Efimov Physics – The Three-Body Problem

Kajsa-My Blomdahl

Stockholms Universitet

kajsamy.blomdahl@fysik.su.se

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The Peculiar Efimov Effect

- Resonant 2-body forces can give rise to a series of bound energy levels in 3-particle systems.
- The size of each Efimov state is much larger than the force-range between the individual particle pairs $\rightarrow$ quantum mechanical state.
- When the two-body s-wave scattering length $a \rightarrow \pm\infty$ the # of bound states is infinite.
- # of 3-body bound states is *reduced* as the two-body interaction is made more attractive.
- Emerge irrespective of the nature of the 2-body forces and can *in principle* be observed in all quantum mechanical systems.

Scattering Length

- The 2-body *s*-wave scattering length characterise the strength of the interparticle interaction. Definition:

$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k} \tag{1}$$

- Negative scattering lengths correspond to an attractive effective interaction.
- Positive scattering lengths correspond to a repulsive effective interaction.

The 2-body Problem

2-body Scattering

Particles with large scattering lengths in the low-energy regime have universal properties.

Universal Properties.. In What Sense?

Depend on the scattering length alone and not on the details of the short-range interaction.

Exemple: 2B Binding Energy For 2 Identical Bosons

$$E_D = \frac{\hbar^2}{2\mu_{2b} a^2}. \tag{2}$$

Universality in 3-Body Systems

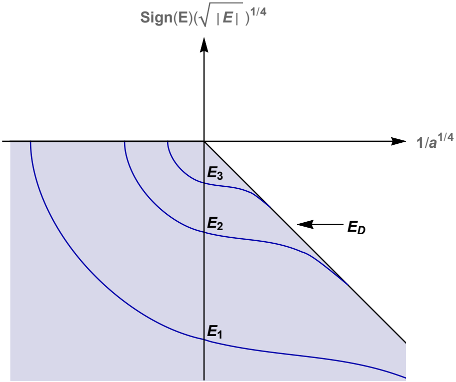


Figure: The energies of the three first Efimov states are plotted as functions of the inverse scattering length a. Three different regions can be identified in the figure. The energy levels scale geometrically: $\frac{E_T^{n+1}}{E_T^n} = e^{2\pi/s_0} \approx 515$, for identical bosons ($s_0 \simeq 1.00624$ for $J = 0^+$ states).

Cause

Emergent Attractive 3-Body Potentials

3-Particles, What Is The Problem?!

Apparently simple, However

- The configuration space for the 3BP is 6D after separating out the center of mass motion.
- 3 additional constants of motion can be provided by conservation of the total angular momentum.
- This leaves a three dimensional Schrödinger equation in the quantum case.

Solving The 3-body Problem: Step 1, Jacobi Coordinates

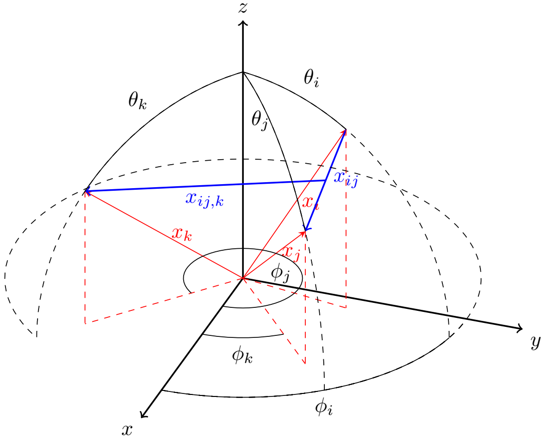


Figure: Spatial positions of three particles.

Solving The 3-body Problem: Step 2, Hyperspherical Coordinates

- Separate internal and external coordinates
- Internal coordinates: 1 hyperradius: controls the size
- Internal coordinates: 2 hyperangles: shape and particle permutations

Solving The 3-body Problem: Step 3, Adiabatic Representation

The Schrödinger equation in hyperspherical coordinates

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V(\rho,\Omega)\right)\psi(\rho,\Omega)=E\psi(\rho,\Omega). \tag{3}$$

- Treat the hyperradius as a parameter!
- $\rightarrow$ 3-body Born-Oppenheimer-like potential

$$H_{ad}\Phi_\nu(\rho;\Omega)=U_\nu(\rho)\Phi_\nu(\rho;\Omega) \tag{4}$$

Solving The 3-body Problem: Step 4, 3-Body Energies

The total wave function,

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho; \Omega), \tag{5}$$

can in this way be represented in terms of adiabatic states, which, in principle, yields an exact representation of the three-body Schrödinger equation if all couplings are included.

Solving The 3-body Problem: Step 4, Continued

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu(\rho) - \frac{1}{2\mu} Q_{\mu\mu}(\rho) \right) F_{n\mu}(\rho) \\ -\frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}(\rho)\frac{\partial}{\partial\rho} + Q_{\mu\nu}(\rho) \right) F_{n\nu}(\rho) = E_n F_{n\mu}(\rho). \tag{6}$$

Emergent Attractive and Repulsive Potentials

In the adiabatic approximation the effective potentials are defined as

$$W_\nu(\rho) = U_\nu(\rho) - \frac{1}{2\mu} Q_{\nu\nu}(\rho) = U_\nu(\rho) - \frac{1}{2\mu} P^2_{\nu\nu}(\rho). \tag{7}$$

These potentials are used for determining the single channel solutions of (6).

Effective Potentials in the Asymptotic Limit

3 free atoms $a < 0$:

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}. \tag{8}$$

Atom-dimer configurations $a > 0$:

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} E_{2b} + \frac{l(l+1)}{2\mu\rho^2}. \tag{9}$$

Effective Potentials in the Intermediate Region

Efimov physics comes into play when $r_0 \ll \rho \ll |a|$! As the scattering length grows in magnitude for $a > 0$ and $a < 0$ the lowest effective potential will converge to the Efimov potential

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}, \tag{10}$$

in which $s_0 \simeq 1.00624$.

Scattering Model

3-body Masses

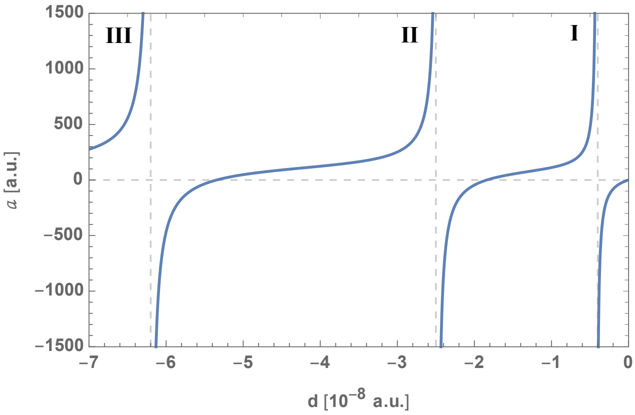
$$m = m(Rubidium - 87) \tag{11}$$

Assuming the Potential Can Be Written As a Sum of 2-B Interactions

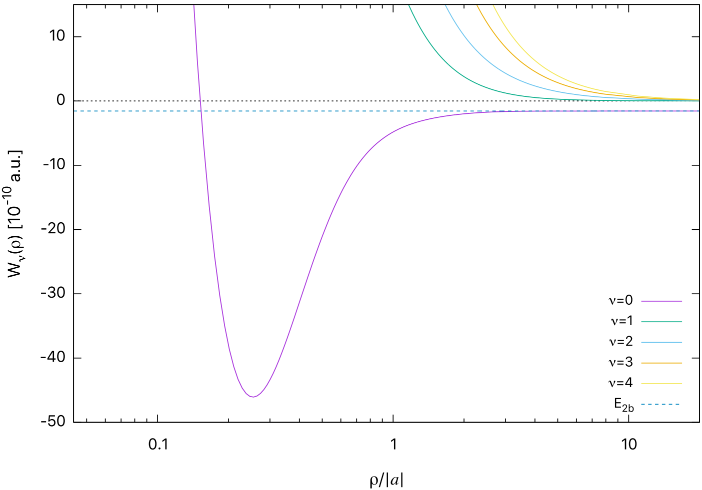
$$V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31}) \tag{12}$$

The 2-body Model Potential

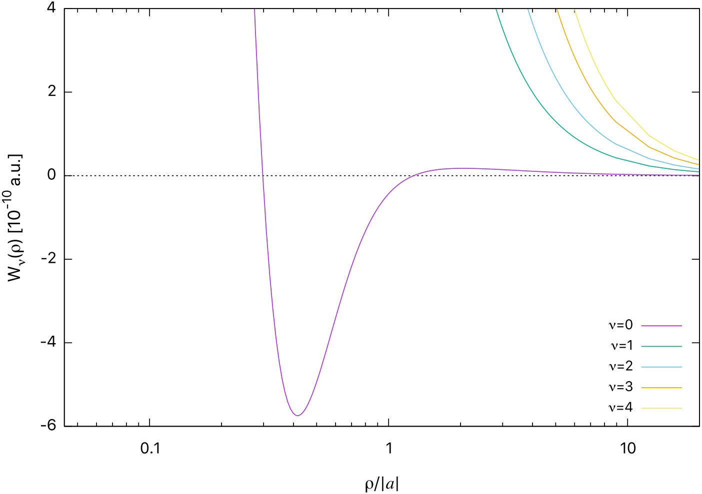
$$v(r) = d\cosh^{-2}(r/r_0) \tag{13}$$


1500
1000
500
0
−500
−1000
−1500
III
II
I
−7
−6
−5
−4
−3
−2
−1
0
a [a.u.]
d [10^−8 a.u.]

$a > 0$



$a < 0$



We expect that the lowest effective potential curve will converge towards

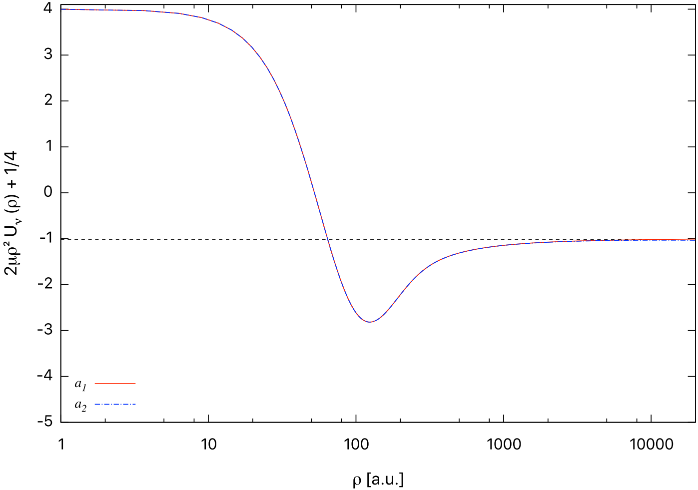
$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}, \tag{14}$$

for $|a| \to \infty$. This behaviour should be evident if the potentials are multiplied by $2\mu\rho^2$ and plotted as

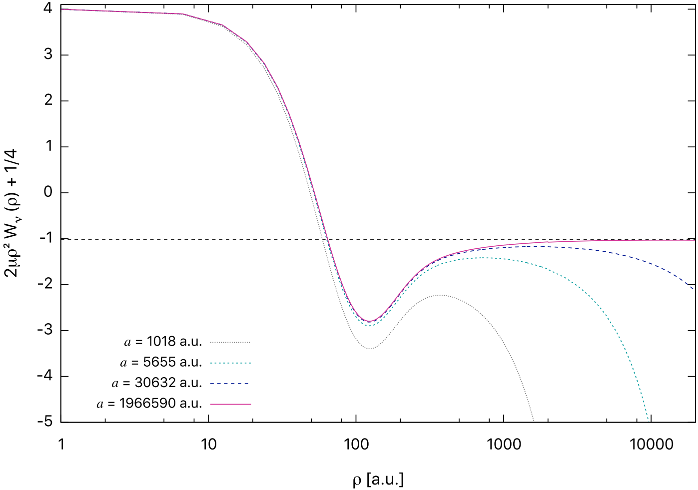
$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4}, \tag{15}$$

since these curves should approach the universal value $-s_0^2(\simeq -1.0125$ for $J = 0^+$ states).

$a \to \pm\infty$, $-s_0^2(\simeq -1.0125$ for $J = 0^+$ states)

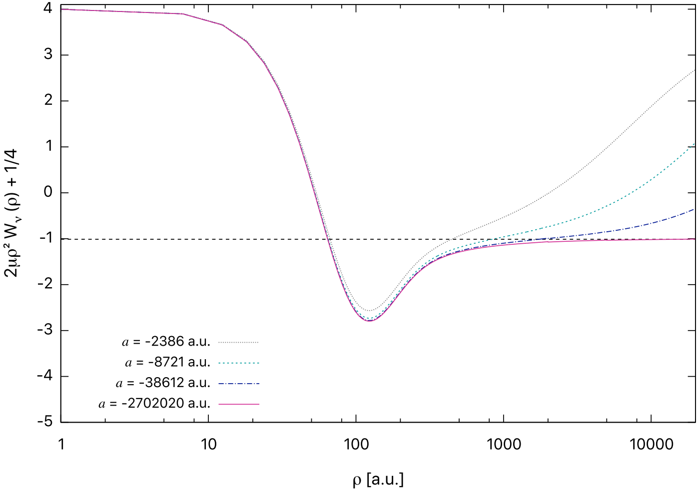


$a > 0$



4
3
2
1
0
-1
-2
-3
-4
-5
1
10
100
1000
10000
2μρ² W_ν (ρ) + 1/4
ρ [a.u.]
a = 1018 a.u.
a = 5655 a.u.
a = 30632 a.u.
a = 1966590 a.u.

$a < 0$



4
3
2
1
0
-1
-2
-3
-4
-5
1
10
100
1000
10000
2μρ² W_ν (ρ) + 1/4
ρ [a.u.]
a = -2386 a.u.
a = -8721 a.u.
a = -38612 a.u.
a = -2702020 a.u.

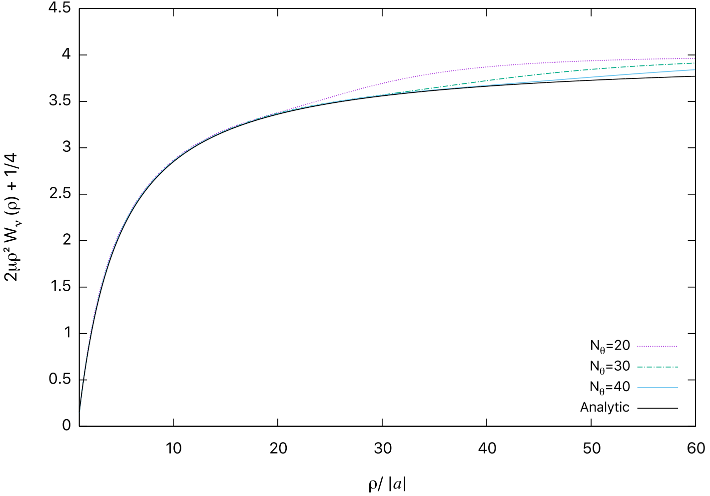
Comparison With the Adiabatic Potential From Solving Faddeev's Equation

For $\rho \gg r_0$ the adiabatic potentials ν_n (which correspond to $\xi(\rho)$) can be determined analytically through the transcendental equation

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right), \tag{16}$$

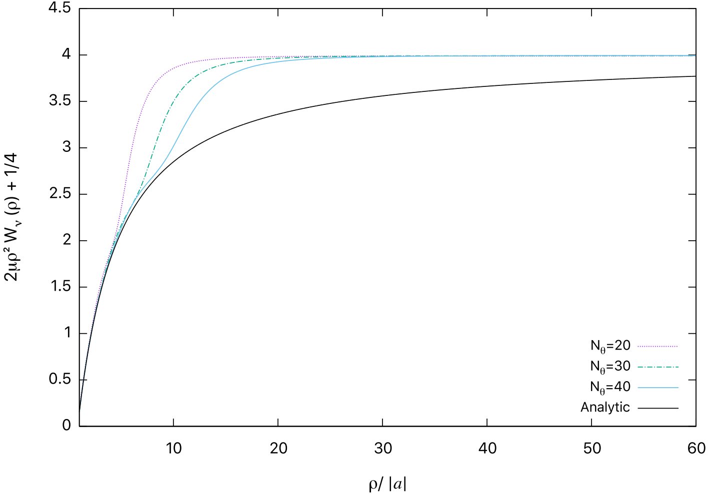
for different scattering lengths a.

$a = -2386$ a.u.



4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
10
20
30
40
50
60
2μρ² W_ν (ρ) + 1/4
ρ/ |a|
N_θ=20
N_θ=30
N_θ=40
Analytic

$a = -8721$ a.u.



4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
10
20
30
40
50
60
2μρ² W_ν (ρ) + 1/4
ρ/ |a|
N_θ=20
N_θ=30
N_θ=40
Analytic