Homework 1 - Introduction to Machine Learning for Engineers

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1 Probability

1. Suppose W is a Gaussian random variable with distribution $N(\mu, \sigma^2)$ and U a uniform random variable over the interval $[a, b]$. Assuming that W and U are independent, what is the expected value $\mathbb{E}[Z]$ and variance $\mathrm{Var}[Z]$ of $Z = 3W + 2U$?

   - **Expected value:**
   $$\mathbb{E}[Z] = \mathbb{E}[3W + 2U] = 3\mathbb{E}[W] + 2\mathbb{E}[U]$$
   Since W is Gaussian with mean μ and U is uniform over $[a, b]$ with mean $\frac{a+b}{2}$:
   $$\mathbb{E}[Z] = 3\mu + 2\left(\frac{a+b}{2}\right) = \mathbf{3\mu + (a + b)}$$
   - **Variance:** if a random variable X, is scaled by a constant a, then the variance of the scaled random variable is a^2 times the variance of the original random variable. Therefore:
   $$\mathrm{Var}[Z] = \mathrm{Var}[3W + 2U] = 3^2\mathrm{Var}[W] + 2^2\mathrm{Var}[U]$$
   Since W is Gaussian with variance σ^2 and U is uniform over $[a, b]$ with variance $\frac{(b-a)^2}{12}$:
   $$\mathrm{Var}[Z] = 9\sigma^2 + 4\left(\frac{(b-a)^2}{12}\right) = 9\sigma^2 + \frac{(b-a)^2}{3}$$

2. Consider the following joint distribution between the random variable X, which takes values T or F, and the random variable Y, which takes values a, b, c, or d.

$P(X, Y)$	$Y = a$	$Y = b$	$Y = c$	$Y = d$
$X = T$	0.1	0.2	0.1	0.1
$X = F$	0.1	0.1	0.2	0.1

   (a) **Marginal distribution P_Y:**
   $$\Pr(Y = a) = \Pr(X = T, Y = a) + \Pr(X = F, Y = a) = 0.1 + 0.1 = \mathbf{0.2}$$
   $$\Pr(Y = b) = \Pr(X = T, Y = b) + \Pr(X = F, Y = b) = 0.2 + 0.1 = \mathbf{0.3}$$
   $$\Pr(Y = c) = \Pr(X = T, Y = c) + \Pr(X = F, Y = c) = 0.1 + 0.2 = \mathbf{0.3}$$
   $$\Pr(Y = d) = \Pr(X = T, Y = d) + \Pr(X = F, Y = d) = 0.1 + 0.1 = \mathbf{0.2}$$

   (b) **Conditional probability $\Pr(X = T \mid Y \in \{b, c, d\})$:** since we are conditioning on Y being in the set $\{b, c, d\}$, we need to find the probability of $X = T$ and Y being in the set $\{b, c, d\}$ and divide it by the probability of Y being in the set $\{b, c, d\}$:
   this is the bayes theorem:
   $$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)}$$
   $$\Pr(Y \in \{b, c, d\}) = \Pr(Y = b) + \Pr(Y = c) + \Pr(Y = d) = 0.3 + 0.3 + 0.2 = 0.8$$
   $$\Pr(X = T \cap Y \in \{b, c, d\}) = \Pr(X = T, Y = b) + \Pr(X = T, Y = c) + \Pr(X = T, Y = d) = 0.2 + 0.1 + 0.1 = 0.4$$
   $$\Pr(X = T \mid Y \in \{b, c, d\}) = \frac{\Pr(X = T \cap Y \in \{b, c, d\})}{\Pr(Y \in \{b, c, d\})} = \frac{0.4}{0.8} = \mathbf{0.5}$$