Title goes here

Some information about a cool theorem or anything i find interesting, related equations (here: Dirac-equation)

$$(i\gamma_\mu \partial^\mu - m)\psi = 0$$

Some more information here

Ehrenfests teorem

Hvorfor kollapser ikke bølgefunksjonen til hverdagslige ting (som f.eks. en stol)? Korrespondanseprinsippet forteller oss at i grensen av store antall partikler og høye energier, må kvantemekanikk være ekvivalent med klassisk fysikk. Paul Ehrenfest fant en måte å relatere tidsutviklingen i forventningsverdien av kvantemekaniske operatorer til forventningsverdien av kraften som virker på systemet. Fra Newtons andre lov er dette forventet, men Ehrenfests teorem gir en matematisk trygghet. En generalisering av teoremet kan skrives på formen

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle O \rangle = -i\hbar \langle [O, \mathcal{H}] \rangle + \left\langle \frac{\partial O}{\partial t} \right\rangle$$

hvor O er en operator som korresponderer til en observerbar størrelse og $\mathcal{H}$ er Hamiltonian til systemet.

The Hubbard Model

The Hubbard model is often called the "Ising model" of strongly correlated quantum systems, and an exact solution to this lattice model is only known in one dimension except in special limits. It is given by the Hamiltonian

$$\mathcal{H} = -\sum_{i,j,\sigma} t_{ij} c^{\dagger}_{i\sigma} c_{j\sigma} + \sum_{i,\sigma} U_i n_{i,\sigma} n_{i,-\sigma}.$$

where the first term is a tight-binding approximation of the system near its Fermi-energy, and the second term describes the on-site energy cost of having doubly occupied lattice sites.

Hubbard-Stratonovich-decoupling

The HS-decoupling is a useful identity if you are troubled by too many interacting fermions in your system and would prefer a theory written in terms of (hopefully) noninteracting bosons. The transformation is based on the identity

$$\mathrm{e}^{-\frac{a}{2}\psi^2} = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} \mathrm{d}\varphi \, \mathrm{e}^{-\left(\frac{\varphi^2}{2a} + i\varphi\psi\right)},$$

where ψ represent fermions through Graßman variables, and φ represent bosonic field(s). This transformation is used in all kinds of quantum field theories. A discrete version can be used to transform the *quantum mechanical* Hubbard model in d dimensions into a *classical* Ising-like model in $d+1$ dimensions.

Superconductivity - BCS theory (1)

The Bardeen-Cooper-Schrieffer theory of superconductivity is the most celebrated microscopic theory describing the macroscopic phenomena of superconductivity. In this theory, an effective attractive potential between electrons in vicinity of the Fermi surface leads to the condensation of "Cooper pairs", reducing the free energy of the system. After some crude simplifications, a model Hamiltonian reads

$$\mathcal{H} = \sum_{k,\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{k,k'} V_{kk'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{k'\downarrow} c_{k'\uparrow},$$

where $V_{kk'}$ is attractive in a thin shell close to the Fermi surface. This attractive potential can for instance be mediated by interacting with quantized lattice vibrations – phonons.

Superconductivity - BCS theory (2)

Since this Hamiltonian is quartic in fermion-operators, solving it exactly is very difficult. However, a mean field treatment is possible by letting $c_{-k\downarrow}c_{k\uparrow} = \langle c_{-k\downarrow}c_{k\uparrow}\rangle$ + fluctuations, and the system might be solved. The self-consistent BCS gap-equation states

$$\Delta_k = -\sum_{k'} V_{kk'}\Delta_{k'}\chi_{k'}$$

$$\chi_k = \frac{1}{\sqrt{\varepsilon_k^2 + \Delta_k^2}}\tanh\left(\frac{\beta}{2}\sqrt{\varepsilon_k^2 + \Delta_k^2}\right),$$

where $\beta = \frac{1}{k_B T}$. This is a "gap"-equation precisely because the condensation of Cooper pairs opens a gap Δ_k in the electronic excitation spectrum. A goal of modern physics is to find systems whose superconducting gap is as large as possible.

Superconductivity - BCS theory (3)

The computation of Δ_k often requires heavy numerical computations, but some further analytic work can be done by assuming a constant attractive potential up to a Debye energy, where it is taken to be zero. Remarkably, it turns out that the ratio of the gap Δ_k at zero temperature to the critical temperature (the temperature where $\Delta_k = 0$) is a purely numerical constant given by

$$\frac{\Delta(T=0)}{k_B T_C} = \pi \mathrm{e}^{-\gamma},$$

where γ is the Euler-Mascheroni constant. This constant fits fairly well with the experimentally observed values for several materials, and is reason to celebrate the BCS - theory as a whole.

Schrieffer-Wolff transformation

We have seen that attractive interactions between electrons give rise to a condensation of Cooper pairs and a finite gap in the electron band structure. Kohn & Luttinger proposed in 1965 a Cooper pairing mechanism with the assumption that the effective interactions between electrons is attractive. This mechanism is obtained by what is now called the Schrieffer-Wolff transformation, a unitary transformation of the interacting Hamiltonian

$$\begin{aligned}\mathcal{H}' &= \mathrm{e}^{-S}\mathcal{H}\mathrm{e}^{S} \\ &= \mathcal{H}_0 + V + [\mathcal{H}, S] + \frac{1}{2}[[\mathcal{H}, S], S] + \dots\end{aligned}$$

By choosing S such that $[\mathcal{H}_0, S] = -V$, we obtain to $\mathcal{O}(V^2)$

$$\mathcal{H}' = \mathcal{H}_0 + \frac{1}{2}[V, S] \equiv \mathcal{H}_0 + V'$$

Although this looks pretty similar to the starting point, this transformation is the first step of obtaining a gap equation for finite T, with unrestricted scattering momenta!

@karlockert

Topological Insulators - 1

These materials are "topological" in the sense that certain quantities of the system is left invariant when it undergoes smooth transformations in the parameter space. An example of such a property is the Chern invariant n.
This number can be calculated as the total Berry flux through the Brillouin zone from each band.

$$H(\boldsymbol{k}) = E_{\boldsymbol{k}} I + \mathbf{d}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma},$$

the Chern number can be expressed in terms of the normalized vector $\hat{\mathbf{d}}_{\boldsymbol{k}} = \mathbf{d}_{\boldsymbol{k}} / |\mathbf{d}_{\boldsymbol{k}}|$ alone;

$$n = \frac{1}{4\pi} \int_{\mathsf{1BZ}} \mathrm{d}^2\boldsymbol{k}\, \hat{\mathbf{d}}_{\boldsymbol{k}} \cdot (\partial_{k_x} \hat{\mathbf{d}}_{\boldsymbol{k}} \times \partial_{k_y} \hat{\mathbf{d}}_{\boldsymbol{k}}).$$

Topological Insulators - 2

Haldane showed in 1988 that for a particular model of the hexagonal lattice – such as Graphene, the hall conductance is quantized in terms of this integer

$$\sigma^{xy} = n\frac{2\pi e^2}{\hbar}$$

where the Chern number $n = 0, \pm 1$ is related to the number of edge states through what is now called the bulk-boundary correspondence. This model of spinless fermions on a hexagonal lattice is famously called the "Haldane model". The generalisation with spin degrees of freedom included led to the prediction of Spin Hall insulators – a two dimensional rendition of a topological insulator.

Field Operators in Many-Body QM

The field operators of a system is written as the single-particle wave function times a creation / annihilation operator which increases or decreases the occupancy of said state, summed over all available states in the system.

$$\psi^\dagger(x,t) = \sum_\alpha \varphi^*_\alpha(x) c^\dagger_\alpha(t)$$

The time-dependency of $c^\dagger_\alpha$ is determined by the Hamiltonian of the system, and is in the Heisenberg picture written as

$$c^\dagger_\alpha(t) = \mathrm{e}^{i\hat{H}t} c^\dagger_\alpha \mathrm{e}^{-i\hat{H}t}$$

(with $\hbar = 1$)

Linear response - The Kubo formula

In linear response theory, the Kubo formula relates a non-equilibrium quantum property with its value at equilibrium. The setup is as follows; start with a qantity O you want to measure, and a time independent Hamiltonian H_0. Then, at time $t = t_0$, a time-dependent (small) perturpation H' is added to the system. The Kubo formula states that the (now time-dependent) observed value for O will be (to linear order)

$$\langle O(t) \rangle = \langle O \rangle_0 - i \int_{t_0}^{t} \mathrm{d}t' \left\langle [\hat{O}(t), \hat{H}'(t')] \right\rangle_0 .$$

This can be highly useful, since it can give relatively good approximations to what will happen when one suddenly makes small changes to the physical system. (Example: how conductivity changes when suddenly changing a weak magnetic field)

Liouville's theorem (1838)

Liouville's theorem is is a key insight in Hamiltonian mechanics, and a fundamental theorem of statistical physics. It states that under canonical transformations, an arbitrary volume of phase space is left invariant.

$$\frac{\partial \rho}{\partial t} + \sum_k \left(\frac{\partial \rho}{\partial q_k} \frac{\partial q_k}{\partial t} + \frac{\partial \rho}{\partial p_k} \frac{\partial p_k}{\partial t} \right) = 0$$

Here, ρ is the phase space density, the number of state configurations inside a given phase space volume. q_k and p_k are canonical coordinates and momenta, respectively. You may recognize the second term as the Poisson bracket $\{\rho, H\}$

Bosonization of fermions in one dimension

Fermions, as we know, are subject to the Pauli Exclusion principle. This is also true in one dimension, with the dramatic consequence that single-particle excitations are singular (no electrons can "move" past another)! The field operator of a given fermion species can be written in terms of bosonic(!) operators $a_\mu^\dagger, a_\mu$ as

$$\psi_\mu(x) = F_\mu \frac{1}{\sqrt{L}} \mathrm{e}^{-i\frac{2\pi}{L}\hat{N}_\mu x} \mathrm{e}^{-i\phi_\mu^\dagger(x)} \mathrm{e}^{-i\phi_\mu(x)}$$

with

$$\phi_\mu^\dagger(x) \equiv -i \sum_{q>0} \left(\frac{2\pi}{L|q|}\right)^{\frac{1}{2}} \mathrm{e}^{-\xi\frac{|q|}{2}} \mathrm{e}^{-iqx} a_{q,\mu}^\dagger.$$

Here, F_μ is a "Klein-operator", increasing the occupancy number N_μ of fermion species μ relative to the vacuum. Luttinger showed that in one dimension any interaction breaks the discontinuity at the Fermi-surface. This renders traditional Fermi-liquid theory inapplicable to these systems.