**№16.4** $\omega_1, \omega_2, .. \omega_{n-1}$ $\qquad \omega_n - ?$

$\det A$

$\det C$

Вековое уравнение: $\det(C - \lambda A) = 0$, где с/з $\lambda_i = \omega_i^2$

$A$ - ортогональна ($AA^T = E$) $\Downarrow$ сущ. обратная = трансп. $\Rightarrow$ $\det A^T \cdot \det(C - \lambda A) = 0$

$\Leftrightarrow \det(A^T(C - \lambda A)) = 0$

$\det(CA^T - \lambda E) = 0$, где $\lambda$ - все те же собств. знач.

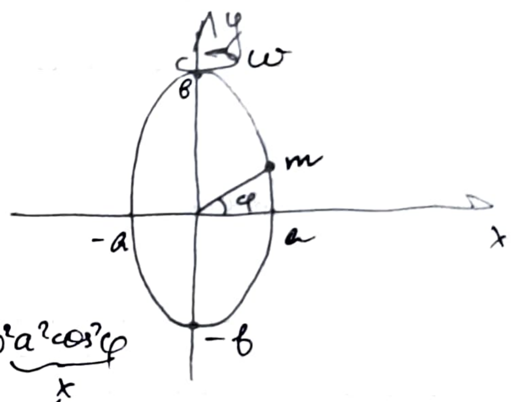
$\Rightarrow \det(CA^T) = \lambda_1 \lambda_2 \cdots \lambda_n = \omega_1^2 \omega_2^2 \cdots \omega_n^2 = \dfrac{\det C}{\det A}$

$$\Rightarrow \boxed{\omega_n = \frac{1}{\prod_{i=1}^{n-1} \omega_i} \sqrt{\frac{\det C}{\det A}}} \to \text{отв.}$$

**№16.16**

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$m, \omega$




Наша координата $\varphi$

$\Rightarrow \begin{cases} x = a\cos\varphi \\ y = b\sin\varphi \end{cases}$

$\Pi = \Pi(\varphi) = mg\underbrace{b\sin\varphi}_{y} - \dfrac{m}{2}\omega^2 \underbrace{a^2\cos^2\varphi}_{x}$

$T = T(\varphi) = \dfrac{m}{2}(\dot{x}^2 + \dot{y}^2) = \dfrac{m}{2}(a^2\cos^2\varphi + b^2\sin^2\varphi)\dot{\varphi}$

$\dfrac{\partial \Pi}{\partial \varphi} = mgb\cos\varphi + m\omega^2 a^2 \cos\varphi \sin\varphi = 0 \Rightarrow \begin{cases} \cos\varphi = 0 \\ gb + \omega^2 a^2 \sin\varphi = 0 \end{cases}$

$\Rightarrow \begin{cases} \sin\varphi = -\dfrac{gb}{\omega^2 a^2} & \text{– п.р.} \\ \cos\varphi = 0 \end{cases} \Rightarrow \begin{cases} \varphi_1 = \dfrac{\pi}{2} \\ \varphi_2 = -\dfrac{\pi}{2} \\ \varphi_3 = -\arcsin\left(\dfrac{gb}{\omega^2 a^2}\right), \ \omega \geq \dfrac{\sqrt{gb}}{a} \\ \varphi_4 = \pi - \arcsin\left(\dfrac{gb}{\omega^2 a^2}\right) \end{cases}$

$\varphi_3$ эквив. $\varphi_4$

1) Теперь исследуем на устойч.:

$\dfrac{\partial^2 \Pi}{\partial \varphi^2} = -mgb\sin\varphi + m\omega^2 a^2 \cos 2\varphi$

$\left.\dfrac{\partial^2 \Pi}{\partial \varphi^2}\right|_{\varphi=\varphi_1} = -mgb - m\omega^2 a^2 < 0 \Rightarrow$ неустойчивое при $\forall \omega \in \mathbb{R}$.

$\left.\dfrac{\partial^2 \Pi}{\partial \varphi^2}\right|_{\varphi=\varphi_2} = mgb - m\omega^2 a^2 = m(gb - \omega^2 a^2)$ : при $\omega^2 > \dfrac{gb}{a^2}$ – неуст.

$\omega^2 < \dfrac{gb}{a^2}$ – устойч.

$\omega^2 a^2 = gb$: $\Pi(\varphi_2) = mgb\sin\varphi - \frac{m}{2} gb\cos^2\varphi = mgb\left(\sin\varphi - \frac{\cos^2\varphi}{2}\right) = \underbrace{-1 \cdot mgb}_{\text{строгий минимум}}$

$mgb\left(\frac{2\sin\varphi - \cancel{x}1 + \sin^2\varphi}{2}\right)$

$-2 \quad -1 + 1$

устойчиво

$$\left.\frac{\partial^2\Pi}{\partial\varphi^2}\right|_{\varphi=\varphi_3} = \frac{mg^2b^2}{\omega^2a^2} + m\omega^2a^2\left(1 - \frac{g^2b^2}{\omega^4a^4}\right) = m\omega^2a^2\cos^2\varphi^3 > 0$$

при $\forall\varphi_3 \neq \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$

При $\omega \geq \frac{\sqrt{gb}}{a}$ реализуется $\rightarrow$

При $\omega = \frac{a}{\sqrt{gb}}$: $\varphi_3 = -\arcsin(1) = -\frac{\pi}{2} \Rightarrow \varphi_3 = \varphi_2 = -\frac{\pi}{2}$

а $\varphi_2$ устойчиво при условии $\omega^2a^2 \leq gb \Rightarrow \varphi_3$ - всегда устойчиво

Причем при $\omega = \frac{\sqrt{gb}}{a}$ происходит смена уст.

г) Колебания у устойчивых положений:

$T = \frac{m}{2}(a^2\sin^2\varphi + b^2\cos^2\varphi)\dot\varphi^2$

$\Pi = mgb\sin\varphi - \frac{m}{2}\omega^2a^2\cos^2\varphi$

$\varphi_2 = -\frac{\pi}{2}$ $\left\{\begin{array}{l}\sin\varphi \sim -1 \\ \cos\varphi \sim \varphi + \frac{\pi}{2}\end{array}\right| \Rightarrow T = \frac{m}{2}\left(a^2 + b^2\left(\varphi_2 + \frac{\pi}{2}\right)^2\right)\dot\varphi^2 \simeq \frac{m}{2}a^2\dot\varphi^2$ $\quad \varphi_2 = -\frac{\pi}{2}$

~~$\Pi = mgb\sin\left(-\frac{\pi}{2}\right) +$~~ $\sin\varphi - \sin a = \cos a \cdot (\varphi - a) - \frac{\sin a}{2} \cdot (\varphi - a)^2$

$\cos\varphi - \cos a = -\sin a \cdot (\varphi - a) - \frac{\cos a}{2}(\varphi - a)^2$

$$\Rightarrow \Pi = mgb\left(\sin\left(-\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right)\left(\varphi + \frac{\pi}{2}\right) - \frac{\sin\left(-\frac{\pi}{2}\right)}{2} \cdot \left(\varphi + \frac{\pi}{2}\right)^2\right) -$$

$$- \frac{m}{2}\omega^2a^2\left(\cos^2\left(-\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\left(\varphi + \frac{\pi}{2}\right) - \frac{\cos\left(-\frac{\pi}{2}\right)}{2}\left(1 + \frac{\pi}{2}\right)^2\right)^2 =$$

$$= -mgb + \left(\frac{mgb}{2} - \frac{m\omega^2a^2}{2}\right)\left(\varphi + \frac{\pi}{2}\right)^2 = \Pi$$

$\Rightarrow \ddot\varphi + \left(\frac{gb}{a^2} - \omega^2\right)\varphi = -\frac{\pi}{2}\left(\frac{gb}{a^2} - \omega^2\right)$ $\quad$ При $\omega = \frac{\sqrt{gb}}{a}$ - колеб. нет.

$\ddot\varphi + \Omega^2\varphi = 0$, где $\Omega^2 = \frac{gb}{a^2} - \omega^2$, $\varphi = \varphi_0\cos(\Omega t) + \frac{\dot\varphi_0}{\Omega}\sin\Omega t - \frac{\pi}{2}$

$\varphi_3 = -\arcsin\left(\frac{gb}{\omega^2a^2}\right)$: $\omega > \frac{\sqrt{gb}}{a} \Rightarrow \left\{\begin{array}{l}\sin\varphi \simeq -\frac{gb}{\omega^2a^2} + \sqrt{1 - \frac{g^2b^2}{\omega^4a^4}}(\varphi - \varphi_3) \\ \cos\varphi \sim \sqrt{1 - \frac{g^2b^2}{\omega^4a^4}} + \frac{gb}{\omega^2a^2}(\varphi - \varphi_3)\end{array}\right.$

Линеаризуем: $\varphi \to -\arcsin\left(\frac{gb}{\omega^2 a^2}\right) - A$

$$(a^2\sin^2\varphi + b^2\cos^2\varphi)\ddot\varphi + (gb + \omega^2 a^2\sin\varphi)\cos\varphi = 0$$

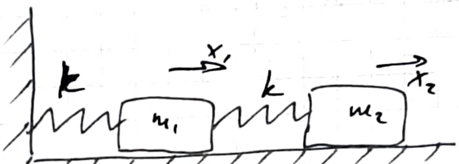
$$(a^2A^2 + b^2(1-A^2))\ddot\varphi + (\varphi-\varphi_3)(gbA^3 + \omega^2 a^2(1-A^2)) +$$

$$+ (gb - \frac{gb}{a^2}a^2)\sqrt{1-A^2}\cdot\ddot\varphi + \left(\frac{1^2 - A^2 + \omega^2(1-A^2)}{A^2 + \frac{b^2}{a^2}(1-A^2)}\right)\varphi =$$

$$= \varphi_3\frac{\omega^2(1-A^2)}{A^2+\frac{b^2}{a^2}(1-A^2)} \Rightarrow \Omega^2 = \frac{a^2\omega^2\left(\omega^4 - \frac{g^2b^2}{a^4}\right)}{a^2\frac{g^2b^2}{a^4} + b^2\left(\omega^4 - \frac{g^2b^2}{a^4}\right)} > 0$$

$$\boxed{\varphi = \varphi_0\cos(\Omega t) + \frac{\dot\varphi_0}{\Omega}\sin(\Omega t) + \varphi_3}$$

№16.34




$n=2 \Rightarrow q = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ $\Pi = \frac{kx_1^2}{2} + \frac{k(x_2-x_1)^2}{2} =$

$= kx_1^2 - kx_1x_2 + \frac{kx_2^2}{2}$

$T = \frac{m_1\dot x_1^2}{2} + \frac{m_2\dot x_2^2}{2}$

$$\Rightarrow C = \begin{pmatrix} k & -\frac{k}{2} \\ -\frac{k}{2} & \frac{k}{2}\end{pmatrix} \qquad A = \begin{pmatrix}\frac{m_1}{2} & 0 \\ 0 & \frac{m_2}{2}\end{pmatrix}$$

$$\det(C-\lambda A) \overset{\det}{=} \begin{pmatrix} k - \frac{\lambda m_1}{2} & -\frac{k}{2} \\ -\frac{k}{2} & \frac{k}{2} - \frac{\lambda m_2}{2}\end{pmatrix} = \frac{1}{4}(2k-\lambda m_1)(k-\lambda m_2) - \frac{k^2}{4} =$$

$\Rightarrow$ ~~$\lambda_{1,2} = \frac{km_2 \pm k\sqrt{m_2^2 - m_1m_2}}{2m_1m_2}$~~

Найдем соотношение между массами: $\lambda_{1,2} = \frac{2m_2 - m_1 \pm \sqrt{(2m_2-m_1)^2 - 4k^2 m_1m_2}}{2m_1m_2}$

$$(C-\lambda A)\bar u_1 = (C-\lambda A)\begin{pmatrix}1\\-1\end{pmatrix} = 0 \Rightarrow \begin{cases} -\frac{m_1\lambda}{2} + k + \frac{k}{2} = 0 \\ -\frac{k}{2} + \frac{m_2\lambda}{2} - \frac{k}{2} = 0\end{cases}$$

$$\Rightarrow \begin{cases} m_1\lambda = 3k \\ m_2\lambda = 2k\end{cases} \Rightarrow \frac{m_1}{m_2} = \frac{3}{2} \Rightarrow m_1 = \frac{3m_2}{2}$$

~~$\Rightarrow \lambda = \frac{km_2 \pm k\sqrt{m_2^2 - m_2^2\frac{3}{2}}}{}$~~

~~$= \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4}k^2 \cdot 6k^2}}{3m_2}$~~

$\lambda = \frac{2m_2 + \frac{3}{2}m_2 \pm \sqrt{\left(2m_2 + \frac{3}{2}m_2\right)^2 - 4k^2\frac{3m_2^2}{2}}}{3m_2^2} =$

$\frac{49}{4} - 6 = \frac{49-24}{4} = \frac{25}{4}$

$$\Rightarrow \lambda_{1,2} = \frac{\left(\frac{7}{2} \pm \frac{5}{2}\right)k}{3m_2}$$

$$\lambda_{1,2} = \frac{\left(\frac{7}{2}\pm\frac{5}{2}\right)k}{3m_2} = \frac{(7\pm5)k}{2\cdot(2m_1)}$$

$$\begin{cases} \lambda_1 = \frac{3k}{m_1} = \frac{2k}{m_2} \\ \lambda_2 = \frac{k}{2m_1} = \frac{k}{3m_2} \end{cases}$$

$$\Rightarrow |C-\lambda_2 A| = \begin{pmatrix} -\frac{3}{2}m_1 & \frac{3}{2}m_2 \\ m_1 & -m_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}m_1 & m_1 \\ m_1 & -\frac{2}{3}m_1 \end{pmatrix}$$

$$(C-\lambda_2 A)\bar{u}_2 = (C-\lambda_2 A)\begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

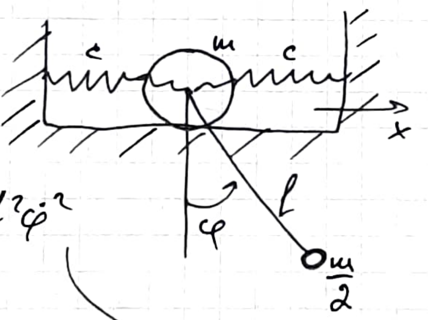
$$\Rightarrow \begin{cases} -\frac{3}{2}u_1^2 + u_2^2 = 0 \\ u_1^2 - \frac{2}{3}u_2^2 = 0 \end{cases} \Rightarrow \bar{u}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\frac{1}{\omega_1} = \sqrt{\frac{m_1}{3k}} = \sqrt{\frac{m_2}{2k}} \Rightarrow \omega_1^2 = \frac{k}{2m_1} = \frac{k}{3m_2} = \lambda_2$$

№16.42

$v, m, c, l, \frac{m}{2}$

$cl = mg$




при $x=0$ в пруж. равновес.

$$\Pi = 2\cdot\frac{cx^2}{2} + \frac{mgl}{2}(1-\cos\varphi)$$

$$T = \frac{m\dot{x}^2}{2} + \frac{m\omega^2}{2} + \frac{1}{2}\frac{mv^2}{2}\left(\frac{\dot{x}}{v}\right)^2 = \frac{1}{2}\left(\left(\frac{3}{2}m\right)\dot{x}^2 + 2\dot{x}l\dot{\varphi}\frac{m\cos\varphi}{2} + ml^2\dot{\varphi}^2\right) + \frac{mv^2}{4}\left(\frac{\dot{x}}{v}\right)^2$$

$$v_m^2 = \dot{x}^2 + 2\dot{x}\dot{\varphi}l + l^2\dot{\varphi}^2$$

$$\frac{\partial\Pi}{\partial x} = 2cx \to x=0$$

$$\frac{\partial\Pi}{\partial\varphi} = mgl\sin\varphi \to \varphi_1 = 0,\ \varphi_2 = \pi$$

$$\Pi = \Pi(x,\varphi) = cx^2 + \frac{mgl}{2}\left(1 - 1 + \frac{\varphi^2}{2}\right) \approx cx^2 + \frac{mgl\varphi^2}{4}$$

$$T = \frac{1}{2}\left(2m\dot{x}^2 + m\dot{x}\dot{\varphi}l + \frac{ml^2}{2}\dot{\varphi}^2\right)$$

$$\Rightarrow C = \begin{pmatrix} 2c & 0 \\ 0 & \frac{mgl}{2} \end{pmatrix} = \begin{pmatrix} \frac{2gm}{l} & 0 \\ 0 & \frac{mgl}{2} \end{pmatrix}$$

$$A = \begin{pmatrix} 2m & \frac{ml}{2} \\ \frac{ml}{2} & \frac{ml^2}{2} \end{pmatrix}$$

$\det(C - \lambda A) = 0 \Rightarrow$

$$\begin{vmatrix} \frac{2gm}{l} - 2\lambda m & -\frac{\lambda m l}{2} \\ -\frac{\lambda m l}{2} & \frac{mgl}{2} - \frac{\lambda m l^2}{2} \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{8}{3}\frac{g}{l}\lambda + \frac{4g^2}{3l^2} = 0.$$

$$\lambda_{12} = \frac{\frac{4}{3}\frac{g}{l} \pm \sqrt{\frac{16}{9}\frac{g^2}{l^2} - \frac{4g^2}{3l^2}}}{1} = \frac{4}{3}\frac{g}{l} \pm \frac{2}{3}\frac{g}{l}$$

$\lambda_1 = \frac{2g}{l} > 0 \quad \lambda_2 = \frac{2}{3}\frac{g}{l} > 0$

Амплитудные вектора:

$\lambda_1 = \frac{2g}{l}$ $(C - \lambda_1 A)\bar{u}^1 = \begin{pmatrix} \frac{2gm}{l} - \frac{4mg}{l} & -mg \\ -mg & \frac{mgl}{2} - \frac{2mgl}{2} \end{pmatrix} \begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} = 0.$

$$\begin{pmatrix} -\frac{2mg}{l} & -mg \\ -mg & -\frac{mgl}{2} \end{pmatrix} \begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} = 0 \Rightarrow \begin{cases} -\frac{2}{l}u_1^1 - u_2^1 = 0 \\ -u_1^1 - \frac{u_2^1 l}{2} = 0 \end{cases}$$

$\Rightarrow$ ~~$u_1^1 = -u_2^1\frac{l}{2} \Rightarrow -\frac{2}{l}\cdot\left(-\frac{l}{2}\right)u_2^1 - u_2^1 = 0.$~~ $u_2^1 = -\frac{2}{l}$ пусть $u_1^1 = 1. \Rightarrow u_2^1 = -\frac{2}{l}$

$\Rightarrow \bar{u}^1 = \begin{pmatrix} 1 \\ -\frac{2}{l} \end{pmatrix}$

$\lambda_2 = \frac{2}{3}\frac{g}{l}$: $(C - \lambda_2 A)\bar{u}^2 = \begin{pmatrix} \frac{2mg}{l} - \frac{4}{3}\frac{mg}{l} & -\frac{mg}{3} \\ -\frac{mg}{3} & \frac{mgl}{2} - \frac{mgl}{3} \end{pmatrix} \begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix} = 0$

($\frac{2}{3}\frac{mg}{l}$; $\frac{1}{6}mgl$)

$$\Rightarrow \begin{cases} \frac{2}{3}\frac{u_1^2}{l} - \frac{u_2^2}{3} = 0 \\ -\frac{u_1^2}{3} + \frac{u_2^2}{6}l = 0 \end{cases} \Rightarrow u_2^2 = \frac{2u_1^2}{l}. \Rightarrow u_1^2 = 1 \Rightarrow u_2^2 = \frac{2}{l}$$

$\Rightarrow \bar{u}^2 = \begin{pmatrix} 1 \\ \frac{2}{l} \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2}{l} \end{pmatrix}(C_1^1\cos\omega_1 t + C_2^1\sin\omega_1 t) + \begin{pmatrix} 1 \\ \frac{2}{l} \end{pmatrix}(C_1^2\cos\omega_2 t + C_2^2\sin\omega_2 t)$$

где $\omega_1 = \sqrt{\frac{2g}{3l}}$, $\omega_2 = \sqrt{\frac{2g}{l}}$

об

№16.112 $T = \frac{(\dot{\theta}_1^2 + \dot{\theta}_2^2)}{2}$ $\Pi = \frac{(\theta_1 + \theta_2)^2 c}{2}$

$Q_1 = \tilde{\theta}_1 = b\dot{\theta}_1$ $\mathcal{L} = T - \Pi = \frac{1}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2 - c\theta_1^2 - 2c\theta_1\theta_2 - c\theta_2^2)$

$Q_2 = \tilde{\theta}_2 = b\dot{\theta}_2$ Р

$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_i} - \frac{\partial\mathcal{L}}{\partial\theta_i} = \tilde{Q}_i \Rightarrow \begin{cases} \frac{d}{dt}(\dot{\theta}_1) + c\theta_1 + c\theta_2 = b\dot{\theta}_1 \\ \frac{d}{dt}(\dot{\theta}_2) + c\theta_2 + c\theta_1 = b\dot{\theta}_2 \end{cases}$

$\Rightarrow \begin{cases} \ddot{\theta}_1 - b\dot{\theta}_1 + c(\theta_1 + \theta_2) = 0 \\ \ddot{\theta}_2 - b\dot{\theta}_2 + c(\theta_1 + \theta_2) = 0 \end{cases} \Rightarrow \theta_2 = -\theta_1 + \frac{b}{c}\dot{\theta}_1 - \frac{\ddot{\theta}_1}{c}$

$\Rightarrow -\ddot{\theta}_1 + \cancel{\frac{b}{c}\theta_1^{(3)}} - \frac{\theta_1^{(4)}}{c} - \cancel{b\dot{\theta}_1} + \frac{b^2}{c}\ddot{\theta}_1 - \cancel{\frac{b}{c}\theta_1^{(3)}} + \cancel{b\dot{\theta}_1} - \ddot{\theta}_1 = 0$

$\Rightarrow \theta_1^{(4)} + (2c - b^2)\ddot{\theta}_1 = 0$ Решим ур-е.

$v = \ddot{\theta}_1 \Rightarrow \ddot{v} + (2c - b^2)v = 0$, $\omega^2 = 2c - b^2$

$\Rightarrow v = c_1' \cos\omega t + c_2'' \sin\omega t$ Поменяем замену и проект.

$\theta_1 = c_0 + c_1 t + c_2\cos\omega t + c_3 \sin\omega t$

$\Rightarrow \theta_2 = -c_0 - c_1 t - c_2\cos\omega t - c_3\sin\omega t + \frac{b}{c}(c_1 - \omega c_2 \sin\omega t + \omega c_3\cos\omega t) + \frac{\omega^2 c_2\cos\omega t + \omega^2 c_3 \sin\omega t}{c}$

$$\Rightarrow \boxed{\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix} c_0 \\ \frac{bc_1}{c} - c_0 \end{pmatrix} + \begin{pmatrix} c_1 \\ -c_1\end{pmatrix} t + \begin{pmatrix} c_2 \\ \frac{\omega}{c}(\omega c_2 + bc_3) - c_2 \end{pmatrix}\cos\omega t + \begin{pmatrix} c_3 \\ \frac{\omega}{c}(\omega c_3 - bc_2) - c_3 \end{pmatrix}\sin\omega t}$$

Об.