1. Consider the system of linear equations below:

$$\begin{aligned} 3x + y &= 6 \\ -2x + 2y + 8z &= -8 \\ 4x + 4y + 8z &= 4 \end{aligned}$$

(a) Express the system in a compact matrix form, Ax = b

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 2 & 8 \\ 4 & 4 & 8 \end{bmatrix} x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} b = \begin{bmatrix} 6 \\ -8 \\ 4 \end{bmatrix}$$

To represent the system in a compact matrix form:

$$\begin{bmatrix} 3 & 1 & 0 \\ -2 & 2 & 8 \\ 4 & 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 4 \end{bmatrix}$$

(b) **Use Gaussian elimination to determine if the system has no solution, one unique solution or infinitely many solutions and justify your answer**

$$\begin{bmatrix} 3 & 1 & 0 & | & 6 \\ -2 & 2 & 8 & | & -8 \\ 4 & 4 & 8 & | & 4 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} -2 & 2 & 8 & | & -8 \\ 3 & 1 & 0 & | & 6 \\ 4 & 4 & 8 & | & 4 \end{bmatrix} R_1 = R_1 + R_2 \begin{bmatrix} 1 & 3 & 8 & | & -2 \\ 3 & 1 & 0 & | & 6 \\ 4 & 4 & 8 & | & 4 \end{bmatrix}$$

$$R_2 = R_2 - 3R_1 \begin{bmatrix} 1 & 3 & 8 & | & -2 \\ 0 & -8 & -24 & | & 12 \\ 4 & 4 & 8 & | & 4 \end{bmatrix} R_3 = R_3 - 4R_1 \begin{bmatrix} 1 & 3 & 8 & | & -2 \\ 0 & -8 & -24 & | & 12 \\ 0 & -8 & -24 & | & 12 \end{bmatrix}$$

$$R_3 = R_3 - R_2 \begin{bmatrix} 1 & 3 & 8 & | & -2 \\ 0 & -8 & -24 & | & 12 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} R_2 = -\frac{1}{8}R_2 \begin{bmatrix} 1 & 3 & 8 & | & -2 \\ 0 & 1 & 3 & | & -1.5 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This system has infinitely many solutions since the last row of the matrix is all zeros. The system is consistent.

(c) **If the system has a solution, what is the solution?**

$$\begin{aligned} x + 3y + 8z &= -2 \\ y + 3z &= -1.5 \end{aligned}$$

$$
\begin{aligned}
x &= -2 - 3y - 8z \\
y &= -1.5 - 3z
\end{aligned}
$$

$$
\begin{aligned}
x &= -2 - 3(-1.5 - 3z) - 8z \\
x &= -2 + 4.5 + 9z - 8z \\
x &= 2.5 + z
\end{aligned}
$$

The solution to the system is:

$$
\begin{aligned}
x &= 2.5 + z \\
y &= -1.5 - 3z \\
z &= z
\end{aligned}
$$

where Z is a free variable and can take any value.

2. **Determine whether the following systems of equations (or matrix equations) described below have no solution, one unique solution, or infinitely many solutions, and justify your answer.**

(a)

$$
\begin{aligned}
ax + by &= c \\
dx + ey &= f
\end{aligned}
$$

where a, b, c, d, e, f are scalars satisfying $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

$$
\frac{a}{d} = \frac{b}{c} = \frac{c}{f}
$$

$$
e = \frac{bd}{a}
$$

$$
\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}
$$

$$
\begin{bmatrix} a & b \\ d & \frac{bd}{a} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}
$$

in matrix argumented form:

$$
\begin{bmatrix} a & b & | & c \\ d & \frac{bd}{a} & | & f \end{bmatrix} R_1 = \frac{1}{a} R_1 \begin{bmatrix} 1 & \frac{b}{a} & | & \frac{c}{a} \\ d & \frac{bd}{a} & | & f \end{bmatrix} R_2 = R_2 - dR_1 \begin{bmatrix} 1 & \frac{b}{a} & | & \frac{c}{a} \\ 0 & 0 & | & f - d\frac{c}{a} \end{bmatrix}
$$

The system has no solution if $f - d\frac{c}{a} \neq 0$, or infinitely many solutions if $f - d\frac{c}{a} = 0$

(b) A homogeneous system of 3 equations in 4 unknowns

$$\begin{aligned} ax + by + cz + dw &= 0 \\ ex + fy + gz + hw &= 0 \\ ix + jy + kz + lw &= 0 \end{aligned}$$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has infinitely many solutions since there will be a lot of free variables.

(c) **Ax = b, where the row-reduced echelon form of the augmented matrix [A — b] looks as follows:**

$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

The system has no solution since the last row of the matrix is all zeros with the last element being 1, this means that the system is inconsistent.

3. Given the matrix A:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

(a) Compute $A^\top A$ and show that it is symmetric

$$A^\top A = \begin{bmatrix} 0 & 1 & -2 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

To show that it is symmetric: $A^\top A = (A^\top A)^\top$

$$\begin{bmatrix} 6 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

(b) Compute $(A^\top A)^{-1}$ using Gaussian Elimination

$$\begin{bmatrix} 6 & -1 & -1 & | & 1 & 0 & 0 \\ -1 & 4 & 0 & | & 0 & 1 & 0 \\ -1 & 0 & 4 & | & 0 & 0 & 1 \end{bmatrix} R_1 = \frac{1}{6}R_1 \begin{bmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} & | & \frac{1}{6} & 0 & 0 \\ -1 & 4 & 0 & | & 0 & 1 & 0 \\ -1 & 0 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 = R_2 + R_1 \begin{bmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} & | & \frac{1}{6} & 0 & 0 \\ 0 & \frac{23}{6} & -\frac{1}{6} & | & \frac{1}{6} & 1 & 0 \\ -1 & 0 & 4 & | & 0 & 0 & 1 \end{bmatrix} R_3 = R_3 + R_1 \begin{bmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} & | & \frac{1}{6} & 0 & 0 \\ 0 & \frac{23}{6} & -\frac{1}{6} & | & \frac{1}{6} & 1 & 0 \\ 0 & -\frac{1}{6} & \frac{17}{6} & | & \frac{1}{6} & 0 & 1 \end{bmatrix}$$

$$R_2 = \frac{6}{23}R_2 \begin{bmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} & | & \frac{1}{6} & 0 & 0 \\ 0 & 1 & -\frac{1}{23} & | & \frac{1}{23} & \frac{6}{23} & 0 \\ 0 & -\frac{1}{6} & \frac{17}{6} & | & \frac{1}{6} & 0 & 1 \end{bmatrix} R_1 = R_1 + \frac{1}{6}R_2 \begin{bmatrix} 1 & 0 & -\frac{1}{23} & | & \frac{1}{23} & \frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{23} & | & \frac{1}{23} & \frac{6}{23} & 0 \\ 0 & -\frac{1}{6} & \frac{17}{6} & | & \frac{1}{6} & 0 & 1 \end{bmatrix}$$

$$R_3 = R_3 + \frac{1}{6}R_2 \begin{bmatrix} 1 & 0 & -\frac{1}{23} & | & \frac{1}{23} & \frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{23} & | & \frac{1}{23} & \frac{6}{23} & 0 \\ 0 & 0 & \frac{17}{23} & | & \frac{1}{23} & \frac{1}{6} & 1 \end{bmatrix} R_3 = \frac{23}{17}R_3 \begin{bmatrix} 1 & 0 & -\frac{1}{23} & | & \frac{1}{23} & \frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{23} & | & \frac{1}{23} & \frac{6}{23} & 0 \\ 0 & 0 & 1 & | & \frac{1}{17} & \frac{3}{17} & \frac{23}{17} \end{bmatrix}$$

$$R_1 = R_1 + \frac{1}{23}R_3 \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{17} & \frac{5}{17} & \frac{23}{17} \\ 0 & 1 & -\frac{1}{23} & | & \frac{1}{23} & \frac{6}{23} & 0 \\ 0 & 0 & 1 & | & \frac{1}{17} & \frac{3}{17} & \frac{23}{17} \end{bmatrix} R_2 = R_2 + \frac{1}{23}R_3 \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{17} & \frac{5}{17} & \frac{23}{17} \\ 0 & 1 & 0 & | & \frac{2}{17} & \frac{9}{17} & \frac{23}{17} \\ 0 & 0 & 1 & | & \frac{1}{17} & \frac{3}{17} & \frac{23}{17} \end{bmatrix}$$

$$(A^\top A)^{-1} = \begin{bmatrix} \frac{1}{17} & \frac{5}{17} & \frac{23}{17} \\ \frac{2}{17} & \frac{9}{17} & \frac{23}{17} \\ \frac{1}{17} & \frac{3}{17} & \frac{23}{17} \end{bmatrix}$$

(c) Let $b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. If $Ax = b$, show that $x = (A^\top A)^{-1}A^\top b$ and obtain the value of x. [Note: $x = (A^\top A)^{-1}A^\top b$ is called the Normal equation]

$$Ax = b, where, A = A^\top A = \begin{bmatrix} 6 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix} b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Ax = b, (A^\top A)x = b$$

$$(A^\top A)^{-1}(A^\top A)x = (A^\top A)^{-1}b$$
$$x = (A^\top A)^{-1}b$$
$$x = \begin{bmatrix} \frac{1}{17} & \frac{5}{17} & \frac{23}{17} \\ \frac{2}{17} & \frac{9}{17} & \frac{23}{17} \\ \frac{1}{17} & \frac{3}{17} & \frac{23}{17} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$x = \begin{bmatrix} 24/17 \\ 25/17 \\ 24/17 \end{bmatrix}$$

4. CMU-Africa is trying to understand the pricing strategy used by Delight Canteen. The Canteen sells various types of food, each with a different price, but only charges a total price for all food types on a student's plate. As a machine learning engineer, you want to help students determine the price of their food based on the quantity (measured in Grams) of each foot type they add to their plate. In other to achieve this, you collect data from 6 of your friends on the quantity of each food type they served and the total price. The observations from your friends are recorded below.

Transaction	Food a (g)	Food b (g)	Food c (g)	Food d (g)	Total Cost (RWF)
1	100	50	150	200	2500
2	50	50	100	300	2300
3	100	150	200	100	3000
4	50	200	300	50	2900
5	200	50	250	50	3100
6	300	50	50	200	4300

From the observations in the table above, you are expected to obtain the price per quantity of each food type (mearsured in RWF/g).

(a) Describe the above using a system of linear equations

$$\begin{aligned} 100a + 50b + 150c + 200d &= 2500 \\ 50a + 50b + 100c + 300d &= 2300 \\ 100a + 150b + 200c + 100d &= 3000 \\ 50a + 200b + 300c + 50d &= 2900 \\ 200a + 50b + 250c + 50d &= 3100 \\ 300a + 50b + 50c + 200d &= 4300 \end{aligned}$$

(b) Write the system of linear equations in a compact matrix form, Ax = b

$$A = \begin{bmatrix} 100 & 50 & 150 & 200 \\ 50 & 50 & 100 & 300 \\ 100 & 150 & 200 & 100 \\ 50 & 200 & 300 & 50 \\ 200 & 50 & 250 & 50 \\ 300 & 50 & 50 & 200 \end{bmatrix} \quad x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad b = \begin{bmatrix} 2500 \\ 2300 \\ 3000 \\ 2900 \\ 3100 \\ 4300 \end{bmatrix}$$

$$Ax = b = \begin{bmatrix} 100 & 50 & 150 & 200 \\ 50 & 50 & 100 & 300 \\ 100 & 150 & 200 & 100 \\ 50 & 200 & 300 & 50 \\ 200 & 50 & 250 & 50 \\ 300 & 50 & 50 & 200 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2500 \\ 2300 \\ 3000 \\ 2900 \\ 3100 \\ 4300 \end{bmatrix}$$

(c) Use Gaussian elimination to obtain a solution for the system.

$$\begin{bmatrix} 100 & 50 & 150 & 200 & | & 2500 \\ 50 & 50 & 100 & 300 & | & 2300 \\ 100 & 150 & 200 & 100 & | & 3000 \\ 50 & 200 & 300 & 50 & | & 2900 \\ 200 & 50 & 250 & 50 & | & 3100 \\ 300 & 50 & 50 & 200 & | & 4300 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 4 & | & 50 \\ 1 & 1 & 2 & 6 & | & 46 \\ 2 & 3 & 4 & 2 & | & 60 \\ 1 & 4 & 6 & 1 & | & 58 \\ 4 & 1 & 5 & 1 & | & 62 \\ 6 & 1 & 1 & 4 & | & 86 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 2 & 1 & 3 & 4 & | & 50 \\ 2 & 3 & 4 & 2 & | & 60 \\ 1 & 4 & 6 & 1 & | & 58 \\ 4 & 1 & 5 & 1 & | & 62 \\ 6 & 1 & 1 & 4 & | & 86 \end{bmatrix} R_2 = R_2 - 2R_1 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & -1 & -1 & -8 & | & -42 \\ 2 & 3 & 4 & 2 & | & 60 \\ 1 & 4 & 6 & 1 & | & 58 \\ 4 & 1 & 5 & 1 & | & 62 \\ 6 & 1 & 1 & 4 & | & 86 \end{bmatrix} R_3 = R_3 - 2R_1 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & -1 & -1 & -8 & | & -42 \\ 0 & 1 & 0 & -10 & | & -32 \\ 1 & 4 & 6 & 1 & | & 58 \\ 4 & 1 & 5 & 1 & | & 62 \\ 6 & 1 & 1 & 4 & | & 86 \end{bmatrix}$$

$$R_4 = R_4 - R_1 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & -1 & -1 & -8 & | & -42 \\ 0 & 1 & 0 & -10 & | & -32 \\ 0 & 3 & 4 & -5 & | & 12 \\ 4 & 1 & 5 & 1 & | & 62 \\ 6 & 1 & 1 & 4 & | & 86 \end{bmatrix} R_5 = R_5 - 4R_1 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & -1 & -1 & -8 & | & -42 \\ 0 & 1 & 0 & -10 & | & -32 \\ 0 & 3 & 4 & -5 & | & 12 \\ 0 & -3 & -3 & -23 & | & -122 \\ 6 & 1 & 1 & 4 & | & 86 \end{bmatrix}$$

$$R_6 = R_6 - 6R_1 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & -1 & -1 & -8 & | & -42 \\ 0 & 1 & 0 & -10 & | & -32 \\ 0 & 3 & 4 & -5 & | & 12 \\ 0 & -3 & -3 & -23 & | & -122 \\ 0 & -5 & -11 & -32 & | & -190 \end{bmatrix} R_2 = -R_2 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & 1 & 1 & 8 & | & 42 \\ 0 & 1 & 0 & -10 & | & -32 \\ 0 & 3 & 4 & -5 & | & 12 \\ 0 & -3 & -3 & -23 & | & -122 \\ 0 & -5 & -11 & -32 & | & -190 \end{bmatrix}$$

$$R_3 = R_3 - R_2 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & 1 & 1 & 8 & | & 42 \\ 0 & 0 & -1 & -18 & | & -74 \\ 0 & 3 & 4 & -5 & | & 12 \\ 0 & -3 & -3 & -23 & | & -122 \\ 0 & -5 & -11 & -32 & | & -190 \end{bmatrix} R_4 = R_4 - 3R_2 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & 1 & 1 & 8 & | & 42 \\ 0 & 0 & -1 & -18 & | & -74 \\ 0 & 0 & 1 & -29 & | & -114 \\ 0 & -3 & -3 & -23 & | & -122 \\ 0 & -5 & -11 & -32 & | & -190 \end{bmatrix}$$

$$R_5 = R_5 + 3R_2 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & 1 & 1 & 8 & | & 42 \\ 0 & 0 & -1 & -18 & | & -74 \\ 0 & 0 & 1 & -29 & | & -114 \\ 0 & 0 & 0 & 1 & | & 4 \\ 0 & -5 & -11 & -32 & | & -190 \end{bmatrix} R_6 = R_6 + 5R_2 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & 1 & 1 & 8 & | & 42 \\ 0 & 0 & -1 & -18 & | & -74 \\ 0 & 0 & 1 & -29 & | & -114 \\ 0 & 0 & 0 & 1 & | & 4 \\ 0 & 0 & -6 & 8 & | & 20 \end{bmatrix}$$

$$R_3 = -R_3 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & 1 & 1 & 8 & | & 42 \\ 0 & 0 & 1 & 18 & | & 74 \\ 0 & 0 & 1 & -29 & | & -114 \\ 0 & 0 & 0 & 1 & | & 4 \\ 0 & 0 & -6 & 8 & | & 20 \end{bmatrix} R_2 = R_2 - R_3 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & 1 & 0 & -10 & | & -32 \\ 0 & 0 & 1 & 18 & | & 74 \\ 0 & 0 & 1 & -29 & | & -114 \\ 0 & 0 & 0 & 1 & | & 4 \\ 0 & 0 & -6 & 8 & | & 20 \end{bmatrix}$$

$$R_2 = R_2 + 10R_4 \begin{bmatrix} 1 & 1 & 2 & 6 & | & 46 \\ 0 & 1 & 0 & 0 & | & 8 \\ 0 & 0 & 1 & 18 & | & 74 \\ 0 & 0 & 1 & -29 & | & -114 \\ 0 & 0 & 0 & 1 & | & 4 \\ 0 & 0 & -6 & 8 & | & 20 \end{bmatrix}$$

$$R_2 = 8, R_4 = 4$$

$$-6R_3 + -8R_4 = 20$$

$$-6R_3 + -8(4) = 20$$

$$-6R_3 + 32 = 20$$

$$-6R_3 = -12$$
$$R_3 = \mathbf{2}$$

$$R_1 + R_2 + 2R_3 + 6R_4 = 46$$
$$R_1 + 8 + 2(2) + 6(4) = 46$$
$$R_1 + 8 + 4 + 24 = 46$$
$$R_1 = 10$$
$$\text{food a } = \mathbf{10}, \text{ food b } = \mathbf{8}, \text{ food c } = \mathbf{2}, \text{ food d } = \mathbf{4}$$

(d) What is the rank of the matrix A of the compact matrix form

$$A = \begin{bmatrix} 100 & 50 & 150 & 200 \\ 50 & 50 & 100 & 300 \\ 100 & 150 & 200 & 100 \\ 50 & 200 & 300 & 50 \\ 200 & 50 & 250 & 50 \\ 300 & 50 & 50 & 200 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 1 & 2 & 6 \\ 2 & 3 & 4 & 2 \\ 1 & 4 & 6 & 1 \\ 4 & 1 & 5 & 1 \\ 6 & 1 & 1 & 4 \end{bmatrix}$$

Rank of A is **4**

(e) What is rank of the augmented matrix [A—b]

$$\begin{bmatrix} 100 & 50 & 150 & 200 & | & 2500 \\ 50 & 50 & 100 & 300 & | & 2300 \\ 100 & 150 & 200 & 100 & | & 3000 \\ 50 & 200 & 300 & 50 & | & 2900 \\ 200 & 50 & 250 & 50 & | & 3100 \\ 300 & 50 & 50 & 200 & | & 4300 \end{bmatrix}$$

Rank of the argumented matrix is **4**

(f)

5. Which of the following sets are subspaces of $\mathbb{R}^3$?
conditions to check if a set is a subspace of $\mathbb{R}^3$:

1. The zero vector is in the set
2. The set is closed under addition
3. The set is closed under scalar multiplication

(a) $U_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 \leq 0 \right\}$

i.

$$\text{zero vector} = \text{let x be } 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} is \in U_1 \text{ since } 0 \leq 0 \text{ and } X_1 \leq 0$$

ii. closed under addition condition:

x and y are two vectors in vector space U_1

let vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and vector $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be in vector space U_1

let $x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$ be in U_1

If $x_1 \leq 0$ and $y_1 \leq 0,$ then $x_1 + y_1 \leq 0$ because $0 + 0 = 0$

iii. closed under scalar multiplication condition:

let vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be in vector space U_1

let c be a scalar

let $cx = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$ be in U_1

If $c < 0$ and $x_1 < 0,$ then $cx_1 > 0,$ which violates the closed under scalar condition for U_1.

U_1 is not a subspace of $\mathbb{R}^3$

(b) $U_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + 2x_2 + x_3 = 0 \right\}$

i. the zero vector is in the set:

let $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ be in vector space U_2

since $0 + 2(0) + 0 = 0$ then x is in U_2

ii. closed under addition condition:

x and y are two vectors in vector space U_2

let vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and vector $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be in vector space U_2

$\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$ is in U_2 if $(x_1 + y_1) + 2(x_2 + y_2) + (x_3 + y_3) = 0$.

Since $x_1 + 2x_2 + x_3 = 0$ and $y_1 + 2y_2 + y_3 = 0,$ then $(x_1 + y_1) + 2(x_2 + y_2) + (x_3 + y_3) = 0$ because $0 + 0 = 0$

iii. closed under scalar multiplication condition:

let vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be in vector space U_2

let c be a scalar

let $cx = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$ be in U_2

If $x_1 + 2x_2 + x_3 = 0$ then $c(x_1 + 2x_2 + x_3) = 0$ because $c(0) = 0$

therefore U_2 is a subspace of $\mathbb{R}^3$

(c)

(d) $U_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_3 = 1, x_2 = 2x_1 \right\}$

i. the zero vector is in the set if:

let $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ be in vector space U_3

since $0 = 1$ and $0 = 2(0)$ then x is not in U_3

therefore U_3 is not a subspace of $\mathbb{R}^3$

(e) $U_4 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_3 = 0 \right\}$

i. the zero vector is in the set if:

$$\text{let } x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ be in vector space } U_4$$

since $0 = 0$ and $X_3 = 0$ then x is in U_4

ii. closed under addition condition:

x and y are two vectors in vector space U_4

let vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and vector $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be in vector space U_4

$\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$ is in U_4 if $x_3 + y_3 = 0$.

Since $x_3 = 0$ and $y_3 = 0$, then $x_3 + y_3 = 0$

iii. closed under scalar multiplication condition:

let vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be in vector space U_4

let c be a scalar

let $cx = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$ be in U_4

If $x_3 = 0$ then $c(x_3) = 0$

therefore U_4 is a subspace of $\mathbb{R}^3$

6. consider the following vectors

$$u = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} \in \mathbb{R}^3$$

and $\alpha, \beta \in \mathbb{R}$

(a) Express w as a linear combination of u and v, of the form,

$$w = \alpha u + \beta v$$

systems of linear equations

$$3\alpha + 5\beta = 3$$
$$2\alpha + 3\beta = 3$$
$$3\alpha + 4\beta = 6$$

$$\begin{bmatrix} 3 & 5 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$$

the augumented matrix:

$$\begin{bmatrix} 3 & 5 & | & 3 \\ 2 & 3 & | & 3 \\ 3 & 4 & | & 6 \end{bmatrix}$$

$$R_1 = R_1 - R_2 \begin{bmatrix} 1 & 2 & | & 0 \\ 2 & 3 & | & 3 \\ 3 & 4 & | & 6 \end{bmatrix} R_2 = R_2 - 2R_1 \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & -1 & | & 3 \\ 3 & 4 & | & 6 \end{bmatrix} R_3 = R_3 - 3R_1 \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & -1 & | & 3 \\ 0 & -2 & | & 6 \end{bmatrix}$$

$$R_3 = R_3 - 2R_2 \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & -1 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix} R_2 = -R_2 \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 1 & | & -3 \\ 0 & 0 & | & 0 \end{bmatrix} R_1 = R_1 - 2R_2 \begin{bmatrix} 1 & 0 & | & 6 \\ 0 & 1 & | & -3 \\ 0 & 0 & | & 0 \end{bmatrix}$$

so, the solution is $\alpha = 6$ and $\beta = -3$

(b) Are u, v, and w linearly independent? What is the rank of the matrix whose columns are u, v, and w ?

i. u,v and w are not linearly independent because w can be expressed as a linear combination of u and v.

ii. the rank of the matrix whose columns are u, v, and w is 2.

(c) Let $x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \in \mathbb{R}^3$. Show that the set of vectors $\{u, v, x\}$ are linearly independent. What is the rank of the matrix whose columns are u, v, and x?

for linear independent vectors, the only solution to the equation $\alpha u + \beta v + \gamma x = 0$

$$u = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}, \quad x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$3\alpha + 5\beta + 3\gamma = 0$$
$$2\alpha + 3\beta + \gamma = 0$$
$$3\alpha + 4\beta + 2\gamma = 0$$

$$\begin{bmatrix} 3 & 5 & 3 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the augumented matrix:

$$\begin{bmatrix} 3 & 5 & 3 & | & 0 \\ 2 & 3 & 1 & | & 0 \\ 3 & 4 & 2 & | & 0 \end{bmatrix}$$

$$R_1 = R_1 - R_2 \begin{bmatrix} 1 & 2 & 2 & | & 0 \\ 2 & 3 & 1 & | & 0 \\ 3 & 4 & 2 & | & 0 \end{bmatrix} R_2 = R_2 - 2R_1 \begin{bmatrix} 1 & 2 & 2 & | & 0 \\ 0 & -1 & -3 & | & 0 \\ 3 & 4 & 2 & | & 0 \end{bmatrix} R_3 = R_3 - 3R_1 \begin{bmatrix} 1 & 2 & 2 & | & 0 \\ 0 & -1 & -3 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix}$$

$$R_3 = R_3 - 2R_2 \begin{bmatrix} 1 & 2 & 2 & | & 0 \\ 0 & -1 & -3 & | & 0 \\ 0 & 0 & -1 & | & 0 \end{bmatrix} R_3 = -R_3 \begin{bmatrix} 1 & 2 & 2 & | & 0 \\ 0 & -1 & -3 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} R_2 = -R_2 \begin{bmatrix} 1 & 2 & 2 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$R_1 = R_1 - 2R_3 \begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} R_2 = R_2 - 3R_3 \begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} R_1 = R_1 - 2R_2 \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

As per the reduced row echelon form, the vectors u, v, x are linearly independent. The rank of the matrix whose columns are u, v, x is 3.

(d) Find a, b, and $c \in \mathbb{R}$ such that:

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

$$u = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}, \quad x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$3a + 5b + 3c = 5$$
$$2a + 3b + c = 2$$
$$3a + 4b + 2c = 1$$

$$\begin{bmatrix} 3 & 5 & 3 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

the augumented matrix:

$$\begin{bmatrix} 3 & 5 & 3 & | & 5 \\ 2 & 3 & 1 & | & 2 \\ 3 & 4 & 2 & | & 1 \end{bmatrix}$$

$$R_1 = R_1 - R_2 \begin{bmatrix} 1 & 2 & 2 & | & 3 \\ 2 & 3 & 1 & | & 2 \\ 3 & 4 & 2 & | & 1 \end{bmatrix} R_2 = R_2 - 2R_1 \begin{bmatrix} 1 & 2 & 2 & | & 3 \\ 0 & -1 & -3 & | & -4 \\ 3 & 4 & 2 & | & 1 \end{bmatrix}$$

$$R_3 = R_3 - 3R_1 \begin{bmatrix} 1 & 2 & 2 & | & 3 \\ 0 & -1 & -3 & | & -4 \\ 0 & -2 & -4 & | & -8 \end{bmatrix} R_3 = R_3 - 2R_2 \begin{bmatrix} 1 & 2 & 2 & | & 3 \\ 0 & -1 & -3 & | & -4 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

$$R_3 = \frac{1}{2}R_3 \begin{bmatrix} 1 & 2 & 2 & | & 3 \\ 0 & -1 & -3 & | & -4 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} R_2 = -R_2 \begin{bmatrix} 1 & 2 & 2 & | & 3 \\ 0 & 1 & 3 & | & 4 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$R_1 = R_1 - 2R_3 \begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 1 & 3 & | & 4 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} R_2 = R_2 - 3R_3 \begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} R_1 = R_1 - 2R_2 \begin{bmatrix} 1 & 0 & 0 & | & -5 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$a =$ **-5**, $b =$ **4** and $c =$ **0**

(e) Let $\mathbf{y} = \begin{bmatrix} 4 \\ 2 \\ k \end{bmatrix} \in \mathbb{R}^3$. Find k such that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{y}$ are linearly dependent.

$$3a + 5b = 4$$
$$2a + 3b = 2$$
$$3a + 4b = k$$

The augumented matrix:

$$\begin{bmatrix} 3 & 5 & | & 4 \\ 2 & 3 & | & 2 \\ 3 & 4 & | & k \end{bmatrix}$$

$$R_1 = R_1 - R_2 = \begin{bmatrix} 1 & 2 & | & 2 \\ 2 & 3 & | & 2 \\ 3 & 4 & | & k \end{bmatrix} R_2 = R_2 - 2R_1 \begin{bmatrix} 1 & 2 & | & 2 \\ 0 & -1 & | & -2 \\ 3 & 4 & | & k \end{bmatrix}$$

$$R_3 = R_3 - 3R_1 \begin{bmatrix} 1 & 2 & | & 2 \\ 0 & -1 & | & -2 \\ 0 & -2 & | & k-6 \end{bmatrix} R_3 = R_3 - 2R_2 \begin{bmatrix} 1 & 2 & | & 2 \\ 0 & -1 & | & -2 \\ 0 & 0 & | & k-2 \end{bmatrix}$$

$$R_1 = R_1 - 2R_2 = \begin{bmatrix} 1 & 0 & | & 6 \\ 0 & -1 & | & -2 \\ 0 & 0 & | & k-2 \end{bmatrix} R_2 = -R_2 \begin{bmatrix} 1 & 0 & | & 6 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & k-2 \end{bmatrix}$$

a = 6, b = 2, k = 2 K = 2

(f) Is the set $\{\mathbf{u}, \mathbf{v}, \mathbf{x}\}$ a basis of $\mathbb{R}^3$? Explain why or why not.
yes there are linearly independent and the rank of the matrix whose columns are u, v, x is 3. Therefore, the set $\{u, v, x\}$ is a basis of $\mathbb{R}^3$.

(g) If $B = \{\mathbf{u}, \mathbf{v}, \mathbf{x}\}$ is a basis of $\mathbb{R}^3$, express $\begin{bmatrix} -1 \\ -6 \\ 6 \end{bmatrix}$ as a linear combination of B.

$$u = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}, \quad x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$x \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 6 \end{bmatrix}$$

$$3x + 5y + 3z = -1$$
$$2x + 3y + z = -6$$
$$3x + 4y + 2z = 6$$

$$\begin{bmatrix} 3 & 5 & 3 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 6 \end{bmatrix}$$

the augumented matrix:

$$\begin{bmatrix} 3 & 5 & 3 & | & -1 \\ 2 & 3 & 1 & | & -6 \\ 3 & 4 & 2 & | & 6 \end{bmatrix}$$

$$R_1 = R_1 - R_2 \begin{bmatrix} 1 & 2 & 2 & | & 5 \\ 2 & 3 & 1 & | & -6 \\ 3 & 4 & 2 & | & 6 \end{bmatrix} R_2 = R_2 - 2R_1 \begin{bmatrix} 1 & 2 & 2 & | & 5 \\ 0 & -1 & -3 & | & -16 \\ 3 & 4 & 2 & | & 6 \end{bmatrix}$$

$$R_3 = R_3 - 3R_1 \begin{bmatrix} 1 & 2 & 2 & | & 5 \\ 0 & -1 & -3 & | & -16 \\ 0 & -2 & -4 & | & -9 \end{bmatrix} R_3 = R_3 - 2R_2 \begin{bmatrix} 1 & 2 & 2 & | & 5 \\ 0 & -1 & -3 & | & -16 \\ 0 & 0 & 2 & | & 23 \end{bmatrix}$$

$$R_3 = \frac{1}{2}R_3 \begin{bmatrix} 1 & 2 & 2 & | & 5 \\ 0 & -1 & -3 & | & -16 \\ 0 & 0 & 1 & | & 11.5 \end{bmatrix} R_2 = -R_2 \begin{bmatrix} 1 & 2 & 2 & | & 5 \\ 0 & 1 & 3 & | & 16 \\ 0 & 0 & 1 & | & 11.5 \end{bmatrix} R_2 = R_2 - 3R_3 \begin{bmatrix} 1 & 2 & 2 & | & 5 \\ 0 & 1 & 0 & | & -18.5 \\ 0 & 0 & 1 & | & 11.5 \end{bmatrix}$$

$$R_1 = R_1 - 2R_3 \begin{bmatrix} 1 & 2 & 0 & | & -18 \\ 0 & 1 & 0 & | & -18.5 \\ 0 & 0 & 1 & | & 11.5 \end{bmatrix} R_1 = R_1 - 2R_2 \begin{bmatrix} 1 & 0 & 0 & | & 19 \\ 0 & 1 & 0 & | & -18.5 \\ 0 & 0 & 1 & | & 11.5 \end{bmatrix}$$

The linear combination of B is $19u - 18.5v + 11.5x = \begin{bmatrix} -1 \\ -6 \\ 6 \end{bmatrix}$