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Exercise 2.1.1: Prove that multiplication of complex numbers is associative. More precisely, let $z = a + bi$, $w = c + di$, and $v = g + hi$, and prove that $z(wv) = (zw)v$.

Proof. We must show that $z(wv) = (zw)v$

Let's start by assuming that they are equal.
$z(wv) = (zw)v$

$(a + bi)((c + di)(g + hi)) = ((a + bi)(c + di))(g + hi)$

$(a + bi)(cg + chi + dgi - dhi) = (ac + adi + bci - bd)(g + hi)$

$acg + achi + adgi - adh + bcgi - bch - bdg - bdhi =$
$acg + adgi + bcgi - bdg + achi - adh - bch - bdhi$

When alligned, we see that:
$acg + achi + adgi - adh + bcgi - bdg - bch - bdhi =$
$acg + achi + adgi - adh + bcgi - bdg - bch - bdhi$

Therefore we can see that $z(wv) = (zw)v$

□

Exercise 2.1.2: Let $z = a + bi$, $w = c + di \in \mathbb{C}$ and prove each of the following statements:

(i) $z + \overline{z}$ is real and $z - \overline{z}$ is imaginary.

Proof. First let's begin with proving $z + \overline{z}$ is real.
Recall that if $z = a + bi$, then $\overline{z} = a - bi$.
Notice that complex numbers are the set $\{a + bi | a, b \in \mathbb{R}\}$

$z + \overline{z} = (a + bi) + (a - bi) = 2a + bi - bi = 2a$

Because $2 \in \mathbb{R}$ and $a \in \mathbb{R}$, and real numbers are closed under multiplication, $2a$ is also real, therefore $z + \overline{z}$ is real.

Next, we will prove that $z - \overline{z}$ is imaginary.
$z - \overline{z} = (a + bi) - (a - bi) = a - a + bi + bi = 2bi$.

Because $2 \in \mathbb{R}$ and $b \in \mathbb{R}$, and real numbers are closed under multiplication, then $2b \in \mathbb{R}$. Imaginary numbers are the set $\{ci | c \in \mathbb{R}\}$. We can see that $2bi \in \{ci | c \in \mathbb{R}\}$, therefore $z - \overline{z}$ is imaginary. □

(ii) $\overline{z + w} = \overline{z} + \overline{w}$.

Proof. $z + w = (a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$

$\overline{z + w} = (a+c) - (b+d)i = a+c-bi-di = (a-bi)+(c-di) = \overline{z}+\overline{w}$ $\square$

(iii) $\overline{zw} = \overline{z}\,\overline{w}$.

Proof. $zw = (a + bi)(c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i$

$\overline{zw} = (ac - bd) - (ad + bc)i = ac - bd - adi - bci =$
$(a - bi)(c - di) = \overline{z}\,\overline{w}$ $\square$

Exercise 2.2.1: Prove that if $\mathbb{F}$ is a field and $a, b \in \mathbb{F}$ with $ab = 0$, then either $a = 0$ or $b = 0$.

Proof. We will demonstrate a proof by contradiction. We will show that for the statement $ab = 0$ if neither $a = 0$ nor $b = 0$, i.e., $a \neq 0$ and $b \neq 0$, then this will result in a contradiction.

Recall from the definition of a field, that every nonzero element $a \in \mathbb{F}$ has a multiplicative inverse a^{-1}, such that $aa^{-1} = 1$

Because $\mathbb{F}$ is a field, and we are assuming that $a \neq 0$ and $b \neq 0$, then we can take the multiplicative inverse of either in the statement $ab = 0$.

$ab = 0$
$a^{-1}ab = 0a^{-1}$
$b = 0$

We have thus contradicted the statement that $b \neq 0$. Let's try this again for a.

$ab = 0$
$ab^{-1}b = 0b^{-1}$
$a = 0$

Again we have contradicted the statement by finding that $a = 0$.

Because we have found a contradiction in statement that if $ab = 0$ then $a \neq 0$ and $b \neq 0$, we have therefore proven the original statement to be true that if $ab = 0$ then either $a = 0$ or $b = 0$ $\square$

Exercise 2.2.2: Prove that $\mathbb{Q}(\sqrt{2})$ is a field. Hint: you should use the fact that $\sqrt{2} \notin \mathbb{Q}$.

Proof. Recall, from Example 2.2.4, $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \in \mathbb{R} | a, b \in \mathbb{Q}\} \subset \mathbb{R}$
Note: $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$ also $\mathbb{Q} \neq \mathbb{Q}(\sqrt{2})$ because $\sqrt{2} \notin \mathbb{Q}$

We will show that $\mathbb{Q}(\sqrt{2})$ is a subfield of $\mathbb{R}$, which because a subfield is also a field itself, means that $\mathbb{Q}(\sqrt{2})$ is itself a field.

We will now demonstrate that $\mathbb{Q}(\sqrt{2})$ is a subfield of $\mathbb{R}$. In order for a subset $\mathbb{K}$ to be a subfield of a field $\mathbb{F}$, the following properties must hold for every $x, y \in \mathbb{K}$.

(i) $\mathbb{K} \subset \mathbb{F}$

(ii) The elements $0, 1 \in \mathbb{F}$ are also in $\mathbb{K}$

(iii) $x + y$, xy, $-x \in \mathbb{K}$

(iv) if $x \neq 0$ then $x^{-1} \in \mathbb{K}$

First, proving (i), Given the problem statement from the book, we see that $b\sqrt{2} \in \mathbb{R}$. Also, within the problem statement we can see $a \in \mathbb{Q}, a + b\sqrt{2} \in \mathbb{R}$. Because $\mathbb{R}$ is a field with the two operations addition and multiplication, and $a \in \mathbb{Q} \subset \mathbb{R}$, then $(a + b\sqrt{2}) \in \mathbb{R}$, hence $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$.

Second, proving (ii), because $\mathbb{Q}$ is a field, $0, 1 \in \mathbb{Q}$. Since $a, b \in \mathbb{Q}$, then if $a = 1$ and $b = 0$, then $1 + 0\sqrt{2} = 1$ and if $a = 0$ and $b = 0$, then $0 + 0\sqrt{2} = 0$, hence $0, 1 \in \mathbb{Q}(\sqrt{2})$.

Third, proving (iii): let $c, d, e, f \in \mathbb{Q}$, $x = c + d\sqrt{2}$, $y = e + f\sqrt{2}$.

$x + y = (c + d\sqrt{2}) + (e + f\sqrt{2}) = ((c + e) + (d + f)\sqrt{2}) \in \mathbb{Q}(\sqrt{2})$

$xy = (c + d\sqrt{2})(e + f\sqrt{2}) = ce + cf\sqrt{2} + de\sqrt{2} + 2df =$
$((ce + 2df) + (cf + de)\sqrt{2}) \in \mathbb{Q}(\sqrt{2})$

$-x = -(c + d\sqrt{2}) = (-c + -d\sqrt{2}) \in \mathbb{Q}(\sqrt{2})$
Also, we can see that $-x$ is the additive inverse such that $-x + x = 0$.
$-x + x = (-c + -d\sqrt{2}) + (c + d\sqrt{2}) = c - c + d\sqrt{2} - d\sqrt{2} = 0 + 0 = 0$

Finally, proving (iv), recall if $x \neq 0$ then $xx^{-1} = 1$.

We will now verify that if $x = c + d\sqrt{2} \neq 0$, then $(c + d\sqrt{2})^{-1} \neq 0 \in \mathbb{Q}(\sqrt{2})$
Observe that since $x \neq 0$, $c + d\sqrt{2} \neq 0$, furthermore $c^2 - 2d^2 \neq 0$.

$x^{-1} = \frac{1}{x} = \frac{1}{c+d\sqrt{2}} = \frac{c-d\sqrt{2}}{(c+d\sqrt{2})(c-d\sqrt{2})} = \frac{c-d\sqrt{2}}{c^2-cd\sqrt{2}+cd\sqrt{2}-2d^2} = \frac{c-d\sqrt{2}}{c^2-2d^2} =$
$(\frac{c}{c^2-2d^2}) + (\frac{-d}{c^2-2d^2})\sqrt{2}$

Because $\frac{c}{c^2-2d^2} \in \mathbb{Q}$ and $\frac{-d}{c^2-2d^2} \in \mathbb{Q}$, therefore $(\frac{c}{c^2-2d^2}) + (\frac{-d}{c^2-2d^2})\sqrt{2} \in \mathbb{Q}(\sqrt{2})$

By showing that $\mathbb{Q}(\sqrt{2})$ is a subfield of $\mathbb{R}$, which is itself a field, we have shown that $\mathbb{Q}(\sqrt{2})$ is a field.

$\square$

Exercise 2.3.1: Prove parts (iii)-(v) of Proposition 2.3.2:
Suppose $\mathbb{F}$ is a field.

(i) Addition and multiplication are commutative, associative operations on $\mathbb{F}[x]$ which restrict to the operations of addition and multiplication on $\mathbb{F} \subset \mathbb{F}[x]$.

(ii) Multiplication distributes over addition: $f(g+h) = fg+gh$ for all $f, g, h \in \mathbb{F}[x]$

(iii) $0 \in \mathbb{F}$ is an additive identity in $\mathbb{F}[x] : f + 0 = f$ for all $f \in \mathbb{F}[x]$.

Proof. Let
$$f \in \mathbb{F}[x], f = \sum_{k=0}^{n} a_k x^k$$
Observe:
$$f + 0 = \sum_{k=0}^{n} (a_k + 0)x^k = \sum_{k=0}^{n} a_k x^k = f$$
□

(iv) Every $f \in \mathbb{F}[x]$ has an additive inverse given by $-f = (-1)f$ with $f + (-f) = 0$.

Proof. Let
$$f \in \mathbb{F}[x], f = \sum_{k=0}^{n} a_k x^k$$
Observe:
$$f+(-1)f = f+(-f) = \sum_{k=0}^{n} (a_k+(-1)a_k)x^k = \sum_{k=0}^{n} (a_k+(-a_k))x^k = \sum_{k=0}^{n} 0x^k = 0$$
□

(v) $1 \in \mathbb{F}$ is the multiplicative identity in $\mathbb{F}[x] : 1f = f$ for all $f \in \mathbb{F}[x]$.

Proof. Let
$$f \in \mathbb{F}[x], f = \sum_{k=0}^{n} a_k x^k$$
Observe:
$$(1)f = \sum_{k=0}^{n} (1)a_k x^k = \sum_{k=0}^{n} a_k x^k = f$$
□