Homework 8

MATH 417: INTRODUCTION TO ABSTRACT ALGEBRA

NAME: DATE:

(Exercises are taken from *Introduction to Abstract Algebra* by Christopher J Leininger.)

Exercise 3.4.1 Let $\tau \in S_n$ and suppose that $\sigma = (k_1\ k_2\ \ldots\ k_j)$ is a j–cycle. Prove that the conjugate of σ by τ is also a j–cycle, and is given by

$$\tau\sigma\tau^{-1} = (\tau(k_1)\ \tau(k_2)\ \ldots\ \tau(k_j)).$$

Further prove that if $\sigma' \in S_n$ is any other j–cycle, then σ and σ' are conjugate. *Hint: For the second part, you should explicitly find a conjugating element $\tau \in S_n$.*

The **cycle structure** of an element $\sigma \in S_n$ denotes the number of cycles of each length in the disjoint cycle representation of σ. We can encode the cycle structure with a **partition of** n**:**

$$n = j_1 + j_2 + \cdots + j_r,$$

where $\{j_r\}$ are positive integers giving the length of the distinct cycles (where we include "1's" for every number that is fixed, which we view as a "1–cycle" i.e. the identity). For example, the cycle structure of $(1\ 2\ 3)(5\ 6) \in S_6$ is $1 + 2 + 3 = 6$, since there is a 1–cycle, a 2–cycle, and a 3–cycle in the disjoint cycle representation.

Exercise 3.4.2 Suppose $\sigma_1, \sigma_2 \in S_n$. Using the previous exercise, prove that σ_1 and σ_2 have the same cycle structure if and only if they are conjugate.

Exercise 3.4.3 Proposition 1.3.9 shows that every permutation is a composition of 2–cycles, and thus the set of all 2–cycles generates S_n (i.e. the subgroup $G < S_n$ generated by the set of all 2–cycles is all of S_n). Prove that $(1\ 2)$ and $(1\ 2\ 3\ \ldots\ n)$ generates S_n; that is, prove

$$H = \langle (1\ 2), (1\ 2\ 3\ \ldots\ n) \rangle = S_n.$$

Hint: Consider $\sigma = (1\ 2)(1\ 2\ 3\ \ldots\ n) \in H$ *and then* $\sigma^k (1\ 2) \sigma^{-k} \in H$ *for* $k \geq 1$. *See also Exercise 3.4.1.*

Exercise 3.4.6 Prove $D_3 \cong S_3$.

Exercise 3.4.8 Let $n \geq 3$. Prove that $R_n = \{I, r, r^2, r^3, \ldots, r^{n-1}\} \subset D_n$, the cyclic subgroup generated by r, is a normal subgroup. This is called the **subgroup of rotations.**

Exercise 3.5.1 Prove *Fermat's Little Theorem:* For every prime $p \geq 2$ and $a \in \mathbb{Z}$, we have $a^p \equiv a \bmod p$. *Hint: Consider the two cases $p|a$ and $p \not| a$, in the latter case thinking about the group $\mathbb{Z}_p^\times$.*

Exercise 3.5.4 Suppose G is a group and $N < G$ is a subgroup with $[G : N] = 2$. Prove that $N \lhd G$ is a normal subgroup.

Exercise 3.5.6 Suppose $K, H < G$ are subgroups of a group G. Prove that for all $g \in G$, $H \cap gK$ is either empty, or is equal to a coset of $K \cap H$ in H. Using this, prove that

$$[H : K \cap H] \leq [G : K].$$

Hint: To prove the inequality, define a function from $H/K \cap H \to G/K$ and prove that it is well-defined and injective.