GALOIS EXTENSIONS AND A CONJECTURE OF OGG

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Abstract. Let $N = pq$, where $p = 2, 3, 5, 7, 13$ and $q \neq p$ is another prime. We propose a general strategy for proving a conjecture of Ogg about isogenies from the new quotient of $J_0(N)$ to the Jacobian of the Shimura curve attached to the indefinite quaternion algebra over $\mathbf{Q}$ of discriminant N. This strategy is based on the results of Helm and Ribet. Using this strategy, we prove a general conditional result toward Ogg's conjecture and discuss in detail the case when $N = 65$.

1. Introduction

Let $p = 2, 3, 5, 7, 13$ and $q \neq p$ be another prime. Let $N = pq$. Let $J_0(N)$ be the Jacobian of the modular curve $X_0(N)$ and J^N be the Jacobian of the Shimura curve X^N attached to a maximal order in the indefinite quaternion algebra over $\mathbf{Q}$ of discriminant N. Let $J_0(N)^{\mathrm{old}}$ be the abelian subvariety of $J_0(N)$ generated by the images of $J_0(q)$ in $J_0(N)$ under the maps induced by the two degeneracy morphisms $X_0(N) \rightrightarrows X_0(q)$ (note that $J_0(p) = 0$). Let $J_0(N)^{\mathrm{new}}$ be the quotient of $J_0(N)$ by $J_0(N)^{\mathrm{old}}$. A conjecture of Ogg predicts that there exists an isogeny $\pi : J_0(N)^{\mathrm{new}} \to J^N$ whose kernel is a specific group arising from the cuspidal divisor subgroup of $J_0(N)$; see [Ogg85]. The fact that $J_0(N)^{\mathrm{new}}$ and J^N are $\mathbf{Q}$-isogenous for general square-free N with an even number of prime factors was proved by Ribet [Rib80], but Ribet's proof does not give any information about the kernels of isogenies.

Ogg's conjecture remains largely open except for a handful of cases when X^N is hyperelliptic [GR04, GM16] or $N = 5 \cdot 13 = 65$ [KP18]. The approach to Ogg's conjecture in [GR04, GM16] relies on explicit calculation of equations defining the Shimura curves, whereas in [KP18] we used the Hecke equivariance of Ribet isogenies and the fact that the Hecke algebra of level 65 is a rather simple ring. In this article we continue exploring avenues that lead to partial results confirming Ogg's conjecture. While we again employ the Hecke algebra, we propose a different approach from [KP18] which has the advantage of being applicable to larger values of N than 65.

Now we outline our approach and state the main results. To simplify the notation, let $J := J_0(N)$ and $J' := J^N$. Let $\mathbf{T}$ be the $\mathbf{Z}$-subalgebra of $\mathrm{End}(J)$ generated by the Hecke operators T_r for primes $r \nmid N$ and U_r for primes $r \mid N$. Let S denote the finite set of maximal ideals of $\mathbf{T}$ that are either Eisenstein, or of residue characteristic 2 or 3. (The Eisenstein ideal $\mathcal{E}$ of $\mathbf{T}$ is the ideal generated by all $T_r - (r+1)$

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for primes $r \nmid N$; the Eisenstein maximal ideals are the maximal ideals containing $\mathcal{E}$.) There is an element $\sigma_S \in \mathbf{T}$ such that for any maximal ideal $\mathfrak{m}$ of $\mathbf{T}$, one has $\sigma_S \in \mathfrak{m}$ if and only if $\mathfrak{m} \in S$ (cf. Lemma 3.2 in [Hel07]). Set $\mathbf{T}_S := \mathbf{T}[\sigma_S^{-1}]$.

J^{new} and J' have purely toric reduction at the primes p and q, and good reduction everywhere else. For $A = J^{\mathrm{new}}$ or J', denote by $M_p(A) = \mathrm{Hom}(\mathcal{A}^0_{\overline{\mathbf{F}}_p}, \mathbf{G}_{m,\overline{\mathbf{F}}_p})$ the character group of A at p. Here $\mathcal{A}$ is the Néron model of A over $\mathbf{Z}_p$, and $\mathcal{A}^0_{\mathbf{F}_p}$ is the connected component of the identity of the special fibre of $\mathcal{A}$ at p. The character group $M_p(A)$ is a free abelian group of rank equal to $\dim(A)$. We similarly define the character group $M_q(A)$ at q. By the Néron mapping property, $\mathbf{T}$ acts on $M_p(A)$ and $M_q(A)$.

A special case of a result of Helm [Hel07, Prop. 8.13] implies that there is an isomorphism of $\mathbf{T}_S$-modules

$$\mathrm{Hom}(J^{\mathrm{new}}, J') \cong_{\mathbf{T}_S} \mathrm{Hom}(M_q(J'), M_q(J^{\mathrm{new}})). \tag{1.1}$$

On the other hand, a special case of a result of Ribet [Rib90, Thm. 4.1] implies that

$$M_q(J') \cong_{\mathbf{T}} M_p(J^{\mathrm{new}}). \tag{1.2}$$

Since the cuspidal divisor group of J is annihilated by the Eisenstein ideal of $\mathbf{T}$, Ogg's conjecture combined with (1.1) and (1.2) implies that

$$M_p(J^{\mathrm{new}}) \cong_{\mathbf{T}_S} M_q(J^{\mathrm{new}}). \tag{1.3}$$

Conversely, if (1.3) is true, then (1.1) and (1.2) imply that there is an isogeny $\pi : J^{\mathrm{new}} \to J'$ whose kernel is supported on the maximal ideals in S.

This offers a natural strategy for proving Ogg's conjecture. First, one needs to prove (1.3). Since the character groups are free $\mathbf{Z}$-modules, this step involves only linear algebra calculations, which may be quite daunting in practice - but we note here that there exist algorithms that allow one to do this at least in principle; cf. section 3. The second step comprises classifying isogenies supported on the maximal ideals in S. This can be achieved by excluding the existence of certain subgroup schemes in $J[\mathfrak{m}^s]$ for $\mathfrak{m} \in S$, a problem which in [KP18] (for $N = 65$) was handled by an ad hoc counting argument.

In this paper we offer a more systematic approach for step 2 based on the non-existence of certain deformations of non-split Galois extensions

$$0 \to \mathbf{Z}/\ell \to \overline{\rho} \to \mu_\ell \to 0. \tag{1.4}$$

More precisely, let $\mathfrak{m} \in S$ and write ℓ for the residue characteristic of $\mathfrak{m}$. We assume that $\ell \geq 5$. Then $\mathfrak{m}$ is a new Eisenstein maximal ideal. By the results of Ohta and Yoo [Oht14, Yoo16], we know that ℓ divides one of the following numbers $(p-1)(q+1)$, $(p+1)(q-1)$ or $(p-1)(q-1)$, i.e., ℓ divides either $p \pm 1$ or $q \pm 1$. However, $\mathfrak{m}$ is new only if $\ell \nmid (p-1)(q-1)$, we are thus left with the cases $\ell \mid p+1$ (which can only happen when $p = 13$) or $\ell \mid q+1$. Furthermore, for ℓ as above which does not divide both $p+1$ and $q+1$ a theorem of Ribet and Yoo [Yoo16] guarantees that $\dim_{\mathbf{F}_\ell} J[\mathfrak{m}] = 2$. In particular the completed Hecke algebra $\mathbf{T}_{\mathfrak{m}}$ is Gorenstein and the action of $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $J[\mathfrak{m}]$ gives rise to an extension

$$0 \to \mathbf{Z}/\ell \to J[\mathfrak{m}] \to \mu_\ell \to 0. \tag{1.5}$$

This extension does not split. Indeed, by a theorem of Vatsal [Vat05], the extension (1.5) splits if and only if $\mu_\ell \subset \mathcal{S}$, where $\mathcal{S}$ denotes the Shimura subgroup of J.

Ignoring the 2-primary and 3-primary torsion, one has $\#\mathcal{S} = (p-1)(q-1)$; cf. [LO91]. Thus for $\ell \nmid (p-1)(q-1)$ we see that $\mu_\ell \not\subset \mathcal{S}$. Hence (1.5) can in fact be viewed as a non-split extension of Galois modules of the form (1.4). We also note that, ignoring the 2-primary and 3-primary torsion, the cuspidal divisor group $\mathcal{C}$ of J and the Eisenstein ideal $\mathcal{E}$ satisfy (cf. [CL97], [Oht14], [Yoo16])

$$\mathbf{T}/\mathcal{E} \cong \mathcal{C} \cong \mathbf{Z}/(p-1)(q-1) \oplus \mathbf{Z}/(p+1)(q-1) \oplus \mathbf{Z}/(p-1)(q+1).$$

This implies that $\mathfrak{m}$ is the unique Eisenstein maximal ideal of residue characteristic ℓ and the constant subgroup scheme of $J[\mathfrak{m}]$ in (1.5) is $\mathcal{C}[\ell]$.

In Theorem 2.9 (and Corollary 2.10) we prove that under the above assumptions on ℓ, the Galois representation $\overline{\rho} := J[\mathfrak{m}]$ does not admit any (non-trivial) reducible (Fontaine-Laffaille) deformations of determinant ϵ, the ℓ-adic cyclotomic character (or its mod ℓ^m reduction). This allows us to prove the following result which is the main theorem of the paper.

Theorem 1.1. *Let S be the set of new Eisenstein maximal ideals. Let $\ell \neq 2, 3$ be a prime such that ℓ divides $p+1$ or $q+1$ but not both. Assume further that*

(i) $M_p(J^{\mathrm{new}}) \cong_{\mathbf{T}_S} M_q(J^{\mathrm{new}})$ *and*
(ii) $J/J[\mathfrak{m}] \cong J$ *for $\mathfrak{m} \in S$ with residue characteristic ℓ.*

Then there is a Ribet isogeny $\pi : J^{\mathrm{new}} \to J'$ such that the ℓ-primary part of $\ker \pi$ *is $\mathcal{C}[\ell]$ as predicted by Ogg's conjecture.*

Let us now explain how to prove Theorem 1.1. Let $\phi : J^{\mathrm{new}} \to J'$ be any Ribet isogeny. Let H_ϕ denote the ℓ-primary part of its kernel, $\ell \neq 2, 3$. Condition (i) guarantees that H_ϕ is only supported on Eisenstein maximal ideals. Let $\mathfrak{m}$ be such with residue characteristic ℓ. If $J[\mathfrak{m}] \subset H_\phi$, then we use the isomorphism (ii) enough times to obtain an isogeny $\pi : J^{\mathrm{new}} \to J'$ with the property $J[\mathfrak{m}] \not\subset H := H_\pi$. On the other hand we have $H \subset J[\mathfrak{m}^s]$ for some $s \in \mathbf{Z}_+$. We claim that H is a proper subscheme of $J[\mathfrak{m}]$. If this is not the case (i.e., H is not a proper subscheme of $J[\mathfrak{m}]$) then we see as in the proof of Proposition 4.5 in [KP18] that without loss of generality we may assume that $s = 2$. The equivalence of (1) and (2) in Lemma 15.1 of [Maz77] implies that since $\dim_{\mathbf{F}_\ell} J[\mathfrak{m}] = 2$ we get $J[\mathfrak{m}^2] \cong \mathbf{T}/\mathfrak{m}^2 \oplus \mathbf{T}/\mathfrak{m}^2$ as $\mathbf{T}$-modules. Hence $H = \mathbf{T}/\mathfrak{m}^{s_1} \oplus \mathbf{T}/\mathfrak{m}^{s_2}$, with $0 \leq s_1 \leq s_2$. Clearly $s_1 = 0$ since otherwise $H \supset H[\mathfrak{m}] = J[\mathfrak{m}]$. Also $s_2 = 2$ as otherwise $H \subset J[\mathfrak{m}]$. Hence H is a Galois stable line (free $\mathbf{T}/\mathfrak{m}^2$-module of rank 1) in $J[\mathfrak{m}^2]$. Let χ_1 be the character by which $G_{\mathbf{Q}}$ acts on this line and write χ_2 for the character by which it acts on the quotient $J[\mathfrak{m}^2]/H$. Then the Galois representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}/\mathfrak{m}^2)$ afforded by $J[\mathfrak{m}^2]$ satisfies the conditions in Corollary 2.10 with $\{\ell_1, \ell_2\} = \{p, q\}$, $\Sigma' = \{p, q, \ell\}$ (we note that ρ is in the image of the Fontaine-Laffaille functor since it arises as a subquotient of the Galois representation afforded by the Tate module of an abelian variety), so it cannot exist. Thus, $H \subsetneq J[\mathfrak{m}]$. Finally, because $J[\mathfrak{m}]$ is non-split, the only $G_{\mathbf{Q}}$-stable subgroup of $J[\mathfrak{m}]$ is its constant subgroup $\mathbf{Z}/\ell$ which comes from the cuspidal divisor group. This is what Ogg's conjecture predicts.

To conclude the introduction, let us briefly comment on the assumptions (i) and (ii) of Theorem 1.1. Assumption (i) can be checked using an explicit matrix representation of generators of $\mathbf{T}$. In the case $N = 65$ we carry out this calculation in section 3. In fact in this case we are able to prove a stronger result, namely that $M_p(J) \cong M_q(J)$ as $\mathbf{T}$-modules without inverting the operators in S. (This also shows that (1.1) is not true without inverting the Eisenstein maximal ideals since the Jacobians J and J' are not isomorphic in this case.) Assumption (ii) follows if

one can prove that $\mathfrak{m}$ is (globally) principal. Indeed, if η is a generator then $J[\mathfrak{m}]$ is the kernel of the isogeny $J \xrightarrow{\cdot\eta} J$. In the case $N = 65$, the prime $\ell = 7$ is the only one which satisfies the conditions of Theorem 1.1 (the other two Eisenstein primes are 2 and 3) and the Eisenstein maximal ideal of residue characteristic 7 is indeed principal (cf. [KP18], section 3). Thus Theorem 1.1 gives an alternative proof that the 7-primary part of Ogg's conjecture for $N = 65$ is true.

2. Non-existence of certain Galois extensions

Let $\ell > 2$ and $\Sigma := \{\ell_1, \ell_2, \dots, \ell_k\}$ be a set of distinct primes such that $\ell \nmid \ell_i(\ell_i - 1)$ for $i = 1, \dots, k$. Write $\Sigma' := \Sigma \cup \{\ell\}$ and $G_{\Sigma'}$ for the absolute Galois group of the maximal Galois extension of $\mathbf{Q}$ unramified outside Σ'.

Consider a representation $\overline{\rho} : G_{\Sigma'} \to \mathrm{GL}_2(\mathbf{F}_\ell)$ which is a non-split extension of the form

$$\overline{\rho} = \begin{bmatrix} 1 & * \\ & \overline{\epsilon} \end{bmatrix},$$

where ϵ will denote the ℓ-adic cyclotomic character (but we will also use ϵ to denote the reduction of the ℓ-adic cyclotomic character mod ℓ^m) and $\overline{\epsilon}$ its mod ℓ reduction.

The main result of this section is Theorem 2.9 (and Corollary 2.10) which asserts the non-existence of certain trace-reducible deformations of $\overline{\rho}$. The proof essentially boils down to showing that there are no (trace-reducible) deformations to $\mathbf{Z}/\ell^2$ as well as no non-trivial (trace-reducible) deformations to the dual numbers $\mathbf{F}_\ell[X]/X^2$. We begin with the $\mathbf{Z}/\ell^2$-case – the harder of the two (Proposition 2.1 below), which we prove in a slightly greater generality than needed for our purposes. We fix once and for all an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell$. Let $m \geq 2$ be an integer.

Proposition 2.1. *Suppose* $\mathrm{val}_\ell(\ell_1^2 - 1) = m - 1$ *(which is equivalent to* $\mathrm{val}_\ell(\ell_1 + 1) = m - 1$*) and* $\mathrm{val}_\ell(\ell_i^2 - 1) = 0$ *(equivalent to* $\mathrm{val}_\ell(\ell_i + 1) = 0$*) for all* $i = 2, 3, \dots, k$*. Then there does not exist a Galois representation* $\rho_m : G_{\Sigma'} \to \mathrm{GL}_2(\mathbf{Z}/\ell^m)$ *such that*

(i) ρ_m *is crystalline in the image of the Fontaine-Laffaille functor at* ℓ*;*
(ii) $\det \rho_m = \epsilon$*;*
(iii) $\mathrm{tr}\, \rho_m = \chi_1 + \chi_2$ *for some Galois characters* $\chi_1, \chi_2 : G_{\Sigma'} \to (\mathbf{Z}/\ell^m)^\times$ *with* $\chi_1 \equiv 1$ *(mod* ℓ*) and* $\chi_2 \equiv \overline{\epsilon}$ *(mod* ℓ*);*
(iv) $\rho_m \equiv \overline{\rho}$ *mod* ℓ*.*

Remark 2.2. Below for brevity we will refer to representations in the image of the Fontaine-Laffaille functor simply as Fontaine-Laffaille representations. All the properties of such representations that we will use are stated e.g. in [BK13].

We prepare the proof of Proposition 2.1 by several lemmas.

Lemma 2.3. *We must have* $\chi_1 = 1$ *and* $\chi_2 = \epsilon$

Proof. It is enough to show that $\chi_1 = 1$ as then $\chi_2 = \epsilon$ by (ii). First note that since ρ_m is a Fontaine-Laffaille representation and the category of these is closed under taking subquotients, so is χ_1. Furthermore, χ_1 is unramified outside Σ'. Hence to prove the claim it is enough to show that the trivial character does not admit any non-trivial Fontaine-Laffaille infinitesimal deformations $\psi : G_{\Sigma'} \to (\mathbf{F}[X]/X^2)^\times$. This in turn can be proven as Proposition 9.5 of [BK13]. □

To prove Proposition 2.1 let us first note that by the main Theorem of [Urb99] if ρ_m whose trace splits as in (iii) exists then it can be conjugated to an upper-triangular representation of the form

$$\rho_m = \begin{bmatrix} \chi_1 & * \\ & \chi_2 \end{bmatrix}.$$

We can treat ρ_m as an element of $H^1(\mathbf{Q}, (\mathbf{Z}/\ell^m)(\chi_1\chi_2^{-1}))$ which does not lie in $H^1(\mathbf{Q}, (\ell\mathbf{Z}/\ell^m\mathbf{Z})(\chi_1\chi_2^{-1}))$, i.e., is of maximal order. This is so, because the extension given by ρ_m reduces mod ℓ to $\overline{\rho}$ which is not split.

For the moment we will work in a slightly greater generality and assume that $\chi_1 = 1$ and $\chi_2 = \chi = \epsilon^n$ for $n \neq 0$, however we apply it only in the case when $n = 1$. Set

$$T = \mathbf{Z}_\ell(-n) = \mathbf{Z}_\ell(\epsilon^{-n}), \quad V = \mathbf{Q}_\ell(-n), \quad W = \mathbf{Q}_\ell/\mathbf{Z}_\ell(-n)$$

and

$$W_M := \ell^{-M}\mathbf{Z}_\ell/\mathbf{Z}_\ell(-n) = \mathbf{Z}_\ell/\ell^M\mathbf{Z}_\ell(-n) = W[\ell^M],$$

where by $W[s]$ we mean the s-torsion. For a place v of $\mathbf{Q}$, and $M = V, W$ or W_M, set $H^1_{\mathrm{ur}}(\mathbf{Q}_v, M) = \ker(H^1(\mathbf{Q}_v, M) \to H^1(I_v, M))$. Then, following [Rub00], section 1.3, we set

$$H^1_f(\mathbf{Q}_v, V) := \begin{cases} H^1_{\mathrm{ur}}(\mathbf{Q}_v, V) & \text{if } v \neq \ell \\ \ker(H^1(\mathbf{Q}_v, V) \to H^1(\mathbf{Q}_v, V \otimes_{\mathbf{Q}_v} B_{\mathrm{cris}})) & \text{if } v = \ell. \end{cases}$$

We define $H^1_f(\mathbf{Q}_v, W)$ as the image of $H^1_f(\mathbf{Q}_v, V)$ in $H^1(\mathbf{Q}_v, W)$. For the finite set Σ of finite places of $\mathbf{Q}$, we then define the global Selmer group (cf. [Rub00], Definition 1.5.1):

$$\mathcal{S}^\Sigma(\mathbf{Q}, W) := \ker(H^1(\mathbf{Q}, W) \to \bigoplus_{v \notin \Sigma} \frac{H^1(\mathbf{Q}_v, W)}{H^1_f(\mathbf{Q}_v, W)}).$$

One defines $\mathcal{S}^\Sigma(\mathbf{Q}, W_M)$ similarly (cf. [Rub00], p. 22).

Lemma 2.4. *One has* $\mathcal{S}^\Sigma(\mathbf{Q}, W_M) = \mathcal{S}^\Sigma(\mathbf{Q}, W)[\ell^M]$.

Proof. By Lemma 1.5.4 of [Rub00], we get that there is a natural surjection of the left-hand side onto the right-hand side. However, the proof of that lemma uses the exact sequence in Lemma 1.2.2(i) in [Rub00] and in our case $W^{G_{\mathbf{Q}}} = 0$, which shows that the surjection is in fact an isomorphism. □

Let us first relate $\mathcal{S}^\Sigma(\mathbf{Q}, W_m)$ to $\mathcal{S}^\emptyset(\mathbf{Q}, W_m)$.

Lemma 2.5. *Suppose* $n \neq 0$, $\mathrm{val}_\ell(\ell_1^{n+1} - 1) = m - 1$ *and* $\mathrm{val}_\ell(\ell_i^{n+1} - 1) = 0$ *for all* $i = 2, 3, \ldots, k$. *Then one has*

$$\#\mathcal{S}^\Sigma(\mathbf{Q}, W_m) \leq \ell^{m-1}\#\mathcal{S}^\emptyset(\mathbf{Q}, W_m).$$

Proof. Fix $s \in \{1, 2, \ldots, k\}$. Since W is unramified at ℓ_s we get $H^1_{\mathrm{ur}}(\mathbf{Q}_{\ell_s}, W) = H^1_f(\mathbf{Q}_{\ell_s}, W)$ (by Lemma 1.3.5(iv) in [Rub00]) as well as $H^1_{\mathrm{ur}}(\mathbf{Q}_{\ell_s}, W_m) = H^1_f(\mathbf{Q}_{\ell_s}, W_m)$ (by Lemma 1.3.8(ii) in [Rub00]) and

$$H^1(I_{\ell_s}, W_m) = \mathrm{Hom}(\mathbf{Z}_\ell(1), W_m) = W_m(-1). \tag{2.1}$$

This gives an upper bound of ℓ^m on the order of the quotient $H^1(\mathbf{Q}_{\ell_s}, W_m)/H^1_f(\mathbf{Q}_{\ell_s}, W_m)$. However, let us now show that the upper bound is in fact ℓ^{m-1} (resp. 1) if $s = 1$ (resp. $s \neq 1$). Indeed, this will follow if we show that the map $H^1(\mathbf{Q}_{\ell_s}, W_m) \to$

$H^1(I_{\ell_s}, W_m)$ is not surjective (resp. is the zero map) if $s = 1$ (resp. $s \neq 1$). To do so consider the inflation-restriction sequence (where we set $G := \mathrm{Gal}(\mathbf{Q}_{\ell_s}^{\mathrm{ur}}/\mathbf{Q}_{\ell_s})$):

$$H^1(G, W_m) \to H^1(\mathbf{Q}_{\ell_s}, W_m) \to H^1(I_{\ell_s}, W_m)^G \to H^2(G, W_m).$$

The last group in the above sequence is zero since $G \cong \hat{\mathbf{Z}}$ and $\hat{\mathbf{Z}}$ has cohomological dimension one. This means that the image of the restriction map $H^1(\mathbf{Q}_{\ell_s}, W_m) \to H^1(I_{\ell_s}, W_m)$ equals $H^1(I_{\ell_s}, W_m)^G$. Let us show that the latter $\mathbf{Z}_\ell$-module is a proper submodule of $H^1(I_{\ell_s}, W_m)$ (resp. is the zero module) if $s = 1$ (resp. $s \neq 1$). Indeed,

$$H^1(I_{\ell_s}, W_m)^G = \mathrm{Hom}_G(\mathbf{Z}_\ell(1), \frac{\frac{1}{\ell^m}\mathbf{Z}_\ell}{\mathbf{Z}_\ell}(-n)) = \mathrm{Hom}_G(\mathbf{Z}_\ell, \frac{\frac{1}{\ell^m}\mathbf{Z}_\ell}{\mathbf{Z}_\ell}(-n-1)).$$

So, $\phi \in H^1(I_{\ell_s}, W_m)$ lies in $H^1(I_{\ell_s}, W_m)^G = \mathrm{Hom}_G(\mathbf{Z}_\ell, \frac{\frac{1}{\ell^m}\mathbf{Z}_\ell}{\mathbf{Z}_\ell}(-n-1))$ if and only if $\phi(x) = \phi(gx) = g \cdot \phi(x) = \epsilon^{-n-1}(g)\phi(x)$ for every $x \in I_{\ell_s}$ and every $g \in G$, i.e., if and only if

$$(\epsilon^{-n-1}(g) - 1)\phi(x) \in \mathbf{Z}_\ell \quad \text{for every } x \in I_{\ell_s},\ g \in G. \tag{2.2}$$

Since Frob_{ℓ_s} topologically generates G, we see that (2.2) holds if and only if it holds for every $x \in I_{\ell_s}$ and for $g = \mathrm{Frob}_{\ell_s}$. We have $\epsilon^{-n-1}(\mathrm{Frob}_{\ell_s}) - 1 = \ell_s^{-n-1} - 1 = \frac{1-\ell_s^{n+1}}{\ell_s^{n+1}}$. Since $\ell_s^{n+1} \in \mathbf{Z}_\ell^\times$, condition (2.2) becomes

$$(1 - \ell_s^{n+1})\phi(x) \in \mathbf{Z}_\ell \quad \text{for every } x \in I_{\ell_s}. \tag{2.3}$$

By our assumption $\mathrm{val}_\ell(1 - \ell_s^{n+1}) = m - 1$ (resp. $\mathrm{val}_\ell(1 - \ell_s^{n+1}) = 0$) if $s = 1$ (resp. $s \neq 1$), which implies that (2.3) is equivalent to $\ell^{m-1}\phi(x) = 0$ (resp. $\phi(x) = 0$) in W_m if $s = 1$ (resp. $s \neq 1$). Using the isomorphism (2.1) we see that this implies that $H^1(I_{\ell_s}, W_m)^G$ is a proper $\mathbf{Z}_\ell$-submodule of $H^1(I_{\ell_s}, W_m)$ as $W_m(-1)$ certainly contains elements not annihilated by ℓ^{m-1}.

Now, by the Poitou-Tate duality (cf. [Rub00], Theorem 1.7.3) we have an exact sequence

$$0 \to \mathcal{S}^\emptyset(\mathbf{Q}, W_m) \to \mathcal{S}^\Sigma(\mathbf{Q}, W_m) \to \bigoplus_{i=1}^k \frac{H^1(\mathbf{Q}_{\ell_i}, W_m)}{H^1_f(\mathbf{Q}_{\ell_i}, W_m)}.$$

As shown above the order of the module on the right is bounded from above by ℓ^{m-1}. This gives the desired inequality. □

Let us record here one consequence of the above proof.

Lemma 2.6. *Suppose $\mathcal{S}^\emptyset(\mathbf{Q}, W_m) = 0$. Assume $n \neq 0$, $\mathrm{val}_\ell(\ell_1^{n+1} - 1) = m - 1$ and $\mathrm{val}_\ell(\ell_i^{n+1} - 1) = 0$ for all $i = 2, 3, \ldots, k$. Then $\mathcal{S}^\Sigma(\mathbf{Q}, W_m)$ is a cyclic $\mathbf{Z}_\ell$-module, i.e., $\mathcal{S}^\Sigma(\mathbf{Q}, W_m) \cong \mathbf{Z}/\ell^s$. Furthermore, $\dim_{\mathbf{F}_\ell} \mathcal{S}^\Sigma(\mathbf{Q}, W_1) = 1$.*

Proof. From the Poitou-Tate duality (and the first isomorphism theorem for modules) we get $\mathcal{S}^\Sigma(\mathbf{Q}, W_m) \subset \frac{H^1(\mathbf{Q}_{\ell_1}, W_m)}{H^1_{\mathrm{ur}}(\mathbf{Q}_{\ell_1}, W_m)} \cong I$, where I is the image of the restriction map $H^1(\mathbf{Q}_{\ell_1}, W_m) \to H^1(I_{\ell_1}, W_m) \cong W_m$. The last module is cyclic. The one-dimensionality statement follows from this and Lemma 2.4. □

From now on set $n = 1$, so $W = \mathbf{Q}_\ell/\mathbf{Z}_\ell(-1)$.

Proposition 2.7. *The Selmer group $\mathcal{S}^\emptyset(\mathbf{Q}, W_m)$ is trivial.*

Proof. It is enough to show that the group $\mathcal{S}^{\emptyset}(\mathbf{Q}, W_1)$ is trivial. Indeed, Lemma 2.4 shows $\mathcal{S}^{\emptyset}(\mathbf{Q}, W_m) = \mathcal{S}^{\emptyset}(\mathbf{Q}, W)[\ell^m]$. So it suffices to show that $\mathcal{S}^{\emptyset}(\mathbf{Q}, W) = 0$. Since the latter module is divisible, it is enough to show that it has no ℓ-torsion, i.e., that $\mathcal{S}^{\emptyset}(\mathbf{Q}, W)[\ell] = \mathcal{S}^{\emptyset}(\mathbf{Q}, W_1) = 0$. It follows from Fontaine-Laffaille theory that $H^1_f(\mathbf{Q}_\ell, W_1) = H^1_{\mathrm{ur}}(\mathbf{Q}_\ell, W_1)$ so that $\mathcal{S}^{\emptyset}(\mathbf{Q}, W_1) = \mathrm{Hom}(\mathrm{Cl}_{\mathbf{Q}(\mu_\ell)}, W_1)^{\mathrm{Gal}(\mathbf{Q}(\mu_\ell)/\mathbf{Q})}$. The latter module is zero by Herbrand's Theorem since the relevant Bernoulli number $B_2 = 1/6$ (see e.g., Theorem 6.17 in [Was97]). □

Proof of Proposition 2.1. Assume that ρ_m as in the proposition exists. We can treat ρ_m as an element of $H^1(\mathbf{Q}, (\mathbf{Z}/\ell^m)(\chi_1\chi_2^{-1}))$ which is not annihilated by ℓ^{m-1} because its mod ℓ reduction is non-split. By Lemma 2.3 we have $\chi_1 = 1$ and $\chi_2 = \epsilon$. Also note that $\mathbf{Z}/\ell^m(\epsilon^{-1}) \cong W_m$. The extension given by ρ_m being unramified away from Σ' and Fontaine-Laffaille (at ℓ) in fact gives rise to an element inside $\mathcal{S}^{\Sigma}(\mathbf{Q}, W_m) \subset H^1(\mathbf{Q}, W_m)$ not annihilated by ℓ^{m-1}. However, combining Lemma 2.5 applied in the case $n = 1$ with Proposition 2.7 we see that $\mathcal{S}^{\Sigma}(\mathbf{Q}, W_m)$ is annihilated by ℓ^{m-1} which leads to a contradiction. □

Proposition 2.8. *Let $\rho' : G_{\Sigma'} \to \mathrm{GL}_2(\mathbf{F}_\ell[X]/X^2)$ be a representation such that*

(i) *ρ' is Fontaine-Laffaille;*
(ii) *$\det \rho' = \bar{\epsilon}$;*
(iii) *$\mathrm{tr}\, \rho' = \chi_1 + \chi_2$ for some Galois characters $\chi_1, \chi_2 : G_{\Sigma'} \to (\mathbf{F}_\ell[X]/X^2)^\times$ with $\chi_1 \equiv 1$ mod X and $\chi_2 \equiv \bar{\epsilon}$ mod X;*
(iv) *$\rho' \equiv \overline{\rho}$ mod X.*

Then ρ' is isomorphic to $\overline{\rho}$ viewed as an $\mathbf{F}_\ell[X]/X^2[G_{\Sigma'}]$-module via the natural inclusion $\mathrm{GL}_2(\mathbf{F}_\ell) \hookrightarrow \mathrm{GL}_2(\mathbf{F}_\ell[X]/X^2)$.

Proof. Using again the main theorem of [Urb99] we conclude that ρ' can be conjugated to a representation of the form $\begin{bmatrix} \chi_1 & * \\ & \chi_2 \end{bmatrix}$. Hence χ_1 and χ_2 as subquotients of ρ' are also Fontaine-Laffaille. Again arguing as in the proof of Proposition 9.5 in [BK13] we get that 1 and $\bar{\epsilon}$ do not admit any non-trivial infinitesimal Fontaine-Laffaille deformations, so we must have $\chi_1 = 1$ and $\chi_2 = \bar{\epsilon}$. This puts us in the setup of section 6 of [BK13] with Assumption 6(ii) satisfied. Hence the claim follows from Proposition 7.2 of [BK13], using Lemma 2.6 above to see that Assumption 6(i) is also satisfied. □

Let $\mathcal{L}$ be the category of local complete Noetherian $\mathbf{Z}_\ell$-algebras with residue field $\mathbf{F}_\ell$. Consider deformations $\rho' : G_{\Sigma'} \to \mathrm{GL}_2(A)$ of $\overline{\rho}$ for A an object of $\mathcal{L}$ which are such that:

- $\det \rho' = \epsilon$;
- ρ' is Fontaine-Laffaille at ℓ.

Since $\overline{\rho}$ has scalar centralizer the above deformation problem is representable (cf. [Ram93], p. 270) by a universal deformation ring R. We write $\sigma : G_{\Sigma'} \to \mathrm{GL}_2(R)$ for the universal deformation.

Let I be the ideal of reducibility of the universal deformation σ, i.e., I is the smallest ideal $I' \subset R$ such that $\mathrm{tr}\, \sigma$ is a sum of characters χ_1 and χ_2 mod I' with the property that χ_1 reduces to 1 and χ_2 reduces to $\bar{\epsilon}$ modulo the maximal ideal $\mathfrak{m}_R$ of R.

Theorem 2.9. *Suppose* $\mathrm{val}_\ell(\ell_1^2-1)=1$ *and* $\mathrm{val}_\ell(\ell_i^2-1)=0$ *for all* $i=2,3,\dots,k$. *Then* $I=\mathfrak{m}_R$.

Proof. It follows from Proposition 2.1 (and universality of R) that R/I does not admit a surjection to $\mathbf{Z}/\ell^2$. Similarly it follows from Proposition 2.8 that R/I does not admit a surjection to $\mathbf{F}[X]/X^2$. Thus I is the maximal ideal by Lemma 3.5 in [Cal06]. □

Let us explain one consequence of Theorem 2.9. If A is any object in $\mathcal{L}$ and $\rho: G_{\Sigma'}\to \mathrm{GL}_2(A)$ is a continuous representation such that

(i) ρ is Fontaine-Laffaille;
(ii) $\det\rho=\epsilon$;
(iii) $\mathrm{tr}\,\rho=\chi_1+\chi_2$ for some Galois characters $\chi_1,\chi_2: G_{\Sigma'}\to A^\times$ with $\chi_1\equiv 1 \bmod \mathfrak{m}_A$ and $\chi_2\equiv\overline{\epsilon} \bmod \mathfrak{m}_A$;
(iv) $\rho=\overline{\rho} \bmod \mathfrak{m}_A$,

then the $\mathbf{Z}_\ell$-algebra map $\phi: R\to A$ whose existence follows from universality of R factors through (by the definition of I) a $\mathbf{Z}_\ell$-algebra map $R\twoheadrightarrow \mathbf{F}_\ell=R/I\xrightarrow{\phi} A$ such that ρ is isomorphic to $\overline{\rho}$ viewed as a $A[G_{\Sigma'}]$-module via ϕ.

Corollary 2.10. *Let* $k=2$. *Suppose* $\mathrm{val}_\ell(\ell_1^2-1)=1$ *and* $\mathrm{val}_\ell(\ell_2^2-1)=0$. *Let* $\mathbf{T}$ *be the Hecke algebra as in section 1 and* $\mathfrak{m}$ *a maximal Eisenstein ideal as in Theorem 1.1. Then there does not exist a Galois representation* $\rho: G_{\Sigma'}\to \mathrm{GL}_2(\mathbf{T}/\mathfrak{m}^2)$ *such that* ρ *satisfies (i)-(iv) as above with* $A=\mathbf{T}/\mathfrak{m}^2$.

Proof. Suppose ρ as in the statement exists. Note that $\mathbf{T}/\mathfrak{m}^2$ is an object of $\mathcal{L}$. Then by universality of R we get a $\mathbf{Z}_\ell$-algebra map $\phi: R\to \mathbf{T}/\mathfrak{m}^2$. Let us first see that this map is surjective. Indeed, viewing $\mathbf{T}$ as the Hecke algebra acting on the space of weight 2 cusp forms of level $\Gamma_0(\ell_1\ell_2)$ we first complete it at the ideal $\mathfrak{m}$ and note that $\mathbf{T}_\mathfrak{m}$ is an element of $\mathcal{L}$ (since $\mathbf{T}_\mathfrak{m}/\mathfrak{m}\mathbf{T}_\mathfrak{m}=\mathbf{T}/\mathfrak{m}\mathbf{T}=\mathbf{F}_\ell$). For every minimal prime $\mathfrak{P}$ of $\mathbf{T}_\mathfrak{m}$ we have a canonical map $\mathbf{T}_\mathfrak{m}\twoheadrightarrow \mathbf{T}_\mathfrak{m}/\mathfrak{P}$ given by sending operators T_r and U_r to the eigenvalues of the corresponding cusp form. It follows from Proposition A.2.3 and A.2.2(2) in [WWE05] that the algebra $\mathbf{T}_\mathfrak{m}$ is generated by the operators T_r for $r\nmid \ell\ell_1\ell_2$. Indeed, our assumptions on the valuations of the ℓ_i imply that the Atkin-Lehner signature denoted in [WWE05] by ϵ equals $(-1,1)$ - this is forced by the condition that the constant term of the relevant Eisenstein series (cf. equation (1.3.1) in [WWE05]) vanishes modulo ℓ. In other words our Hecke algebra $\mathbf{T}_\mathfrak{m}$ equals the Hecke algebra denoted in [WWE05] by $\mathbb{T}_U^{(-1,1),0}$, which in turn equals $\mathbb{T}^{(-1,1),0}$ by Proposition A.2.3 in [WWE05]. It then follows from Proposition A.2.2 that this last Hecke algebra is generated by T_r for $r\nmid \ell\ell_1\ell_2$. Thus the intersection of all the minimal primes $\bigcap_\mathfrak{P}\mathfrak{P}$ equals 0 as it consists of all the operators T such that $Tf=0$ for all eigenforms f of $\mathbf{T}_\mathfrak{m}$. Hence in particular $\mathbf{T}_\mathfrak{m}$ injects into $\prod_\mathfrak{P}\mathbf{T}_\mathfrak{m}/\mathfrak{P}=\tilde{\mathbf{T}}_\mathfrak{m}$, where $\tilde{\mathbf{T}}_\mathfrak{m}$ is the normalization of $\mathbf{T}_\mathfrak{m}$.

We claim that the combined map $R\to\prod_\mathfrak{P}\mathbf{T}_\mathfrak{m}/\mathfrak{P}=\tilde{\mathbf{T}}_\mathfrak{m}\supset\mathbf{T}_\mathfrak{m}$ surjects onto $\mathbf{T}_\mathfrak{m}$. This is a standard argument, which we summarize here in our situation. First arguing as in the proof of Proposition 7.13 in [BK13] using Theorem 2.9 above for the cyclicity of R/I we conclude that R is generated by the set $\{\mathrm{tr}\,\sigma(\mathrm{Frob}_r)\mid r\notin\Sigma'\}$. Since each of these traces is mapped to T_r under the map $R\to\tilde{\mathbf{T}}_\mathfrak{m}$ we see that the image is contained in $\mathbf{T}_\mathfrak{m}$. In fact, it equals $\mathbf{T}_\mathfrak{m}$ as we showed above that $\mathbf{T}_\mathfrak{m}$ is generated by T_r with $r\nmid\ell\ell_1\ell_2$.

Having established the surjectivity of $R \to \mathbf{T}_{\mathfrak{m}}$ we now use (iii) above and the definition of I to conclude that the induced surjection $\phi : R \twoheadrightarrow \mathbf{T}/\mathfrak{m}^2$ factors through a $\mathbf{Z}_\ell$-algebra map $R \twoheadrightarrow R/I \xrightarrow{\phi} \mathbf{T}/\mathfrak{m}^2$. However, $R/I \cong \mathbf{F}_\ell$ by Theorem 2.9 implying that $\mathfrak{m} = \mathfrak{m}^2$, which is absurd. □

3. Character groups of $J_0(65)$ as Hecke modules

In this section $J := J_0(65)$. In this case, $J = J^{\mathrm{new}}$. Let M_p denote the character group of J at p as defined in the introduction. For $p = 5, 13$, M_p is a free abelian group of rank $\dim(J) = 5$. By the Néron mapping property, the action of the Hecke algebra $\mathbf{T}$ on J extends canonically to an action on the Néron model $\mathcal{J}$ of J over $\mathbf{Z}_p$. For $p = 5, 13$, $\mathbf{T}$ acts faithfully on $\mathcal{J}^0_{\mathbf{F}_p}$, and hence also on M_p (because J has purely toric reduction at p). The main result of this section is the fact that M_5 and M_{13} are isomorphic as $\mathbf{T}$-modules. The proof is based on explicit calculations with Brandt matrices; cf. [Gro87].

Remark 3.1. The algebra $\mathbf{T} \otimes \mathbf{Q}$ is semi-simple of dimension 5 over $\mathbf{Q}$. Since $\mathbf{T} \otimes \mathbf{Q}$ acts faithfully on $M_p \otimes \mathbf{Q}$, $p = 5, 13$, which is also 5-dimensional over $\mathbf{Q}$, one easily concludes that $M_p \otimes \mathbf{Q}$ is free over $\mathbf{T} \otimes \mathbf{Q}$ of rank 1. Thus, $M_5 \otimes \mathbf{Q} \cong M_{13} \otimes \mathbf{Q}$ as $\mathbf{T}$-modules, but the isomorphism over $\mathbf{Z}$ is more subtle.

Proposition 3.2. *There are isomorphisms of* $\mathbf{T}$*-modules* $M_5 \cong M_{13} \cong \mathbf{T}$.

Proof. The following `Magma` routine computes the action of T_n on M_5 for a given positive integer n:

```
> B5:= BrandtModule(5, 13);
> M5:= CuspidalSubspace(B);
> Sn:=HeckeOperator(M5, n);
```

The result is an explicit matrix $S_n \in M_5(\mathbf{Z})$. Repeating the same process with the roles of 5 and 13 interchanged, we get another matrix $S'_n \in M_5(\mathbf{Z})$ by which T_n acts on M_{13} (with respect to implicit $\mathbf{Z}$-bases chosen by the program).

A calculation with discriminants shows that $\mathbf{T}$, as a free $\mathbf{Z}$-module of rank 5, is generated by the Hecke operators $T_1, T_2, T_3, T_5, T_{11}$; cf. [KP18, Sec. 3]. We have

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 0 & -1 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 & -1 \\ -1 & 2 & 1 & 0 & -2 \\ 2 & -1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, \quad S_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$S_{11} = \begin{bmatrix} 0 & 3 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 1 & -2 & -1 & 2 & 1 \\ -2 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

In `Magma`, the action of Hecke operators on M_p is defined to be from the right, i.e., as on row vectors. Let $v = [1,0,0,0,0] \in M_5$, and

$$A := \begin{bmatrix} vS_1 \\ vS_2 \\ vS_3 \\ vS_5 \\ vS_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 & 0 \end{bmatrix}.$$

One easily verifies that $\det(A) = 1$, hence

$$M_5 = \mathbf{Z}vS_1 + \mathbf{Z}vS_2 + \mathbf{Z}vS_3 + \mathbf{Z}vS_5 + \mathbf{Z}vS_{11} = v\mathbf{T}.$$

Thus, $M_5 \cong \mathbf{T}$ is a free $\mathbf{T}$-module of rank 1. A similar calculation with M_{13}, gives

$$A' := \begin{bmatrix} vS'_1 \\ vS'_2 \\ vS'_3 \\ vS'_5 \\ vS'_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -2 & -1 & 0 & 2 \end{bmatrix}.$$

In this case, $\det(A) = -1$, hence again $M_{13} = v\mathbf{T}$. □

Remark 3.3. If the level $N = pq$ is as in Ogg's conjecture (i.e., $p = 2,3,5,7,13$ and $p \neq q$), then $J_0(N)$ has purely toric reduction at q, but it is unlikely that in general M_q is a free $\mathbf{T}$-module. Thus, the fact that M_5 and M_{13} are free $\mathbf{T}$-modules is a coincidence (a priori, there is no reason for this to happen). To emphasize this point, we note that the dual $M_5^* = \mathrm{Hom}(M_5, \mathbf{Z})$ of M_5 with induces action of $\mathbf{T}$ is not a free $\mathbf{T}$-module. (On M_5^* the Hecke operator T_n acts by the transpose of the matrix by which it acts on M_5). Indeed, otherwise we get $\mathbf{T} \cong \mathrm{Hom}(\mathbf{T}, \mathbf{Z})$, which implies that the localization of $\mathbf{T}$ at any maximal ideal is Gorenstein in contradiction to [KP18, Prop. 3.7].

Remark 3.4. The proof of Proposition 3.2 is rather ad hoc. Suppose more generally that we are given two $\mathbf{T}(N)$-modules M, M' for a Hecke algebra of some level N such that M, M' are free of the same finite rank over $\mathbf{Z}$ and $M \otimes_{\mathbf{Z}} \mathbf{Q} \cong_{\mathbf{T}(N)} M' \otimes_{\mathbf{Z}} \mathbf{Q}$. Also, suppose we are able to compute efficiently the matrices S_n, S'_n by which T_n acts on M and M', respectively. The question of the integral isomorphism $M \cong_{\mathbf{T}(N)} M'$ is equivalent to the existence of an invertible matrix $S \in \mathrm{GL}_r(\mathbf{Z})$ such that $SS_nS^{-1} = S'_n$ for all $n \geq 1$; here $r = \mathrm{rank}_{\mathbf{Z}}(M)$. In fact, it is enough to find such S that works for all n up to an explicit bound depending on N (the Sturm bound). Despite the elementary nature of this question, computationally it is challenging. The problem of integral conjugacy of matrices is a classical problem related to class groups of orders in number fields (see [LM33]), and there are algorithms that solve this problem (see [Sar79], [Gru80]), but currently these algorithms do not seem to be implemented in any of the standard computational programs, such as `Magma`. (Given two $m \times m$ matrices A and B with rational or integral entries, `Magma` currently can test whether A is conjugate to B in $\mathrm{GL}_m(\mathbf{Z})$ only if $m = 2$.)

References

BK13. Tobias Berger and Krzysztof Klosin, *On deformation rings of residually reducible Galois representations and $R = T$ theorems*, Math. Ann. **355** (2013), no. 2, 481–518. MR 3010137

Cal06. Frank Calegari, *Eisenstein deformation rings*, Compos. Math. **142** (2006), no. 1, 63–83. MR 2196762

CL97. Seng-Kiat Chua and San Ling, *On the rational cuspidal subgroup and the rational torsion points of $J_0(pq)$*, Proc. Amer. Math. Soc. **125** (1997), no. 8, 2255–2263. MR 1396972

GM16. Josep González and Santiago Molina, *The kernel of Ribet's isogeny for genus three Shimura curves*, J. Math. Soc. Japan **68** (2016), no. 2, 609–635. MR 3488137

GR04. Josep González and Victor Rotger, *Equations of Shimura curves of genus two*, Int. Math. Res. Not. (2004), no. 14, 661–674. MR 2038166

Gro87. Benedict H. Gross, *Heights and the special values of L-series*, Number theory (Montreal, Que., 1985), CMS Conf. Proc., vol. 7, Amer. Math. Soc., Providence, RI, 1987, pp. 115–187. MR 894322

Gru80. Fritz J. Grunewald, *Solution of the conjugacy problem in certain arithmetic groups*, Word problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976), Stud. Logic Foundations Math., vol. 95, North-Holland, Amsterdam-New York, 1980, pp. 101–139. MR 579942

Hel07. David Helm, *On maps between modular Jacobians and Jacobians of Shimura curves*, Israel J. Math. **160** (2007), 61–117. MR 2342491

KP18. Krzysztof Klosin and Mihran Papikian, *On Ribet's isogeny for $J_0(65)$*, Proc. Amer. Math. Soc. **146** (2018), no. 8, 3307–3320. MR 3803657

LM33. Claiborne G. Latimer and C. C. MacDuffee, *A correspondence between classes of ideals and classes of matrices*, Ann. of Math. (2) **34** (1933), no. 2, 313–316. MR 1503108

LO91. San Ling and Joseph Oesterlé, *The Shimura subgroup of $J_0(N)$*, Astérisque (1991), no. 196-197, 6, 171–203 (1992), Courbes modulaires et courbes de Shimura (Orsay, 1987/1988). MR 1141458

Maz77. B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 33–186 (1978). MR 488287

Ogg85. A. P. Ogg, *Mauvaise réduction des courbes de Shimura*, Séminaire de théorie des nombres, Paris 1983–84, Progr. Math., vol. 59, Birkhäuser Boston, Boston, MA, 1985, pp. 199–217. MR 902833

Oht14. Masami Ohta, *Eisenstein ideals and the rational torsion subgroups of modular Jacobian varieties II*, Tokyo J. Math. **37** (2014), no. 2, 273–318. MR 3304683

Ram93. R. Ramakrishna, *On a variation of Mazur's deformation functor*, Compositio Math. **87** (1993), no. 3, 269–286. MR 1227448

Rib80. Kenneth Ribet, *Sur les variétés abéliennes à multiplications réelles*, C. R. Acad. Sci. Paris Sér. A-B **291** (1980), no. 2, A121–A123. MR 604997

Rib90. K. A. Ribet, *On modular representations of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *arising from modular forms*, Invent. Math. **100** (1990), no. 2, 431–476. MR 1047143

Rub00. Karl Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton University Press, Princeton, NJ, 2000, Hermann Weyl Lectures. The Institute for Advanced Study. MR 1749177

Sar79. R. A. Sarkisjan, *The conjugacy problem for collections of integral matrices*, Mat. Zametki **25** (1979), no. 6, 811–824, 956. MR 540237

Urb99. E. Urban, *On residually reducible representations on local rings*, J. Algebra **212** (1999), no. 2, 738–742. MR 1676863

Vat05. V. Vatsal, *Multiplicative subgroups of $J_0(N)$ and applications to elliptic curves*, J. Inst. Math. Jussieu **4** (2005), no. 2, 281–316. MR 2135139

Was97. Lawrence C. Washington, *Introduction to cyclotomic fields*, second ed., Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1997. MR 1421575

WWE05. Preston Wake and Carl Wang-Erickson, *The Eistenstein ideal with squarefree level*, Preprint (2005), arXiv: 1804.06400.

Yoo16. Hwajong Yoo, *The index of an Eisenstein ideal and multiplicity one*, Math. Z. **282** (2016), no. 3-4, 1097–1116. MR 3473658

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