# Theoretical and Numerical Studies of Efimov States

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1. Hi, my name is Kajsa-My Blomdahl, I study Efimov Physics, which is encapsulte a number of effects that appear in the quantum 3BP.

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1 Introduction
2 Theoretical Approach
3 Effective Potentials
4 Numerical Approach
5 Scattering Model
6 Results

1. To understand important features of the quantum 3BP I will start by introduce a few concepts from quantum scattering of 2 particles.

# Two-body (2-b) Interactions



- Atomic collisions in the ultra cold regime



Two-body (2-b) Interactions

- Atomic collisions in the ultra cold regime

1. Atomic interactions are, essentially, pair-wise and short-ranged, which means that they interact when they are close to each other.

# Two-body (2-b) Interactions

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- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as $s$-waves, $p$-waves and $d$-waves etc.



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1. At very low energies, atoms behave like point particles and have quantized orbital angular momenta l. The quantum numbers l = 0,1,2, associated with an atom, are referred to as s-waves, p-waves and d-waves, and so on

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1. In the ultracold regime s-wave collisions dominate (because higher partial waves are reflected by the centrifugal barrier in the SE)

# Two-body (2-b) Interactions

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- 2-b scattering in this regime is governed by a parameter called the s-wave scattering length $a$



Two-body (2-b) Interactions

- Atomic collisions in the ultra cold regime
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1. Two-body scattering in this regime is solely governed by a single parameter called the s-wave scattering length a, or just scattering length for short

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# Scattering Length

- Definition:

$$a = \lim_{k \to 0} -\frac{\tan \delta_0(k)}{k}$$



Scattering Length

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1. The s-wave scattering length is defined in the low-energy limit as

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Scattering Length

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1. where $\delta$ is the s-wave phase shift of the outgoing wave (and $k$ is the wave number $= \sqrt{2\mu E}/\hbar$)

# Scattering Length



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- Characterizes the strength of the interparticle interaction



Scattering Length

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1. The scattering length characterizes the strength of the interaction In the absence of an interaction, the phase shift is simply zero and the outgoing scattered wave is in phase with the incoming wave. Any interaction will cause a dephasing between the outgoing and incoming waves. The strongest dephasing occur when is $\pi/2$ $a$ will then diverge.

# Scattering Length



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1. The sign of $a$ carries information about wether the effective interaction is attractive or repulsive. If the two-body interaction has no bound states $a$ is negative. However if the interaction has one or more bound state $a$ can be both positve and negative.

# Scattering Length

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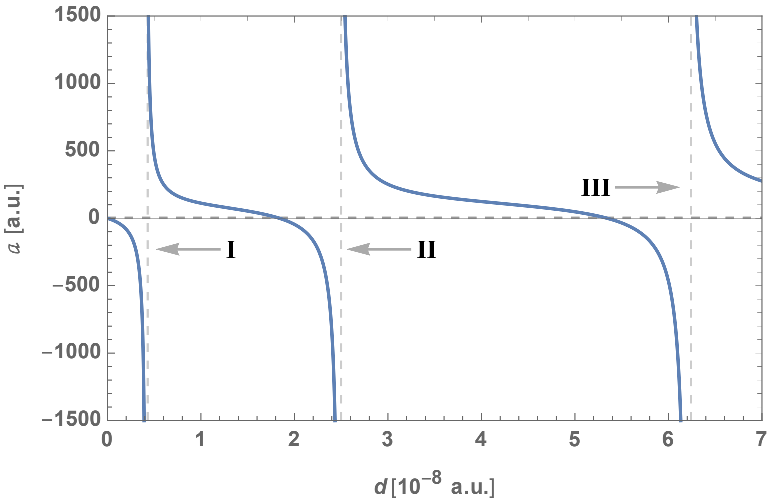




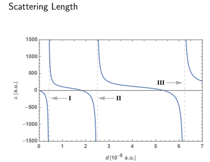

1. To illustrate this I have finetuned a model two body potential by changing the depth of the potential. Here we have $a$ on the y-axis and the depth $d$ of the attractive 2b-potential on the x-axis.

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1. Negative scattering lengths correspond to attractive effective interactions, meaning the scattered wave is being pulled in by the potential

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# Scattering Length

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- Characterizes the strength of the interparticle interaction
- The sign of $a$ and the effective interaction
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- For $|a| \to \infty$ the interaction is called resonant



Scattering Length

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- For $|a| \to \infty$ the interaction is called resonant

1. When the magnitude of $a \to \infty$ we say that the interaction is resonant. In this case the interaction is fully characterized by the scattering length, which is much larger than the interaction range of the particles.

# Universality in 2-b systems

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**2-b scattering**

Particles with large $|a|$ in the low-energy regime have universal properties



Universality in 2-b systems

2-b scattering
Particles with large $|a|$ in the low-energy regime have universal properties

1. Particles with large scattering lengths in the low-energy regime are interesting because they have universal properties.

# Universality in 2-b systems

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**2-b scattering**

Particles with large $|a|$ in the low-energy regime have universal properties

**Universal properties ... In what sense?**

Depend on the scattering length alone and not on the details of the short-range interaction



Universality in 2-b systems

2-b scattering
Particles with large $|a|$ in the low-energy regime have universal properties
Universal properties ... In what sense?
Depend on the scattering length alone and not on the details of the short-range interaction

1. What do we mean by universal? It means that they depend on the scattering alone and not on the details of the short-range interction, which means all bosons behave in the same way it does not matter what atomic species we look at

# Universality in 2-b systems

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**2-b scattering**

Particles with large $|a|$ in the low-energy regime have universal properties

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**Example: 2-b binding energy for 2 identical bosons**

$$E_D = \frac{\hbar^2}{ma^2}$$



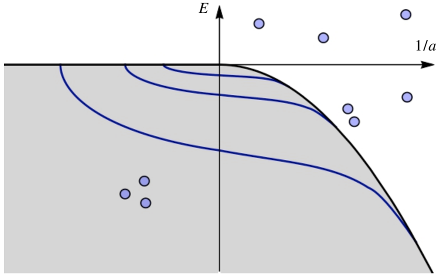
Universality in 2-b systems

2-b scattering
Particles with large $|a|$ in the low-energy regime have universal properties
Universal properties ... In what sense?
Depend on the scattering length alone and not on the details of the short-range interaction
Example: 2-b binding energy for 2 identical bosons
$E_D = \frac{\hbar^2}{ma^2}$

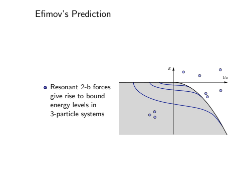
1. For example:

# Efimov's Prediction

- Resonant 2-b forces give rise to bound energy levels in 3-particle systems






1. In the 1970 Vitaly Efimov predicted that resonant can give rise to a series of bound energy levels in 3-particle systems, which we now call Efimov states

# Efimov's Prediction

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Scattering Length
Universality
The Efimov Effect

Theoretical Approach

Effective Potentials

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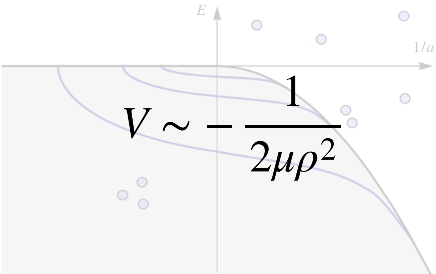
Scattering Model

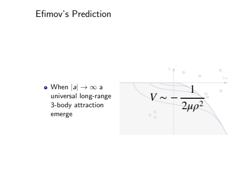
Results

- When $|a| \to \infty$ a universal long-range 3-body attraction emerge




$$V \sim -\frac{1}{2\mu\rho^2}$$





1. When the short-ranged two-body forces approached resonance, he found a universal long-range three-body attraction emerging, giving rise to an infinite number of trimer states with binding energies obeying a discrete scaling law at resonance.

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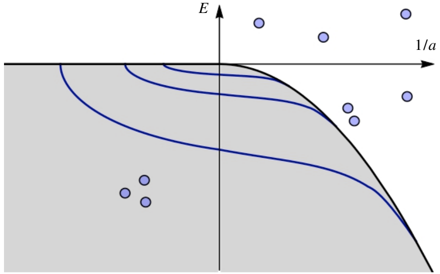
# Efimov's Prediction

- Scale transformation constant

$$\lambda = e^{\pi/s_0} \approx 22.7$$






Efimov's Prediction
Scale transformation constant
$\lambda = e^{\pi/s_0} \approx 22.7$

1. The Efimov states have universal properties. For three identical bosons are that the size and energy of successive trimer states in the resonant limit are related by a scale transformation with a constant $\lambda = e^{\pi/s_0} \approx 22.7$

# Efimov's Prediction

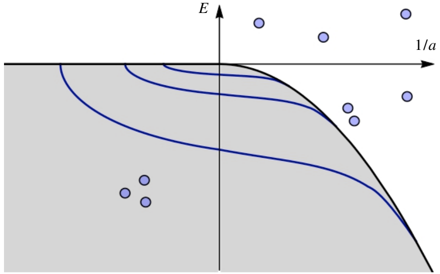




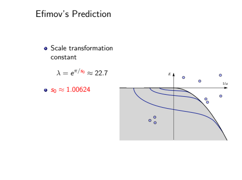
- Scale transformation constant

$$\lambda = e^{\pi/s_0} \approx 22.7$$

- $s_0 \approx 1.00624$





1. Is a universal constant in Efimov physics, which I will return to later

# Efimov's Prediction

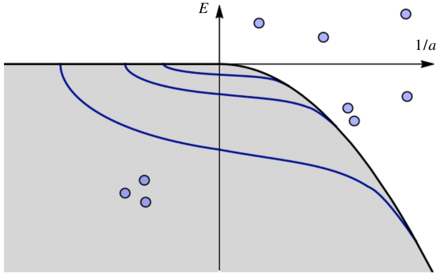


- Scale transformation constant

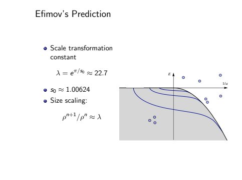
$$\lambda = e^{\pi/s_0} \approx 22.7$$

- $s_0 \approx 1.00624$
- Size scaling:

$$\rho^{n+1}/\rho^{n} \approx \lambda$$





1. The ratio of the size of two succesive Efimov states is given by the constant lambda

# Efimov's Prediction



- Scale transformation constant

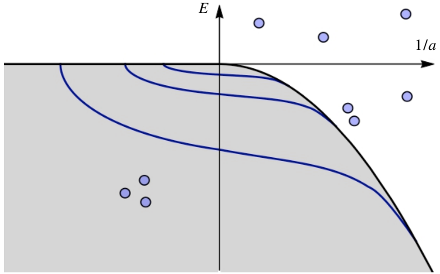
$$\lambda = e^{\pi/s_0} \approx 22.7$$

- $s_0 \approx 1.00624$
- Size scaling:

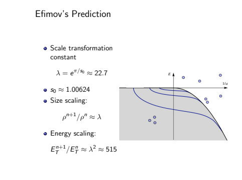
$$\rho^{n+1}/\rho^{n} \approx \lambda$$

- Energy scaling:

$$E_T^{n+1}/E_T^{n} \approx \lambda^2 \approx 515$$





1. while the ratio of the energy of two succesive Efimov states scale geometrically with lambda

# The Peculiar Efimov Effect

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1. The Efimov effect is remarkable in many ways

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- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect



The Peculiar Efimov Effect

- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect

1. Because the size of each Efimov state is much larger than the force-range between the individual pairs it means we are dealing with a pure quantum mechanical effect

# The Peculiar Efimov Effect

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- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
- When $a \rightarrow \pm\infty$ the $\#$ of ES $\rightarrow \infty$



The Peculiar Efimov Effect

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1. When the magnitude of scattering length approach infty, there is an infinite number of Efimov states

# The Peculiar Efimov Effect



- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is reduced as the 2-b interaction is made more attractive



The Peculiar Efimov Effect

- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
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1. # of 3-b bound states is *reduced* as the 2-b interaction is made more attractive.

# The Peculiar Efimov Effect



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- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
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- The effect is universal and can *in principle* be observed in any QM system



The Peculiar Efimov Effect

- The size of each Efimov state (ES) $\gg$ the interaction range ($r_0$) between the individual particle pairs $\rightarrow$ QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is reduced as the 2-b interaction is made more attractive
- The effect is universal and can *in principle* be observed in any QM system

1. The effect is universal, which means that the states emerge irrespective of the nature of the 2-b forces and can in principle be observed in all quantum mechanical systems

# Theoretical Approach





1. The 3BP is famous for being hard to solve

Theoretical Approach

Q: 3-Particles, What Is The Problem?



Theoretical Approach

Q: 3-Particles, What Is The Problem?

1. So why is the problem of 3 so complex?

# Theoretical Approach

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Q: 3-Particles, What Is The Problem?

A: The configuration space (CS) for the 3BP is 9D and highly non-trivial ...



1. Well, the configuration space for the 3BP is 9D and highly non-trivial ...

Theoretical Approach

Q: 3-Particles, What Is The Problem?
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# Theoretical Approach

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Q: 3-Particles, What Is The Problem?

A: The configuration space (CS) for the 3BP is 9D and highly non-trivial ...

Solution: Reduce the number of dimensions!



Theoretical Approach

Q: 3-Particles, What Is The Problem?
A: The configuration space (CS) for the 3BP is 9D and highly non-trivial ...
Solution: Reduce the number of dimensions!

1. So what we want to do is to reduce the dimensionality of the problem

## Step 1: Relative Coordinates

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- Separate out CoM by introducing relative coordinates



Step 1: Relative Coordinates
- Separate out CoM by introducing relative coordinates

1. The first step to reduce the number of D is to separate out the CoM by introducing relative coordinates.

## Step 1: Relative Coordinates

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- Separate out CoM by introducing relative coordinates
- CS $\rightarrow$ 6D



1. CoM motion decouples from the internal motion in the SE the configuration space is effectively reduced to 6D

Step 1: Relative Coordinates

- Separate out CoM by introducing relative coordinates
- CS $\rightarrow$ 6D

# Step 1: Relative Coordinates

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Introduction

Theoretical Approach

Step 1

Step 2

Step 3

Effective Potentials

Numerical Approach

Scattering Model

Results

- Separate out CoM by introducing relative coordinates
- Choice: Mass-normalized Jacobi coordinates
- CS $\rightarrow$ 6D



1. For internal coordinates we choose a Mass-Normalized Jacobi coordinates

Step 1: Relative Coordinates

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# Step 1: Relative Coordinates

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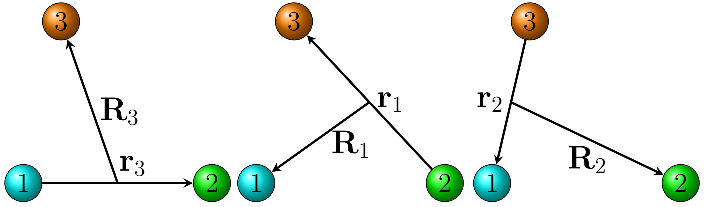




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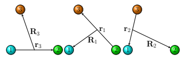

1. **r** is the vector that connects two of the particles and **R** connects the CoM of these two particles with the third particle
2. Particle permutations are easily performed using these coord.

# Step 2: Hyperspherical Coordinates

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Step 2: Hyperspherical Coordinates

1. Step 2 in simplifying the problem of three particles is to introduce hyperspherical coordinates

## Step 2: Hyperspherical Coordinates



- Combine $\mathbf{r}_k$and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$



Step 2: Hyperspherical Coordinates

- Combine $\mathbf{r}_k$and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$

1. The general idea is to combine the components of the two Jacobi vectors into a single six-dimensional position vector q, which represents a point in R6.

# Step 2: Hyperspherical Coordinates

- Combine $\mathbf{r}_k$and $\mathbf{R}_k$ into a 6D position vector in $\mathbb{R}^6$
- Hyperspherical coordinates: $\rho$ and $\Omega$

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$



Step 2: Hyperspherical Coordinates

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1. The hyperspherical coord. of this point are given by the hyperradius  and five hyperangles $\Omega$.
2. The hyperradial coordinate is both rotationally and permutationally invariant and is defined as the square root of the sum of the squared Jacobi vectors.
3. The hyperangles can be defined in many ways and I will not go in to the details here.

# Step 2: Hyperspherical Coordinates

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- Separate internal and external coordinates



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- Separate internal and external coordinates

1. At any instant, three particles form a plane in R3. We can define the internal motion of the particles within this plane in terms of the hyperradial coordinate (size) and two of the angles(shape and particle permutation). The three other angles relate rotations of this plane in a space fixed system.
2. The three other angles relate rotations of this plane in a space fixed system.

# Step 2: Hyperspherical Coordinates

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- Separate internal and external coordinates
- For $J = 0$ only internal coordinates matter $\rightarrow$ 3D Schrödinger equation (SE)



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$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

- Separate internal and external coordinates
- For $J = 0$ only internal coordinates matter $\rightarrow$ 3D Schrödinger equation (SE)

1. When the orbital angular momenta J=0 only the internal coordinates matter and we are left with a 3D SE for the internal motion

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Numerical Approach Exact rep.

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Step 3: The Adiabatic Representation

Numerical Approach Exact rep.

1. Now we move on to the final step (in the simplification of the 3BP)

# Step 3: The Adiabatic Representation

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

Numerical Approach Exact rep.



Step 3: The Adiabatic Representation

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Numerical Approach Exact rep.

1. After a clever rescaling of the wfn, the hyperspherical SE can be written like this

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- The hyperspherical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

Numerical Approach Exact rep.



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Numerical Approach Exact rep.

1. Where lambda is the generalized angular momenta

# Step 3: The Adiabatic Representation



- The hyperspherical SE:

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- Trick: Treat $\rho$ as an adiabatic parameter!

Numerical Approach Exact rep.



Step 3: The Adiabatic Representation

- The hypersperical SE:

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- Trick: Treat $\rho$ as an adiabatic parameter!

Numerical Approach Exact rep.

1. Now, the trick is to treat the hyperradius as an adiabatic parameter!
2. That is, we fix $\rho$ in a Born-Oppenheimer like manner

# Step 3: The Adiabatic Representation

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

- Trick: Treat $\rho$ as an adiabatic parameter!
- Solve the adiabatic eq.

$$H_{ad}\Phi_\nu(\rho;\Omega)=U_\nu(\rho)\Phi_\nu(\rho;\Omega)$$

Numerical Approach Exact rep.



Step 3: The Adiabatic Representation
- The hyperspherical SE:

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- Trick: Treat $\rho$ as an adiabatic parameter!
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$$H_{ad}\Phi_\nu(\rho;\Omega)=U_\nu(\rho)\Phi_\nu(\rho;\Omega)$$

Numerical Approach Exact rep.

1. And solve the remaining adiabatic eigenvalue equation

# Step 3: The Adiabatic Representation

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- The hyperspherical SE:

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$

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- $\rightarrow$ 3-body BO-like potential

Numerical Approach | Exact rep.



Step 3: The Adiabatic Representation
- The hyperspherical SE:
$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\Lambda^2+\frac{15}{4}}{2\mu\rho^2}+V\right)\psi=E\psi$$
- Trick: Treat $\rho$ as an adiabatic parameter!
- Solve the adiabatic eq.
$$H_{ad}\Phi_\nu(\rho;\Omega)=U_\nu(\rho)\Phi_\nu(\rho;\Omega)$$
- $\rightarrow$ 3-body BO-like potential
Numerical Approach | Exact rep.

1. In this way we obtain the three-body equivalent of a BO potential.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

SE



Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

SE

1. Now, in this way the total wfn can be represented by a sum of adiabatic states

# Step 3: The Adiabatic Representation cont.

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- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho; \Omega)$$

SE



Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho; \Omega)$$

1. (shown in red)

# Step 3: The Adiabatic Representation cont.



- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

SE



Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

SE

1. If we substitute this sum into the 3-b SE (klick on link)

# Step 3: The Adiabatic Representation cont.

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- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_nF_{n\mu}$$

SE



Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_nF_{n\mu}$$

SE

1. We will get an exact representation of the 3-body SE if all couplings are included

# Step 3: The Adiabatic Representation cont.

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- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_n F_{n\mu}$$

SE



Step 3: The Adiabatic Representation cont.
The total wave function is represented in terms of adiabatic states
The hyperradial eigenvalue equation

1. The focus of my work has been on this part of this eq.

# Step 3: The Adiabatic Representation cont.



- Th… a…

**Three-body effective potentials**

$$W_\nu(\rho) = U_\nu(\rho) - \frac{1}{2\mu} Q_{\nu\nu}(\rho) = U_\nu(\rho) - \frac{1}{2\mu} P^2_{\nu\nu}(\rho)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_n F_{n\mu}$$





Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu}Q_{\mu\mu}\right)F_{n\mu} - \frac{1}{2\mu}\left(\sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu}\right)F_{n\nu} = E_n F_{n\mu}$$

SE

1. In the adiabatic approximation we define the three-body effective potentials are defined as
2. These potentials are used for determining the single channel solutions of (below)

# Step 3: The Adiabatic Representation cont.



- Th… a…

**Three-body effective potentials**

$$W_\nu(\rho) = U_\nu(\rho) - \frac{1}{2\mu} Q_{\nu\nu}(\rho) \approx U_\nu(\rho) - \cancel{\frac{1}{2\mu} P^2_{\nu\nu}(\rho)}$$

- The hyperradial eigenvalue equation

$$\left( -\frac{1}{2\mu}\frac{\partial^2}{\partial\rho^2} + U_\mu - \frac{1}{2\mu} Q_{\mu\mu} \right) F_{n\mu} - \frac{1}{2\mu}\left( \sum_{\nu\neq\mu} 2P_{\mu\nu}\frac{\partial}{\partial\rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

SE



Step 3: The Adiabatic Representation cont.

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho,\Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho)\Phi_\nu(\rho;\Omega)$$

- The hyperradial eigenvalue equation

1. In this talk I will not explain how the second term is calculated, in the calculations I have performed it is small and we will ignore it from here on

# Convergence In the Asymptotic Limit

Kajsa-My Blomdahl





Convergence In the Asymptotic Limit

1. (The following discussion concerns short-ranged two-body interactions, where $|a| \gg r_0$)
2. The behavoiur of the 3-b potentials in the asymptotic limit, (i.e., when the hyperradius is much larger than the magnitude a) depend on the sign of a.

# Convergence In the Asymptotic Limit

**For $a < 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$



Convergence In the Asymptotic Limit

For $a < 0$:

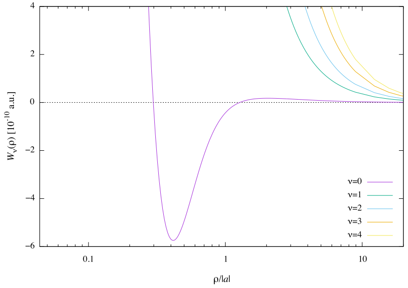
$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$

1. When a has a negative sign there is no weakly bound dimer and the lowest effective potential will converge to the three-body continuum channels, i.e., the kinetic energy for three free particles.

# Convergence In the Asymptotic Limit

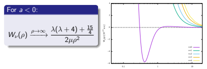
**For $a < 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$






Convergence In the Asymptotic Limit

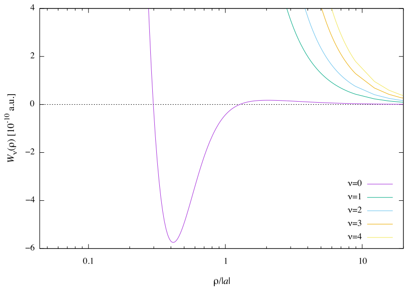


# Convergence In the Asymptotic Limit



**For $a < 0$:**

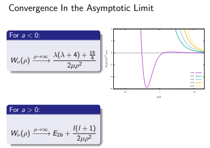
$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$




**For $a > 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} E_{2b} + \frac{l(l+1)}{2\mu\rho^2}$$



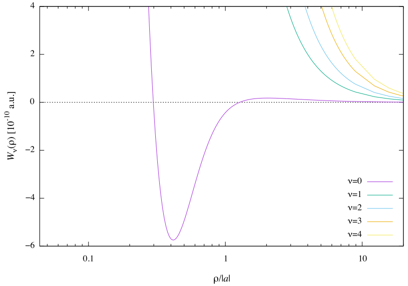


1. However, for systems of 3 identical bosons with a pair-wise attraction that is strong enough to support 2-body bound states, one 3-body effective potential curve will converge asymptotically to each two-body bound state.

# Convergence In the Asymptotic Limit

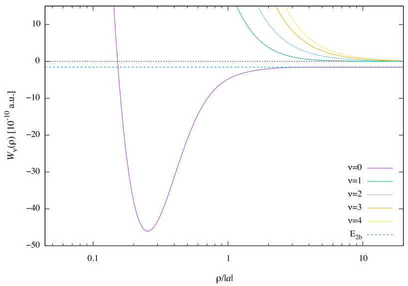
**For $a < 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} \frac{\lambda(\lambda+4)+\frac{15}{4}}{2\mu\rho^2}$$




**For $a > 0$:**

$$W_\nu(\rho) \xrightarrow{\rho\to\infty} E_{2b} + \frac{l(l+1)}{2\mu\rho^2}$$

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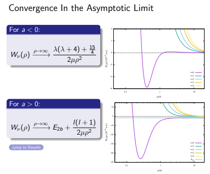

# Convergence In the Intermediate Region



## Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

Jump to Results



Convergence In the Intermediate Region
Intermediate Interaction Range
$r_0 \ll \rho \ll |a|$
Jump to Results

1. Efimov physics comes into play in the intermediate region.
2. In this region the three-body effective potentials are modified by the Efimov physics. It can lead to both attractive and repulsive effective potentials.

# Convergence In the Intermediate Region

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## Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

## The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

Jump to Results



Convergence In the Intermediate Region
Intermediate Interaction Range
$r_0 \ll \rho \ll |a|$
The Lowest Potential's Convergent Form (3 Identical Bosons)
$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$
Jump to Results

1. For 3 identical bosons the effective potential will be attractive and is responsible for the Efimov Effect!

# Convergence In the Intermediate Region

Kajsa-My Blomdahl



## Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

## The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

Jump to Results



Convergence In the Intermediate Region
Intermediate Interaction Range
$r_0 \ll \rho \ll |a|$
The Lowest Potential's Convergent Form (3 Identical Bosons)
$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$
Jump to Results

1. And is responsible for the Efimov Effect!

# Convergence In the Intermediate Region

Kajsa-My Blomdahl



## Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

## The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

## Universal Constant (3 Identical Bosons)

$$s_0 \simeq 1.00624$$

Jump to Results



Convergence In the Intermediate Region
Intermediate Interaction Range
$r_0 \ll \rho \ll |a|$
The Lowest Potential's Convergent Form (3 Identical Bosons)
$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$
Universal Constant (3 Identical Bosons)
$s_0 \simeq 1.00624$
Jump to Results

1. The universal constant $s_0$ sets the size and energy scaling of succeding Efimov states.

# Analytic Model



Jump to Results



Analytic Model

Jump to Results

1. We can obtain a similair result analytically by instead of solving the SE we solve the coupled Faddeev equations

# Analytic Model



- The adiabatic potentials $\nu_n$ can be determined analytically through the transcendental equation

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Jump to Results



Analytic Model

- The adiabatic potentials $\nu_n$ can be determined analytically through the transcendental equation

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Jump to Results

1. We then obtain the adiabatic potential $\nu$ through the transcendental eq.

# Analytic Model

- The adiabatic potentials $\nu_n$ can be determined analytically through the transcendental equation

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Jump to Results



Analytic Model

- The adiabatic potentials $\nu_n$ can be determined analytically through the transcendental equation

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Jump to Results

1. This adiabatic potential is a function of $\rho/a$

# Analytic Model



- The adiabatic potentials $\nu_n$ can be determined analytically through the transcendental equation

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right)-\frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right)=\sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

- Three-body effective potential

$$W_\nu(\rho/a)=\frac{\left(\nu_n(\rho/a)-\frac{1}{4}\right)}{2\mu\rho^2}$$

Jump to Results



Analytic Model

- The adiabatic potentials $\nu_n$ can be determined analytically through the transcendental equation

$$\sqrt{\nu_n}\cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right)-\frac{8}{\sqrt{3}}\sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right)=\sqrt{2}\frac{\rho}{a}\sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

- Three-body effective potential

$$W_\nu(\rho/a)=\frac{\left(\nu_n(\rho/a)-\frac{1}{4}\right)}{2\mu\rho^2}$$

Jump to Results

1. It is related to the 3-body potential
2. In the result section I will compare these solutions to my numerically calculated potentials.

# Numerical Approach; B-spline Collocation

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$$\text{Task} = ?$$



Numerical Approach; B-spline Collocation
Task = ?

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$$\text{Task} = \text{Find } W_\nu(\rho)!$$



Numerical Approach; B-spline Collocation

Task = Find $W_\nu(\rho)$!

# Numerical Approach; B-spline Collocation

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First: Find a Basis



Numerical Approach; B-spline Collocation

First: Find a Basis

# Numerical Approach; B-spline Collocation



First: Find a Basis

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$



Numerical Approach; B-spline Collocation

First: Find a Basis

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$

# Numerical Approach; B-spline Collocation

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First: Find a Basis

Then: Expand

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$



Numerical Approach; B-spline Collocation

First: Find a Basis
Then: Expand

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$

# Numerical Approach; B-spline Collocation

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First: Find a Basis

Then: Expand

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_{\nu}(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$



Numerical Approach; B-spline Collocation

First: Find a Basis
Then: Expand
$\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
$\Phi_{\nu}(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$

# Numerical Approach; B-spline Collocation

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First: Find a Basis

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$

Then: Expand

- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$

Next: Substitute $\Phi_\nu$ into Eq. (4)



Numerical Approach; B-spline Collocation

First: Find a Basis
Then: Expand
Next: Substitute $\Phi_\nu$ into Eq. (4)

- $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$

# Numerical Approach; B-spline Collocation

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| | | |
|---|---|---|
| First: | Find a Basis | $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$ |
| Then: | Expand | $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$ |
| Next: | Substitute $\Phi_\nu$ into Eq. (4) | $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$ |



Numerical Approach; B-spline Collocation

First: Find a Basis — $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
Then: Expand — $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
Next: Substitute $\Phi_\nu$ into Eq. (4) — $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$

# Numerical Approach; B-spline Collocation

| | | |
|---|---|---|
| First: | Find a Basis | $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$ |
| Then: | Expand | $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$ |
| Next: | Substitute $\Phi_\nu$ into Eq. (4) | $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$ |
| Finally: | Solve the Generalized Eigenvalue Eq. | |



Numerical Approach; B-spline Collocation

First: Find a Basis $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
Then: Expand $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
Next: Substitute $\Phi_\nu$ into Eq. (4) $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$
Finally: Solve the Generalized Eigenvalue Eq.

# Numerical Approach; B-spline Collocation



| | | |
|---|---|---|
| First: | Find a Basis | $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$ |
| Then: | Expand | $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$ |
| Next: | Substitute $\Phi_\nu$ into Eq. (4) | $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$ |
| Finally: | Solve the Generalized Eigenvalue Eq. | $W(\rho) \approx U(\rho)$ |



Numerical Approach; B-spline Collocation

First: Find a Basis $\varphi_{lm} = \varphi_{1l}(\theta)\varphi_{2m}(\phi)$
Then: Expand $\Phi_\nu(\rho;\theta,\phi) = \sum_{l,m}^{L,M} c_{lm}\varphi_{lm}$
Next: Substitute $\Phi_\nu$, into Eq. (4) $\mathbf{H}_{\mathrm{ad}}\mathbf{c} = U\mathbf{B}\mathbf{c}$
Finally: Solve the Generalized Eigenvalue Eq. $W(\rho) \approx U(\rho)$

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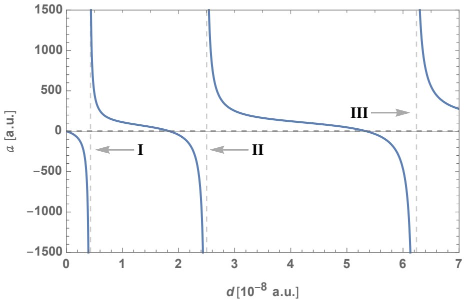
# Scattering Model

**Masses**

$m = m(Rubidium - 87)$

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└Scattering Model

└Scattering Model

Scattering Model

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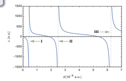


# Scattering Model

Kajsa-My Blomdahl

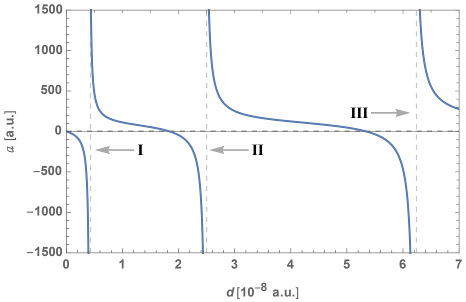


## Masses

$m = m(Rubidium - 87)$

## Assumption

$V(\rho, \theta, \psi) =$
$v(r_{12}) + v(r_{23}) + v(r_{31})$






Scattering Model
Masses
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# Scattering Model

Kajsa-My Blomdahl



**Masses**

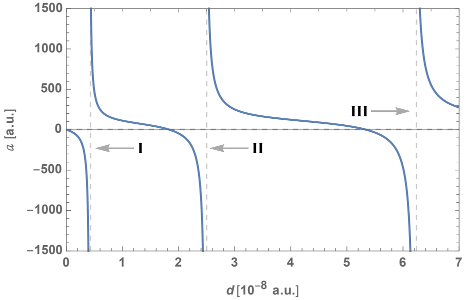
$m = m(Rubidium - 87)$

**Assumption**

$V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31})$

**2B Model Potential**

$v(r) = d \cosh^{-2}(r/r_0)$

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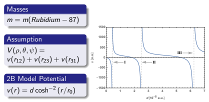


# Scattering Model



**Masses**

$m = m(Rubidium - 87)$

**Assumption**

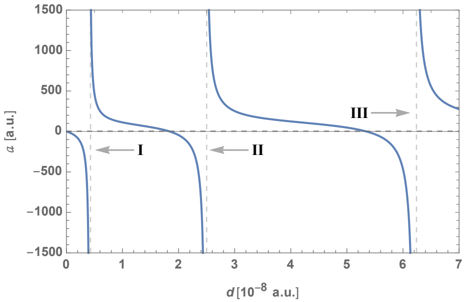
$V(\rho, \theta, \psi) =$
$v(r_{12}) + v(r_{23}) + v(r_{31})$

**2B Model Potential**

$v(r) = d \cosh^{-2}(r/r_0)$

**Interaction Range**

$r_0 = 55$ a.u.






Scattering Model

Masses

$m = m(Rubidium - 87)$

Assumption

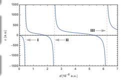
$V(\rho, \theta, \psi) =$
$v(r_{12}) + v(r_{23}) + v(r_{31})$

2B Model Potential

$v(r) = d \cosh^{-2}(r/r_0)$

Interaction Range

$r_0 = 55$ a.u.



# Convergence and Accuracy

- For $a \to \pm\infty$ we expect that the lowest effective potential curve will converge towards the Efimovian form
- This behaviour is easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as

$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4} \tag{1}$$

since these curves should approach the universal value $-s_0^2 (\simeq -1.0125)$ in the intermediate region

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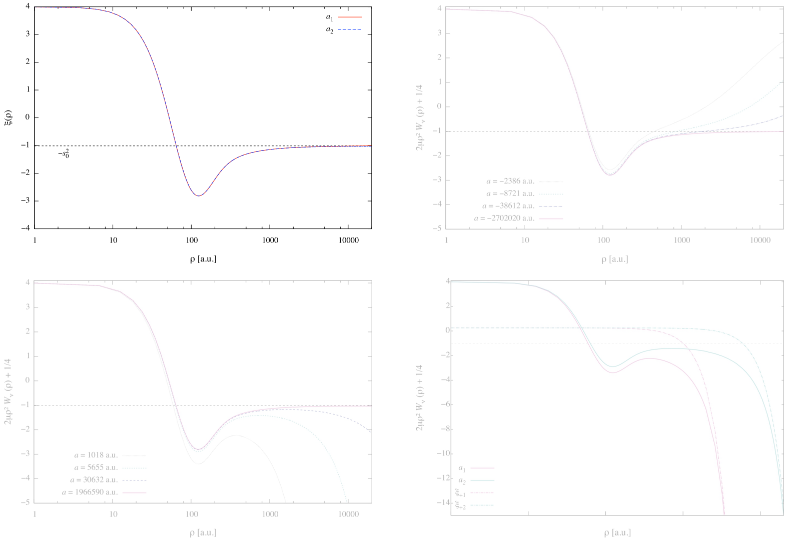
To Figures

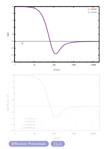
# Efimov-like Potentials $\xi(\rho)$ for Different $a$






Effective Potentials $\xi(\rho)$





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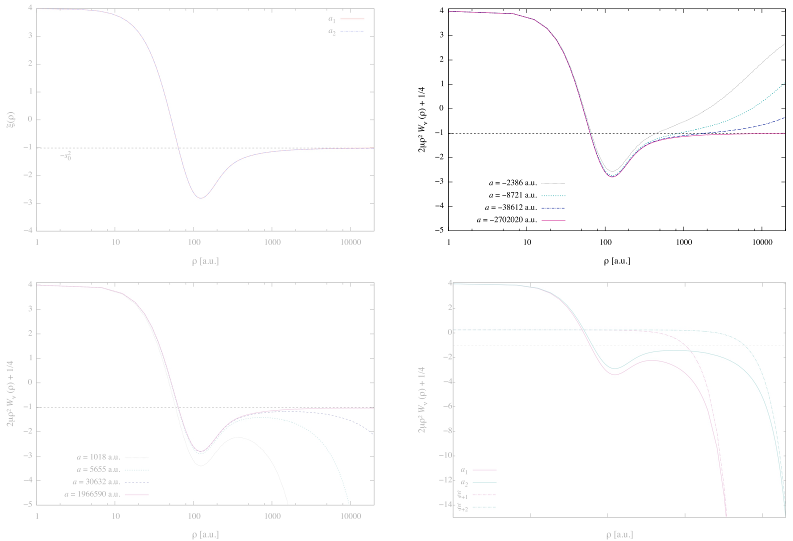
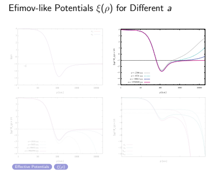



Effective Potentials $\xi(\rho)$





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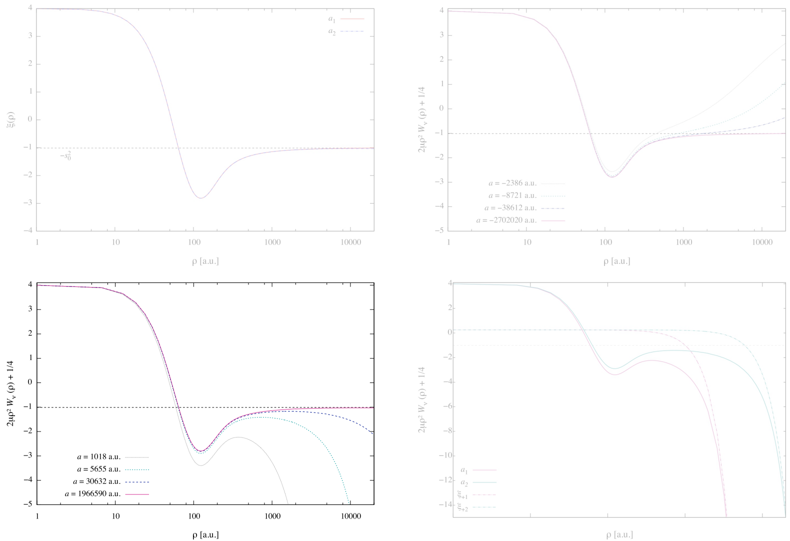


Effective Potentials $\xi(\rho)$

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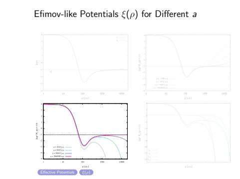

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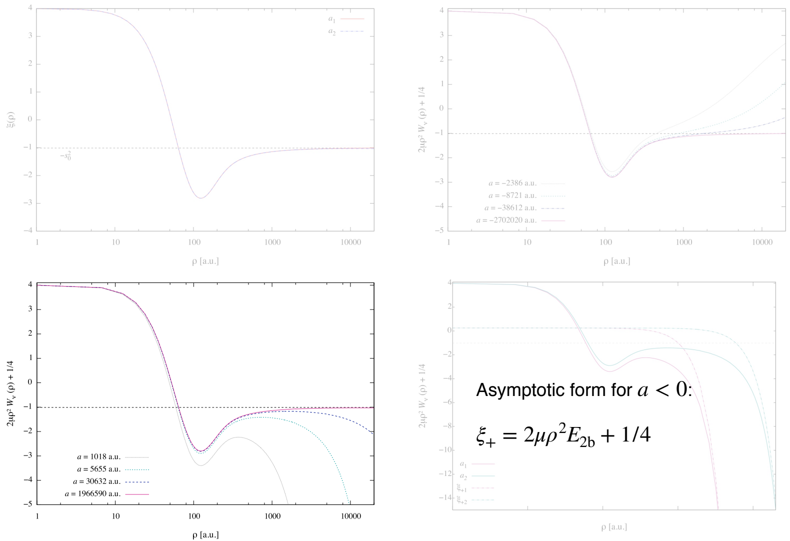
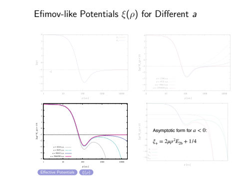



Effective Potentials
$\xi(\rho)$





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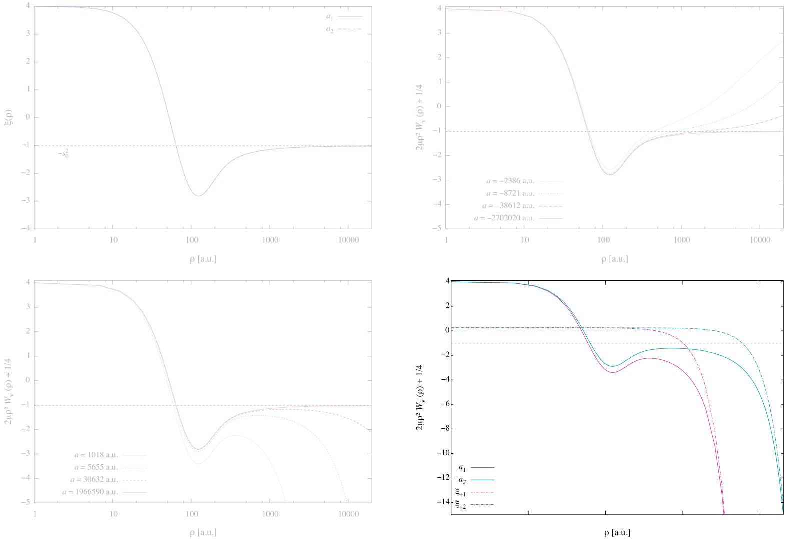
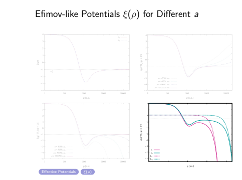


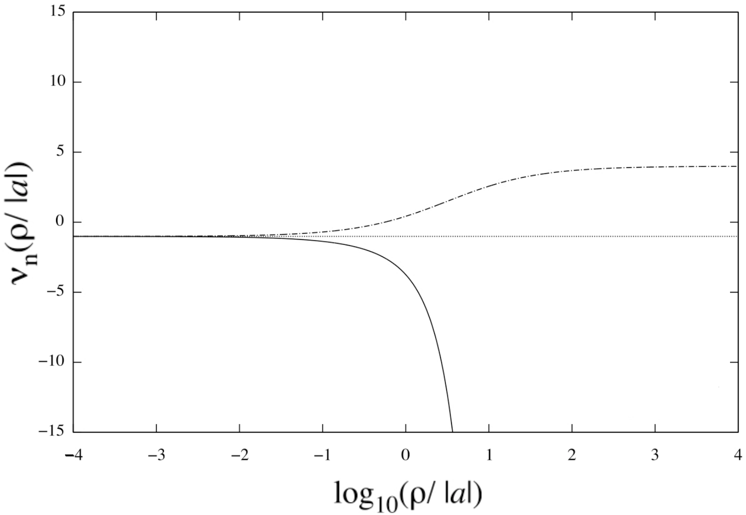

Effective Potentials $\xi(\rho)$





# Comparison to the Analytical Model (1)



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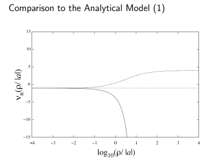
Results
Comparison to the Analytical Model
Comparison to the Analytical Model (1)



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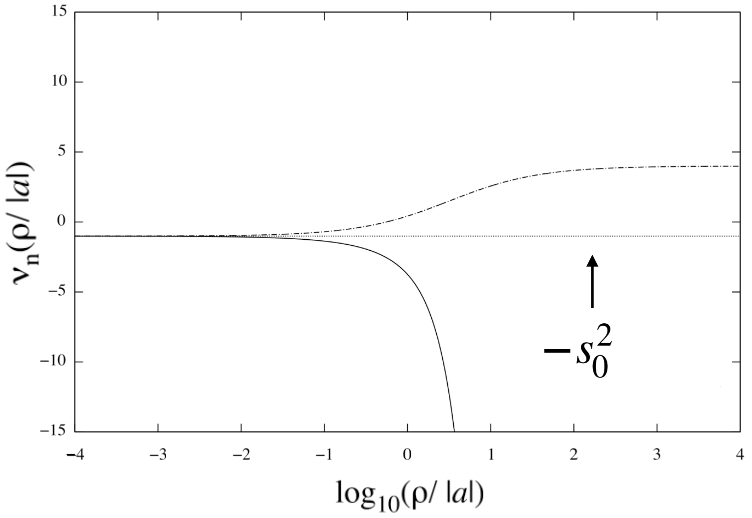


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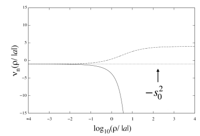
Comparison to the Analytical Model (1)



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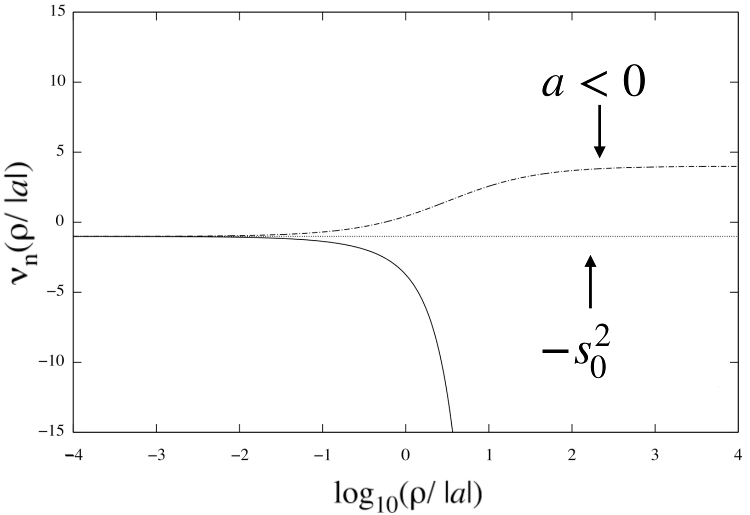




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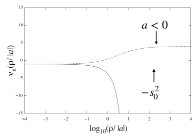

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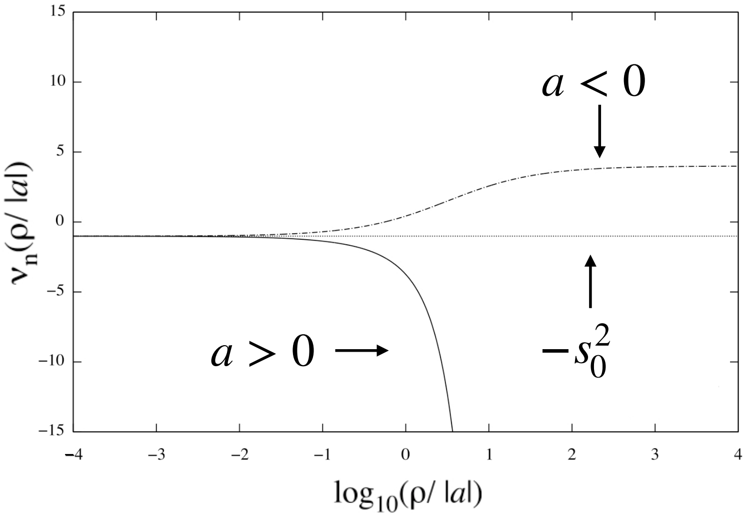
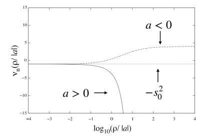
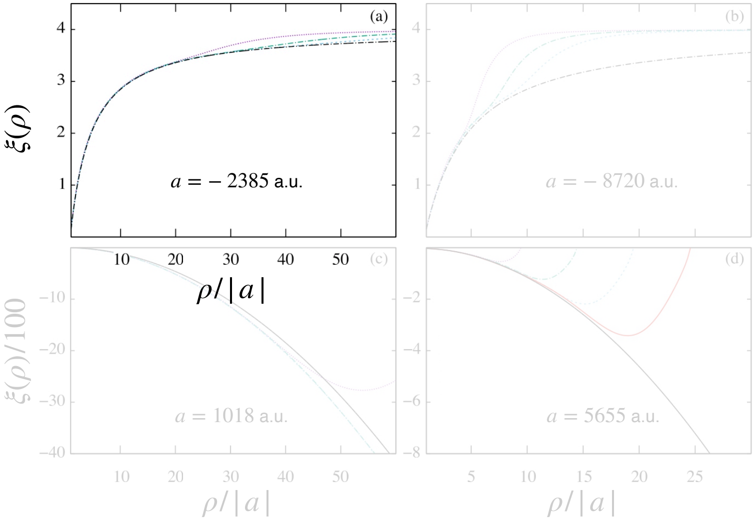




# Comparison to the Analytical Model (2)



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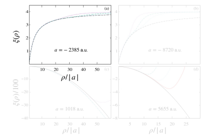
Results
└Comparison to the Analytical Model
└Comparison to the Analytical Model (2)

Comparison to the Analytical Model (2)



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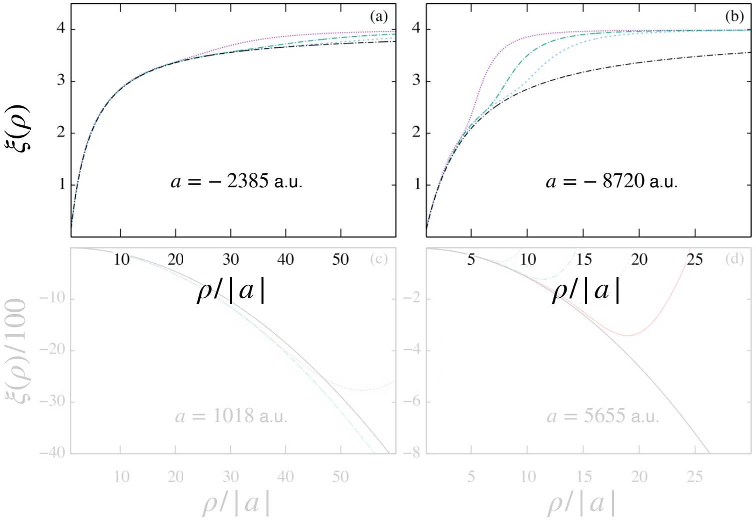


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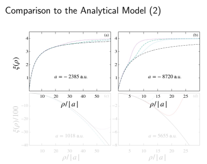

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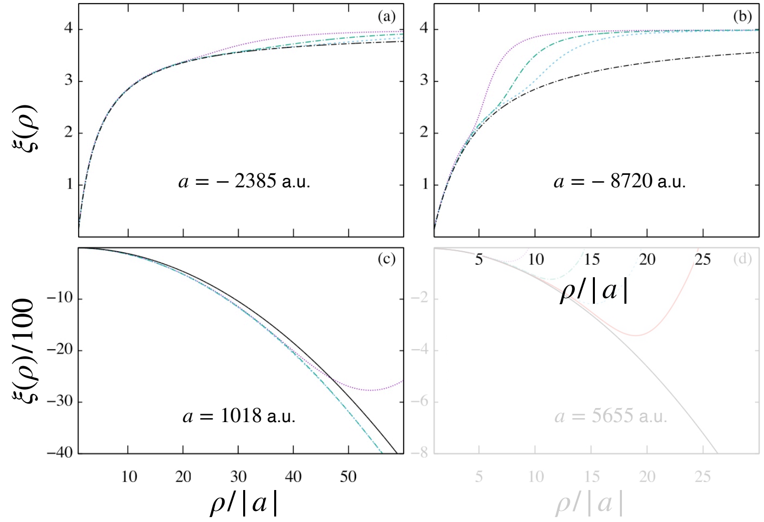




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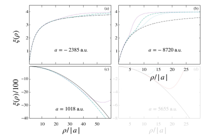

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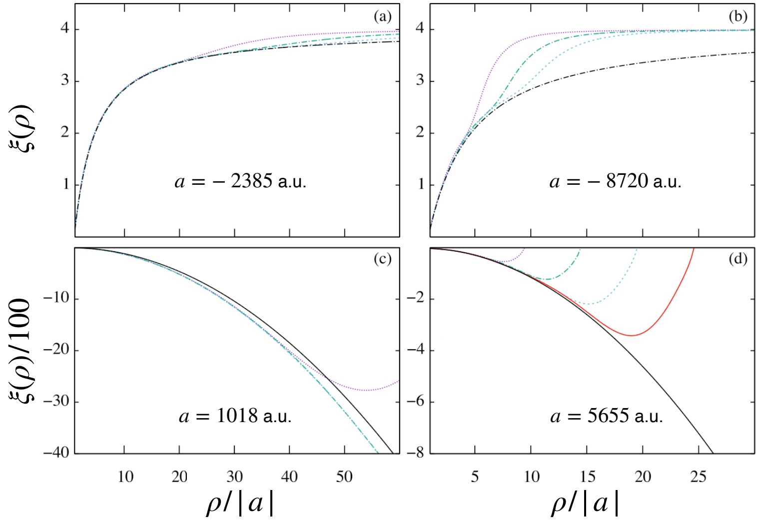




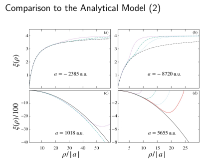

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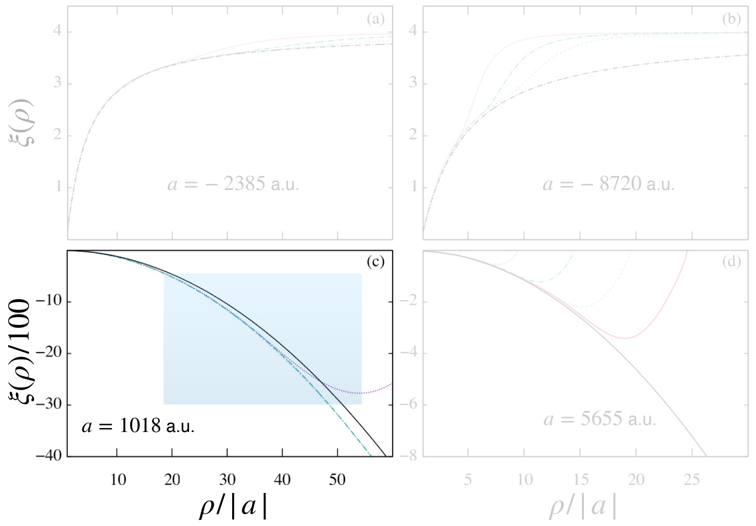


Kajsa-My Blomdahl



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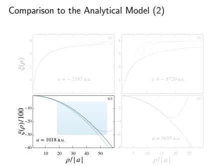
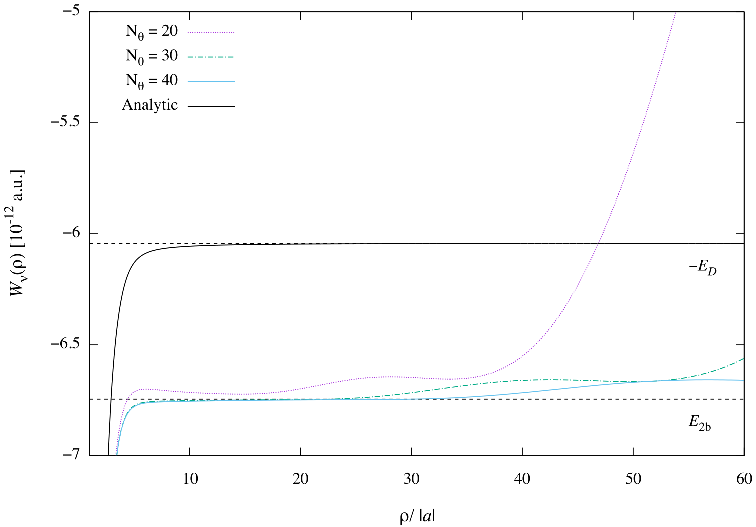


# Comparison to the Analytical Model (3)



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Results
Comparison to the Analytical Model
Comparison to the Analytical Model (3)