Algorithms, Probability and Computing: Special Assignment 1

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Exercise 1

Avoiding points.

Exercise 2

Exercise 3

(a) $a_0 = 7, a_1 = 1 + 2 * a_0 = 15$
$\forall n \geq 2:$

$$
\begin{aligned}
a_n &= 1 + 2 * \sum_{i=1}^{n} a_{i-1} && (1)\\
a_{n-1} &= 1 + 2 * \sum_{i=1}^{n} a_{i-1} && (2)\\
\Rightarrow a_n - a_{n-1} &= 1 + 2 * \sum_{i=1}^{n} a_{i-1} - (1 + 2 * \sum_{i=1}^{n} a_{i-1}) = 2 * a_{n-1} && (3)\\
\Rightarrow a_n &= 3 * a_{n-1} && (4)\\
&= \dots && (5)\\
&= 3^{n-1} * a_1 && (6)\\
&= 3^{n-1} * 15 && (7)\\
& && (8)
\end{aligned}
$$

(b) $L_n := \mathit{length\ of\ a\ central\ path\ in\ a\ BST\ of\ size\ n\ nodes}$
$l_n := \mathbb{E}(L_n)$
$l_0 = 0, l_1 = 1, l_2 = 1/2 \cdot 2 + 1/2 \cdot 1 = 3/2$
$\forall n \geq 3:$

$$
\begin{aligned}
l_n &= \sum_{i=1}^{n} (\mathbb{E}[L_n | root = i] \cdot \Pr[root = i]) && (9)\\
&= \sum_{i=1}^{n} (\mathbb{E}[L_n | root = i] \cdot 1/n) && (10)\\
&= 1/n \cdot \sum_{i=1}^{n} (\mathbb{E}[L_n | root = i]) && (11)\\
&= 1/n \cdot \sum_{i=1}^{n} (\mathbb{E}[L_{i-1}] + 1) && (12)\\
&= 1 + 1/n \cdot \sum_{i=1}^{n} (l_{i-1}) && (13)\\
& && (14)
\end{aligned}
$$

Furthermore, $l_{n-1} = 1/n \cdot \sum_{i=1}^{n-1}(l_{i-1})$

$$n \cdot l_n - (n-1) \cdot l_{n-1} = n + \sum_{i=1}^{n}(l_{i-1}) - ((n-1) + \sum_{i=1}^{n-1}(l_{i-1}) \quad (15)$$
$$= 1 + l_{n-1} \quad (16)$$
$$\Rightarrow n \cdot l_n = 1 + n \cdot l_{n-1} \quad (17)$$
$$\Rightarrow l_n = 1/n + l_{n-1} \quad (18)$$
$$= 1/n + 1/(n-1) + \cdots + l_3 + l_2 \quad (19)$$
$$= 1/n + 1/(n-1) + \cdots + l_3 + 3/2 \quad (20)$$
$$= 1/n + 1/(n-1) + \cdots + l_3 + 1/2 + 1 \quad (21)$$
$$= H_n \quad (22)$$
$$(23)$$

We see that this also holds for $n = 2$ and $n = 1$ and therefore: $\forall n \in \mathbb{N} - \{0\}$.

(c) $P_0^{(1)} = P_1^{(1)} = P_2^{(1)} = 0, P_3^{(1)} = 1/3, P_4^{(1)} = 1/4 \cdot 1/3 = 1/12$
$\forall n \geq 5:$

$$P_n^{(1)} = \sum_{i=1}^{n}((P_n^{(1)}|root = i) \cdot \Pr[root = i]) \quad (24)$$
$$= \sum_{i=1}^{n}((P_n^{(1)}|root = i) \cdot 1/n \quad (25)$$
$$= 1/n \cdot \sum_{i=1}^{n}((P_n^{(1)}|root = i) \quad (26)$$
$$= 1/n \cdot \sum_{i=1}^{n} P_{i-1}^{(1)} \quad (27)$$
$$(28)$$

Furthermore, $P_{n-1}^{(1)} = 1/(n-1) \cdot \sum_{i=1}^{n-1}((P_{i-1}^{(1)}|root = i)$

$$\Rightarrow n \cdot P_n^{(1)} - (n-1) \cdot P_{n-1}^{(1)} = \sum_{i=1}^{n} P_{i-1}^{(1)} - (\sum_{i=1}^{n-1} P_{i-1}^{(1)}) \quad (29)$$
$$= P_{n-1}^{(1)} \quad (30)$$
$$\Rightarrow n \cdot P_n^{(1)} = n \cdot P_{n-1}^{(1)} \quad (31)$$
$$\Rightarrow P_n^{(1)} = P_{n-1}^{(1)} \quad (32)$$
$$= P_{n-2}^{(1)} = \cdots = P_4^{(1)} \quad (33)$$
$$= 1/12 \quad (34)$$

We see that this also holds for $n = 4$ and therefore: $\forall n \in \mathbb{N}, n \geq 4$.

(d)

Exercise 4