CS 38 Introduction to Algorithms **Problem Set 1**

Problem 1

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(a) Show that there exist strictly increasing functions $f, g : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $f \notin O(g)$ and $g \notin O(f)$.

Let us consider the functions f and g ans defined below:

$$f(x) = \begin{cases} 2x & \text{if } n \text{ is odd} \\ 2x+1 & \text{if } n \text{ is even} \end{cases}$$

$$g(x) = \begin{cases} 2x+1 & \text{if } n \text{ is odd} \\ 2x & \text{if } n \text{ is even} \end{cases}$$

For $m, n \in \mathbb{Z}_+$, when $m > n$, then $2(m) > 2(n)$ and $2(m) + 1 > 2(n) + 1$. Additionally, for $m, n, o \in \mathbb{Z}_+$, if $m > n > o$, then $2(m) > 2(n) + 1 > 2(o)$ and $2(m) + 1 > 2(n) > 2(o) + 1$. Thus, we have that f and g are strictly increasing.

When x is odd, $f(x) = 2x$ and $g(x) = 2x + 1$, so for x odd, $f(x) < g(X)$. When x is even, $f(x) = 2x + 1$ and $g(x) = 2x$, so $f(x) > g(x)$. Since these functions are strictly increasing but alternate between which one is greater, we have that neither one dominates the other as x asymptotically approaches infinity. Therefore, we have that $f \notin O(g)$ and $g \notin O(f)$. □

(b) Prove or disprove: For any finite collection of functions $\{f_1, f_2, \cdots, f_k\}$, from $\mathbb{Z}_+ \to \mathbb{Z}_+$, there is a function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that:

$f_i \in O(g) \forall i$

If $g' : \mathbb{Z}_+ \to \mathbb{Z}_+$ is such that $f_i \in O(g')$, then $g \in O(g')$

Let us choose $g = \text{MAX}(f_1, f_2, \cdots, f_k)$ to be the maximum value of any f_i for a given input. By this definition we have that $\forall i$, $f_i \leq g$. From this we find that $f_i \in O(g) \forall i$.

If we have some $g' : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $f_i \in O(g') \forall i$, then there exists some $N \in \mathbb{Z}_+$ and a real scalar c such that $cg'(n) \geq f_i(n)$ $\forall i$ and $\forall n \geq N$. Because this is true for each individual f_i, then it holds for the maximum of the f_i at any input n too. Therefore we will have that there exists a real scalar c and some $N \in \mathbb{Z}_+$ such that $cg'(n) \geq \text{MAX}(f_1(n), f_2(n), \cdots f_k(n))$ $\forall n \geq N$. Therefore $g \in O(g')$

Since for any finite collection of functions mapping from and to the positive integers we can easily define the maximum of the set of functions upon a given input, we have shown that for all such collects, there does exists a function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ that satisfies the stated conditions. □

(c)

Let us take the countably infinite set defined by $\{f_i | f_i = n^i\}$. We desire to show that there is no $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\forall i \in \mathbb{Z}_+ f_i \in O(g)$ and if there exists a second such function $h : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\forall i \in \mathbb{Z}_+, f_i \in O(h)$, then $g \in O(h)$. In order to do this, we will use proof by contradiction. Let us assume that there exits such a function g with the aforementioned properties. We will then take $h = \lceil g/n \rceil$.

Let us first show that $\forall i, f_i \in O(h)$. By our earlier assumption, we know that $\forall i, f_i \in O(g)$. Therefore, there exists a constant c and a $n_0 \in \mathbb{Z}_+$ such that $\forall n \geq n_0, f_i(n) \leq cg(n)$. Dividing by n on both sides and taking the ceiling we have that $\forall n \geq n_0, f_{i+1}(n) \leq c \lceil g(n)/n \rceil = ch(n)$. Thus all $f_i \in O(h)$.

Under our initial assumption, this implies that we ought to have $g \in O(h)$. We will show that this is false and therefore that g cannot exists. If $g \in O(h)$ then there exists a scalar c and some $n_0 \in \mathbb{Z}_+$ such that $\forall n \geq n_0, g(n) \leq ch(n)$. Taking the definition we chose for $h(n)$, this can be rewritten as $g(n) \leq c \lceil g(n)/n \rceil$. This simplifies to $\lceil n \rceil \leq c$. However, no finite constant c can be an upper bound for a linear term n (i.e. $n \notin O(1)$). Therefore, we have that g does not in fact satisfy the second condition. Thus, there is no such g that satisfies the two conditions set forth in the case of a countably infinite set of functions from $\mathbb{Z}_+ \to \mathbb{Z}_+$. $\square$

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