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Exercise 1.1.1: Given functions $\sigma : A \to B$ and $\tau : B \to C$, prove that if $\tau \circ \sigma$ is injective, then so is σ.

Proof. We will demonstrate a proof by contradiction. Assume σ is not injective, then by definition there exists two distinct values a_1, $a_2 \in A$ such that $\sigma(a_1) = \sigma(a_2)$. This implies that $\tau(\sigma(a_1)) = \tau(\sigma(a_2))$ which contradicts the assumption that $x \in A$, $\tau(\sigma(x))$ is an injective function. Hence a_1 and a_2 cannot exist and σ must be an injective function. $\square$

Exercise 1.2.2: Decide which of the following are equivalence relations on the set of natural numbers $\mathbb{N}$. For those that are, prove it. For those that are not, explain why.

(i) $x \sim y$ if $\|x - y\| \leq 3$

Proof. We will demonstrate a proof by counterexample. We will demonstrate that there exists a situation such that $x \sim y$ and $y \sim z$ but $x \sim z$ is false. Let $x, y, z \in \mathbb{N}$ and let $x = 2$, $y = 4$, and $z = 6$.

We can see that for the case of $x \sim y$ that $\|2 - 4\| \leq 3$ is true.
We can see that for the case of $y \sim z$ that $\|4 - 6\| \leq 3$ is true.
However, we can see that for the case of $x \sim z$ that $\|2 - 6\| \leq 3$ is false.
Hence, the statement is not transitive, and therefore it is not an equivalence relation. $\square$

(ii) $x \sim y$ if $\|x - y\| \geq 3$

Proof. For any $x \in \mathbb{N}, x - x = 0$. Furthermore $\|x - x\| = 0$. Clearly 0 is less than 3, hence the statement $\|x - x\| \geq 3$ is false, hence $x \sim x$ is false, which means $x \sim y$ is not reflexive and therefore not an equivalence relation. $\square$

(iii) $x \sim y$ if x and y have the same digit in the 1's place (expressed in base 10).

Proof. Let the set D be the set of decimal digits, which would be exactly $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Note that $D \subset \mathbb{N}$. Let $x_1, y_1, z_1 \in D$ be the digit in the one's place of $x, y, z \in \mathbb{N}$ respectively. which means that a statement such as $x_1 = y_1$ would be read as x and y have the same digit in the one's place.

First, for any natural number $n \in \mathbb{N}, n$ is the same as itself, hence $x_1 = x_1$ and $y_1 = y_1$, and therefore $x \sim x$ and $y \sim y$ so it is reflexive.

Second, observe that if $x \sim y$, then $x_1 = y_1$, furthermore x_1 is the same number as y_1, but then y_1 is the same number as x_1, and hence $y \sim x$, so it is symmetric.

Third, if $x \sim y$ then $x_1 = y_1$ and if $y \sim z$ then $y_1 = z_1$, then x_1 is the same as y_1 and y_1 is the same as z_1, so x_1 is the same as z_1, and hence $x \sim z$, and therefore it is transitive.

Finally, because we have shown that the relation is reflexive, symmetric, and transitive, by definition this is an equivalence relation. $\square$

(iv) $x \sim y$ if $x \geq y$

Proof. We will demonstrate a proof by counterexample. Let $x = 2$ and $y = 1$. In this scenario, $x \sim y$ because $2 \geq 1$, however $y \sim x$ is false, because that would imply $1 \geq 2$, which is absurd, and therefore a counterexample showing that the relation is not symmetric. Because this relation is not symmetric, it is therefore not an equivalence relation. $\square$

Exercise 1.2.3: Prove that for any $f : X \to Y$, the relation $\sim_f$ defined in Example 1.2.6 is an equivalence relation. Show that for any $x \in X$, the equivalence class of x is precisely $[x] = f^{-1}(f(x))$.

Note: f^{-1} refers to the *preimage* of *fiber* of f, which is defined on page 4 of your textbook. You should *not* assume that f is an invertible function.

Recall that Example 1.2.6 states: Suppose $f : X \to Y$ is a function. Define a relation $\sim_f$ on X by $x \sim_f y$ if $f(x) = f(y)$.

Proof. First, we will show that the relation is an equivalence relation.

Because x is the same as x and f is the same as f, we can see that $f(x) = f(x)$, so therefore we can see that the relation is reflexive.

If $f(x) = f(y)$, then $f(x)$ is the same as $f(y)$, so $f(y) = f(x)$, and therefore symmetric.

If $f(x) = f(y)$ and $f(y) = f(z)$ where $z \in X$, then $f(x)$ is the same as $f(y)$, but then $f(y)$ is the same as $f(z)$, hence $f(x) = f(z)$ therefore the relation is transitive.

Because the relation is reflexive, symmetric, and transitive, it is therefore an equivalence relation.

Next, we will show that for any $x \in X$, the equivalence class of x is precisely $[x] = f^{-1}(f(x))$.

By definition $[x]$ is the set $\{y \in X | y \sim_f x\}$, which when substituting the definition of $\sim_f$ is $[x] = \{y \in X | f(y) = f(x)\}$. With the preimage of f being defined as $f^{-1}(Y) = \{y \in X | f(y) \in Y\}$. For a given x, $f(x)$ will yield a singular value from Y. In other words, for any given x, $f(x)$ would give us a singleton set for that particular x input, and if we have some item a in a singlton set containing b, i.e. $a \in \{b\}$, we can conclude that $a = b$. Hence we can conclude that when we take the preimage of $f(x)$, we can write the resulting set as follows $f^{-1}(f(x)) = \{y \in X | f(y) \in \{f(x\}\} = \{y \in X | f(y) = f(x)\}$. We can see that this set that we have found for $f^{-1}(f(x))$ is eactly the same as the set we found for $[x]$ We can see that these sets are precisely the same, so therefore $[x]$ is precisely $f^{-1}(f(x))$. $\square$

Exercise 1.2.4: Prove Proposition 1.2.12: If $\sim$ is an equivalence relation on X, then the function $\pi : X \to X/\sim$ defined by $\pi(x) = [x]$ is a surjective map, and the equivalence relation $\sim_\pi$ determined by π is precisely $\sim$.

Proof. First we will show that the function is a surjective map. By definition, for any $[x] \in X/\sim$ then $\pi(x) = [x]$ where $x \in X$, therefore the function $\pi : X \to X/\sim$ is a surjective map.

Next, we will show that equivalence relation $\sim_\pi$ determined by π is precisely $\sim$. Recall that $X/\sim = \{[x] \mid x \in X\}$ also, $x \sim_\pi y$ implies $\pi(x) = \pi(y)$. Also note that $\pi(x) = [x]$ and $\pi(y) = [y]$ where $x, y \in X$. If $x \sim_\pi y$ then $\pi(x) = \pi(y)$. Furthermore $[x] = [y]$. Given that the definition of an equivalence class is $[a] = \{b \in A \mid b \sim a\}$, by this definition, we see that $x \sim y$, which shows that the equivalence relation $\sim_\pi$ is precisely $\sim$ $\square$

Exercise 1.3.1: Let $\sigma \in S_8$ be the permutation given by the 2×8 matrix

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 2 & 7 & 6 & 8 & 5 \end{pmatrix}$$

Express $\sigma, \sigma^2, \sigma^3$ and σ^{-1} in disjoint cycle notation.

For σ:
$\sigma(1) = 4$
$\sigma(2) = 1$
$\sigma(3) = 3$
$\sigma(4) = 2$
$\sigma(5) = 7$
$\sigma(6) = 6$
$\sigma(7) = 8$
$\sigma(8) = 5$

In disjoint cycle notation:

$$\sigma = \begin{pmatrix}1 & 4 & 2\end{pmatrix} \quad \begin{pmatrix}3\end{pmatrix} \quad \begin{pmatrix}6\end{pmatrix} \quad \begin{pmatrix}5 & 7 & 8\end{pmatrix}$$

For σ^2:
$\sigma(\sigma(1)) = \sigma(4) = 2$
$\sigma(\sigma(2)) = \sigma(1) = 4$
$\sigma(\sigma(3)) = \sigma(3) = 3$
$\sigma(\sigma(4)) = \sigma(2) = 1$
$\sigma(\sigma(5)) = \sigma(7) = 8$
$\sigma(\sigma(6)) = \sigma(6) = 6$

$\sigma(\sigma(7)) = \sigma(8) = 5$
$\sigma(\sigma(8)) = \sigma(5) = 7$

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 3 & 1 & 8 & 6 & 5 & 7 \end{pmatrix}$$

In disjoint cycle notation:

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix} \begin{pmatrix} 6 \end{pmatrix} \begin{pmatrix} 5 & 8 & 7 \end{pmatrix}$$

For σ^3:
$\sigma(\sigma(\sigma(1))) = \sigma(\sigma(4)) = \sigma(2) = 1$
$\sigma(\sigma(\sigma(2))) = \sigma(\sigma(1)) = \sigma(4) = 2$
$\sigma(\sigma(\sigma(3))) = \sigma(\sigma(3)) = \sigma(3) = 3$
$\sigma(\sigma(\sigma(4))) = \sigma(\sigma(2)) = \sigma(1) = 4$
$\sigma(\sigma(\sigma(5))) = \sigma(\sigma(7)) = \sigma(8) = 5$
$\sigma(\sigma(\sigma(6))) = \sigma(\sigma(6)) = \sigma(6) = 6$
$\sigma(\sigma(\sigma(7))) = \sigma(\sigma(8)) = \sigma(5) = 7$
$\sigma(\sigma(\sigma(8))) = \sigma(\sigma(5)) = \sigma(7) = 8$

$$\sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

In disjoint cycle notation:

$$\sigma^3 = (1) \quad (2) \quad (3) \quad (4) \quad (5) \quad (6) \quad (7) \quad (8)$$

For σ^{-1}, we must find the inverse of σ, such that $\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1}$ is the identity.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 3 & 1 & 8 & 6 & 5 & 7 \end{pmatrix}$$

Let's verify that this is correct by showing that $\sigma \circ \sigma^{-1}$ is the identity.
$\sigma^{-1}(1) = 2$
$\sigma^{-1}(2) = 4$
$\sigma^{-1}(3) = 3$
$\sigma^{-1}(4) = 1$
$\sigma^{-1}(5) = 8$
$\sigma^{-1}(6) = 6$
$\sigma^{-1}(7) = 5$
$\sigma^{-1}(8) = 7$

For $\sigma \circ \sigma^{-1}$
$\sigma(\sigma^{-1}(1)) = \sigma(2) = 1$
$\sigma(\sigma^{-1}(2)) = \sigma(4) = 2$
$\sigma(\sigma^{-1}(3)) = \sigma(3) = 3$
$\sigma(\sigma^{-1}(4)) = \sigma(1) = 4$
$\sigma(\sigma^{-1}(5)) = \sigma(8) = 5$
$\sigma(\sigma^{-1}(6)) = \sigma(6) = 6$
$\sigma(\sigma^{-1}(7)) = \sigma(5) = 7$

$\sigma(\sigma^{-1}(8)) = \sigma(7) = 8$

$$\sigma \circ \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

We can see that we have correctly found σ^{-1} in disjoint cycle notation to be:

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix} \begin{pmatrix} 6 \end{pmatrix} \begin{pmatrix} 5 & 8 & 7 \end{pmatrix}$$

Exercise 1.3.2: Consider $\sigma = \begin{pmatrix} 3 & 4 & 8 \end{pmatrix} \begin{pmatrix} 5 & 7 & 6 & 9 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 9 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 7 & 4 \end{pmatrix}$ in S_9 expressed in disjoint cycle notation. Compute $\sigma \circ \tau$ and $\tau \circ \sigma$, expressing both in disjoint cycle notation.

In matrix form:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 4 & 8 & 7 & 9 & 6 & 3 & 5 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 5 & 2 & 1 & 6 & 4 & 8 & 3 \end{pmatrix}$$

For σ:
$\sigma(1) = 1$
$\sigma(2) = 2$
$\sigma(3) = 4$
$\sigma(4) = 8$
$\sigma(5) = 7$
$\sigma(6) = 9$
$\sigma(7) = 6$
$\sigma(8) = 3$
$\sigma(9) = 5$

For τ:
$\tau(1) = 9$
$\tau(2) = 7$
$\tau(3) = 5$
$\tau(4) = 2$
$\tau(5) = 1$
$\tau(6) = 6$
$\tau(7) = 4$
$\tau(8) = 8$
$\tau(9) = 3$

For $\sigma \circ \tau$:
$\sigma(\tau(1)) = \sigma(9) = 5$
$\sigma(\tau(2)) = \sigma(7) = 6$
$\sigma(\tau(3)) = \sigma(5) = 7$
$\sigma(\tau(4)) = \sigma(2) = 2$
$\sigma(\tau(5)) = \sigma(1) = 1$
$\sigma(\tau(6)) = \sigma(6) = 9$
$\sigma(\tau(7)) = \sigma(4) = 8$

$\sigma(\tau(8)) = \sigma(8) = 3$
$\sigma(\tau(9)) = \sigma(3) = 4$

In matrix form:

$$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 2 & 1 & 9 & 8 & 3 & 4 \end{pmatrix}$$

In disjoint cycle notation:

$$\sigma \circ \tau = \quad (1 \quad 5) \quad (2 \quad 6 \quad 9 \quad 4) \quad (3 \quad 7 \quad 8)$$

For $\tau \circ \sigma$:
$\tau(\sigma(1)) = \tau(1) = 9$
$\tau(\sigma(2)) = \tau(2) = 7$
$\tau(\sigma(3)) = \tau(4) = 2$
$\tau(\sigma(4)) = \tau(8) = 8$
$\tau(\sigma(5)) = \tau(7) = 4$
$\tau(\sigma(6)) = \tau(9) = 3$
$\tau(\sigma(7)) = \tau(6) = 6$
$\tau(\sigma(8)) = \tau(3) = 5$
$\tau(\sigma(9)) = \tau(5) = 1$

$$\tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 2 & 8 & 4 & 3 & 6 & 5 & 1 \end{pmatrix}$$

In disjoint cycle notation:

$$\tau \circ \sigma = \quad (1 \quad 9) \quad (2 \quad 7 \quad 6 \quad 3) \quad (4 \quad 8 \quad 5)$$