Kajsa-My Blomdahl



# Theoretical and Numerical Studies of Efimov States

Kajsa-My Blomdahl

Stockholms Universitet

*kajsamy.blomdahl@fysik.su.se*

September 8, 2019



2019-09-08

Theoretical and Numerical Studies of Efimov States

Kajsa-My Blomdahl

Stockholms Universitet

kajsamy.blomdahl@fysik.su.se

September 8, 2019

1. Hi! So the topic of my talk is "Theoretical and numerical studies of Efimov states"
2. Efimov states are the central part of Efimov Physics, which is the physical theory of a quantum effect that appears in systems of three particles.

# Outline







└Outline

Outline

1. Introduction
2. Theoretical Approach
3. Effective Potentials
4. Numerical Approach
5. Scattering Model
6. Results
7. Conclusion and Outlook
8. Supplemental

1. You can find and follow the outline of my presentation in the margin during my talk.

# Objective

Efimov states: Giant trimers with universal properties





Objective

Efimov states: Giant trimers with universal properties

1. Efimov states are giant trimers with universal properties, which can be formed even when all three of the two-body subsystems are unbound.

# Objective





Efimov states: Giant trimers with universal properties

Aim: Build a program that calculates these states





Objective

Efimov states: Giant trimers with universal properties
Aim: Build a program that calculates these states

1. My aim has so far been to build a program that calculates these states numerically ...
2. ... for the purpose of investigating the limitations of the analytical theory of Efimov physics.

# Objective

Efimov states: Giant trimers with universal properties

Aim: Build a program that calculates these states

Purpose: Investigate the limitations of the analytical theory





Objective

Efimov states: Giant trimers with universal properties
Aim: Build a program that calculates these states
Purpose: Investigate the limitations of the analytical theory

1. My aim has so far been to build a program that calculates these states numerically ...
2. ... for the purpose of investigating the limitations of the analytical theory of Efimov physics.

# Two-body (2-b) Interactions

Kajsa-My Blomdahl









1. To understand important features of the quantum 3BP I will introduce a few concepts from the theory of quantum scattering of 2 particles.

# Two-body (2-b) Interactions



- Atomic collisions in the ultra cold regime





Two-body (2-b) Interactions

- Atomic collisions in the ultra cold regime

1. Atomic interactions are, essentially, pair-wise and short-ranged, which means that they interact when they are close to each other.

# Two-body (2-b) Interactions

- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as *s*-waves, *p*-waves and *d*-waves etc.





Two-body (2-b) Interactions

- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as *s*-waves, *p*-waves and *d*-waves etc.

1. At very low energies, atoms behave like point particles and have quantized orbital angular momenta l. The quantum numbers l = 0,1,2, associated with an atom, are referred to as s-waves, p-waves and d-waves, and so on

# Two-body (2-b) Interactions

- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as *s*-waves, *p*-waves and *d*-waves etc.





Two-body (2-b) Interactions

- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as *s*-waves, *p*-waves and *d*-waves etc.

1. In the ultracold regime s-wave collisions dominate (because higher partial waves are reflected by the centrifugal barrier in the SE)

# Two-body (2-b) Interactions



- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2,$ are referred to as $s$-waves, $p$-waves and $d$-waves etc.
- 2-b scattering in this regime is governed by a parameter called the s-wave scattering length $a$





Two-body (2-b) Interactions

- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2,$ are referred to as $s$-waves, $p$-waves and $d$-waves etc.
- 2-b scattering in this regime is governed by a parameter called the s-wave scattering length $a$

1. Two-body scattering in this regime is solely governed by a single parameter called "the s-wave scattering length" a, or just "the scattering length" for short

# Scattering Length







Scattering Length

1. The scattering length is defined in the low-energy limit by the following ...

# Scattering Length

- Definition:

$$a = \lim_{k\to 0} -\frac{\tan\delta_0(k)}{k}$$





Scattering Length

- Definition: $a = \lim_{k\to 0} -\frac{\tan\delta_0(k)}{k}$

1. The scattering length is defined in the low-energy limit by the following ...

- Definition:

$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$$





Scattering Length

- Definition:

$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$$

1. ... Here $\delta$ is the s-wave phase shift of the outgoing scattered wave (and $k$ is the wave number $= \sqrt{2\mu E}/\hbar$)

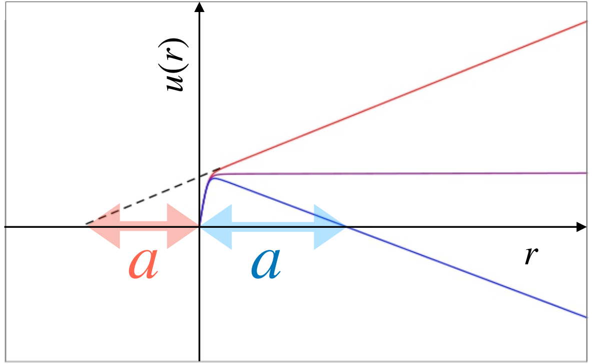
# Scattering Length

Kajsa-My Blomdahl



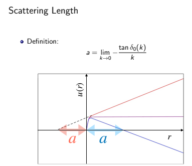
- Definition:

$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$$







1. For zero-energy scattering, $a$ is simply the distance to the intercept of the tangent of the radial wave-function

# Scattering Length

- Definition:

$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction





Scattering Length

- Definition: $a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$
- Characterizes the strength of the interparticle interaction

1. The scattering length characterizes the strength of the interaction ...
2. In the absence of an interaction, the phase shift is simply zero and the outgoing scattered wave is in phase with the incoming wave.
3. Any interaction will cause a dephasing between the outgoing and incoming waves. The strongest dephasing occurs when $\delta$ is $\pi/2$ and $a$ will then diverge.

# Scattering Length

Kajsa-My Blomdahl



- Definition:

$$a = \lim_{k \to 0} - \frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of $a$ and the *effective* interaction





Scattering Length

- Definition:

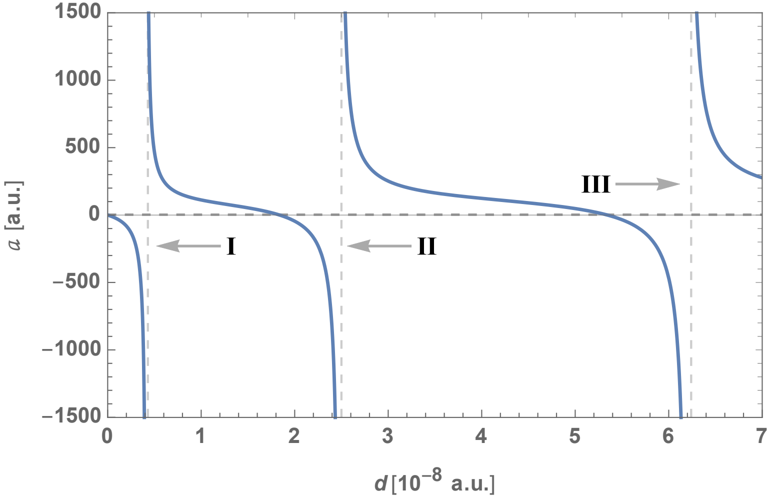
$$a = \lim_{k \to 0} - \frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of $a$ and the *effective* interaction

1. The sign of $a$ carries information about wether the effective interaction is attractive or repulsive.
2. If the two-body interaction has no bound states $a$ is negative ...
3. However if the interaction has one or more bound states $a$ can be both positive and negative.

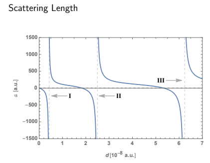
# Scattering Length









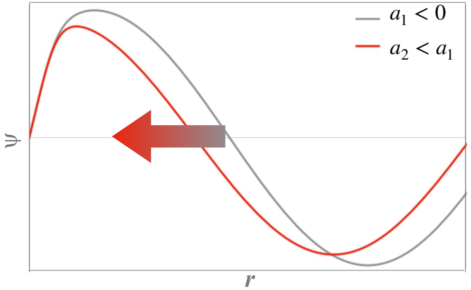

1. To illustrate this, I have used a model two-body potential and fine-tuned $a$ by changing the depth of this potential. Here we have $a$ on the vertical axis and the depth $d$ of the attractive 2b-potential on the horizontal axis.
2. The asymptotes in this figure represent resonances
3. First we have the case when there are no 2-b bound states and $a$ is negative. When the depth is increased the magnitude of $a$ increases.
4. When we move across the resonance a weakly bound dimer is formed and the scattering length changes sign.
5. If we now increase the depth further $a$ will decrease until it reach zero, where it again changes sign and the process is repeated.

# Scattering Length

- Definition:

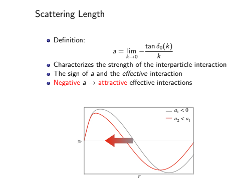
$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of $a$ and the *effective* interaction
- Negative $a$ $\rightarrow$ attractive effective interactions







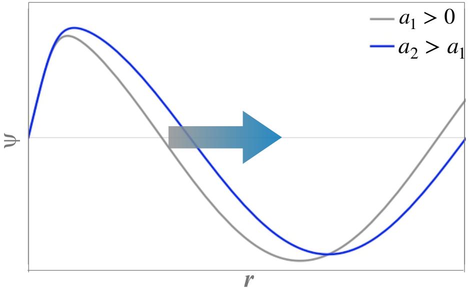
1. Negative scattering lengths correspond to an attractive effective interaction, meaning that the scattered wave is being pulled in by the potential

# Scattering Length

- Definition:

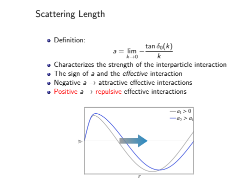
$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of $a$ and the *effective* interaction
- Negative $a$ $\rightarrow$ attractive effective interactions
- Positive $a$ $\rightarrow$ repulsive effective interactions







1. Positive scattering lengths correspond to a repulsive effective interaction, meaning that the scattered wave is being pushed out by the potential

# Scattering Length

- Definition:

$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of $a$ and the *effective* interaction
- Negative $a \rightarrow$ attractive effective interactions
- Positive $a \rightarrow$ repulsive effective interactions
- For $|a| \rightarrow \infty$ the interaction is called resonant





Scattering Length

- Definition:

$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of $a$ and the *effective* interaction
- Negative $a \rightarrow$ attractive effective interactions
- Positive $a \rightarrow$ repulsive effective interactions
- For $|a| \rightarrow \infty$ the interaction is called resonant

1. When the magnitude of $a \rightarrow \infty$ we say that the interaction is resonant. In this case the interaction is fully characterized by the scattering length, which is much larger than the interaction range of the particles.

# Universality in 2-b systems





**2-b scattering**

Particles with large $|a|$ in the low-energy regime have universal properties





Universality in 2-b systems

2-b scattering
Particles with large $|a|$ in the low-energy regime have universal properties

1. Particles with large scattering lengths in the low-energy regime are interesting because they have universal properties.

Kajsa-My Blomdahl

Introduction
Two-body Interactions
Scattering Length
Universality
The Efimov Effect
Theoretical Approach
Effective Potentials
Numerical Approach
Scattering Model
Results
Conclusion and Outlook
Supplemental

**2-b scattering**

Particles with large $|a|$ in the low-energy regime have universal properties

**Universal properties … In what sense?**

Depend on the scattering length alone and not on the details of the short-range interaction





Universality in 2-b systems

2-b scattering
Particles with large $|a|$ in the low-energy regime have universal properties

Universal properties … In what sense?
Depend on the scattering length alone and not on the details of the short-range interaction

1. What do we mean by universal? Well we mean that they depend on the scattering length alone and not on the details of the short-range interaction.
2. … which turn means that all bosons behave in the same way, it does not matter what specific atomic species we are looking at

# Universality in 2-b systems



**2-b scattering**

Particles with large $|a|$ in the low-energy regime have universal properties

**Universal properties ... In what sense?**

Depend on the scattering length alone and not on the details of the short-range interaction

▸ Results

**Example: 2-b binding energy for 2 identical bosons**

$$E_D = \frac{\hbar^2}{ma^2}$$





Universality in 2-b systems

2-b scattering
Particles with large $|a|$ in the low-energy regime have universal properties
Universal properties ... In what sense?
Depend on the scattering length alone and not on the details of the short-range interaction
Example: 2-b binding energy for 2 identical bosons
$E_D = \frac{\hbar^2}{ma^2}$

1. One of the universal properties is for example the binding energy of the dimer ...
2. ... which has a $1/a^2$ dependency on the scattering length

# Universality in 2-b systems

**2-b scattering**

Particles with large $|a|$ in the low-energy regime have universal properties

**Universal properties … In what sense?**

Depend on the scattering length alone and not on the details of the short-range interaction

▸ Results

**Example: 2-b binding energy for 2 identical bosons**

$$E_D = \frac{\hbar^2}{ma^2}$$





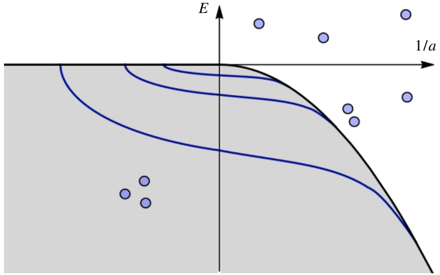
Universality in 2-b systems
2-b scattering
Particles with large |a| in the low-energy regime have universal properties
Universal properties … In what sense?
Depend on the scattering length alone and not on the details of the short-range interaction
Results
Example: 2-b binding energy for 2 identical bosons
$E_D = \frac{\hbar^2}{ma^2}$

1. … which has a $1/a^2$ dependency on the scattering length

# Efimov's Prediction

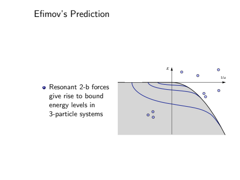


- Resonant 2-b forces give rise to bound energy levels in 3-particle systems







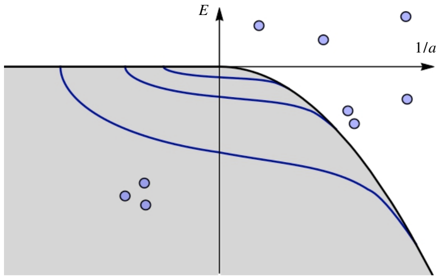
1. In the 1970 Vitaly Efimov predicted that resonant 2-b forces can give rise to a series of bound energy levels in the 3-particle spectra. These bound states are now called Efimov states.

Kajsa-My Blomdahl



# Efimov's Prediction

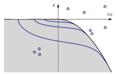
- Resonant 2-b forces give rise to bound energy levels in 3-particle systems








Efimov's Prediction
Resonant 2-b forces give rise to bound energy levels in 3-particle systems

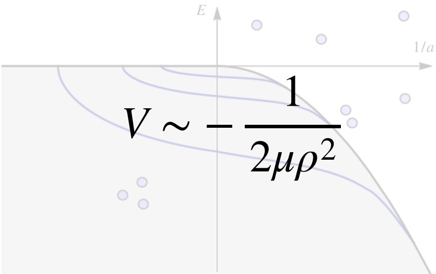


1. In the figure to the right we have a three-body energy spectrum, which I have plotted with the inverse of the scattering length on the horizontal axis and the energy on the vertical axis ...
2. We can identify three different regions in this figure
3. ... above the zero-energy line we have 3 free particles. The black parabolic curve is the energy of the universal dimer. Below the zero-energy line, in the white region, we have states where two particles form a dimer and one is far away.
4. ... and the grey region is the trimers region, where we have 3-b bound states... In the middle we are at resonance.

Kajsa-My Blomdahl

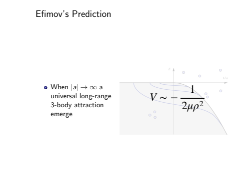


- When $|a| \to \infty$ a universal long-range 3-body attraction emerge







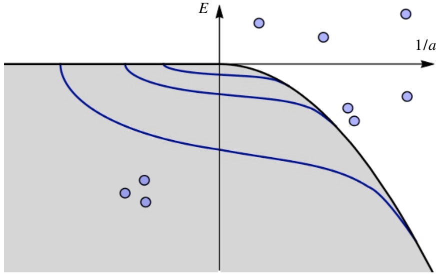
1. Approaching resonance, Efimov found a universal long-range three-body attraction emerging ...
2. ... giving rise to an infinite number of trimer states with binding energies obeying a discrete scaling law at resonance.

# Efimov's Prediction



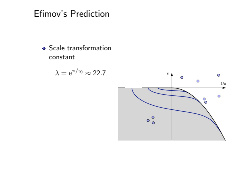
- Scale transformation constant

$$\lambda = \mathrm{e}^{\pi/s_0} \approx 22.7$$







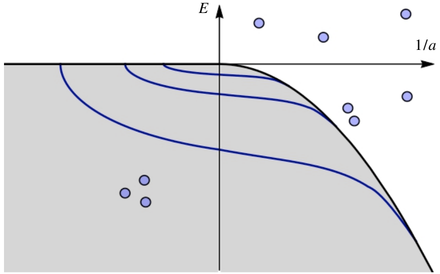
1. The Efimov states have universal properties. For three identical bosons, the size and energy of each successive trimer state are related by a scale transformation with a constant, which in the resonant limit is given by $\lambda = e^{\pi/s_0} \approx 22.7$

# Efimov's Prediction

- Scale transformation constant

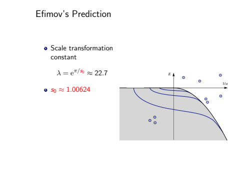
$$\lambda = e^{\pi/s_0} \approx 22.7$$

- $s_0 \approx 1.00624$







1. Later on when I talk about the universal constant, it is most often this $s_0$ that is referred to, and it will appear in the equations I will show later.

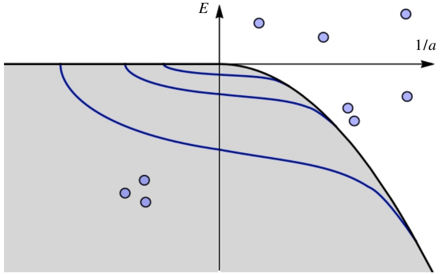
# Efimov's Prediction

- Scale transformation constant

$$\lambda = e^{\pi/s_0} \approx 22.7$$

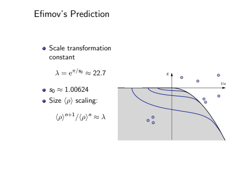
- $s_0 \approx 1.00624$
- Size $\langle\rho\rangle$ scaling:

$$\langle\rho\rangle^{n+1}/\langle\rho\rangle^{n} \approx \lambda$$








1. The symmetry of the asymptotic Efimov spectrum is characterized by a discrete scale invariance where the size and binding energies each form a geometric progression ..
2. the size of an excited state is larger than the previous state by the factor $\lambda$ ...
3. and the binding energies of two consecutive states scale like one over lambda squared

# Efimov's Prediction



- Scale transformation constant

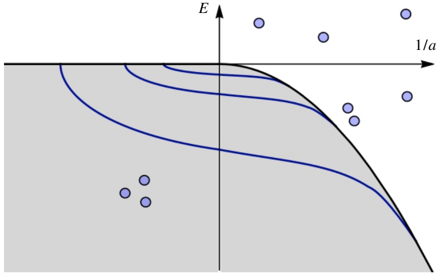
$$\lambda = e^{\pi/s_0} \approx 22.7$$

- $s_0 \approx 1.00624$
- Size $\langle \rho \rangle$ scaling:

$$\langle \rho \rangle^{n+1} / \langle \rho \rangle^{n} \approx \lambda$$

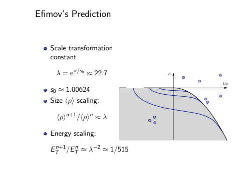
- Energy scaling:

$$E_T^{n+1}/E_T^{n} \approx \lambda^{-2} \approx 1/515$$







1. The symmetry of the asymptotic Efimov spectrum is characterized by a discrete scale invariance where the size and binding energies each form a geometric progression ..
2. the size of an excited state is larger than the previous state by the factor $\lambda$ ...
3. and the binding energies of two consecutive states scale like one over lambda squared

Kajsa-My
Blomdahl







The Peculiar Efimov Effect

1. The Efimov effect is remarkable in many ways

# The Peculiar Efimov Effect





- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect





The Peculiar Efimov Effect

- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect

1. Because the size of each Efimov state is much larger than the force-range between the individual pairs it means we are dealing with a pure quantum mechanical effect

# The Peculiar Efimov Effect



- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$





The Peculiar Efimov Effect

- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$

1. When the magnitude of scattering length approach infty, there is an infinite number of Efimov states

# The Peculiar Efimov Effect

Kajsa-My Blomdahl



- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is reduced as the 2-b interaction is made more attractive





The Peculiar Efimov Effect

- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is reduced as the 2-b interaction is made more attractive

1. # of 3-b bound states is *reduced* as the 2-b interaction is made more attractive.

# The Peculiar Efimov Effect



- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is reduced as the 2-b interaction is made more attractive
- The effect is universal and can *in principle* be observed in any QM system





The Peculiar Efimov Effect

- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is reduced as the 2-b interaction is made more attractive
- The effect is universal and can *in principle* be observed in any QM system

1. The effect is universal, which means that the states emerge irrespective of the nature of the 2-b forces and can in principle be observed in all quantum mechanical systems

# Theoretical Approach







Theoretical Approach

1. The 3BP is famous for being hard to solve

Kajsa-My Blomdahl



Q: 3-Particles, What Is The Problem?





Theoretical Approach

Q: 3-Particles, What Is The Problem?

1. So why is the problem of 3 so complex?

Q: 3-Particles, What Is The Problem?

A: The configuration space (CS) for the 3BP is 9D and highly non-trivial ...





Theoretical Approach

Q: 3-Particles, What Is The Problem?
A: The configuration space (CS) for the 3BP is 9D and highly non-trivial ...

1. Well, the configuration space for the 3BP is 9D and highly non-trivial ...

Kajsa-My Blomdahl



Q: 3-Particles, What Is The Problem?

A: The configuration space (CS) for the 3BP is 9D and highly non-trivial ...

Solution: Reduce the number of dimensions!





Theoretical Approach

Q: 3-Particles, What Is The Problem?
A: The configuration space (CS) for the 3BP is 9D and highly non-trivial ...
Solution: Reduce the number of dimensions!

1. So what we want to do is to reduce the dimensionality of the problem

# Step 1: Relative Coordinates

Kajsa-My Blomdahl



- Separate out CoM by introducing relative coordinates





Step 1: Relative Coordinates

- Separate out CoM by introducing relative coordinates

1. The first step to reduce the number of D is to separate out the CoM by introducing relative coordinates.

# Step 1: Relative Coordinates



- Separate out CoM by introducing relative coordinates
- CS $\rightarrow$ 6D





Step 1: Relative Coordinates

- Separate out CoM by introducing relative coordinates
- CS $\rightarrow$ 6D

1. CoM motion decouples from the internal motion in the SE the configuration space is effectively reduced to 6D

Kajsa-My Blomdahl



- Separate out CoM by introducing relative coordinates
- Choice: Hyperspherical coordinates *via* mass-normalized Jacobi coordinates
- CS $\rightarrow$ 6D





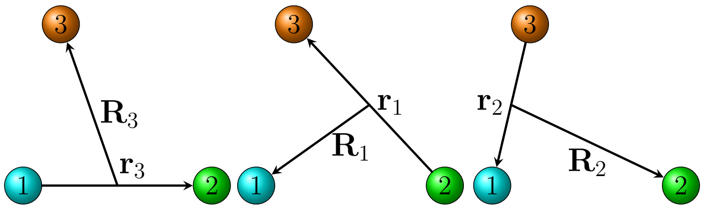
Step 1: Relative Coordinates

- Separate out CoM by introducing relative coordinates
- Choice: Hyperspherical coordinates *via* mass-normalized Jacobi coordinates
- CS $\rightarrow$ 6D

1. To later introduce internal coordinates we will go via the relative mass-normalized Jacobi coordinates

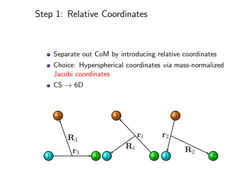
## Step 1: Relative Coordinates

- Separate out CoM by introducing relative coordinates
- Choice: Hyperspherical coordinates *via* mass-normalized Jacobi coordinates
- CS $\rightarrow$ 6D








1. (lower case) **r** is the vector that connects two of the particles and (upper case) **R** connects the CoM of these two particles with the third particle
2. why? particle permutations are easily performed using these coord.

# Step 2: Hyperspherical Coordinates







Step 2: Hyperspherical Coordinates

1. Step 2 in simplifying the problem of three particles is to introduce hyperspherical coordinates

# Step 2: Hyperspherical Coordinates



- Combine $\mathbf{r}_k$ and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$





Step 2: Hyperspherical Coordinates

- Combine $\mathbf{r}_k$ and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$

1. The general idea is to combine the components of the two Jacobi vectors into a single six-dimensional position vector q, which represents a point in R6.

# Step 2: Hyperspherical Coordinates



- Combine $\mathbf{r}_k$and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$
- Hyperspherical coordinates: $\rho$ and $\Omega$

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$





Step 2: Hyperspherical Coordinates

- Combine $\mathbf{r}_k$and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$
- Hyperspherical coordinates: $\rho$ and $\Omega$

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

1. The hyperspherical coord. of this point are given by the hyperradius $\rho$ and five hyperangles $\Omega$.
2. The hyperradial coordinate is both rotationally and permutationally invariant and is defined as the square root of the sum of the squared Jacobi vectors.
3. The hyperangles can be defined in many ways and I will not go in to the details here.

## Step 2: Hyperspherical Coordinates

- Combine $\mathbf{r}_k$and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$
- Hyperspherical coordinates: $\rho$ and $\Omega$

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

- Separate internal and external coordinates





Step 2: Hyperspherical Coordinates

- Combine $\mathbf{r}_k$and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$
- Hyperspherical coordinates: $\rho$ and $\Omega$

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

- Separate internal and external coordinates

1. At any instant, three particles form a plane in R3. We can define the internal motion of the particles within this plane in terms of the hyperradial coordinate (size) and two of the angles(shape and particle permutation)
2. The three other angles relate rotations of this plane in a space fixed system.

# Step 2: Hyperspherical Coordinates



- Combine $\mathbf{r}_k$ and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$
- Hyperspherical coordinates: $\rho$ and $\Omega$

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

- Separate internal and external coordinates
- For $J = 0$ only internal coordinates matter $\rightarrow$ 3D Schrödinger equation (SE)





Step 2: Hyperspherical Coordinates

- Combine $\mathbf{r}_k$ and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$
- Hyperspherical coordinates: $\rho$ and $\Omega$

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

- Separate internal and external coordinates
- For $J = 0$ only internal coordinates matter $\rightarrow$ 3D Schrödinger equation (SE)

1. When the orbital angular momenta J=0 only the internal coordinates matter and we are left with a 3D SE for the internal motion

# Step 3: The Adiabatic Representation

Kajsa-My Blomdahl



Numerical Approach Exact rep.





Step 3: The Adiabatic Representation
Numerical Approach Exact rep.

1. Now we move on to the final step (in the simplification of the 3BP)
2. After a clever rescaling of the wfn, the hyperspherical SE can be written in the following way:

Kajsa-My Blomdahl



- The hyperspherical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

Numerical Approach | Exact rep.



2019-09-08



Step 3: The Adiabatic Representation
- The hyperspherical SE:
$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$
Numerical Approach | Exact rep.

1. Now we move on to the final step (in the simplification of the 3BP)
2. After a clever rescaling of the wfn, the hyperspherical SE can be written in the following way:

# Step 3: The Adiabatic Representation

- The hypersperical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

Numerical Approach  Exact rep.





Step 3: The Adiabatic Representation

- The hypersperical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

Numerical Approach  Exact rep.

1. Here the operator lambda squared is the so called squared grand angular momentum operator, which contains dependencies of the two hyperangular coordinates

Kajsa-My Blomdahl



# Step 3: The Adiabatic Representation

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

Numerical Approach   Exact rep.





Step 3: The Adiabatic Representation

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

Numerical Approach   Exact rep.

1. and V are the two-body interactions which depend on both the hyperradial and hyperangular coordinates

## Step 3: The Adiabatic Representation

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

- Trick: Treat $\rho$ as an adiabatic parameter!

Numerical Approach Exact rep.





Step 3: The Adiabatic Representation
The hyperspherical SE:
$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$
Trick: Treat $\rho$ as an adiabatic parameter!
Numerical Approach Exact rep.

1. Now, the trick is to treat the hyperradius as an adiabatic parameter!
2. That is, we fix $\rho$ in a Born-Oppenheimer like manner

# Step 3: The Adiabatic Representation

Kajsa-My Blomdahl



- The hyperspherical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

- Trick: Treat $\rho$ as an adiabatic parameter!
- Solve the adiabatic eq.

$$H_{ad}\Phi_\nu(\rho;\Omega)=U_\nu(\rho)\Phi_\nu(\rho;\Omega)$$

Numerical Approach Exact rep.





Step 3: The Adiabatic Representation
The hyperspherical SE:
$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$
Trick: Treat $\rho$ as an adiabatic parameter!
Solve the adiabatic eq.
$$H_{ad}\Phi_\nu(\rho;\Omega)=U_\nu(\rho)\Phi_\nu(\rho;\Omega)$$
Numerical Approach Exact rep.

1. And solve the remaining adiabatic eigenvalue equation

# Step 3: The Adiabatic Representation

- The hyperspherical SE:

$$\left( -\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right)\psi = E\psi$$

- Trick: Treat $\rho$ as an adiabatic parameter!
- Solve the adiabatic eq.

$$H_{ad}\Phi_\nu(\rho;\Omega) = U_\nu(\rho)\Phi_\nu(\rho;\Omega)$$

- $\rightarrow$ 3-body BO-like potential

Numerical Approach | Exact rep.





Step 3: The Adiabatic Representation

- The hyperspherical SE:

$$\left( -\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right)\psi = E\psi$$

- Trick: Treat $\rho$ as an adiabatic parameter!
- Solve the adiabatic eq.

$$H_{ad}\Phi_\nu(\rho;\Omega) = U_\nu(\rho)\Phi_\nu(\rho;\Omega)$$

- $\rightarrow$ 3-body BO-like potential

Numerical Approach | Exact rep.

1. In this way we obtain the three-body equivalent of a BO potential.

# Step 3: The Adiabatic Representation cont.



- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho; \Omega)$$

SE C&O





Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho; \Omega)$$

SE C&O

1. Now, in this way the total wfn can be represented by a sum of adiabatic states

# Step 3: The Adiabatic Representation cont.

Kajsa-My Blomdahl



- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

SE C&O





Step 3: The Adiabatic Representation cont.
The total wave function is represented in terms of adiabatic states
$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$
SE C&O

1. (shown in red)

## Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

SE C&O





Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

SE C&O

1. If we substitute this sum into the 3-b SE (klick on link)

# Step 3: The Adiabatic Representation cont.



- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_n F_{n\mu}$$

SE C&O





Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_n F_{n\mu}$$

SE C&O

1. We will obtain the this hyperradial eq.
2. Here, the objects P and Q are non-adiabatic couplings.
3. This is an exact representation of the 3-body SE if all couplings are included.

# Step 3: The Adiabatic Representation cont.

Kajsa-My Blomdahl



- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_n F_{n\mu}$$

SE C&O





Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_n F_{n\mu}$$

SE C&O

1. The focus of my work has been on this part of this eq.

# Step 3: The Adiabatic Representation cont.



- Th… a…

**Three-body effective potentials**

$$W_\nu(\rho) = U_\nu(\rho) - \frac{1}{2\mu} Q_{\nu\nu}(\rho) = U_\nu(\rho) - \frac{1}{2\mu} P_{\nu\nu}^2(\rho)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_n F_{n\mu}$$

SE C&O






Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_n F_{n\mu}$$

SE C&O


1. In the adiabatic approximation these three-body effective potentials W are defined in the following way
2. These effective potentials can be used to determine the single channel solutions of the equation below

# Step 3: The Adiabatic Representation cont.

Kajsa-My Blomdahl



- Th… a…

**Three-body effective potentials**

$$W_\nu(\rho) = U_\nu(\rho) - \frac{1}{2\mu} Q_{\nu\nu}(\rho) \approx U_\nu(\rho) - \cancel{\frac{1}{2\mu} P^2_{\nu\nu}(\rho)}$$

- The hyperradial eigenvalue equation

$$\left( -\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu} Q_{\mu\mu} \right) F_{n\mu} - \frac{1}{2\mu}\left( \sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

SE C&O






Step 3: The Adiabatic Representation cont.


- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$


- The hyperradial eigenvalue equation

$$\left( -\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu} Q_{\mu\mu} \right) F_{n\mu} - \frac{1}{2\mu}\left( \sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

SE C&O


1. In this talk I will not explain how the second term is calculated, for large hyperradii it is small and we will ignore it from here on

# Convergence In the Asymptotic Limit







1. (The following discussion concerns short-ranged two-body interactions, where $|a| \gg r_0$)
2. The behavoiur of the 3-b potentials in the asymptotic limit, (i.e., when the hyperradius is much larger than the magnitude a) depend on the sign of a.

# Convergence In the Asymptotic Limit



**For $a < 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$





Convergence In the Asymptotic Limit

For $a < 0$:

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$

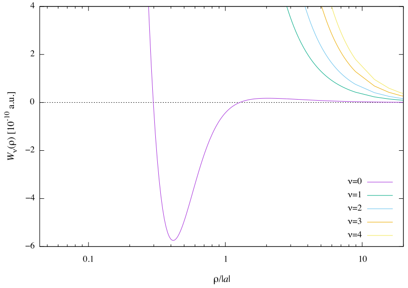
1. When a has a negative sign there is no weakly bound dimer and the lowest effective potential will converge to the three-body continuum channels, i.e., the kinetic energy for three free particles.

# Convergence In the Asymptotic Limit



**For $a < 0$:**

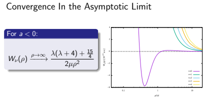
$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$






2019-09-08

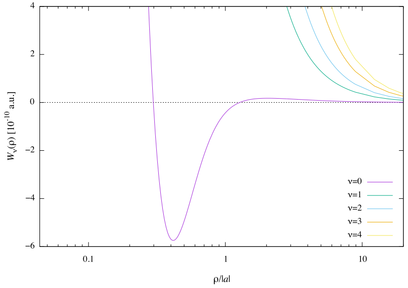




# Convergence In the Asymptotic Limit

**For $a < 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$

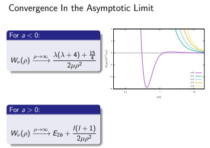



**For $a > 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} E_{2b} + \frac{l(l+1)}{2\mu\rho^2}$$








1. However, for systems of 3 identical bosons with a pair-wise attraction that is strong enough to support 2-body bound states, one 3-body effective potential curve will converge asymptotically to each two-body bound state.

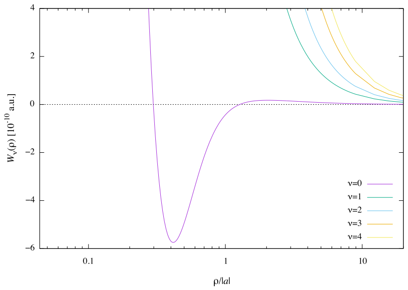
# Convergence In the Asymptotic Limit

Kajsa-My Blomdahl



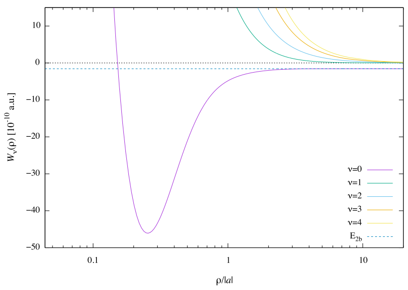
**For $a < 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$




**For $a > 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} E_{2b} + \frac{l(l+1)}{2\mu\rho^2}$$

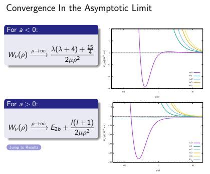



Jump to Results



2019-09-08





# Convergence In the Intermediate Region



**Intermediate Interaction Range**

$$r_0 \ll \rho \ll |a|$$

Jump to Results





Convergence In the Intermediate Region
Intermediate Interaction Range
$r_0 \ll \rho \ll |a|$
Jump to Results

1. Efimov physics comes into play in the intermediate region.
2. In this region the three-body effective potentials are modified by the Efimov physics. It can lead to both attractive and repulsive effective potentials.

# Convergence In the Intermediate Region



## Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

## The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

Jump to Results





Convergence In the Intermediate Region
Intermediate Interaction Range
$r_0 \ll \rho \ll |a|$
The Lowest Potential's Convergent Form (3 Identical Bosons)
$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$
Jump to Results

1. For 3 identical bosons the effective potential will be attractive and is responsible for the Efimov Effect!

# Convergence In the Intermediate Region



## Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

## The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

Jump to Results





Convergence In the Intermediate Region

Intermediate Interaction Range

$r_0 \ll \rho \ll |a|$

The Lowest Potential's Convergent Form (3 Identical Bosons)

$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$

Jump to Results

1. And is responsible for the Efimov Effect!

# Convergence In the Intermediate Region

Kajsa-My Blomdahl



## Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

## The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

## Universal Constant (3 Identical Bosons)

$$s_0 \simeq 1.00624$$

Jump to Results





Convergence In the Intermediate Region
Intermediate Interaction Range
$r_0 \ll \rho \ll |a|$
The Lowest Potential's Convergent Form (3 Identical Bosons)
$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$
Universal Constant (3 Identical Bosons)
$s_0 \simeq 1.00624$
Jump to Results

1. The universal constant $s_0$ sets the size and energy scaling of succeding Efimov states.

# Analytic Model





Jump to Results





Analytic Model

Jump to Results

1. I indicated in the beginning that we can obtain a similar result analytically. If we instead of solving the SE solve the coupled Faddeev equations..

Kajsa-My Blomdahl

Introduction
Theoretical Approach
Effective Potentials
The Asymptotic Limit
Intermediate Region
Analytical Model
Numerical Approach
Scattering Model
Results
Conclusion and Outlook
Supplemental

- The analytical formulation yields similar adiabatic potentials $\nu_n$ through:

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Jump to Results





Analytic Model

- The analytical formulation yields similar adiabatic potentials $\nu_n$ through:

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Jump to Results

1. ..we obtain the adiabatic potentials $\nu$ through this transcendental eq.

# Analytic Model



- The analytical formulation yields similar adiabatic potentials $\nu_n$ through:

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Jump to Results





Analytic Model

- The analytical formulation yields similar adiabatic potentials $\nu_n$ through:

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Jump to Results

1. These adiabatic potentials are functions of $\rho/a$
2. In the intermediate region, when $a$ is much larger than $\rho$, the right hand side of this equation vanish and the resulting transcendental equation has the following solutions

# Analytic Model

- The analytical formulation yields similar adiabatic potentials $\nu_n$ through:

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right)-\frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right)=\sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

**Solution for $|a| \to \infty$**

$\sqrt{-\nu_0(0)} = s_0$ and $\sqrt{\nu_n(0)} = s_n$

Jump to Results





Analytic Model
The analytical formulation yields similar adiabatic potentials $\nu_n$ through:
$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right)-\frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right)=\sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$
Solution for $|a| \to \infty$
$\sqrt{-\nu_0(0)} = s_0$ and $\sqrt{\nu_n(0)} = s_n$
Jump to Results

1. (Solutions shown on screen)

# Analytic Model



- The analytical formulation yields similar adiabatic potentials $\nu_n$ through:

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right)-\frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right)=\sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

**Solution for $|a| \to \infty$**

$$\sqrt{-\nu_0(0)} = s_0 \text{ and } \sqrt{\nu_n(0)} = s_n$$

Jump to Results





Analytic Model

- The analytical formulation yields similar adiabatic potentials $\nu_n$ through:

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right)-\frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right)=\sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Solution for $|a| \to \infty$

$\sqrt{-\nu_0(0)} = s_0$ and $\sqrt{\nu_n(0)} = s_n$

Jump to Results

1. Where the lowest adiabatic potential takes on the value of the universal constant $s_0$.

# Analytic Model

- The analytical formulation yields similar adiabatic potentials $\nu_n$ through:

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right)-\frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right)=\sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

**Solution for $|a| \to \infty$**

$\sqrt{-\nu_0(0)} = s_0$ and $\sqrt{\nu_n(0)} = s_n$

- Correponding three-body effective potentials

$$\widetilde{W}_\nu(\rho/a)=\frac{(\nu_n(\rho/a)-\frac{1}{4})}{2\mu\rho^2}$$

Jump to Results





Analytic Model
The analytical formulation yields similar adiabatic potentials $\nu_n$ through:
Solution for $|a| \to \infty$
Correponding three-body effective potentials
Jump to Results

1. So, the adiabatic potentials $\nu$ can be related to the 3-body effective potentials in the following way
2. Okey, so if there is an analytical solution why do we want to solve this numerically? Well the reason is that the analytical formulation is valid only at large hyperradii, while the numerical potentials are valid over the whole hyperradial range.
3. In the result section I will compare these solutions to my numerically calculated potentials.

# Numerical Approach; B-spline Collocation

Kajsa-My Blomdahl



$$\text{Task} = ?$$





1. So what was task?
2. Well, to solve the quantum 3BP!
3. And how do we do this?

Numerical Approach; B-spline Collocation
Task = ?

Kajsa-My Blomdahl



$$\text{Task} = \text{Find } W_\nu(\rho)!$$





Numerical Approach; B-spline Collocation

Task = Find $W_\nu(\rho)$!

1. A good start is to find the 3-b effective potentials!
2. This means that we need to solve the adiabatic eigenvalue eq.

# Numerical Approach; B-spline Collocation



First: Expand
$\Phi_\nu(\rho; \theta, \phi)$





Numerical Approach; B-spline Collocation

First: Expand
$\Phi_\nu(\rho; \theta, \phi)$

1. The first step in solving this equation is to expand the solutions (the angular wave functions) in a suitable basis.

Kajsa-My Blomdahl



First: Expand $\Phi_\nu(\rho;\theta,\phi)$

- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$





Numerical Approach; B-spline Collocation

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$

1. The first step in solving this equation is to expand the solutions (the angular wave functions) in a suitable basis.

# Numerical Approach; B-spline Collocation



Basis: B-splines

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$





Numerical Approach; B-spline Collocation

Basis: B-splines

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

$\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$

1. I have used a tensor product of one-dimensional B-splines for generating a base function in the two angular dimensions

# Numerical Approach; B-spline Collocation



Basis: B-splines

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$





Numerical Approach; B-spline Collocation

Basis: B-splines

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$

1. I have used a tensor product of one-dimensional B-splines for generating a base function in the two angular dimensions

# Numerical Approach; B-spline Collocation



Basis: B-splines

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

Then: Substitute $\Phi_\nu$ into the Adiabatic Eq.

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$





Numerical Approach; B-spline Collocation
Basis: B-splines
First: Expand $\Phi_\nu(\rho;\theta,\phi)$
Then: Substitute $\Phi_\nu$ into the Adiabatic Eq.
$\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
$\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$

1. The next step is to substitute this expansion it into the adiabatic eq.
2. The adiabatic equation can then be written in the following matrix form

Basis: B-splines

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

Then: Substitute $\Phi_\nu$ into the Adiabatic Eq.

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
- $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$





Numerical Approach; B-spline Collocation
Basis: B-splines
First: Expand $\Phi_\nu(\rho;\theta,\phi)$
Then: Substitute $\Phi_\nu$ into the Adiabatic Eq.
$\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
$\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
$\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$

1. The adiabatic equation can then be written in the following matrix form

# Numerical Approach; B-spline Collocation



Basis: B-splines

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

Then: Substitute $\Phi_\nu$ into the Adiabatic Eq.

Finally: Solve the Generalized Eigenvalue Eq.

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
- $\mathbf{H}_{\text{ad}}\mathbf{c} = U\mathbf{Bc}$





Numerical Approach; B-spline Collocation

Basis: B-splines

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

Then: Substitute $\Phi_\nu$ into the Adiabatic Eq.

Finally: Solve the Generalized Eigenvalue Eq.

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
- $\mathbf{H}_{\text{ad}}\mathbf{c} = U\mathbf{Bc}$

1. And lastly we solve this Generalized eigenvalue eq.
2. and retrieve the eigenvalues, which are our effective 3-body potentials

# Numerical Approach; B-spline Collocation



Basis: B-splines

First: Expand $\Phi_\nu(\rho;\theta,\phi)$

Then: Substitute $\Phi_\nu$ into the Adiabatic Eq.

Finally: Solve the Generalized Eigenvalue Eq.

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
- $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$
- $W(\rho) \approx U(\rho)$





Numerical Approach; B-spline Collocation

Basis: B-splines
First: Expand $\Phi_\nu(\rho;\theta,\phi)$
Then: Substitute $\Phi_\nu$ into the Adiabatic Eq.
Finally: Solve the Generalized Eigenvalue Eq.

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
- $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$
- $W(\rho) \approx U(\rho)$

1. And lastly we solve this Generalized eigenvalue eq.
2. and retrieve the eigenvalues, which are our effective 3-body potentials

Kajsa-My Blomdahl



**Masses**

$m = m(^{87}\mathrm{Rb})$





Scattering Model
Masses
$m = m(^{87}\mathrm{Rb})$

1. For the scattering model I have used masses corresponding to Rb-87 (boson)(Z=37, N=50)

# Scattering Model



**Masses**

$m = m(^{87}\mathrm{Rb})$

**Assumption**

$V(\rho, \theta, \psi) =$
$v(r_{12}) + v(r_{23}) + v(r_{31})$





Scattering Model

Masses

$m = m(^{87}\mathrm{Rb})$

Assumption

$V(\rho, \theta, \psi) =$
$v(r_{12}) + v(r_{23}) + v(r_{31})$

1. I have used the assumption that the total potential can be written as a sum of 2-b potentials

# Scattering Model



**Masses**

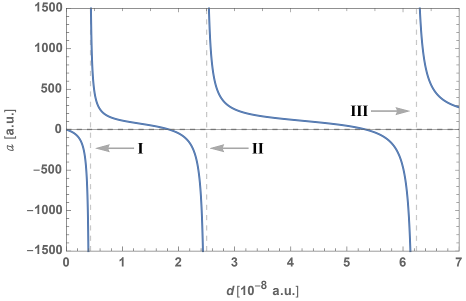
$m = m(^{87}\mathrm{Rb})$

**Assumption**

$V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31})$

**2B Model Potential**

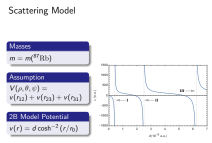
$v(r) = d\cosh^{-2}(r/r_0)$






2019-09-08





1. I have used the following model potential because I can calculate the scattering length from it

# Scattering Model

Kajsa-My Blomdahl



## Masses

$m = m(^{87}\mathrm{Rb})$

## Assumption

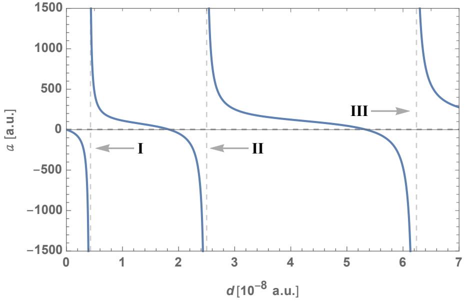
$V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31})$

## 2B Model Potential

$v(r) = d \cosh^{-2}(r/r_0)$

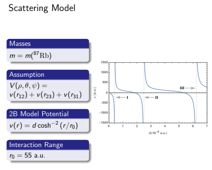
## Interaction Range

$r_0 = 55$ a.u.







1. And set the interaction range to 55 a.u.

# Convergence and Accuracy

Kajsa-My Blomdahl



- For $a \to \pm\infty$ we expect convergence towards the Efimovian form

To Figures





Convergence and Accuracy

- For $a \to \pm\infty$ we expect convergence towards the Efimovian form

To Figures

1. And now to the results!
2. For $a \to \pm\infty$ we expect that the lowest effective potential curve will converge towards the Efimovian form

# Convergence and Accuracy

Kajsa-My Blomdahl



- For $a \to \pm\infty$ we expect convergence towards the Efimovian form
- Easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as

$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4}$$

To Figures





Convergence and Accuracy

- For $a \to \pm\infty$ we expect convergence towards the Efimovian form
- Easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as

$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4}$$

To Figures

1. This behaviour is easier to recognize if the potentials are multiplied by this factor $2\mu\rho^2$ and plot them as
2. Since these curves should approach the universal value $-s_0^2 (\simeq -1.0125)$ in the intermediate region

# Convergence and Accuracy



- For $a \to \pm\infty$ we expect convergence towards the Efimovian form
- Easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as

$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4}$$

- Should approach the universal value $-s_0^2(\simeq -1.0125)$ in the intermediate region

To Figures





Convergence and Accuracy

- For $a \to \pm\infty$ we expect convergence towards the Efimovian form
- Easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as

$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4}$$

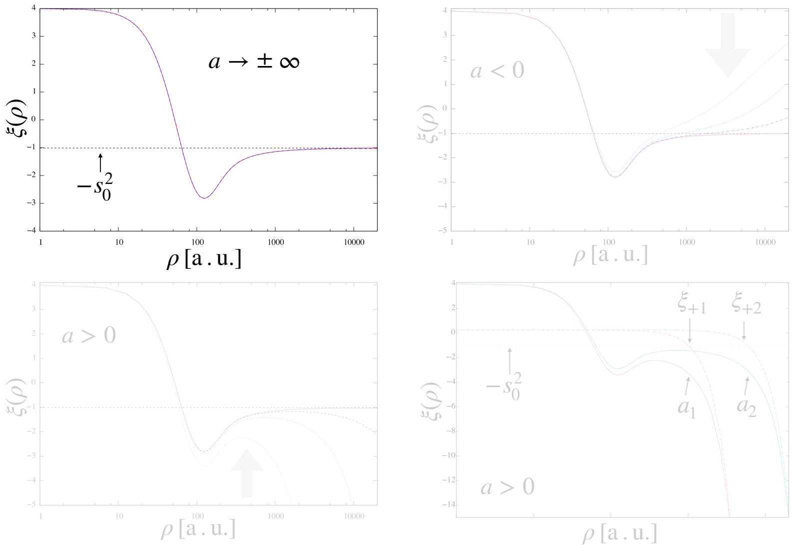
- Should approach the universal value $-s_0^2(\simeq -1.0125)$ in the intermediate region

To Figures

1. This behaviour is easier to recognize if the potentials are multiplied by this factor $2\mu\rho^2$ and plot them as
2. Since these curves should approach the universal value $-s_0^2(\simeq -1.0125)$ in the intermediate region

# Efimov-like Potentials $\xi(\rho)$ for Different $a$

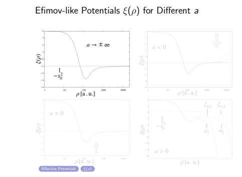





Effective Potentials $\xi(\rho)$

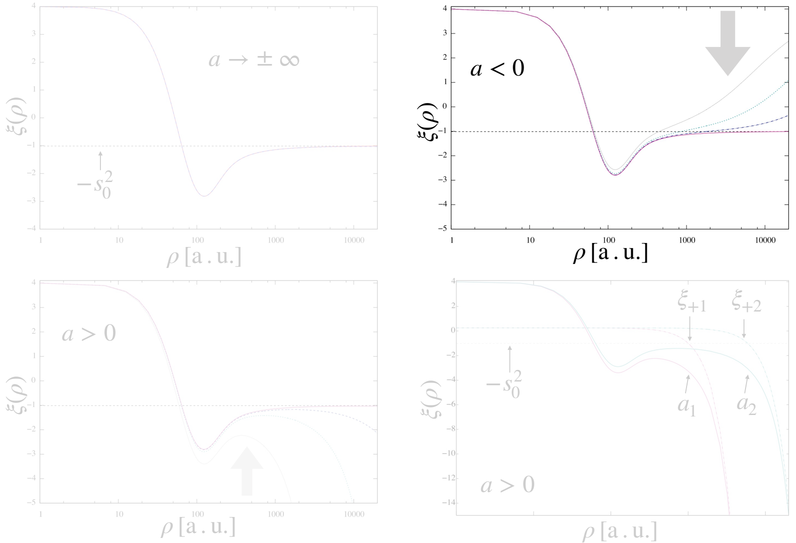






1. When the scattering length diverge we see that the potential curve $\xi$ converge to the expected value at large hyperradii (note the log scale on the $\rho$-axis)
2. For two-body potentials where $|a|$ is large but finite we expect that the effective potentials are to some extent affected by Efimov physics in the intermediate range ($r_0 \ll \rho \ll |a|$) and that the lowest effective potentials obtained with a larger magnitude of a exhibit closer resemblance with the true Efimov potential

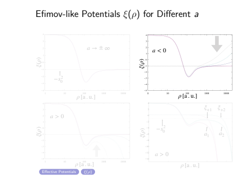
# Efimov-like Potentials $\xi(\rho)$ for Different $a$




Effective Potentials $\xi(\rho)$



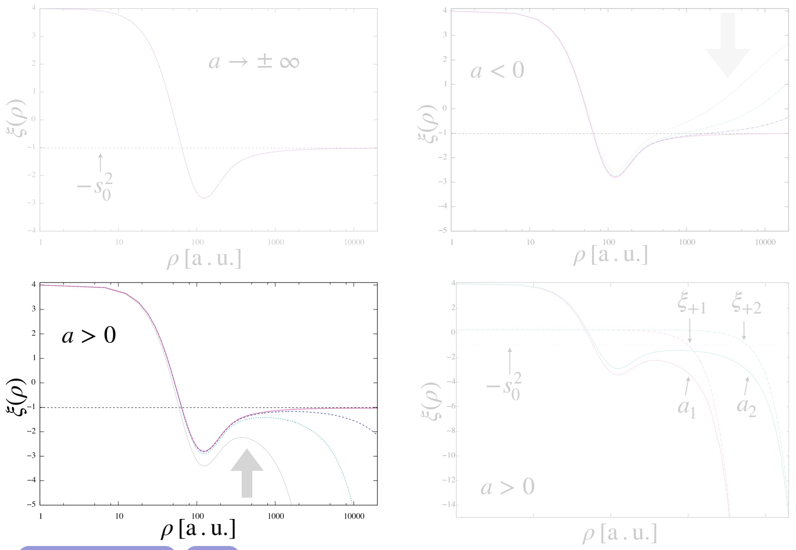




1. This is indeead what we see for negative $a$. Here I have plotted 4 curves with $|a|$ ranging from 2000 to 3 M a.u. At large hyperradii (in the asymptotic limit) the curves start to converge to 4 which corresponds to the first eigenvalue of the angular kinetic energy operator.

# Efimov-like Potentials $\xi(\rho)$ for Different $a$

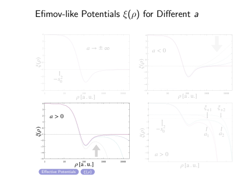





Effective Potentials $\xi(\rho)$

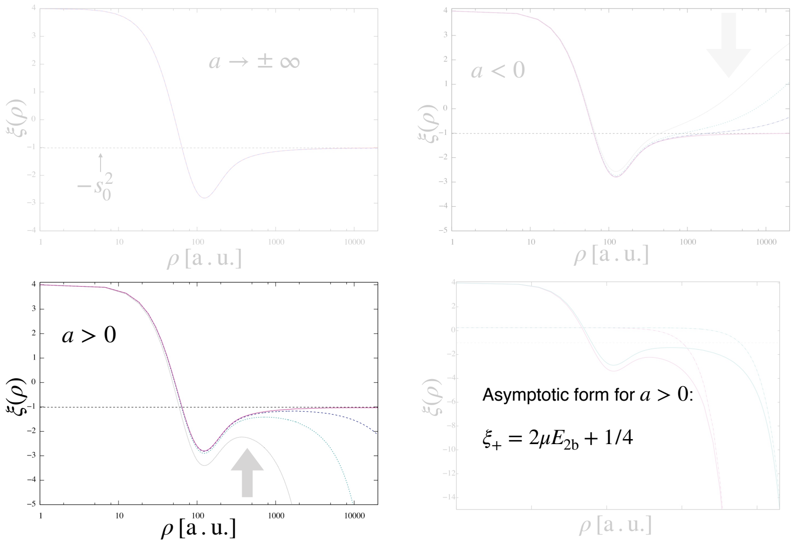






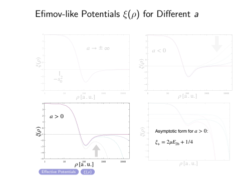
1. For large positive $a$ we see a flattening behaviour of the curves from below which tend to get closer to the universal value as $a$ increase.

# Efimov-like Potentials $\xi(\rho)$ for Different $a$





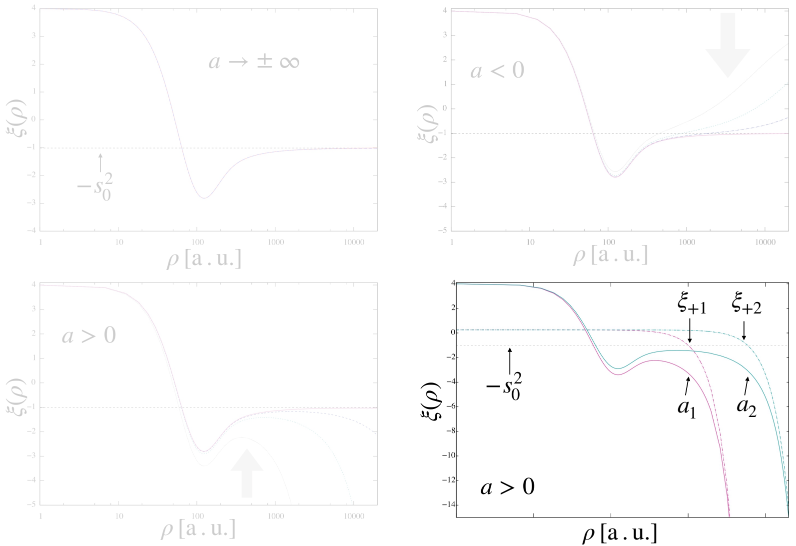


1. However, in this case the curves become parabolic in the asymptotic range since the effective potential converge to the energy of the two body bound state.

# Efimov-like Potentials $\xi(\rho)$ for Different $a$






Effective Potentials $\xi(\rho)$

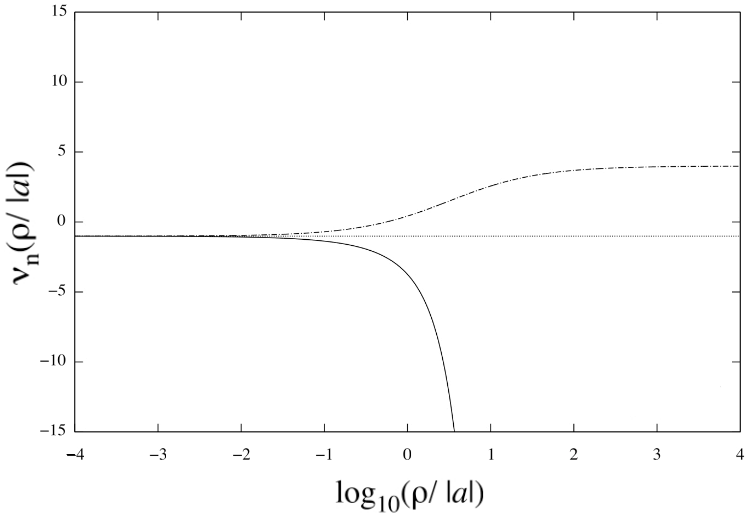




1. To show that this is indeed the case I have plotted the corresponding $E_{2b}$ curves for two of the $\xi$ potentials.
2. Show the curves on the screen.

# Comparison to the Analytical Model (1)

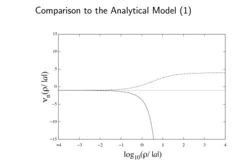





◂ Analytic Potential ▸ $a < 0$ ▸ $a > 0$

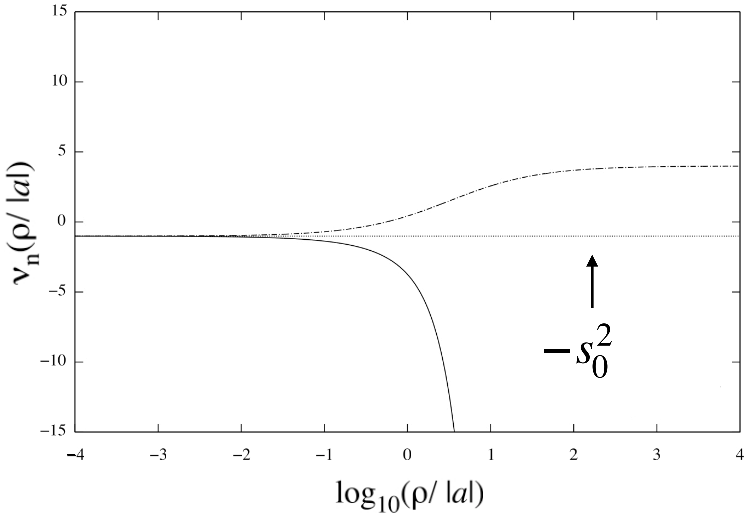






1. With this figure I want to show the equivalent lowest potentials for positive and negative *a* calculated from the transcendental Faddeev eq.

# Comparison to the Analytical Model (1)

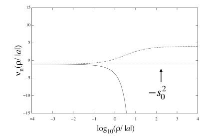



◂ Analytic Potential ▸ $a < 0$ ▸ $a > 0$





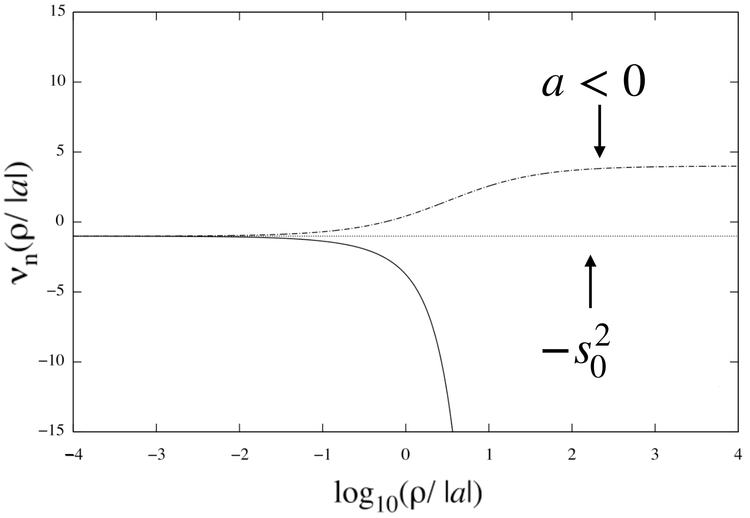
Comparison to the Analytical Model (1)



1. The universal value is shown as a this dotted line. And we can se that when $a > \rho$, the adiabatic potential for both positive and negative $a$ converge to this universal value.

# Comparison to the Analytical Model (1)



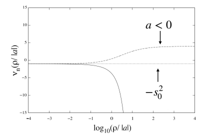



◂ Analytic Potential ▸ $a < 0$ ▸ $a > 0$



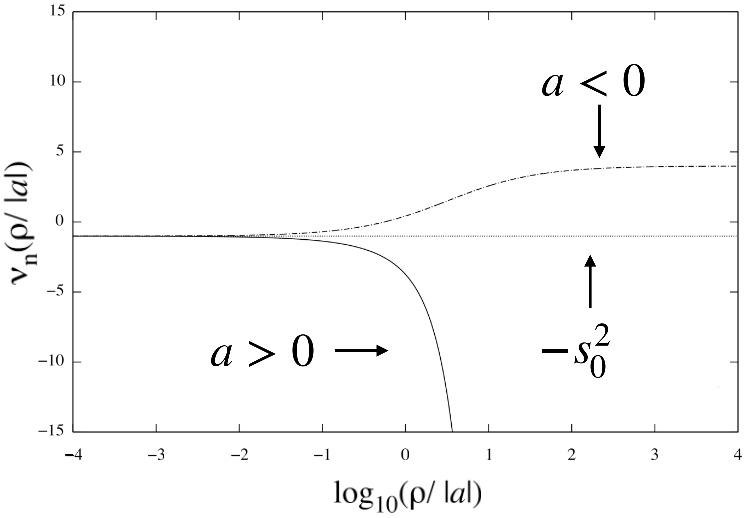


Comparison to the Analytical Model (1)



1. When $\rho > |a|$ we can observe that the curve for negative $a$ approach the value 4, which again corresponds to the eigenvalues of the angular kinetic energy operator for three particles.

# Comparison to the Analytical Model (1)



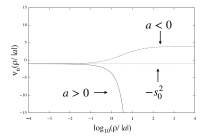



◂ Analytic Potential ▸ $a < 0$ ▸ $a > 0$





Comparison to the Analytical Model (1)

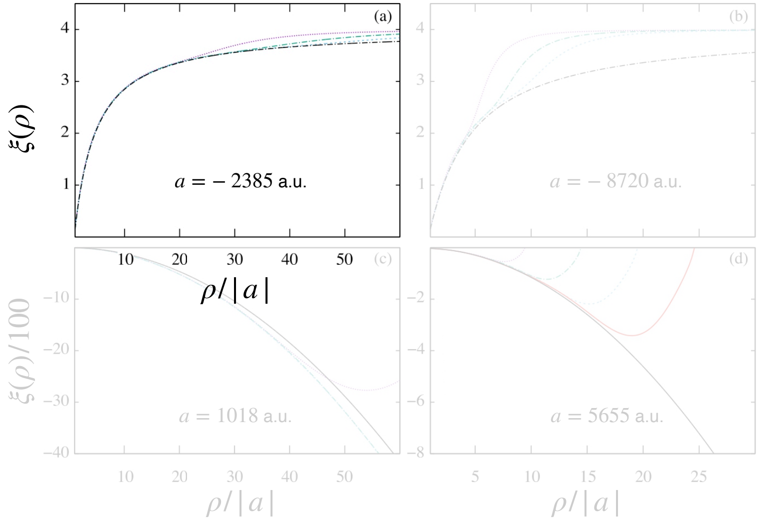


1. For postive $a$, we intead observe a parabolic behaviour like that corresponding to the form of the energy of the 2-b bound state when $\rho > a$.

# Comparison to the Analytical Model (2)

Kajsa-My Blomdahl

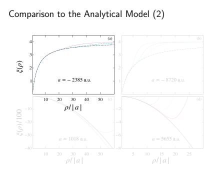





$\nu_0(\rho/a)$ $E_D$

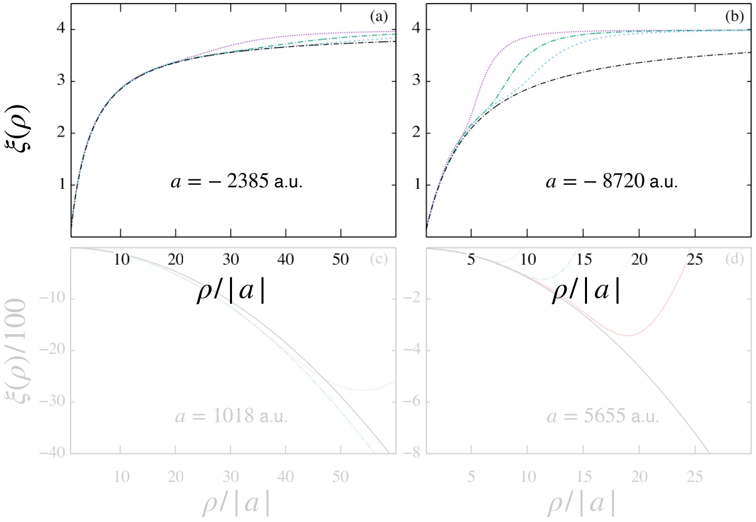






1. I have compared my numerically calculate potentials with the potential obtained solving the transcendental Faddeev eq.
2. I will show you the results for two negative and two positive $a$.
3. In this figure I have plotted the analytic potential (black dotted line) for negative $a$. The coloured curves are numerical potentials calculated with an increasing number of B-splines in each direction. As might be expected the range of convergence is larger for potentials calculated with a largest number of B-splines.

# Comparison to the Analytical Model (2)



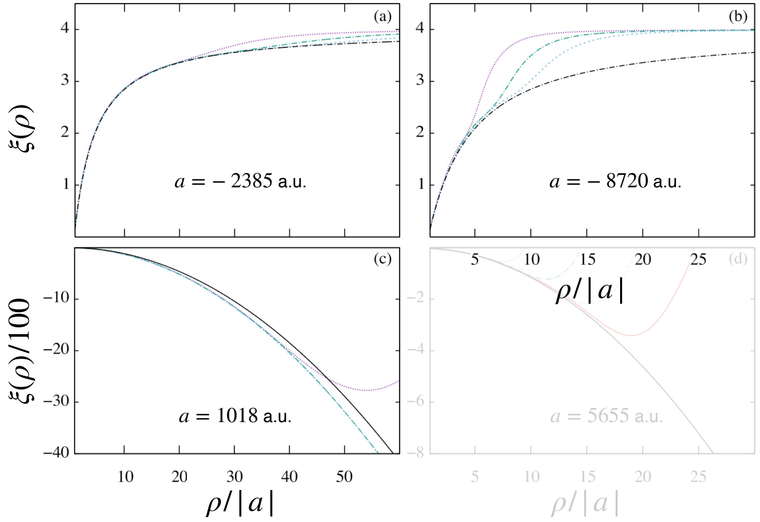

◂ $\nu_0(\rho/a)$ $E_D$





1. We can observe that for larger magnitudes of *a* the range of convergence decreases.

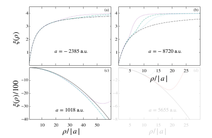
# Comparison to the Analytical Model (2)




◂ $\nu_0(\rho/a)$ $E_{\mathrm{D}}$



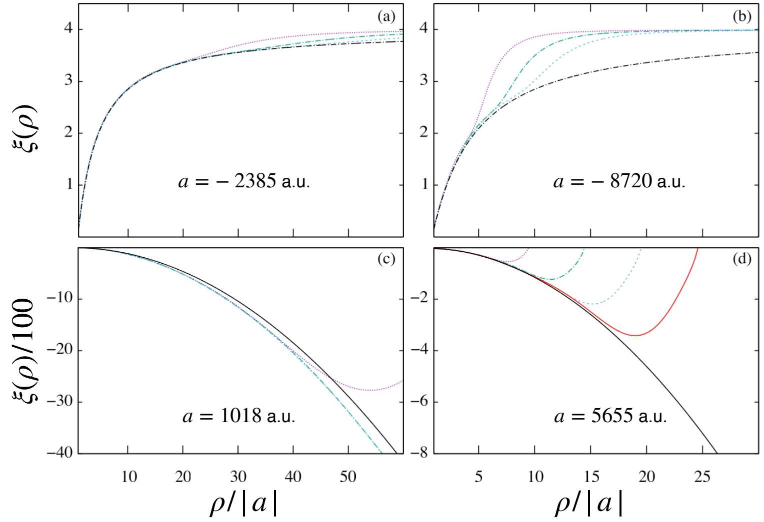




1. No we look at positive $a$. Here we have the analytic curve in black. All numerically calculated potentials have converged over the whole hyppreradial range apart from the one calculated with the least number of B-splines.
2. However, there seem to be a small difference between the numerical and analytical results.

# Comparison to the Analytical Model (2)

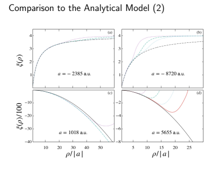





$\nu_0(\rho/a)$ $E_D$

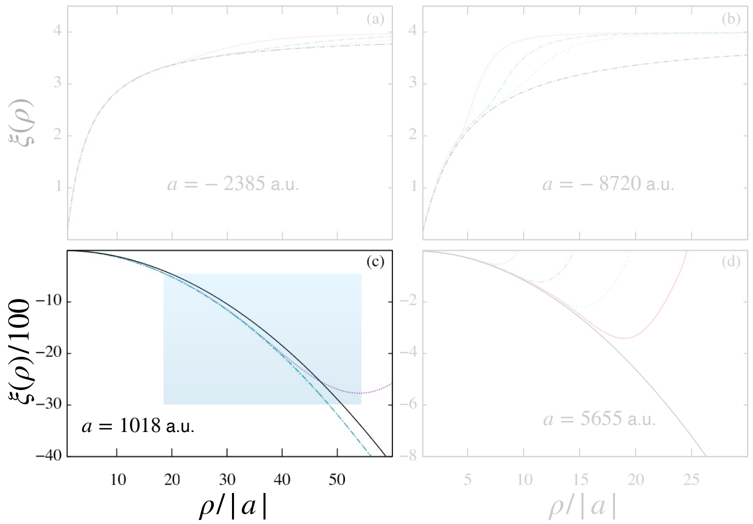






1. For larger *a* the analytical and numerical results show greater similarity.
2. However, again the hyperradial range of convergence have become smaller.
3. By comparing the panels to left and right, it appears that the hyperradial range for convergence generally is larger for the states with smaller $|a|$. We suspect that this discrepancy is due to the knot point placement, but this has not been verified yet.

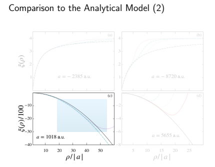
# Comparison to the Analytical Model (2)




◂ $\nu_0(\rho/a)$ $E_D$

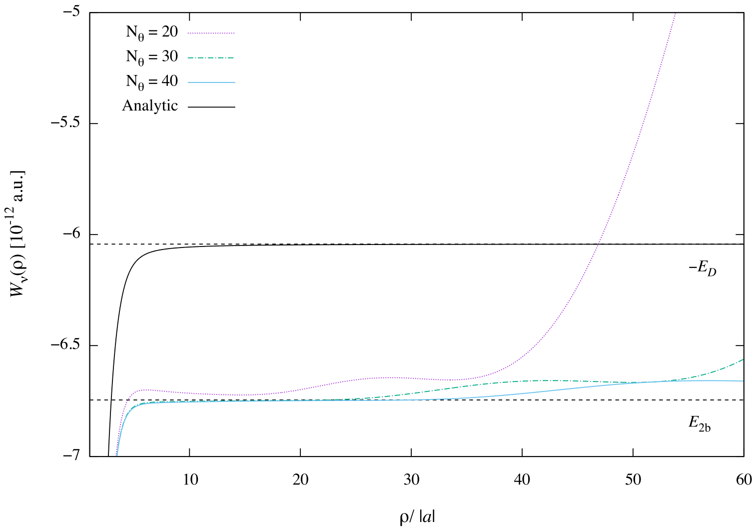






1. Now, how do we this?
2. Well, the slightly different form in the parabolic divergence of the curves is due to a discrepancy between the exact two-body energy and the energy of the universal dimer.

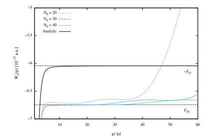
# Comparison to the Analytical Model (3)








Comparison to the Analytical Model (3)



1. Here I have plotted the actual three-body effective potentials $W$, the corresponding analytic potential together with the two-body energy $E_{2b}$ and the energy of the universal dimer.
2. We can see that the analytic potential goes to

Kajsa-My Blomdahl



# Conclusion and Outlook

Conclusion: The program works! But convergence at large hyperradii can be improved ...





Conclusion and Outlook

Conclusion: The program works! But convergence at large hyperradii can be improved ...

1. In conclusion, the program works! However, the convergence at large hyperradii can be improved. We need to look closer into how the placement of the B-splines on the angular grid can be used to optimize the numerical convergence. (coalescences points)

# Conclusion and Outlook



Conclusion: The program works! But convergence at large hyperradii can be improved ...





Conclusion and Outlook

Conclusion: The program works! But convergence at large hyperradii can be improved ...

1. And now to the outlook..

Conclusion: The program works! But convergence at large hyperradii can be improved ...

Outlook:
- Use $W(\rho)$ to solve the Hyperradial eq. $\rightarrow E_{\mathrm{T}}$





Conclusion and Outlook

Conclusion: The program works! But convergence at large hyperradii can be improved ...
Outlook: • Use $W(\rho)$ to solve the Hyperradial eq. $\rightarrow E_{\mathrm{T}}$

1. First of all, I want to extend the program so that it calculates the actual three-body energy eigenvalues. To do this I will use the three-body effective potentials to solve the hyperradial eigenvalue eq.
2. And secondly, I want implement a two-body potential model with van der Waals-tail that resembles alkali atoms

# Conclusion and Outlook



Conclusion: The program works! But convergence at large hyperradii can be improved ...

Outlook:
- Use $W(\rho)$ to solve the Hyperradial eq. $\rightarrow E_{\mathrm{T}}$
- Implement with a van der Waals potential





Conclusion and Outlook

Conclusion: The program works! But convergence at large hyperradii can be improved ...
Outlook: Use $W(\rho)$ to solve the Hyperradial eq. $\rightarrow E_{\mathrm{T}}$
Implement with a van der Waals potential

1. First of all, I want to extend the program so that it calculates the actual three-body energy eigenvalues. To do this I will use the three-body effective potentials to solve the hyperradial eigenvalue eq.
2. And secondly, I want implement a two-body potential model with van der Waals-tail that resembles alkali atoms
3. And now to the final questions: what can we use this for?

Kajsa-My Blomdahl



Conclusion: The program works! But convergence at large hyperradii can be improved ...

Outlook:
- Use $W(\rho)$ to solve the Hyperradial eq. $\rightarrow E_{\mathrm{T}}$
- Implement with a van der Waals potential
- Explore the temperature dependences of Efimov states in ${}^{39}\mathrm{K}$





Conclusion and Outlook

Conclusion: The program works! But convergence at large hyperradii can be improved ...

Outlook:
- Use $W(\rho)$ to solve the Hyperradial eq. $\rightarrow E_{\mathrm{T}}$
- Implement with a van der Waals potential
- Explore the temperature dependences of Efimov states in ${}^{39}\mathrm{K}$

1. I want to explore the temperature dependences of Efimov states in systems of Potassium-39 (19 protons 20 neutrons)
2. And why do I want to study K-39? Well, most experimental findings concerning of Efimov states has been carried out using ultra cold clouds of alkali atoms and for all elements apart from K-39, experiment and theory goes hand in hand.
3. However, K-39 behaves differently in several ways. And very resent experimental findings (concerning the temperature dependency of the shape and position of an Efimov resonance) are even in stark contrast with the current theory. (So there is room for new insights to be gained in the interesting realm of universal three-body physics.)

Thank you for listening!





Thank you for listening!

# B-splines



- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$





B-splines

- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$

1. A basis spline, or B-spline, of order $k$ is a piecewise polynomial function of degree $(k-1)$ defined on a collection of points, which we call knot points.

# B-splines

Kajsa-My Blomdahl



- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$
- Knot points: $t_i$, where $t_i < t_{i+1}$





B-splines

- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$
- Knot points: $t_i$, where $t_i < t_{i+1}$

1. A basis spline, or B-spline, of order $k$ is a piecewise polynomial function of degree $(k-1)$ defined on a collection of points, which we call knot points.

## B-splines



- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$
- Knot points: $t_i$, where $t_i < t_{i+1}$
- B-splines are local





B-splines

- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$
- Knot points: $t_i$, where $t_i < t_{i+1}$
- B-splines are local

1. The B-splines are local in the sense that they will be non-zero only in a limited region of space.

# B-splines

- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$
- Knot points: $t_i$, where $t_i < t_{i+1}$
- B-splines are local
- Definition, $k=1$:

$$B_{i,k=1}(x) \doteq \begin{cases} 1, & \text{if} \quad t_i \leq x < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$





B-splines

- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$
- Knot points: $t_i$, where $t_i < t_{i+1}$
- B-splines are local
- Definition, $k=1$:

$$B_{i,k=1}(x) \doteq \begin{cases} 1, & \text{if} \quad t_i \leq x < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

1. B-splines of the first order are defined by the following

# B-splines



- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$
- Knot points: $t_i$, where $t_i < t_{i+1}$
- B-splines are local
- Definition, $k=1$:

$$B_{i,k=1}(x) \doteq \begin{cases} 1, & \text{if} \quad t_i \leq x < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

- Cox-de-Boor recursion formula:

$$B_{i,k}(x) \doteq \frac{x - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(x) + \frac{t_{i+k} - x}{t_{i+k} - t_{i+1}} B_{i+1,k-1}(x)$$





B-splines

- $B_{i,k} \rightarrow$ piecewise polynomial function of degree $k-1$
- Knot points: $t_i$, where $t_i < t_{i+1}$
- B-splines are local
- Definition, $k=1$:

$$B_{i,k=1}(x) \doteq \begin{cases} 1, & \text{if} \quad t_i \leq x < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

- Cox-de-Boor recursion formula:

$$B_{i,k}(x) \doteq \frac{x - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(x) + \frac{t_{i+k} - x}{t_{i+k} - t_{i+1}} B_{i+1,k-1}(x)$$

1. B-splines of the first order are defined by the following
2. The higher order B-splines can be generated recursively by the Cox-de-Boor recursion formula

# B-splines; Pros and cons



**Pros**

- Can be generated anywhere on the grid
- Easy to differentiate

**Cons**

- Knot-point placement greatly affect convergence





B-splines; Pros and cons

Pros
- Can be generated anywhere on the grid
- Easy to differentiate

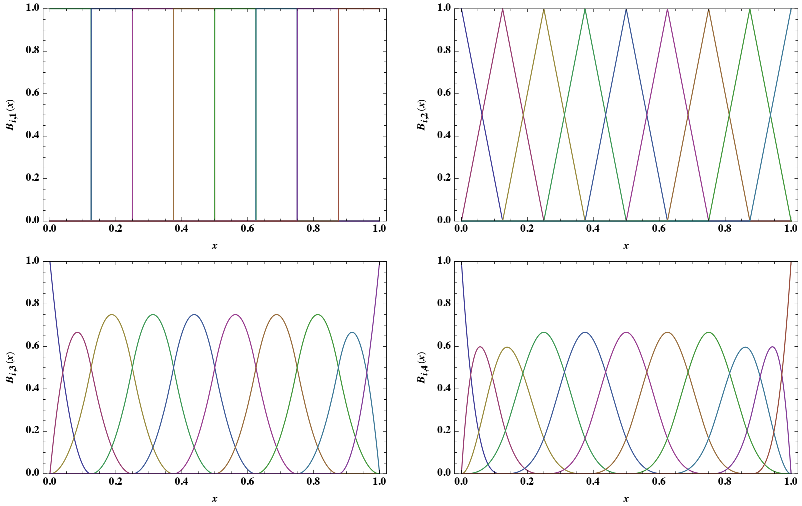
Cons
- Knot-point placement greatly affect convergence

1. B-splines are flexible to use since they can be generated anywhere on the grid and..
2. ..they are easy to differentiate
3. Main difficulty when using B-splines is to determine the number of knots to use and where they should be placed on the grid to get good numerical convergence

# B-splines, $k = 1 - 4$


**Kajsa-My Blomdahl**



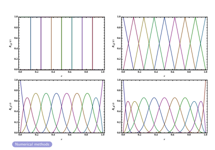





Numerical methods





1. In this figure I have plotted B-splines of order $k = 1 - 4$