Generalized form of Snell's law in anisotropic media: A practical approach

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A generalized form of Snell's law in anisotropic media

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ABSTRACT

We have reformulated the law governing the refraction of rays at a planar interface separating two anisotropic media in terms of slowness surfaces. Equations connecting ray directions and phase-slowness angles are derived using geometrical properties of the gradient operator in slowness space. A numerical example shows that, even in weakly anisotropic media, the ray trajectory governed by the anisotropic Snell's law is significantly different from that obtained using the isotropic form. This could have important implications for such considerations as imaging (e.g., migration) and lithology analysis (e.g., amplitude variation with offset).

Expressions are shown specifically for compressional (qP) waves but they can easily be extended to SH waves by equating the anisotropic parameters (i.e., $\varepsilon = \delta \Rightarrow \gamma$) and to qSV and converted waves by similar means.

The analytic expressions presented are more complicated than the standard form of Snell's law. To facilitate practical application, we include our Mathematica code.

INTRODUCTION

The laws governing ray bending have been investigated since the time of Ptolemy (2nd century A.D.). Prior to the work of Snell (1591–1626), ray bending had been described by equating the ratio of angles, and not their sines, to the ratio of velocities, quite accurate nevertheless for small angles. Snell's law extended the validity to all angles of incidence. However, it assumes both media to be isotropic. In the present paper, we extend Snell's expression to encompass anisotropic media.

For simplicity, we assume horizontally layered media and a dependence of velocity on angle of incidence that is the same for all azimuths, that is, the media considered exhibit transverse isotropy (TI) with a vertical symmetry axis (TIV). Although this case has 2-D geometry, the general methodology is equally applicable to three dimensions.

We proceed by two different routes: (1) by applying the continuity of k_x, the component of the wave number, $\mathbf{k}$, tangential to the interface between two media, and (2) via Fermat's principle, or the stationarity of traveltime between the source located in one medium and receiver located in the other medium. Both routes (the continuity conditions and the stationary traveltime concept) yield the same expression for the refraction law, i.e., Snell's law.

Thus, Snell's law in horizontally layered media and for a given frequency can conveniently be restated as the requirement that k_x be continuous across the boundary. This property holds for both isotropic and anisotropic media regardless of the type of body wave generated at the boundary (longitudinal or transverse) and forms the basis for our strategy of calculating reflected and transmitted angles. The approach following Fermat's principle can be formulated through the calculus of variations (e.g., Slawinski and Webster, 1999).

Present exploration methods of data acquisition and processing offer the potential to investigate certain subtle characteristics of the subsurface. Consequently, rigorous and widely applicable tools are needed. A refraction law for anisotropic media contributes to more refined and efficient exploration techniques.

METHOD AND RESULTS

The formulation presented in the next two sections (geometrical and mathematical) states equations for a general velocity anisotropy expressed in terms of 3-D Cartesian coordinates.

Geometrical formulation

Snell's law can be illustrated using phase-slowness surfaces for both incident and transmitted media (e.g., Auld, 1973;

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Helbig, 1994). The geometrical construction is facilitated by the fact that the phase-slowness vectors of the incident, reflected, and refracted waves are coplanar. Their being coplanar follows from the kinematic requirement that boundary conditions be satisfied at all times and at every point of the interface. For TI therefore, and without any loss of generality, it is possible to choose a Cartesian coordinate system such that all the phase-slowness vectors lie in the xz-plane. The familiar case of isotropic media is considered in Figure 1.

Mathematical formulation

The geometrical approach illustrated in Figure 1 for the isotropic case (i.e., spherical slowness surfaces) is extended to include more general scenarios where the slowness surface is an arbitrary surface in slowness space. Consider two anisotropic media separated by a planar, horizontal interface. Let the phase-slowness surface in the upper medium be given by the level surface of a function $f(x, y, z)$ in slowness space spanned by Cartesian coordinates x, y, and z:

$$f(x, y, z) = a. \quad (1)$$

Similarly, let the phase-slowness surface in the lower medium be given by the level surface of a function $g(x, y, z)$ in slowness space:

$$g(x, y, z) = b. \quad (2)$$

In slowness space, x, y, and z have dimensions of slowness, as does r, which we use below as a general symbol for slowness.

Consider a ray incident on the boundary from above. All phase-slowness vectors (for incident, reflected, and transmitted waves) must be coplanar so, without loss of generality, we can take them to lie in the xz-plane (Figure 2). Denoting the phase-slowness vector as $\mathbf{m}$, the continuity conditions require that

$$\mathbf{m}_i \cdot \bar{\mathbf{x}} = \mathbf{m}_r \cdot \bar{\mathbf{x}} = \mathbf{m}_t \cdot \bar{\mathbf{x}}, \quad (3)$$

where $\bar{\mathbf{x}}$ is a unit vector in the x-direction and subscripts i, r, and t refer to incident, reflected, and transmitted waves, respectively.

The group(ray)-slowness vector, $\mathbf{w}$, is normal to the phase-slowness surface. Using the property that the gradient points

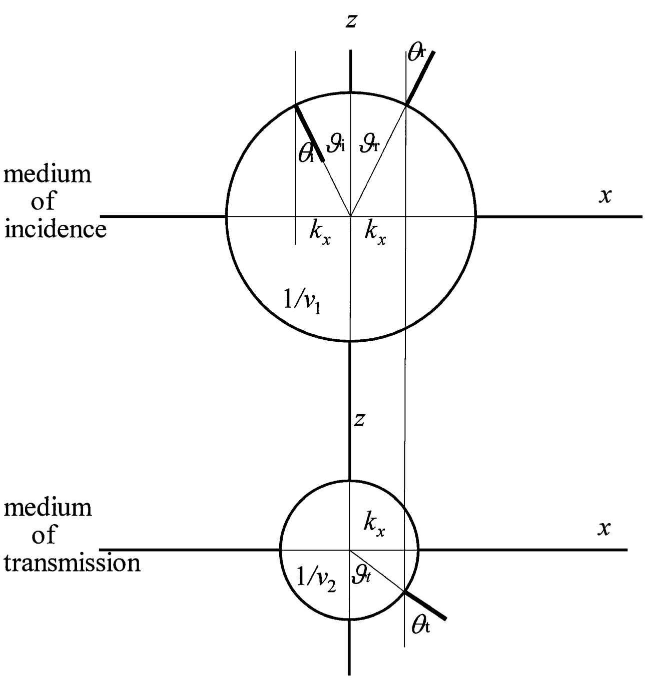

FIG. 1. The geometrical construction yielding reflection and transmission angles of slowness vectors in an isotropic medium using the phase-slowness curve. The curves represent phase slowness in the medium of incidence and transmission, whereas the Cartesian axes x and z correspond to the horizontal and vertical slownesses, respectively. The same concept applies in an anisotropic medium except that the xz-plane cross-section of the phase-slowness surface does not, in general, form a sphere. The thin lines within the circles (radii) are collinear with the phase-slowness vectors; the thick lines, normal to the phase-slowness surface correspond to the group-slowness vectors. The symbols ϑ_i, ϑ_r and ϑ_t are the angles between phase-slowness vectors for incident, reflected, and transmitted waves and the normal to the interface; θ_i, θ_r and θ_t are the angles between group-slowness vectors for incident, reflected, and transmitted waves and the normal to the interface (i.e., ray angles). In the isotropic case, $\vartheta_i = \theta_i$, $\vartheta_r = \theta_r$, and $\vartheta_t = \theta_t$.

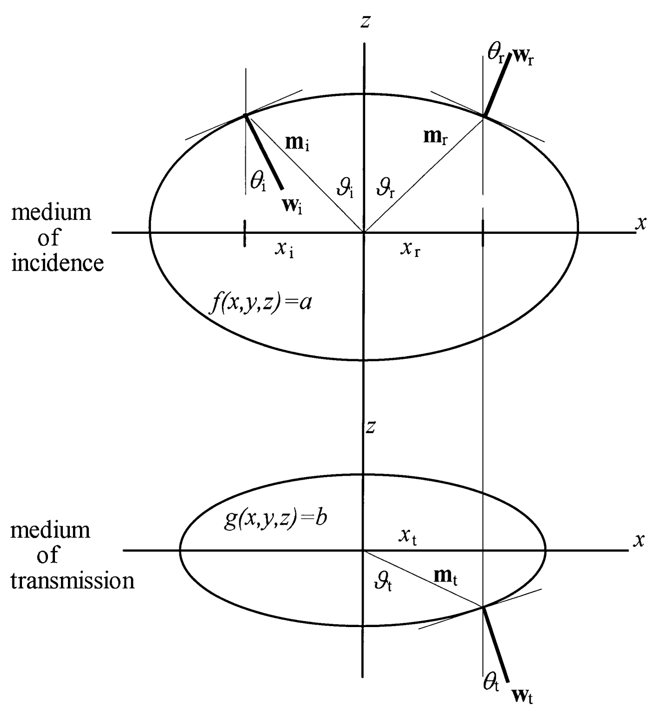

FIG. 2. The geometrical construction yielding reflection and transmission angles of slowness vectors in an anisotropic medium using the phase-slowness curve. The curves represent phase slowness in the medium of incidence and transmission, whereas the Cartesian axes x and z correspond to the horizontal and vertical slownesses, respectively. This is an illustration of ray angles for incident, reflected, and transmitted rays in anisotropic media separated by a horizontal, planar interface using phase-slowness surfaces described by functions f and g. The $\mathbf{m}$ vectors correspond to phase-slowness and $\mathbf{w}$ vectors to group slowness; θ and ϑ correspond to ray angle and phase angle, respectively for incident, reflected, and transmitted waves. Note that $\vartheta_i \neq \theta_i$, $\vartheta_r \neq \theta_r$, and $\vartheta_t \neq \theta_t$ (cf. Figure 1).

in the direction in which f increases most rapidly and that it is normal to any surface of constant f gives

$$\mathbf{w}_i \parallel \nabla f(x, y, z)|_{(x_i, y_i, z_i)}, \tag{4}$$

i.e., the ray vector, $\mathbf{w}$, is parallel to the gradient. Normalizing, and choosing the function f to have a minimum at the origin $O(0,0,0)$ and to be monotonically increasing outwards yields

$$\bar{\mathbf{w}}_i = -\frac{\nabla f(x, y, z)|_{(x_i, y_i, z_i)}}{|\nabla f(x, y, z)\|_{(x_i, y_i, z_i)}}, \tag{5}$$

where the negative sign ensures that the incident unit ray vector, $\bar{\mathbf{w}}_i$, points towards the boundary. From the definition of dot product, the cosine of the angle of incidence (the angle between the ray vector and the normal to the interface) is given by

$$\cos\theta_i = (-\bar{\mathbf{z}}) \cdot \bar{\mathbf{w}}_i = \frac{\bar{\mathbf{z}} \cdot \nabla f(x, y, z)|_{(x_i, y_i, z_i)}}{|\nabla f(x, y, z)\|_{(x_i, y_i, z_i)}}, \tag{6}$$

where $\bar{\mathbf{z}}$ is a unit vector in the z-direction.

The normalized transmitted ray vector is given by

$$\bar{\mathbf{w}}_t = \frac{\nabla g(x, y, z)|_{(x_t, y_t, z_t)}}{|\nabla g(x, y, z)\|_{(x_t, y_t, z_t)}}, \tag{7}$$

and thus the cosine of the angle of transmission (again with respect to the interface normal) is given by

$$\cos\theta_t = (-\bar{\mathbf{z}}) \cdot \bar{\mathbf{w}}_t = -\frac{\bar{\mathbf{z}} \cdot \nabla g(x, y, z)|_{(x_t, y_t, z_t)}}{|\nabla g(x, y, z)\|_{(x_t, y_t, z_t)}}. \tag{8}$$

In evaluating the above expression, one uses the facts that the horizontal phase-slowness components are equal (i.e., $x_t = -x_i$), and that y_i and y_t are zero by the choice of the coordinate system. Hence, z can be found by substituting x and y into equations (1) or (2). Note that physical solutions must have $\bar{\mathbf{w}}_t \cdot (-\bar{\mathbf{z}}) \geq 0$.

While the phase-slowness vectors, $\mathbf{m}$, are coplanar for the incident, reflected, and transmitted waves, the ray vectors, $\mathbf{w}$, need not lie in the same plane. Their directions are determined by the normals to the phase-slowness surfaces. They will, however, remain in the same plane if the phase-slowness surfaces are rotationally symmetric about the z-axis. If the phase-slowness surface does not possess rotational symmetry, the incident, reflected and transmitted group vectors need not be coplanar. In such a case the angle of deviation, χ, from the sagittal plane, assumed to coincide with the xz-plane and containing all phase-slowness vectors, $\mathbf{m}$, can be found by considering the projection, $\mathbf{w}_{xz}$, of the ray vector, $\mathbf{w}$, on this plane:

$$\mathbf{w}_{xz} = [\mathbf{w} \cdot \mathbf{x}, 0, \mathbf{w} \cdot \mathbf{z}]. \tag{9}$$

From the definition of scalar (dot) product, it follows that

$$\cos\chi = \frac{\mathbf{w} \cdot \mathbf{w}_{xz}}{|\mathbf{w}\|\mathbf{w}_{xz}|}. \tag{10}$$

The geometrical formulation presented above can be adapted for incident, reflected, or transmitted rays. Other concepts, such as total internal reflection, also emerge naturally from this formulation.

QUASI-COMPRESSIONAL WAVE

To illustrate the approach presented above, we derive a refraction law for quasi-compressional (qP) waves. In equation (11), qP-wave phase velocity is expressed in terms of two anisotropic parameters, δ and ε (Thomsen, 1986). In the mathematical formulation, we have expressed angles of incidence and transmission as a function of the ray parameter, m_{x0} or x_0, common to both media.

The phase velocity, v_{qP}, of a qP-wave in a weakly anisotropic medium is given by Thomsen (1986) as

$$v_{qP}(\xi) = \alpha(1 + \delta\sin^2\xi\cos^2\xi + \varepsilon\cos^4\xi), \tag{11}$$

where α is the velocity for propagation perpendicular to the interface, and δ and ε are anisotropic parameters. Here, the phase angle, ξ, is the phase latitude, the complement of the phase colatitude, ϑ, used by Thomsen (1986).

The slowness curve, $m \equiv r$, in the medium of incidence can be expressed as

$$r(\xi) = \frac{1}{v_{qP}(\xi)} = \frac{1}{\alpha(1 + \delta\sin^2\xi\cos^2\xi + \varepsilon\cos^4\xi)}. \tag{12}$$

In the 2-D TIV case, using a standard expression for the normal to a curve expressed in polar coordinates, we use an equation relating ray angle of incidence, θ, and the phase latitude, ξ:

$$\theta = \arctan\frac{\frac{dr}{d\xi} - r\tan\xi}{\frac{dr}{d\xi}\tan\xi + r}, \tag{13}$$

(e.g., Anton, 1984).

Knowing the characteristics of the medium of incidence $(\alpha_i, \delta_i, \varepsilon_i)$ (using the subscript i or t to refer to the medium of incidence or transmission, respectively), and given the ray angle of incidence, θ_i, equation (13) still can not, in general, be solved explicitly for the corresponding phase angle, ξ_i. It can, however, be solved numerically (see Appendix A). Once the phase angle, ξ_i, is found, the corresponding ray parameter, x_0, can be calculated as

$$x_0 = \frac{\cos\xi_i}{v_{qP}(\xi_i)} = \frac{\cos\xi_i}{\alpha_i\left(1 + \delta_i\sin^2\xi_i\cos^2\xi_i + \varepsilon_i\cos^4\xi_i\right)}. \tag{14}$$

Having found the ray parameter, x_0, which is continuous across the interface, we calculate the phase latitude in the medium of transmission, ξ_t, from

$$\begin{aligned} x_0 &= \frac{\cos\xi_i}{v_{qP}(\xi_i)} = \frac{\cos\xi_t}{v_{qP}(\xi_t)} \\ &= \frac{\cos\xi_t}{\alpha_t\left(1 + \delta_t\sin^2\xi_t\cos^2\xi_t + \varepsilon_t\cos^4\xi_t\right)}. \end{aligned} \tag{15}$$

Equation (15) can be rewritten as a quartic in $\cos\xi_t$ and solved for the phase angle, ξ_t;

$$\alpha_t x_0(\varepsilon_t - \delta_t)\cos^4\xi_t + \alpha_t x_0\delta_t\cos^2\xi_t - \cos\xi_t + \alpha_t x_0 = 0, \tag{16}$$

where physically acceptable values of cos ξ_t must be real and in the range [0, 1].

Having found the phase latitude in the medium of transmission, ξ_t, and knowing that the transmitted ray is normal to the slowness curve, we can calculate the ray angle of transmission,

θ_t, explicitly from equation (13). The above method was performed based on the weak-anisotropy assumption, given the properties of the media of incidence and transmission.

We can also proceed using a more general mathematical formulation. TIV media (i.e., 2-D velocity anisotropy) can be characterized by a slowness curve. Consequently, for such cases, the former approach is sufficient. For potentially more complicated media (i.e., exhibiting a 3-D velocity anisotropy), a more general method, involving slowness surfaces (as opposed to slowness curves) is necessary. To illustrate its use based on the TI-media case, let $g(r,\xi)$ be a function in slowness space defined by

$$g(r,\xi) = \frac{1}{r} - \alpha_t\left(\delta_t \sin^2\xi \cos^2\xi + \varepsilon_t \cos^4\xi\right), \tag{17}$$

where r is the radius of the slowness surface (i.e., the magnitude of the slowness). The slowness curve is given by the level surface of $g(r,\xi)$, i.e., by the set of points (r,ξ) for which

$$g(r,\xi) = \alpha_t. \tag{18}$$

The gradient can be expressed as

$$\nabla g(r,\xi) = \mathbf{r}\frac{\partial g}{\partial r} + \Xi \frac{1}{r}\frac{\partial g}{\partial \xi}, \tag{19}$$

where the angle is measured with respect to the x-axis, $\mathbf{r}$ is the radial unit vector, and Ξ is the azimuthal unit vector (i.e., perpendicular to the radius).

The propagation (ray, group) vector is always perpendicular to the slowness surface (i.e., its direction is parallel to the gradient of g). For g given by expression (17), the gradient can be written as follows:

$$\nabla g(r,\xi) = \mathbf{r}\left(-\frac{1}{r^2}\right) + \Xi\left(\frac{\alpha_t \sin(2\xi)\left[\delta_t \cos(2\xi) - 2\varepsilon_t \cos^2\xi\right]}{r}\right). \tag{20}$$

In polar coordinates, the Cartesian unit vector, $\mathbf{z}$, can be expressed as

$$\mathbf{z} = \mathbf{r}\sin\xi + \Xi\cos\xi. \tag{21}$$

This form is used in the desired dot product, i.e., equation (8). The transmitted group angle, θ_t, which the group slowness vector makes with the normal to the interface, can be expressed in terms of the given transmitted phase angle, ξ_t, calculated using equation (16), as

$$\cos\theta_t = \frac{\alpha_t \cos\xi_t \sin(2\xi_t)\left(\delta_t \cos(2\xi_t) - 2\varepsilon_t \cos^2\xi_t\right) - \dfrac{\sin\xi_t}{r(\xi_t)}}{\sqrt{\dfrac{1}{[r(\xi_t)]^2} + \alpha_t \sin(2\xi_t)\left[\delta_t \cos(2\xi_t) - 2\varepsilon_t \cos^2\xi_t\right]^2}}, \tag{22}$$

where,

$$r(\xi_t) = \frac{1}{\alpha_t\left(1 + \delta_t \sin^2\xi_t \cos^2\xi + \varepsilon_t \cos^4\xi_t\right)}. \tag{23}$$

Numerical example

Consider a planar horizontal interface between two anisotropic media. In the upper medium, the vertical wave speed is $\alpha_1 = 3000$ m/s, and the anisotropic parameters are $\varepsilon_1 = -0.2$ and $\delta_1 = 0.1$. In the lower medium, the vertical wave speed is $\alpha_2 = 4000$ m/s, and anisotropic parameters are $\varepsilon_2 = 0.15$ and $\delta_2 = -0.2$. The ray strikes the interface from above (Figure 3) at an incidence angle of $\theta_i = 30°$. The calculations (Appendix A) yield the results in Table I.

Notice that based on vertical wave speeds, as would be the case for α_1 and α_2 obtained from "zero-offset" vertical seismic profiles (VSP), an incidence angle of 30° would yield (by the standard form of Snell's law) a refraction angle of 41.81°. This is significantly different from the result of 64.01° obtained based on the weak-anisotropy assumption.

Particularly complicated phenomena can be observed in the case of converted waves (qP to qSV). For instance, at a flat horizontal interface, a ray can bend towards or away from the normal, depending on the angle of incidence (e.g., Slawinski, 1996). In particular cases, such bending leads to nonuniqueness of raypath, i.e., given a source and receiver, there are several stationary points of the traveltime function and consequently several ray trajectories.

There are several ways to confirm the correctness of the solution (i.e., to verify that the results obtained using an algorithm

Table 1. Group and phase angles and velocities of incidence and transmission.

	Incidence	Transmission
Group angles	30.00°	64.01°
Phase angles	35.57°	51.53°
Group velocities	3013 m/s	4133 m/s
Phase velocities	2998 m/s	4036 m/s

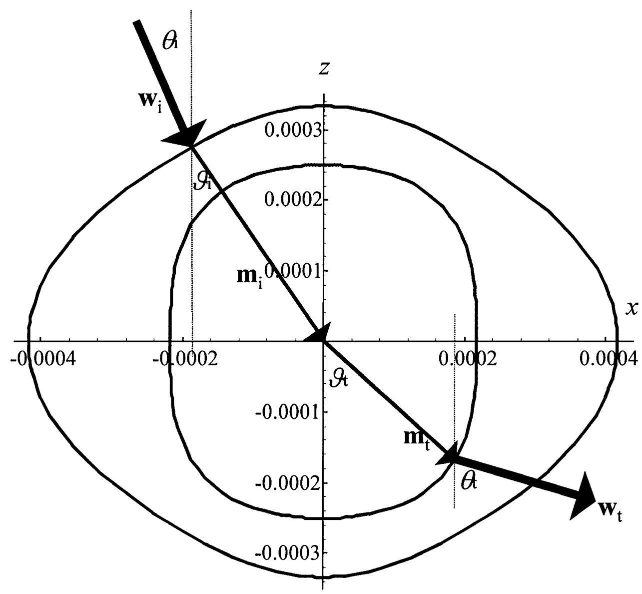

FIG. 3. Quasi-compressional (qP) wave slowness curves (i.e., cross-sections of corresponding slowness surfaces in a vertical plane). Parameters are $\alpha_1 = 3000$ m/s, $\alpha_2 = 4000$ m/s, $\varepsilon_1 = -0.2$, $\varepsilon_2 = 0.15$, $\delta_1 = 0.1$, $\delta_2 = -0.2$. The outer curve corresponds to the "slower" medium of incidence. The inner curve corresponds to the "faster" medium of transmission.

are in agreement with certain fundamental requirements). The fulfillment of those requirements constitutes necessary conditions for the validity of the method. First, the phase and group angles, ϑ, and θ, as well as the magnitudes of phase and group velocities, v and V, must satisfy the following equation in either medium:

$$\cos(\theta - \vartheta) = \left|\frac{v(\vartheta)}{V(\theta)}\right|. \tag{24}$$

This can be confirmed graphically with the help of Figure 4, from which it is clear that

$$\cos\zeta = v/V. \tag{25}$$

Second, the phase angles and phase velocities must satisfy the following equation across the interface:

$$\frac{\sin(\vartheta_1)}{v_1(\vartheta_1)} = \frac{\sin(\vartheta_2)}{v_2(\vartheta_2)}, \tag{26}$$

where the subscripts 1 and 2 correspond to the upper and lower media, respectively. Expression (26) is the standard form of Snell's law, which is always valid for phase angles and phase velocities. Third, Fermat's principle of stationary time must be satisfied (e.g., Helbig, 1994). This might not be obvious from a quick inspection, and it might require a reformulation in terms of the Euler-Lagrange equation (e.g., Slawinski and Webster, 1999).

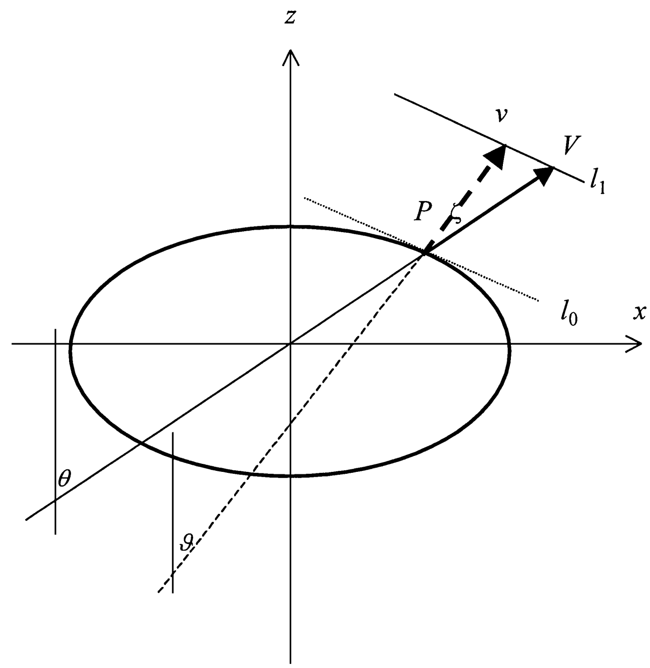

FIG. 4. The phase-velocity vector, v, is orthogonal to the wavefront. The group-velocity vector, V, in an anisotropic yet homogeneous medium is collinear with the ray under consideration. For an infinitesimal time increment, greatly exaggerated here (l_0 to l_1), V is the hypotenuse of a right triangle, and the enclosed angle ζ is equal to $\theta - \vartheta$.

The approach involving exact formulas

In principle, it is possible to carry out all the derivations described in this paper using exact equations. For qP-wave phase velocity, the form equivalent to equation (11) is (Thomsen, 1986)

$$F(r, \xi) = \frac{1}{r} - \alpha\sqrt{1 + \varepsilon\cos^2\xi + D(\xi)} = 0, \tag{27}$$

where $D(\xi)$ is given by

$$D(\xi) \equiv \frac{1}{2}\left(1 - \frac{\beta^2}{\alpha^2}\right) \times \left\{\sqrt{1 + \frac{4(2\delta - \varepsilon)\sin^2\xi\cos^2\xi}{1 - \frac{\beta^2}{\alpha^2}} + \frac{4\varepsilon\left(1 - \frac{\beta^2}{\alpha^2} + \varepsilon\right)\cos^4\xi}{\left(1 - \frac{\beta^2}{\alpha^2}\right)^2}} - 1\right\}. \tag{28}$$

DISCUSSION AND CONCLUSIONS

In this paper, we have illustrated the derivation and application of the general Snell's law using compressional (qP) waves. The same approach, however, can be applied in an analogous manner to both shear and converted waves (Slawinski, 1996). For SH-waves, whose velocity dependence can be described in terms of a single parameter, γ, one can quite conveniently use an exact expression without the weak-anisotropy approximation. Notably, the exact expression for a refraction law with elliptical velocity dependence (equivalent to SH-waves in TI media) was presented as a result of a VSP study in the Sahara desert (Dunoyer de Segonzac and Laherrere, 1959). Also, one can easily reduce expressions provided in this paper to the weak-anisotropy SH case by equating parameters ε and δ.

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APPENDIX A

MATHEMATICA CODE FOR *qP* WAVES

Mathematica software can be used to investigate ray bending at the interface between two (TI) anisotropic media. Given angle of incidence and parameters of the medium of incidence and the medium of transmission, it provides the angle of transmission, s. Note that Mathematica software is case-sensitive.

Qi = *angle of incidence in radians*
VJ = *vertical speed in the medium of incidence*
VD = *vertical speed in the medium of transmission*
EJ = *anisotropic parameter in the medium of incidence*, ε
DJ = *anisotropic parameter in the medium of incidence, ε*
ED = *anisotropic parameter in the medium of transmission*, δ
DD = *anisotropic parameter in the medium of transmission*, δ
R = Qi *(initial guess for the numerical solution)*

the next step expresses the slowness curve in the medium of incidence:

```
ri = 1/(VJ*(1+DJ*Sin[zi]^2*Cos[zi]^2+EJ*Cos[zi]^4))
```

the next step calculates the derivative w.r.t. phase angle in slowness curve in the medium of incidence:

```
dri = Sin[2*zi]*(EJ−DJ*Cos[2*zi]+EJ*Cos[2*zi])/
(VJ*(1+EJ*Cos[zi]^4+DJ*Cos[zi]^2*Sin[zi]^2)^2)
```

next steps calculate the ray parameter, x:

```
FindRoot[Cot[Qi]==
  (dri−ri*Tan[zi])/(dri*Tan[zi]+ri), {zi,R,−Pi,Pi}]
zif = Abs[zi/.%]
x = N[Abs[Cos[zif]/
  (VJ*(1+DJ*Sin[zif]^2*Cos[zif]^2+EJ*Cos[zif]^4))]]
```

the next step calculates the phase latitude in the medium of transmission, z:

```
FindRoot[VD*x*(ED−DD)*C^4+VD*x*DD*C^2−
  C+VD*x == 0, {C, 0.5, 0, 1.5}]
z = ArcCos[C/.%]
dz = 2*z
r = 1/(VD*(1+DD*Sin[z]^2*Cos[z]^2+ED*Cos[z]^4))
t = VD*Cos[z]*Sin[dz]*(Abs[DD*Cos[dz]−2*ED*Cos[z]^2])
 −Sin[z]/r
b = Sqrt[1/r^2+(VD*Sin[dz]*(DD*Cos[dz]
  −2*ED*Cos[z]^2))^2]
```

the next step calculates the angle of transmission, s

```
s = ArcCos[Abs[t/b]]
```