ETH Zurich
Institute of Theoretical Computer Science
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HS 2017
Graded Homework 2

Randomized Algorithms and Probabilistic Methods

This is the second graded homework exercise set.

Regulations:

- *There will be a total of* ***three*** *special exercise sets during this semester.*
- *You are expected to solve them carefully and then write a nice and complete exposition of your solutions using* ***LaTeX****. Please submit your solutions as a pdf file to ppfister@inf.ethz.ch (filename: [nethz user name]_ghw2.pdf).*
- *You are welcome to discuss the exercises with your colleagues, but we expect each of you to hand in your own,* ***individual writeup****. Write down your name and* ***list all your collaborators*** *in the beginning of your writeup.*
- *Your solutions will be* ***graded****. Each graded homework will account for 10% of your final grade for the course (so 30% of the grade in total).*

Due date: Tuesday, November 14th, 2017 at 12:00

Exercise 1 *(7 points)*

We throw m balls into n bins uniformly at random independently from each other. The bins are labelled $b_1, \ldots, b_n$ and lined up in a row. We say that two balls are *neighbours* if they land in adjacent bins (we consider b_1 and b_n to be adjacent as well). Moreover, we say that a family of disjoint pairs of balls is a *neighbour-matching* if the two balls of each pair are neighbours. The size of such a neighbour-matching is given by its number of pairs of balls.

Assume that $m \gg n$, and prove that for every fixed $\varepsilon > 0$, whp there is a neighbour-matching of size at least $(1-\varepsilon)m/2$.

Exercise 2 *(7 points)*

Let the 0-1 matrix $A \in \{0,1\}^{n \times n}$and $\varepsilon > 0$ be given. Show that for n sufficiently large there exists a vector $q \in \{-1,1\}^n$ such that

$$\|Aq\|_\infty < \sqrt{(6+\varepsilon)n \ln n}.$$

Note: For any n-dimensional vector $x \in \mathbb{R}^n$ the *infinity norm* is defined by $\|x\|_\infty = \max_{i \in [n]} |x_i|$.

Exercise 3 *(10 points)*

For a graph $G = (V, E)$ a subset of edges $E' \subset E$ is called *edge-dominating* if every edge $e \in E$ shares at least one vertex with some edge of E'. The edge-domination number $\gamma(G)$ is the minimum size $|E'|$ of an edge-dominating set $E' \subset E$. If $H_1 = (V, E_1), H_2 = (V, E_2)$ are graphs on the same vertex set, then for their union $H = (V, E_1 \cup E_2)$ it is easy to see that $\gamma(H) \leq \gamma(H_1) + \gamma(H_2)$ and you may use this fact without proof.

Let G be a graph with $\gamma(G) = 100000$ and let H be a random subgraph of G created by adding each edge of G to H with probability $1/2$ independently of the other edges. Show that $\Pr[\gamma(H) \leq 40000] < 1/1000$.

Exercise 4 *(5+5 points)*

Consider the random graph $G \sim G_{n,p}$.

1. Show that for $p \gg n^{-2/3}$ the following holds with high probability. Each subset of $n/2$ vertices of G contains a triangle.

Now fix some $0 < \epsilon < 1$. We say that a graph H on n vertices is almost triangular if it contains a set of pairwise vertex-disjoint triangles that cover at least $(1-\epsilon)n$ vertices.

2. Find a weak threshold for $G \sim G_{n,p}$ to be almost triangular (for fixed $0 < \epsilon < 1$).