Department of Physics, NTNU

Homework 6
TFY4210/FY8302 Quantum theory of solids
Spring 2021.

Problem 1

In this problem, we will consider a system defined by a Hamiltonian

$$H = H_0 + V = \sum_{k,\sigma} \varepsilon_k \; c^\dagger_{k,\sigma} c_{k,\sigma} + V$$

where V is a two-body interaction term that renders the problem not exactly solvable. In class, we have seen how we can express the single-particle electron Green's function for such a system as

$$\begin{aligned} G(k, t-t') &= -i\langle \Psi(0)|T\left[\hat{c}_k(t)\hat{c}^\dagger_k(t')\right]|\Psi(0)\rangle \\ &= -i\frac{\langle \phi_0|T\left[c_k(t)c^\dagger_k(t')S(\infty,-\infty)\right]|\phi_0\rangle}{\langle \phi_0|S(\infty,-\infty)|\phi_0\rangle} \end{aligned}$$

where $S(\infty, -\infty)$ is given by

$$S(\infty,-\infty) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T\left[V(t_1)\cdots V(t_n)\right]$$

and T is the time-ordering operator. The notation is otherwise the same as we have used in class. Assume now that the perturbation is given by the Hubbard-interaction

$$V = U\sum_i n_{i\uparrow}n_{i\downarrow} = \frac{U}{2}\sum_{i,\sigma} n_{i\sigma}n_{i-\sigma}$$

a) Write the Hubbard-interaction on the form

$$V = \sum_{k,k',q,\sigma,\sigma'} \tilde{V}(q,\sigma,\sigma') \; c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k',\sigma'} c_{k,\sigma}$$

thereby specifying $\tilde{V}(q,\sigma,\sigma')$. (Note: this interaction is spin-dependent, contrary to what the case is for the standard density-density Coulomb-interaction or the effective electron-electron interaction mediated by phonons.)

b) Use the resulting interaction and calculate the leading order correction of the denominator in the second expression for $G(k, t-t')$, and give a diagrammatic representation for it along the same lines that we used for the electron-phonon coupling in class.

c) Calculate the leading order correction of the numerator in the second expression for $G(k, t-t')$, and give a diagrammatic representation for it along the same lines that we used for the electron-phonon coupling in class.

d) Show that, to leading order in V, the denominator cancels the disconnected diagrams appearing in the numerator.