Lecture notes FY8305

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Digitialized from “FAG 74986 Funksjonal-integral metoder HØST 1996”. Used as lecture notes for self-study in the course “FY8305 - Functional Integral Methods in Condensed Matter Physics”. **Link to course page**

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1 Uncertainties/mistakes

- Something weird in the derivation of (2.28).
- The indices in equation (4.2) seems off.
- In the proof in section 4.1 i think there should be a creation operator on one of the last lines in the proof, not annihilation operator as it stands in the notes. $a\mathrm{e}^{\varphi a^\dagger}|0\rangle$ instead of $a\mathrm{e}^{\varphi a}|0\rangle$
- It is a somewhat unclear purpose of Equations (4.31) to (4.35).
- The ordering of headlines and equations in section 4 is a bit weird, and some things seem a bit unmotivated.
- Double check the labeling on states/ sub indices in (5.19) and the previous few eqs.

2 Short recap of second quantization for fermions and bosons

Pages 2-10 in lecture notes.

Notation: $\mu =$ set of quantum numbers that define a one-particle state.

2.1 Many particle basis

Ex 1.

$$\mu = (\vec{k}, \sigma) : \text{Wave number, spin}$$
$$\mu = (i, \sigma) : \text{Lattice point, spin}$$
$$\mu = (n, i) : \text{Orbital, lattice point}$$

A many-particle basis can be written $|\phi\rangle = |n_\mu, n_\nu, \ldots, n_{\mu_N}\rangle$. Many particle states are built by combining many one-particle states, but where the one-particle states are not necessarily independent. If one of the set of quantum numbers, μ_i, are changed, this scattering will generally have consequences for the distribution of quantum numbers for the remaining sets $\{\mu_j\}_{j \neq i}$. We generally imagine that many-particle states can be built as a linear combination of $|\phi\rangle$'s;

$$|\Psi\rangle = \sum_{n_{\mu_1}, \ldots, n_{\mu_N}} \phi_{\mu_1, \ldots, n_{\mu_N}} |\mu_1, \ldots, n_{\mu_N}\rangle . \tag{2.1}$$

A definite one-state vector $|n_\mu, \ldots, n_{\mu_N}\rangle$ can be demanded from a vacuum state (where there is no filled one-particle states) $|0\rangle$ via creation operators.

bosons :	$a^\dagger_\mu$
fermions :	$c^\dagger_\mu$

A quanta in a one-particle state can be destroyed by the annihilation operators.

bosons :	a_μ
fermions :	c_μ

These operators satisfy some commutation relations:

$$[a_\mu, a_\nu] = [a^\dagger_\mu, a^\dagger_\nu] = 0 \tag{2.2}$$
$$[a_\mu, a^\dagger_\nu] = \delta_{\mu\,\nu} \tag{2.3}$$
$$[A, B] = AB - BA \tag{2.4}$$
$$\{c^\dagger_\mu, c^\dagger_\nu\} = \{c_\mu, c_\nu\} = 0 \tag{2.5}$$
$$\{c_\mu, c^\dagger_\nu\} = \delta_{\mu\,\nu} \tag{2.6}$$
$$\{A, B\} = AB + BA \tag{2.7}$$

These will automatically satisfy the Pauli principle as well, which gives symmetri/ antisymmetric solutions by exchange, dependent if the particles are bosons/fermions.

2.2 From classical formulation to second quantization of one-particle operators

For one-particle operators we usually have a kinetic energy function on a form like

$$T = \sum_i T\left(\vec{r}_i, \vec{p}_i\right) = \sum_i T\left(\vec{r}_i, \frac{\partial}{\partial r}\right) \tag{2.8}$$

Ex 2. External electrostatic potential:

$$T = \sum_i V_{\text{ext}}\left(\vec{r}_i\right) \tag{2.9}$$

Ex 3. Kinetic energy:

$$T = \sum_i \frac{p^2}{2m} = -\sum_i \frac{\hbar^2}{2m}\nabla_i^2 \tag{2.10}$$

Ex 4. Crystal-potential:

$$T = \sum_i \sum_j v_{\text{cryst}}\left(\vec{r}_i, \vec{R}_j\right) \tag{2.11}$$

Second quantization by an operator on this form can be written

$$T = \sum_{\mu,\nu} T_{\mu\nu} c_\mu^\dagger c_\nu, \tag{2.12}$$

where

$$T_{\mu\,\nu} = \left\langle \mu \middle| T\left(\vec{r}, \vec{p}\right) \middle| \nu \right\rangle. \tag{2.13}$$

Note: The matrix element of one-particle operators are determined by matrix elements in the Hilbert space of one-particle states.

2.3 From classical formulation to second quantization of two-particle operators

Typically, we consider pair-potentials

$$V = \sum_{i,j} V\left(\vec{r}_i, \vec{r}_j\right). \tag{2.14}$$

Ex 5. Exchange interaction of two charges

$$V = \frac{e^2}{2} \sum_{i \neq j} \frac{1}{\left|\vec{r}_i - \vec{r}_j\right|} \tag{2.15}$$

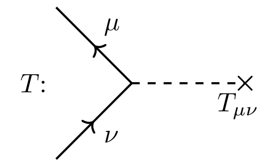

Figure 1: Scattering from an external potential $v_{\mu\nu}c^\dagger_\mu c_\nu$

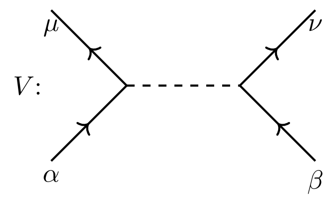

Figure 2: Exchange interaction between two particles.

The second quantization versions of these are

$$V = \sum_{\mu,\ldots,\beta} V_{\mu\nu\alpha\beta} c^\dagger_\mu c^\dagger_\nu c_\alpha c_\beta, \tag{2.16}$$

where again

$$V_{\mu\nu\alpha\beta} = \langle \mu\nu | V(\vec{r}_i, \vec{r}_j) | \beta\alpha \rangle \tag{2.17}$$

Note: The matrix element of two-particle operators are determined by matrix elements in the Hilbert room of two-particle states.

The Hamiltonian:

$$H = T + V \tag{2.18}$$

$$T = -\sum_i \frac{\hbar^2}{2m} \nabla_i^2 \tag{2.19}$$

So far, we have just presented second quantization for fermion operators, but an equivalent statement will of course hold for the second quantization version of the Hamiltonian for an interacting, material, bosonic system, which has the same identical form as (2.18). Notice that each term in H has just as many $c^\dagger_\mu$ as c_ν.

2.4 Statistical mechanics

Assume that we know the spectrum E_N^n for an interacting many-particle system, defined by a state $|\psi_N\rangle_n$, where N is the number of particles in the system and n is an index that indicates what excited state $|\psi_N\rangle_n$ the system is in. $|\psi_N\rangle$ is also assumed to be known, such that the matrix product of observables can be calculated:

$$H\,|\psi\rangle_n = E_N\,|\psi\rangle_n\,. \tag{2.20}$$

To do statistical mechanics, we need to introduce temperature. We do this by using the canonical partition function

$$Z_N = \sum_n \mathrm{e}^{-\beta E_N^n}. \tag{2.21}$$

Note, in (2.21) we sum over states, <u>not</u> the energy levels E_N^n.

$$\begin{aligned} Z &= \sum_n {}_n\langle\psi_N|\mathrm{e}^{-\beta H}|\psi_N\rangle_n \\ &= \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right) = \mathrm{Tr}\left(S^{-1}S\mathrm{e}^{-\beta H}\right) \\ &= \mathrm{Tr}\left(S\mathrm{e}^{-\beta H}S^{-1}\right) \\ &= \sum_{n'} {}_{n'}\langle\phi_N|\mathrm{e}^{-\beta H}|\phi_N\rangle_{n'}\,. \end{aligned} \tag{2.22}$$

We see in (2.22) that we can use an <u>arbitrary</u> basis to calculate the partition function. The most convenient basis is often a basis where the Hamiltonian is diagonal, but not always.

We write the statistical mean value of an operator as

$$\begin{aligned} \left\langle\hat{O}\right\rangle &\equiv \frac{1}{Z}\,\mathrm{Tr}\left(\hat{O}\mathrm{e}^{-\beta H}\right) \\ &= \frac{1}{Z}\sum_n {}_n\langle\psi_N|\hat{O}\mathrm{e}^{-\beta H}|\psi_N\rangle_n \\ &= \frac{1}{Z}\sum_{n,n'} {}_n\langle\psi_N|\hat{O}|\psi_N\rangle_{n'}\,\underbrace{{}_{n'}\langle\psi_N|\mathrm{e}^{-\beta H}|\psi_N\rangle_n}_{\delta_{nn'}\mathrm{e}^{-\beta E_{n'}}}. \end{aligned} \tag{2.23}$$

Thus, we have

$$\left\langle\hat{O}\right\rangle = \frac{1}{Z}\sum_n \underbrace{{}_n\langle\psi_N|\hat{O}|\psi_N\rangle_n}_{\text{QM matrix element}}\,\mathrm{e}^{-\beta E_N^n}. \tag{2.24}$$

Notice how the temperature, T only appears in the last factor in (2.24). Let us now consider the ground state ($n=0$) in the low temperature limit with energy E_0 corresponding to the state $|\psi_N\rangle_0$.

$$\begin{aligned} \left\langle\hat{O}\right\rangle &\simeq \frac{1}{Z_{\beta=\infty}}\mathrm{e}^{-\beta E_0}\,{}_0\langle\psi_N|\hat{O}|\psi_N\rangle_0 \\ &= \frac{\mathrm{e}^{-\beta E_0}}{\mathrm{e}^{-\beta E_0}}\,{}_0\langle\psi_N|\hat{O}|\psi_N\rangle_0\,, \end{aligned}$$

such that

$$\left\langle\hat{O}\right\rangle \overset{\beta\to\infty}{=} {}_0\langle\psi_N|\hat{O}|\psi_N\rangle_0\,. \tag{2.25}$$

We now have a way to calculate the statistical mean value in the ground state at zero temperature. Let us now now assume that the energy spectrum is such that the ground state is separated from excited states by a gap (band insulators, semiconductors, superconductors). This way, we can express the excited state energies as

$$E_N^1 = E_N^0 + \Delta_N \tag{2.26}$$

such that

$$E_N^2, E_N^3, \dots \geq E_N^1. \tag{2.27}$$

This way, we get from (2.24)

$$\begin{aligned}
\left\langle \hat{O} \right\rangle &= \frac{1}{Z} \sum_n {}_n \langle \psi_N | \hat{O} | \psi_N \rangle_n \, \mathrm{e}^{-\beta E_N^n} \\
&= \frac{\sum_n {}_n \langle \psi_N | \hat{O} | \psi_N \rangle_n \, \mathrm{e}^{-\beta E_N^n}}{\sum_n \mathrm{e}^{-\beta E_N^n}} \\
&= \cdots \\
&= \frac{{}_0 \langle \psi_N | \hat{O} | \psi_N \rangle_0 \, \mathrm{e}^{-\beta E_N^0 \left(1 + \mathrm{e}^{-\beta \Delta} \dots\right)}}{\mathrm{e}^{-\beta E_N^0} \left(1 + \mathrm{e}^{-\beta \Delta} \dots\right)}
\end{aligned} \tag{2.28}$$

and we find that as $\beta\Delta >> 1$, $\hat{O} \simeq {}_0 \langle \psi_N | \hat{O} | \psi_N \rangle_0$. In semiconductors we find $\Delta \sim 10\text{mev} \sim 1000\text{K}$.

3 Coherent states and introduction to Grassmann variables

Pages 10-17 in lecture notes.

3.1 Coherent states

A coherent state (both for fermions and bosons) is defined as an eigenstate to an annihilation operator

$$a_\mu |\phi\rangle = \varphi_\mu |\phi\rangle \qquad \textbf{Bosons} \tag{3.1}$$

$$c_\mu |\psi\rangle = \xi_\mu |\psi\rangle \qquad \textbf{Fermions} \tag{3.2}$$

Both $|\psi\rangle$ and $|\phi\rangle$ must contain a component with the least (≥ 0) quantum number (quant), but it is clear that neither $|\psi\rangle$ nor $|\phi\rangle$ can be states with a sharply defined number of particles. They are therefore also “hard to destroy”. This also explains why we chose to define them as eigenstates of the annihilation operators, not the creation operators. We will get back to the creation of these coherent states.

We will first look at the bosonic case:

3.1.1 Bosonic case

$$a_\mu |\phi\rangle = \varphi_\mu |\phi\rangle \tag{3.3}$$

$$\begin{aligned}
[a_\mu, a_\nu] = 0 & \\
& \Rightarrow (a_\mu a_\nu - a_\nu a_\mu) |\phi\rangle = 0 \\
& = (\varphi_\mu \varphi_\nu - \varphi_\nu \varphi_\mu) |\phi\rangle \\
& \qquad \Rightarrow [\varphi_\mu, \varphi_\nu] = 0.
\end{aligned} \tag{3.4}$$

Equation (3.4) will always be satisfied if $\varphi_\mu \in \mathbb{C}$. **The eigenvalues to coherent boson states can be chosen as complex numbers. This is something we can state without knowing anything about how these states are constructed.**

3.1.2 Fermionic case

$$c_\mu |\psi\rangle = \xi_\mu |\psi\rangle \tag{3.5}$$

$$\begin{aligned}
\{c_\mu, c_\nu\} = 0 & \\
& \Rightarrow (c_\mu c_\nu + c_\nu c_\mu) |\psi\rangle = 0 \\
& = (\xi_\mu \xi_\nu + \xi_\nu \xi_\mu) |\psi\rangle \\
& \qquad \Rightarrow \{\xi_\mu, \xi_\nu\} = 0.
\end{aligned} \tag{3.6}$$

If $\xi_\mu \in \mathbb{C}$, (3.6) will only be satisfied if $\{\xi_\mu\} = 0$, trivial eigenvalues. **The eigenvalues for coherent fermion states must be chosen as anti-commuting numbers, Grassmann-variables.**

3.2 Grassmann variables

3.2.1 Fundamentals

Equation (3.6) states the fundamental property of Grassmann variables, and it immediately follows that

$$\xi_\mu^2 = 0, \tag{3.7}$$

the squares of the Grassmann variables vanish! Similarly we have that $\xi^n = \xi^2\xi^{n-2} = 0, n \geq 2$. An arbitrary series expansion in Grassmann variables

$$\begin{aligned} f(\xi) &= \sum_n c_n \xi^n \\ &= c_0 + c_1\xi + \dots \\ &= c_0 + c_1\xi \end{aligned} \tag{3.8}$$

is linear. We can also consider $f(\xi^*) = c_0 + c_1\xi^*$, where $(\xi^*)^* = \xi$. An arbitrary function of ξ, ξ^* can be written on the forms

$$A(\xi, \xi^*) = c_0 + c_1\xi + c_2\xi^* + c_3\xi\xi^* \tag{3.9}$$

$$= c_0 + c_1\xi + c_2\xi^* + d_3\xi^*\xi \tag{3.10}$$

We will now look into some of the properties of functions of Grassmann variables.

3.2.2 Differentiation

Differentiation with respect to Grassman variables follows

$$\frac{\partial \xi}{\partial \xi} = 1 \qquad \frac{\partial \xi}{\partial \xi^*} = 0 \tag{3.11}$$

$$\frac{\partial \xi^*}{\partial \xi} = 0 \qquad \frac{\partial \xi^*}{\partial \xi^*} = 1 \tag{3.12}$$

$$\frac{\partial (\xi\xi^*)}{\partial \xi} = \xi^* \tag{3.13}$$

$$\frac{\partial (\xi\xi^*)}{\partial \xi^*} = -\frac{\partial (\xi^*\xi)}{\partial \xi^*} = -\xi \tag{3.14}$$

$$\frac{\partial f(\xi)}{\partial \xi^*} = 0 \tag{3.15}$$

$$\frac{\partial f(\xi)}{\partial \xi} = c_1 = \frac{\partial f(\xi^*)}{\partial \xi^*}, \tag{3.16}$$

and for functions defined as in (3.9), we have

$$\frac{\partial}{\partial \xi} A(\xi, \xi^*) = c_1 + c_3 \xi^* = c_1 - d_3 \xi^* \tag{3.17}$$

$$\frac{\partial}{\partial \xi^*} A(\xi, \xi^*) = c_2 - c_3 \xi = c_2 + d_3 \xi. \tag{3.18}$$

3.2.3 Integration

Integrating with respect to Grassmann variables are motivated from the properties of "normal" Riemann integrals, that if $f(x = \pm\infty) = 0$, then

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\mathrm{d}f}{\mathrm{d}x} = 0. \tag{3.19}$$

Equivalently we define

$$\int \mathrm{d}\xi \cdot 1 = \int \mathrm{d}\xi \, \frac{\mathrm{d}\xi}{\mathrm{d}\xi} = 0 \tag{3.20}$$

$$\int \mathrm{d}\xi^* = 0, \tag{3.21}$$

in other words, the integral of a total differential is zero.

$$\int \mathrm{d}\xi \, \xi = \int \mathrm{d}\xi^* \, \xi^* = 1 \tag{3.22}$$

is a "normalization" criteria. These relations define what we mean by Grassmann-integration.

Now, we have

$$\int \mathrm{d}\xi \, f(\xi) = c_1 \tag{3.23}$$

$$\frac{\partial f}{\partial \xi} = c_1 \tag{3.24}$$

$$\begin{aligned} \int \mathrm{d}\xi \, A(\xi, \xi^*) &= \int \mathrm{d}\xi \left(x_0 + c_1 \xi + c_2 \xi^* + c_3 \xi \xi^* \right) \\ &= c_1 + c_3 \xi^*. \end{aligned} \tag{3.25}$$

As we can see by comparing (3.25) with (3.17), "integration" = "derivation". The somewhat hand wavy definition of integration is motivated by the fact that it gives results that reminds us about results from the theory for complex functions.

3.2.4 The number operator

Generally, we have that

$$a_\mu^\dagger \left|\phi\right\rangle \neq \left|\phi\right\rangle \qquad\qquad c_\mu^\dagger \left|\xi\right\rangle \neq \left|\xi\right\rangle , \tag{3.26}$$

and so

$$a_\mu^\dagger a_\mu \left|\phi\right\rangle \neq N_\mu^\phi \left|\phi\right\rangle \tag{3.27}$$

$$c_\mu^\dagger c_\mu \left|\xi\right\rangle \neq N_\mu^\xi \left|\xi\right\rangle . \tag{3.28}$$

The coherent states are not eigenstates of the counting operator. $\left|\phi\right\rangle , \left|\xi\right\rangle$ are not states with a fixed number of "quants".

3.2.5 Algebra

Consider a vector space with the following additional properties:

$$\begin{array}{ll} 1) & (xy)z = x(yz) \\ 2) & x(y+z) = xy + xz \\ 3) & (x+y)z = xz + yz \\ 4) & \alpha xy = x\alpha y. \end{array}$$

In Abelian algebra, $xy = yx$, while in Grassmann algebra $xy = -yx$. Complex numbers are generators for the Abelian algebra over the field $\mathbb{G}$ of commuting numbers. Grassmann numbers are generators for the algebra over the field $\mathbb{G}$ of anticommuting numbers.

4 Construction of coherent states for bosons, and its properties

Pages 18-26 in lecture notes.

4.1 Construction

The definition of a coherent boson state $|\phi\rangle$ is stated in (3.1). We make an ansatz – a qualified guess – that $|\phi\rangle$ can be created as

$$|\phi\rangle = \mathrm{e}^{\sum_\mu \varphi_\mu a_\mu^\dagger} |0\rangle . \tag{4.1}$$

We claim that

$$a_\mu \mathrm{e}^{\sum_\nu \varphi_\nu a_\nu^\dagger} |0\rangle = \varphi_\mu \mathrm{e}^{\sum_\nu \varphi_\nu a_\nu^\dagger} |0\rangle \tag{4.2}$$

Proof.

$$a \sum_{n=0} \frac{\left(\varphi a^\dagger\right)^n}{n!} |0\rangle = a \sum_{n=1} \frac{\varphi^n}{n!} \left(a^\dagger\right)^n |0\rangle .$$

Note that $\left(a^\dagger\right)^n a |0\rangle = 0$, and so we wish to "commute a through".

$$\begin{aligned}
\left[a, f\left(a^\dagger\right)\right] &= \sum_{n=1} c_n \left[a, \left(a^\dagger\right)^n\right] \\
&= \sum_{n=1} n c_n \left(a^\dagger\right)^{n-1} \\
f\left(a^\dagger\right) &= \sum_{n=0}^{\infty} c_n \left(a^\dagger\right)^n \\
&\implies \left[a, f\left(a^\dagger\right)\right] = \frac{\partial}{\partial a^\dagger} f\left(a^\dagger\right) .
\end{aligned}$$

More generally:

$$\begin{aligned}
\left[g(a), f(a^\dagger)\right] &= g\left(\frac{\partial}{\partial a^\dagger}\right) f\left(a^\dagger\right) \\
\left[a, \left(a^\dagger\right)^n\right] &= n \left(a^\dagger\right)^{n-1} \\
\left[(a)^m, \left(a^\dagger\right)^n\right] &= \frac{n!}{(n-m)!} \left(a^\dagger\right)^{n-m} \\
\left\{c, f\left(c^\dagger\right)\right\} &= \frac{\partial}{\partial c^\dagger} f\left(c^\dagger\right) \\
\left\{g(c), f(c^\dagger)\right\} &= g\left(\frac{\partial}{\partial c^\dagger}\right) f\left(c^\dagger\right)
\end{aligned}$$

To find out what the commutator $\left[a,\left(a^\dagger\right)^n\right]$ is, we use that

$$[A,BC]=[A,B]C+B[A,C], \tag{4.3}$$

with $A=a, B=a^\dagger, C=\left(a^\dagger\right)^{n-1}$ to get

$$\begin{aligned}
\left[a,\left(a^\dagger\right)^n\right] &= \left(a^\dagger\right)^{n-1}+a^\dagger\left[a,\left(a^\dagger\right)^{n-1}\right]\\
&= a\left(a^\dagger\right)^{n-1}\\
\implies a\left(a^\dagger\right)^n|0\rangle &= n\left(a^\dagger\right)^{n-1}|0\rangle\\
\implies a\mathrm{e}^{\varphi a^\dagger}|0\rangle &= \sum_{n=1}^{\infty}\frac{\varphi^n n\left(a^\dagger\right)^{n-1}}{n!}|0\rangle\\
&= \varphi\sum_{n=1}^{\infty}\frac{\left(\varphi a^\dagger\right)^{n-1}}{(n-1)!}|0\rangle\\
&= \varphi\mathrm{e}^{\varphi a^\dagger}|0\rangle .
\end{aligned}$$

□

Then, for more modes (quantum numbers), we get

$$|\phi\rangle=\mathrm{e}^{\sum_\mu\varphi_\mu a_\mu^\dagger}|0\rangle \tag{4.4}$$

with the φ_μ's satisfying

$$a_\mu|\phi\rangle=\varphi_\mu|\phi\rangle . \tag{4.5}$$

Coherent states: "difficult to destroy".

We also could have done this in a more direct way: Assume

$$|\phi\rangle=\prod_\mu f_\mu\left(\varphi,a_\mu^\dagger\right)|0\rangle \tag{4.6}$$

with

$$a_\mu=\varphi_\mu|\phi\rangle . \tag{4.7}$$

Then, we need that

$$a_\mu f_\mu\left(\varphi,a_\mu^\dagger\right)|0\rangle=\varphi_\mu f_\mu\left(\varphi,a_\mu^\dagger\right)|0\rangle , \tag{4.8}$$

but

$$a_\mu f_\mu|0\rangle=[a_\mu,f_\mu]|0\rangle \tag{4.9}$$

$$\implies \frac{\partial}{\partial a_\mu^\dagger}f_\mu=\varphi_\mu f_\mu \tag{4.10}$$

$$\implies \frac{\mathrm{d}f_\mu}{f_\mu}=\varphi_\mu\,\mathrm{d}a_\mu^\dagger \tag{4.11}$$

$$\ln f_\mu=\varphi_\mu a_\mu^\dagger \tag{4.12}$$

$$f_\mu=\mathrm{e}^{\varphi_\mu a_\mu^\dagger}. \tag{4.13}$$

4.2 Properties

We will now look at some of the properties of coherent bosonic states.

$$a^\dagger_\mu \left|\phi\right\rangle = a^\dagger_\mu \mathrm{e}^{\sum_\mu \varphi_\nu a^\dagger_\nu} \left|0\right\rangle \tag{4.14}$$
$$= \frac{\partial}{\partial \varphi_\mu} \left|\phi\right\rangle \tag{4.15}$$

$$\left\langle\phi\right| = \left\langle 0\right| \mathrm{e}^{\sum_\mu \varphi^*_\mu a_\mu} \implies \tag{4.16}$$
$$\left\langle\phi\right| a_\mu = \frac{\partial}{\partial \varphi^*_\mu} \left\langle\phi\right| \tag{4.17}$$

The overlap of two coherent bosonic states are

$$\langle\phi|\sigma\rangle = \langle 0|\mathrm{e}^{\sum_\mu \varphi^*_\mu a_\mu} \mathrm{e}^{\sum_\nu \sigma_\nu a^\dagger_\nu}|0\rangle\,. \tag{4.18}$$

Now define

$$A = \sum_\mu \varphi^*_\mu a_\mu$$
$$B = \sum_\nu \sigma_\nu a_\nu$$

such that

$$\langle\phi|\sigma\rangle = \langle 0|\mathrm{e}^A \mathrm{e}^B|0\rangle\,. \tag{4.19}$$

We see that

$$\langle 0|\mathrm{e}^B \mathrm{e}^A|0\rangle = 1. \tag{4.20}$$

Baker-Hausdorff:

$$e^{A+B} = \mathrm{e}^A \mathrm{e}^B \mathrm{e}^{-\frac{1}{2}[A,B]} \tag{4.21}$$
$$= \mathrm{e}^B \mathrm{e}^A \mathrm{e}^{-\frac{1}{2}[B,A]}, \tag{4.22}$$

where $[A,B]$ commutes with A, B.

$$\implies \mathrm{e}^A \mathrm{e}^B = \mathrm{e}^B \mathrm{e}^A \mathrm{e}^{[A,B]} \tag{4.23}$$
$$[A,B] = \sum_{\mu,\nu} \varphi^*_\mu \sigma_\nu \underbrace{\left[a_\mu, a^\dagger_\nu\right]}_{\delta_{\mu\nu}} \tag{4.24}$$
$$= \sum_\mu \varphi^*_\mu \sigma_\mu \tag{4.25}$$
$$\implies \tag{4.26}$$
$$\langle\phi|\sigma\rangle = \mathrm{e}^{\sum_\mu \varphi^*_\mu \sigma_\mu} \underbrace{\langle 0|\mathrm{e}^B \mathrm{e}^A|0\rangle}_{=1} \tag{4.27}$$
$$= \mathrm{e}^{\sum_\mu \varphi^*_\mu \sigma_\mu}, \tag{4.28}$$

the states are not orthogonal!

For the normalization of $|\phi\rangle$ we have

$$\langle\phi|\phi\rangle = \mathrm{e}^{\sum_\mu \varphi_\mu^* \varphi_\mu} \tag{4.29}$$

$$= \mathrm{e}^{\langle N\rangle}. \tag{4.30}$$

$\langle N\rangle$ is the average number of particles in the state $|\phi\rangle$

$$\frac{\langle\phi|\sum_\mu a_\mu^\dagger a_\mu|\phi\rangle}{\langle\phi|\phi\rangle} = \langle N\rangle \tag{4.31}$$

$$= \sum_\mu \varphi_\mu^* \varphi_\mu \tag{4.32}$$

$$\left\langle (\Delta N)^2 \right\rangle = \frac{1}{\langle\phi|\phi\rangle}\left[\langle\phi|\hat{N}^2|\phi\rangle - \left(\langle\phi|\hat{N}|\phi\rangle\right)^2\right] \tag{4.33}$$

$$= \frac{1}{\langle\phi|\phi\rangle}\left[\langle\phi|\sum_{\mu,\nu} a_\mu^\dagger a_\mu a_\nu^\dagger a_\nu|\phi\rangle - \left(\sum_\mu \varphi_\mu^* \varphi_\mu\right)^2\right] \tag{4.34}$$

$$= \sum_\mu \varphi_\mu^* \varphi_\mu = N. \tag{4.35}$$

4.2.1 Coherent states for a one-bosonic oscillator

Following the construction from (4.1), we have

$$|z\rangle = \mathrm{e}^{za^\dagger}\,|0\rangle \tag{4.36}$$

$$= \sum_{n=0}^{\infty} \frac{z^n \left(a^\dagger\right)^n}{n!}\,|0\rangle = \sum_n \frac{z^n}{\sqrt{n!}}\frac{\left(a^\dagger\right)}{\sqrt{n!}}\,|0\rangle \tag{4.37}$$

$$= \sum_n \frac{z^n}{\sqrt{n!}}\,|n\rangle \tag{4.38}$$

$$\frac{1}{2\pi i}\int \mathrm{d}z\,\mathrm{d}z^*\,\mathrm{e}^{-z^*z}\,|z\rangle\langle z| \tag{4.39}$$

$$= \frac{1}{2\pi i}\int \mathrm{d}z\,\mathrm{d}z^*\,\mathrm{e}^{-zz^*}\sum_{n,m}\frac{z^n}{\sqrt{n!}}\frac{(z^*)^m}{\sqrt{m!}}\,|m\rangle\langle m| \tag{4.40}$$

$$= \frac{1}{2\pi i}\sum_{n,m}\frac{1}{\sqrt{n!m!}}\,|n\rangle\langle m|\int \mathrm{d}z\,\mathrm{d}z^*\,\mathrm{e}^{-zz^*} z^n\,(z^*)^m \tag{4.41}$$

$$= \frac{1}{\pi}\sum_{n,m}\frac{1}{\sqrt{n!m!}}\,|n\rangle\langle m|\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\left(x^2+y^2\right)}\,(x+iy)^n)\,(x-iy)^m\,. \tag{4.42}$$

Now, the integral in (4.42) is

$$\begin{aligned}
&\frac{1}{\pi}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\left(x^2+y^2\right)}\left(x+iy)^n\right)(x-iy)^m \\
&=\frac{1}{\pi}\int \mathrm{d}\rho\,\rho\,\mathrm{d}\theta\,\mathrm{e}^{-\rho^2}\left(\rho\mathrm{e}^{i\theta}\right)^n\left(\rho\mathrm{e}^{-i\theta}\right)^m \\
&=\frac{1}{\pi}\underbrace{\int \mathrm{d}\theta\,\mathrm{e}^{i\theta(n-m)}}_{2\pi\delta_{nm}}\int \mathrm{d}\rho\,\rho^{n+m+1}\mathrm{e}^{-\rho^2} \\
&=2\delta_{nm}\int \mathrm{d}\rho\,\rho^{2n+1}\mathrm{e}^{-\rho^2} \\
&=\delta_{nm}\int_0^\infty \mathrm{d}r\,r^{-\frac{1}{2}}r^{\frac{2n+1}{2}}e^{-r}=\int_0^\infty \mathrm{d}r\,r^n\mathrm{e}^{-r} \\
&=\delta_{nm}n!,
\end{aligned} \tag{4.43}$$

such that (4.39) becomes

$$\begin{aligned}
\frac{1}{2\pi i}\int \mathrm{d}z\,\mathrm{d}z^*\,\mathrm{e}^{-z^*z}\,|z\rangle\langle z| &= \sum_{n,m}\frac{1}{\sqrt{n!m!}}\,|n\rangle\langle m|\cdot\delta_{nm}n! \\
&= \sum_n |n\rangle\langle n| = 1.
\end{aligned}$$

Coherent states stays coherent under the propagation of time, but with a different label:

$$\begin{aligned}
\mathrm{e}^{\frac{-iHt}{\hbar}}\,|z\rangle &= \mathrm{e}^{\frac{-i\hbar\omega a^\dagger a t}{\hbar}}\,|z\rangle = \mathrm{e}^{-i\omega t a^\dagger a}\,|z\rangle \\
&= \mathrm{e}^{-i\omega t a^\dagger a}\sum_n \frac{z^n}{\sqrt{n!}}\,|n\rangle = \sum_n \frac{z^n}{\sqrt{n!}}\mathrm{e}^{-i\omega t a^\dagger a}\,|n\rangle \\
&= \sum_n \frac{z^n}{\sqrt{n!}}\mathrm{e}^{-i\omega t n}\,|n\rangle = \sum_n \frac{\left(z\mathrm{e}^{-i\omega t}\right)^n}{\sqrt{n!}}\,|n\rangle \\
&= \left|z\mathrm{e}^{-i\omega t}\right\rangle
\end{aligned}$$

The propagator in this basis is thus very simple:

$$\langle z_n|\mathrm{e}^{\frac{-iHt}{\hbar}}|z_0\rangle = \left\langle z_n\middle|z_0\mathrm{e}^{-i\omega t}\right\rangle \tag{4.44}$$

$$= \mathrm{e}^{z_n^* z_0 \mathrm{e}^{-i\omega t}} \tag{4.45}$$

4.2.2 The completeness relation for coherent boson states

$$
\begin{aligned}
&\int \left(\prod_\mu \frac{\mathrm{d}\varphi_\mu^* \, \mathrm{d}\varphi_\mu}{2\pi i} \right) \mathrm{e}^{-\sum_\mu \varphi_\mu^* \varphi_\mu} \, |\phi\rangle\langle\phi| \\
&= \int \left(\prod_\mu \frac{\mathrm{d}\varphi_\mu^* \, \mathrm{d}\varphi_\mu}{2\pi i} \right) \mathrm{e}^{-\sum_\mu \varphi_\mu^* \varphi_\mu} \mathrm{e}^{\sum_\mu \varphi_\mu a_\mu^\dagger} \, |0\rangle\langle 0| \, \mathrm{e}^{-\sum_\mu a_\mu \varphi_\mu^*} \\
&= \int \left(\prod_\mu \frac{\mathrm{d}\varphi_\mu^* \, \mathrm{d}\varphi_\mu}{2\pi i} \right) \mathrm{e}^{-\sum_\mu \varphi_\mu^* \varphi_\mu} \\
&\cdot \sum_{n_\mu=0}^{\infty} \frac{\varphi_\mu^{n_\mu}}{\sqrt{n_\mu!}} \sum_{n_\nu=0}^{\infty} \frac{\varphi_\nu^{n_\nu}}{\sqrt{n_\nu!}} \cdots |n_\mu, n_\nu, \ldots\rangle \\
&\cdot \sum_{m_\mu=0}^{\infty} \frac{\left(\varphi_\mu^*\right)^{m_\mu}}{\sqrt{m_\mu!}} \sum_{m_\nu=0}^{\infty} \frac{\left(\varphi_\nu^*\right)^{m_\nu}}{\sqrt{m_\nu!}} \cdots \langle m_\mu, m_\nu, \ldots| \\
&= \underbrace{\sum_{n_\mu=0}^{\infty} \sum_{n_\nu=0}^{\infty}}_{\sum_{\{n_\mu\}}} \cdots |n_\mu, n_\nu, \ldots\rangle\langle n_\mu, n_\nu, \ldots| \\
&= I
\end{aligned}
$$

5 Coherent states for fermions

5.1 Construction

We will also in the fermionic case make an ansatz on the construction of coherent fermionic states, somewhat similar to (4.1):

$$|\psi\rangle = \mathrm{e}^{-\sum_\mu \xi_\mu c_\mu^\dagger} |0\rangle \tag{5.1}$$
$$= \prod_\mu \left(1 - \xi_\mu c_\mu^\dagger\right) |0\rangle \tag{5.2}$$

It is simpler to show that the ansatz satisfies the definition (3.2) ($c_\mu |\psi\rangle = \xi_\mu |\psi\rangle$) for fermions than it was for bosons. Use the fact that $\xi_\mu^2 = 0$ to express the expansion of the exponential function.

$$c_\mu \prod_\nu \left(1 - \xi_\nu c_\nu^\dagger\right) |0\rangle = c_\mu |\psi\rangle \tag{5.3}$$

affects one of the products:

$$c_\mu \left(1 - \xi_\mu c_\mu^\dagger\right) |0\rangle = +\xi_\mu \underbrace{c_\mu c_\mu^\dagger}_{1 - c_\mu \dagger c_\mu} |0\rangle \tag{5.4}$$
$$= +\xi_\mu |0\rangle \tag{5.5}$$
$$= \xi_\mu \left(1 - \xi_\mu c_\mu^\dagger\right) |0\rangle \tag{5.6}$$
$$\Longrightarrow$$
$$c_\mu |\psi\rangle = \xi_\mu |\psi\rangle\,, \tag{5.7}$$

where we used the anticommutation relations $\{\xi_\mu, c_\mu\} = \left\{\xi_\mu, c_\mu^\dagger\right\} = 0$.

5.2 Properties

5.2.1 Creation operator

Acting with the creation operator on a coherent fermion state:

$$\begin{aligned} c_\mu^\dagger |\psi\rangle &= c_\mu^\dagger \left(1 - \xi_\mu c_\mu^\dagger\right) \prod_{\nu \neq \mu} \left(1 - \xi_\nu c_\nu^\dagger\right) |0\rangle \\ &= -\frac{\partial}{\partial \xi_\mu} \left(1 - \xi_\mu c_\mu^\dagger\right) \prod_{\nu \neq \mu} \left(1 - \xi_\nu c_\nu^\dagger\right) |0\rangle \\ &= -\frac{\partial}{\partial \xi_\mu} |\psi\rangle\,. \end{aligned}$$

Similarly, on the "bra" vectors:

$$\begin{aligned}\langle\psi|\, c_\mu &= \prod_\mu \langle 0|\left(1+\xi_\mu^* c_\mu\right) c_\mu \\ &= \frac{\partial}{\partial \xi_\mu^*}\prod_\mu \langle 0|\left(1+\xi_\mu^* c_\mu\right) \\ &= \frac{\partial}{\partial \xi_\mu^*}\langle\psi|\end{aligned}$$

NB: Note the plus sign in the product.

5.2.2 Overlap

The overlap between two coherent fermion states:

$$\begin{aligned}\langle\psi|\psi'\rangle &= \langle 0|\prod_{\mu,\nu}\left(1+\xi_\nu^* c_\nu\right)\left(1-\xi_\mu c_\mu^\dagger\right)|0\rangle \\ &= \prod_{\mu,\nu}\langle 0|\left(1+\xi_\nu^* c_\nu \xi_\mu c_\mu^\dagger\right)|0\rangle \\ &= \prod_{\nu\neq\mu}\cdot 1\prod_\mu\left(1+\xi_\mu^*\xi_\mu\right) \\ &= \prod_\mu\left(1+\xi_\mu^*\xi_\mu\right) \\ \text{Re-exponentiation} \implies \langle\psi|\psi'\rangle &= \mathrm{e}^{\sum_\mu \xi_\mu^*\xi_\mu}.\end{aligned}$$

We have used $\{c_\mu, \xi_\mu\} = 0$.

5.2.3 Completeness relation

The completeness relation for fermion coherent states is

$$\int\prod_\mu \mathrm{d}\xi_\mu^*\,\mathrm{d}\xi_\mu \mathrm{e}^{-\sum_\mu \xi_\mu^*\xi_\mu}\,|\xi\rangle\langle\xi| = 1. \tag{5.8}$$

Proof. **For one mode:**

$$
\begin{aligned}
&\int \mathrm{d}\xi\, \mathrm{d}\xi^*\, \mathrm{e}^{-\xi^*\xi} \mathrm{e}^{-\xi c^\dagger}\, |0\rangle\langle 0|\, \mathrm{e}^{-c\xi^*} \\
&= \int \mathrm{d}\xi\, \mathrm{d}\xi^*\, (1-\xi^*\xi)\left(1-\xi c^\dagger\right) |0\rangle\langle 0|\, (1-c\xi^*) \\
&= \int \mathrm{d}\xi\, \mathrm{d}\xi^* \left[1-\xi^*\xi-\xi^\dagger\right] |0\rangle\langle 0|\, (1+\xi^* c) \\
&= \int \mathrm{d}\xi\, \mathrm{d}\xi^* \left[(1-\xi^*\xi)\, |0\rangle - \xi c^\dagger\, |0\rangle\right] \left[\langle 0| + \xi^*\, \langle 0|\, c\right] \\
&= \int \mathrm{d}\xi\, \mathrm{d}\xi^* \left[(1-\xi^*\xi)\, |0\rangle\langle 0| - \xi c^\dagger\, |0\rangle\langle 0| \right. \\
&\qquad\qquad \left. +\ (1-\xi^*\xi)\, |0\rangle\, \xi^*\, \langle 0|\, c - \xi c^\dagger\, |0\rangle\, \xi^*\, \langle 0|\, c\right] \\
&= \int \mathrm{d}\xi\, \mathrm{d}\xi^* \left[(1-\xi^*\xi)\, |0\rangle\langle 0| - \xi\, |1\rangle\langle 0| \right. \\
&\qquad\qquad \left. +\ (1-\xi^*\xi)\, \xi^*\, |0\rangle\langle 1| + \xi\xi^*\, |1\rangle\langle 1|\right] \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = 1
\end{aligned}
$$

For multiple modes:

$$
\begin{aligned}
&\int \prod_\mu \mathrm{d}\xi_\mu\, \mathrm{d}\xi^*{}_\mu\, \mathrm{e}^{-\sum_\mu \xi_\mu^*\xi_\mu}\, |\xi\rangle\langle \xi| \\
&= \int \prod_\mu \mathrm{d}\xi_\mu\, \mathrm{d}\xi^*\, \mu \mathrm{e}^{-\sum_\mu \xi_\mu^*\xi_\mu} \mathrm{e}^{-\sum_\mu \xi_\mu c_\mu^\dagger}\, |0\rangle\langle 0|\, \mathrm{e}^{-\sum_\mu c_\mu \xi_\mu^*} \\
&= \int \left(\prod_\mu \mathrm{d}\xi_\mu^*\, \mathrm{d}\xi_\mu\right) \left(\prod_\mu (1-\xi_\mu^*\xi_\mu)\right) \left(\prod_\mu (1-\xi_\mu c_\mu^\dagger)\right) \\
&\qquad\qquad \times |0\rangle\langle 0| \left(\prod_\mu (1+\xi_\mu^* c_\mu)\right)
\end{aligned}
$$

We can treat $\xi_\mu^*\xi_\mu, \xi_\mu c_\mu^\dagger$ etc. as ordinary numbers when we change places, since they commute.

□

The trace of an operator:

$$
\begin{aligned}
\operatorname{Tr} A &= \sum_n \langle n|A|n\rangle \\
&= \int \prod_\mu \mathrm{d}\xi_\mu^* \, \mathrm{d}\xi_\mu \mathrm{e}^{-\sum_\mu \xi_\mu^* \xi_\mu} \sum_n \langle n|\xi\rangle \, \langle \xi|A|n\rangle \\
&= \int \prod_\mu \mathrm{d}\xi_\mu^* \, \mathrm{d}\xi_\mu \mathrm{e}^{-\sum_\mu \xi_\mu^* \xi_\mu} \underbrace{\sum_n \langle -\xi|A|n\rangle \, \langle n|\xi\rangle}_{\langle -\xi|A|\xi\rangle} \\
&= \int \prod_\mu \mathrm{d}\xi_\mu^* \, \mathrm{d}\xi_\mu \mathrm{e}^{-\sum_\mu \xi_\mu^* \xi_\mu} \, \langle -\xi|A|\xi\rangle \qquad (5.9)
\end{aligned}
$$

$\hat{N} = \sum_\mu c_\mu^\dagger c_\mu$is the number operator, as usual. What is the mean value of this operator in a fermion coherent state?

$$
\frac{\langle \xi|\hat{N}|\xi\rangle}{\langle \xi|\xi\rangle} = \sum_\mu \frac{\langle \xi|c_\mu^\dagger c_\mu|\xi\rangle}{\langle \xi|\xi\rangle} \qquad (5.10)
$$

$$
= \sum_\mu \xi_\mu^* \xi_\mu \qquad (5.11)
$$

This is neither a real nor complex number! It is therefore meaningless to talk about the mean value of number of fermions in a coherent state.

In (5.9) we used a property that is not true in general, but is under the integral.

$$
\begin{aligned}
\langle \psi|\xi\rangle &= c_0 + c_1 \xi \\
\langle \xi|\psi\rangle &= d_0 + d_1 \xi^*
\end{aligned}
$$

Terms linear in ξ, ξ^* is zero under Grassmann integration

$$
\begin{aligned}
|\xi\rangle &\equiv \mathrm{e}^{\xi c^\dagger} |0\rangle \\
|-\xi\rangle &= \mathrm{e}^{-\xi c^\dagger} |0\rangle \\
&\neq -|\xi\rangle
\end{aligned}
$$

Such that

$$
\langle \psi|\xi\rangle \, \langle \xi|\psi\rangle \neq \langle -\xi|\psi\rangle \, \langle \psi|\xi\rangle \,, \qquad (5.12)
$$

but it comes out correct in the integral. We used this as

$$
\int \mathrm{d}\xi \, \mathrm{d}\xi^* \, \langle \psi|\xi\rangle \, \langle \xi|\psi\rangle = \int \mathrm{d}\xi \, \mathrm{d}\xi^* \, \langle -\xi|\psi\rangle \, \langle \psi|\xi\rangle \qquad (5.13)
$$

The reason for this fundamental difference between Bosonic and fermionic coherent states lies in the Pauli exclusion principle and the definition of coherent states.

With a given set of one-particle states, together with the Pauli princliple, a physical state must have a fixed, determinable number of particles, and cannot be an eigenstate of a annihilation operator. The fermionic coherent states therefore lay outside the Hilbert space of physical states, and need not represent observable states. For bosons, the symmetric property means that even with a given set of quantum numbers, physical states can be an eigenstate of the annihilation operator. This is because each one particle state can assume an arbitrary number of quants. Boson coherent states are thus physical. They are in fact physical states naturally occurring when taking the classical limit of a quantum field theory. Also in lasers.

When we considered the trace (eq (5.9)) of an operator A for both for both fermionic and bosonic coherent states, we had to consider the matrix elements

$$\begin{aligned} &\langle\phi|A|\phi\rangle && \textbf{(Bosons)} \\ &\langle-\psi|A|\psi\rangle && \textbf{(Fermions)} \end{aligned}$$

For bosons: Assume that $A(a_\mu^\dagger, a_\mu)$ are normal ordinals ; all a_μ are placed to the right of $a_\mu^\dagger$-

$$a_\mu \left|\phi\right\rangle = \varphi_\mu \left|\phi\right\rangle \tag{5.14}$$
$$\left(a_\mu\right)^n \left|\phi\right\rangle = \left(\varphi_\mu\right)^n \left|\phi\right\rangle \tag{5.15}$$
$$A\left(a_\mu\right) \left|\phi\right\rangle = A\left(\varphi_\mu\right) \left|\phi\right\rangle , \tag{5.16}$$

thus

$$\langle\phi|A\left(a_\mu^\dagger, a_\mu\right)|\phi'\rangle = A\left(\varphi_\mu^*, \varphi_{\mu'}\right) \langle\phi|\phi'\rangle \tag{5.17}$$
$$= A\left(\varphi_\mu^*, \varphi_{\mu'}\right) \mathrm{e}^{\sum_\mu \varphi_\mu^* \varphi_{\mu'}} . \tag{5.18}$$

Similarly,

$$\langle\psi|A\left(c_\mu^\dagger, c_\mu\right)|\psi'\rangle = A\left(\xi_\mu^*, \xi_\mu'\right) \mathrm{e}^{\sum_\mu \xi_\mu^* \xi_\mu'} \tag{5.19}$$

Thus, the calculation of expectation values reduces to quadratures; multiple integrals over $(\varphi_\mu^*, \varphi_\mu)$ or (ξ_μ^*, ξ_μ).

6 Free electron gas

We start with the Hamiltonian

$$\begin{aligned}\mathcal{H} &= \sum_{k,\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} \\ &= \sum_\sigma \int \mathrm{d}x\, \psi^\dagger_\sigma(x) \varepsilon(\nabla) \psi_\sigma(x). \end{aligned} \tag{6.1}$$

The partition function is

$$\mathcal{Z} = \int \mathcal{D}\left[\varphi^*(\tau)\right] \mathcal{D}\left[\varphi(\tau)\right] \mathrm{e}^{\mathcal{S}} \tag{6.2}$$

where $\varphi_\lambda(0) = -\varphi_\lambda(\beta)$ (antiperiodic for fermions) and

$$\mathcal{S} = -\sum_\lambda \int_0^\beta \mathrm{d}\tau \left[\varphi^*_\lambda \frac{\partial \varphi_\lambda}{\partial \tau} + \mathcal{H}\left(\{\varphi^*_\lambda, \varphi_\lambda\}\right)\right] \tag{6.3}$$

Now choose quantum numbers $\lambda = (k, \sigma)$ because $\mathcal{H}$ is diagonal in the plane wave basis. Then,

$$\mathcal{S} = -\sum_{k,\sigma} \int_0^\beta \mathrm{d}\tau\, \varphi^*_{k\sigma}(\tau) \left(\frac{\partial}{\partial \tau} + \varepsilon_k\right) \varphi_{k\sigma}(\tau) \tag{6.4}$$

where $\{\varphi_{k\sigma}(\tau)\}$ are Grassman variables. $\mathcal{Z}$ now becomes a Gaussion integral over Grassmann variables, which we have seen earlier. By direct insertion of this result, we find

$$\mathcal{Z} = \mathrm{e}^{\mathrm{Tr}\ln(\partial_\tau + \varepsilon_k)} = \prod_{k,\sigma} \left(1 + \mathrm{e}^{-\beta\varepsilon_k}\right) \tag{6.5}$$

with

$$\mathrm{Tr} = \sum_{k,\sigma} \int_0^\beta \mathrm{d}\tau \cdot \mathrm{tr} \tag{6.6}$$

where "tr" here is the trace of the operator $\ln\left(\partial_\tau + \varepsilon_k\right)$

$$\mathrm{tr} \ln\left(\partial_\tau + \varepsilon_k\right) = \sum_n \langle n | \ln\left(\partial_\tau + \varepsilon_k\right) | n \rangle\,. \tag{6.7}$$

To be able to get a <u>local</u> expression for $\ln\left(\partial_\tau + \varepsilon_k\right)$, the choice of a plane wave basis for $|n\rangle$ is convenient.

$$|n\rangle = u_{nk} = \frac{1}{\sqrt{\beta}} \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega_n \tau)} \tag{6.8}$$

where

$$\omega_n = \frac{(2n+1)\,\pi}{\beta}. \tag{6.9}$$

The reason for this choice of ω_n is that we see that this ensures $u_{nk}(\tau)$ to have the same antiperiodic properties as $\varphi_\lambda(\beta)$ When we take the trace only over such states, the requirement $\varphi_\lambda(0) = -\varphi_\lambda(\beta)$ is automatically satisfied.

$$\begin{aligned}
&\sum_n \langle n|\ln(\partial_\tau + \varepsilon_k)|n\rangle \\
&= \frac{1}{\beta}\sum_{\omega_n} \mathrm{e}^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)} \ln(\partial_\tau + \varepsilon_k)\, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)}.
\end{aligned} \tag{6.10}$$

Before we continue, we investigate the trace of an arbitrary operator

$$\operatorname{tr}\ln A = \sum_n \langle n|A|n\rangle\,. \tag{6.11}$$

$\ln A$ is defined by its series expansion

$$\begin{aligned}
\ln A &= \ln(1 + A - 1) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(A-1)^k\,,
\end{aligned} \tag{6.12}$$

such that

$$\operatorname{tr}\ln A = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \operatorname{tr}\left[(A-1)^k\right]. \tag{6.13}$$

Define $B = A - 1$. Now choose S such that $S^{-1}BS = S^{-1}AS - 1 = D - 1$, i.e. such that A is diagonalized.

$$\begin{aligned}
\operatorname{tr}\left(B^k\right) &= \operatorname{tr}\left[(D-1)^k\right] \\
&= \sum_m (\lambda_m - 1)^k \implies \\
\operatorname{tr}\ln A &= \sum_m \sum_k \frac{(-1)^{k+1}}{k}(\lambda_m - 1)^k \\
&= \sum_m \ln(1 + \lambda_m - 1) \\
&= \sum_m \ln\lambda_m = \ln\left(\prod_m \lambda_m\right) \\
\implies \operatorname{tr}\ln A &= \ln\det A
\end{aligned} \tag{6.14}$$