FC-0306-2 (QRMT): Assignment 1

1. Write down the algorithmic steps for the following ruler and (collapsible) compass constructions, and argue their correctness
   (a) the perpendicular to a given line through a given point on the line
   (b) the perpendicular bisector of a line segment
   (c) the parallel to a given line through a given point
   (d) bisection of an angle
   (e) a parallelogram given one of its sides as a line segment and a vertex on the opposite side
2. Argue the existence of a "non-collapsible compass" given a ruler and a "collapsible compass".
3. Show how you may construct the natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ using ruler and compass. Also give the constructions for addition, subtraction, multiplication and computing ratios.
4. Consider Euclid's GCD algorithm given as:

$$gcd(a, b) = \begin{cases} a & \text{if } a = b \\ gcd(a - b, b) & \text{if } a > b \\ gcd(a, b - a) & \text{if } b > a \end{cases}$$

   Show that $gcd(a, b)$ is *constructible*.
5. Argue that the rationals $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z} \text{ and } gcd(p, q) = 1\}$ are *constructible*.
6. Read the definition of a *Field* from here and argue that $(\mathbb{Q}, +, \times)$ forms a *Field*.
7. Show that $\mathbb{Q}$ is a *linearly ordered set*, that is

(a) $a \leq a$ for all $a \in \mathbb{Q}$ (*refexivity*)

(b) For $a, b \in \mathbb{Q}$, $a \leq b$ and $b \leq a$ implies $a = b$ (*antisymmetry*)

(c) For $a, b, c \in \mathbb{Q}$, $a \leq b$ and $b \leq c$ implies $a \leq c$ (*transitivity*)

(d) For $a, b \in \mathbb{Q}$, either $a \leq b$ or $b \leq a$ (*comparibility* or the *trichotomy law*)

8. Argue that $\sqrt{a}$ for $a \in \mathbb{N}$ is *constructible* and *irrational*.

9. Argue that an angle cannot be trisected using a ruler and compass in general. So rationals have holes.

10. Read about the definition of real numbers $\mathbb{R}$ using *Dedekind cuts* from here and show that the *Dedekind cuts* form a *Field* under addition and multiplication.

11. Argue that every bounded subset of reals, $S \subset \mathbb{R}$, has a least upper bound. So reals do not have holes.

12. Read from here the Archimedes' method of trisecting an angle using a marked ruler and a compass called the *Neusis construction* (under the heading "With a marked ruler"). Explain

    (a) why it is not a pure ruler and compass construction?

    (b) why is there necessarily a loss of precision – even conceptually speaking – in the Archimedes' construction unlike in a pure ruler and compass one.