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Conic Assignment

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Problem Statement:

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$

SOLUTION:

Given:

Equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

Equation of line is

$$\frac{x}{a}+\frac{y}{b}=1 \tag{2}$$

To Find

To find the intersection points and area of shaded region shown in figure

STEP-1

The given ellipse can be expressed as conics with parameters,

$$\mathbf{V}=\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \tag{3}$$

$$\mathbf{u}=0 \tag{4}$$

$$f=-(a^2b^2) \tag{5}$$

STEP-2

the given line equation can be written as

$$\mathbf{x}=\begin{pmatrix} a \\ 0 \end{pmatrix}+k\begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix} \tag{6}$$

STEP-3

The points of intersection of the line,

$$L: \quad \mathbf{x}=\mathbf{q}+\kappa\mathbf{m} \quad \kappa\in\mathbb{R} \tag{7}$$

with the conic section,

$$\mathbf{x}^\top\mathbf{V}\mathbf{x}+2\mathbf{u}^\top\mathbf{x}+f=0 \tag{8}$$

are given by

$$\mathbf{x}_i=\mathbf{q}+\kappa_i\mathbf{m} \tag{9}$$

where,

$$\kappa_i=\frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right. \\ \left.\pm\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2-\left(\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{10}$$

On substituting

$$\mathbf{q}=\begin{pmatrix} a \\ 0 \end{pmatrix} \tag{11}$$

$$\mathbf{m}=\begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix} \tag{12}$$

With the given ellipse as in eq(3),(4),(5),
The value of κ ,

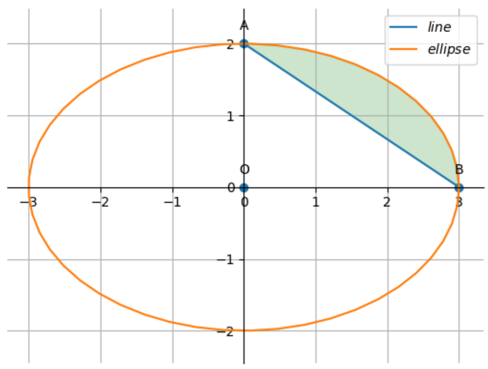
$$\kappa=0,-6 \tag{13}$$

by substituting eq(13) in eq(6)we get the points of intersection of line with ellipse

$$\mathbf{A}=\begin{pmatrix} a \\ 0 \end{pmatrix} \tag{14}$$

$$\mathbf{B}=\begin{pmatrix} 0 \\ b \end{pmatrix} \tag{15}$$

Result



From the figure,

Total area of portion is given by,

$$\implies A = \int_0^a \int_{\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} dydx \tag{16}$$

Where $\frac{b}{a}(a-x)$ is equation of line and $\frac{b}{a}\sqrt{a^2-x^2}$ is the equation of ellipse

$\implies A = \int_0^a \frac{b}{a}(a-x) - \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$$\implies A = \int_0^a \frac{b}{a}(a-x)\,dx - \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx \tag{17}$$

on solving equation(19)

$\implies A = \frac{\pi ab}{4} - \frac{ab}{2}$

Get the python code of the figures from

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https://github.com/kkousar/KOUSAR_FWC/blob/main/circle_Assignment/code/circle.py
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The area of the smaller region is ,

$$A = \frac{ab}{2}(\frac{\pi}{2} - 1) \tag{18}$$

Construction

Points	**coordinates**
B	$\begin{pmatrix} a \\ 0 \end{pmatrix}$
A	$\begin{pmatrix} 0 \\ b \end{pmatrix}$