# Algorithms, Probability and Computing: Special Assignment 1

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# Exercise 1

For this exercise, we assume that the relevant random binary search tree holds the set of elements $S$ of size $n$. Without loss of generality, we will assume that $S = \{1, n\}$ and in particular $i \in S \Rightarrow rank(i) = i$. Let's us the following recursive notation to express the structure of a binary search tree: $(node)(leftSubtree)(rightSubtree)$.

a) By direct application of definitions:
$\mathbb{E}[Z_2^{(1)}] = \frac{1}{2} \cdot \mathbb{E}[Z_2^{(1)}|root = 1] + \frac{1}{2} \cdot \mathbb{E}[Z_2^{(1)}|root = 2] = \frac{1}{2} + \frac{1}{2} = 1$
$\mathbb{E}[Z_2] = \mathbb{E}[\frac{1}{2-1} \cdot \sum_{i=1}^{2-1}(Z_2^{(i)})] = \mathbb{E}[Z_2^1] = 1$

$$\begin{aligned}
\mathbb{E}[Z_3^{(1)}] &= \frac{1}{6} \cdot \mathbb{E}[Z_3^{(1)}|(1)(2)(3)] + \frac{1}{6} \cdot \mathbb{E}[Z_3^{(1)}|(1)(3)(2)] + \frac{1}{3} \cdot \mathbb{E}[Z_3^{(1)}|(2)(1)(3)] && (1)\\
&+ \frac{1}{6} \cdot \mathbb{E}[Z_3^{(1)}|(3)(1)(2)] + \frac{1}{6} \cdot \mathbb{E}[Z_3^{(1)}|(3)(2)(1)] && (2)\\
&= \frac{1}{2} \cdot (1 + 2 + 2 + 1 + 1) = \frac{7}{6} && (3)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[Z_3] &= \mathbb{E}[\frac{1}{3-1} \cdot \sum_{i=1}^{3-1} Z_3^{(i)}] = \frac{1}{2} \cdot \mathbb{E}[Z_3^{(1)}] + \frac{1}{2} \cdot \mathbb{E}[Z_3^{(2)}] && (4)\\
&= \frac{1}{2} \cdot (\frac{7}{6} + \frac{1}{6} \cdot \mathbb{E}[Z_3^{(2)}|(1)(2)(3)] + \frac{1}{6} \cdot \mathbb{E}[Z_3^{(2)}|(1)(3)(2)] + \frac{1}{3} \cdot \mathbb{E}[Z_3^{(2)}|(2)(1)(3)] && (5)\\
&+ \frac{1}{6} \cdot \mathbb{E}[Z_3^{(2)}|(3)(1)(2)] + \frac{1}{6} \cdot \mathbb{E}[Z_3^{(1)}|(3)(2)(1)]) && (6)\\
&= \frac{1}{2} \cdot (\frac{7}{6} + \frac{7}{6}) && (7)\\
&= \frac{7}{6} && (8)
\end{aligned}$$

b) We learnt from a) that $\mathbb{E}[Z_2^{(1)}] = 1$ and $\mathbb{E}[Z_3^{(1)}] = \frac{7}{6}$.
For $n \geq 4$:

$$\begin{aligned}
\mathbb{E}[Z_n^{(1)}] &= \sum_{i=1}^{n} (\Pr(root = i) \cdot \mathbb{E}[Z_n^{(1)}|root = i]) && (9)\\
&= \frac{1}{n} \cdot (\sum_{i=1}^{n} \mathbb{E}[Z_n^{(1)}|root = i])) && (10)\\
&= \frac{1}{n} \cdot (\mathbb{E}[Z_n^{(1)}|root = 1] + \mathbb{E}[Z_n^{(1)}|root = 2] + \sum_{i=3}^{n} \mathbb{E}[Z_n^{(1)}|root = i])) && (11)
\end{aligned}$$

If 1 is the root, we know that 2 is in the right subtree of the root. Furthermore we can observe that the distance between 1 and 2 corresponds to the depth of 2 in the right subtree incremented by one. If 2 is the root, we know that 1 is in its left subtree. Furthermore we know that $S = 1, n$, which implies that $S \cap (1, 2) = \emptyset$. Hence 1 is the only node in the left subtree of 2. Therefore there distance is equal to 1. In all other cases, 1 and 2 are in the left subtree of the root and there distance is equal to their distance in the left subtree of the root as the root cannot be on the path between 1 and 2 as it is larger than both.

Moreover we learnt that $D_n^{(i)} = H_i - H_{n-i+1} - 2$ [0].

$$\begin{aligned}
\mathbb{E}[Z_n^{(1)}] &= \frac{1}{n} \cdot (D_{n-1}^{(1)} + 1 + 1 + \sum_{i=3}^{n} \mathbb{E}[Z_{i-1}^{(1)}]) && (12)\\
&= \frac{1}{n} \cdot (H_1 + H_{n-1-1+1} + \sum_{i=3}^{n} \mathbb{E}[Z_{i-1}^{(1)}]) && (13)\\
&= \frac{1}{n} \cdot (H_1 + H_{n-1} + \sum_{i=3}^{n} \mathbb{E}[Z_{i-1}^{(1)}]) && (14)\\
n \cdot \mathbb{E}[Z_n^{(1)}] - (n-1) \cdot \mathbb{E}[Z_{n-1}^{(1)}] &= H_{n-1} - H_{n-2} + \mathbb{E}[Z_{n-1}^{(1)}] && (15)\\
\Leftrightarrow n \cdot \mathbb{E}[Z_n^{(1)}] &= n \cdot \mathbb{E}[Z_{n-1}^{(1)}] + \frac{1}{n-1} && (16)\\
\Leftrightarrow \mathbb{E}[Z_n^{(1)}] &= \mathbb{E}[Z_{n-1}^{(1)}] + \frac{1}{n(n-1)} && (17)\\
&= \frac{1}{n(n-1)} + \frac{1}{(n-1)(n-2)} + ... + \frac{1}{4 \cdot 3} + \mathbb{E}[Z_3^{(1)}] && (18)\\
&= \sum_{i=4}^{n} \frac{1}{i(i-1)} + \mathbb{E}[Z_3^{(1)}] && (19)\\
&= \sum_{i=4}^{n} \frac{i-(i-1)}{i(i-1)} + \mathbb{E}[Z_3^{(1)}] && (20)\\
&= \sum_{i=4}^{n} (\frac{1}{i-1} - \frac{1}{i}) + \mathbb{E}[Z_3^{(1)}] && (21)\\
&= \sum_{i=4}^{n} \frac{1)}{i-1} - \sum_{i=4}^{n} \frac{1}{i} + \mathbb{E}[Z_3^{(1)}] && (22)\\
&= \sum_{i=3}^{n-1} \frac{1}{i} - \sum_{i=4}^{n} \frac{1}{i} + \mathbb{E}[Z_3^{(1)}] && (23)\\
&= \frac{1}{3} - \frac{1}{n} + \mathbb{E}[Z_3^{(1)}] && (24)\\
&= \frac{1}{3} - \frac{1}{n} + \frac{7}{6} && (25)\\
&= \frac{3}{2} - \frac{1}{n} && (26)
\end{aligned}$$

We see that $\frac{3}{2} - \frac{1}{2} = 1$ and $\frac{3}{2} - \frac{1}{3} = \frac{7}{6}$, therefore we can conclude that

$$\forall n \geq 2 : \mathbb{E}[Z_n^{(1)}] = \frac{3}{2} - \frac{1}{n}$$

c)

d)

[0] APC script 2015 Edition, Chapter 1.4 Expected Depth of Individual Keys, p.18 [0] APC script, Chapter 2.1 Point/Line Relative to a Convex Polygon, paragraph 'Inside/On/Outside a Cnvex Polygon', p. 39 [1] APC script, Chapter 2.1 Point/Line Relative to a Convex Polygon, paragraph 'A Line Hitting a Convex Polygon', p. 40 [2] APC script, Chapter 2.3 Planar Point Location - More exmaples, paragrahp 'Closest Point in the Plane - the Post offce Problem', p.55

# Exercise 2

a) Analogously to the proof of Lemma 1.5 [0], we will proceed with an exhaustive case distinction in order to prove that

$$\mathbb{E}[A_i^j] = \frac{p(j)}{\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l)}$$

(i) $i = j$

$$\begin{aligned}
\mathbb{E}[A_i^{(j)}] &= E[A_j^{(j)}] && (27)\\
&= 1 \text{ (definition of ancestor indicator variable)} && (28)\\
&= \frac{p(j)}{p(j)} \text{ } (p(j) > 0 \text{ is given}) && (29)\\
&= \frac{p(j)}{\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l)} \text{ (as } i = j) && (30)\\
& && (31)
\end{aligned}$$

(ii) $i = 1$ and $j = n$

Note that $i$ and $j$ will always end up in different subtrees of the root, except for the cases in which either of both is the root. Hence, if none of them is the root, the indicator variable is equal to 0. If $i$ is the root, $j$ is a child of $i$ and $j$ is different from $i$ because $n \geq 2$. Therefore $j$ cannot be an ancestor of $i$ if $i$ is the root. Consequently, the only case is which $j$ is an ancestor of $i$ is when $j$ is the root of the tree.

$$\begin{aligned}
\mathbb{E}[A_i^{(j)}] &= \Pr(root = j) && (32)\\
&= \frac{p(j)}{\sum_{l=1}^{n} p(l)} \text{ (definition of the weighted binary tree)} && (33)\\
&= \frac{p(j)}{\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l)} \text{ (as } i = 1 \text{ and } j = n) && (34)\\
& && (35)
\end{aligned}$$

(iii) $j = 1$ and $i = n$

This case is symmetric to case (ii).

(iv) $i \neq j$ and $n > j - i + 1$ and $root \in \{min\{i,j\}, max\{i,j\}\}$

If $i$ is the root, $j$ cannot be an ancestor of as we know that $i \neq j$. If $root \in \{min\{i,j\}+1, max\{i,j\}-1\}$, we know that i and j will always end up in different subtrees of the root. Hence, if none of them is the root, the indicator variable is equal to 0. Consequently, the only case is which $j$ is an ancestor of $i$ is when $j$ is the root of the tree.

$$\begin{aligned}
\mathbb{E}[A_i^{(j)}] &= \Pr(root = j | root \in \{min\{i,j\}, max\{i,j\}\}) && (36)\\
&= \frac{\Pr(root = j)}{\Pr(root \in \{min\{i,j\}, max\{i,j\}\})} \text{ (definition of conditional probability)} && (37)\\
&= \frac{\frac{p(j)}{\sum_{l=1}^{n} p(l)}}{\frac{\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l)}{\sum_{l=1}^{n} p(l)}} \text{ (definition of the weighted binary tree)} && (38)\\
&= \frac{p(j)}{\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l)} && (39)
\end{aligned}$$

(v) $n > j - i + 1$ and $root \notin \{min\{i,j\}, max\{i,j\}\}$
As the root must either be smaller than $i$ and smaller than $j$ or larger than $i$ and larger than $j$, we know that $i$ and $j$ will always end up in the same subtree of the root. Note that within the left subtree the ranks of the keys will remain the same. Within the right subtree, the ranks are all decreased by $k$ where $k = root$. Hence the difference of ranks between $i$ and $j$ remains the same in both subtrees.
Therefore, we can now look at the respective subtree containing $i$ and $j$ in an isolated fashion, as we know that the current root does not contribute anything to the notion we look at. In the base case in which $n = 2$, we can either apply (i) if $i$ and $j$ are equal and (ii) if they're not. For $n > 2$, we can inductively apply cases (i), (ii), (iii), (iv) and (v).

b) We want to prove that

$$\forall x, y \in \mathbb{R} \; s.t. \; 0 \leq x < y : \frac{x}{y} \leq ln(y) - ln(y - x)$$

As we know that $x < y$, we can formulate the following

$$\exists \, \delta \in \mathbb{R} \; s.t. \; \delta > 0 \; and \; x = y - \delta$$

We know that INSERT NAME HERE $\forall x \in \mathbb{R} : 1 + x \leq e^x$. With $z = ln(\frac{\delta}{y})$ we get the following:

$$
\begin{align}
& 1 + ln(\frac{\delta}{y}) \leq e^{ln(\frac{\delta}{y})} \tag{40} \\
\Leftrightarrow \; & 1 - ln(\frac{y}{\delta}) \leq \frac{\delta}{y} \tag{41} \\
\Leftrightarrow \; & 1 - \frac{\delta}{y} \leq ln(\frac{y}{\delta}) \tag{42} \\
\Leftrightarrow \; & \frac{y - \delta}{y} \leq ln(\frac{y}{y - y + \delta}) \tag{43} \\
\Leftrightarrow \; & \frac{x}{y} \leq ln(\frac{y}{y - x - \delta + \delta}) \tag{44} \\
\Leftrightarrow \; & \frac{x}{y} \leq ln(\frac{y}{y - x}) \tag{45} \\
\Leftrightarrow \; & \frac{x}{y} \leq ln(y) - ln(y - x) \tag{46} \\
& \tag{47}
\end{align}
$$

c) We want to prove that

$$\forall i \in [n] : \mathbb{E}[D_n^{(i)}] \leq 2 \cdot ln(\frac{1}{p(i)})$$

$$\mathbb{E}[D_n^{(i)}] = \sum_{j=1}^{n} \mathbb{E}[A_i^j] \text{ (definition of Depth } D_n^{(i)}) \tag{48}$$

$$= \sum_{j=1, j\neq i}^{n} \frac{p(j)}{\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l)} \text{ (application of a))} \tag{49}$$

$$\leq \sum_{j=1, j\neq i}^{n} ln(\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l)) - ln(\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l) - p(j)) \text{ (application of b))} \tag{50}$$

$$\leq (\sum_{j=1, j\neq i}^{n} ln(\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l))) - ((\sum_{j=1, j\neq i}^{n} ln(\sum_{l=min\{i,j\}}^{max\{i,j\}} p(l) - p(j))) \tag{51}$$

$$\leq \sum_{j=1}^{i-1} ln(\sum_{l=j}^{i} p(l))) + \sum_{j=i+1}^{n} ln(\sum_{l=i}^{j} p(l))) - \sum_{j=1}^{i-1} ln(\sum_{l=j}^{i} p(l) - p(j))) - \sum_{j=i+1}^{n} ln(\sum_{l=i}^{j} p(l) - p(j))) \tag{52}$$

$$\leq \sum_{j=1}^{i-1} ln(\sum_{l=j}^{i} p(l))) + \sum_{j=i+1}^{n} ln(\sum_{l=i}^{j} p(l))) - \sum_{j=1}^{i-1} ln(\sum_{l=j+1}^{i} p(l))) - \sum_{j=i+1}^{n} ln(\sum_{l=i}^{j-1} p(l))) \tag{53}$$

$$\leq \sum_{j=1}^{i-1} ln(\sum_{l=j}^{i} p(l))) + \sum_{j=i+1}^{n} ln(\sum_{l=i}^{j} p(l))) - \sum_{j=2}^{i} ln(\sum_{l=j}^{i} p(l))) - \sum_{j=i}^{n-1} ln(\sum_{l=i}^{j} p(l))) \tag{54}$$

$$\leq ln(\sum_{l=1}^{i} p(l)) - ln(p(i)) + ln(\sum_{l=i+1}^{n} p(l)) - ln(p(i)) \tag{55}$$

$$\leq ln(\frac{(\sum_{l=1}^{i} p(l))}{p(i)} \cdot \frac{\sum_{l=i+1}^{n} p(l)}{p(i)}) \tag{56}$$

$$\tag{57}$$

The definition of the weighted binary tree tells us that $\sum_{l=1}^{n} p(l) = 1$ as well as $\forall i, p(i) > 0$. It follows immediately that $\sum_{l=1}^{i} p(l) \leq \sum_{l=1}^{n} p(l) = 1$ and analogously $\sum_{l=i+1}^{n} p(l) \leq \sum_{l=1}^{n} p(l) = 1$. Hence we can conclude:

$$\mathbb{E}[D_n^{(i)}] \leq ln(\frac{1}{p(i)} \cdot \frac{1}{p(i)}) \tag{58}$$

$$\leq 2 \cdot ln(\frac{1}{p(i)}) \tag{59}$$

# Exercise 3

a) We determine the maximum number of permutations, short $maxP(n)$ of inversions on a set of $n$ element by induction.

- Claim: $\forall n \geq 2 : maxP(n) = \frac{n(n-1)}{2}$
- Base case: n = 2:
  There is exactly 1 permutation if the second element is smaller than the first.

$$\frac{n(n-1)}{2} = \frac{2 \cdot 1}{2} = 1$$

- Induction step: $n \to n + 1$
  The induction hypothesis tells us that we have $maxP(n) = \frac{n(n-1)}{2}$ for the set of n numbers. We cannot create more than $n$ inversions by adding the $n + 1$th element as there are only $n + 1$ elements and there

cannot be more than one inversion per pair of elements. We can create exactly $n$ inversions by either positioning an element larger than the previous maximum in the first position or an element being smaller than the previous minimum in last position. Hence

$$\begin{aligned} maxP(n+1) &= maxP(n) + n && (60)\\ &= \frac{n(n-1)}{2} + n \text{ (induction hypothesis)} && (61)\\ &= \frac{n(n-1+2)}{2} && (62)\\ &= \frac{(n+1)n}{2} && (63)\\ & && (64) \end{aligned}$$

b) We are given a set of $n$ non-vertical lines $L$ and $a \in \mathbb{R}$. We assume that the lines are stored in the fashion $l_i \in L : l_i = (x_i, y_i)$ and in general position, i.e. no three lines intersect in the same point. CHECK THIS

- Sort the lines according to their slopes, i.e. $pos(l_i) < pos(l_j) \Leftrightarrow x_i <= x_j$. After sorting,$l_i$ represents the $i$ th line from the input, and $pos(l_i)$ the position of the line in the sorted order.
- For each line $l_i$, store $values[pos(l_i)] = a \cdot x_i + y_i$

c)

d)