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The Inverse Problem of Flobenius-Perron Equations in 1D Difference Systems

—— *1D Map Idealization* ——

Shinji KOGA

Osaka Kyoiku University, Osaka 543

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We derive the 1D difference systems for an arbitrary invariant density by solving the Flobenius-Perron equation inversely. We find the variety of the analytical forms of the 1D systems in the cases of the elemental functions representing the invariant densities. We also confirm that the existence and uniqueness of the invariant densities for the derived 1D maps are assured. On the basis of the generality of the formula, we infer the possibility of the 1D map idealization as one of the methods for analyzing the noisy data.

§ 1. Introduction

From an experimental point of view, it is desirable to construct a dynamical system which realizes the experimental data[1),2)] at least statistically. Phenomenologically, it is natural to inquire the physical fundamental processes underlying the pertinent phenomenon at first. It is, however, also intriguing to find a dynamical system which possesses the similar statistical properties, especially the invariant density or the correlation function.

As a first step toward this goal, we find a simple formula between the 1D difference system and the invariant density by solving the Flobenius-Perron equation inversely in this paper.

Experimentally, it is not necessary to find a 1D difference system on purpose, when the system is governed by the 1D difference system, because the density and the difference system can be found simultaneously in this case. However, once the procedure for obtaining the 1D difference equation for a given density is established, this procedure provides us with the *idealization* based on the 1D difference system for the more complicated system with many degrees of freedom. The *idealization* implies that the stationary statistical properties are realized by considering the 1D system, even if the temporal evolution of the system to be considered cannot be described by the 1D system. According to this *idealization*, it is possible to construct several appropriate 1D difference equations from arbitrary experimental data.

Let us consider the experimental data which may be either the discrete or continuous time series. When the time series is continuous, we prepare the discrete data by cutting it storoboscopically. Then we construct the stationary density defined on the finite interval by means of a box-counting method. As we shall see later, we shall find solvable ordinary differential equations for the 1D maps, when the stationary density is given. Therefore we can construct several corresponding 1D maps whose stationary densities coincide with the density given by the experimental data.

Theoretically, this inverse problem is closely related to the ergodicity problem. Namely, when the sufficiently smooth invariant density (absolutely continuous measure) whose support is the total interval is given as the preassigned quantity, we inevitably require beforehand that the system must exhibit the ergodic property. In this case, the existence and uniqueness of the invariant density is of great importance for the 1D map obtained by solving the inverse problem. These properties of the invariant density are proved for the maps which satisfy several conditions,[5] e.g.,

$$\inf\left|\frac{dF(x)}{dx}\right|>1\,.$$

We refer to Ref. 5) for the details of the theorem denoted by $T2$. Accordingly, we only have to examine whether the maps obtained in the inverse problem satisfy the above conditions in order to assure that the empirical density which is realized for almost all initial conditions in the meaning of the Lebesgue measure is the preassigned invariant density. We note here the existence and uniqueness can be verified even if the above conditions are not satisfied, when the conjugation relation holds for the map whose invariant density exists and is unique.

In this paper we shall discuss several concrete examples of the 1D maps for the absolutely continuous measures. We find that all examples are the ergodic transformations, and that they satisfy the above-mentioned conditions. Therefore the corresponding empirical densities naturally coincide with the preassigned densities in the meaning the Lebesgue measure.

§ 2. Formulation

Let us consider the Flobenius-Perron equation given by

$$f_{n+1}(x)=\int_0^1 \delta(x-F(y))f_n(y)dy\,, \quad (n=0,1,2,\cdots) \tag{2·1}$$

where the 1D difference equation is written as

$$x_{n+1}=F(x_n)\,. \quad (n=0,1,2,\cdots) \tag{2·2}$$

Equation (2·1) implies that the probability density evolves toward the stationary invariant density $f(x)$, provided that all eigenvalues except unity are located within the unit circle. Therefore, if the Flobenius-Perron operator is asymptotically stable,[3] then we have

$$f(x)=\int_0^1 \delta(x-F(y))f(y)dy\,. \tag{2·3}$$

Since we consider the inverse problem[4] for the function $F(x)$, we make an attempt to derive the equation for $F(x)$, by changing the form of Eq. (2·3). For this purpose, we restrict ourselves to the cases such that

$$F(1-x)=F(x)\,, \quad \text{(type 1)} \tag{2·4}$$

$$F(x+1/2)=F(x). \quad \text{(type 2)} \tag{2·5}$$

In type 1, the map is symmetric about the value $x=1/2$. Type 2 corresponds to the translationally symmetric map. These restrictions are mainly due to the difficulty for the arbitrary form of the map $F(x)$. Namely it is difficult to obtain the inverse image of $F(x)$, as we shall see later.

We next substitute $F(x)$ into Eq. (2·3),

$$f(F(x))=\int_0^1 \delta(F(x)-F(y))f(y)dy. \tag{2·6}$$

The delta function can be written as

$$\delta(F(x)-F(y))=\frac{\delta(x-y)+\delta(1-x-y)}{\left|\frac{dF(x)}{dx}\right|} \tag{2·7}$$

in type 1, and

$$\delta(F(x)-F(y))=\frac{\delta(x-y)+\delta(x+1/2-y)}{\left|\frac{dF(x)}{dx}\right|} \tag{2·8}$$

in type 2. We note that the explicit forms of the delta function such as Eqs. (2·7) and (2·8) would be difficult to obtain in the other cases. By substituting Eqs. (2·7) and (2·8) into Eq. (2·6), we find that

$$\frac{dF(x)}{dx}=\frac{f(x)+f(1-x)}{f(F(x))} \tag{2·9}$$

in type 1, and

$$\frac{dF(x)}{dx}=\frac{f(x)+f(x+1/2)}{f(F(x))} \tag{2·10}$$

in type 2. Here we have assumed that the gradient of $F(x)$ is positive in the interval $[0, 1/2]$ for both types. More specifically, we assume that

$$F(0)=0 \tag{2·11}$$

for both types. Equations (2·9) and (2·10) are the ordinary differential equations for $F(x)$, so that the analytical expressions of $F(x)$ can be derived for the appropriate density $f(x)$. As far as the analytic forms of the invariant densities are concerned, it seems that the conjugation relation is the only one method for the particular types of the 1D maps.[5] Therefore, the formulae (2·9) and (2·10) are important for deriving a considerably large class of the analytically expressed maps for the analytical invariant density.

We should note here that Eqs. (2·9) and (2·10) must be integrated for the interval $[0, 1/2]$, but not for $[0, 1]$. The behavior of $F(x)$ for $[1/2, 1]$ can be found from the properties of Eqs. (2·4) and (2·5).

We furthermore find that

$$F(1/2-0)=1 \tag{2·12}$$

for both cases by integrating Eqs. (2·9) and (2·10).

Before we proceed to our discussion for the concrete forms of $f(x)$, we derive the Lyapunov exponent λ described by

$$\lambda=\int_0^1 f(x)\log\left|\frac{dF(x)}{dx}\right|dx\,. \tag{2·13}$$

Since the forms of $dF(x)/dx$ are obtained from Eqs. (2·9) and (2·10), after simple calculations, we find that

$$\lambda=\int_0^1 dx g(x)\log g(x)-\int_0^1 dx f(x)\log f(x)+\log 2\,, \tag{2·14}$$

where

$$g(x)=\frac{f(x)+f(1-x)}{2} \tag{2·15}$$

for type 1, and

$$g(x)=\frac{f(x)+f(x+1/2)}{2} \tag{2·16}$$

for type 2. Here we have assumed that

$$f(x+1)=f(x)$$

in type 2. It follows from Eq. (2·13) that the Lyapunov exponent λ is different from log2 by the difference between the information for $f(x)$ and $g(x)$ defined by Eqs. (2·14) and (2·15). If $f(x)=f(1-x)$ or $f(x)=f(x+1/2)$, then we find that λ equals log2.

§ 3. Examples of the 1D maps

We discuss the concrete examples in the following. As we have stated in the Introduction, we have to confirm whether the obtained 1D maps satisfy the conditions of the theorem for the existence and uniqueness of the invariant density. All examples discussed below show the difference equations expressed by some elemental functions except for the Gaussian distribution.

We begin with the density of sinusoidal type written as

$$f(x)=a\sin(\pi x/m)\,,\quad (m\geq 1) \tag{3·1}$$

where

$$a=\frac{\pi}{m(1-\cos(\pi/m))}\,. \tag{3·2}$$

Then it follows from Eq. (2·9) in type 1 that

$$\int_0^y f(y)dy=\int_0^x (f(x)+f(1-x))dx\,, \tag{3·3}$$

where

$$y=F(x)\,, \tag{3·4}$$

so that we obtain

$$F(x)=\begin{cases} \dfrac{m}{\pi}\cos^{-1}\left[\cos\left(\dfrac{\pi x}{m}\right)-\cos\left(\dfrac{\pi(1-x)}{m}\right)+\cos\left(\dfrac{\pi}{m}\right)\right], & (0<x<1/2) \\ \dfrac{m}{\pi}\cos^{-1}\left[\cos\left(\dfrac{\pi(1-x)}{m}\right)-\cos\left(\dfrac{\pi x}{m}\right)+\cos\left(\dfrac{\pi}{m}\right)\right]. & (1/2<x<1) \end{cases} \tag{3·5}$$

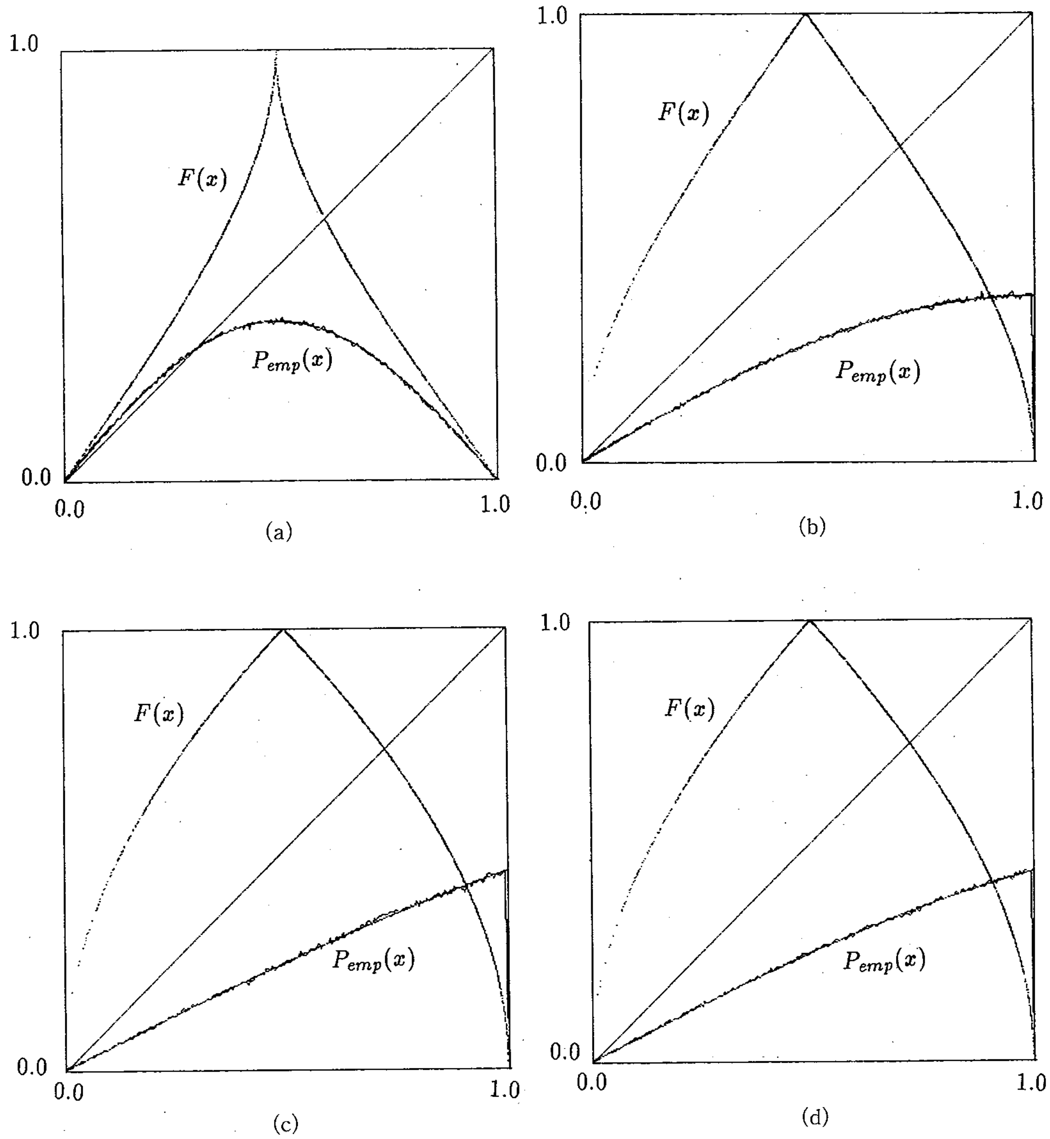

Fig. 1. The behaviors of the type 1 1D map given by Eq. (3·5), the given sine density, and the empirical density obtained by the box-counting method with the 100 boxes for the 1000000 iterations. Figures (a)~(d) correspond to the cases $m=1, 2, 3, 4$, respectively.

In Fig. 1 (a), we show the behaviors of the 1D map given by Eq. (3·5) and the empirical density for $m=1$. We also show the density given by Eq. (3·1) in Fig. 1. However, since the coincidence between the empirical and given densities is good, the difference cannot be seen appreciably.

In Figs. 1 (b)~(d), we show the similar behaviors for $m=2, 3, 4$.

It follows from Eq. (3·5) that

$$\frac{dF(x)}{dx}=-\frac{m}{\pi}\frac{dh(x)}{dx}\Big/\sqrt{1-h(x)^2}>1 \quad \text{for} \quad 0<x<1/2\,, \quad (m\geq 1)$$

where

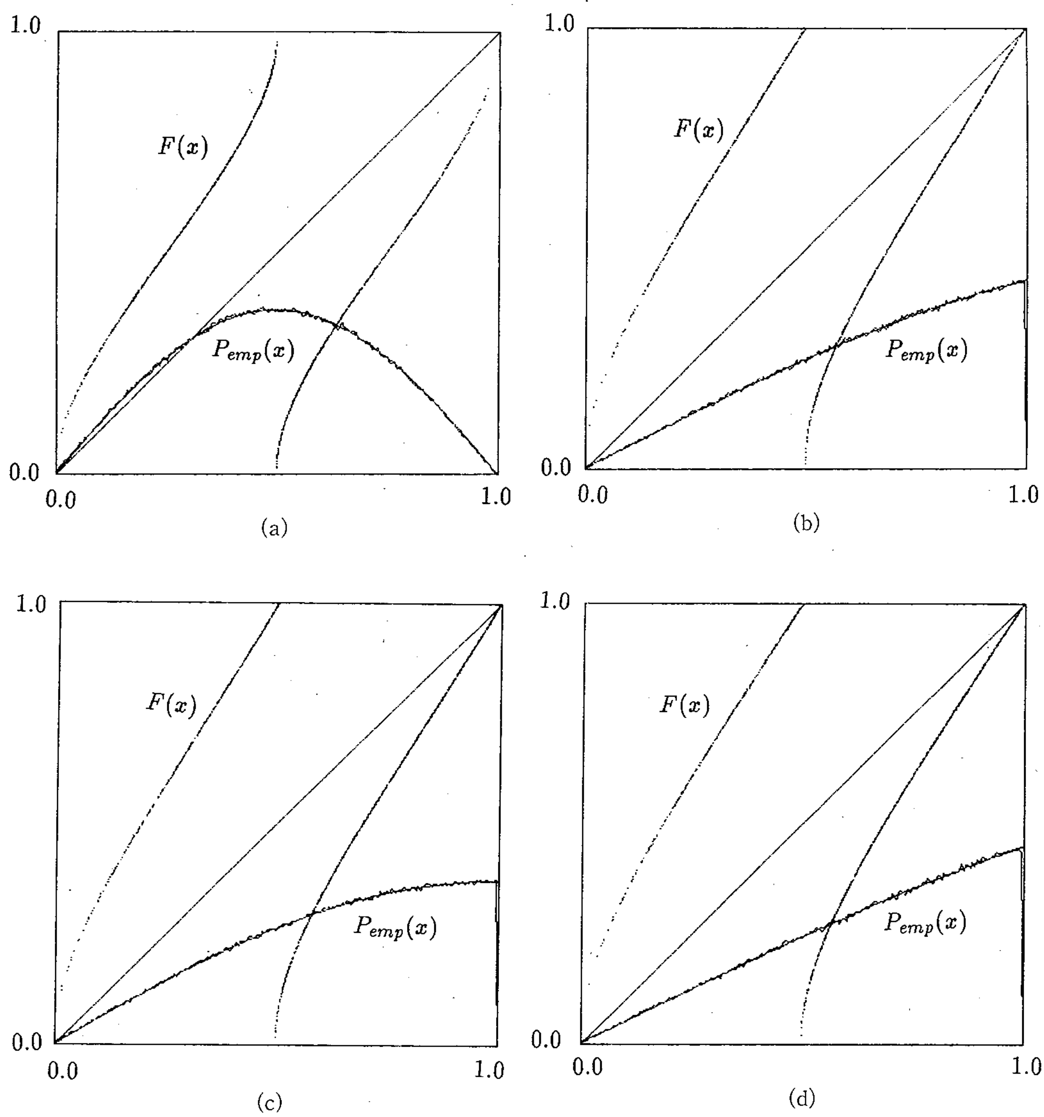

Fig. 2. The behaviors of the type 2 1D map given by Eq. (3·6) and the empirical density for the case of the sine density. The details are similar to Fig. 1.

$$h(x)=\cos\left(\frac{\pi x}{m}\right)-\cos\left(\frac{\pi(1-x)}{m}\right)+\cos\left(\frac{\pi}{m}\right).$$

Moreover we can select the two subintervals in the theorem, so that we conclude that the unique invariant density for the map given by Eq. (3·5) exists and coincides with the preassigned density given by Eq. (3·1). In what follows we have calculated $dF(x)/dx$ analogously for all examples except for the Gaussian distribution, so that we find that all examples satisfy the conditions of the theorem. Therefore we only mention the explicit forms of the derived 1D maps, and exhibit the behaviors of the 1D maps graphically in the following figures.

We next consider type 2 for Eq. (3·1). In this case, we find that

$$F(x)=\begin{cases}\dfrac{m}{\pi}\cos^{-1}\left[\cos\left(\dfrac{\pi x}{m}\right)+\cos\left(\dfrac{\pi(x+1/2)}{m}\right)-\cos\left(\dfrac{\pi}{2m}\right)\right], & (0<x<1/2)\\ \dfrac{m}{\pi}\cos^{-1}\left[\cos\left(\dfrac{\pi(x-1/2)}{m}\right)+\cos\left(\dfrac{\pi x}{m}\right)-\cos\left(\dfrac{\pi}{2m}\right)\right]. & (1/2<x<1)\end{cases}\tag{3·6}$$

In Figs. 2 (a)~(d), we show the 1D map given by Eq. (3·6), the empirical density, and the given density for $m=1, 2, 3, 4$.

In Figs. 3 and 4, we show the curves similar to Figs. 1 and 2 for the cosine density described by

$$f(x)=\frac{\pi}{m\sin(\pi/m)}\cos\left(\frac{\pi x}{m}\right). \quad (m\geq 2)\tag{3·7}$$

Figure 3 corresponds to type 1, where the 1D map is given by

$$F(x)=\begin{cases}\dfrac{m}{\pi}\sin^{-1}\left[\sin\left(\dfrac{\pi x}{m}\right)-\sin\left(\dfrac{\pi(1-x)}{m}\right)+\sin\left(\dfrac{\pi}{m}\right)\right], & (0<x<1/2)\\ \dfrac{m}{\pi}\sin^{-1}\left[\sin\left(\dfrac{\pi(1-x)}{m}\right)-\sin\left(\dfrac{\pi x}{m}\right)+\sin\left(\dfrac{\pi}{m}\right)\right]. & (1/2<x<1)\end{cases}\tag{3·8}$$

Figure 4 corresponds to type 2, where the 1D map is

$$F(x)=\begin{cases}\dfrac{m}{\pi}\sin^{-1}\left[\sin\left(\dfrac{\pi x}{m}\right)+\sin\left(\dfrac{\pi(x+1/2)}{m}\right)-\sin\left(\dfrac{\pi}{2m}\right)\right], & (0<x<1/2)\\ \dfrac{m}{\pi}\sin^{-1}\left[\sin\left(\dfrac{\pi(x-1/2)}{m}\right)+\sin\left(\dfrac{\pi x}{m}\right)-\sin\left(\dfrac{\pi}{2m}\right)\right]. & (1/2<x<1)\end{cases}\tag{3·9}$$

As Figs. 3 and 4 show, each empirical density almost equals the given density for $m=2, 3, 4, 5$.

In the following, we consider the limiting case such that m tends to infinity. In this case, we have for the sine density

$$f(x)=2x\,,\tag{3·10}$$

and we find that

$$F(x)=\sqrt{2x}\,,\quad (\text{type 1}\quad 0<x<1/2)\tag{3·11}$$

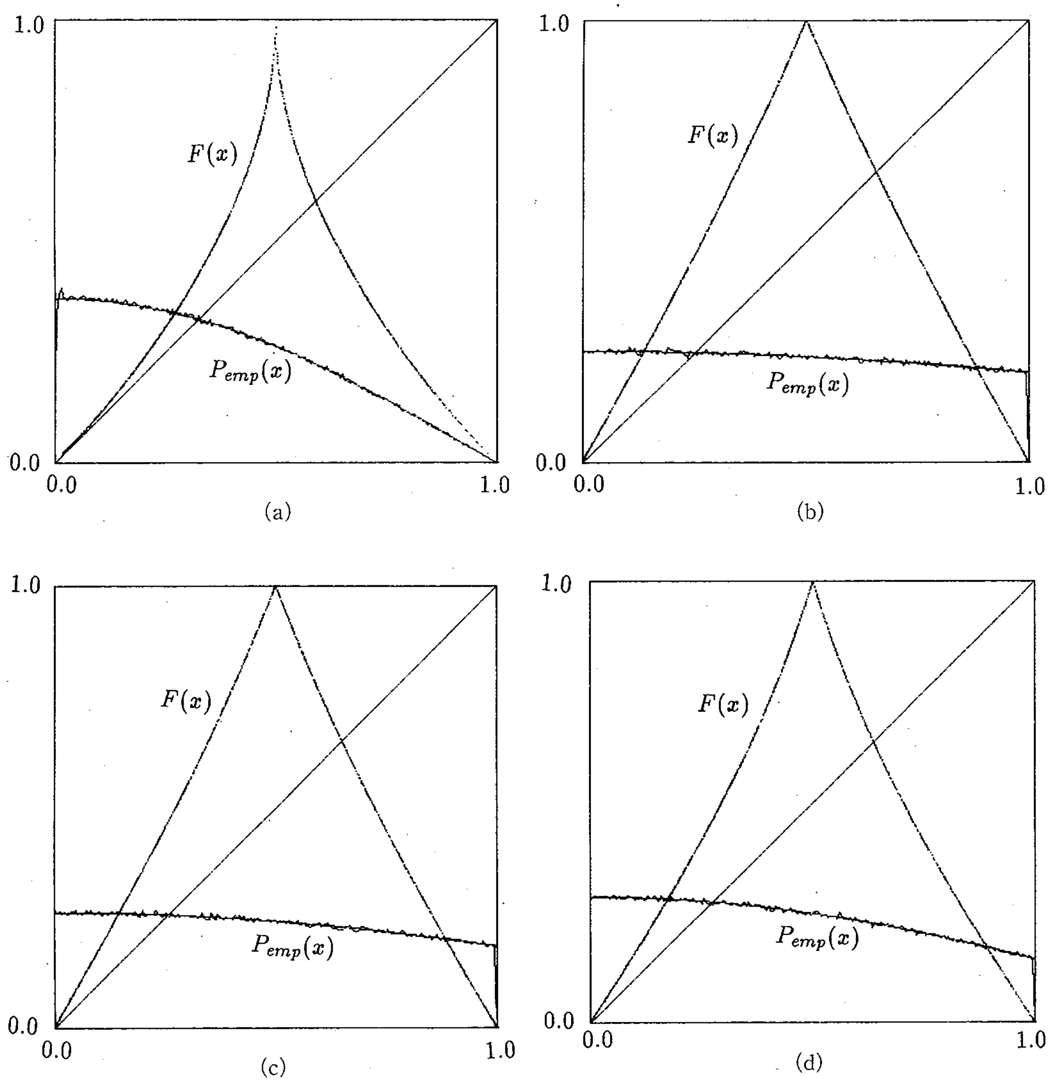

Fig. 3. The behaviors of the type 1 1D map given by Eq. (3·8) and the empirical density for the case of the cosine density.

$$F(x)=\sqrt{x^2+(x+1/2)^2-1/4}\ . \quad (\text{type 2} \quad 0<x<1/2) \tag{3·12}$$

We also have for the cosine density

$$f(x)=1 \tag{3·13}$$

and

$$F(x)=2x\ , \quad (\text{type 1} \quad 0<x<1/2) \tag{3·14}$$

$$F(x)=2x\ . \quad (\text{type 2} \quad 0<x<1/2) \tag{3·15}$$

The calculations for the algebraic function representing the density are also

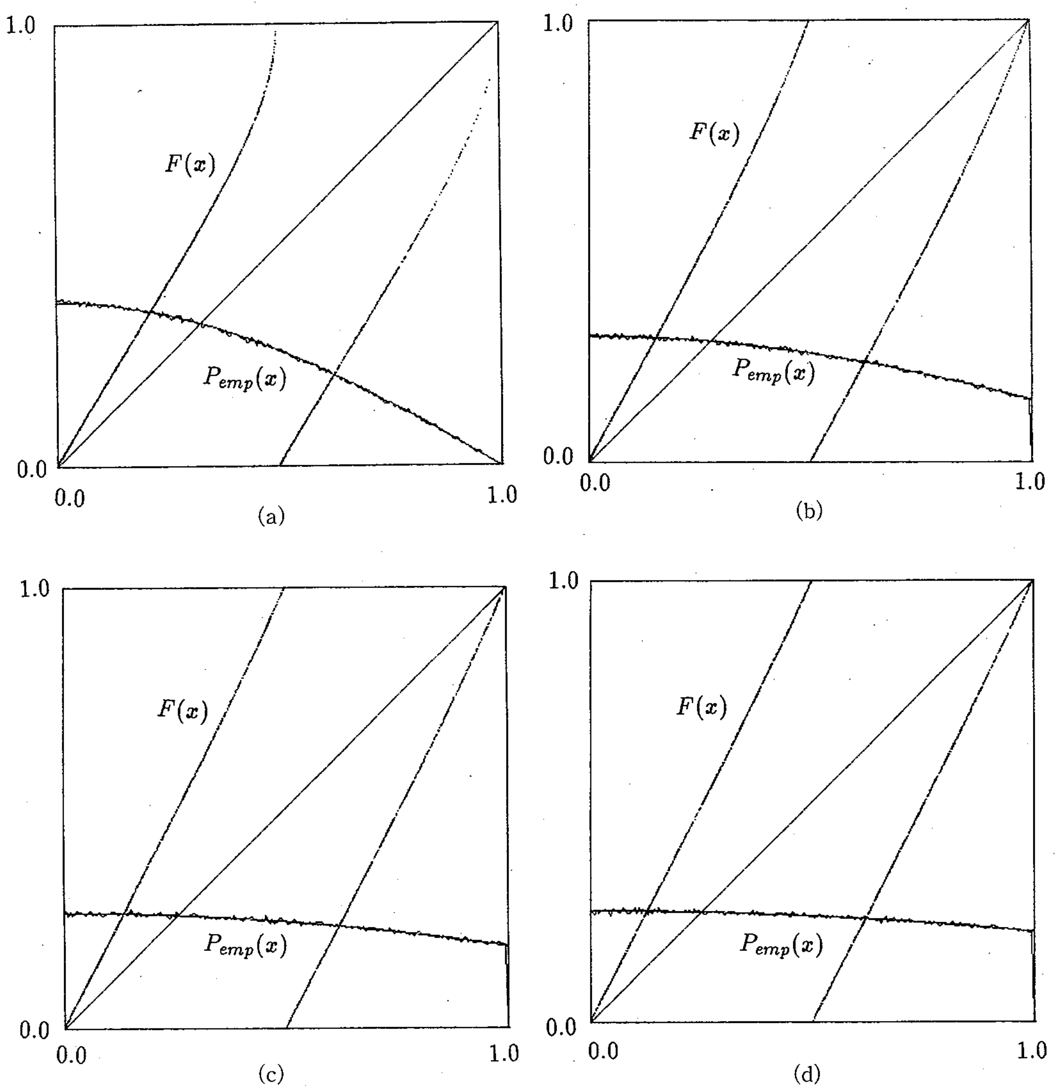

Fig. 4. The behaviors of the type 2 1D map given by Eq. (3·9) and the empirical density for the case of the cosine density.

analogous. We take as a first example a line density described by

$$f(x)=a+bx\,,\quad (0<a<2) \tag{3·16}$$

where

$$b=2-2a\,. \tag{3·17}$$

In this case, we have

$$F(x)=\frac{-a+\sqrt{a^2+4bx}}{b}\,,\quad (\text{type 1}\quad 0<x<1/2) \tag{3·18}$$

$$F(x)=\frac{-a+\sqrt{a^2+2b(1+a)x+2b^2x^2}}{b}. \quad (\text{type 2} \quad 0<x<1/2) \tag{3·19}$$

The second example of the algebraic density is as follows:

$$f(x)=ax+bx^3, \quad (0<a<4) \tag{3·20}$$

where

$$b=4-2a. \tag{3·21}$$

Then we have

$$F(x)=\sqrt{\frac{-a+\sqrt{a^2+b(4ax+b+bx^4-b(1-x)^4)}}{b}},$$

$$(\text{type 1} \quad 0<x<1/2) \tag{3·22}$$

$$F(x)=\sqrt{\frac{-a+\sqrt{a^2+b(2ax^2+bx^4+2ax_1{}^2-a/2+bx_1{}^4-b/16)}}{b}},$$

$$(\text{type 2} \quad 0<x<1/2) \tag{3·23}$$

where

$$x_1=x+1/2.$$

All examples stated above possess the property

$$F(1/2-0)=1. \tag{3·24}$$

Therefore, the support for the chaos is the total interval $[0, 1]$. Since it is well known that the logistic map except for $r=4$ shows the rather singular behavior of the invariant density which stands on the part of the total interval, all examples which have been considered may correspond to the case $r=4$. We should note here that the density

$$f(x)=\frac{1}{\pi\sqrt{x(1-x)}}$$

yields the logistic map for $r=4$ for type 1 actually. Although the logistic map does not satisfy the conditions of the theorem, this map is the ergodic transformation, since the conjugation relation for the tent map holds.

We have also calculated the 1D maps which cannot be expressed by elemental functions (expressible in terms of error functions) in the case of the Gaussian density which is written as

$$f(x)=a\exp(-(x-1/2)^2/b), \tag{3·25}$$

where

$$a=\frac{1}{\int_0^1 \exp(-(x-1/2)^2/b)dx}. \tag{3·26}$$

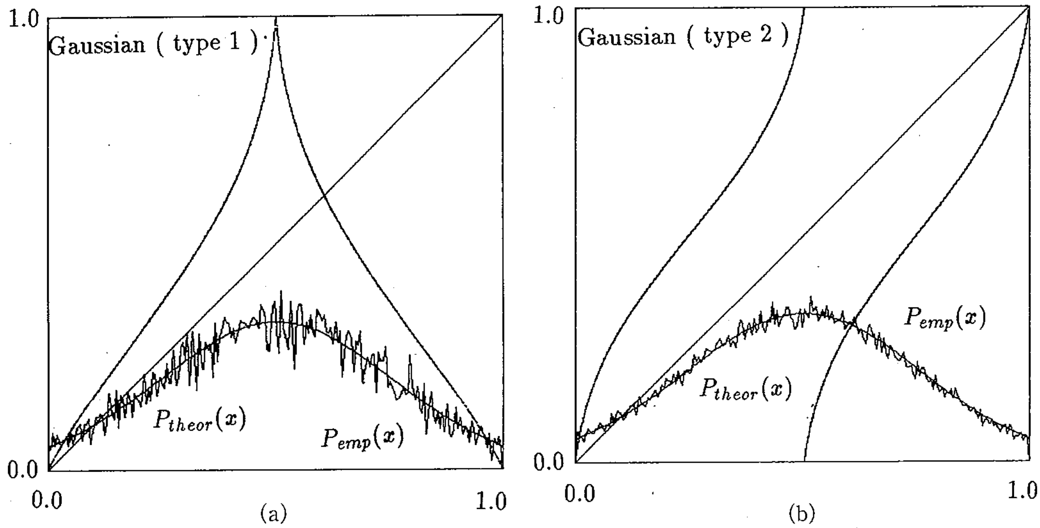

Fig. 5. The behaviors of the 1D map and the empirical density in the case of the Gaussian density for both types for $b=0.13$.

We have employed the Newton method in order to obtain the 1D map by 1000 iterations for each step. The results are shown in Figs. 5 (a) and (b) for type 1 and type 2, respectively. As Fig. 5 shows, the derivative of $F(x)$ is greater than unity in the interval $[0, 1/2]$ for both types, these maps are the ergodic transformations, and the empirical densities and the preassigned densities must coincide with each other. However, the difference is a little large compared with the before-mentioned maps for both types. This is mainly due to the numerical difficulty in the process of computing Eqs. $(2\cdot9)$ and $(2\cdot10)$.

§ 4. Concluding remarks

We have discussed the inverse problem for the Flobenius-Perron equation. Although we restrict the problem to the specific two types of the 1D map, we have obtained several classes of the 1D maps represented by the elemental functions except for the Gaussian distribution. It is, of course, possible to add more 1D maps expressible in terms of elemental functions, because the formulae $(2\cdot9)$ and $(2\cdot10)$ can be integrated for more abundant functions. All examples considered above reveal that they satisfy the conditions of the theorem stated in the Introduction, so that the empirical densities for all examples must be close to the preassigned densities in the sense of the Lebesgue measure.

As we have stated in the Introduction, the *idealization* by means of the 1D map can be carried out by considering Eqs. $(2\cdot9)$ and $(2\cdot10)$. Here we do not take into account the problem of whether the experimental system can be described by the 1D system or not. Accordingly, this 1D map idealization is not appropriate for understanding the underlying physical processes. It is, however, important to mention that considering the 1D map idealization is the trigger to research the inverse problem of finding the governing dynamical system constructed from experimental data.

We also note on the problem of the time correlation. Since the correlation depends on the dynamical law, there exist a variety of the behaviors of the correlations for the single given density. The details of the correlations will be discussed elsewhere.

Another important problem remains to be solved. What is the behavior of the 1D map like for the choice of the singular density which is piecewisely continuous? This problem may be related to the intermittency phenomenon where the 1D map is tangential to the line $y=x$ at a certain point, and must be clarified as the future problem.

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