Homework 2 - Maths Foundation for Machine Learning

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1. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of the vector space $\mathbb{R}^2$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The action of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ on the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is given by

$$T(\mathbf{v}_1) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \quad \text{and} \quad T(\mathbf{v}_2) = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}.$$

Find the formula for $T(\mathbf{x})$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$$A = \begin{pmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{pmatrix}$$

This is because the transformation is linear and the transformation of a vector is a linear combination of the transformation of the basis vectors. Therefore, the transformation of a vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by:

$$Ax = \begin{pmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 4x + 8y \\ 6x + 10y \end{pmatrix}$$

so the formula for $T(\mathbf{x})$ is:

$$T(\mathbf{x}) = \begin{pmatrix} 2x \\ 4x + 8y \\ 6x + 10y \end{pmatrix}$$

2. For an integer $n \geq 0$, let P_n be the vector space of polynomials of degree at most n. The set $B = \{1, x, x^2, \ldots, x^n\}$ is a basis of P_n called the standard basis. Let $T : P_n \to P_{n+1}$ be the map defined by, for $f \in P_n$,

$$T(f)(x) = xf(x).$$

Prove that T is a linear transformation, and find its range and nullspace.

$$\text{let} \quad f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \text{ this is a polynomial of degree at most } n$$

Then the transformation of f to $T(f)$ is:

$$T(f)(x) = x(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) = a_0 x + a_1 x^2 + a_2 x^3 + \ldots + a_n x^{n+1}$$

This is a polynomial of degree at most $n + 1$, therefore T is a linear transformation. The range of T is the set of all polynomials of degree at most $n + 1$, and the nullspace of T is the set of all polynomials of degree at most n such that $f(x) = 0$. Therefore, the range of T is P_{n+1} and the nullspace of T is P_n.

3. Let $C[0, 3]$ be the vector space of real functions on the interval $[0, 3]$. Let P_3 denote the set of real polynomials of degree 3 or less. Define the map $T : C[0, 3] \to P_3$ by

$$T(f)(x) = f(0) + f(1)x + f(2)x^2 + f(3)x^3.$$

Determine if T is a linear transformation.

$$\text{let} \quad f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \text{ this is a polynomial of degree at most 3}$$

Then the transformation of f to $T(f)$ is:

$$T(f)(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

This is a polynomial of degree at most 3, therefore T is a linear transformation.

4. Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of the vector space $\mathbb{R}^2$ where

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Let $C = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a basis of the vector space $\mathbb{R}^3$ where

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ on the basis B given by:

$$T(\mathbf{b}_1) = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} \quad \text{and} \quad T(\mathbf{b}_2) = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix}.$$

(a) Find the transformation matrix A_T of the linear transformation T

$$A_T = \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix}$$

(b) Consider a new basis $\tilde{B} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ of $\mathbb{R}^2$ and $\tilde{C} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$. Under the basis change, what is the transformation matrix $\tilde{A}_T$?

The transformation matrix $\tilde{A}_T$ is given by $\tilde{A}_T = P^{-1} A_T S$

let us get the transformation matrix S from the basis $\tilde{B}$ to the basis B:

$$B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so:

$$S = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$$

we have the transformation matrix A_T as:

$$A_T = \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix}$$

we need to find the transformation matrix P from the basis C to the basis $\tilde{C}$:

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -\frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(c) What is the image and kernel of T

The image of T is the span of the columns of A_T, and the kernel of T is the nullspace of A_T

$$\text{The image of } T = \text{span} \left\{ \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right\}$$

$$\text{The kernel of } T = \text{nullspace} \left\{ \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix} \right\}$$

$$\text{The kernel of } T = \text{nullspace} \left\{ \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix} \right\} = \text{nullspace} \left\{ \begin{pmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{pmatrix} \right\} = \text{nullspace} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{\mathbf{0}\}$$

$$\text{The kernel of } T = \{\mathbf{0}\}$$

$$\text{The image of } T = \text{span} \left\{ \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right\}$$

$$\text{The image of } T = \text{span} \left\{ \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right\}$$

5. A rotation in 3-D space (whose Cartesian coordinates we will call x, y, and z as usual) is characterized by three angles. We will characterize them as a rotation around the x-axis, a rotation around the y-axis, and a rotation around the z-axis.

   (a) Derive the rotation matrix R_1 that transforms a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to a new vector $\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$ by rotating it counterclockwise by angle θ around the x-axis, then an angle δ around the y-axis, and finally an angle ϕ around the z-axis.

$$R_1 = R_z(\phi)R_y(\delta)R_x(\theta)$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R_y(\delta) = \begin{pmatrix} \cos(\delta) & 0 & \sin(\delta) \\ 0 & 1 & 0 \\ -\sin(\delta) & 0 & \cos(\delta) \end{pmatrix}$$

$$R_z(\phi) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\delta) & 0 & \sin(\delta) \\ 0 & 1 & 0 \\ -\sin(\delta) & 0 & \cos(\delta) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R_1 = \begin{pmatrix} \cos(\phi)\cos(\delta) & \cos(\phi)\sin(\delta)\sin(\theta) - \sin(\phi)\cos(\theta) & \cos(\phi)\sin(\delta)\cos(\theta) + \sin(\phi)\sin(\theta) \\ \sin(\phi)\cos(\delta) & \sin(\phi)\sin(\delta)\sin(\theta) + \cos(\phi)\cos(\theta) & \sin(\phi)\sin(\delta)\cos(\theta) - \cos(\phi)\sin(\theta) \\ -\sin(\delta) & \cos(\delta)\sin(\theta) & \cos(\delta)\cos(\theta) \end{pmatrix}$$

   (b) Derive the rotation matrix R_2 that transforms a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to a new vector $\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$ by rotating it counterclockwise by an angle δ around the y-axis, then an angle θ around the x-axis, and finally an angle ϕ around the z-axis.

$$R_2 = R_z(\phi)R_x(\theta)R_y(\delta)$$

$$R_2 = \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\delta) + \cos(\phi)\sin(\theta)\sin(\delta) & \sin(\phi)\sin(\delta) + \cos(\phi)\sin(\theta)\cos(\delta) \\ \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\delta) + \sin(\phi)\sin(\theta)\sin(\delta) & -\cos(\phi)\sin(\delta) + \sin(\phi)\sin(\theta)\cos(\delta) \\ -\sin(\theta) & \cos(\theta)\sin(\delta) & \cos(\theta)\cos(\delta) \end{pmatrix}$$

   (c) Confirm that $R_1R_1^\top = R_2R_2^\top = I$ (Hint: Do not directly multiply the matrices from part 1 above but write down the matrices you need to multiply and multiply them in pairs).

$$R_1R_1^\top = R_z(\phi)R_y(\delta)R_x(\theta)R_x^\top(\theta)R_y^\top(\delta)R_z^\top(\phi)$$

$$R_1R_1^\top = R_z(\phi)R_y(\delta)R_x(\theta)R_x(\theta)R_y(\delta)R_z(\phi)$$

$$R_1R_1^\top = R_z(\phi)R_y(\delta)R_y(\delta)R_z(\phi)$$

$$R_1R_1^\top = R_z(\phi)R_z(\phi)$$

$$R_1R_1^\top = R_z(\phi)R_z(\phi) = I$$

$$R_2R_2^\top = R_z(\phi)R_x(\theta)R_y(\delta)R_y^\top(\delta)R_x^\top(\theta)R_z^\top(\phi)$$

$$R_2R_2^\top = R_z(\phi)R_x(\theta)R_y(\delta)R_y(\delta)R_x(\theta)R_z(\phi)$$

$$R_2R_2^\top = R_z(\phi)R_x(\theta)R_x(\theta)R_z(\phi)$$

$$R_2R_2^\top = R_z(\phi)R_z(\phi) = I$$

$$R_2R_2^\top = I$$

$$R_1R_1^\top = R_2R_2^\top = I$$

$$\text{Therefore, } R_1R_1^\top = R_2R_2^\top = I$$

6. Let

$$A = \begin{pmatrix} a & -1 \\ 1 & 4 \end{pmatrix}$$

   be a 2×2 matrix, where a is some real number. Suppose that the matrix A has an eigenvalue 3.

   (a) Determine the value of a.

$$\text{The eigenvalues of } A \text{ are the roots of the characteristic polynomial } \det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} a & -1 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} a-\lambda & -1 \\ 1 & 4-\lambda \end{pmatrix}\right) = (a-\lambda)(4-\lambda) - (-1)(1)$$

$$(a-\lambda)(4-\lambda) - (-1)(1) = \lambda^2 - 4\lambda - a\lambda + 4a + 1$$

$$\lambda^2 - 4\lambda - a\lambda + 4a + 1 = \lambda^2 - (4+a)\lambda + 4a + 1$$

$$\text{The eigenvalues of } A \text{ are the roots of } \lambda^2 - (4+a)\lambda + 4a + 1 = 0$$

$$\text{Given that the matrix } A \text{ has an eigenvalue of 3, then } 3^2-(4+a)3+4a+1=0$$
$$9-3(4+a)+4a+1=0$$
$$9-12-3a+4a+1=0$$
$$-3a+4a-2=0=a=2$$
$$\text{Therefore, the value of } a \text{ is } a=\mathbf{2}$$

(b) Does the matrix A have eigenvalues other than 3?

$$\text{our matrix A is } A=\begin{pmatrix}2 & -1\\ 1 & 4\end{pmatrix}$$
$$\text{The eigenvalues of } A \text{ are the roots of } \lambda^2-(4+2)\lambda+4(2)+1=0$$
$$\lambda^2-6\lambda+9=0$$
$$\text{we expand the equation } (\lambda-3)(\lambda-3)=0$$
$$\text{Therefore, the matrix } A \text{ has only one eigenvalue of } \mathbf{3}$$

7. Suppose G_{k+2} is the average of the two previous numbers G_{k+1} and G_k. That is:

$$G_{k+2}=\frac{1}{2}G_{k+1}+\frac{1}{2}G_k.$$

Let:

$$\begin{pmatrix}G_{k+2}\\ G_{k+1}\end{pmatrix}=A\begin{pmatrix}G_{k+1}\\ G_k\end{pmatrix},$$

(a) Find the eigenvalues and eigenvectors of A.

$$\text{The eigenvalues of } A \text{ are the roots of the characteristic polynomial } \det(A-\lambda I)=0$$
$$A=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 0\end{pmatrix}.$$
$$\det(A-\lambda I)=\det\left(\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 0\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right)=\det\left(\begin{pmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\ 1 & -\lambda\end{pmatrix}\right)=\left(\frac{1}{2}-\lambda\right)(-\lambda)-\left(\frac{1}{2}\right)$$
$$\left(\frac{1}{2}-\lambda\right)(-\lambda)-\left(\frac{1}{2}\right)=\lambda^2-\frac{1}{2}\lambda-\frac{1}{2}$$
$$\text{The eigenvalues of } A \text{ are the roots of } \lambda^2-\frac{1}{2}\lambda-\frac{1}{2}=0$$

multiplying by 2 to clear the fractions:

$$2\lambda^2-\lambda-1=0$$
$$\text{we expand the equation } (2\lambda+1)(\lambda-1)=0$$
$$\text{Therefore, the eigenvalues of } A \text{ are } \lambda=-\frac{1}{2} \text{ and } \lambda=1$$
$$\text{The matrix } A \text{ has two eigenvalues } \lambda_1=-\frac{1}{2} \text{ and } \lambda_2=1$$
$$\text{The eigenvectors of } A \text{ are the solutions to the equation } (A-\lambda I)\mathbf{v}=\mathbf{0}$$
$$\text{For } \lambda=-\frac{1}{2}, \text{ the eigenvector is :}$$
$$\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}v_1\\ v_2\end{pmatrix}=-\frac{1}{2}\begin{pmatrix}v_1\\ v_2\end{pmatrix}$$
$$\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}v_1\\ -\frac{1}{2}v_2\end{pmatrix}$$
$$\frac{1}{2}v_1+\frac{1}{2}v_2=-\frac{1}{2}v_1$$

multiplying by 2 to clear the fractions:

$$v_1+v_2=-v_1$$
$$v_1+v_2+v_1=0$$
$$2v_1+v_2=0$$
$$v_2=-2v_1$$

lets represent V_1 in terms of v_2:

$$v_1=-\frac{1}{2}v_2$$

so, the eigenvector for $\lambda=-\frac{1}{2}$ is:

$$\begin{pmatrix}-\frac{1}{2}\\ 1\end{pmatrix}$$
$$\text{For } \lambda=1, \text{ the eigenvector is :}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\frac{1}{2}v_1 + \frac{1}{2}v_2 = v_1$$
$$v_1 + v_2 = 2v_1$$
$$v_1 + v_2 - 2v_1 = 0$$
$$-v_1 + v_2 = 0$$
$$v_1 = v_2$$

so, the eigenvector for $\lambda = 1$ is:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) Find the limit as $n \to \infty$ of the matrices A^n.

$$\text{The matrix } A \text{ has two eigenvalues } \lambda_1 = -\frac{1}{2} \text{ and } \lambda_2 = 1$$
$$\text{The eigenvectors of } A \text{ are } \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\text{The matrix } A \text{ can be diagonalized as } A = PDP^{-1}$$
$$\text{where } P = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$
$$\text{The limit as } n \to \infty \text{ of the matrices } A^n \text{ is } \lim_{n\to\infty} A^n = P \lim_{n\to\infty} D^n P^{-1}$$
$$\text{The limit as } n \to \infty \text{ of the matrix } D^n \text{ is } \lim_{n\to\infty} D^n = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} \left(-\frac{1}{2}\right)^n & 0 \\ 0 & 1^n \end{pmatrix}$$
$$\text{The limit as } n \to \infty \text{ of the matrix } D^n \text{ is } \lim_{n\to\infty} D^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\text{The limit as } n \to \infty \text{ of the matrices } A^n \text{ is } \lim_{n\to\infty} A^n = P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P^{-1}$$
$$\text{The limit as } n \to \infty \text{ of the matrices } A^n \text{ is } \lim_{n\to\infty} A^n = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$
$$\begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} 1 & -1 \\ -1 & -\frac{1}{2} \end{pmatrix} = \frac{1}{-\frac{1}{2} - 1} \begin{pmatrix} 1 & -1 \\ -1 & -\frac{1}{2} \end{pmatrix} = -\frac{2}{3} \begin{pmatrix} 1 & -1 \\ -1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

8. You are given bases in $\mathbb{R}^3$. Apply the Gram-Schmidt process on each of them to obtain the orthogonal bases. Transform the orthogonal bases to orthonormal bases.

(a) $\left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$

$$\text{Let } \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$
$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$
$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}{\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{0}{12} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$
$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}{\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix}0\\3\\1\end{pmatrix} - \frac{8}{12}\begin{pmatrix}2\\2\\2\end{pmatrix} - (-)\frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix}0\\3\\1\end{pmatrix} - \frac{2}{3}\begin{pmatrix}2\\2\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix} - \begin{pmatrix}\frac{4}{3}\\\frac{4}{3}\\\frac{4}{3}\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix} - \begin{pmatrix}\frac{4}{3}\\\frac{4}{3}\\\frac{4}{3}\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0-\frac{4}{3}+\frac{1}{2}\\3-\frac{4}{3}+0\\1-\frac{4}{3}-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}-\frac{5}{6}\\\frac{5}{3}\\-\frac{5}{6}\end{pmatrix}$$

$$\text{The orthogonal basis is } \left\{\begin{pmatrix}2\\2\\2\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}-\frac{5}{6}\\\frac{5}{3}\\-\frac{5}{6}\end{pmatrix}\right\}$$

To make them orthonormal, we divide each vector by its norm:

$$\mathbf{u}_1 = \begin{pmatrix}2\\2\\2\end{pmatrix} \quad \text{norm} = \sqrt{2^2+2^2+2^2} = \sqrt{12} = 2\sqrt{3}$$

$$\mathbf{u}_1 = \frac{1}{2\sqrt{3}}\begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix} \quad \text{norm} = \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix}-\frac{5}{6}\\\frac{5}{3}\\-\frac{5}{6}\end{pmatrix} \quad \text{norm} = \sqrt{\left(-\frac{5}{6}\right)^2+\left(\frac{5}{3}\right)^2+\left(-\frac{5}{6}\right)^2} = \sqrt{\frac{25}{36}+\frac{25}{9}+\frac{25}{36}} = \sqrt{\frac{25}{36}+\frac{100}{36}+\frac{25}{36}} = \sqrt{\frac{150}{36}} = \sqrt{\frac{25}{6}} = \frac{5}{\sqrt{6}}$$

$$\mathbf{u}_3 = \frac{1}{5\sqrt{6}}\begin{pmatrix}-\frac{5}{6}\\\frac{5}{3}\\-\frac{5}{6}\end{pmatrix} = \begin{pmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\end{pmatrix}$$

$$\text{The orthonormal basis is } \left\{\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\end{pmatrix}\right\}$$

(b) $\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}\right\}$

$$\text{Let } \mathbf{v}_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}2\\3\\1\end{pmatrix}$$

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - (-)\frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{pmatrix} = \begin{pmatrix}0+\frac{1}{2}\\1-\frac{1}{2}\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \begin{pmatrix}2\\3\\1\end{pmatrix} - \frac{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}\begin{pmatrix}1\\-1\\0\end{pmatrix} - \frac{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}}{\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}}\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix}2\\3\\1\end{pmatrix} - (-)\frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} - 5\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix}2\\3\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix} - \begin{pmatrix}\frac{5}{2}\\\frac{5}{2}\\0\end{pmatrix} = \begin{pmatrix}2\\3\\1\end{pmatrix} - \begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix} - \begin{pmatrix}\frac{5}{2}\\\frac{5}{2}\\0\end{pmatrix} = \begin{pmatrix}2+\frac{1}{2}-\frac{5}{2}\\3-\frac{1}{2}-\frac{5}{2}\\1-0\end{pmatrix} = \begin{pmatrix}2-2\\3-3\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\text{The orthogonal basis is } \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

To make them orthonormal, we divide each vector by its norm:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{norm} = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad \text{norm} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 0^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$$

$$\mathbf{u}_2 = \frac{1}{\frac{\sqrt{2}}{2}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{norm} = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$$

$$\mathbf{u}_3 = \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{The orthonormal basis is } \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$