Homework 0 - Introduction to Probabilistic Graphical Models

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1 Probability

1. A fair coin is tossed 10 times. The sample space for each trial is Head, Tail and the trials are independent. What is the probability of having:

$$P(H) = \frac{1}{2}, \quad P(T) = \frac{1}{2}$$

The probability of getting exactly k heads is given by the binomial distribution:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

where $n = 10$ and $p = \frac{1}{2}$.

(a) Zero Tail

$$P(0T) = \binom{10}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$$

(b) 6 heads

$$P(6H) = \binom{10}{6} \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4 = \frac{210}{1024}$$

(c) At least three heads

$$P(X \geq 3) = 1 - P(X < 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2) = 1 - \left(\frac{1}{1024} + \frac{10}{1024} + \frac{45}{1024}\right) = \frac{968}{1024}$$

(d) At least three Heads given the first trail was a Head!

$$P(X \geq 3|X_1 = H) = 1 - P(X < 3|X_1 = H) = 1 - P(X = 0|X_1 = H) - P(X = 1|X_1 = H) - P(X = 2|X_1 = H) = 1 - \left(\frac{1}{512} + \frac{9}{512} + \frac{36}{512}\right) =$$

$$= 1 - \left(\frac{1}{512} + \frac{9}{512} + \frac{36}{512}\right) = \frac{466}{512}$$

2. Assuming the probability that it rains on Monday is 0.45; the probability that it rains on Wednesday is 0.4; and the probability that it rains on Wednesday given that rained on Monday is 0.6. What is the probability that:

(a) It rains on both days

$$P(M \cap W) = P(W|M)P(M) = 0.6 \times 0.45 = 0.27$$

(b) Rain will come next Monday, given that it has just finished raining today (Wednesday)

$$P(M|W) = \frac{P(W|M)P(M)}{P(W)} = \frac{0.6 \times 0.45}{0.4} = 0.675$$

3. Let X denote the outcome of a random experiment with possible values {-4, -3, -2, -1, 0, 1, 2, 3, 4} according to the following probability law:

$$P(X = x) = \begin{cases} ck^2 & \text{if } x \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

$$\sum_{x \in \{-4,-3,-2,-1,0,1,2,3,4\}} P(X = x) = 1$$

$$\sum_{x \in \{-4,-3,-2,-1,0,1,2,3,4\}} ck^2 = 1$$

$$c \sum_{x \in \{-4,-3,-2,-1,0,1,2,3,4\}} k^2 = 1$$

$$c \sum_{k=-4}^{4} k^2 = 1$$

To find the value of c, we can use the formula for the sum of squares of the first n natural numbers:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=-4}^{4} k^2 = \sum_{k=1}^{4} k^2 + \sum_{k=1}^{4} k^2 = \frac{4(4+1)(2\times 4+1)}{6} + \frac{4(4+1)(2\times 4+1)}{6} = 30$$

$$c \times 30 = 1$$

$$c = \frac{1}{30}$$

(b) Compute the expectation and variance of X.

$$E[X] = \sum_{x\in\{-4,-3,-2,-1,0,1,2,3,4\}} xP(X=x) = \sum_{x\in\{-4,-3,-2,-1,0,1,2,3,4\}} x \times \frac{1}{30}x^2 = \frac{1}{30} \sum_{x\in\{-4,-3,-2,-1,0,1,2,3,4\}} x^3$$

$$E[X] = \frac{1}{30} \sum_{x\in\{-4,-3,-2,-1,0,1,2,3,4\}} x^3 = \frac{1}{30} \times 0 = 0$$

$$E[X^2] = \sum_{x\in\{-4,-3,-2,-1,0,1,2,3,4\}} x^2P(X=x) = \sum_{x\in\{-4,-3,-2,-1,0,1,2,3,4\}} x^2 \times \frac{1}{30}x^2 = \frac{1}{30} \sum_{x\in\{-4,-3,-2,-1,0,1,2,3,4\}} x^4$$

$$E[X^2] = \frac{1}{30} \sum_{x\in\{-4,-3,-2,-1,0,1,2,3,4\}} x^4 = \frac{1}{30} \times 0 = 0$$

$$Var[X] = E[X^2] - E[X]^2 = 0 - 0 = 0$$

4. ou just built a new COVID test with the following properties:

- If a person has COVID, the test is positive with 0.95 probability.
- If a person does not have COVID, the test can still be positive with 0.05 probability.

You are told that a random person has COVID with probability 0.001. You just use your test on a random person and it turns out to be positive. What is the probability that the person really has COVID?

$$P(C) = 0.001, \quad P(\bar{C}) = 0.999$$

$$P(T|C) = 0.95, \quad P(T|\bar{C}) = 0.05$$

Bayes theorem:

$$P(C|T) = \frac{P(T|C)P(C)}{P(T)}$$

Probability of a positive test(Total Probability):

$$P(T) = P(T|C)P(C) + P(T|\bar{C})P(\bar{C}) = 0.95 \times 0.001 + 0.05 \times 0.999 = 0.0509$$

so:

$$P(C|T) = \frac{0.95 \times 0.001}{0.0509} = 0.0186$$