

Randomized Algorithms and Probabilistic Methods: Graded Homework 3

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Exercise 1

a)

For $k \in \{1, \dots, n\}$, we know that there are $\binom{n}{k}$ subgraphs of G with k vertices. For one of those subgraphs to be fully connected, i.e. K_k , all of the $\binom{k}{2}$ possible edges have to be part of the edge set of G . Defining X_k to be the number of cliques of size k in G , its expectation equals $\binom{n}{k}(1/2)^{\binom{k}{2}}$, as every edge is selected with probability $1/2$.

We observe that $\Pr[X \geq k] = \Pr[X_k > 0]$ because if a clique of size $k+l, l \geq 1$ exists in G , G also has cliques of size k . As X_k can only take on positive integer values, $\mathbb{E}[X_k]$ has to be greater or equal $\Pr[X \geq k]$. For some $c \in \mathbb{N}_{>0}$, independent of n :

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{\infty} \Pr[X \geq k] = \sum_{k=1}^n \Pr[X \geq k] \\ &= \sum_{k=1}^{c \cdot \lceil \log(n) \rceil} \Pr[X \geq k] + \sum_{i=c \cdot \lceil \log(n) \rceil + 1}^n \Pr[X \geq k] \\ &\leq \sum_{k=1}^{c \cdot \lceil \log(n) \rceil} 1 + \sum_{i=c \cdot \lceil \log(n) \rceil + 1}^n \Pr[X \geq k] \\ &\leq c \cdot \lceil \log(n) \rceil + \sum_{k=c \cdot \lceil \log(n) \rceil + 1}^n \mathbb{E}[X_k] \\ &= c \cdot \lceil \log(n) \rceil + \sum_{k=c \cdot \lceil \log(n) \rceil + 1}^n \binom{n}{k} (1/2)^{\binom{k}{2}}\end{aligned}$$

It is left to show that $\sum_{k=c \cdot \lceil \log(n) \rceil + 1}^n \binom{n}{k} (1/2)^{\binom{k}{2}}$ is in $\mathcal{O}(\log(n))$. $\binom{n}{k}$ can be easily bounded by n^k from above. Also, we observe the following:

$$\begin{aligned}k \geq c \cdot \lceil \log(n) \rceil + 1 &\Rightarrow \binom{k}{2} = \frac{k(k-1)}{2} \geq \frac{k \cdot c \cdot \log(n)}{2} \\ &\Rightarrow (1/2)^{\binom{k}{2}} \leq (1/2)^{\frac{k \cdot c \cdot \log(n)}{2}} = n^{-k \cdot c/2}\end{aligned}$$

Combining those two bounds we obtain:

$$\begin{aligned}
\sum_{k=c \cdot \lceil \log(n) \rceil + 1}^n \binom{n}{k} (1/2)^{\binom{k}{2}} &\leq \sum_{k=c \cdot \lceil \log(n) \rceil + 1}^n n^k n^{-k \cdot c/2} = \sum_{k=c \cdot \lceil \log(n) \rceil + 1}^n n^{k(1-c/2)} \\
&\leq \sum_{k=4 \cdot \lceil \log(n) \rceil + 1}^n n^{-k} \text{ for } c = 4 \\
&\leq \sum_{i=1}^n n^{-i} = H_n \leq \log(n+1) \leq 2 \cdot \log(n) \text{ for } n \geq 3
\end{aligned}$$

We conclude that $\mathbb{E}[X] \leq 4 \cdot \log(n) + 2 \cdot \log(n)$. Equivalently, for $c' = 6$ we have $\mathbb{E}[X] \leq c' \log(n)$, which is what we intended to show. For $n \in \{1, 2\}$ we can fix the constant to be a very large number.

b)

Assuming $\mathbb{E}[X] = \Theta(\log(n))$ we can express the desired probability's negation in a convenient fashion.

$$\begin{aligned}
&\Pr[X \notin [(1 - \alpha(n))\mathbb{E}[X], (1 + \alpha(n))\mathbb{E}[X]]] \\
&= \Pr[X < (1 - \alpha(n))\mathbb{E}[X] \wedge X > (1 + \alpha(n))\mathbb{E}[X]] \\
&= \Pr[X - \mathbb{E}[X] < -\alpha(n)\mathbb{E}[X] \wedge X - \mathbb{E}[X] > \alpha(n)\mathbb{E}[X]] \\
&= \Pr[|X - \mathbb{E}[X]| > \alpha(n)\mathbb{E}[X]] \\
&\leq \Pr[|X - \mathbb{E}[X]| \geq \alpha(n)\mathbb{E}[X]]
\end{aligned}$$

We recognize a form from Lemma 8.2. In order to make use of it, we need to ensure its conditions are satisfied. Let $\Omega = \Omega_1 \times \dots \times \Omega_N$ with $N = n$ and each Ω_i describing whether all of the $n - 1$ edges of vertex i are included or not. It follows directly that the effect of each dimension c_i is at most 1. Let $\omega \in \Omega$ be a selection of such vertices and $X(\omega)$ its maximum clique size. Setting $r = X(\omega)$ trivially fulfills $r \leq X(\omega)$. Let us define J to be the subset of $\{1, \dots, n\}$ that indicates which vertices compose a $X(\omega)$ -clique - arbitrarily in case of ties. Then, for each $\omega' \in \Omega$ with $\forall i \in J, w'_i = w_i$, the size of the maximum clique in ω' can only increase, i.e. $X(\omega') \geq X(\omega) = r$. Moreover:

$$\sum_{i \in J} c_i^2 \leq \sum_{i \in J} 1 = |J| = X(\omega) = r$$

Thereby the necessary conditions hold true and Lemma 8.2.'s application yields:

$$\begin{aligned}
&\Pr[X \notin [(1 - \alpha(n))\mathbb{E}[X], (1 + \alpha(n))\mathbb{E}[X]]] \\
&\leq 4e^{-\Omega\left(\frac{(\alpha(n)\mathbb{E}[X])^2}{\mathbb{E}[X] + \alpha(n)\mathbb{E}[X]}\right)} = 4e^{-\Omega\left(\frac{\alpha(n)^2 \mathbb{E}[X]}{1 + \alpha(n)}\right)}
\end{aligned}$$

We define $\alpha(n) = \log(n)^{-1/4} \in o(1)$.

$$\lim_{n \rightarrow \infty} \frac{\alpha(n)^2 \log(n)}{1 + \alpha(n)} = \lim_{n \rightarrow \infty} \frac{\log(n)}{\sqrt{\log(n)} \cdot 1} = \lim_{n \rightarrow \infty} \sqrt{\log(n)} \rightarrow \infty$$

It follows directly that $\Pr[X \notin [(1 - \alpha(n))\mathbb{E}[X], (1 + \alpha(n))\mathbb{E}[X]]]$ tends to 0 for large enough n .

$$\begin{aligned} & \Pr[X \in [(1 - \alpha(n))\mathbb{E}[X], (1 + \alpha(n))\mathbb{E}[X]]] \\ &= 1 - \Pr[X \notin [(1 - \alpha(n))\mathbb{E}[X], (1 + \alpha(n))\mathbb{E}[X]]] \\ &\rightarrow 1 \end{aligned}$$

Exercise 2

We define $(Y_t)_{t \in \mathbb{N}}$ to be the amount of 'missing mutants', i.e. $n - |X_t|$. Hence we are looking to bound $\mathbb{E}[T]$ for Y_t from above. We define u_t and v_t to be the u and v of the t -th round of the evolutionary process. As $G = K_n$ and $V_o = n/2$, we know that $\Pr[u_t \in V_o] = \Pr[u_t \notin V_o] = 1/2$.

The process' drawing process weighs vertices from V_t with r . Given that $Y_t = y$, there are $|X_t| = n - y$ such vertices. Hence:

$$\Pr[v_t \in X_t] = \frac{r(n - y)}{r(n - y) + 1 \cdot y} \text{ and } \Pr[v_t \notin X_t] = \frac{1 \cdot y}{r(n - y) + 1 \cdot y}$$

As u_t is drawn after v_t , there are only $n - 1$ vertices to choose from. In particular, we observe that y is at most $n/2$ because X_t will always include V_o . Hence the size of $X_t \setminus V_o$ is given by $|X_t| - |V_o| = n - y - n/2 = n/2 - y$.

$$\Pr[u_t \in X_t \setminus V_o] = \frac{n/2 - y}{n - 1} \text{ and } \Pr[u_t \notin X_t] = \frac{y}{n - 1}$$

Moreover, in order for Y_{t+1} to decrease over Y_t , v_t has to be part of X_t and u_t be a node not yet contained in X_t . Hence:

$$\begin{aligned} \Pr[Y_{t+1} = Y_t - 1] &= \Pr[v_t \in X_t \wedge u_t \notin X_t] \\ &= \Pr[v_t \in X_t] \Pr[u_t \notin X_t | v_t \in X_t] \\ &= \frac{r(n - y)}{r(n - y) + y} \frac{y}{n - 1} \end{aligned}$$

On the other hand, for Y_{t+1} to increase over Y_t , v_t has to be excluded from X_t and u_t be already contained in X_t . Hence:

$$\begin{aligned} \Pr[Y_{t+1} = Y_t + 1] &= \Pr[v_t \notin X_t \wedge u_t \in X_t \setminus V_o] \\ &= \Pr[v_t \notin X_t] \Pr[u_t \in X_t \setminus V_o | v_t \notin X_t] \\ &= \frac{y}{r(n - y) + y} \frac{n/2 - y}{n - 1} \end{aligned}$$

We will now inspect the drift in order to find a fitting function g to apply Theorem 9.11. We observe that Y_t can only remain, increase by one or decrease

by one.

$$\begin{aligned}
\mathbb{E}[Y_{t+1}|Y_t = y] &= y \Pr[Y_{t+1} = Y_t] + (y-1) \Pr[Y_{t+1} = Y_t - 1] + (y+1) \Pr[Y_{t+1} = Y_t + 1] \\
&= y - \Pr[Y_{t+1} = Y_t - 1] + \Pr[Y_{t+1} = Y_t + 1] \\
&= y - (\Pr[Y_{t+1} = Y_t - 1] - \Pr[Y_{t+1} = Y_t + 1]) \\
&= y - \frac{y}{(r(n-y) + y)(n-1)} (r(n-y) - (n/2 - y)) \\
&= y - \frac{y(y(1-r) + n(r-1) + n/2)}{(r(n-y) + y)(n-1)} \\
&= y - \frac{y((r-1)(n-y) + n/2)}{(r(n-y) + y)(n-1)} \\
&\leq y - \frac{y((r-1)n/2 + n/2)}{(rn + y)(n-1)} \text{ with } (n-y \geq n/2, n-y \leq n) \\
&= y - \frac{y \cdot r \cdot n/2}{(rn + y)(n-1)} \\
&\leq y - \frac{y \cdot r}{2(rn + y)} \text{ with } (n-1 \leq n) \\
&\leq y - \frac{y \cdot r}{2(r+1)n} \text{ with } (y \leq n) \\
&\leq \frac{c \cdot y}{n} \text{ with } (c \geq \frac{r}{2(r+1)})
\end{aligned}$$

We observe that $\int_1^{n/2} \frac{c \cdot y}{n} dy = [c \cdot y \log(n)]_1^{n/2} = c \cdot n \log(n)$ and $\frac{1}{\frac{c-1}{n}} = \frac{n}{c}$. Hence, we can apply Theorem 9.12 with $Y_0 = n/2, h(y) = \frac{c \cdot y}{n}$:

$$\mathbb{E}[T] \leq \frac{c}{n} + c \cdot n \log(n)$$

In conclusion, we have shown that for Y_t $\mathbb{E}[T] \in \mathcal{O}(n \log(n))$. According to the definition of Y_t , it $X_t = V$ if $Y_t = 0$. Hence the bound on the process end also holds for X_t .

Exercise 3

Our goal is to apply Theorem 9.11, followed by Theorem 9.10. In particular, we define Theorem 9.11's $C := y, X_0 = n$ and

$$g(i) = \begin{cases} \frac{\sqrt{i}(x+y)}{xy}, & i > y \\ 0, & i \leq y \end{cases}$$

Our main concern is to show that Theorem 9.11's inequality holds. In other words, we want to find a $c > 0$, s.t. for all $i \in \mathbb{N}, i > y$:

$$\begin{aligned}
& \sum_{m > c} \Pr[X_{t+1} = m | X_t = i] g(m) \leq g(i) - c \\
& \Leftrightarrow \Pr[X_{t+1} = i - y | X_t = i] g(i - y) + \Pr[X_{t+1} = \min(i + x, n) | X_t = i] g(\min(i + x, n)) \leq g(i) - c \\
& \Leftrightarrow \frac{x}{x + y} g(i - y) + \frac{y}{x + y} g(i + x) \leq g(i) - c \\
& \Leftrightarrow \frac{x}{x + y} g(i - y) + \frac{y}{x + y} \frac{\sqrt{i + x}(x + y)}{xy} - \frac{\sqrt{i}(x + y)}{xy} \leq -c \\
& \Leftrightarrow \frac{x}{x + y} \frac{\sqrt{i - y}(x + y)}{xy} + \frac{\sqrt{i + x}}{x} - \frac{\sqrt{i}}{x} - \frac{\sqrt{i}}{y} \leq -c
\end{aligned}$$

We can replace $g(i - y)$ by assuming that $i - y > y$ because it only makes the inequality harder to satisfy. We observe that the initial condition holds true if:

$$\frac{\sqrt{i + x} - \sqrt{i}}{x} - \frac{\sqrt{i} - \sqrt{i - y}}{y} \leq -c$$

With $f(i) = \sqrt{i}$, we can apply the Mean Value Theorem for both of those terms. In other words, we know that there exist $\xi_1 \in]i, i + x[$ and $\xi_2 \in]i - y, i[$ s.t. $f'(\xi_1) = \frac{\sqrt{i + x} - \sqrt{i}}{x}$ and $f'(\xi_2) = \frac{\sqrt{i} - \sqrt{i - y}}{y}$. Note that our g is concave and ξ_1, ξ_2 are distinct. Multiplying and dividing our equation by the same term, we obtain that the initial condition holds true if:

$$(\xi_1 - \xi_2) \frac{f'(\xi_1) - f'(\xi_2)}{\xi_1 - \xi_2} \leq -c$$

As our $f'(i) = \frac{1}{2\sqrt{i}}$ is continuous for $i > k$, we can apply the Mean Value Theorem once again. Hence there is a $\xi_3 \in]\xi_2, \xi_1[$ such that $f''(\xi_3) = \frac{f'(\xi_1) - f'(\xi_2)}{\xi_1 - \xi_2}$, with $f''(i) = -\frac{1}{4\sqrt{i^3}}$. A look into the intervals from which ξ_1 and ξ_2 originate, we know that ξ_1 is less than $i + x$ and ξ_2 more than $i - y$. Therefore $\xi_1 - \xi_2$ can be bounded from above by $(i + x) - (i - y) = x + y$. Additionally, $\xi_3 \leq \xi_1 \leq i + x \leq n + x$. Combining this knowledge we can state that the initial condition is satisfied if:

$$\begin{aligned}
& -(x + y) \frac{1}{4\sqrt{(n + x)^3}} \leq -c \\
& \Leftrightarrow c \leq (x + y) \frac{1}{4\sqrt{(n + x)^3}}
\end{aligned}$$

Therefore we can apply Theorem 9.11 and state that:

$$\mathbb{E}[T_y] \leq g(n)/y \leq \frac{\sqrt{n}(x + y)}{xy} \frac{4\sqrt{(n + x)^3}}{x + y} \leq \sqrt{n} \sqrt{4(n + x)^3} \leq k \cdot n^2$$

for some large enough k , independent of n . In order to apply Theorem 9.10, we define p_0 to be the probability to 'go left', i.e. $\frac{x}{x + y}$ and $C = y$. We observe that $\mathbb{E}[T_y | X_0 = n]$ is always finite as there is no way of 'getting stuck' on the right-hand side of y and the Markov chain itself is finite. Therefore $\sum_{m > y} \Pr[X_{t+1} = m | X_t = i] \mathbb{E}[T_y | X_0 = m]$ can be bounded by some large B for all $i \leq y$. This insight allows us to apply Theorem 9.10, i.e. $\mathbb{E}[T] = \mathbb{E}[T_y] + \mathcal{O}(1) \leq k \cdot n^2 + \mathcal{O}(1)$.

Exercise 4

(a)

Given a subgraph H of G on ν vertices and μ balls, we define X to be the number of edges in H and X_i to be the indicator variable telling us whether a particular edge i is included in H . We know that there are 2μ edge candidates as each ball can have at most two edges and no edges between bins exist. The probability of an edge candidate being included in H is the probability of the bin from the edge in G being part of H . This latter is the amount of bins in H divided by the amount of bins in G . Hence:

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1}^{2\mu} \mathbb{E}[X_i] \text{ (LOE)} \\ &= \sum_{i=1}^{2\mu} \frac{\nu - \mu}{n} = \frac{2\mu(\nu - \mu)}{n} \\ &\leq \frac{2\nu^2}{n}\end{aligned}$$

We will now apply a Union Bound to make use of our bound for $\mathbb{E}[X]$. Additionally, we note that there are at most $\binom{n}{\nu} \binom{\nu}{\mu}$ ways of constructing subgraphs H from G s.t. H has ν vertices and μ balls.

$$\begin{aligned}&\Pr[\text{some subgraph has 3 more edges than vertices}] \\ &= \Pr\left[\bigcup_{H \subseteq G} H \text{ has 3 more edges than vertices}\right] \\ &\leq \sum_{H \subseteq G} \Pr[H \text{ has at least } \nu + 3 \text{ edges} | H \text{ has } \nu \text{ vertices}] \text{ (U.B.)} \\ &\leq \sum_{\nu=1}^{1000 \log(\log(n))} \sum_{H \subseteq G} \Pr[H \text{ has at least } \nu + 3 \text{ edges} | H \text{ has } \nu \text{ vertices}] \\ &\leq \sum_{\nu=1}^{1000 \log(\log(n))} \sum_{\mu=1}^{\nu} \sum_{H \subseteq G} \Pr[H \text{ has at least } \nu + 3 \text{ edges} | H \text{ has } \nu \text{ vertices and } \mu \text{ balls}] \\ &\leq \sum_{\nu=1}^{1000 \log(\log(n))} \sum_{\mu=1}^{\nu} \binom{n}{\nu} \binom{\nu}{\mu} \Pr[\text{subgraph has at least } \nu + 3 \text{ edges} | \text{has } \nu \text{ vertices and } \mu \text{ balls}]\end{aligned}$$

We can now use Markov's inequality saying that for $X \geq 0, t > 0$ $\Pr[X \geq t] \leq \mathbb{E}[X]/t$. In this particular instance we get that:

$$\begin{aligned}\Pr[\text{subgraph has } \nu \text{ vertices, } \nu \text{ balls and at least } \nu + 3 \text{ edges}] &\leq \frac{\mathbb{E}[X]}{\nu + 3} \\ &\leq \frac{2\nu^2}{n(\nu + 3)} \leq \frac{2\nu}{n}\end{aligned}$$

Including this bound in our essential probability, we obtain:

$$\begin{aligned}
& \Pr[\text{some subgraph has 3 more edges than vertices}] \\
& \leq \sum_{\nu=1}^{1000 \log(\log(n))} \sum_{\mu=1}^{\nu} \binom{n}{\nu} \binom{\nu}{\mu} \frac{2\nu}{n} \\
& \leq \sum_{\nu=1}^{1000 \log(\log(n))} \sum_{\mu=1}^{\nu} n^{\nu} \nu^{\mu} \frac{2\nu}{n} \\
& \leq \sum_{\nu=1}^{1000 \log(\log(n))} \nu n^{\nu} \nu^{\nu} \frac{2\nu}{n} \\
& \leq \frac{1}{n} \nu_{\max} \nu_{\max} n^{\nu_{\max}} \nu_{\max}^{\nu_{\max}} 2\nu_{\max} \text{ with } \nu_{\max} = 1000 \log(\log(n)) \\
& \leq \frac{1}{n} \cdot \mathcal{O}(\log(n)^{\log(n)}) \\
& \rightarrow 0 \text{ for } n \rightarrow \infty
\end{aligned}$$

(b)

(c)

- (i) By definition of the construction of the tree, we know that we have exactly one leaf, a ball with label 6, per bin and one bin per leaf. In other words $t(h) = \nu(h)$.

The presence of the ball of height h , connected to bin u , implies the presence of a leaf associated with u . Moreover, the subtrees starting in balls labeled 7 to $h-1$ connected to u will contain leaves. Hence: $t(h) = 1 + \sum_{i=7}^{h-1} t(i)$.

Claim 1. $\forall h \in \mathbb{N}, h \geq 7 : t(h) = 2^{h-7}$

Proof. We observe that for $h = 7$, there is one leaf, i.e. $t(7) = 1 = 2^{7-7}$. We will assume our hypothesis for some $h \geq 7$ and will show it holds for $h+1$ as well.

$$\begin{aligned}
t(h+1) &= 1 + \sum_{i=7}^h t(i) \\
&= 1 + \sum_{i=7}^h 2^{i-7} \text{ (induction hypothesis)} \\
&= 1 + (2^{h+1-7} - 1) = 2^{h+1-7}
\end{aligned}$$

□

- (ii) Knowing that the witness graph is a tree, we know that we can add up, for a bin, the leaf, the ball of greatest height h and all balls in subtrees rooted in 7 to $h-1$ connected to the bin, without double-counting or missing a ball. Hence $\mu(h) = 2 + \sum_{i=7}^{h-1} \mu(i)$.

Claim 2. $\forall h \in \mathbb{N}, h \geq 7 : \mu(h) = 2^{h-6}$

Proof. We observe that for $h = 7$, there is one leaf as well as the ball labeled 7, i.e. $\mu(7) = 2 = 2^{7-6}$. We will assume our hypothesis for some $h \geq 7$ and will show it holds for $h + 1$ as well.

$$\begin{aligned}\mu(h+1) &= 2 + \sum_{i=7}^h \mu(i) \\ &= 2 + \sum_{i=7}^h 2^{i-6} \text{ (induction hypothesis)} \\ &= 2 + 2(2^{h+1-7} - 1) = 2^{h+1-6}\end{aligned}$$

□

(iii) See (i).

(d)