Theoretical and Numerical Studies of Efimov States

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August 26, 2019

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   - Scattering length
   - Universality
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Efimov's Prediction

- Resonant 2-body forces can give rise to a series of bound energy levels in 3-particle systems.
- When the short-ranged two-body forces approached resonance, he found a universal long-range three-body attraction emerging, giving rise to an infinite number of trimer states with binding energies obeying a discrete scaling law at resonance.

The Peculiar Efimov Effect

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- The size of each Efimov state is much larger than the force-range between the individual particle pairs $\rightarrow$ quantum mechanical state.
- When the 2B s-wave scattering length $a \rightarrow \pm\infty$ the # of bound states is infinite.
- # of 3B bound states is *reduced* as the 2B interaction is made more attractive.
- Emerge irrespective of the nature of the 2B forces and can *in principle* be observed in all quantum mechanical systems.

Scattering Length

- The 2B s-wave scattering length characterise the strength of the interparticle interaction. Definition:

$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k} \tag{1}$$

- Negative scattering lengths correspond to an attractive effective interaction.
- Positive scattering lengths correspond to a repulsive effective interaction.

Universality In 2B Systems

2B Scattering

Particles with large scattering lengths in the low-energy regime have universal properties.

Universal Properties.. In What Sense?

Depend on the scattering length alone and not on the details of the short-range interaction.

Exemple: 2B Binding Energy For 2 Identical Bosons

$$E_D = \frac{\hbar^2}{2\mu_{2b} a^2}. \tag{2}$$

Universality in 3B Systems

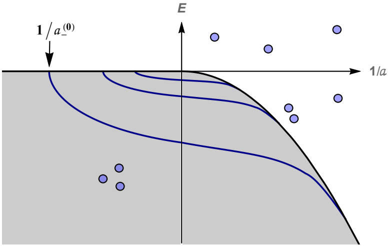


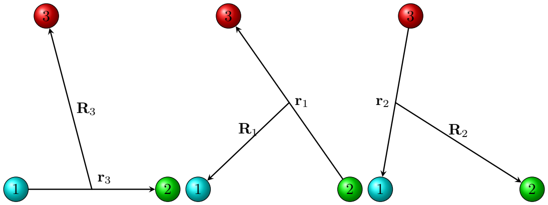
Figure: The energies of the three first Efimov states are plotted as functions of the inverse scattering length a. Three different regions can be identified in the figure. The energy levels scale geometrically: $\frac{E_T^{n+1}}{E_T^n} = e^{2\pi/s_0} \approx 515$, for identical bosons ($s_0 \simeq 1.00624$ for $J = 0^+$ states).

3-Particles, What Is The Problem?!

Apparently simple, However

- The configuration space for the 3BP is 6D after separating out the center of mass motion.
- 3 additional constants of motion can be provided by conservation of the total angular momentum.
- This leaves a three dimensional Schrödinger equation in the quantum case.

Jacobi Coordinates



Solving The 3-body Problem: Step 2, Hyperspherical Coordinates

- Separate internal and external coordinates
- Internal coordinates: 1 hyperradius: controls the size
- Internal coordinates: 2 hyperangles: shape and particle permutations

Solving The 3-body Problem: Step 3, Adiabatic Representation

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The Schrödinger equation in hyperspherical coordinates

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V(\rho,\Omega)\right)\psi(\rho,\Omega)=E\psi(\rho,\Omega). \quad (3)$$

- Treat the hyperradius as a parameter!
- $\rightarrow$ 3-body Born-Oppenheimer-like potential

$$H_{ad}\Phi_\nu(\rho;\Omega)=U_\nu(\rho)\Phi_\nu(\rho;\Omega) \quad (4)$$

Numerical Approach

Solving The 3-body Problem: Step 4, 3-Body Energies

The total wave function,

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho; \Omega), \tag{5}$$

can in this way be represented in terms of adiabatic states, which, in principle, yields an exact representation of the three-body Schrödinger equation if all couplings are included.

Solving The 3-body Problem: Step 4, Continued

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial \rho^2} + U_\mu(\rho) - \frac{1}{2\mu} Q_{\mu\mu}(\rho) \right) F_{n\mu}(\rho) - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}(\rho)\frac{\partial}{\partial\rho} + Q_{\mu\nu}(\rho) \right) F_{n\nu}(\rho) = E_n F_{n\mu}(\rho). \quad (6)$$

Emergent Attractive and Repulsive Potentials

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In the adiabatic approximation the effective potentials are defined as

$$W_\nu(\rho) = U_\nu(\rho) - \frac{1}{2\mu} Q_{\nu\nu}(\rho) = U_\nu(\rho) - \frac{1}{2\mu} P^2_{\nu\nu}(\rho). \tag{7}$$

These potentials are used for determining the single channel solutions of (6).

Convergence In the Asymptotic Limit

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For $a < 0$:

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$

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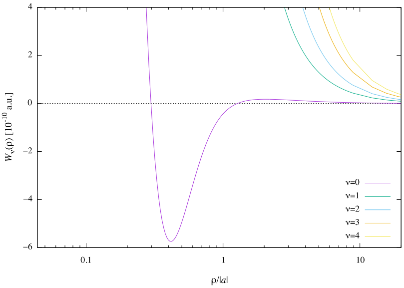

Convergence In the Asymptotic Limit

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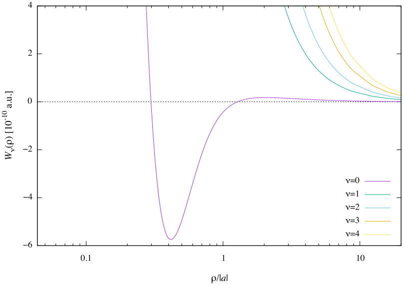

For $a > 0$:

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} E_{2b} + \frac{l(l+1)}{2\mu\rho^2}$$

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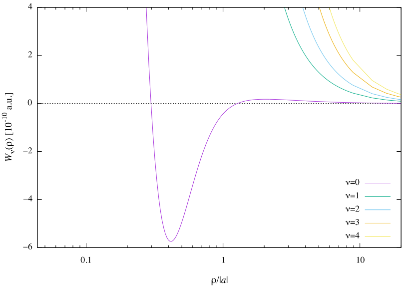

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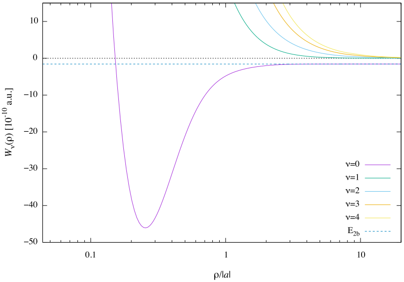

Convergence In the Intermediate Region

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Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

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Convergence In the Intermediate Region

Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

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Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

Universal Constant (3 Identical Bosons)

$$s_0 \simeq 1.00624$$

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$$\text{Task} = ?$$

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Numerical Approach; B-spline Collocation

$$\text{Task} = \text{Find } W_\nu(\rho)!$$

Numerical Approach; B-spline Collocation

First: Find a Basis

Numerical Approach; B-spline Collocation

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First: Find a Basis

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$

Numerical Approach; B-spline Collocation

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First: Find a Basis

Then: Expand

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$

Numerical Approach; B-spline Collocation

First: Find a Basis

Then: Expand

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$

Numerical Approach; B-spline Collocation

First: Find a Basis

Then: Expand

Next: Substitute Φ_ν into Eq. (4)

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$

Numerical Approach; B-spline Collocation

First:	Find a Basis	$\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
Then:	Expand	$\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
Next:	Substitute Φ_ν into Eq. (4)	$\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$

Numerical Approach; B-spline Collocation

First: Find a Basis

Then: Expand

Next: Substitute Φ_ν into Eq. (4)

Finally: Solve the Generalized Eigenvalue Eq.

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
- $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$

Numerical Approach; B-spline Collocation

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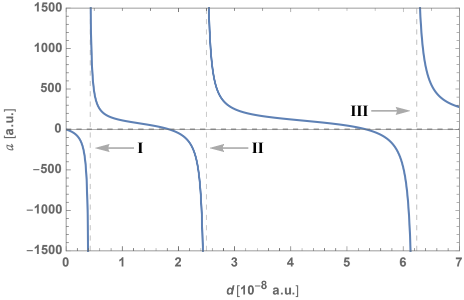
First:	Find a Basis	$\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
Then:	Expand	$\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
Next:	Substitute Φ_ν into Eq. (4)	$\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$
Finally:	Solve the Generalized Eigenvalue Eq.	$W(\rho) = U(\rho)$

Scattering Model

Masses

$$m = m(Rubidium - 87)$$



Scattering Model

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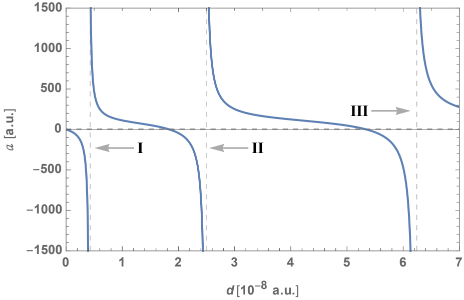
$m = m(Rubidium - 87)$

Assumption

$V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31})$



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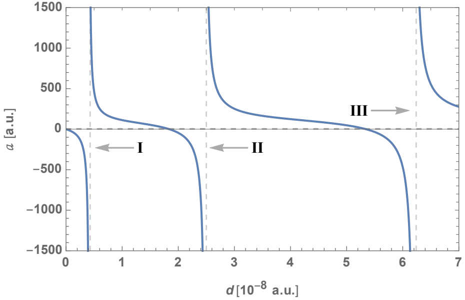
$$V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31})$$

2B Model Potential

$$v(r) = d\cosh^{-2}(r/r_0)$$

Interaction Range

$r_0 = 55$ a.u.



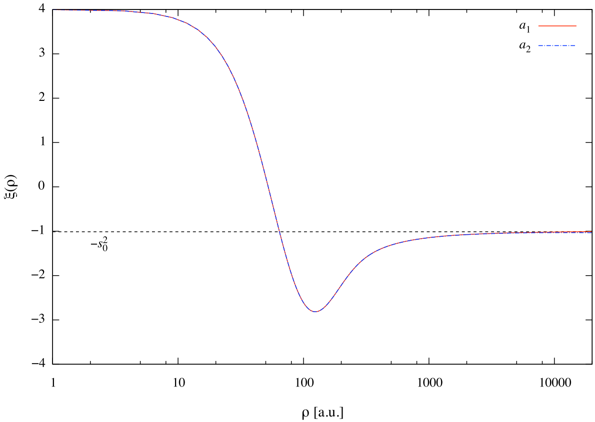
Convergence and Accuracy

- For $a \to \pm\infty$ we expect that the lowest effective potential curve will converge towards the Efimovian form
- This behaviour is easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as

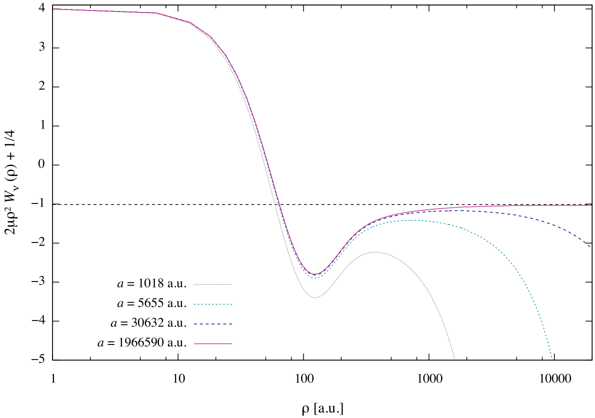
$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4} \tag{8}$$

since these curves should approach the universal value $-s_0^2 (\simeq -1.0125)$ in the intermediate region

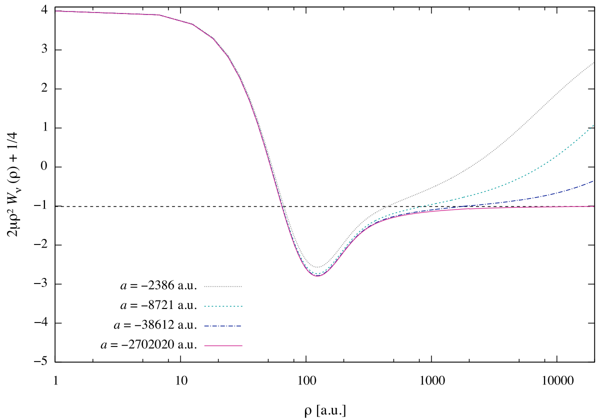
$a \to \pm\infty$, $-s_0^2 (\simeq -1.0125$ for $J = 0^+$ states$)$



$a > 0$


$2\mu\rho^2 W_\nu(\rho) + 1/4$
4
3
2
1
0
−1
−2
−3
−4
−5
1
10
100
1000
10000
ρ [a.u.]
a = 1018 a.u.
a = 5655 a.u.
a = 30632 a.u.
a = 1966590 a.u.

$a < 0$


4
3
2
1
0
−1
−2
−3
−4
−5
2μρ² W_ν (ρ) + 1/4
a = −2386 a.u.
a = −8721 a.u.
a = −38612 a.u.
a = −2702020 a.u.
1
10
100
1000
10000
ρ [a.u.]

Comparison With the Adiabatic Potential From Solving Faddeev's Equation

For $\rho \gg r_0$ the adiabatic potentials ν_n (which correspond to $\xi(\rho)$) can be determined analytically through the transcendental equation

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right), \tag{9}$$

for different scattering lengths a.

Faddeev