Department of Physics, NTNU

Homework 8
TFY4210/FY8916 Quantum theory of solids
Spring 2021.

Problem 1

The creation and destruction operator of a Cooper-pair is given by

$$\begin{aligned} b_{\mathbf{k}} &= c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} \\ b^{\dagger}_{\mathbf{k}} &= c^{\dagger}_{\mathbf{k},\uparrow} c^{\dagger}_{-\mathbf{k},\downarrow} \end{aligned}$$

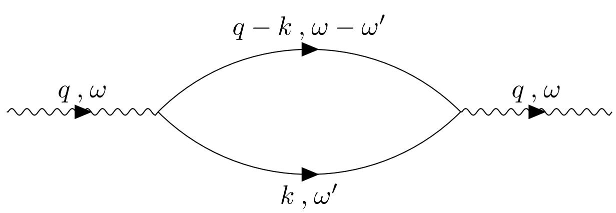
a) Compute the commutators $\left[b_{\mathbf{k}}, b^{\dagger}_{\mathbf{k}'}\right]$ and $[b_{\mathbf{k}}, b_{\mathbf{k}'}]$, and the anti-commutator $\{b_{\mathbf{k}}, b_{\mathbf{k}'}\}$. Compare your results with what you find if the operators had been boson-operators.

b) Imagine that you had two different condensed matter systems, one with a relatively strong attraction between electrons, and one with a relatively weak attraction between electrons. Which one of these two system would you think a description of Cooper-pairs as bosons would be the best approximation?

Problem 2

In Problem 1 we considered creation an destruction operators for a Cooper-pair. Consider now a bosonic *pair-fluctuation field* $\phi(\mathbf{q}, \omega)$ which can split into two electrons, and two electrons can recombine into the field $\phi(\mathbf{q}, \omega)$.

A Feynman-diagram for such a process is given in the Figure below.



$$= \chi_0(\mathbf{q}, \omega)\ K(\mathbf{q}, \omega)\ \chi_0(\mathbf{q}, \omega)$$

The wavy line is the Green's function for the free field $\phi(\mathbf{q}, \omega)$, denoted $\chi_0(\mathbf{q}, \omega)$. Denote the bubble-diagram by $K(\mathbf{q}, \omega)$. Consider now a Dyson-equation for the Green's function $\chi(\mathbf{q}, \omega)$ for the pairing field

$$\begin{aligned} \chi(\mathbf{q}, \omega)^{-1} &= \chi_0(\mathbf{q}, \omega)^{-1} - K(\mathbf{q}, \omega) \\ \chi_0(\mathbf{q}, \omega)^{-1} &= V \end{aligned}$$

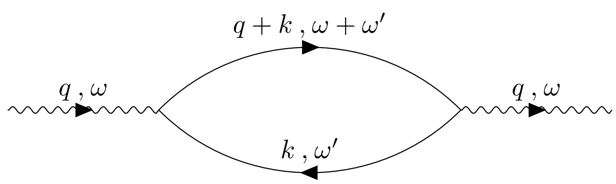
Here, $V =$ is a constant attractive interaction that works in a thin shell around the Fermi-surface.

a) Use the Feynman-rules to compute the integral over ω' in the bubble-diagram, taking care to express your answer in terms of $T = 0$ Fermi-distribution $\theta(\varepsilon_F - \varepsilon_{\mathbf{k}})$ and its complementary distribution $\theta(\varepsilon_{\mathbf{k}} - \varepsilon_F)$, and corresponding quantities with $\mathbf{k} \to \mathbf{k} + \mathbf{q}$.

b) Generalize this to finite temperatures $T > 0$ by replacing the θ-functions by finite-temperature Fermi-distributions and simplify the expression by letting $\mathbf{q} \to 0$.

c) $\chi(\mathbf{q}, \omega)$ may be given a physical interpretation as a *pair-susceptibility*, i.e. the ability an electron system has to form pairs of electrons, Cooper-pairs. The divergence of this susceptibility indicates an instability of the Fermi-sea of electrons. Find the temperature at which this instability takes place by explicitly computing the remaining $\mathbf{k}$-sum. (Hint: Consider the case $\mathbf{q} = 0$ and convert the sum over $\mathbf{k}$ to an energy integral and approximate the single-particle density of states by its value on the Fermi-surface.)

Note how very differently the particle-particle bubble $K(\mathbf{q}, \omega)$ behaves from the particle-hole bubble $\Pi(\mathbf{q}, \omega)$ we considered in class.



$= D_0(\mathbf{q}, \omega)\ \Pi(\mathbf{q}, \omega)\ D_0(\mathbf{q}, \omega)$

There, we found that $\lim_{\mathbf{q}\to 0} \Pi(\mathbf{q}, \omega = 0) = -2N(\varepsilon_F)$. This is essentially T-independent.