Project 3: Riemann Sums of a Double Integral

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My objective in this project is to explore double integrals as a limit of Riemann prisms.

$$\iint_R f(x,y)\,\mathrm{dA} = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y \quad (1)$$

In particular, I will be examining a function over a region defined as:

$$f(x,y) = 4 - x^2 + y\ , \qquad R = [-2,2] \times [0,2] \quad (2)$$

1 Approximation with Rectangular Prisms

I would like to use 8 rectangular prisms to approximate the volume of the solid region under $f(x,y)$ over the rectangular region R. From the definition of a double integral, I would like to say that $m = 4$ and $n = 2$, or that I will have 4

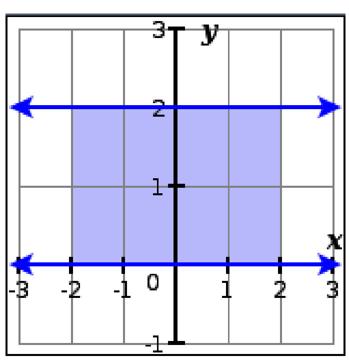

Figure 1: The region R of integration.

y \ x	-1.5	-0.5	0.5	1.5
0.5	2.25	4.25	4.25	2.25
1.5	3.25	5.25	5.25	3.25

Table 1: $f(x, y)$ at sample points.

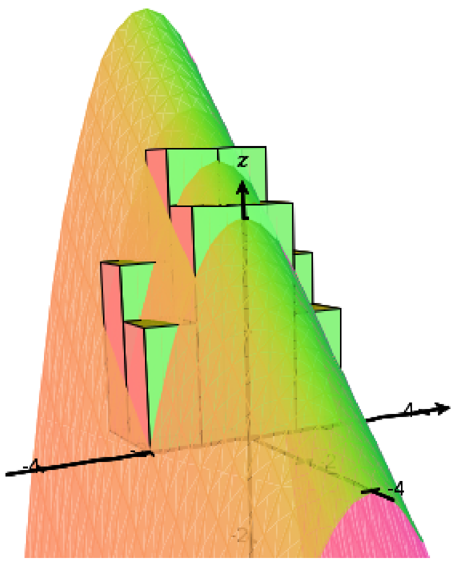

Figure 2: The surface $z = f(x, y)$ with 8 Riemann prisms approximating a volume between f and R.

partitions of my x-interval and 2 partitions of my y-interval to constitute the 8 prisms. The limits of integration over R are $-2 \leq x \leq 2$ and $0 \leq y \leq 2$, so

$$\Delta A = \Delta x \Delta y = (\frac{b-a}{m})(\frac{d-c}{n}) = (\frac{2-(-2)}{4})(\frac{2-0}{2}) = 1 \tag{3}$$

Because $\Delta A = 1$, my approximation will be

$$V \approx \sum_{i=1}^{4} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*) \tag{4}$$

or, in other words, the volume will be approximately equal to the sum of the function evaluated at the sample points in my partitions. Because it is generally more accurate, and there is no need for the precision of, say, Simpson's Rule, I will opt for the Midpoint Rule in choosing my sample points. The sample points and the values for $f(x, y)$ are summarized in Table 1.

Summing these values, $V \approx 30$.

2 Volume as a Double Integral

Because our particular function $f(x,y) \geq 0$ over the region of integration $R = [-2,2] \times [0,2]$, we can interpret the double integral over this region as the volume under the surface $z = f(x,y)$. Therefore,

$$V = \int_{-2}^{2} \int_{0}^{2} (4 - x^2 + y)\,\mathrm{d}y\,\mathrm{d}x = \int_{-2}^{2} \left[4y - x^2 y + \frac{y^2}{2} \right]_{y=0}^{y=2} \mathrm{d}x$$

$$= \int_{-2}^{2} 10 - 2x^2\,\mathrm{d}x = 10x - \frac{2}{3}x^3 \Big|_{-2}^{2} = \left[\Big(20 - \frac{2}{3}(8)\Big) - \Big(-20 - \frac{2}{3}(-8)\Big) \right]$$

$$= 40 - \frac{32}{3} = 29\frac{1}{3}$$

Depending on how accurately we needed to know the volume, the value obtained from sampling midpoints of f compared to the exact value found in the double integral is fairly close.

3 Varying Approximations of Volume

If I were to increase the number of prisms in the y-direction, my approximation to the volume would remain 30. The reason for this has to do with these equations:

$$\frac{\partial f}{\partial x} = -2x \qquad\qquad \frac{\partial f}{\partial y} = 1$$

Because the rate of change of f along the y-axis is 1, a constant, this value does not depend on a particular value of y, and so we are not necessarily losing information about the behavior of f in between sample points in y in the same way we are when $\frac{\partial f}{\partial x} = -2x$.

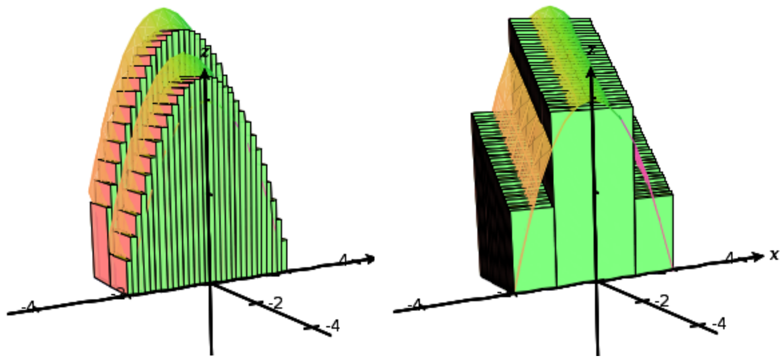

Figure 3: Approximating V when (a) $m = 30$ and (b) $n = 30$.

This is readily apparent when I alternately allow $m = 30$ and $n = 30$ in Figure 3. Note how the abundance of prisms in the y-direction does not appear to fill in the gaps beneath the surface; in comparison with Figure 3, there is just as much empty space below and above f.

This is not the case when there are many prisms in the x-direction, as we can see that the volume of the prisms begins to approach the true volume under f. In fact, when $m = 30$, $n = 2$, $V \approx 29.3452$, which is significantly closer to the actual value $V = 29\frac{1}{3}$.