Неделя 11

№23.2
$$\begin{cases} \hat{q} = \alpha q + \beta p \\ \hat{p} = \gamma q + \delta p \end{cases}$$

Крит. каконич. преоб.:

$$\sum \hat{p}_i \delta \hat{q}_i - c \sum p_i \delta_i = -\delta F(q, p, t)$$

$$\Rightarrow \begin{cases} \sum \hat{p}_j \frac{\partial \hat{q}_j}{\partial q_i} - c p_i = -\frac{\partial F}{\partial q_i} \\ \sum \hat{p}_j \frac{\partial \hat{q}_j}{\partial p_i} = -\frac{\partial F}{\partial p_i} \end{cases} \Rightarrow \begin{cases} (\gamma q + \delta p) \cdot \alpha - cp = -\frac{\partial F}{\partial q} \\ (\gamma q + \delta p) \beta = -\frac{\partial F}{\partial p} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial^2 F}{\partial p \partial q} = -\delta \alpha + c \\ \frac{\partial^2 F}{\partial q \partial p} = -\gamma \beta \end{cases} \Rightarrow c = \delta \alpha - \gamma \beta \neq 0$$

$\Rightarrow$ нужная возможность всегда сущ. при невырожденности

$$\frac{\partial F}{\partial p} = -\gamma \beta \cdot q - \delta \beta p \Rightarrow F = -\gamma \beta q p - \frac{\delta \beta}{2} p^2 + f(q)$$

$$\frac{\partial F}{\partial q} = -\gamma \beta p + f(q) = -\gamma \alpha q - \delta \alpha p + cp$$

$$f'(q) = -\underbrace{(\delta \alpha - \gamma \beta)}_{c} p + cp - \gamma \alpha q \Rightarrow f(q) = -\gamma \alpha \frac{q^2}{2}$$

$$\Rightarrow F = -\frac{1}{2} (\alpha \gamma q^2 + 2\beta \gamma q p + \beta \delta p^2) \text{ - производящая ф-я.}$$

Валентность и производящая ф-я найдены $\Rightarrow$ преобр. каноничн.

№23.6
$$\begin{cases} \hat{q} = p \exp q \\ \hat{p} = q + \exp(-q) + \ln p \end{cases}$$

т. к. преобразования каноничные?

$$\begin{cases} (q + \exp(-q) + \ln p) p \exp q - cp = -\frac{\partial F}{\partial q} \\ (q + \exp(-q) + \ln p) \exp q = -\frac{\partial F}{\partial p} \end{cases}$$

$$\frac{\partial^2 F}{\partial p \partial q} = -\left(\frac{1}{p} p \exp(q) + \ln p \exp q - c + q \exp q + 1\right)$$

$$\frac{\partial^2 F}{\partial q \partial p} = -((1 - \exp(-q)) \exp q + (q + \exp(-q) + \ln p) \exp q) =$$

$$= -(1 - \exp(-q) + q + \exp(-q) + \ln p) \exp q =$$

$$= -\exp q (1 + q + \ln p)$$

$$\frac{\partial^2 F}{\partial p \partial q} = \frac{\partial^2 F}{\partial q \partial p} \Rightarrow c - \exp q - \ln p \exp q - q \exp q - 1 =$$

$$= -\exp q - q \exp q - \ln p \exp q \Rightarrow \underline{c = 1}$$

№23.18 $L(q,\dot q,t)$

$$\tilde q_i=\tilde q_i(q,t) \qquad i=\overline{1,n}$$

$$\tilde L(\tilde q_i,\dot{\tilde q}_i,t)=L(q(\tilde q,t),\dot q(\tilde q,\dot{\tilde q},t),t)$$

$$p_i=\frac{\partial L}{\partial \dot q_i}(q,p,t)=\frac{\partial \tilde L}{\partial \dot q_i}(\tilde q(q,t),\dot{\tilde q}(q,\dot q,t),t)=\sum_{k=1}^{n}\frac{\partial \tilde L}{\partial \dot{\tilde q}_k}\cdot\frac{\partial \dot{\tilde q}_k}{\partial \dot q_i}$$

$$d\dot{\tilde q}_k=\sum_{i=1}^{n}\left(\frac{\partial \dot{\tilde q}_k}{\partial q_i}dq_i+\frac{\partial \dot{\tilde q}_k}{\partial \dot q_i}d\dot q_i\right)+\frac{\partial \dot{\tilde q}_k}{\partial t}dt$$

$$d\tilde q_k=\sum_{i=1}^{n}\left(\frac{\partial \tilde q_k}{\partial q_i}dq_i\right)+\frac{\partial \tilde q_k}{\partial t}dt$$

$$\tilde p_i=\sum_{j=1}^{n}p_j\frac{\partial q_j}{\partial \tilde q_i}$$

Критерий каноничности:

$$\sum_{k=1}^{n}\tilde p_k\,\delta\tilde q_k=c\sum_{k=1}^{n}p_k\delta q_k-\sum_{k=1}^{n}\frac{\partial F}{\partial q_k}\delta q_k+\frac{\partial F}{\partial p_k}\delta p_k$$

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}p_j\underbrace{\frac{\partial q_j}{\partial \tilde q_k}\cdot\frac{\partial \tilde q_k}{\partial q_i}}_{\delta_{ij}}\delta q_i=\sum_{k=1}^{n}\left[\left(cp_k-\frac{\partial F}{\partial q_k}\right)\delta q_k-\frac{\partial F}{\partial p_k}\delta p_k\right]$$

$$\begin{cases}\sum_{j=1}^{n}p_j\delta_{ij}=cp_i-\frac{\partial F}{\partial q_i}\\ \frac{\partial F}{\partial p_i}=0\end{cases}\Rightarrow\begin{cases}\frac{\partial F}{\partial q_i}=(c-1)p_i\\ \frac{\partial F}{\partial p_i}=0\end{cases}\Rightarrow \frac{\partial^2F}{\partial q_i\partial p_j}=0 \quad \frac{\partial^2F}{\partial p_i\partial p_j}=0 \quad \frac{\partial^2F}{\partial q_i\partial q_j}=0 \quad \frac{\partial^2F}{\partial p_j\partial p_i}=0$$

Из равенства производных при $\forall i,j=\overline{1,n}$ $\boxed{\frac{\partial^2F}{\partial p_i\partial q_j}=(c-1)\delta_{ij}}$

$c=1$ $\Rightarrow\begin{cases}\frac{\partial F}{\partial q_i}=0\\ \frac{\partial F}{\partial p_i}=0\end{cases}\Rightarrow F=f(t)$

№23.39 $u=\frac{pq^3}{2t}$

$$\begin{cases}\tilde q=\frac{1}{q^2}+\ln(tpq^3)\\ \tilde p=pq^3(1+t\exp\frac{1}{q^2})\end{cases}\Rightarrow\begin{cases}pq^3(1+t\exp\frac{1}{q^2})\left(-\frac{2}{q^3}+\frac{3tpq^2}{tpq^3}\right)-cp=-\frac{\partial F}{\partial q}\\ pq^3(1+t\exp\frac{1}{q^2})\left(\frac{tq^3}{tpq^3}\right)=-\frac{\partial F}{\partial p}\end{cases}$$

$$\begin{cases}pq^3(1+t\exp\frac{1}{q^2})\left(\frac{3}{q}-\frac{2}{q^3}\right)-cp=-\frac{\partial F}{\partial q}\\ q^3(1+t\exp\frac{1}{q^2})=-\frac{\partial F}{\partial p}\end{cases}$$

$$\frac{\partial^2F}{\partial p\partial q}=c-q^3(1+t\exp\frac{1}{q^2})\left(\frac{3}{q}-\frac{2}{q^3}\right)$$

$$\frac{\partial^2F}{\partial q\partial p}=-3q^2(1+t\exp\frac{1}{q^2})-q^3\left(-\frac{2t}{q^3}\exp\frac{1}{q^2}\right)=2t\exp\frac{1}{q^2}-3q^2(1+t\exp\frac{1}{q^2})$$

Приравняем вторые производные:

$\Rightarrow\ c\cdot q^3(1+t\exp\frac{1}{q^2})\left(\frac{3}{q}-\frac{2}{q^3}\right)=2t\exp\frac{1}{q^2}-3q^2(1+t\exp\frac{1}{q^2})$

$c=(1+t\exp\frac{1}{q^2})(3q^2-2-3q^2)+2t\exp\frac{1}{q^2}\quad\Rightarrow\ \underline{c=-2}$

$\frac{\partial F}{\partial p}=-q^3(1+t\exp\frac{1}{q^2})\quad\Rightarrow F=-pq^3(1+t\exp\frac{1}{q^2})+f(q)$

$\frac{\partial F}{\partial q}=-3pq^2(1-t\exp\frac{1}{q^2})-pq^3\left(-\frac{2t}{q^3}\exp\frac{1}{q^2}\right)+f'(q)=$

$=-3pq^2(1+t\exp\frac{1}{q^2})+2pt\exp\frac{1}{q^2}+f'(q).$

$f'(q)+2pt\exp\frac{1}{q^2}-3pq^2(1+t\exp\frac{1}{q^2})=-2p-p(1+t\exp\frac{1}{q^2})(3q^2-2)$

$f'(q)+2pt\exp\frac{1}{q^2}-3pq^2(1+t\exp\frac{1}{q^2})=-2p-3q^2p(1+t\exp\frac{1}{q^2})+$
$+2p(1+t\exp\frac{1}{q^2}).$

$f'(q)=0\ \Rightarrow\ f(q)=const\ \Rightarrow F(q,p,t)=-pq^3(1+t\exp\frac{1}{q^2})$

$pq^3(1+t\exp\frac{1}{q^2})d\tilde{q}-\tilde{H}dt=-2(pdq-Hdt)-dF(q,p,t)$

$pq^3(1+t\exp\frac{1}{q^2})-\tilde{H}=-cH-\frac{\partial F}{\partial t}$

$\tilde{H}=cH+pq^3\left(\frac{1}{t}+\exp\frac{1}{q^2}\right)+\frac{\partial F}{\partial t}$

$\tilde{H}=-\cancel{\frac{pq^3}{t}}+pq^3\frac{1}{t}+pq^3\exp\frac{1}{q^3}-pq^3\exp\frac{1}{q^2}=0\quad\Rightarrow \underline{\tilde{H}=0}.$

№23.86
$$\begin{cases}\tilde{q}_1=\frac{1}{2}(\ln(p_1+9q_2)-q_1)\\ \tilde{q}_2=\frac{1}{2}(\ln(p_2+9q_1)-3q_2)\\ \tilde{p}_1=-2(p_1+9q_2)\\ \tilde{p}_2=-\frac{2}{3}(p_2+9q_1)\end{cases}\qquad \begin{cases}p_1=e^{q_1+2\tilde{q}_1}-9q_2\\ p_2=e^{3q_2+2\tilde{q}_2}-9q_1\end{cases}$$

$$\Rightarrow\begin{cases}p_1=e^{q_1+2\tilde{q}_1}-9q_2\\ p_2=e^{3q_2+2\tilde{q}_2}-9q_1\\ \tilde{p}_1=-2e^{q_1+2\tilde{q}_1}\\ \tilde{p}_2=-\frac{2}{3}e^{3q_2+2\tilde{q}_2}\end{cases}$$

Критерий каноничности для $(q,\tilde{q})$:
$$\begin{cases}\tilde{p}_i=-\frac{\partial S}{\partial \tilde{q}_i}\\ cp_i=\frac{\partial S}{\partial q_i}\end{cases}$$

$$\begin{cases}2e^{q_1+2\tilde{q}_1}=\frac{\partial S}{\partial \tilde{q}_1}\\ \frac{2}{3}e^{3q_2+2\tilde{q}_2}=\frac{\partial S}{\partial \tilde{q}_2}\\ ce^{q_1+2\tilde{q}_1}-9cq_2=\frac{\partial S}{\partial q_1}\\ ce^{3q_2+2\tilde{q}_2}-9cq_1=\frac{\partial S}{\partial q_2}\end{cases}$$

Найдем смешанные производные S:

$\frac{\partial^2 S}{\partial q_1\partial \tilde{q}_1}=2e^{q_1+2\tilde{q}_1}$; $\frac{\partial^2 S}{\partial q_2\partial \tilde{q}_1}=0$

$\frac{\partial^2 S}{\partial \tilde{q}_2\partial q_1}=0$; $\frac{\partial^2 S}{\partial q_1\partial \tilde{q}_2}=0$

$$\frac{\partial^2 S}{\partial q_2 \partial q_1} = -9c \; ; \quad \frac{\partial^2 \varphi}{\partial \hat{q}_1 \partial q_1} = 2c\, e^{q_1 + 2\hat{q}_1}$$

$$\frac{\partial^2 S}{\partial \hat{q}_2 \partial q_1} = 0 \; ; \quad \frac{\partial^2 S}{\partial q_1 \partial q_2} = -9c$$

$$\frac{\partial^2 S}{\partial \hat{q}_1 \partial q_2} = 0 \; ; \quad \frac{\partial^2 S}{\partial \hat{q}_2 \partial q_2} = 2c\, e^{3q_2 + 2\hat{q}_2}$$

$$\Rightarrow \begin{cases} 2 = 2c \\ 2 = 2c \\ -9c = -9c \end{cases} \Rightarrow c = 1$$

$$S = e^{q_1 + 2\hat{q}_1} - 9 q_1 q_2 + \frac{1}{3} e^{3q_2 + 2\hat{q}_2}$$

Теперь проверим:

$$\frac{\partial S}{\partial q_1} = 2e^{q_1 + 2\hat{q}_1} \qquad \frac{\partial S}{\partial q_1} = e^{q_1 + 2\hat{q}_1} - 9q_2$$

$$\frac{\partial S}{\partial q_2} = \frac{2}{3} e^{3q_2 + 2\hat{q}_2} \; ; \quad \frac{\partial S}{\partial q_2} = e^{3q_2 + 2\hat{q}_2} - 9q_1$$

$\Rightarrow$ Преобразование каноническое