

Theoretical and Numerical Studies of Efimov States

Kajsa-My Blomdahl

Stockholms Universitet

kajsammy.blomdahl@fysik.su.se

September 3, 2019

2019-09-03

1. Hi, my name is Kajsa-My Blomdahl, I study Efimov Physics, which is encapsulate a number of effects that appear in the quantum 3BP.

1 Introduction

2 Theoretical Approach

3 Effective Potentials

4 Numerical Approach

5 Scattering Model

6 Results

2019-09-03

1. To understand important features of the quantum 3BP I will start by introduce a few concepts from quantum scattering of 2 particles.

Two-body (2-b) Interactions

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length
Universality
The Efimov
Effect

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- Atomic collisions in the ultra cold regime

2019-09-03

└ Introduction
 └ Two-body Interactions
 └ Two-body (2-b) Interactions

• Atomic collisions in the ultra cold regime

1. Atomic interactions are, essentially, pair-wise and short-ranged, which means that they interact when they are close to each other.

Two-body (2-b) Interactions

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov

Effect

Theoretical
Approach

Effective
Potentials

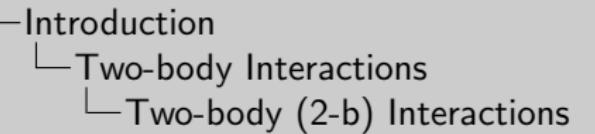
Numerical
Approach

Scattering
Model

Results

- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as *s-waves*, *p-waves* and *d-waves* etc.

2019-09-03



- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as *s-waves*, *p-waves* and *d-waves* etc.

1. At very low energies, atoms behave like point particles and have quantized orbital angular momenta l . The quantum numbers $l = 0, 1, 2$, associated with an atom, are referred to as *s-waves*, *p-waves* and *d-waves*, and so on

Two-body (2-b) Interactions

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov
Effect

Theoretical
Approach

Effective
Potentials

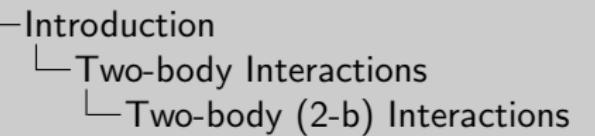
Numerical
Approach

Scattering
Model

Results

- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as **s-waves**, **p-waves** and **d-waves** etc.

2019-09-03



- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as **s-waves**, **p-waves** and **d-waves** etc.

1. In the ultracold regime s-wave collisions dominate (because higher partial waves are reflected by the centrifugal barrier in the SE)

Two-body (2-b) Interactions

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov

Effect

Theoretical
Approach

Effective
Potentials

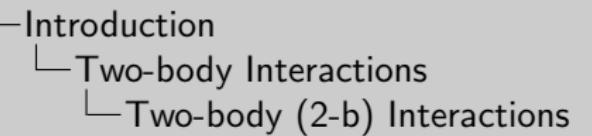
Numerical
Approach

Scattering
Model

Results

- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as *s-waves*, *p-waves* and *d-waves* etc.
- 2-b scattering in this regime is governed by a parameter called the **s-wave scattering length a**

2019-09-03



- Atomic collisions in the ultra cold regime
- Quantized orbital angular momenta $l = 0, 1, 2$, are referred to as *s-waves*, *p-waves* and *d-waves* etc.
- 2-b scattering in this regime is governed by a parameter called the **s-wave scattering length a**

1. Two-body scattering in this regime is solely governed by a single parameter called the **s-wave scattering length a** , or just scattering length for short

Scattering Length

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov

Effect

Theoretical
Approach

Effective
Potentials

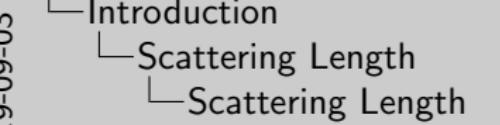
Numerical
Approach

Scattering
Model

Results

- Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$



2019-09-03

1. The s-wave scattering length is defined in the low-energy limit as

Scattering Length

• Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

Scattering Length

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov

Effect

Theoretical
Approach

Effective
Potentials

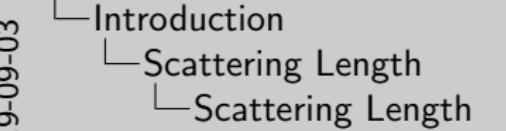
Numerical
Approach

Scattering
Model

Results

- Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$



2019-09-03

1. where δ is the s-wave phase shift of the outgoing wave (and k is the wave number = $\sqrt{2\mu E}/\hbar$)

Scattering Length

• Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

Scattering Length

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality
The Efimov
Effect

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction

2019-09-03
└ Introduction
 └ Scattering Length
 └ Scattering Length

Scattering Length

• Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

• Characterizes the strength of the interparticle interaction

1. The scattering length characterizes the strength of the interaction. In the absence of an interaction, the phase shift is simply zero and the outgoing scattered wave is in phase with the incoming wave. Any interaction will cause a dephasing between the outgoing and incoming waves. The strongest dephasing occurs when $\pi/2$; a will then diverge.

Scattering Length

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality
The Efimov
Effect

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

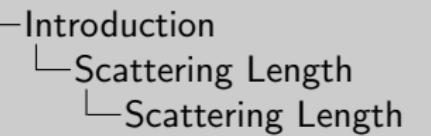
Results

- Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of a and the effective interaction

2019-09-03



♦ Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

♦ Characterizes the strength of the interparticle interaction

♦ The sign of a and the effective interaction

1. The sign of a carries information about whether the effective interaction is attractive or repulsive. If the two-body interaction has no bound states a is negative. However if the interaction has one or more bound state a can be both positive and negative.

Scattering Length

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov
Effect

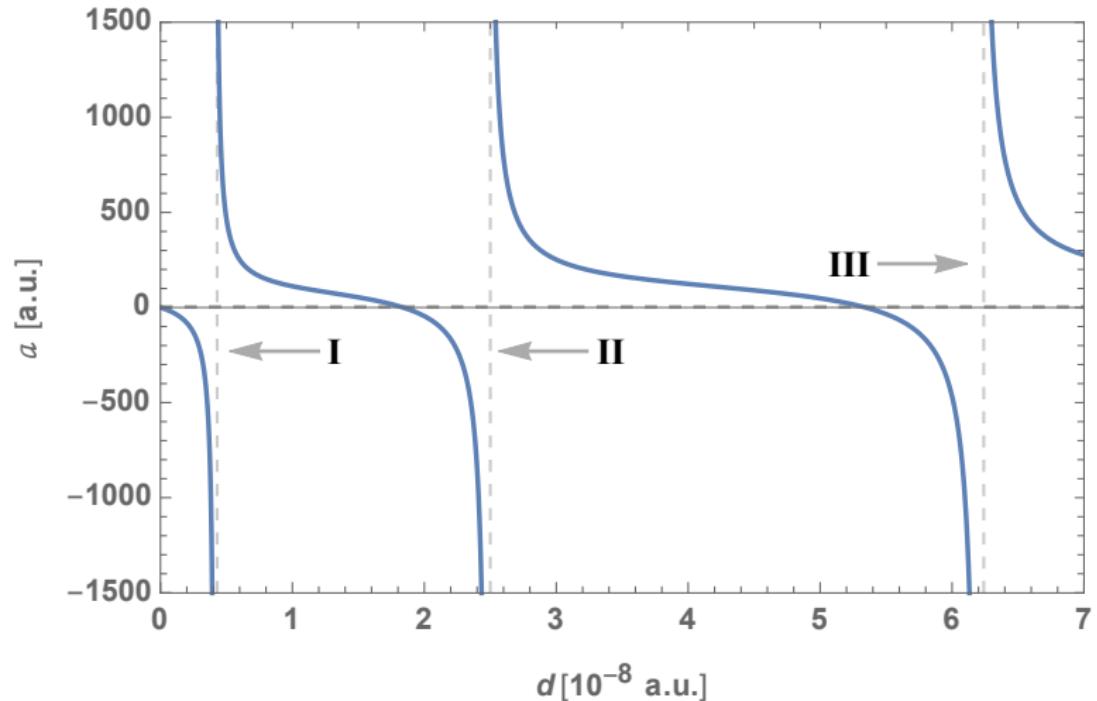
Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

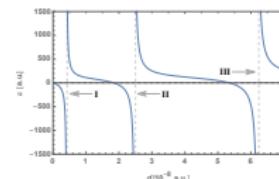
Results



2019-09-03

└ Introduction
 └ Scattering Length
 └ Scattering Length

Scattering Length



1. To illustrate this I have finetuned a model two body potential by changing the depth of the potential. Here we have a on the y-axis and the depth d of the attractive 2b-potential on the x-axis.

Scattering Length

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov

Effect

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of a and the effective interaction
- **Negative $a \rightarrow$ attractive** effective interactions

2019-09-03

- └ Introduction
 - └ Scattering Length
 - └ Scattering Length

Scattering Length

• Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of a and the effective interaction
- **Negative $a \rightarrow$ attractive** effective interactions

1. Negative scattering lengths correspond to attractive effective interactions, meaning the scattered wave is being pulled in by the potential

Scattering Length

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov

Effect

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

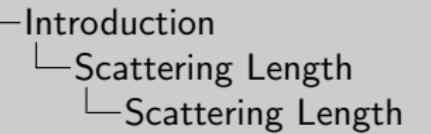
Results

- Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of a and the effective interaction
- Negative $a \rightarrow$ attractive effective interactions
- Positive $a \rightarrow$ repulsive** effective interactions

2019-09-03



- Definition: $a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$
- Characterizes the strength of the interparticle interaction
- The sign of a and the effective interaction
- Negative $a \rightarrow$ attractive effective interactions
- Positive $a \rightarrow$ repulsive** effective interactions

- Positive scattering lengths correspond to repulsive effective interactions, meaning the scattered wave is being pushed out by the potential

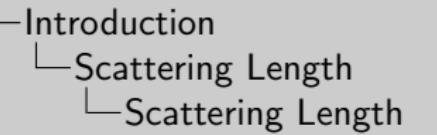
Scattering Length

- Definition:

$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$

- Characterizes the strength of the interparticle interaction
- The sign of a and the effective interaction
- Negative $a \rightarrow$ attractive effective interactions
- Positive $a \rightarrow$ repulsive effective interactions
- For $|a| \rightarrow \infty$ the interaction is called **resonant**

2019-09-03



Scattering Length

- Definition:
$$a = \lim_{k \rightarrow 0} -\frac{\tan \delta_0(k)}{k}$$
- Characterizes the strength of the interparticle interaction
- The sign of a and the effective interaction
- Negative $a \rightarrow$ attractive effective interactions
- Positive $a \rightarrow$ repulsive effective interactions
- For $|a| \rightarrow \infty$ the interaction is called **resonant**

1. When the magnitude of $a \rightarrow \infty$ we say that the interaction is resonant. In this case the interaction is fully characterized by the scattering length, which is much larger than the interaction range of the particles.

Universality in 2-b systems

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov
Effect

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

2-b scattering

Particles with large $|a|$ in the low-energy regime have universal properties

2019-09-03

- └ Introduction
- └ Universality
- └ Universality in 2-b systems

2-b scattering
Particles with large $|a|$ in the low-energy regime have universal properties

1. Particles with large scattering lengths in the low-energy regime are interesting because they have universal properties.

Universality in 2-b systems

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov
Effect

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

2-b scattering

Particles with large $|a|$ in the low-energy regime have universal properties

Universal properties ... In what sense?

Depend on the scattering length alone and not on the details of the short-range interaction

2019-09-03

- └ Introduction
- └ Universality
- └ Universality in 2-b systems

2-b scattering
Particles with large $|a|$ in the low-energy regime have universal properties
Universal properties ... In what sense?
Depend on the scattering length alone and not on the details of the short-range interaction

1. What do we mean by universal? It means that they depend on the scattering alone and not on the details of the short-range interaction, which means all bosons behave in the same way it does not matter what atomic species we look at

Universality in 2-b systems

Kajsa-My
Blomdahl

[Introduction](#)

Two-body
Interactions

Scattering
Length

Universality

The Efimov
Effect

[Theoretical
Approach](#)

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

2-b scattering

Particles with large $|a|$ in the low-energy regime have universal properties

Universal properties ... In what sense?

Depend on the scattering length alone and not on the details of the short-range interaction

► Results

Example: 2-b binding energy for 2 identical bosons

$$E_D = \frac{\hbar^2}{ma^2}$$

2019-09-03

- └ Introduction
- └ Universality
- └ Universality in 2-b systems

1. For example:

2-b scattering
Particles with large $|a|$ in the low-energy regime have universal properties

Universal properties ... In what sense?
Depend on the scattering length alone and not on the details of the short-range interaction

Example: 2-b binding energy for 2 identical bosons

$$E_D = \frac{\hbar^2}{ma^2}$$

Efimov's Prediction

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions
Scattering
Length
Universality

The Efimov Effect

Theoretical Approach

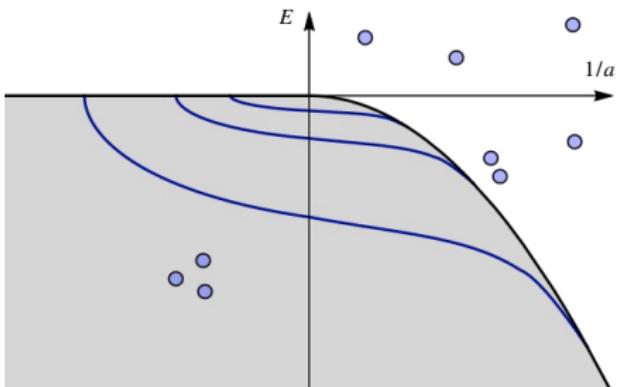
Effective
Potentials

Numerical Approach

Scattering Model

Results

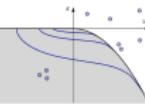
- Resonant 2-b forces give rise to bound energy levels in 3-particle systems



2019-09-03

└ Introduction
 └ The Efimov Effect
 └ Efimov's Prediction

Resonant 2-b forces give rise to bound energy levels in 3-particle systems



- In the 1970 Vitaly Efimov predicted that resonant can give rise to a series of bound energy levels in 3-particle systems, which we now call Efimov states

Efimov's Prediction

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions
Scattering
Length
Universality
The Efimov
Effect

Theoretical Approach

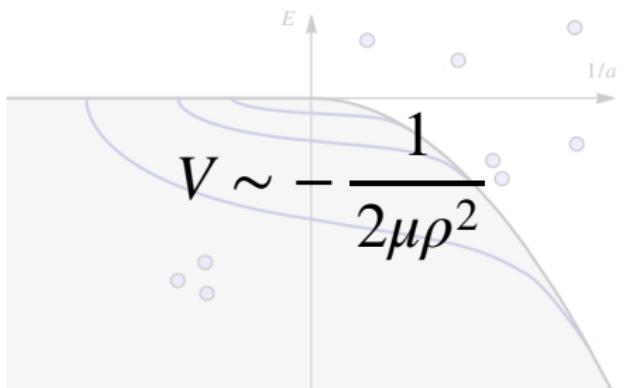
Effective
Potentials

Numerical Approach

Scattering Model

Results

- When $|a| \rightarrow \infty$ a universal long-range 3-body attraction emerge



2019-09-03

└ Introduction
 └ The Efimov Effect
 └ Efimov's Prediction

When $|a| \rightarrow \infty$ a universal long-range 3-body attraction emerge

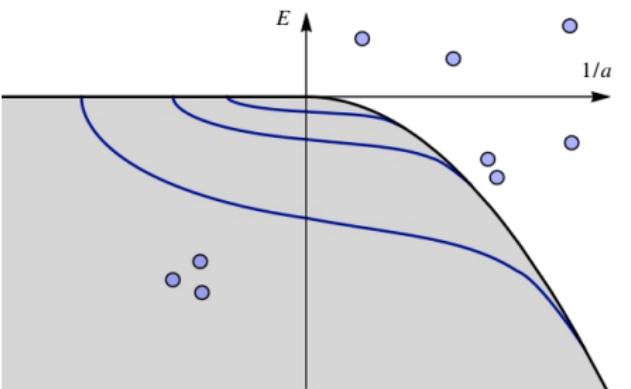
$$V \sim -\frac{1}{2\mu\rho^2}$$

- When the short-ranged two-body forces approached resonance, he found a universal long-range three-body attraction emerging, giving rise to an infinite number of trimer states with binding energies obeying a discrete scaling law at resonance.

Efimov's Prediction

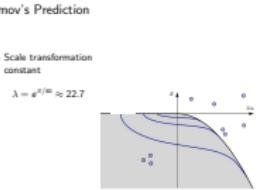
- Scale transformation constant

$$\lambda = e^{\pi/s_0} \approx 22.7$$



2019-09-03

- └ Introduction
 - └ The Efimov Effect
 - └ Efimov's Prediction



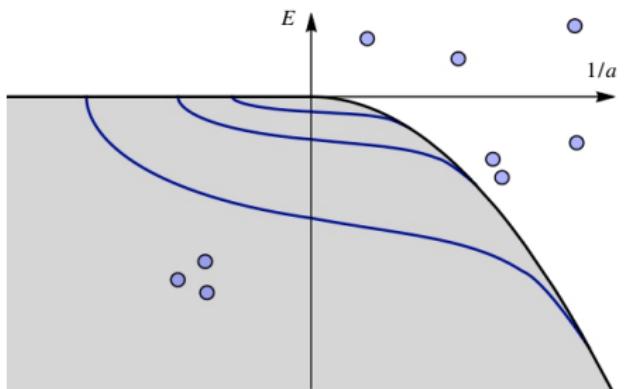
1. The Efimov states have universal properties. For three identical bosons are that the size and energy of successive trimer states in the resonant limit are related by a scale transformation with a constant $\lambda = e^{\pi/s_0} \approx 22.7$

Efimov's Prediction

- Scale transformation constant

$$\lambda = e^{\pi/s_0} \approx 22.7$$

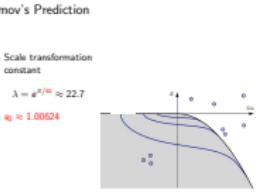
- $s_0 \approx 1.00624$



2019-09-03

- Introduction
- The Efimov Effect
- Efimov's Prediction

1. Is a universal constant in Efimov physics, which I will return to later



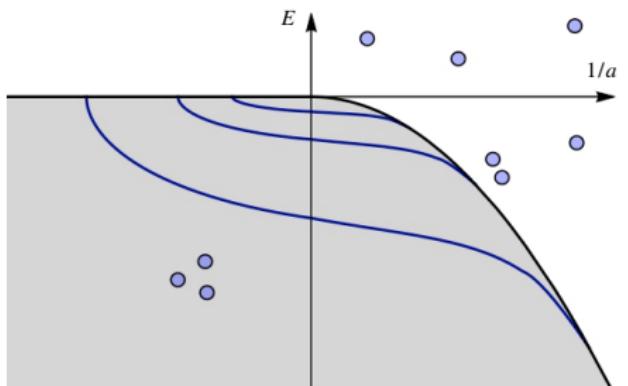
Efimov's Prediction

- Scale transformation constant

$$\lambda = e^{\pi/s_0} \approx 22.7$$

- $s_0 \approx 1.00624$
- Size scaling:

$$\rho^{n+1}/\rho^n \approx \lambda$$

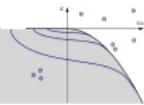


2019-09-03

- └ Introduction
 - └ The Efimov Effect
 - └ Efimov's Prediction

Efimov's Prediction

- Scale transformation constant
- $\lambda = e^{\pi/s_0} \approx 22.7$
- $s_0 \approx 1.00624$
- Size scaling:
 - $\rho^{n+1}/\rho^n \approx \lambda$



1. The ratio of the size of two successive Efimov states is given by the constant lambda

Efimov's Prediction

- Scale transformation constant

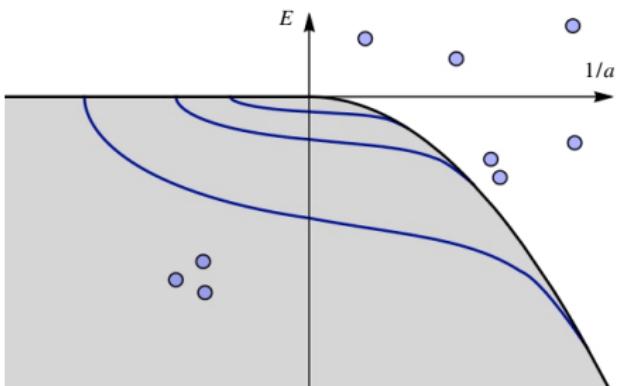
$$\lambda = e^{\pi/s_0} \approx 22.7$$

- $s_0 \approx 1.00624$
- Size scaling:

$$\rho^{n+1}/\rho^n \approx \lambda$$

- Energy scaling:

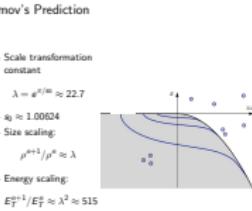
$$E_T^{n+1}/E_T^n \approx \lambda^2 \approx 515$$



2019-09-03

- └ Introduction
 - └ The Efimov Effect
 - └ Efimov's Prediction

1. while the ratio of the energy of two successive Efimov states scale geometrically with lambda



The Peculiar Efimov Effect

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions
Scattering
Length
Universality

The Efimov Effect

Theoretical Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

2019-09-03

- └ Introduction
 - └ The Efimov Effect
 - └ The Peculiar Efimov Effect

1. The Efimov effect is remarkable in many ways

The Peculiar Efimov Effect

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions
Scattering
Length

Universality
The Efimov
Effect

Theoretical Approach

Effective
Potentials

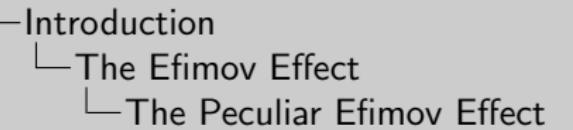
Numerical
Approach

Scattering
Model

Results

- The size of each Efimov state (ES) \gg the interaction range (r_0) between the individual particle pairs \rightarrow QM effect

2019-09-03



• The size of each Efimov state (ES) \gg the interaction range (r_0) between the individual particle pairs \rightarrow QM effect

1. Because the size of each Efimov state is much larger than the force-range between the individual pairs it means we are dealing with a pure quantum mechanical effect

The Peculiar Efimov Effect

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions
Scattering
Length
Universality
The Efimov
Effect

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The size of each Efimov state (ES) \gg the interaction range (r_0) between the individual particle pairs \rightarrow QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$

2019-09-03

└ Introduction

 └ The Efimov Effect

 └ The Peculiar Efimov Effect

- The size of each Efimov state (ES) \gg the interaction range (r_0) between the individual particle pairs \rightarrow QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$

1. When the magnitude of scattering length approach infinity, there is an infinite number of Efimov states

The Peculiar Efimov Effect

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions
Scattering
Length
Universality
The Efimov
Effect

Theoretical
Approach

Effective
Potentials

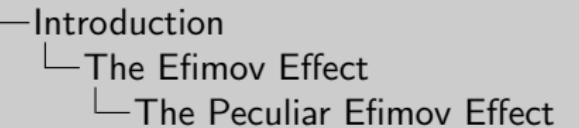
Numerical
Approach

Scattering
Model

Results

- The size of each Efimov state (ES) \gg the interaction range (r_0) between the individual particle pairs \rightarrow QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is **reduced** as the 2-b interaction is made more attractive

2019-09-03



- The size of each Efimov state (ES) \gg the interaction range (r_0) between the individual particle pairs \rightarrow QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is **reduced** as the 2-b interaction is made more attractive

1. # of 3-b bound states is *reduced* as the 2-b interaction is made more attractive.

The Peculiar Efimov Effect

Kajsa-My
Blomdahl

Introduction

Two-body
Interactions

Scattering
Length

Universality

The Efimov
Effect

Theoretical Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The size of each Efimov state (ES) \gg the interaction range (r_0) between the individual particle pairs \rightarrow QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is reduced as the 2-b interaction is made more attractive
- The effect is universal and can *in principle* be observed in any QM system

Introduction

The Efimov Effect

The Peculiar Efimov Effect

2019-09-03

- The size of each Efimov state (ES) \gg the interaction range (r_0) between the individual particle pairs \rightarrow QM effect
- When $a \rightarrow \pm\infty$ the # of ES $\rightarrow \infty$
- The # of ES is reduced as the 2-b interaction is made more attractive
- The effect is universal and can *in principle* be observed in any QM system

1. The effect is universal, which means that the states emerge irrespective of the nature of the 2-b forces and can in principle be observed in all quantum mechanical systems

Theoretical Approach

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1
Step 2
Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

2019-09-03 └ Theoretical Approach

└ Theoretical Approach

1. The 3BP is famous for being hard to solve

Theoretical Approach

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1
Step 2
Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

Q: 3-Particles, What Is The Problem?

2019-09-03

└ Theoretical Approach

└ Theoretical Approach

1. So why is the problem of 3 so complex?

Theoretical Approach

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1
Step 2
Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

Q: 3-Particles, What Is The Problem?

A: The configuration space (CS) for the 3BP is 9D
and highly non-trivial ...

2019-09-03

└ Theoretical Approach

└ Theoretical Approach

1. Well, the configuration space for the 3BP is 9D and highly non-trivial ...

Q: 3-Particles, What Is The Problem?
A: The configuration space (CS) for the 3BP is 9D
and highly non-trivial ...

Theoretical Approach

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1
Step 2
Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

Q: 3-Particles, What Is The Problem?

A: The configuration space (CS) for the 3BP is 9D
and highly non-trivial ...

Solution: Reduce the number of dimensions!

2019-09-03

└ Theoretical Approach

└ Theoretical Approach

Q: 3-Particles, What Is The Problem?
A: The configuration space (CS) for the 3BP is 9D
and highly non-trivial ...
Solution: Reduce the number of dimensions!

Step 1: Relative Coordinates

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- Separate out CoM by introducing relative coordinates

2019-09-03

Theoretical Approach
└ Step 1
 └ Step 1: Relative Coordinates

• Separate out CoM by introducing relative coordinates

1. The first step to reduce the number of D is to separate out the CoM by introducing relative coordinates.

Step 1: Relative Coordinates

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- Separate out CoM by introducing relative coordinates
- $CS \rightarrow 6D$

2019-09-03

Theoretical Approach
└ Step 1
 └ Step 1: Relative Coordinates

• Separate out CoM by introducing relative coordinates
• $CS \rightarrow 6D$

1. CoM motion decouples from the internal motion in the SE the configuration space is effectively reduced to 6D

Step 1: Relative Coordinates

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- Separate out CoM by introducing relative coordinates
- Choice: Mass-normalized Jacobi coordinates
- CS → 6D

2019-09-03

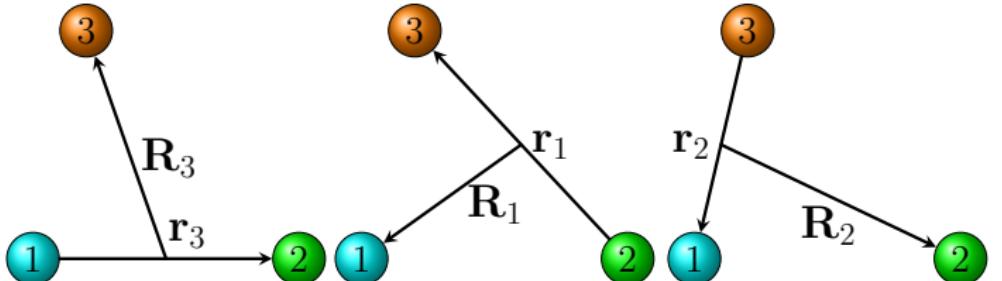
Theoretical Approach
└ Step 1
 └ Step 1: Relative Coordinates

- Separate out CoM by introducing relative coordinates
- Choice: Mass-normalized Jacobi coordinates
- CS → 6D

1. For internal coordinates we choose a Mass-Normalized Jacobi coordinates

Step 1: Relative Coordinates

- Separate out CoM by introducing relative coordinates
- Choice: Mass-normalized **Jacobi coordinates**
- $CS \rightarrow 6D$



9 / 22

2019-09-03

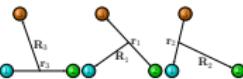
Theoretical Approach

Step 1

Step 1: Relative Coordinates

Step 1: Relative Coordinates

- Separate out CoM by introducing relative coordinates
- Choice: Mass-normalized **Jacobi coordinates**
- $CS \rightarrow 6D$



Step 2: Hyperspherical Coordinates

Kajsa-My
Blomdahl

Introduction
Theoretical Approach
Step 1
Step 2
Step 3

Effective Potentials

Numerical Approach

Scattering Model

Results

2019-09-03

Theoretical Approach
└ Step 2
 └ Step 2: Hyperspherical Coordinates

1. Step 2 in simplifying the problem of three particles is to introduce hyperspherical coordinates

Step 2: Hyperspherical Coordinates

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- Combine \mathbf{r}_k and \mathbf{R}_k into a 6D position vector in \mathbb{R}^6

2019-09-03

Theoretical Approach
└ Step 2
 └ Step 2: Hyperspherical Coordinates

Combine \mathbf{r}_k and \mathbf{R}_k into a 6D position vector in \mathbb{R}^6

1. The general idea is to combine the components of the two Jacobi vectors into a single six-dimensional position vector \mathbf{q} , which represents a point in \mathbb{R}^6 .

Step 2: Hyperspherical Coordinates

Kajsa-My
Blomdahl

Introduction
Theoretical Approach
Step 1
Step 2
Step 3

Effective Potentials

Numerical Approach

Scattering Model

Results

- Combine \mathbf{r}_k and \mathbf{R}_k into a 6D position vector in \mathbb{R}^6
- Hyperspherical coordinates: ρ and Ω

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

2019-09-03

Theoretical Approach
└ Step 2
 └ Step 2: Hyperspherical Coordinates

• Combine \mathbf{r}_k and \mathbf{R}_k into a 6D position vector in \mathbb{R}^6
• Hyperspherical coordinates: ρ and Ω

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

1. The hyperspherical coord. of this point are given by the hyperradius and five hyperangles Ω .
2. The hyperradial coordinate is both rotationally and permutationally invariant and is defined as the square root of the sum of the squared Jacobi vectors.
3. The hyperangles can be defined in many ways and I will not go in to the details here.

Step 2: Hyperspherical Coordinates

Kajsa-My
Blomdahl

Introduction

Theoretical Approach

Step 1
Step 2
Step 3

Effective Potentials

Numerical Approach

Scattering Model

Results

- Combine \mathbf{r}_k and \mathbf{R}_k into a 6D position vector in \mathbb{R}^6
- Hyperspherical coordinates: ρ and Ω

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

- Separate internal and external coordinates

2019-09-03

Theoretical Approach

Step 2

Step 2: Hyperspherical Coordinates

- Combine \mathbf{r}_k and \mathbf{R}_k into a 6D position vector in \mathbb{R}^6
- Hyperspherical coordinates: ρ and Ω

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$
- Separate internal and external coordinates

1. At any instant, three particles form a plane in R3. We can define the internal motion of the particles within this plane in terms of the hyperradial coordinate (size) and two of the angles(shape and particle permutation). The three other angles relate rotations of this plane in a space fixed system.
2. The three other angles relate rotations of this plane in a space fixed system.

Step 2: Hyperspherical Coordinates

Kajsa-My
Blomdahl

Introduction

Theoretical Approach

Step 1

Step 2

Step 3

Effective Potentials

Numerical Approach

Scattering Model

Results

- Combine \mathbf{r}_k and \mathbf{R}_k into a 6D position vector in \mathbb{R}^6
- Hyperspherical coordinates: ρ and Ω

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$

- Separate internal and external coordinates
- For $J = 0$ only **internal coordinates** matter → 3D Schrödinger equation (SE)

2019-09-03

Theoretical Approach

Step 2

Step 2: Hyperspherical Coordinates

- Combine \mathbf{r}_k and \mathbf{R}_k into a 6D position vector in \mathbb{R}^6
- Hyperspherical coordinates: ρ and Ω

$$\rho = (\mathbf{r}^2 + \mathbf{R}^2)^{1/2}$$
- Separate internal and external coordinates
- For $J = 0$ only **internal coordinates** matter → 3D Schrödinger equation (SE)

1. When the orbital angular momenta $J=0$ only the internal coordinates matter and we are left with a 3D SE for the internal motion

Step 3: The Adiabatic Representation

Kajsa-My
Blomdahl

Introduction
Theoretical Approach
Step 1
Step 2
Step 3
Effective Potentials
Numerical Approach
Scattering Model
Results

Numerical Approach Exact rep.

2019-09-03

Theoretical Approach
└ Step 3
 └ Step 3: The Adiabatic Representation

1. Now we move on to the final step (in the simplification of the 3BP)

Step 3: The Adiabatic Representation

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

Numerical Approach Exact rep.

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation

• The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

1. After a clever rescaling of the wfn, the hyperspherical SE can be written like this

Step 3: The Adiabatic Representation

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

Numerical Approach Exact rep.

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation

The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

1. Where lambda is the generalized angular momenta

Step 3: The Adiabatic Representation

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

- Trick: Treat ρ as an adiabatic parameter!

Numerical Approach Exact rep.

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation

The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

Trick: Treat ρ as an adiabatic parameter!

1. Now, the trick is to treat the hyperradius as an adiabatic parameter!
2. That is, we fix ρ in a Born-Oppenheimer like manner

Step 3: The Adiabatic Representation

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

- Trick: Treat ρ as an adiabatic parameter!
- Solve the adiabatic eq.

$$H_{ad}\Phi_\nu(\rho; \Omega) = U_\nu(\rho)\Phi_\nu(\rho; \Omega)$$

Numerical Approach

Exact rep.

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation

The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

Trick: Treat ρ as an adiabatic parameter!

Solve the adiabatic eq.

$$H_{ad}\Phi_\nu(\rho; \Omega) = U_\nu(\rho)\Phi_\nu(\rho; \Omega)$$

Step 3: The Adiabatic Representation

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

- Trick: Treat ρ as an adiabatic parameter!
- solve the adiabatic eq.

$$H_{ad}\Phi_\nu(\rho; \Omega) = U_\nu(\rho)\Phi_\nu(\rho; \Omega)$$

- → 3-body BO-like potential

Numerical Approach

Exact rep.

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation

The hyperspherical SE:

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial\rho^2} + \frac{\Lambda^2 + \frac{15}{4}}{2\mu\rho^2} + V \right) \psi = E\psi$$

Trick: Treat ρ as an adiabatic parameter!

solve the adiabatic eq.

→ 3-body BO-like potential

1. In this way we obtain the three-body equivalent of a BO potential.

Step 3: The Adiabatic Representation cont.

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

SE

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation cont.

Step 3: The Adiabatic Representation cont.

• The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

1. Now, in this way the total wfn can be represented by a sum of adiabatic states

Step 3: The Adiabatic Representation cont.

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The total wave function is represented in terms of **adiabatic states**

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

SE

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation cont.

Step 3: The Adiabatic Representation cont.

♦ The total wave function is represented in terms of **adiabatic states**

$$\psi_A(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{A\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

Step 3: The Adiabatic Representation cont.

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

SE

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation cont.

• The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

1. If we substitute this sum into the 3-b SE (klick on link)

Step 3: The Adiabatic Representation cont.

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial \rho^2} + U_{\mu} - \frac{1}{2\mu} Q_{\mu\mu} \right) F_{n\mu} - \frac{1}{2\mu} \left(\sum_{\nu \neq \mu} 2P_{\mu\nu} \frac{\partial}{\partial \rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

SE

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation cont.

• The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

• The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial \rho^2} + U_{\mu} - \frac{1}{2\mu} Q_{\mu\mu} \right) F_{n\mu} - \frac{1}{2\mu} \left(\sum_{\nu \neq \mu} 2P_{\mu\nu} \frac{\partial}{\partial \rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

Step 3: The Adiabatic Representation cont.

Kajsa-My
Blomdahl

Introduction

Theoretical Approach

Step 1

Step 2

Step 3

Effective Potentials

Numerical Approach

Scattering Model

Results

- The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial \rho^2} + \textcolor{red}{U_{\mu}} - \frac{1}{2\mu} Q_{\mu\mu} \right) F_{n\mu} - \frac{1}{2\mu} \left(\sum_{\nu \neq \mu} 2P_{\mu\nu} \frac{\partial}{\partial \rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

SE

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation cont.

2019-09-03

• The total wave function is represented in terms of adiabatic states

$$\psi_n(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_{\nu}(\rho; \Omega)$$

• The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial \rho^2} + \textcolor{red}{U_{\mu}} - \frac{1}{2\mu} Q_{\mu\mu} \right) F_{n\mu} - \frac{1}{2\mu} \left(\sum_{\nu \neq \mu} 2P_{\mu\nu} \frac{\partial}{\partial \rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

Step 3: The Adiabatic Representation cont.

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Step 1

Step 2

Step 3

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

- Three-body effective potentials

$$W_\nu(\rho) = U_\nu(\rho) - \frac{1}{2\mu} Q_{\nu\nu}(\rho) = U_\nu(\rho) - \frac{1}{2\mu} P_{\nu\nu}^2(\rho)$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial \rho^2} + \textcolor{red}{U_\mu} - \frac{1}{2\mu} \textcolor{red}{Q_{\mu\mu}} \right) F_{n\mu} - \frac{1}{2\mu} \left(\sum_{\nu \neq \mu} 2P_{\mu\nu} \frac{\partial}{\partial \rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

SE

2019-09-03

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation cont.

• The total wave function is represented in terms of adiabatic states

$$\psi_A(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_\nu(\rho; \Omega)$$

• The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial \rho^2} + \textcolor{red}{U_\mu} - \frac{1}{2\mu} \textcolor{red}{Q_{\mu\mu}} \right) F_{n\mu} - \frac{1}{2\mu} \left(\sum_{\nu \neq \mu} 2P_{\mu\nu} \frac{\partial}{\partial \rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

Step 3: The Adiabatic Representation cont.

Kajsa-My
Blomdahl

Introduction

Theoretical Approach

Step 1

Step 2

Step 3

Effective Potentials

Numerical Approach

Scattering Model

Results

- Three-body effective potentials

$$W_\nu(\rho) = U_\nu(\rho) - \frac{1}{2\mu} Q_{\nu\nu}(\rho) \approx U_\nu(\rho) - \frac{1}{2\mu} \cancel{P_{\nu\nu}^2(\rho)}$$

- The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial \rho^2} + \cancel{U_\mu} - \frac{1}{2\mu} \cancel{Q_{\mu\mu}} \right) F_{n\mu} - \frac{1}{2\mu} \left(\sum_{\nu \neq \mu} 2P_{\mu\nu} \frac{\partial}{\partial \rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\mu}$$

SE

Theoretical Approach

Step 3

Step 3: The Adiabatic Representation cont.

2019-09-03

• The total wave function is represented in terms of adiabatic states

$$\psi_A(\rho, \Omega) = \sum_{\nu=0}^{\infty} F_{n\nu}(\rho) \Phi_\nu(\rho; \Omega)$$

• The hyperradial eigenvalue equation

$$\left(-\frac{1}{2\mu} \frac{\partial^2}{\partial \rho^2} + \cancel{U_\nu} - \frac{1}{2\mu} \cancel{Q_{\nu\nu}} \right) F_{n\nu} - \frac{1}{2\mu} \left(\sum_{\mu \neq \nu} 2P_{\mu\nu} \frac{\partial}{\partial \rho} + Q_{\mu\nu} \right) F_{n\nu} = E_n F_{n\nu}$$

Convergence In the Asymptotic Limit

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

The Asymptotic
Limit

Intermediate
Region

Analytical
Model

Numerical
Approach

Scattering
Model

Results

2019-09-03

└ Effective Potentials
 └ The Asymptotic Limit
 └ Convergence In the Asymptotic Limit

1. (The following discussion concerns short-ranged two-body interactions, where $|a| \gg r_0$)
2. The behaviour of the 3-b potentials in the asymptotic limit,
(i.e., when the hyperradius is much larger than the magnitude a) depend on the sign of a.

$$\boxed{\text{For } a < 0: \quad W_\nu(\rho) \xrightarrow{\rho \rightarrow \infty} \frac{\lambda(\lambda + 4) + \frac{15}{4}}{2\mu\rho^2}}$$

Convergence In the Asymptotic Limit

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

The Asymptotic
Limit

Intermediate
Region

Analytical
Model

Numerical
Approach

Scattering
Model

Results

For $a < 0$:

$$W_\nu(\rho) \xrightarrow{\rho \rightarrow \infty} \frac{\lambda(\lambda + 4) + \frac{15}{4}}{2\mu\rho^2}$$

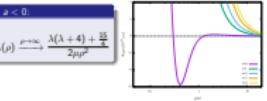
2019-09-03

Effective Potentials

The Asymptotic Limit

Convergence In the Asymptotic Limit

- When a has a negative sign there is no weakly bound dimer and the lowest effective potential will converge to the three-body continuum channels, i.e., the kinetic energy for three free particles.



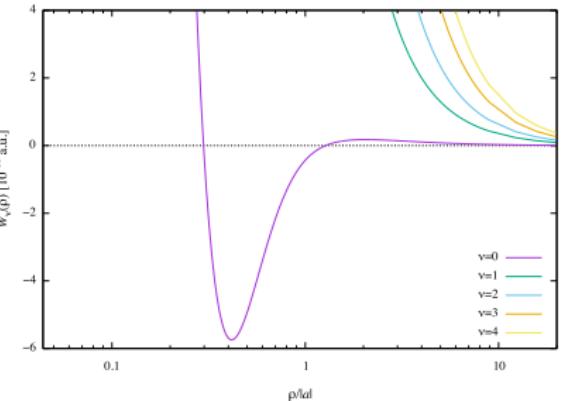
Convergence In the Asymptotic Limit

Kajsa-My
Blomdahl

Introduction
Theoretical Approach
Effective Potentials
The Asymptotic Limit
Intermediate Region
Analytical Model
Numerical Approach
Scattering Model
Results

For $a < 0$:

$$W_\nu(\rho) \xrightarrow{\rho \rightarrow \infty} \frac{\lambda(\lambda + 4) + \frac{15}{4}}{2\mu\rho^2}$$



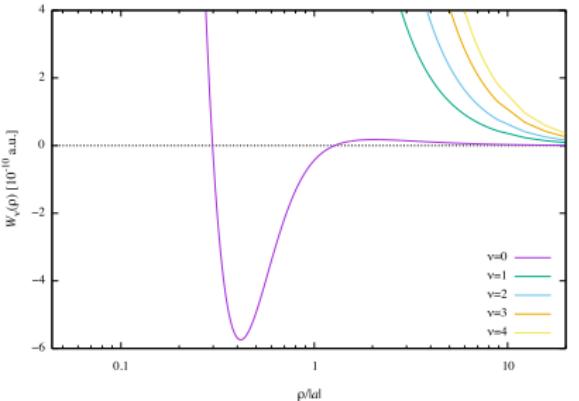
2019-09-03

Effective Potentials
The Asymptotic Limit
Convergence In the Asymptotic Limit

Convergence In the Asymptotic Limit

For $a < 0$:

$$W_\nu(\rho) \xrightarrow{\rho \rightarrow \infty} \frac{\lambda(\lambda + 4) + \frac{15}{4}}{2\mu\rho^2}$$



For $a > 0$:

$$W_\nu(\rho) \xrightarrow{\rho \rightarrow \infty} E_{2b} + \frac{l(l+1)}{2\mu\rho^2}$$

2019-09-03

Effective Potentials

The Asymptotic Limit

Convergence In the Asymptotic Limit

For $a < 0$:

$$W_\nu(\rho) \xrightarrow{\rho \rightarrow \infty} \frac{\lambda(\lambda + 4) + \frac{15}{4}}{2\mu\rho^2}$$

For $a > 0$:

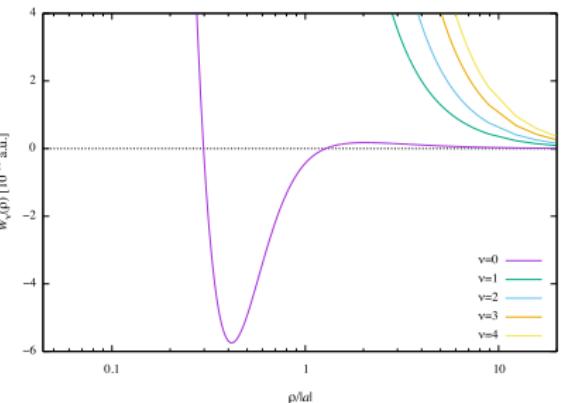
$$W_\nu(\rho) \xrightarrow{\rho \rightarrow \infty} E_{2b} + \frac{l(l+1)}{2\mu\rho^2}$$

1. However, for systems of 3 identical bosons with a pair-wise attraction that is strong enough to support 2-body bound states, one 3-body effective potential curve will converge asymptotically to each two-body bound state.

Convergence In the Asymptotic Limit

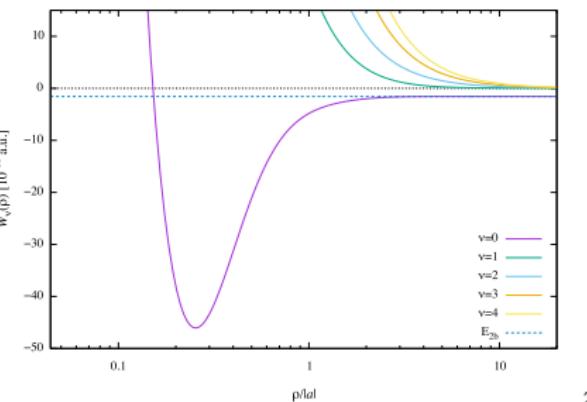
For $a < 0$:

$$W_\nu(\rho) \xrightarrow{\rho \rightarrow \infty} \frac{\lambda(\lambda + 4) + \frac{15}{4}}{2\mu\rho^2}$$



For $a > 0$:

$$W_\nu(\rho) \xrightarrow{\rho \rightarrow \infty} E_{2b} + \frac{l(l+1)}{2\mu\rho^2}$$

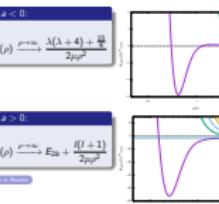


2019-09-03

Effective Potentials

The Asymptotic Limit

Convergence In the Asymptotic Limit



Convergence In the Intermediate Region

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

The Asymptotic
Limit

Intermediate
Region

Analytical
Model

Numerical
Approach

Scattering
Model

Results

Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

Jump to Results

2019-09-03

└ Effective Potentials
 └ Intermediate Region
 └ Convergence In the Intermediate Region

Intermediate Interaction Range
 $r_0 \ll \rho \ll |a|$

Jump to Results

1. Efimov physics comes into play in the intermediate region.
2. In this region the three-body effective potentials are modified by the Efimov physics. It can lead to both attractive and repulsive effective potentials.

Convergence In the Intermediate Region

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

The Asymptotic
Limit

Intermediate
Region

Analytical
Model

Numerical
Approach

Scattering
Model

Results

Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

Jump to Results

2019-09-03

Effective Potentials

Intermediate Region

Convergence In the Intermediate Region

Convergence In the Intermediate Region

Intermediate Interaction Range

$r_0 \ll \rho \ll |a|$

The Lowest Potential's Convergent Form (3 Identical Bosons)

$s_0^2 + \frac{1}{4}$

$2\mu\rho^2$

Jump to Results

1. For 3 identical bosons the effective potential will be attractive and is responsible for the Efimov Effect!

Convergence In the Intermediate Region

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

The Asymptotic
Limit

Intermediate
Region

Analytical
Model

Numerical
Approach

Scattering
Model

Results

Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

Jump to Results

2019-09-03

Effective Potentials
└ Intermediate Region
 └ Convergence In the Intermediate Region

Convergence In the Intermediate Region

Intermediate Interaction Range
 $r_0 \ll \rho \ll |a|$
The Lowest Potential's Convergent Form (3 Identical Bosons)
 $W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$

Jump to Results

1. And is responsible for the Efimov Effect!

Convergence In the Intermediate Region

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

The Asymptotic
Limit

Intermediate
Region

Analytical
Model

Numerical
Approach

Scattering
Model

Results

Intermediate Interaction Range

$$r_0 \ll \rho \ll |a|$$

The Lowest Potential's Convergent Form (3 Identical Bosons)

$$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$$

Universal Constant (3 Identical Bosons)

$$s_0 \simeq 1.00624$$

Jump to Results

2019-09-03

Effective Potentials

Intermediate Region

Convergence In the Intermediate Region

Convergence In the Intermediate Region

Intermediate Interaction Range

$r_0 \ll \rho \ll |a|$

The Lowest Potential's Convergent Form (3 Identical Bosons)

$W_\nu(\rho) = -\frac{s_0^2 + \frac{1}{4}}{2\mu\rho^2}$

Universal Constant (3 Identical Bosons)

$s_0 \simeq 1.00624$

Jump to Results

Analytic Model

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

The Asymptotic
Limit

Intermediate
Region

Analytical
Model

Numerical
Approach

Scattering
Model

Results

Jump to Results

2019-09-03

└ Effective Potentials
 └ Analytical Model
 └ Analytic Model

1. We can obtain a similar result analytically by instead of solving the SE we solve the coupled Faddeev equations

- The adiabatic potentials ν_n can be determined analytically through the transcendental equation

$$\sqrt{\nu_n} \cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}} \sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a} \sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Analytic Model

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

The Asymptotic
Limit

Intermediate
Region

**Analytical
Model**

Numerical
Approach

Scattering
Model

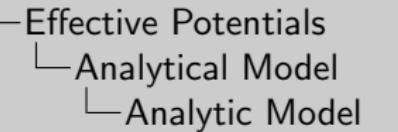
Results

- The adiabatic potentials ν_n can be determined analytically through the transcendental equation

$$\sqrt{\nu_n} \cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}} \sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a} \sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Jump to Results

2019-09-03



- We then obtain the adiabatic potential ν through the transcendental eq.

Analytic Model

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

The Asymptotic
Limit

Intermediate
Region

Analytical
Model

Numerical
Approach

Scattering
Model

Results

- The adiabatic potentials ν_n can be determined analytically through the transcendental equation

$$\sqrt{\nu_n} \cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}} \sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2} \frac{\rho}{a} \sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

[Jump to Results](#)

2019-09-03

Effective Potentials
└ Analytical Model
 └ Analytic Model

1. This adiabatic potential is a function of ρ/a

• The adiabatic potentials ν_n can be determined analytically through the transcendental equation

$$\sqrt{\nu_n} \cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}} \sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2} \frac{\rho}{a} \sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

Next

- The adiabatic potentials ν_n can be determined analytically through the transcendental equation

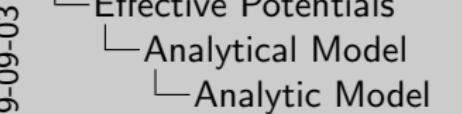
$$\sqrt{\nu_n} \cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}} \sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a} \sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

- Three-body effective potential

$$W_\nu(\rho/a) = \frac{(\nu_n(\rho/a) - \frac{1}{4})}{2\mu\rho^2}$$

[View Details](#)

Analytic Model



2019-09-03

- The adiabatic potentials ν_n can be determined analytically through the transcendental equation

$$\sqrt{\nu_n} \cos\left(\sqrt{\nu_n}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}} \sin\left(\sqrt{\nu_n}\frac{\pi}{6}\right) = \sqrt{2}\frac{\rho}{a} \sin\left(\sqrt{\nu_n}\frac{\pi}{2}\right)$$

- Three-body effective potential

$$W_\nu(\rho/a) = \frac{(\nu_n(\rho/a) - \frac{1}{4})}{2\mu\rho^2}$$

[Jump to Results](#)

Task = ?

Task = Find $W_\nu(\rho)!$

2019-09-03

└ Numerical Approach

 └ Numerical Approach; B-spline Collocation

1. Well, a good start is to find the 3-b effective potentials

Numerical Approach; B-spline Collocation

Numerical Approach; B-spline Collocation

Kajsa-My
Blomdahl

Introduction
Theoretical
Approach
Effective
Potentials
**Numerical
Approach**
Scattering
Model
Results

First: Choose a basis

2019-09-03

└ Numerical Approach

└ Numerical Approach; B-spline Collocation

1. To do this I have chosen to work with a B-spline basis for the two angular coordinates

First: Choose a basis

Numerical Approach; B-spline Collocation

Kajsa-My
Blomdahl

Introduction
Theoretical
Approach
Effective
Potentials
**Numerical
Approach**
Scattering
Model
Results

First: Choose a basis

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$

2019-09-03

└ Numerical Approach

└ Numerical Approach; B-spline Collocation

First: Choose a basis
• $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$

Numerical Approach; B-spline Collocation

Kajsa-My
Blomdahl

Introduction
Theoretical
Approach
Effective
Potentials
**Numerical
Approach**
Scattering
Model
Results

First: Choose a basis

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$

Then: Expand

2019-09-03

└ Numerical Approach

└ Numerical Approach; B-spline Collocation

First: Choose a basis
Then: Expand

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$

1. And put it into the adiabatic eq.

Numerical Approach; B-spline Collocation

Kajsa-My
Blomdahl

Introduction
Theoretical
Approach
Effective
Potentials
Numerical
Approach
Scattering
Model
Results

First: Choose a basis

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$

Then: Expand

- $\Phi_\nu(\rho; \theta, \phi) = \sum_{l,m}^{L,M} c_{lm} \varphi_{Im}$

2019-09-03

└ Numerical Approach

└ Numerical Approach; B-spline Collocation

First: Choose a basis
Then: Expand

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho; \theta, \phi) = \sum_{l,m}^{L,M} c_{lm} \varphi_{Im}$

1. And finally solve the Generalized eigenvalue eq.

Numerical Approach; B-spline Collocation

Kajsa-My
Blomdahl

Introduction
Theoretical
Approach
Effective
Potentials
Numerical
Approach
Scattering
Model
Results

First: Choose a basis

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$

Then: Expand

- $\Phi_\nu(\rho; \theta, \phi) = \sum_{I,M}^L c_{Im} \varphi_{Im}$

Next: Substitute Φ_ν
into the

Adiabatic Eq.

2019-09-03

Numerical Approach

Numerical Approach; B-spline Collocation

First: Choose a basis
Then: Expand
Next: Substitute Φ_ν into the Adiabatic Eq.

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(\rho; \theta, \phi) = \sum_{I,M}^L c_{Im} \varphi_{Im}$

Numerical Approach; B-spline Collocation

Kajsa-My
Blomdahl

Introduction
Theoretical
Approach
Effective
Potentials
Numerical
Approach
Scattering
Model
Results

First: Choose a basis

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$

Then: Expand

- $\Phi_\nu(\rho; \theta, \phi) = \sum_{I,m}^{L,M} c_{Im} \varphi_{Im}$

Next: Substitute Φ_ν
into the

Adiabatic Eq.

2019-09-03

Numerical Approach

Numerical Approach; B-spline Collocation

- First: Choose a basis
- Then: Expand
- Next: Substitute Φ_ν into the Adiabatic Eq.
- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$
- $\Phi_\nu(x; \theta, \phi) = \sum_{I,m}^{L,M} c_{Im} \varphi_{Im}$
- $\mathbf{H}_{ad}\mathbf{c} = \mathbf{UBc}$

Numerical Approach; B-spline Collocation

Kajsa-My
Blomdahl

Introduction
Theoretical
Approach
Effective
Potentials
Numerical
Approach
Scattering
Model
Results

First: Choose a basis

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$

Then: Expand

- $\Phi_\nu(\rho; \theta, \phi) = \sum_{I,m}^{L,M} c_{Im} \varphi_{Im}$

Next: Substitute Φ_ν
into the

Adiabatic Eq.

Finally: Solve the
Generalized
Eigenvalue Eq.

2019-09-03

Numerical Approach

Numerical Approach; B-spline Collocation

- First: Choose a basis
 - Then: Expand
 - Next: Substitute Φ_ν into the Adiabatic Eq.
 - Finally: Solve the Generalized Eigenvalue Eq.
- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$
 - $\Phi_\nu(\rho; \theta, \phi) = \sum_{I,m}^{L,M} c_{Im} \varphi_{Im}$
 - $\mathbf{H}_{ad}\mathbf{c} = U\mathbf{B}\mathbf{c}$

Numerical Approach; B-spline Collocation

Kajsa-My
Blomdahl

Introduction
Theoretical
Approach
Effective
Potentials
Numerical
Approach
Scattering
Model
Results

First: Choose a basis

- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$

Then: Expand

- $\Phi_\nu(\rho; \theta, \phi) = \sum_{I,m}^{L,M} c_{Im} \varphi_{Im}$

Next: Substitute Φ_ν
into the

Adiabatic Eq.

- $\mathbf{H}_{ad}\mathbf{c} = \mathbf{U}\mathbf{B}\mathbf{c}$

Finally: Solve the
Generalized
Eigenvalue Eq.

- $W(\rho) \approx U(\rho)$

2019-09-03

Numerical Approach

Numerical Approach; B-spline Collocation

- First: Choose a basis
 - Then: Expand
 - Next: Substitute Φ_ν into the Adiabatic Eq.
 - Finally: Solve the Generalized Eigenvalue Eq.
- $\varphi_{Im} = \varphi_{1I}(\theta)\varphi_{2m}(\phi)$
 - $\Phi_\nu(x; \theta, \phi) = \sum_{I,m}^{L,M} c_{Im} \varphi_{Im}$
 - $\mathbf{H}_{ad}\mathbf{c} = \mathbf{U}\mathbf{B}\mathbf{c}$
 - $W(\rho) \approx U(\rho)$

Scattering Model

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

Masses

$$m = m(\text{Rubidium} - 87)$$

2019-09-03

└ Scattering Model

└ Scattering Model

Masses
 $m = m(\text{Rubidium} - 87)$

1. For the scattering model I have used masses corresponding to Rb-87 (boson)(Z=37, N=50)

Scattering Model

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

Masses

$$m = m(\text{Rubidium} - 87)$$

Assumption

$$\begin{aligned} V(\rho, \theta, \psi) = \\ v(r_{12}) + v(r_{23}) + v(r_{31}) \end{aligned}$$

2019-09-03

Scattering Model

Scattering Model

Masses

$m = m(\text{Rubidium} - 87)$

Assumption

$V(\rho, \theta, \psi) =$

$v(r_{12}) + v(r_{23}) + v(r_{31})$

Scattering Model

Masses

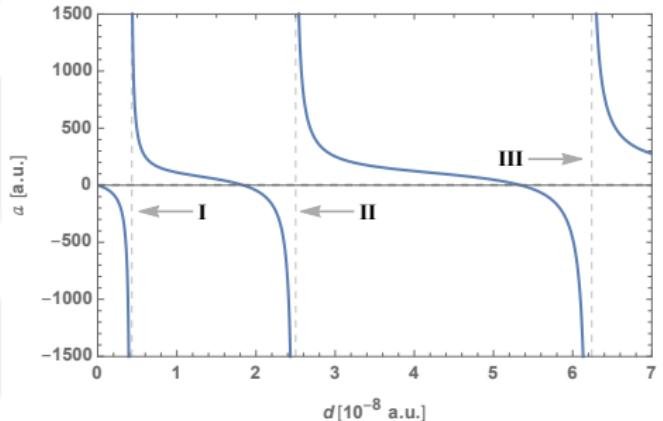
$$m = m(\text{Rubidium} - 87)$$

Assumption

$$V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31})$$

2B Model Potential

$$v(r) = d \cosh^{-2}(r/r_0)$$



2019-09-03

Scattering Model

Scattering Model

Masses
 $m = m(\text{Rubidium} - 87)$

Assumption
 $V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31})$

2B Model Potential
 $v(r) = d \cosh^{-2}(r/r_0)$

1. I have used the following model potential because I can calculate the scattering length from it

Scattering Model

Masses

$$m = m(\text{Rubidium} - 87)$$

Assumption

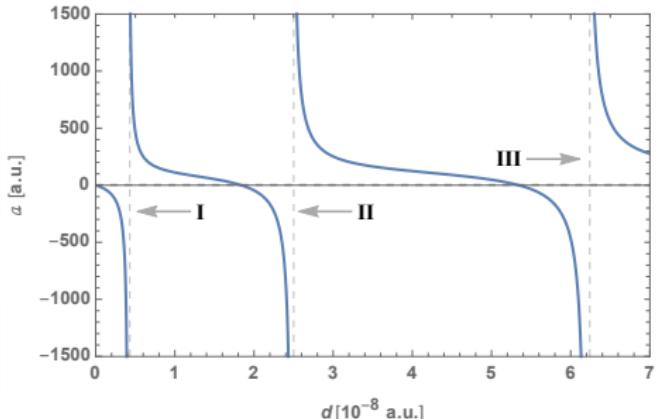
$$V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31})$$

2B Model Potential

$$v(r) = d \cosh^{-2}(r/r_0)$$

Interaction Range

$$r_0 = 55 \text{ a.u.}$$



2019-09-03

Scattering Model

Scattering Model

Masses
 $m = m(\text{Rubidium} - 87)$

Assumption
 $V(\rho, \theta, \psi) = v(r_{12}) + v(r_{23}) + v(r_{31})$

2B Model Potential
 $v(r) = d \cosh^{-2}(r/r_0)$

Interaction Range
 $r_0 = 55 \text{ a.u.}$

1. And set the interaction range to 55 a.u.

Convergence and Accuracy

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

Convergence
and Accuracy

Comparison to
the Analytical
Model

- For $a \rightarrow \pm\infty$ we expect convergence towards the Efimovian form

To Figures

2019-09-03

Results

Convergence and Accuracy

Convergence and Accuracy

And now to the results! For $a \rightarrow \pm\infty$ we expect that the lowest effective potential curve will converge towards the Efimovian form

For $a \rightarrow \pm\infty$ we expect convergence towards

Convergence and Accuracy

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

Convergence
and Accuracy

Comparison to
the Analytical
Model

- For $a \rightarrow \pm\infty$ we expect convergence towards the Efimovian form
- Easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as

$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4}$$

To Figures

2019-09-03

Results

- Convergence and Accuracy
- Convergence and Accuracy

- For $a \rightarrow \pm\infty$ we expect convergence towards the Efimovian form
- Easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as

$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4}$$

This behaviour is easier to recognize if the potentials are multiplied by this factor $2\mu\rho^2$ and plot them as Since these curves should approach the universal value $-s_0^2 (\simeq -1.0125)$ in the intermediate region

Convergence and Accuracy

Kajsa-My
Blomdahl

Introduction

Theoretical
Approach

Effective
Potentials

Numerical
Approach

Scattering
Model

Results

Convergence
and Accuracy

Comparison to
the Analytical
Model

- For $a \rightarrow \pm\infty$ we expect convergence towards the Efimovian form
- Easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as

$$\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4}$$

- Should approach the universal value $-s_0^2 (\simeq -1.0125)$ in the intermediate region

To Figures

2019-09-03

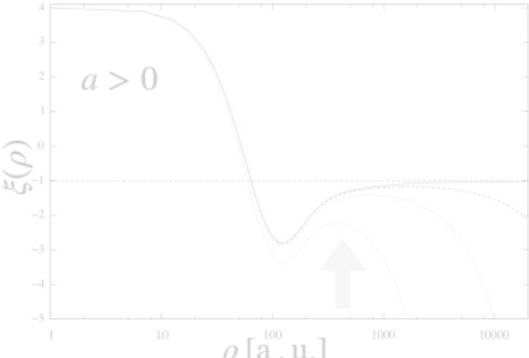
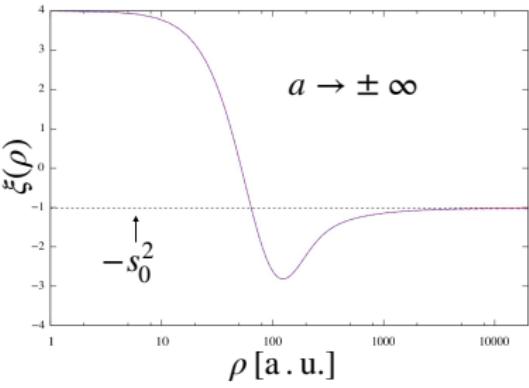
Results

Convergence and Accuracy

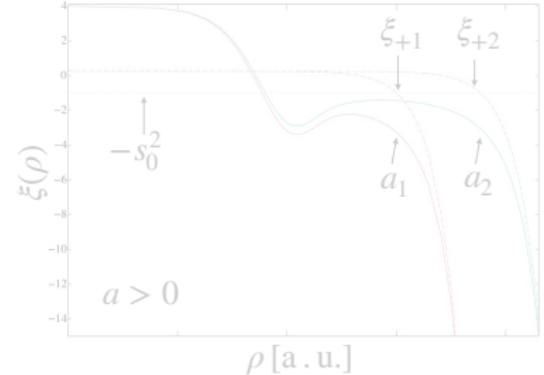
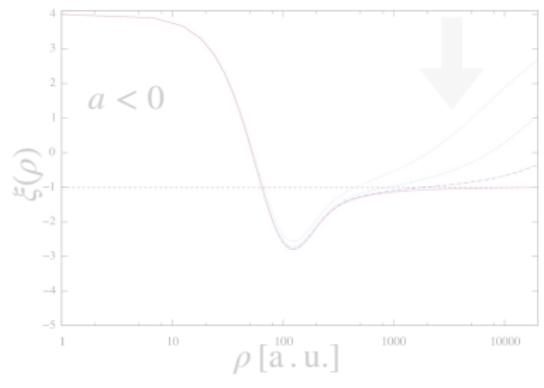
Convergence and Accuracy

- For $a \rightarrow \pm\infty$ we expect convergence towards the Efimovian form
- Easier to recognize if the potentials are multiplied by $2\mu\rho^2$ and plotted as
- $\xi(\rho) = 2\mu\rho^2 W_\nu(\rho) + \frac{1}{4}$
- Should approach the universal value $-s_0^2 (\simeq -1.0125)$ in the intermediate region

Efimov-like Potentials $\xi(\rho)$ for Different a



Effective Potentials

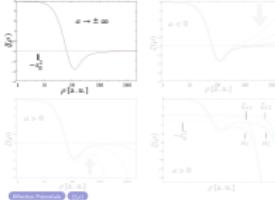
 $\xi(\rho)$ 

Results

Convergence and Accuracy

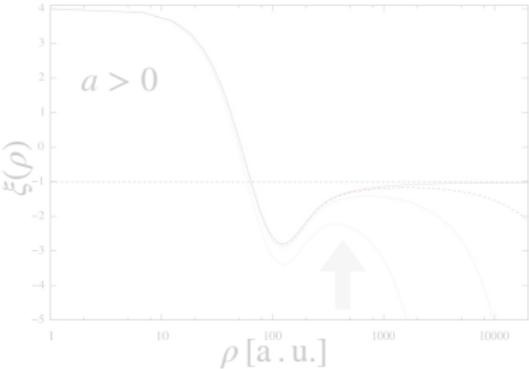
Efimov-like Potentials $\xi(\rho)$ for Different a

2019-09-03

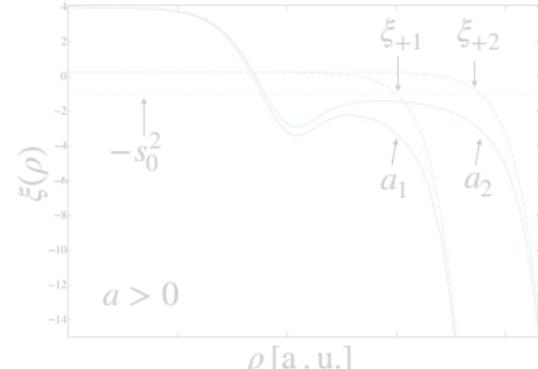
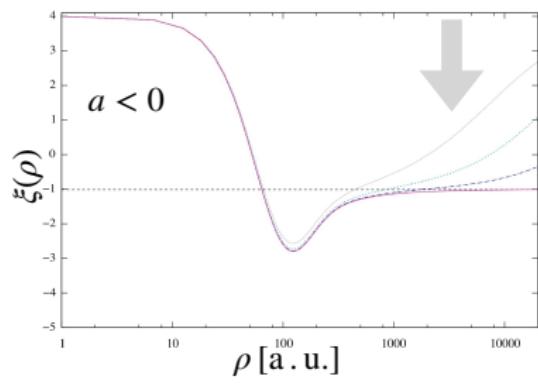
Efimov-like Potentials $\xi(\rho)$ for Different a 

- When the scattering length diverge we see that the potential curve ξ converge to the expected value at large hyperradii (note the log scale on the ρ -axis)
- For two-body potentials where $|a|$ is large but finite we expect that the effective potentials are to some extent affected by Efimov physics in the intermediate range ($r_0 \ll \rho \ll |a|$) and that the lowest effective potentials obtained with a larger magnitude of a exhibit closer resemblance with the true Efimov potential

Efimov-like Potentials $\xi(\rho)$ for Different a



Effective Potentials

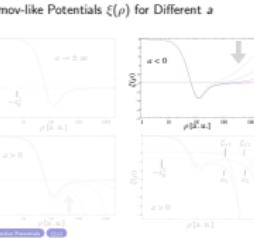
 $\xi(\rho)$ 

Results

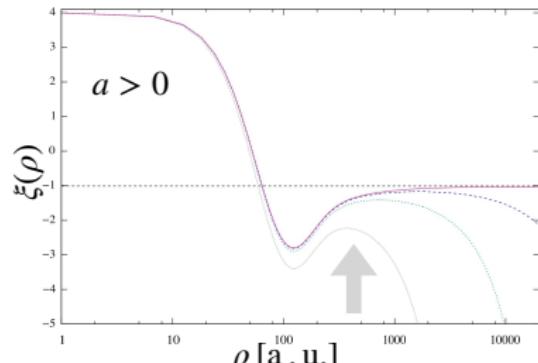
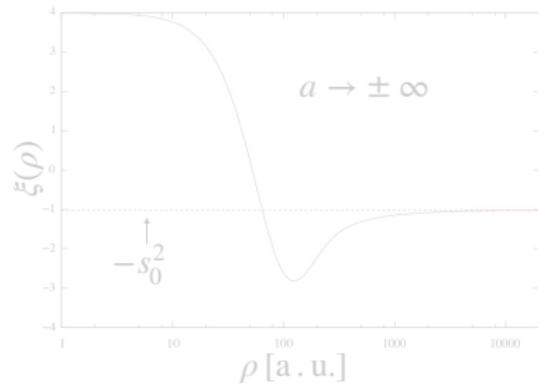
Convergence and Accuracy

Efimov-like Potentials $\xi(\rho)$ for Different a

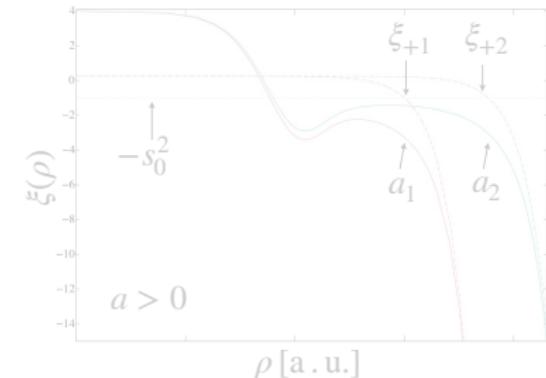
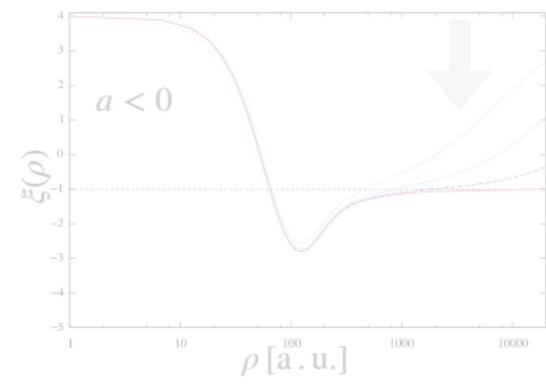
1. This is indeed what we see for negative a . Here I have plotted 4 curves with $|a|$ ranging from 2000 to 3 M a.u. (the curves start to converge to the kinetic energy in the asymptotic limit)



Efimov-like Potentials $\xi(\rho)$ for Different a



Effective Potentials

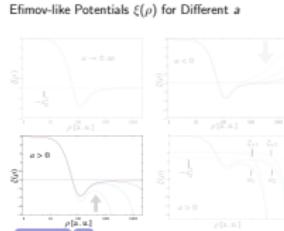
 $\xi(\rho)$ 

2019-09-03

Results

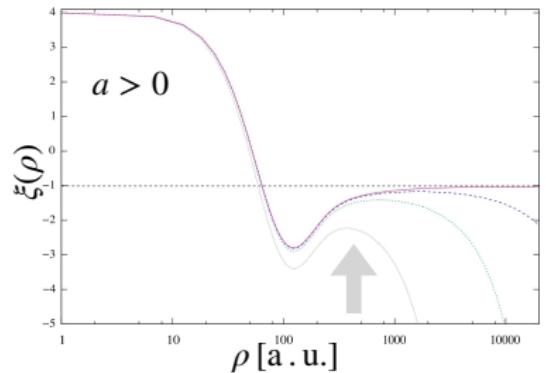
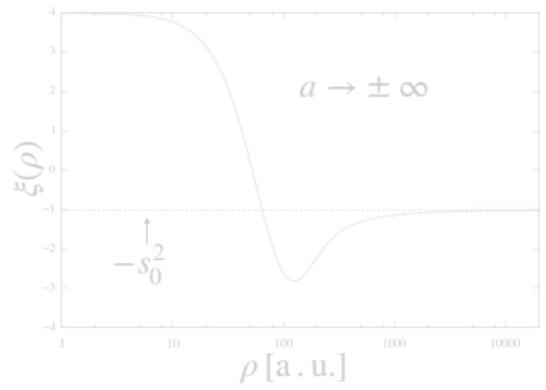
Convergence and Accuracy

Efimov-like Potentials $\xi(\rho)$ for Different a

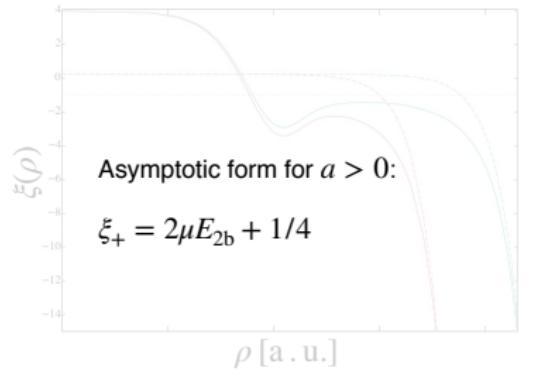
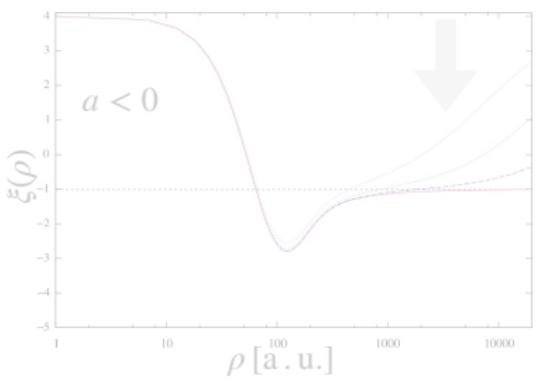


1. This is also the case for large positive a . We see a flattening behaviour of the curves from below which tend to get closer to the universal value as a increase. However, here the curves will start to converge to the energy of the two body bound state. Here I have plotted 4 curves with $|a|$ ranging from 2000 to 3 M a.u. (the curves start to converge to the kinetic energy in the asymptotic limit)

Efimov-like Potentials $\xi(\rho)$ for Different a



Effective Potentials

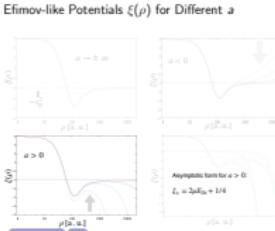
 $\xi(\rho)$ 

Results

Convergence and Accuracy

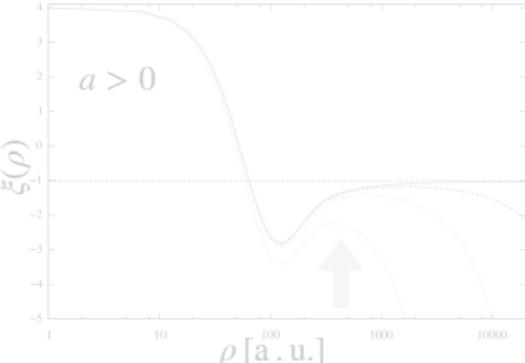
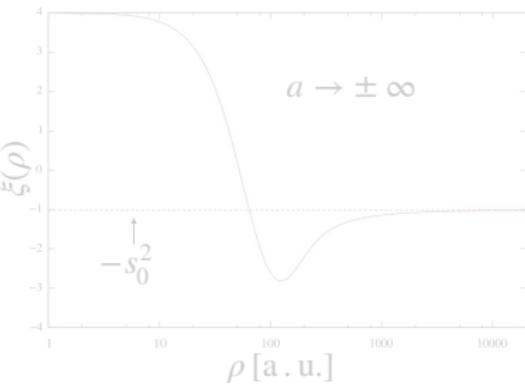
Efimov-like Potentials $\xi(\rho)$ for Different a

2019-09-03

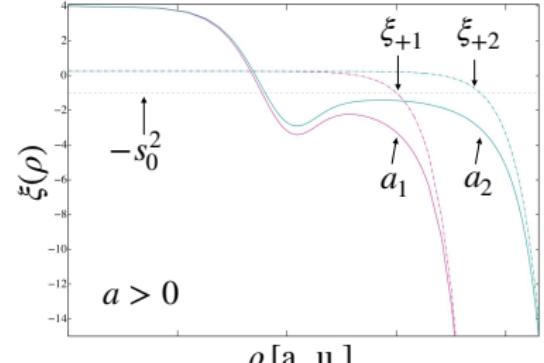
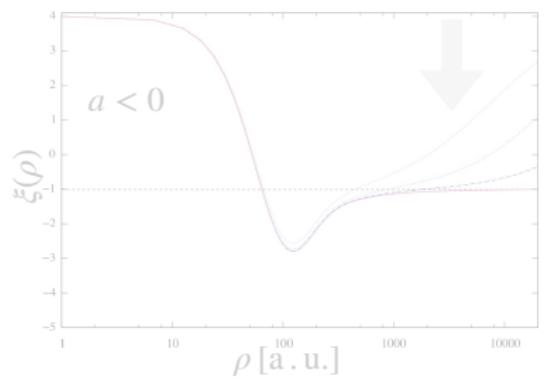


1. However, in this case the curves will start to converge to the energy of the two body bound state. Which here corresponds to

Efimov-like Potentials $\xi(\rho)$ for Different a



Effective Potentials

 $\xi(\rho)$ 

Effective Potentials

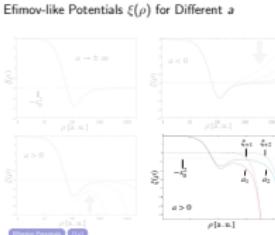
 $\xi(\rho)$

Results

Convergence and Accuracy

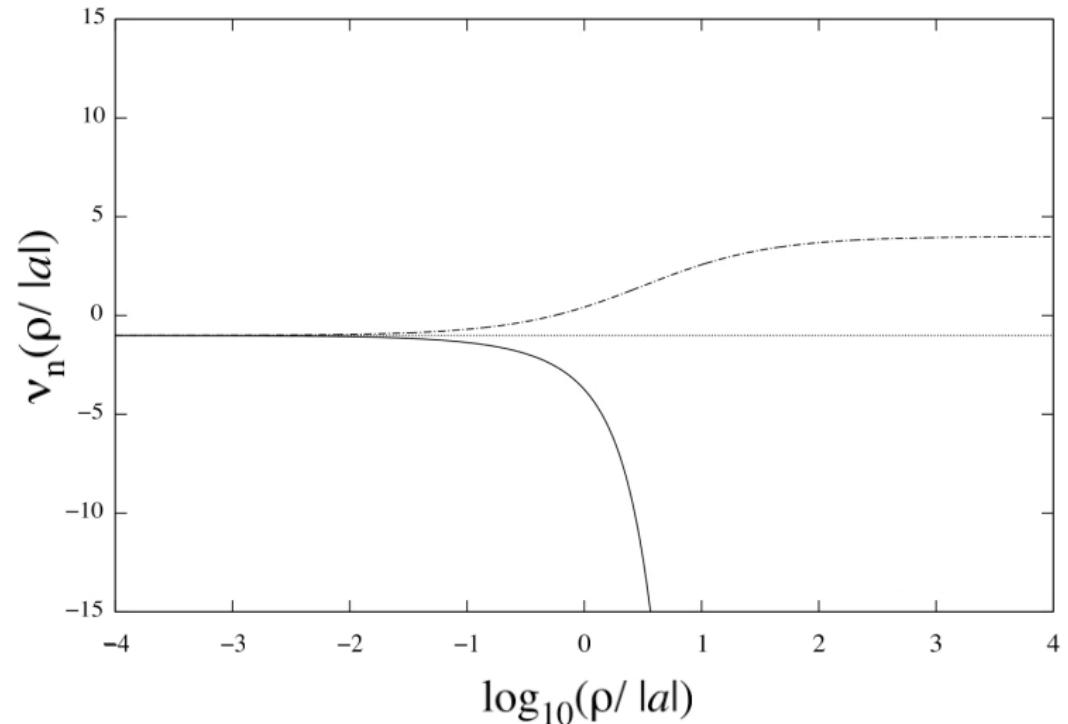
Efimov-like Potentials $\xi(\rho)$ for Different a

2019-09-03



1. Here I have plotted the two curves with lowest a together with the curves ξ corresponding to the 2-b energies and we observe that the curves indeed converge to the 2-b energy in the asymptotic limit.

Comparison to the Analytical Model (1)



◀ Analytic Potential

▶ $a < 0$

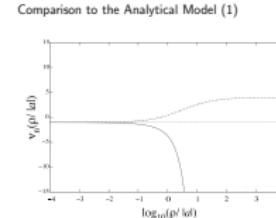
▶ $a > 0$

2019-09-03

Results

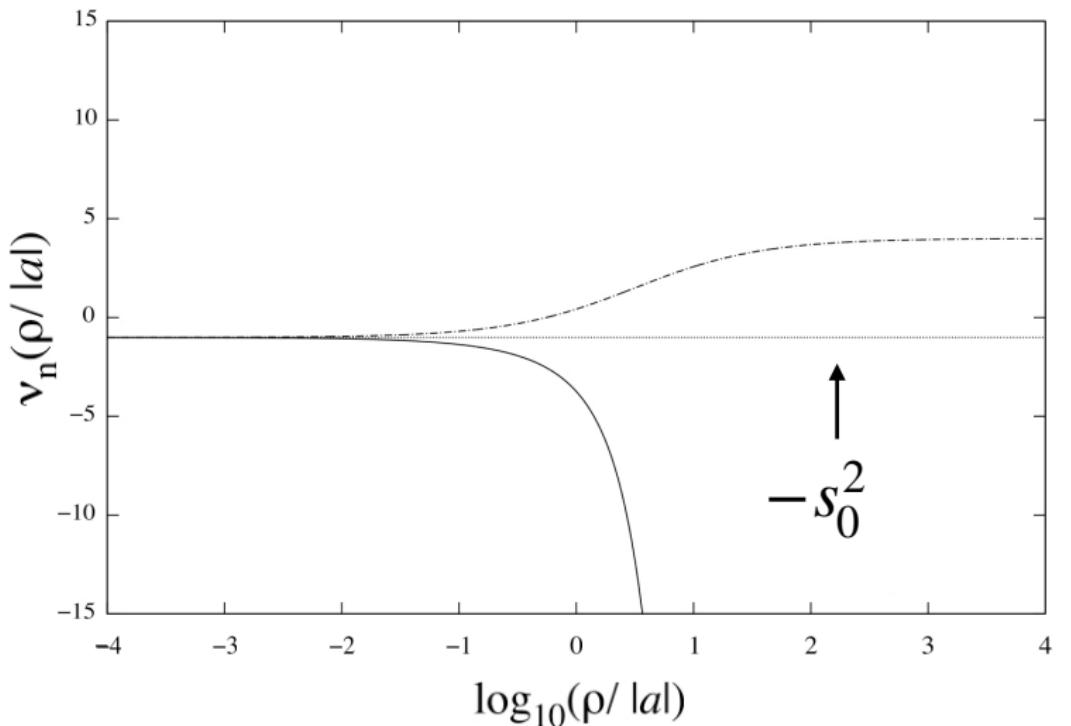
Comparison to the Analytical Model

Comparison to the Analytical Model (1)



1. With this figure I want to show the equivalent lowest potentials for positive and negative a calculated from the transcendental Faddeev eq.

Comparison to the Analytical Model (1)

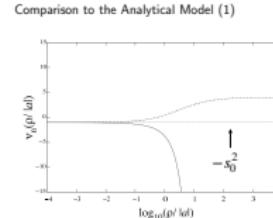


2019-09-03

Results

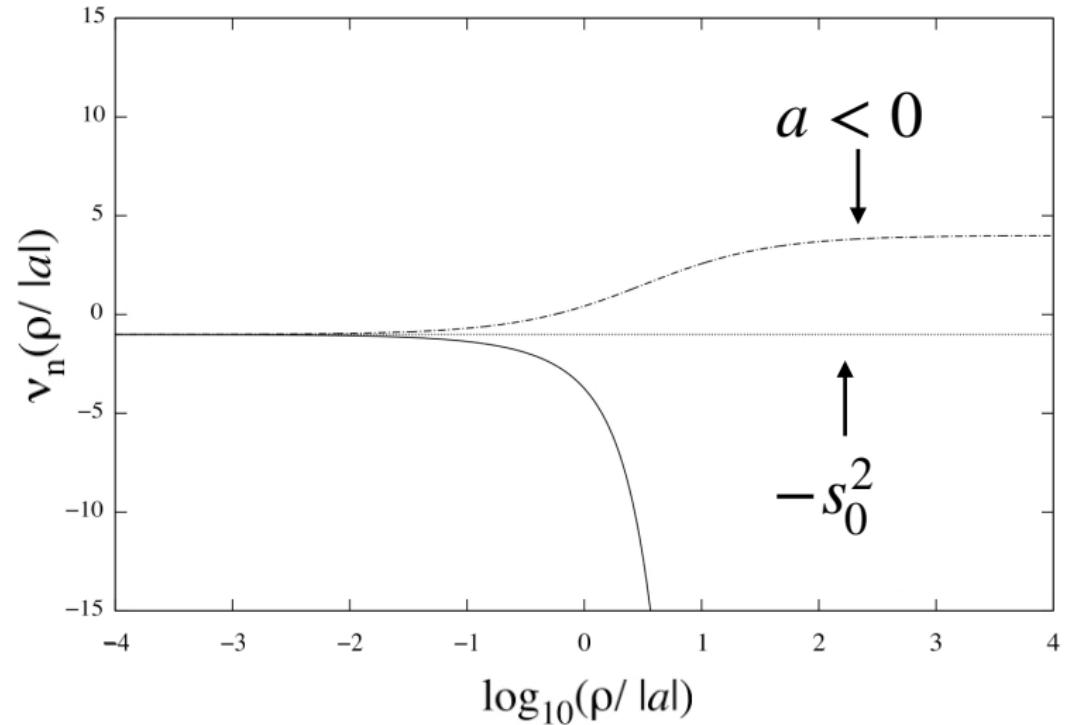
Comparison to the Analytical Model

Comparison to the Analytical Model (1)



1. The universal value is shown as a this dotted line. And we can see that when $a > \rho$, the adiabatic potential for both positive and negative a converge to this universal value.

Comparison to the Analytical Model (1)



2019-09-03

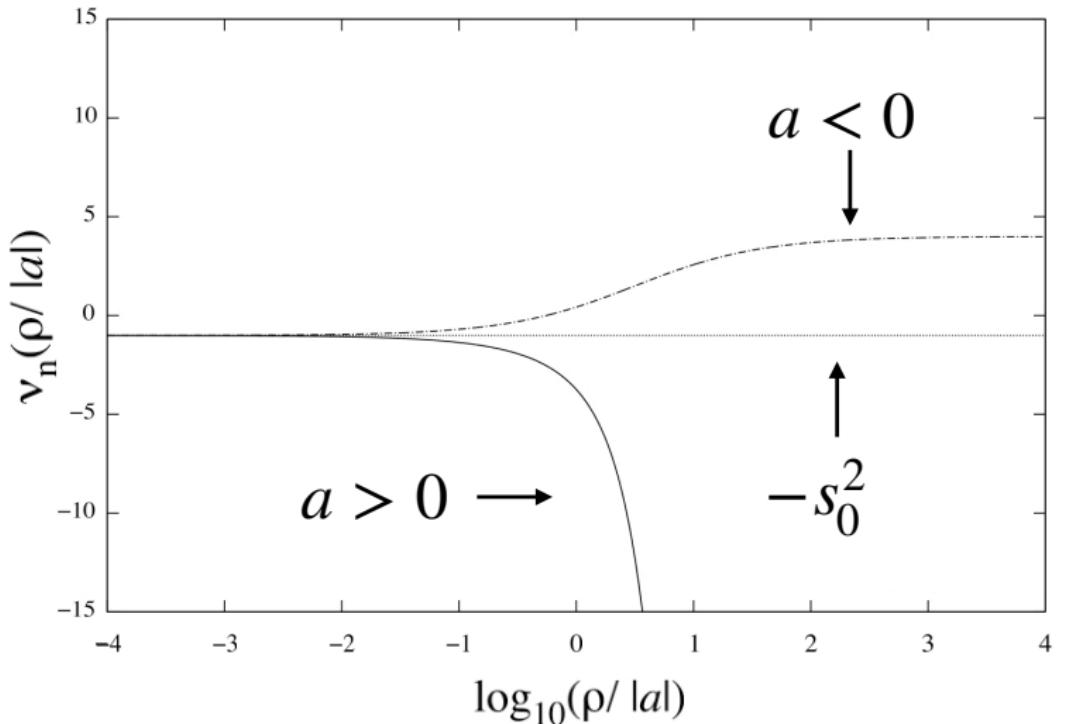
- └ Results
 - └ Comparison to the Analytical Model
 - └ Comparison to the Analytical Model (1)

Comparison to the Analytical Model (1)

The figure shows a plot of the numerical potential $V_n(p/|a|)$ versus $\log_{10}(p/|a|)$ for $a < 0$. The curve follows the same general shape as the main plot, starting at $V_n \approx -0.5$ for large negative x , crossing zero at $x \approx -0.5$, reaching a minimum of approximately -12 at $x = 0$, and then rising towards a horizontal dashed line at $V_n \approx 4$ for large positive x .

1. When $\rho > |a|$ we can observe that the curve for negative a approach the value corresponding to the kinetic energy for three particles.

Comparison to the Analytical Model (1)

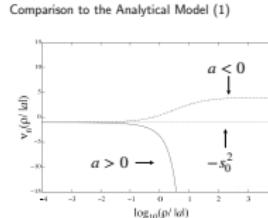


2019-09-03

Results

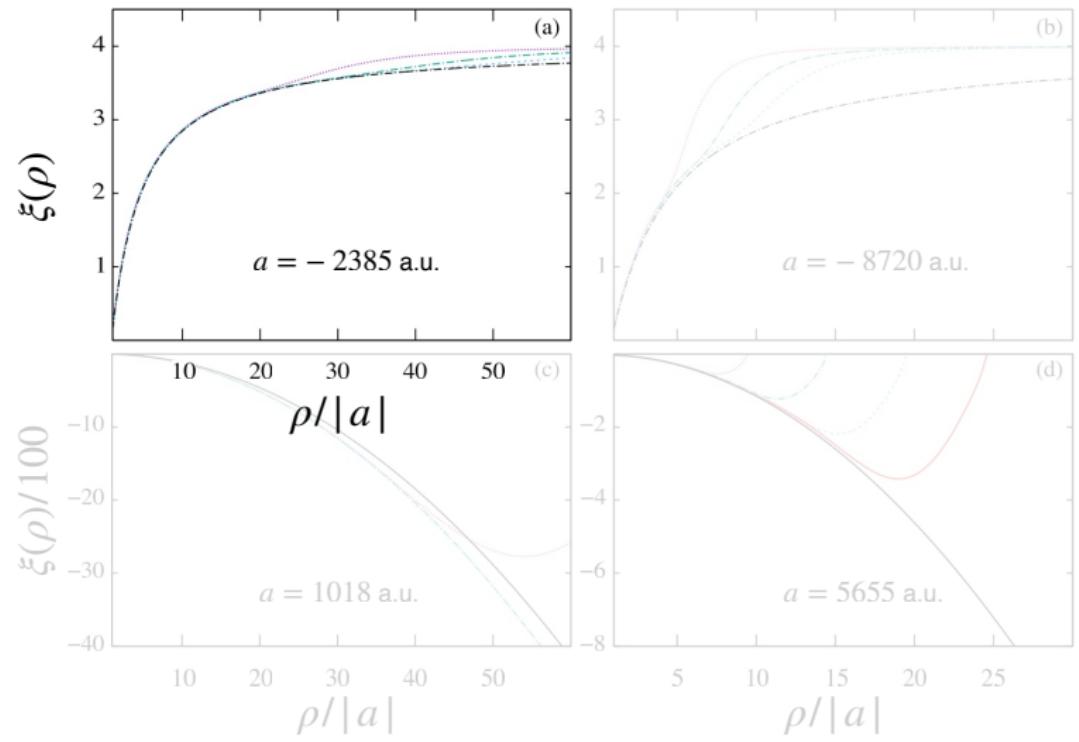
Comparison to the Analytical Model

Comparison to the Analytical Model (1)



1. For positive a , we instead observe a parabolic behaviour like that corresponding to the form of the energy of the 2-b bound state when $\rho > a$.

Comparison to the Analytical Model (2)



2019-09-03

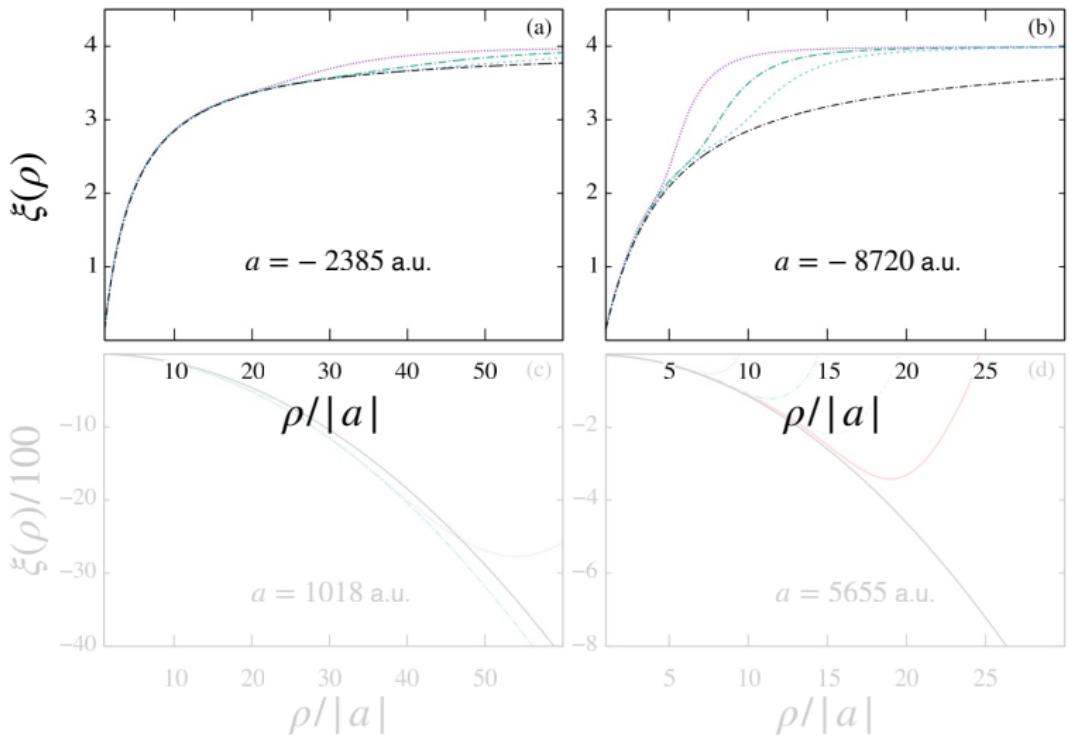
- Results
- Comparison to the Analytical Model
- Comparison to the Analytical Model (2)

Comparison to the Analytical Model (2)

The inset figure contains two subplots. Both have the y-axis labeled $\xi(\rho)$ and the x-axis labeled $\rho/|a|$.

- Left Plot:** Labeled $a = -2385 \text{ a.u.}$. The y-axis ranges from 0 to 4, and the x-axis ranges from 0 to 50. It shows a black dotted line (analytical) and several colored curves (numerical).
- Right Plot:** Labeled $a = -8720 \text{ a.u.}$. The y-axis ranges from 0 to 4, and the x-axis ranges from 0 to 50. It shows a black dotted line (analytical) and several colored curves (numerical).

Comparison to the Analytical Model (2)



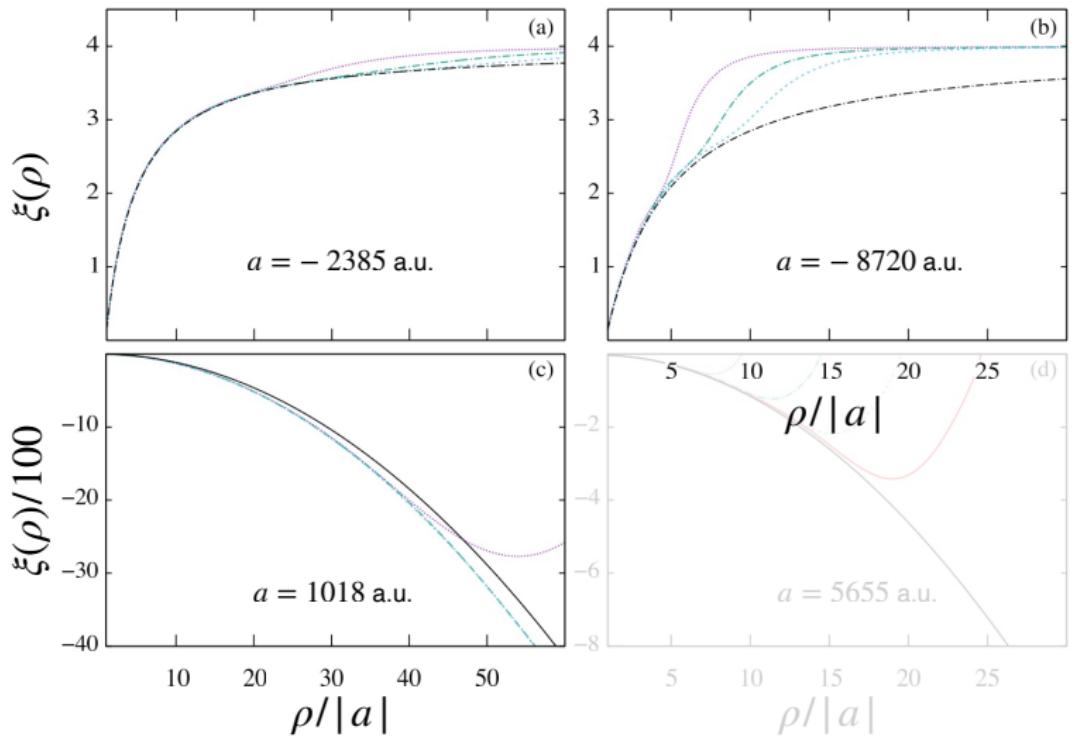
2019-09-03

- └ Results
 - └ Comparison to the Analytical Model
 - └ Comparison to the Analytical Model (2)

Comparison to the Analytical Model (2)

The grid contains four subplots showing the function $\xi(\rho)$ versus $\rho/|a|$ for different values of a . The top-left plot is for $a = -2385 \text{ a.u.}$, the top-right for $a = -8720 \text{ a.u.}$, the bottom-left for $a = 1018 \text{ a.u.}$, and the bottom-right for $a = 5655 \text{ a.u.}$. Each plot shows multiple curves that converge to a single value as $\rho/|a|$ increases. The axes scales vary between plots.

Comparison to the Analytical Model (2)



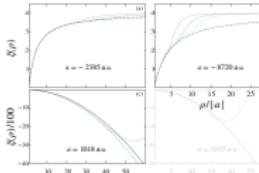
[◀ \$v_0\(\rho/a\)\$](#) [E_D](#)

2019-09-03

Results

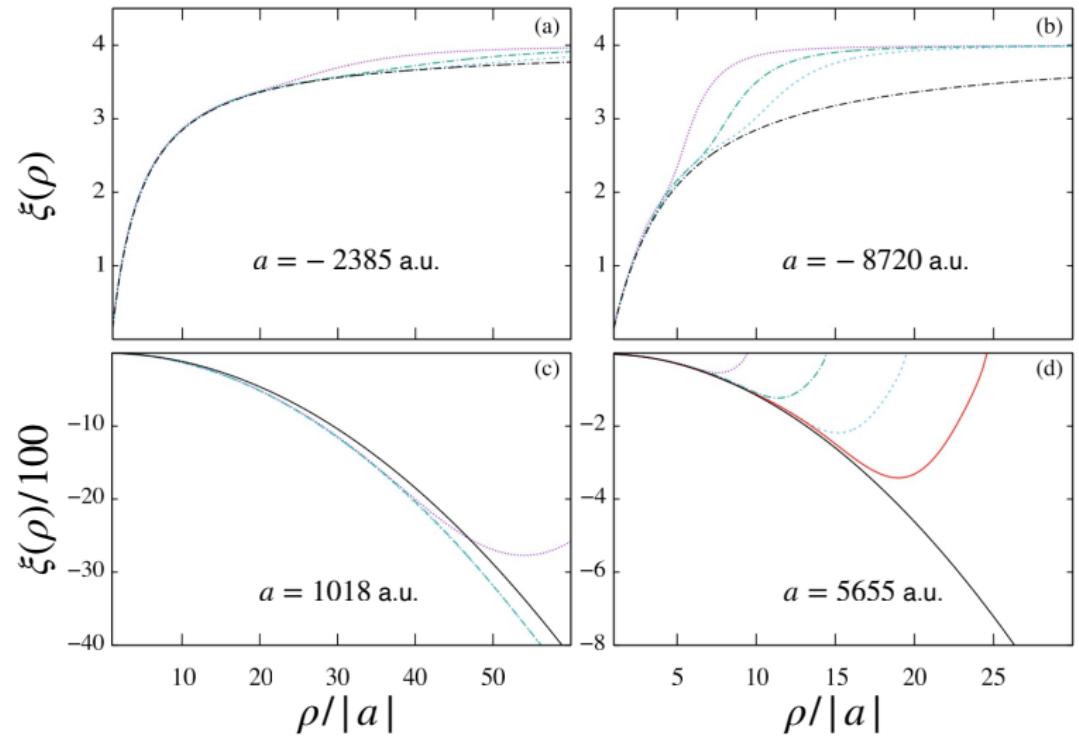
- Comparison to the Analytical Model
- Comparison to the Analytical Model (2)

Comparison to the Analytical Model (2)



1. Now we look at positive a . Here we have the analytic curve in black. All numerically calculated potentials have converged over the whole hyperradial range apart from the one calculated with the least number of B-splines.
2. However, there seem to be a small difference between the numerical and analytical results.

Comparison to the Analytical Model (2)



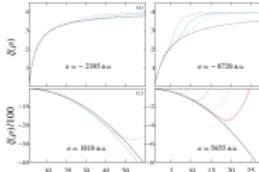
◀ $\nu_0(\rho/a)$ E_D

2019-09-03

Results

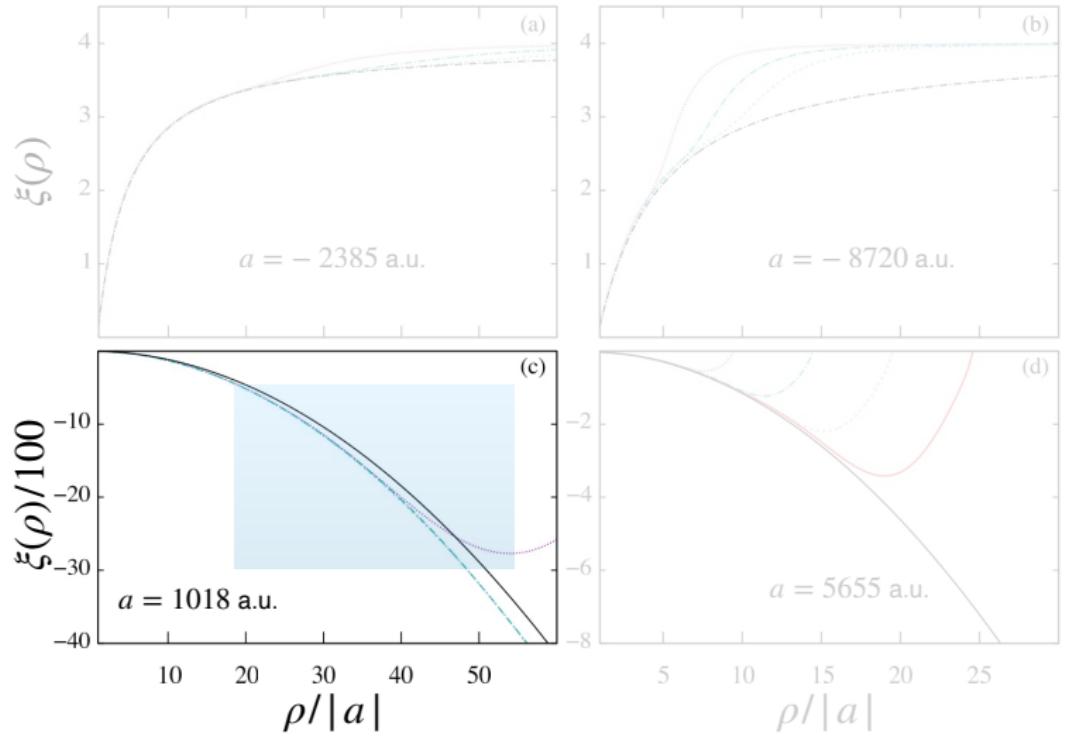
- Comparison to the Analytical Model
- Comparison to the Analytical Model (2)

Comparison to the Analytical Model (2)



- For larger a the analytical and numerical results show greater similarity.
- However, again the hyperradial range of convergence have become smaller.
- By comparing the panels to left and right, it appears that the hyperradial range for convergence generally is larger for the states with smaller $|a|$. We suspect that this discrepancy is due to the knot point placement, but this has not been verified yet.

Comparison to the Analytical Model (2)



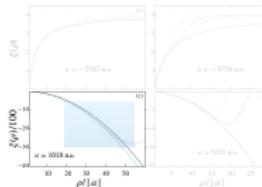
◀ $v_0(\rho/a)$ E_D

2019-09-03

Results

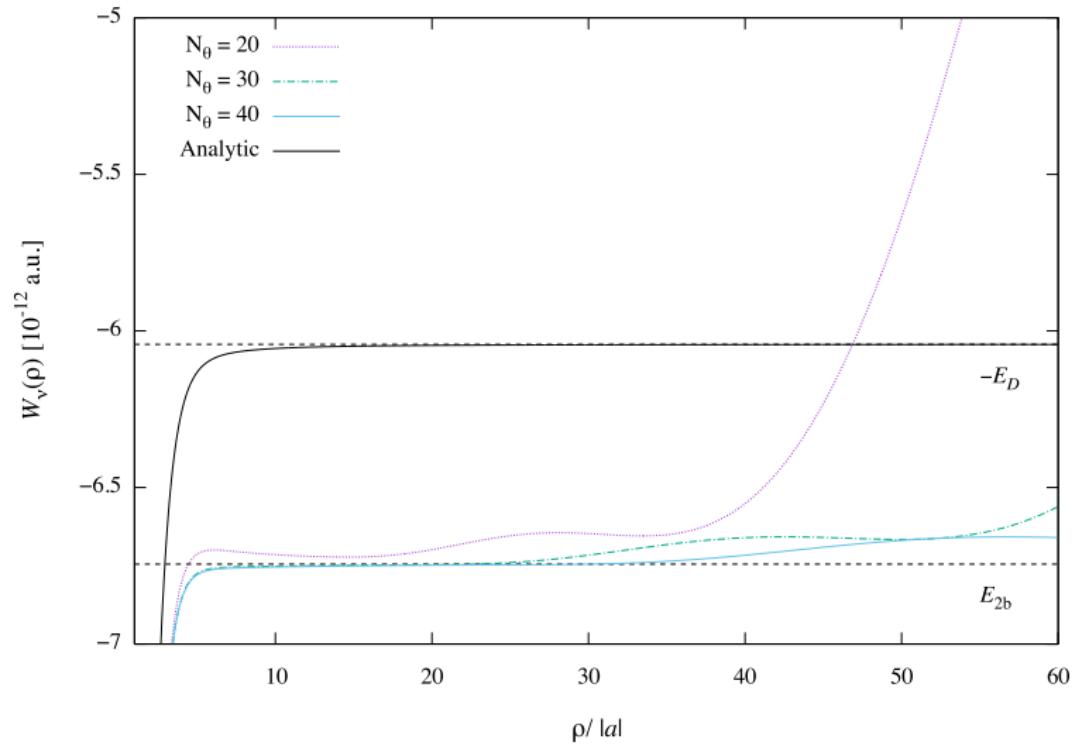
Comparison to the Analytical Model

Comparison to the Analytical Model (2)



1. Now, how do we this?
2. Well, the slightly different form in the parabolic divergency of the curves is due to a discrepancy between the exact two-body energy and the energy of the universal dimer.

Comparison to the Analytical Model (3)



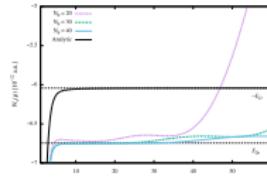
2019-09-03

Results

Comparison to the Analytical Model

Comparison to the Analytical Model (3)

Comparison to the Analytical Model (3)



1. Here I have plotted the actual three-body effective potentials W , the corresponding analytic potential together with the two-body energy E_{2b} and the energy of the universal dimer.
2. We can see that the analytic potential goes to