ALGORITHM FOR SATELLITE ORBIT DETERMINATION FROM DOPPLER SHIFT MEASUREMENTS

The principle of the method, which is somewhat similar to the "variation of geocentric distances" in the orbit determination of minor planets, is as follows. Suppose we are given three sets of frequency measurements obtained simultaneously by three tracking stations:

$$\nu_i\ (t_1),\quad \nu_i\ (t_2),\quad \ldots,\quad \nu_i\ (t_n)\quad . \qquad (i = 1,\ 2,\ 3)$$

These sets can be converted into corresponding sets of radial velocities $\dot{\rho}$ of the satellite by the equation

$$\dot{\rho} = c\,(1 - \nu/\nu_0)\quad ,$$

where c is the velocity of propagation, ν the apparent frequency at the receiver, and ν_0 the absolute frequency of the transmitter.

If we knew the distance ρ_i of the satellite from the i-th station at some particular time $t_{(i)}$, we could compute this distance for any other time of the observations by integrating the radial velocity $\dot{\rho}_i$. Again, if we knew the three distances ρ_1, ρ_2, ρ_3 from the three stations at times t_I and t_{II}, it would be easy to compute the corresponding geocentric position vectors $\underline{r}_I$ and $\underline{r}_{II}$, which (as in a boundary value problem) determine a Keplerian orbit. On the other hand, by approximating the true relative motion of the satellite near the times of closest approach to the tracking stations by a uniform, rectilinear motion, we can get good approximations for the minimum distances d_i. Denoting now by $t_{(i)}$ the time of closest approach to the i-th station, we have the basic unknowns

$$d_1 = \rho_1\,(t_{(1)}),\qquad d_2 = \rho_2\,(t_{(2)}),\qquad d_3 = \rho_3(t_{(3)})\quad .$$

We start with the values d_i belonging to the rectilinear approximation and improve them in an iterative procedure (to be described later) until agreement has been achieved between the observed radial velocities and the radial velocities belonging to a Keplerian motion. The remainder of this paper is devoted to a detailed description of this method.

In order to have an analytical representation of the radial velocities, we fit three least-squares polynomials of the 5th or 7th degree to the three sets,

$$\dot{\rho}_i(t_1), \quad \dot{\rho}_i(t_2), \quad \ldots, \quad \dot{\rho}_i(t_n) \;,$$

so that our representation will be

$$\dot{\rho}_i(t) = a_{i0} + a_{i1} t + \ldots + a_{i7} t^7 \;.$$

At the time of closest approach we have $\dot{\rho}_i(t_{(i)}) = 0$; that is, $t_{(i)}$ can be obtained by solving this equation. Let us rewrite these polynomials in powers of $t - t_{(i)}$:

$$\dot{\rho}_i(t) = b_{i1}(t - t_{(i)}) + \ldots + b_{i7}(t - t_{(i)})^7 \;.$$

Then

$$\begin{aligned} \rho_i(t_I) &= d_i + \frac{b_{i1}}{2}(t_I - t_{(i)})^2 + \ldots + \frac{b_{i7}}{8}(t_I - t_{(i)})^8 \\ \rho_i(t_{II}) &= d_i + \frac{b_{i1}}{2}(t_{II} - t_{(i)})^2 + \ldots + \frac{b_{i7}}{8}(t_{II} - t_{(i)})^8 \end{aligned} \qquad (1)$$

Now let us consider the radial velocities and minimum distances resulting from the assumption of a rectilinear motion. If the ρ_{i0} are the minimum distances and v_i the assumed uniform velocity relative to the i-th station, we have the relations,

$$\rho_i = \sqrt{\rho_{i0}^2 + v_i^2 (t - t_{(i)})^2}$$

and

$$\dot{\rho}_i = \frac{v_i^2 (t - t_{(i)})}{\sqrt{\rho_{i0}^2 + v_i^2 (t - t_{(i)})^2}} = \frac{A(t - t_{(i)})}{\sqrt{1 + B_i (t - t_{(i)})^2}} \;,$$

where

$$A_i = v_i^2 / \rho_{i0}, \quad B_i = v_i^2 / \rho_{i0}^2 \;,$$

or

$$\rho_{i0} = A_i / B_i \;. \qquad (2)$$

We are looking for those values of A_i and B_i that give the best least-squares fit to the observed radial velocities in a short interval of time. The value ρ_{i0} shall be the first approximation to d_i.

following: for the circular approximation there are three unknowns, and these could be determined with reliability only in a longer interval of time. In a longer interval, however, the circular approximation might be worse than the rectilinear in a shorter one.

Returning to the rectilinear approximation, we simplify the equations of condition by writing:

$$A_i = \alpha_i + \xi_i \quad , \qquad B_i = \beta_i - 2\eta_i/\alpha_i \quad , \tag{3}$$

where $\alpha_i = b_{i1}$ and $\beta_i = -2b_{i3}/b_{i1}$ are (already known) approximate values for A_i and B_i. With the notation

$$\gamma_{ij} = \frac{t_j - t_{(i)}}{\sqrt{1 + \beta_i (t_j - t_{(i)})^2}} \quad \text{and} \quad \lambda_{ij} = \dot{\rho}_i(t_j) - \alpha_i \gamma_{ij} \qquad (j = 1, \ldots, n) \; ,$$

the normal equations become

$$\begin{aligned} \left[\sum_{j=1}^{n} \gamma_{ij}^{2} \right] \xi_i + \left[\sum_{j=1}^{n} \gamma_{ij}^{4} \right] \eta_i &= \left[\sum_{j=1}^{n} \gamma_{ij} \lambda_{ij} \right] \\ \left[\sum_{j=1}^{n} \gamma_{ij}^{4} \right] \xi_i + \left[\sum_{j=1}^{n} \gamma_{ij}^{6} \right] \eta_i &= \left[\sum_{j=1}^{n} \gamma_{ij}^{3} \lambda_{ij} \right] . \end{aligned} \tag{4}$$

The solution ξ_i, η_i of these equations defines through equations (3) and (2) a better ρ_{i0} approximation. In a numerical example, the accuracies of α_i/β_i and A_i/B_i with respect to d_i were about 7 percent and 1 percent respectively. An iteration of the solution of equations (4), starting with A_i and B_i rather than with α_i and β_i, did not improve the accuracy significantly.

Now we can compute the approximate values of the ρ_i's for any time in the interval of observations, especially for the arbitrarily selected moments t_I and t_{II}, as shown by equation (1). So that the uncertainties of the observations and the inaccuracy of the d_i will have less effect on the orbit, it would be advisable to use a possibly long time-interval for the orbit determination. However, since the ionospheric effects on the Doppler-shift are stronger at low altitudes, we are limited in our choice of the intervals t_I, t_{II}.

The next step is to compute the geocentric position vectors $\underline{r}_I$ and $\underline{r}_{II}$ of the satellite. Let the position vector of the i-th station be

$$\underline{R}_i = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}$$

the rectangular components referring to the equatorial coordinate system. If these coordinates at sidereal time 0 in Greenwich are denoted by $X_i°$, $Y_i°$, $Z_i°$, then at sidereal time θ we have

$$X_i = X_i° \cos \theta - Y_i° \sin \theta \quad ,$$

$$Y_i = X_i° \sin \theta + Y_i° \cos \theta \quad ,$$

$$Z_i = Z_i° .$$

To carry out the computation, we need the following additional quantities: the side vectors of the station triangle,

$$\underline{D}_1 = \underline{R}_3 - \underline{R}_2 , \quad \underline{D}_2 = \underline{R}_1 - \underline{R}_3 \; ;$$

the unit vector normal to this triangle,

$$\underline{e} = \frac{1}{\Delta} (\underline{D}_1 \times \underline{D}_2), \quad \text{where} \quad \Delta = \sqrt{(\underline{D}_1 \times \underline{D}_2)^2} \quad ;$$

the scalars,

$$f_1 = \tfrac{1}{2} (R_1^2 - \rho_1^2), \; f_2 = \tfrac{1}{2} (R_2^2 - \rho_2^2) , \quad f_3 = \tfrac{1}{2} (R_3^2 - \rho_3^2) \quad ;$$

the vectors,

$$\underline{f} = \frac{1}{\Delta} \left\{ (f_1 - f_3) \, \underline{D}_1 + (f_2 - f_3) \, \underline{D}_2 \right\} , \quad \underline{F} = \underline{e} \times \underline{f} \; ;$$

and the scalars,

$$H = \underline{R}_3 \cdot \underline{e} , \quad G = \underline{R}_3 \cdot \underline{F} - f_3 , \quad F^2 = \underline{F}^2 \quad .$$

Needless to say, the scalars Δ and H are independent of time. The position vector $\underline{r}$ of a point whose three distances from the three endpoints of $\underline{R}_i$ are given is then

$$\underline{r} = \underline{F} + J \, \underline{e} \quad ,$$

where

$$J = H + \sqrt{H^2 + 2G - F^2} \quad .$$

All the quantities involved could be written in a symmetrical form, but the present form is more appropriate for the computation.

Once the two vectors,

$$\underline{r}_I = \begin{pmatrix} x_I \\ y_I \\ z_I \end{pmatrix}$$

and

$$\underline{r}_{II} = \begin{pmatrix} x_{II} \\ y_{II} \\ z_{II} \end{pmatrix} \quad ,$$

are obtained, the orbital elements can be determined in the following way, in accordance with Gauss. The basic unknown is considered to be the sector-triangle ratio y. For this quantity and another unknown,

$$x = \sin^2 \frac{g}{2} \quad ,$$

where

$$g = \frac{1}{2} (E_{II} - E_I)$$

and E is the eccentric anomaly, there are two equations with "known" coefficients,

$$y^3 - y^2 - hy - h/9 = 0 \quad , \tag{5}$$

and

$$x = \frac{m}{y^2} - \ell \quad . \tag{6}$$

The coefficients may be expressed as

$$m = \frac{\tau^2}{k^3} \quad , \qquad \ell = \frac{1}{2}\left(\frac{r_I + r_{II}}{k} - 1\right) \quad ,$$

where

$$k = \sqrt{2(r_I \, r_{II} + \underline{r}_I \cdot \underline{r}_{II})}$$

and

$$\tau = \sqrt{\mu} \, (t_{II} - t_I) \quad ,$$

μ denoting the product of the constant of gravity and the mass of the earth, and

$$h = \frac{m}{\frac{5}{6} + \ell + \xi} \quad , \tag{7}$$

where ξ is a transcendental function of x, represented with sufficient accuracy by

$$\xi = 0.057143 \, x^2 \, (1 + 0.5778 \, x + 0.36 \, x^2) \quad . \tag{8}$$

To solve (5) and (6), we first neglect the very small quantity ξ in h; that is, we start with

$$h = \frac{m}{\frac{5}{6} + \ell} \quad .$$

We solve equation (5) for y, or what is the same, the equation

$$z = \frac{1}{1 - h} \left\{ \frac{10}{9} h - (2 + z) z^2 \right\} \quad ,$$

for $z = y - 1$ by the Newtonian recursive formula

$$z_{k+1} = \frac{\frac{10}{9} h + 2 (1 + z_k) z_k^2}{(1 - h) + (4 + 3z_k) z_k} \quad .$$

We compute x, ξ, then h by equations (6), (7), and (8), and repeat until y reaches its final value. Because of the rapid convergence of the Gaussian procedure, the final value will usually be reached in the second approximation, taking seven decimal places into consideration and for $2g < 60°$. Actually, the same limitations hold for the accuracy of equation (8).

The usual notations for twice the area of the sector of the ellipse and for twice the area of the triangle contained between the two radius vectors $\underline{r}_I$ and $\underline{r}_{II}$ are, respectively, (r_I, r_{II}) and $[r_I, r_{II}]$. The theory of Keplerian motion gives

$$(r_I, r_{II}) = \sqrt{p}\, \tau$$

and obviously

$$[r_I, r_{II}] = \sqrt{(\underline{r}_I \times \underline{r}_{II})^2} \quad .$$

From

$$y = \frac{(r_I, r_{II})}{[r_I, r_{II}]}$$

we have

$$p = \frac{y^2}{\tau^2} [r_I, r_{II}]^2$$

as the parameter of the ellipse.

On the other hand we know that

$$\underline{r}_I \times \underline{r}_{II} = \left[r_I, r_{II}\right] \begin{pmatrix} \sin i \sin \Omega \\ -\sin i \cos \Omega \\ \cos i \end{pmatrix}$$

The position of the orbit plane is therefore determined by

$$\operatorname{tg} \Omega = - \frac{y_I z_{II} - z_I y_{II}}{z_I x_{II} - x_I z_{II}}$$

and

$$\operatorname{tg} i = \frac{\sqrt{(y_I z_{II} - z_I y_{II})^2 + (z_I x_{II} - x_I z_{II})^2}}{x_I y_{II} - y_I x_{II}}$$

or by some equivalent relations using the value of $\left[r_I, r_{II}\right]$.

From the equation of the ellipse,

$$r = \frac{p}{1 + e \cos v} ,$$

we have

$$e \cos v_I = \frac{p}{r_I} - 1$$

and

$$e \cos v_{II} = \frac{p}{r_{II}} - 1$$

The difference $v_{II} - v_I$ of the true anomalies is also known, because

$$\underline{r}_I \cdot \underline{r}_{II} = r_I r_{II} \cos (v_{II} - v_I), \left[r_I, r_{II}\right] = r_I r_{II} \sin(v_{II} - v_I) .$$

Hence

$$e \sin v_I = \frac{e \cos v_I \cos (v_{II} - v_I) - e \cos v_{II}}{\sin (v_{II} - v_I)} =$$

$$= \frac{\left(\frac{p}{r_I} - 1\right) \cos (v_{II} - v_I) - \left(\frac{p}{r_{II}} - 1\right)}{\sin (v_{II} - v_I)}$$

and e, v_I, v_{II} can be computed. Then

$$a = \frac{p}{1-e^2} \quad .$$

The true anomalies serve as auxiliary quantities in the orbit computation. First, we can use, for instance, v_I to compute ω, the argument of perigee, because the angle $u = \omega + v$ is determined unambiguously by the relations

$$r \cos u = x \cos \Omega + y \sin \Omega \quad ,$$

$$r \sin u = \frac{z}{\sin i} \quad .$$

Furthermore, the corresponding eccentric anomalies are given by

$$\cos E = \frac{e + \cos v}{1 + e \cos v} \; , \; \sin E = \frac{\sqrt{1 - e^2} \sin v}{1 + e \cos v} \quad .$$

Finally, Kepler's equation,

$$M = E - e \sin E \quad ,$$

yields the mean anomalies.

The orbit elements just derived are only approximations to the true elements of the satellite, because we started with approximate values of the unknowns d_i. It is to be expected that the angles ω and M (also E, or v, but not so much u) in the orbit plane will be most affected by errors. Now let us compare the observed radial velocities with the radial velocities which result from these orbit elements.

From

$$\underline{\rho}_i = \underline{r} - \underline{R}_i$$

follows

$$\dot{\rho}_i = \frac{1}{\rho_i} (\underline{\rho}_i \cdot \dot{\underline{r}} - \dot{\underline{R}}_i \cdot \underline{r}) \; ;$$

therefore we need $\dot{\underline{r}}$ and $\dot{\underline{R}}_i$ in addition to $\underline{r}$ and $\underline{R}_i$. The vector $\underline{r}$ can be represented by

$$\underline{r} = \underline{A} \; (\cos E - e) + \underline{B} \sin E \quad ,$$

where $\underline{A} = a\underline{P}$, $B = a\sqrt{1 - e^2} \; \underline{Q}$. The unit vectors $\underline{P}$ and $\underline{Q}$ in the direction of perigee ($v = 0°$) and in the direction of $v = 90°$, respectively, have the components

$$\underline{P} = \begin{pmatrix} \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i \\ \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i \\ \sin \omega \sin i \end{pmatrix}$$

and

$$\underline{Q} = \begin{pmatrix} -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i \\ -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i \\ \cos\omega\sin i \end{pmatrix} .$$

Then the velocity vector will be

$$\dot{\underline{r}} = \left\{ -\underline{A}\sin E + \underline{B}\cos E \right\} \dot{E} ,$$

where

$$\dot{E} = \frac{n}{1 - e\cos E} ,$$

as follows by differentiating Kepler's equation.

The velocity vectors of the stations are simply

$$\dot{\underline{R}}_i = \begin{pmatrix} -\dot{\theta}\, Y_i \\ +\dot{\theta}\, X_i \\ 0 \end{pmatrix} ,$$

$\dot{\theta}$ denoting the angular velocity of the rotation of the earth.

For the comparison mentioned above, there are two possibilities, either for a single time near the times of closest approach, or for several times during an interval. The latter procedure involves again the least-squares method. We suggest the use of the first procedure for the first iteration, and the second one for the second iteration.

In particular, we have to compute the orbit elements and from these the radial velocities for the initial set

$$\rho_{10}, \rho_{20}, \rho_{30} ;$$

furthermore, varying these minimal distances by small amounts for the sets

$$\begin{gathered} \rho_{10} + \delta_1, \rho_{20}, \rho_{30} \\ \rho_{10}, \rho_{20} + \delta_2, \rho_{30} \\ \rho_{10}, \rho_{20}, \rho_{30} + \delta_3 . \end{gathered}$$

Let the radial velocities corresponding to each of these sets for a certain time T be denoted by

$$\dot{\rho}_{i,0} , \dot{\rho}_{i,1} , \dot{\rho}_{i,2} , \dot{\rho}_{i,3} ,$$

the observed values on the contrary by $\dot{\rho}_i$. We seek three unknowns, ϑ_1, ϑ_2, ϑ_3, so that

$$\rho_{10} + \vartheta_1 \delta_1, \quad \rho_{20} + \vartheta_2 \delta_2, \quad \rho_{30} + \vartheta_3 \delta_3$$

are the corrected values of ρ_{10}, ρ_{20}, ρ_{30}. We have the following system of equations of linear interpolation:

$$(\dot{\rho}_{i,1} - \dot{\rho}_{i,0}) \vartheta_1 + (\dot{\rho}_{i,2} - \dot{\rho}_{i,0}) \vartheta_2 + (\dot{\rho}_{i,3} - \dot{\rho}_{i,0}) \vartheta_3 = (\dot{\rho}_i - \dot{\rho}_{i,0}). \quad (i = 1, 2, 3)$$

The corrected values $\rho_{i0} + \vartheta_i \delta_i$ define a new set of orbital elements, which can be improved further by the least-squares variant of this procedure.