Неделя 8

№20.1 $\bar{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ $\bar{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\bar{M} = \bar{r} \times \bar{p} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} x_2 p_3 - x_3 p_2 \\ x_3 p_1 - x_1 p_3 \\ x_1 p_2 - x_2 p_1 \end{pmatrix} = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}$$

$$(M_i, p_j) = \frac{\partial M_i}{\partial q^T} \cdot \frac{\partial p_j}{\partial p} - \frac{\partial M_i}{\partial p^T} \cdot \frac{\partial p_j}{\partial q} = \left(\frac{\partial M_i}{\partial x_1}, \frac{\partial M_i}{\partial x_2}; \frac{\partial M_i}{\partial x_3}\right) \begin{pmatrix} \frac{\partial p_j}{\partial p_1} \\ \frac{\partial p_j}{\partial p_2} \\ \frac{\partial p_j}{\partial p_3} \end{pmatrix} -$$

$$- \left(\frac{\partial M_i}{\partial p_1}, \frac{\partial M_i}{\partial p_2}, \frac{\partial M_i}{\partial p_2}\right) \begin{pmatrix} \frac{\partial p_j}{\partial x_1} \\ \frac{\partial p_j}{\partial x_2} \\ \frac{\partial p_j}{\partial x_3} \end{pmatrix}$$

$\frac{\partial p_j}{\partial q} = 0$. т.к.

$$\Rightarrow (M_1, p_1) = (0; p_3; -p_2) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - (0; -x_3; x_2) \cdot (0) = 0$$

$$(M_1, p_2) = (0; p_3, -p_2) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = p_3$$

$$(M_1, p_3) = (0, p_3, -p_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -p_2$$

$$(M_2, p_1) = (-p_3; 0; p_1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -p_3$$

$$(M_2, p_2) = (-p_3; 0; p_1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$(M_2, p_3) = (-p_3, 0; p_1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = p_1$$

$$(M_3; p_1) = (p_2, -p_1, 0) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = p_2$$

$$(M_3; p_2) = (p_2, -p_1, 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -p_1$$

$$(M_3; p_3) = (p_2, -p_1, 0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$(M_i, M_j) = \frac{\partial M_i}{\partial q^T} \cdot \frac{\partial M_j}{\partial p} - \frac{\partial M_i}{\partial p^T} \cdot \frac{\partial M_j}{\partial q} = \left(\frac{\partial M_i}{\partial x_n}\right) \left(\frac{\partial M_j}{\partial p_k}\right) - \left(\frac{\partial M_i}{\partial p_n}\right) \left(\frac{\partial M_j}{\partial x_k}\right)$$

для $i = j$: $(M_i, M_j) = 0. \Rightarrow (M_1; M_1) = (M_2, M_2) = (M_3; M_3) = 0.$

$$(M_1, M_2) = (0; p_3; -p_2) \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} - (0; -x_3, x_2) \begin{pmatrix} -p_3 \\ 0 \\ p_1 \end{pmatrix} = x_1 p_2 - x_2 p_1 = M_3$$

$$(M_1, M_3) = (0; p_3, -p_2) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} - (0, -x_3, x_2) \begin{pmatrix} p_2 \\ -p_1 \\ 0 \end{pmatrix} = x_1 p_3 - x_3 p_1 = -M_2$$

$$(M_2, M_1) = -M_3; \quad (M_2; M_3) = M_1; \quad (M_3; M_1) = M_2; \quad (M_3, M_2) = -M_1$$

$$(H^2, H_j) = \frac{\partial(H^2)}{\partial q^T}\cdot\frac{\partial H_j}{\partial p} - \frac{\partial(H^2)}{\partial p^T}\cdot\frac{\partial H_j}{\partial q} = \frac{\partial(H_1^2+H_2^2+H_3^2)}{\partial q^T}\cdot\frac{\partial H_j}{\partial p} - \frac{\partial(\Sigma V)}{\partial p^T}\cdot\frac{\partial H_j}{\partial q} =$$

$$= 2H_1(H_1, H_j) + 2H_2(H_2; H_j) + 2H_3(H_3; H_j)$$

$$(H^2, H_1) = 2H_1(H_1, H_1) + 2H_2(H_2 H_1) + 2H_3(H_3, H_1) = 0 + 2(-H_3H_2 + H_3H_2) = 0$$

$(H^2, H_2) = 0$ - аналогично. и $(H^2, H_3) = 0$

$$(x_i, H_j) = \left(\frac{\partial x_i}{\partial x_1}, \frac{\partial x_i}{\partial x_2}, \frac{\partial x_i}{\partial x_3}\right)\begin{pmatrix}\frac{\partial H_j}{\partial p_1}\\ \frac{\partial H_j}{\partial p_2}\\ \frac{\partial H_j}{\partial p_3}\end{pmatrix} - \overbrace{\left(\frac{\partial x_i}{\partial p_1}, \frac{\partial x_i}{\partial p_2}, \frac{\partial x_i}{\partial p_3}\right)}^{\to 0}\cdot\left(\ \right) = \frac{\partial H_j}{\partial p_i}$$

При $i=j$ $(x_i, H_j) = 0$.

$\Rightarrow (x_1, H_2) = x_3$

$(x_1, H_3) = -x_2$

$(x_2, H_1) = -x_3$ $\quad$ $(x_3, H_1) = x_2$

$(x_2, H_3) = x_1$ $\quad$ $(x_3, H_2) = -x_1$

№20.3 $\varphi = \varphi(q^2+p^2)$ $\quad$ $\psi = \operatorname{arctg}\left(\frac{p}{q}\right)$

$$(\varphi, \psi) = \frac{\partial\varphi}{\partial q^T}\cdot\frac{\partial\psi}{\partial p} - \frac{\partial\varphi}{\partial p^T}\cdot\frac{\partial\psi}{\partial q} = (\varphi'(q^2+p^2)\cdot 2q)\frac{1}{1+\frac{p^2}{q^2}}\cdot\frac{1}{q} -$$

$$-(\varphi'(p^2+q^2)\cdot 2p)\frac{1}{1+\frac{p^2}{q^2}}\left(-\frac{p}{q^2}\right) = \varphi'\left(\frac{2}{1+(p/q)^2} + \frac{2p}{1+(p/q)^2}\frac{p}{q^2}\right) =$$

$$= 2\varphi'\left(\frac{1+(p/q)^2}{1+(p/q)^2}\right) = 2\varphi'.$$

№20.17 $\begin{cases}\varphi_1 = p_1^2 + q_2^2\\ \varphi_2 = p_2^2 + q_1^2\\ \varphi_3\ (\varphi_1, \varphi_2)\end{cases}$ $\quad$ $H = p_1p_2 + q_1q_2$

$\boxed{\frac{\partial\varphi}{\partial t} + (\varphi, H) = 0}$ - необх. усл. для I инт.

для $\varphi_1, \varphi_2, \varphi_3$ $\frac{\partial\varphi}{\partial t} = 0 \Rightarrow$ проверим скобки Пуассона

$$(\varphi_1, H) = \frac{\partial\varphi_1}{\partial q^T}\frac{\partial H}{\partial p} - \frac{\partial\varphi_1}{\partial p^T}\frac{\partial H}{\partial q} -$$

$$= (0, 2q_2)\begin{pmatrix}p_2\\ p_1\end{pmatrix} - (2p_1, 0)\begin{pmatrix}q_2\\ q_1\end{pmatrix} = 2q_2p_1 - 2p_1q_2 = 0 \ \text{и} \to \text{да}.$$

$$(\varphi_2, H) = (2q_1, 0)\begin{pmatrix}p_2\\ p_1\end{pmatrix} - (0, 2p_2)\begin{pmatrix}q_2\\ q_1\end{pmatrix} = 2q_1p_2 - 2q_1p_2 = 0 \ \text{и} \to \text{да}$$

$$\varphi_3 = (\varphi_1, \varphi_2) = \frac{\partial\varphi_1}{\partial q^T}\frac{\partial\varphi_2}{\partial p} - \frac{\partial\varphi_1}{\partial p^T}\frac{\partial\varphi_2}{\partial q} = (0, 2q_2)\begin{pmatrix}0\\ 2p_2\end{pmatrix} - (2p_1, 0)\begin{pmatrix}2q_1\\ 0\end{pmatrix} =$$

$$= 4q_2p_2 - 4q_1p_1$$

$\Rightarrow (\varphi_3, H) = (-4p_1, 4p_2)\begin{pmatrix} p_2 \\ p_1 \end{pmatrix} - (4q_1, 4q_2)\begin{pmatrix} q_2 \\ q_1 \end{pmatrix} =$

$= -4p_1p_2 + 4p_1p_2 - (-4q_1q_2 + 4q_1q_2) = 0$ // → дз

$\Rightarrow$ скобки Пуассона от ~~перв~~. интегр. сами явл. перв. инт. по т. Якоби-Пуассона

№20.27 $\quad \varphi_i(q_i, p_i) = \alpha_i \qquad i = \overline{1,n}$

$$H = f(t)\,\frac{\sum\limits_{i=1}^{n} \gamma_i \varphi_i(q_i, p_i)}{\sum\limits_{i=1}^{n} \delta_i \varphi_i(q_i, p_i)} \qquad \gamma_i, \delta_i - \text{пост.}$$

Нужно показать, что φ_i – перв. инт. φ_i явно не зав. от t

$\Rightarrow \dfrac{\partial \varphi_i}{\partial t} = 0 \quad \forall i$

Обозначим $\quad \Sigma_I = \sum\limits_{i=1}^{n} \gamma_i \varphi_i(q_i, p_i)$; $\Sigma_{II} = \sum\limits_{i=1}^{n} \delta_i \varphi_i(q_i, p_i)$

$\dfrac{\partial \varphi^i}{\partial q_i} = \varphi^i_{,q} \qquad \dfrac{\partial \varphi^i}{\partial p_i} = \varphi^i_{,p}$

$$(\varphi^i, H) = \frac{\partial H}{\partial q^T}\frac{\partial \varphi^i}{\partial p} - \frac{\partial H}{\partial p^T}\frac{\partial \varphi^i}{\partial q} = \frac{\partial H}{\partial q_i}\cdot \varphi^i_{,p} - \frac{\partial H}{\partial p^T}\varphi^i_{,q} =$$

$$= f(t)\,\frac{\gamma_i \varphi^i_{,p}\Sigma_{II} - \delta_i \varphi^i_{,q}\cdot\Sigma_I}{(\Sigma_{II})^2}\cdot\varphi^i_{,p} - f(t)\,\frac{\gamma_i\varphi^i_{,p}\cdot\Sigma_{II} - \delta_i\varphi^i_{,p}\Sigma_I}{(\Sigma_{II})^2}\cdot\varphi^i_{,q} =$$

$$= \frac{f(t)}{(\Sigma_{II})^2}\varphi^i_{,p}\varphi^i_{,q}\left(\gamma_i\Sigma_{II} - \delta_i\Sigma_I + \delta_i\Sigma_I - \gamma_i\Sigma_{II}\right) = 0$$

$\Rightarrow \varphi^i$ – первый инт. для $\forall i = \overline{1,n}$

№20.30 $\quad \left.\begin{matrix} q(q_0, p_0, t) \\ p(q_0, p_0, t) \end{matrix}\right\} \to$ решение системы с гамильтонианом H_1

$(H_1, H_2) = 0$

$\dfrac{\partial H_1}{\partial p}\dfrac{\partial H_2}{\partial q} - \dfrac{\partial H_1}{\partial q}\dfrac{\partial H_2}{\partial p} = 0 \quad \Rightarrow H_1$ и H_2 – зависимые ф-ии

$f(H_1, H_2) = f(H_1, H_2(H_1)) = H_3(H_1)$

$$\begin{cases} \dot q_3 = \dfrac{\partial H_3}{\partial p} = \dfrac{\partial f}{\partial H_1}\cdot\dfrac{\partial H_1}{\partial p} = \dfrac{\partial f}{\partial H_1}\,\dot q(q_0, p_0, t) \\[2ex] \dot p_3 = -\dfrac{\partial H_3}{\partial q} = -\dfrac{\partial f}{\partial H_1}\dfrac{\partial H_1}{\partial q} = \dfrac{\partial f}{\partial H_1}\,\dot p(q_0, p_0, t) \end{cases}$$

H_1 и H_2 – первые интегралы первой сист. $((H_1, H_1) = (H_1, H_2) = 0) \Rightarrow$

$$\Rightarrow \frac{\partial f}{\partial u_1} = \frac{\partial f(u_1, u_2)}{\partial u_1} = const \Rightarrow \begin{cases} q_3 = q(p_0, q_0, \frac{\partial f}{\partial u_1}, t) \\ p_3 = p(p_0, q_0, \frac{\partial f}{\partial u_1}, t) \end{cases}$$

№ 20.36 $q_i = r\varphi_i$

Лагранжиан: $T = \sum\limits_{i=1}^{n} \frac{m_i}{2}(\dot{q}_i)^2$; $\Pi = \sum\limits_{i=1}^{n-1} \frac{c_i}{2}(q_{i+1} - q_i)^2 + \frac{c_n}{2}(q_n - q_1)^2$

$$L = \sum_{i=1}^{n} \frac{m_i}{2}(\dot{q}_i)^2 - \sum_{i=1}^{n-1} \frac{c_i}{2}(q_{i+1} - q_i)^2 - \frac{c_n}{2}(q_n - q_1)^2$$

Преобразование $\begin{cases} \hat{q}_i = q_i + \alpha \\ \hat{t} = t \end{cases}$ $i = \overline{1, n}$

$\dot{\hat{q}}_i = \dot{q}_i$

$$\hat{L} = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{\hat{q}}_i)^2 - \frac{1}{2}\sum_{i=1}^{n-1} c_i(\hat{q}_{i+1} - \hat{q}_i)^2 = L \quad \text{ч.т.д.}$$

Проверим, ~~проверяю~~ выполняются ли условие теоремы Эмми Нётер:

1) Обратное преобр.: $\begin{cases} q_i = \hat{q}_i - \alpha \\ t = \hat{t} \end{cases}$, $i = \overline{1, n}$

2) Дифференцируемость по α очевидна

3) Тождественно при $\alpha = 0$.

$\begin{cases} q_i = \hat{q}_i \\ t = \hat{t} \end{cases} \to$ тожд.

4) dW - инвар. к преобраз.

$$d\hat{W} = \hat{L}\, d\hat{t} = L\, dt = dW$$

Условие выполнены => величина Φ - перв. инт.

$$\Phi = p^T \frac{\partial q}{\partial \alpha}\Big|_{\alpha=0} - H(q, p, t)\cancel{\frac{\partial \hat{t}}{\partial \alpha}}\Big|_{\alpha=0} = (m_1\dot{q}_1; m_2\dot{q}_2, \dots, m_n\dot{q}_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} =$$

$$= \sum_{i=1}^{n} m_i \dot{q}_i = const \to \text{ч.т.д.}$$