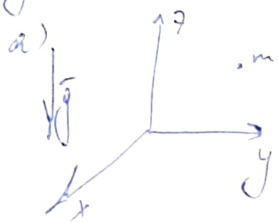
Неделя 12

№24.1 a)



$$q_1 = x,\quad q_2 = y,\quad q_3 = z \Rightarrow H = \frac{p_1^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz.$$

Ур-е Г-Я: $\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right) + mgz = 0$

Будем искать полный интеграл в виде:

$$S(x, y, z, \alpha_x, \alpha_y, \alpha_z, t) = S_0(t, \alpha) + S_1(x, \alpha) + S_y(y, \alpha) + S_z(z, \alpha)$$

$$\Rightarrow \frac{\partial S_0}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial S_x}{\partial x}\right)^2 + \left(\frac{\partial S_y}{\partial y}\right)^2 + \left(\frac{\partial S_z}{\partial z}\right)^2\right) + mgz = 0$$

Функции зависят от своих переменных:

$$\begin{cases} \frac{\partial S_x}{\partial x} = \alpha_x \\ \frac{\partial S_y}{\partial y} = \alpha_y \\ \frac{1}{2m}\left(\frac{\partial S_z}{\partial z}\right)^2 + mgz = \alpha_z \\ \frac{\partial S_0}{\partial t} = -\alpha_z - \frac{1}{2m}(\alpha_x^2 + \alpha_y^2) \end{cases}$$

$S_x = \alpha_x x$

$S_y = \cancel{\alpha_y} \alpha_y y$

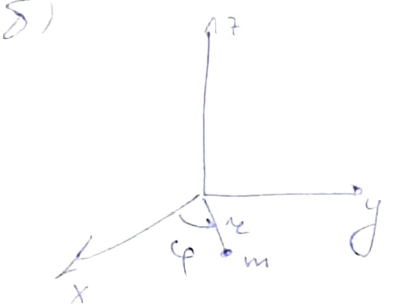
$S_0 = -\left(\alpha_z + \frac{\alpha_x^2}{2m} + \frac{\alpha_y^2}{2m}\right)t$

$S_z = \sqrt{2m}\int\sqrt{\alpha_z - mgz}\,dz = \sqrt{2m}\cdot\left(-\frac{2}{3mg}\right)(\alpha_z - mgz)^{3/2}$

$$S = -\left(\alpha_z + \frac{\alpha_x^2}{2m} + \frac{\alpha_y^2}{2m}\right)t + \alpha_x x + \alpha_y y - \frac{2\sqrt{2m}}{3mg}(\alpha_z - mgz)^{3/2}$$

$$\begin{cases} \beta_x = \frac{\partial S}{\partial \alpha_x} = -\frac{\alpha_x}{m}t + x \\ \beta_y = \frac{\partial S}{\partial \alpha_y} = -\frac{\alpha_y}{m}t + y \\ \beta_z = \frac{\partial S}{\partial \alpha_z} = -t - \frac{\sqrt{2m}}{mg}(\alpha_z - mgz)^{1/2} \end{cases} \Rightarrow \begin{cases} x(t) = \beta_x + \frac{\alpha_x}{m}t \\ y(t) = \beta_y + \frac{\alpha_y}{m}t \\ z(t) = \frac{\alpha_z}{mg} - \frac{g}{2}(\beta_z + t)^2 \end{cases}$$

б)



$$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + \frac{p_z^2}{2m} + mgz$$

Ур-е Гамильтона-Якоби:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + mgz = 0$$

Будем искать полный интеграл в виде:

$$S = S_0(t, \alpha) + S_r(r, \alpha) + S_\varphi(\varphi, \alpha) + S_z(z, \alpha)$$

$$\frac{\partial S_0}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\varphi}{\partial \varphi}\right)^2 + \left(\frac{\partial S_z}{\partial z}\right)^2\right] + mgz = 0.$$

$$\begin{cases} \frac{1}{2m}\left(\frac{\partial S_z}{\partial z}\right)^2 + mgz = \alpha_z \\ \frac{\partial S_\varphi}{\partial \varphi} = \alpha_\varphi \\ \left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{\alpha_\varphi}{r^2} = \alpha_r \\ \frac{\partial S_0}{\partial t} = -\alpha_z - \frac{\alpha_r}{2m} \end{cases} \Rightarrow \begin{cases} S_\varphi = \alpha_\varphi \cdot \varphi \\ S_0 = -\left(\alpha_z + \frac{1}{2m}\alpha_r\right)t \\ S_z = \sqrt{2m}\int\sqrt{\alpha_z - mgz}\,dz = \\ = \frac{2\sqrt{2m}}{3mg}(\alpha_z - mgz)^{3/2} \\ S_r = \int\frac{\sqrt{\alpha_r r^2 - \alpha_\varphi^2}}{r}\,dr \end{cases}$$

$\Rightarrow$ Получили: $S = -\left(\alpha_z + \frac{\alpha_r}{2m}\right)t + \alpha_\varphi \varphi - \frac{2\sqrt{2m}}{3mg}(\alpha_z - mgz)^{3/2} + \int\frac{\sqrt{\alpha_r r^2 - \alpha_\varphi^2}}{r}\,dr$

$$\beta_r = \frac{\partial S}{\partial \alpha_r} = -\frac{t}{2m} + \int\frac{r}{2\sqrt{\alpha_r r^2 - \alpha_\varphi^2}}\,dr = -\frac{t}{2m} + \frac{\sqrt{\alpha_r r^2 - \alpha_\varphi^2}}{2\alpha_r}$$

$$\left(\beta_r + \frac{t}{2m}\right)^2 \cdot 4\alpha_r = r^2 - \frac{\alpha_\varphi^2}{\alpha_r} \Rightarrow r = \sqrt{4\alpha_r\left(\beta_r + \frac{t}{2m}\right)^2 + \frac{\alpha_\varphi^2}{\alpha_r}}$$

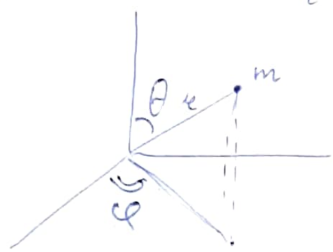
$$\beta_z = \frac{\partial S}{\partial \alpha_z} = \frac{\sqrt{2m}}{mg}(\alpha_z - mgz)^{1/2} \rightarrow z = +\frac{\alpha_z}{mg} - \frac{g}{2}(\beta_z + t)^2$$

$$\beta_\varphi = \frac{\partial S}{\partial \alpha_\varphi} = \varphi + \int\frac{1}{r}\frac{-\alpha_\varphi}{\sqrt{\alpha_r r^2 - \alpha_\varphi^2}}\,dr = \varphi - \alpha_\varphi\int\frac{1}{r^2}\frac{dr}{\sqrt{\alpha_r - \frac{\alpha_\varphi^2}{r^2}}} =$$

$$= \varphi + \alpha_\varphi\int\frac{d\left(\frac{1}{r}\right)}{\sqrt{\alpha_r - \frac{\alpha_\varphi^2}{r^2}}} = \varphi + \int\frac{d\left(\frac{\alpha_\varphi}{r}\right)}{\sqrt{\alpha_r - \left(\frac{\alpha_\varphi}{r}\right)^2}} = \varphi - \arccos\left(\frac{\alpha_\varphi}{r\sqrt{\alpha_r}}\right)$$

$$\cos(\varphi - \beta_\varphi) = \frac{\alpha_\varphi}{r\sqrt{\alpha_r}}$$

№24.9 $\quad \Pi(r) = -\frac{\mu_1}{r} + \frac{\mu_2}{r^2}$ $\qquad \varphi = const$, $\alpha_\varphi = 0$



$$L = \frac{mv^2}{2} - \Pi(r)$$

$$v^2 = v_r^2 + v_\varphi^2 + v_\theta^2$$

$$v_r = \dot r \qquad v_\theta = r\dot\theta \qquad v_\varphi = r\sin\theta\,\dot\varphi$$

$$L = \frac{m}{2}\left[\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\varphi^2\right] - \Pi(r)$$

$$\begin{cases} p_r = \frac{\partial L}{\partial \dot r} = m\dot r \\ p_\theta = \frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta \\ p_\varphi = \frac{\partial L}{\partial \dot\varphi} = mr^2\sin^2\theta\,\dot\varphi \end{cases} \Rightarrow \quad H = \bar p^T \dot{\bar q} - L(\bar q, \dot{\bar q}, t) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} - \frac{\mu_1}{r} + \frac{\mu_2}{r^2}$$

Ур-е Г-Я: $\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] - \frac{\mu_1}{r} + \frac{\mu_2}{r^2} = 0$

Делим переменные: $S = S_0(t,\alpha) + S_r(r,\alpha) + S_\theta(\theta,\alpha) + S_\varphi(\varphi,\alpha)$

$$\frac{\partial S_0}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S_\varphi}{\partial \varphi}\right)^2\right] - \frac{\mu_1}{r} + \frac{\mu_2}{r^2} = 0$$

С учетом того, что $\frac{\partial S_\varphi}{\partial \varphi} = \alpha_\varphi \overset{\text{пусть}}{=} 0 \Rightarrow$

$$\Rightarrow \frac{\partial S_0}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2\right] - \frac{\mu_1}{r} + \frac{\mu_2}{r^2} = 0$$

$$\begin{cases} \frac{\partial S_\theta}{\partial \theta} = \alpha_\theta \\ \frac{1}{2m}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2\right] - \frac{\mu_1}{r} + \frac{\mu_2}{r^2} = \alpha_r \\ \frac{\partial S_0}{\partial t} = -\alpha_r \end{cases} \Rightarrow \quad \begin{aligned} &S_\theta = \alpha_\theta \theta \\ &S_0 = -\alpha_r t \\ &S_r = \int \sqrt{2m\left(\alpha_r + \frac{\mu_1}{r} - \frac{\mu_2}{r^2}\right) - \frac{\alpha_\theta^2}{r^2}}\, dr \end{aligned}$$

$$S = -\alpha_r t + \alpha_\theta\theta + \int \sqrt{2m\left(\alpha_r + \frac{\mu_1}{r} - \frac{\mu_2}{r^2}\right) - \frac{\alpha_\theta^2}{r^2}}\, dr$$

Тогда $\varphi = \beta_0$

$$\beta_r = \frac{\partial S}{\partial \alpha_r} = -t + \int \frac{m}{\sqrt{2m\left(\alpha_r + \frac{\mu_1}{r} - \frac{\mu_2}{r^2}\right) - \frac{\alpha_\theta^2}{r^2}}}\, dr$$

Получили $r(t)$:

$$\beta_\theta = \frac{\partial S}{\partial \alpha_\theta} = \theta + \int \frac{-\alpha_\theta / r^2}{\sqrt{2m\left(\alpha_r + \frac{\mu_1}{r} - \frac{\mu_2}{r^2}\right) - \frac{\alpha_\theta^2}{r^2}}}\, dr = \theta + \alpha_\theta \int \frac{d\left(\frac{1}{r}\right)}{\sqrt{2m\alpha_r + \frac{2m\mu_1}{r} - \frac{2m\mu_2 + \alpha_\theta^2}{r^2}}}$$

$$= \left| \begin{aligned} &u = \tfrac{1}{r} \\ &a = 2m\mu_2 + \alpha_\theta^2 \\ &b = 2m\mu_1 \\ &c = 2m\alpha_r \end{aligned} \right| = \theta + \alpha_\theta \int \frac{du}{\sqrt{c + bu - au^2}} = \theta + \alpha_\theta \int \frac{du}{\sqrt{c - a\left(u^2 - \frac{b}{a}u + \frac{b^2}{4a^2}\right) + \frac{b^2}{4a}}} =$$

$$= \theta + \alpha_\theta \int \frac{du}{\sqrt{c+\frac{b^2}{4a} - a\left(u-\frac{b}{2a}\right)^2}} = \theta + \frac{\alpha_\theta}{\sqrt a}\int \frac{d\left(u-\frac{b}{2a}\right)}{\sqrt{\frac{b^2+4ac}{4a^2} - \left(u-\frac{b}{2a}\right)^2}} =$$

$$= \theta - \frac{\alpha_\theta}{\sqrt a}\arccos\left[\frac{u - b/2a}{\frac{\sqrt{b^2+4ac}}{2a}}\right]$$

$$\frac{u\cdot 2a - b}{\sqrt{b^2+4ac}} = \cos\left(\frac{\sqrt a}{\alpha_\theta}(\theta-\beta_\theta)\right)$$

$$\frac{1}{r} = u = \frac{b}{2a} + \frac{\sqrt{b^2+4ac}}{2a}\cos\left(\frac{\sqrt a}{\alpha_\theta}(\theta-\beta_\theta)\right)$$

$$r(\theta) = \frac{2a/b}{1+\sqrt{1+\frac{4ac}{b^2}}\cos\left(\frac{\sqrt a}{\alpha_\theta}(\theta-\beta_\theta)\right)}$$

Переобозначим:

$$p = \frac{2a}{b} = \frac{2m\mu_2+\alpha_\theta^2}{m\mu_1}$$

и $e = \sqrt{1+\frac{4ac}{b^2}} = \sqrt{1+2m\alpha_r\frac{m\mu_2+\alpha_\theta^2}{(m\mu_1)^2}}$

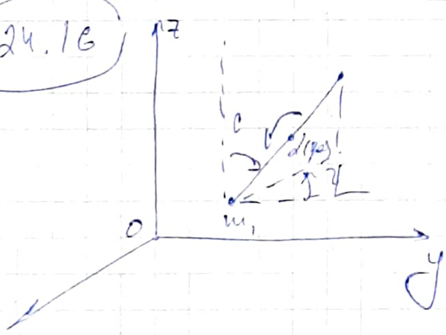
$$\omega = \frac{\sqrt a}{\alpha_\theta} = \sqrt{1+\frac{2m\mu_2}{\alpha_\theta^2}} \qquad \theta_0 = \beta_\theta$$

$$r(\theta) = \frac{p}{1+e\cos(\omega(\theta-\theta_0))}$$

$$\beta_r + \frac{t}{m} = \int \frac{dr}{\sqrt{2m\left(\alpha_r + \frac{\mu_1}{r} - \frac{\mu_2}{r^2}\right) - \frac{\alpha_\theta^2}{r^2}}}$$

Из этого можно выразить $r(t)$

№24.16



$\mu = \frac{m_1 m_2}{m_1+m_2}$ — приведённая масса

$$H = \frac{1}{2(m_1+m_2)}\left(p_x^2+p_y^2+p_z^2\right) + \frac{1}{2\mu}\left[p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\varphi^2\right] - \gamma\frac{m_1m_2}{r}$$

Ур-е Г-Я:

$$\frac{\partial S}{\partial t} + \frac{1}{2(m_1+m_2)}\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2\right] + \frac{1}{2\mu}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] - \gamma\frac{m_1m_2}{r} = 0.$$

Разделяем переменные:

$$S = S_0(t,\alpha) + S_x(x,\alpha) + S_y(y,\alpha) + S_z(z,\alpha) + S_r(r,\alpha) + S_\theta(\theta,\alpha) + S_\varphi(\varphi,\alpha)$$

$$\frac{\partial S_0}{\partial t} + \frac{1}{2(m_1+m_2)}\left[\left(\frac{\partial S_x}{\partial x}\right)^2 + \left(\frac{\partial S_y}{\partial y}\right)^2 + \left(\frac{\partial S_z}{\partial z}\right)^2\right] + \frac{1}{2\mu}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S_\varphi}{\partial \varphi}\right)^2\right] - \gamma\frac{m_1 m_2}{r} = 0.$$

$$\begin{cases} \dfrac{\partial S_x}{\partial x} = \alpha_x \\ \dfrac{\partial S_y}{\partial y} = \alpha_y \\ \dfrac{\partial S_z}{\partial z} = \alpha_z \\ \dfrac{\partial S_\varphi}{\partial \varphi} = \alpha_\varphi \\ \left(\dfrac{\partial S_\theta}{\partial \theta}\right)^2 + \dfrac{\alpha_\varphi^2}{\sin^2\theta} = \alpha_\theta \\ \dfrac{1}{2\mu}\left[\left(\dfrac{\partial S_r}{\partial r}\right)^2 + \dfrac{\alpha_\theta}{r^2}\right] - \gamma\dfrac{m_1 m_2}{r} = \alpha_r \\ \dfrac{\partial S_0}{\partial t} = -\dfrac{\alpha_x^2+\alpha_y^2+\alpha_z^2}{2(m_1+m_2)} - \alpha_r \end{cases} \Rightarrow \begin{cases} S_x = \alpha_x x \\ S_y = \alpha_y y \\ S_z = \alpha_z z \\ S_\varphi = \alpha_\varphi \varphi \\ S_\theta = \displaystyle\int\sqrt{\alpha_\theta - \frac{\alpha_\varphi^2}{\sin^2\theta}}\,d\theta \\ S_r = \displaystyle\int\sqrt{2\mu\alpha_r + \frac{2\mu\gamma m_1 m_2}{r} - \frac{\alpha_\theta}{r^2}}\,dr \\ S_0 = -\left(\dfrac{\alpha_x^2+\alpha_y^2+\alpha_z^2}{2(m_1+m_2)} + \alpha_r\right)t \end{cases}$$

$$\Rightarrow S = -\left(\frac{\alpha_x^2+\alpha_y^2+\alpha_z^2}{2(m_1+m_2)} + \alpha_r\right)t + \alpha_x x + \alpha_y y + \alpha_z z + \alpha_\varphi \varphi + \int\sqrt{\alpha_\theta - \frac{\alpha_\varphi^2}{\sin^2\theta}}\,d\theta + \int\sqrt{2\mu\alpha_r + \frac{2\mu\gamma m_1 m_2}{r} - \frac{\alpha_\theta}{r^2}}\,dr$$

$$\beta_x = \frac{\partial S}{\partial \alpha_x} = -\frac{\alpha_x}{m_1+m_2}t + x$$

$$\beta_y = \frac{\partial S}{\partial \alpha_y} = -\frac{\alpha_y}{m_1+m_2}t + y$$

$$\beta_z = \frac{\partial S}{\partial \alpha_z} = -\frac{\alpha_z}{m_1+m_2}t + z$$

$$\beta_r = \frac{\partial S}{\partial \alpha_r} = -t + \int\frac{\mu\,dr}{\sqrt{2\mu\alpha_r + \frac{2\mu\gamma m_1 m_2}{r} - \frac{\alpha_\theta}{r^2}}}$$

$$\beta_\theta = \frac{\partial S}{\partial \alpha_\theta} = \int\frac{d\theta}{2\sqrt{\alpha_\theta - \frac{\alpha_\varphi^2}{\sin^2\theta}}} - \frac{1}{2}\int\frac{dr}{r^2\sqrt{2\mu\alpha_r + \frac{2\mu\gamma m_1 m_2}{r} - \frac{\alpha_\theta}{r^2}}}$$

$$\beta_\varphi = \frac{\partial S}{\partial \alpha_\varphi} = \varphi - \int\frac{\alpha_\varphi\,d\theta}{\sin^2\theta\sqrt{\alpha_\theta - \alpha_\varphi^2/\sin^2\theta}}$$

№2.433) $H = p_1 q_2 + p_2 q_1$

преобразование: $\begin{cases} q_1 = \hat{q}_1 + \hat{q}_2 \\ q_2 = \hat{p}_1 - \hat{p}_2 \\ p_1 = \hat{p}_1 + \hat{p}_2 \\ p_2 = \hat{q}_2 - \hat{q}_1 \end{cases} \Rightarrow \begin{cases} \hat{p}_1 = \frac{1}{2}(p_1 + q_2) \\ \hat{p}_2 = \frac{1}{2}(p_1 - q_2) \\ \hat{q}_1 = \frac{1}{2}(q_1 - p_2) \\ \hat{q}_2 = \frac{1}{2}(q_1 + p_2) \end{cases}$

Докажем каноничность преобразования:

$$\hat{p}_1 \delta \hat{q}_1 + \hat{p}_2 \delta \hat{q}_2 = c p_1 \delta q_1 + c p_2 \delta q_2 - \delta F$$

$$\frac{1}{4}(p_1 + q_2)(\delta q_1 - \delta p_2) + \frac{1}{4}(p_1 - q_2)(\delta q_1 + \delta p_2) = c p_1 \delta q_1 + c p_2 \delta q_2 - \delta F$$

$\begin{cases} \delta q_1: & \frac{p_1}{2} = c p_1 - \frac{\partial F}{\partial q_1} \\ \delta q_2: & 0 = c p_2 - \frac{\partial F}{\partial q_2} \\ \delta p_1: & 0 = -\frac{\partial F}{\partial p_1} \\ \delta p_2: & -\frac{q_2}{2} = -\frac{\partial F}{\partial p_2} \end{cases} \Rightarrow \begin{cases} \frac{\partial F}{\partial q_1} = (c - \frac{1}{2}) p_1 \\ \frac{\partial F}{\partial q_2} = c p_2 \\ \frac{\partial F}{\partial p_1} = 0 \\ \frac{\partial F}{\partial p_2} = \frac{q_2}{2} \end{cases}$ $\Rightarrow$ при $c = \frac{1}{2}$ $\exists F = \frac{p_2 q_2}{2}$, удовл. теор. $\Rightarrow$ преоб. каноническое.

$$\Rightarrow \hat{H} = \frac{1}{2} H = \frac{1}{2}\left[\hat{p}_1^2 - \hat{p}_2^2 + \hat{q}_2^2 - \hat{q}_1^2\right]$$

Ур-е Г-Я: $\frac{\partial S}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial S}{\partial \hat{q}_1}\right)^2 - \left(\frac{\partial S}{\partial \hat{q}_2}\right)^2 + \hat{q}_2^2 - \hat{q}_1^2\right] = 0$

Делим переменные: $\hat{S} = \hat{S}_0(t, \alpha) + \hat{S}_1(\hat{q}_1, \alpha) + \hat{S}_2(\hat{q}_2, \alpha)$

$$\frac{\partial \hat{S}_0}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \hat{S}_1}{\partial \hat{q}_1}\right)^2 - \hat{q}_1^2\right] - \frac{1}{2}\left[\left(\frac{\partial S_2}{\partial \hat{q}_2}\right)^2 - \hat{q}_2^2\right] = 0$$

$\begin{cases} \frac{1}{2}\left[\left(\frac{\partial \hat{S}_1}{\partial \hat{q}_1}\right)^2 - \hat{q}_1^2\right] = \alpha_1 \\ \frac{1}{2}\left[\left(\frac{\partial \hat{S}_2}{\partial \hat{q}_2}\right)^2 - \hat{q}_2^2\right] = \alpha_2 \\ \frac{\partial \hat{S}_0}{\partial t} = \alpha_2 - \alpha_1 \end{cases} \Rightarrow \begin{cases} \hat{S}_1 = \int \sqrt{2\alpha_1 + \hat{q}_1^2}\, d\hat{q}_1 \\ \hat{S}_2 = \int \sqrt{2\alpha_2 + \hat{q}_2^2}\, d\hat{q}_2 \\ \hat{S}_0 = (\alpha_2 - \alpha_1) t \end{cases}$

$$\Rightarrow \hat{S} = (\alpha_2 - \alpha_1) t + \int \sqrt{2\alpha_1 + \hat{q}_1^2}\, d\hat{q}_1 + \int \sqrt{2\alpha_2 + \hat{q}_2^2}\, d\hat{q}_2$$

$$\Rightarrow \beta_1 = \frac{\partial \hat{S}}{\partial \alpha_1} = -t + \int \frac{d\hat{q}_1}{\sqrt{2\alpha_1 + \hat{q}_1^2}} = -t + sh^{-1}\left(\frac{\hat{q}_1}{\sqrt{2\alpha_1}}\right)$$

$$\hat{q}_1 = c_1 sh(\beta_1 + t)$$

$$\beta_2 = -\frac{\partial S}{\partial \alpha_2} = -t - \int \frac{d\hat{q}_2}{\sqrt{2\alpha_2 + \hat{q}_2^2}} = -t - sh^{-1}\left(\frac{\hat{q}_2}{\sqrt{2\alpha_2}}\right)$$

$$\hat{q}_2 = c_2 sh(\beta_2 + t)$$

$$\tilde{H} = \frac{1}{2}\left(\tilde{p}_1^2 - \tilde{p}_2^2 + \tilde{q}_2^2 - \tilde{q}_1^2\right)$$

$$\begin{cases} \dot{\tilde{q}}_1 = \frac{\partial \tilde{H}}{\partial \tilde{p}_1} = \tilde{p}_1 \\ \dot{\tilde{q}}_2 = \frac{\partial \tilde{H}}{\partial \tilde{p}_2} = -\tilde{p}_2 \end{cases} \Rightarrow \begin{cases} \tilde{p}_1 = c_1 \operatorname{ch}(\beta_1 + t) \\ \tilde{p}_2 = -c_2 \operatorname{ch}(\beta_2 + t) \\ \tilde{q}_1 = c_1 \operatorname{sh}(\beta_1 + t) \\ \tilde{q}_2 = c_2 \operatorname{sh}(\beta_2 + t) \end{cases}$$

Обратный переход:

$$\begin{cases} q_1 = c_1 \operatorname{sh}(\beta_1 + t) + c_2 \operatorname{sh}(\beta_2 + t) \\ q_2 = c_1 \operatorname{ch}(\beta_1 + t) + c_2 \operatorname{ch}(\beta_2 + t) \\ p_1 = c_1 \operatorname{ch}(\beta_1 + t) - c_2 \operatorname{ch}(\beta_2 + t) \\ p_2 = -c_1 \operatorname{sh}(\beta_1 + t) + c_2 \operatorname{sh}(\beta_2 + t) \end{cases}$$

№24.42 $H = p_1^2 + q_1^2 + \frac{p_2^2 + p_3^2}{q_2^2 + q_3^2}$

Ур-е Г-Я: $\frac{\partial S}{\partial t} + \left(\frac{\partial S}{\partial q_1}\right)^2 + q_1^2 + \frac{\left(\frac{\partial S}{\partial q_2}\right)^2 + \left(\frac{\partial S}{\partial q_3}\right)^2}{q_2^2 + q_3^2} = 0.$

Разделим переменные:

$$S = S_0(t, \alpha) + S_1(q_1, \alpha) + S_2(q_2, \alpha) + S_3(q_3, \alpha)$$

$$\frac{\partial S_0}{\partial t} + \left(\frac{\partial S_1}{\partial q_1}\right)^2 + q_1^2 + \frac{\left(\frac{\partial S_2}{\partial q_2}\right)^2 + \left(\frac{\partial S_3}{\partial q_3}\right)^2}{q_2^2 + q_3^2} = 0$$

$$\begin{cases} \frac{\partial S_0}{\partial t} = -h \\ \left(\frac{\partial S_1}{\partial q_1}\right)^2 + q_1^2 = \alpha_1 \\ \left(\frac{\partial S_2}{\partial q_2}\right)^2 - (h - \alpha_1) q_2^2 + \left(\frac{\partial S_3}{\partial q_3}\right)^2 - (h - \alpha_1) q_3^2 = 0. \end{cases}$$

$$\begin{cases} S_0 = -ht \\ S_1 = \int \sqrt{\alpha_1 - q_1^2}\, dq_1 \\ \left(\frac{\partial S_2}{\partial q_2}\right)^2 - (h - \alpha_1) q_2^2 = \alpha_{23} \\ \left(\frac{\partial S_3}{\partial q_3}\right)^2 - (h - \alpha_1) q_3^2 = -\alpha_{23} \end{cases} \Rightarrow \begin{cases} S_0 = -ht \\ S_1 = \int \sqrt{\alpha_1 - q_1^2}\, dq_1 \\ S_2 = \int \sqrt{(h - \alpha_1) q_2^2 + \alpha_{23}}\, dq_2 \\ S_3 = \int \sqrt{(h - \alpha_1) q_2^2 - \alpha_{23}}\, dq_3 \end{cases}$$

$$S = -ht + \int \sqrt{\alpha_1 - q_1^2}\, dq_1 + \int \sqrt{(h - \alpha_1) q_2^2 + \alpha_{23}}\, dq_2 + \int \sqrt{(h - \alpha_1) q_3^2 - \alpha_{23}}\, dq_3$$

$$\Rightarrow 2\beta_1 = 2\frac{\partial S}{\partial \alpha_1} = \int \frac{dq_1}{\sqrt{\alpha_1 - q_1^2}} - \int \frac{q_2^2\, dq_2}{\sqrt{(h-\alpha_1)q_2^2 + \alpha_{23}}} - \int \frac{q_3^2\, dq_3}{\sqrt{(h-\alpha_1)q_3^2 - \alpha_{23}}}$$

$$2\beta_{23} = 2\frac{\partial S}{\partial \alpha_{23}} = \int \frac{dq_2}{\sqrt{(h-\alpha_1)q_2^2 + \alpha_{23}}} - \int \frac{dq_3}{\sqrt{(h-\alpha_1)q_3^2 - \alpha_{23}}}$$

$$2(\beta + t) = \int \frac{q_2^2\, dq_2}{\sqrt{(h-\alpha_1)q_2^2 + \alpha_{23}}} + \int \frac{q_3^2\, dq_3}{\sqrt{(h-\alpha_1)q_3^2 - \alpha_{23}}}$$