frac{5}{6}\\\frac{5}{3}\\-\frac{5}{6}\end{pmatrix} = \begin{pmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\end{pmatrix}$$

$$\text{The orthonormal basis is } \left\{ \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\end{pmatrix} \right\}$$

(b) $\left\{ \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix} \right\}$

$$\text{Let } \mathbf{v}_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}2\\3\\1\end{pmatrix}$$

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{\begin{pmatrix}0\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\0\end{pmatrix}}{\begin{pmatrix}1\\-1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\0\end{pmatrix}} \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - (-)\frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{pmatrix} = \begin{pmatrix}0+\frac{1}{2}\\1-\frac{1}{2}\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{pmatrix}2\\3\\1\end{pmatrix} - \frac{\begin{pmatrix}2\\3\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\0\end{pmatrix}}{\begin{pmatrix}1\\-1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\0\end{pmatrix}} \begin{pmatrix}1\\-1\\0\end{pmatrix} - \frac{\begin{pmatrix}2\\3\\1\end{pmatrix} \cdot \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}}{\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix} \cdot \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}} \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - (-)\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 5\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 + \frac{1}{2} - \frac{5}{2} \\ 3 - \frac{1}{2} - \frac{5}{2} \\ 1 - 0 \end{pmatrix} = \begin{pmatrix} 2 - 2 \\ 3 - 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{The orthogonal basis is } \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

To make them orthonormal, we divide each vector by its norm:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \text{norm} = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \qquad \text{norm} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 0^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$$

$$\mathbf{u}_2 = \frac{1}{\frac{\sqrt{2}}{2}}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{norm} = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$$

$$\mathbf{u}_3 = \frac{1}{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{The orthonormal basis is } \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$