Algorithms, Probability and Computing: Special Assignment 2

Due 6 December 2016

Kevin Klein

Exercise 1

(a) For the sake of contradiction, let's assume $\tau(F) < \mu(F)$.
Say $\tau(F) = min\ |T| = k$. Sizes of sets must be integers, hence $k \in \mathbb{N}$. It follows directly that $\mu(F) = max\ |M| \geq k+1$. The latter expresses that $k+1$ flags can be simultaneously represented without assigning a dancer to more than one flag. These $k+1$ flags of the Olympic matching are by definition pairwise dstinct, meaning that no colour is present in more than one of those flags.
By definition, an Olympic traversal has to pass all flags and thereby also the $k+1$ flags in the Olympic matching. However, as those flags are pairwise distinct, the necessary amount of colours to traverse the matching is $k+1$. This number can only increase when adding colours to traverse flags which are not part of the Olympic matching. Hence $\tau(F) \geq k+1$.
This contradicts are assumption. Therefore we have proven that $\mu(F) \leq \tau(F)$.

(b) Firstly, we need to show that such a $\tau^*(F)$ exists. In order to do so we demonstrate that LP-T is bounded and feasible, which implies existance of an optimal solution.
The cost function increases with every component of x. This is to say that it will attain minimal value if the components of x are 'as small as possible'. However, due to the constraint that $x \geq 0$, we know that the components of x must be at least 0. Thus, we can formulate the lower bound $1_n^T 0_n = 0$ for the minimized result. It follows that LP-T is bounded.
In order to show that LP-T is feasible, we will look at the structure of A. The matrix solely consists of $0's$ and $1's$. Using the information that each flag comprises at least one colour, we can tell that every row holds at least one 1. In order to satisfy the constraint that the dotproduct of each row with x is larger or equal 1, we simply set all components of x to 1. Furthermore, $x = 1_n$ also trivially satisfies the remaining constraint, namely $x \leq 0$. Thererfore x witnesses the feasibility of LP-T.
We now know that LP-T is bounded, feasible and thereby has an optimal solution $\tau^*(F)$.
When closeely observing the definition of the traversal T and $\tau(F)$, it becomes apparent, that we can express it as an integer linear program with strong resemblance to LP-T. Instead of declaring T as a set, we can express T equivalently as the vector x, indicating $x_i = \mathbb{I}[c_i \in T]$. It follows directly that $\tau(F) = |T| = \sum_{i=1}^n x_i = 1_n^T x$. The constraint that the traversal x must include at least one colour of each flag, can be enforced by counting the number of colours each flag and the traversal have in common and making sure this count is always at least 1. This corresponds to the dotproduct of a row of A, representing the colours in a flag, and the traversal x. We can thereby formulate $\tau(F)$ as the optimal value resulting from:

$$minimize\ 1_n^T x,\ subject\ to\ Ax \geq 1_m,\ x \in \{0,1\}^n$$

We can argue analogously that this integer linear program has an optimal value.
The integer LP we came to formulate is only more restrictive than LP-T. Hence, a traversal x leading to the optimal value for the integer linear program, will yield the same value for LP-T. Yet, LP-T might allow for an even better solution, as we will show in c). We have proven that $\tau^*(F) \leq \tau(F)$.

(c) Consider the flags $f_1 = \{c1, c2\},\ f_2 = \{c1, c3\},\ f_3 = \{c2, c3\}$.
In other words:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

We see that no colour is contained in all three flags. Hence the traversal will require at least two colours. We write $\tau(F) \geq 2$. Let us inspect $x_s = \begin{bmatrix} 0.5 & 0.5 & 0.5 \end{bmatrix}^T$. Firstly, our x_s satisfies $x \geq 0_n$. Secondly

$$A \cdot x_s = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \geq 1_m.$$

Thereby, we can conclude that x_s is a feasible solution for LP-T. Furthermore, the optimal value $\tau^*(F)$ must be smaller or equal to the value that the cost function yields for x_s. This follows from the simple fact that the cost function is minimizing. Hence $\tau^*(F) \leq 1_n^T \cdot x_s = 1.5$.
It follows directly that $\tau^*(F) < \tau(F)$.

(d) In order to prove $\mu(F) \leq \tau^*(F)$ we will first show $\mu(F) \leq \mu^*(F)$ followed by $\mu^*(F) \leq \tau^*(F)$.
We define $\mu^*(F)$ to be the optimal value of the following linear program LP-D:

$$maximize\ 1_m^T y\ subject\ to\ A^T y \leq 1,\ y \geq 0$$

We observe that LP-D corresponds to the definition of the dual of LP-T. As we have shown that LP-T is bounded and has optimal value $\tau^*(F)$, the Strong Duality Theorem tells us that LP-D has optimal value $\mu^*(F) = \tau^*(F)$.
Similarly as in (b), we will translate the meaning of an Olympic matching M to an integer linear program and show that this integer linear program is only more constrained than LP-D.
Instead of declaring M as a set, we can express M as the vector y, indicating $y_i = \mathbb{I}[f_i \in M]$. It follows directly that $\mu(F) = |M| = \sum_{i=1}^m y_i = 1_m^T y$. The constraint that the matching y must assign each colour to at most one flag, i.e. the flags are pairwise distinct, can be enforced by counting the number of flags each colour appears in and making sure this count is always at most 1. This corresponds to the dotproduct between a row of A^T, representing for all flags whether they contain a specific colour, and the matching y. We can thereby formulate $\mu(F)$ as the optimal value resulting from:

$$maximize\ 1_m^T y\ subject\ to\ A^T y \leq 1,\ y \in \{0,1\}^m$$

$y = 1_m$ yields an upper lower bound for $\mu(F)$, namely $1_m^T 1_m = m$. $y = 0_m$ yields an obvious feasible solution. Hence the integer linear program has an optimal solution $\mu(F)$.
The integer LP we came to formulate is only more restrictive than LP-D. Hence, a matching y leading to the optimal value for the integer linear program, will yield the same value for LP-D. Yet, LP-D might allow for an even better, i.e. larger, solution, as it is less restrictive and maximizing the cost function. We have proven that $\mu(F) \leq \mu^*(F)$.

We have proven that $\mu(F) \leq \tau^*(F)$.

(e) Let's inspect the probablity to intersect a specific flag when drawing a colour according to p. A flag is intersected when drawing any of the colours containted in the flag. Using p, this gives us:

$$\Pr[f_i] = \frac{\sum_{j=1}^n \mathbb{I}[c_j \in f_i] \cdot x_j^*}{\sum_{j=1}^n x_j^*} = \frac{A_i x^*}{\sum_{j=1}^n x_j^*} = \frac{A_i x^*}{\tau^*(F)}$$

The probability of not intersecting a certain flag upon drawing a colour is hence:

$$1 - \Pr[f_i] = 1 - \frac{A_i x^*}{\tau^*(F)}$$

The probability of not intersecting a certain flag upon drawing s colours, possibly with repetitions, is hence:

$$(1 - \Pr[f_i])^s = (1 - \frac{A_i x^*}{\tau^*(F)})^s$$

After such handy clarifications, let's look at what we actually want to show.

$$\begin{aligned}
\mathbb{E}[\#flags\ not\ intersected\ by\ s\ draws] &= \mathbb{E}[\sum_{i=1}^{m} \mathbb{I}[f_i\ not\ intersected\ by\ s\ draws]] && (1)\\
&= \sum_{i=1}^{m} \mathbb{E}[\mathbb{I}[f_i\ not\ intersected\ by\ s\ draws]] \quad (linearity\ of\ expectation) && (2)\\
&= \sum_{i=1}^{m} \Pr[f_i\ not\ intersected\ by\ s\ draws] && (3)\\
&= \sum_{i=1}^{m} (1 - \frac{A_i x^*}{\tau^*(F)})^s && (4)\\
&\leq \sum_{i=1}^{m} (1 - \frac{1}{\tau^*(F)})^s \quad (Ax^* \geq 1_m) && (5)\\
&\leq m \cdot (1 - \frac{1}{\tau^*(F)})^s && (6)\\
& && (7)
\end{aligned}$$

We now have a term to the power of s. We can apply $1 + x \leq e^x$ for the term.

$$\begin{aligned}
\mathbb{E}[\#flags\ not\ intersected\ by\ s\ draws] &\leq m \cdot (e^{-\frac{1}{\tau^*(F)}})^s && (8)\\
&\leq m \cdot e^{-\frac{s}{\tau^*(F)}} && (9)
\end{aligned}$$

We will investigate wthe necessary condition for this expression to be less or equal 1.

$$\begin{aligned}
& m \cdot e^{-\frac{s}{\tau^*(F)}} \leq 1 && (10)\\
\Leftrightarrow\ & e^{-\frac{s}{\tau^*(F)}} \leq \frac{1}{m} && (11)\\
\Leftrightarrow\ & -\frac{s}{\tau^*(F)} \leq ln(\frac{1}{m}) && (12)\\
\Leftrightarrow\ & -s \leq \tau^*(F) \cdot ln(\frac{1}{m}) \qquad (tau^*(F) \geq 1) && (13)\\
\Leftrightarrow\ & s \geq \tau^*(F) \cdot ln(m) && (14)
\end{aligned}$$

In other words, we know that for any s larger or equal $\lceil \tau^*(F) \cdot ln(m) \rceil$, the expected number of not intersected flags is smaller or equal 1. Therfore we know that we can traverse $m - 1$ flags with $\lceil \tau^*(F) \cdot ln(m) \rceil$ many colours. Hence, with $\lceil \tau^*(F) \cdot ln(m) \rceil + 1$ colours, we are able to traverse all flags by adding one colour from the remaining flag. We know that an Olympic traversal must be possible with $\lceil \tau^*(F) \cdot ln(m) \rceil + 1$ colours, yet this could be enhanced.
We know that we can possibly not intersect $n - 1$ with s draws by drawing s timess the same colour. In other words, the probability of not intersecting > 1 flags is not zero for any s. Hence for any s having expected ≤ 1, there must also be ways of not intersecting 0 flags.

$$\begin{aligned}
\mathbb{E}[\#flags\ not\ intersected\ by\ s\ draws] &= \sum_{i=0}^{n-1} i \cdot \Pr[\#flags\ not\ intersected\ by\ s\ draws = i] && (15)\\
&= 0 \cdot p_0 + \sum_{i=1}^{n-1} i \cdot p_i && (16)\\
&\geq 0 \cdot p_0 + (1 + \epsilon)(1 - p_0) \quad for\ some\ \epsilon > 0 && (17)\\
&\geq (1 + \epsilon)(1 - p_0) && (18)
\end{aligned}$$

We have:

$$(1+\epsilon)(1-p_0) \leq \mathbb{E}[\#flags\ not\ intersected\ by\ s\ draws] \leq 1$$

Trivially, p_0 cannot be 0. This means that there is a way to intersect all flags with $s = \lceil \tau^*(F) \cdot ln(m) \rceil$ colours. Furthermore, we have $s = \lceil \tau^*(F) \cdot ln(m) \rceil \leq \tau^*(F) \cdot ln(m) + 1$. As $\tau(F)$ is defined to be the minimal amount of colours to intersect all flags, we have:

$$\tau(F) \leq \lceil \tau^*(F) \cdot ln(m) \rceil \leq \tau^*(F) \cdot ln(m) + 1$$

.

(f) (i) We want to show that the columns of A are linearly independent.
Thanks to the special structure of A, we know that each row contains exactly two 1's and all other components equal 0.
If $m < n$, we know that $rank(A) \leq dim(rowspace(A)) \leq m$. We also know that the number of linearly independent colums equal $rank(A)$. Hence $\#lin.\ ind.\ colums \leq m < n$. Therefore the columns are linearly dependent.
If $m \geq n$, we proceed by contradiction. Let's assume that A has n linear independent columns and hence $rank(A) = n$. The 'rank-nullity theorem' tells us that $n = rank(A) + dim(kernel(A))$. Therefore the dimension of the kernel must be 0 and $Av = 0$ can only hold for $v = 0$.
Say there are k light colours and hence $n-k$ dark colours. We construct the vector:

$$v_0 \in \{0,1\}^n,\ v_0[i] = \begin{cases} 1 & if\ c_i\ bright \\ -1 & otherwise \end{cases}$$

As we know that each row of A contains a 1 for exactly one bright and one dark colour and all other elements 0, all of the multiplications between rows and v_0 will return 0. In other words, $v_0 \neq 0,\ Av_0 = 0$. Therefore the dimension of A's kernel must be at least 1. This contradicts our assumption that A has rank n. Thereby, A cannot have n linearly independent columns. The columns of A are linearly dependent in both cases.

(ii) In order to show that for each square submatrix S of A, $det|S| \in \{-1,0,1\}$ holds, we will proceed by induction. We will say that S is of size i by i.
Base case: i = 1

$$\forall i \in \{1,..,m\},\ \forall j \in \{1,..,n\}\ A[i,j] \in \{0,1\}\ \Rightarrow det|S| \in \{0,1\}$$

Induction step: $i \rightarrow i+1$
In order to make use of our induction hypothesis, we want to compute $det|S|$, with S of size $i+1$ by $i+1$, by its minors, of size i by i. Thanks to the fact that S is a submatrix of A, we can assume that each row of S has either no, one or two 1's. Furthermore we know that we can pick any row to calculate $det|S|$ by the row's corresponding minors. We make the following case distinction:

a) There is one row in S with all 0's.
$det|S|$ will be calculated by adding the determinant of all minors of this row, scaled by the corrsponding elements of the row. As the latter are all 0, it follows immediately that $det|S| = 0$.

b) There is no row in S with all 0's, there is at least one row in S with exactly one 1.
$det|S|$ will be calculated by adding the determinant of all minors of such a row, scaled by the corrsponding elements of the row. Hence all minors but one, call it M, can be ignored, because the coefficient of the determinant of the minor will be 0 for all but for M. Our induction hypotehsis tells us that $det|M| \in \{-1,0,1\}$ because M is of size i by i and by definition a submatrix of A. Hence $det|S| = (-1)^k \cdot 1 \cdot det|M| \in \{-1,0,1\}$ where $k \in \{1,..,n\}$ is the index of the coefficient in the selected row.

c) There are only rows in S with exactly two 1's.
We have two crucial indications about this S. The first one is that it is a submatrix of A. This implies that it has to fulfill the requirement that if two colours are chosen in one row, one of them must be light and the orther dark. We do not reassign colours, hence this must still hold. The second one is that we have two colours in each row of S.
Hence S represents a legitimate incidence matrix according to the definition of incidence matrices in (f).
Yet we came to prove in (f)(i), that all incidence matrices as described in (f) do have linearly dependent columns. In the case of square matrices, this implies that they are not 'full-rank' or 'invertible'. This again implies that the determinant must be 0. We can conclude $det|S| = 0$.

The case distinction is exhaustive and leads to the desired result for each case. Hence we have proven the statement.

(iii) In order to prove that every basic feasible solution is integral, we first look at the definition of a basic feasible solution. Any basic feasbile solution must satisfy n linearly independent constraints with equality. We say that a basic feasible solution $\tilde{x}$ solves $\tilde{A}x = \tilde{b}$ where $\tilde{A}$ and $\tilde{b}$ represent n linearly independent constraints from either $Ax \geq 1_m$ or $x \geq 0$.
Cramer's rule tells us that the j-th component of our basic feasible solution can be computed by:

$$\tilde{x_j} = \frac{det|\tilde{A_j}|}{det|\tilde{A}|}, \qquad j \in \{1, .., n\}$$

where $\tilde{A_j}$ corresponds to $\tilde{A}$ and replacing its jth column by $\tilde{b}$. In order to argue that all $\tilde{x_j}$ are integers, we will look at both numerator and denominator of the fraction.

i. $det|\tilde{A_j}|$
$\tilde{A}$ contains only 0's and 1's. So does $\tilde{b}$. Hence $\tilde{A_j}$ only contains 0's and 1's. When calculating the determinant we will only add, multiply and adapt the sign of elements. Adding, multiplying and changing signs of integers will result in an integer. Therefore $det|\tilde{A_j}| \in \mathbb{Z}$.

ii. $det|\tilde{A}|$
We know that the rows originating from the constraint $x \geq 0$ contain exaclty one 1. For all those rows, we recursively compute minors, which always consist of the determinant of a minor multiplied by either 1 or -1. When we run out of those rows, we only have rows left from the constraint $Ax \geq 1_m$. Thusly, those remaining rows stem from A. In other words, the matrix of remaining rows is a submatrix of A. As we have shown in (ii), this submatrix yields a determinant of either -1, 0 or 1. Therefore $det|\tilde{A}|$ is of the form $(-1)^k \cdot det|M|,\ det|M \in \{-1, 0, 1\},\ k \in \mathbb{Z}$.
As $\tilde{A}$ contains n linearly independent rows, we know that its determinant cannot equal 0. It follows that $det|\tilde{A}| \in \{-1, 1\}$.

We come to the conclusion that $\forall j \in \{1, .., n\},\ \tilde{x_j} = det|\tilde{A_j}|\ or\ \tilde{x_j} = -det|\tilde{A_j}|$. As $det|\tilde{A_j}| \in \mathbb{Z},\ x_j \in \mathbb{Z}$. Hence x is integral.

(iv) We know from (b) that $\tau^*(F)$ is the optimal value of the feasible and bounded linear program LP-T.
Knowledge from the lecture allows us to deduce that a basic feasible solution can attain this optimal value $\tau^*(F)$. We have shown in iii), that all basic feasible solutions are integral.
Combining this knowledge, we know that there is an integral optimal basic feasible solution.
We have argued in b) that $\tau(F)$ is the optimal value of an integer linear program, which is only more restrive than LP-T. Precisely, it only allows components of the solution to be either 0 or 1. As we already know that we can reach the optimal value of LP-T with an integral solution, it remains to show that such an optimal basic solution does not contain integers other than 0 and 1, in orde

prove that the optimal solution for the integer LP is an optimal solution for LP-T.
We will prove this via contradiction.
Let's assume that the optimal basic feasible solution x of LP-T has a component j' of value $k \notin \{0,1\}$. As $x \geq 0$ holds for all feasible solutions, k cannot be negative. To be explicit, $k \in \{2,3,..\} = \mathbb{N} \setminus \{0,1\}$. Let's look at the constraints $Ax \geq 1$. For each row i in A in which the j'th component is 0, the value of $x_{j'}$ does not matter. For all others, the constraint corresponds to the following:

$$A_i x \geq 1 \Leftrightarrow \sum_{j=1}^{m} A[i][j] * x[j] = 1 \cdot k + \sum_{j=1, j \neq j'}^{m} A[i][j] * x[j] \geq 1$$

We notice that all constraints of the latter form, we could as well replace k by 1 and still satisfy the constraints. For all other constraints ignore the j'th component, an adaption of k would not change their satifaction. No other component would have to be adapted due to a change $k = 1$. The cost function is simply adding up the components of x, hence $k = 1$ gives us a better result. Our assumption of having an optimal solution is contradicted.g
We now know that we can add the constraint $x \in \{0,1\}^n$ to LP-T without influencing the optimal value $\tau^*(F)$. Yet this corresponds exactly to the integer Linear Program with optimal value $\tau(F)$, as formulated in (b). Hence $\tau^*(F) = \tau(F)$.

(v) Firstly, we turn our problems of finding M and T into graph problems. Let the colours be represented as the set of nodes V. As a flag consists of precisely two colours, we will represent the flags as the set of edges E. Furthermore, we are aware of the fact that no two light and no two dark colours, i.e. nodes, will be contained in an edge, i.e. not connected by an edge. This corresponds to the definition of a bipartite graph with disjoint sets representing light and dark colours.
The problem of finding an Olympic matching M, namely selecting a maximal number of flags, while never using a colour twice translates to finding a maximum matching of our graph. This is the case, because the maximum matching corresponds a maximum number of edges, representing flags, with no two edges being adjacent, i.e. no colour being part of two flags.
The problem of finding an Olympic traversal T, namely selecting a minimal number of colours, such that all flags contain at least one of those colours translates to finding the vertex cover of our graph. This is the case, because in the vertex cover, we select a minimum number of nodes, representing colours, such that all edges are connecting one of the selected nodes, i.e. each flag is repreented by at least one colour.
As we have shown existence of the existence of optimal solutions to integer linear programs in (b) and (d), $\mu(F)$ and $\tau(F)$ represent the sizes of a maximum matching and a minimum vertex cover respectively and exist. Fortunately, graph theory's fundamental Koenig's theorem tells us that the size of the maximum matching of a bipartite graph equal the size of a minimum vertex cover. This directly implies $\mu(F) = \tau(F)$.