Homework 1 - Introduction to Probabilistic Graphical Models

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1 Bayesian Networks

1. Consider a simple Markov Chain structure $X \to Y \to Z$, where all variables are binary. You are required to:
   (a) Write a code (using your preferred programming language) that generates a distribution (not necessarily a valid BN one) over the 3 variables.

   [in the notebook]

   (b) Write a code that verifies whether a distribution is a valid BN distribution.

   [in the notebook]

   (c) Using your code, generate 10000 distributions and compute the fraction of distributions that are valid BN distributions.

   [in the notebook]

2. Given the following Bayesian Network

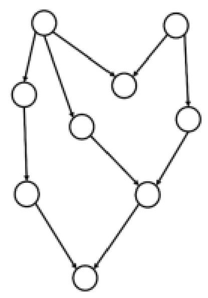

Figure 1: A Bayesian network.

Figure 1: Bayesian Network

   (a) Propose a topological ordering of this graph
   In Figure 2, the topological ordering is:
      i. $A \to B \to C \to D \to E \to F \to G \to H \to I$
      ii. $A \to B \to C \to E \to D \to F \to G \to I \to H$

   (b) Let **X** be a random vector that is Markov with respect to the graph. We assume that the random variables X, are binary. Write all the local conditional independence

   X_A has no parents, so no independence condition applies here.

   X_B has no parents, so no independence condition applies here.

   X_C is conditionally independent of all other nodes given its parent X_A :

$$X_C \perp \{X_B, X_D, X_E, X_F, X_G, X_H, X_I\} \mid X_A$$

   X_D is conditionally independent of all other nodes given its parent X_A :

$$X_D \perp \{X_B, X_C, X_E, X_F, X_G, X_H, X_I\} \mid X_A$$

   X_E is conditionally independent of all other nodes given its parents X_C and X_D :

$$X_E \perp \{X_A, X_B, X_F, X_G, X_H, X_I\} \mid \{X_C, X_D\}$$

   X_F is conditionally independent of all other nodes given its parent X_B :

$$X_F \perp \{X_A, X_C, X_D, X_E, X_G, X_H, X_I\} \mid X_B$$

   X_G is conditionally independent of all other nodes given its parents X_E and X_F :

$$X_G \perp \{X_A, X_B, X_C, X_D, X_H, X_I\} \mid \{X_E, X_F\}$$

   X_H is conditionally independent of all other nodes given its parent X_D :

$$X_H \perp \{X_A, X_B, X_C, X_E, X_F, X_G, X_I\} \mid X_D$$

   X_I is conditionally independent of all other nodes given its parents X_G and X_H :

$$X_I \perp \{X_A, X_B, X_C, X_D, X_E, X_F\} \mid \{X_G, X_H\}$$

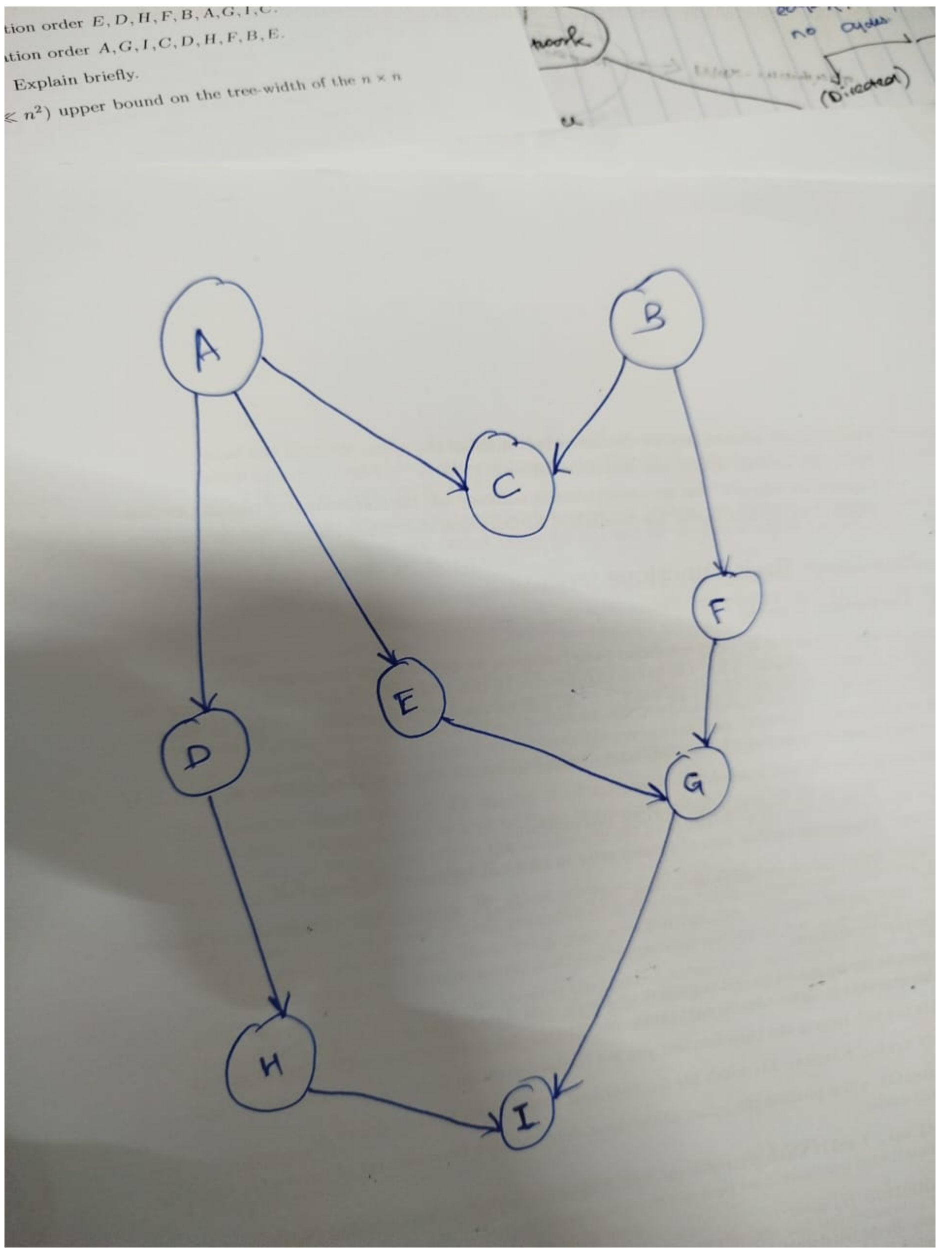

Figure 2: Bayesian Network

3. State True or False, and briefly justify your answer within 3 lines. The statements are either direct consequences of theorems in Koller and Friedman (2009, Ch. 3), or have a short proof. In the follows, P is a distribution and G is a BN structure.

   (a) If $A \perp B \mid C$ and $A \perp C \mid B$, then $A \perp B$ and $A \perp C$. (Suppose the joint distribution of A, B, C is positive.) (This is a general probability question not related to BNs.)

   - **False**. Conditional independence does not imply marginal independence. For example, A and B can be dependent but become independent given C.

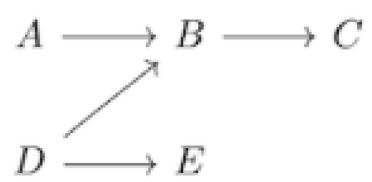

Figure 2: A Bayesian network.

Figure 3: Bayesian Network

   (b) In Figure 2, $E \perp C \mid B$

   (c) in Figure 2, $A \perp E \mid C$

   In figure 3, Recall the definitions of local and global independences of G and independences of P.

$$I_l(G) = \{(X \perp \text{NonDescendants}_G(X) \mid \text{Parents}_G(X))\} \tag{1}$$

$$I(G) = \{(X \perp Y \mid Z) : \text{d-separated}_G(X, Y \mid Z)\} \tag{2}$$

$$I(P) = \{(X \perp Y \mid Z) : P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z)\} \tag{3}$$

   (d) In Figure 3, relation 1 is true.

   (e) In Figure 3, relation 2 is true.

   (f) In Figure 3, relation 3 is true.

   (g) If G is an I-map for P, then P may have extra conditional independencies than G.

   (h) Two BN structures G_1 and G_2 are I-equivalent if they have the same skeleton and the same set of v-structures.

$$
\begin{tikzcd}
P \text{ factorizes over } \mathcal{G} \arrow[r, Rightarrow, "(1)"] & \mathcal{I}(\mathcal{G}) \subseteq \mathcal{I}(P) \arrow[r, Rightarrow, "(2)"] & \mathcal{I}_\ell(\mathcal{G}) \subseteq \mathcal{I}(P) \arrow[ll, Rightarrow, bend left, "(3)"]
\end{tikzcd}
$$

Figure 3: Some relations in Bayesian networks.

(i) If G_1 is an I-map of distribution P, and G_1 has fewer edges than G_2, then G_2 is not a minimal I-map of P.

(j) The P-map of a distribution, if it exists, is unique.