Total and Effective Stress Analysis

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Stress is a tensor that can be represented by a matrix in Cartesian co-ordinates:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{1}$$

In the standard deformation theory, the stress tensor is symmetric such that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{zy} = \sigma_{yz}$, and $\sigma_{xz} = \sigma_{zx}$. In this situation, stresses are often written in vector notation, with only six components:

$$\sigma = \left[\sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \sigma_{xy}\ \sigma_{yx}\ \sigma_{zx}\right]^T \tag{2}$$

According to Terzaghi's principles, stresses are divided into effective stresses, σ', and pore pressures, σ_w:

$$\sigma = \sigma' + \sigma_w \tag{3}$$

Pore pressures are due to pressure of water in the pores. Water is considered not to sustain any shear stresses. Hence, the total shear stresses are equal to effective shear stresses. Moreover, water is isotropic, i.e., the stresses are equal in all directions, so all pore pressure components are equal, p_w.

$$\sigma_w = \left[p_w\ p_w\ p_w\ 0\ 0\ 0\right]^T \tag{4}$$

Material models are generally expressed as a relationship between infinitesimal increment of effective stress and infinitesimal increments of strain. In such relationships, the incremental effective stress is represented by stress rates.

$$\dot{\sigma}' = \left[\dot{\sigma}'_{xx}\ \dot{\sigma}'_{yy}\ \dot{\sigma}'_{zz}\ \dot{\sigma}_{xy}\ \dot{\sigma}_{yx}\ \dot{\sigma}_{zx}\right]^T \tag{5}$$

However, water does not sustain any shear stress, and therefore the effective shear stresses are equal to the total shear stress:

$$\sigma_{xx} = \sigma'_{xx} + p_w \tag{6a}$$
$$\sigma_{yy} = \sigma'_{yy} + p_w \tag{6b}$$
$$\sigma_{zz} = \sigma'_{zz} + p_w \tag{6c}$$
$$\sigma_{xy} = \sigma'_{xy} \tag{6d}$$
$$\sigma_{yz} = \sigma'_{yz} \tag{6e}$$
$$\sigma_{zx} = \sigma'_{zx} \tag{6f}$$

Further distinction is made between steady state pore stress, p_{steady}, and excess pore stress, p_{excess}:

$$p_w = p_{steady} + p_{excess} \tag{7}$$

Steady state pore pressures are considered to be input data, i.e. generated on the basis of phreatic levels. Excess pore pressures are generated during plastic calculations for the case of undrained (A and B) material behaviour. Since the time derivative of the steady state component equals zero, it follows:

$$\dot{p_w} = \dot{p}_{excess} \tag{8}$$

Hooke's law can be inverted to obtain:

$$\begin{bmatrix} \dot{\varepsilon}^e_{xx} \\ \dot{\varepsilon}^e_{yy} \\ \dot{\varepsilon}^e_{zz} \\ \dot{\gamma}^e_{xy} \\ \dot{\gamma}^e_{yz} \\ \dot{\gamma}^e_{zx} \end{bmatrix} = \frac{1}{E'} \begin{bmatrix} 1 & -\upsilon' & -\upsilon' & 0 & 0 & 0 \\ -\upsilon' & 1 & -\upsilon' & 0 & 0 & 0 \\ -\upsilon' & -\upsilon' & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+2\upsilon' & 0 & 0 \\ 0 & 0 & 0 & 0 & 2+2\upsilon' & 0 \\ 0 & 0 & 0 & 0 & 0 & 2+2\upsilon' \end{bmatrix} \begin{bmatrix} \dot{\sigma}_{xx} \\ \dot{\sigma}_{yy} \\ \dot{\sigma}_{zz} \\ \dot{\sigma}_{xy} \\ \dot{\sigma}_{yz} \\ \dot{\sigma}_{zx} \end{bmatrix} \tag{9}$$

Substituting Eq. 6 gives:

$$\begin{bmatrix} \dot{\varepsilon}^e_{xx} \\ \dot{\varepsilon}^e_{yy} \\ \dot{\varepsilon}^e_{zz} \\ \dot{\gamma}^e_{xy} \\ \dot{\gamma}^e_{yz} \\ \dot{\gamma}^e_{zx} \end{bmatrix} = \frac{1}{E'} \begin{bmatrix} 1 & -\upsilon' & -\upsilon' & 0 & 0 & 0 \\ -\upsilon' & 1 & -\upsilon' & 0 & 0 & 0 \\ -\upsilon' & -\upsilon' & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+2\upsilon' & 0 & 0 \\ 0 & 0 & 0 & 0 & 2+2\upsilon' & 0 \\ 0 & 0 & 0 & 0 & 0 & 2+2\upsilon' \end{bmatrix} \begin{bmatrix} \dot{\sigma}_{xx} - \dot{p_w} \\ \dot{\sigma}_{yy} - \dot{p_w} \\ \dot{\sigma}_{zz} - \dot{p_w} \\ \dot{\sigma}_{xy} \\ \dot{\sigma}_{yz} \\ \dot{\sigma}_{zx} \end{bmatrix} \tag{10}$$

Considering slightly incompressible water, the rate of excess pore pressure is written as:

$$\dot{p_w} = \frac{K_w}{n}(\dot{\epsilon}^e_{xx} + \dot{\epsilon}^e_{yy} + \dot{\epsilon}^e_{zz}) \tag{11}$$

in which K_w is the bulk modulus of the water and n is the soil porosity.
The inverted form of Hooke's law may be written in terms of the total stress rates and the undrained parameters E_u and υ_u:

$$\begin{bmatrix} \dot{\varepsilon}^e_{xx} \\ \dot{\varepsilon}^e_{yy} \\ \dot{\varepsilon}^e_{zz} \\ \dot{\gamma}^e_{xy} \\ \dot{\gamma}^e_{yz} \\ \dot{\gamma}^e_{zx} \end{bmatrix} = \frac{1}{E_u} \begin{bmatrix} 1 & -\upsilon_u & -\upsilon_u & 0 & 0 & 0 \\ -\upsilon_u & 1 & -\upsilon_u & 0 & 0 & 0 \\ -\upsilon_u & -\upsilon_u & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+2\upsilon_u & 0 & 0 \\ 0 & 0 & 0 & 0 & 2+2\upsilon_u & 0 \\ 0 & 0 & 0 & 0 & 0 & 2+2\upsilon_u \end{bmatrix} \begin{bmatrix} \dot{\sigma}_{xx} \\ \dot{\sigma}_{yy} \\ \dot{\sigma}_{zz} \\ \dot{\sigma}_{xy} \\ \dot{\sigma}_{yz} \\ \dot{\sigma}_{zx} \end{bmatrix} \tag{12}$$

where:

$$E_u = 2G(1+\upsilon_u) \tag{13}$$

$$\upsilon_u = \frac{\upsilon' + \mu(1+\upsilon')}{1+2\mu(1+\upsilon')} \tag{14}$$

$$\mu = \frac{1}{3n}\frac{K_w}{K'} \tag{15}$$

$$K' = \frac{E'}{3(1-2\upsilon')} \tag{16}$$

Hence, for undrained behaviour in *PLAXIS* the effective parameters G and υ' are transferred into undrained parameters E_u and υ_u according to Eqs. 13 and 14. The index *u* denotes auxiliary parameter for undrained soil. Fully incompressible behaviour is obtained for $\upsilon_u = 0.5$. However, taking $\upsilon_u = 0.5$ leads to singularity of the stiffness matrix. In order to avoid numerical problems caused by extremely low compressibility, υ_u is, by default, taken as 0.495, which makes the undrained soil body slightly compressible.

Types of Analysis

Time Independent Analysis:

- Total stress analysis
  - Undrained
- Effective stress analysis
  - Drained
  - Undrained

Total Stress Analysis

Undrained behaviour can be simulated using a total stress analysis with parameters specified as undrained. In that case, stiffness is modelled using an undrained Young's Modulus E_u and an undrained Poisson ratio φ_u, and strength is modelled using an undrained shear strength s_u and $\varphi = \varphi_u = 0°$. Typically, for undrained Poisson ratio a value close to 0.5 is selected (between 0.495 to 0.499). A value of 0.5 exactly is not possible, since this would lead to the singularity of the matrix. The disadvantage of the undrained total stress analysis is that no distinction is made between effective stresses and pore pressures. Hence, all output referring to effective stresses should now be interpreted as total stresses and all pore pressures are equal to zero.

Undrained Effective Stress Analysis with Effective Strength Parameters

Effective strength parameters c' and ϕ' can be used to model the material's undrained shear strength (Undrained (A)).
In this case, the development of pore-pressure plays a crucial role in providing the right effective stress path that leads to failure at realistic value of undrained shear strength (c_u or s_u). However, most soil models are not capable of providing the right effective stress path in undrained loading. As a result, they will produce the wrong undrained shear strength if the material strength has been specified on the basis of effective strength parameters. If the soil is modelled as an undrained material using the effective strength parameters ϕ' and c', the drainage type has to be set to Undrained (A). In this case, PLAXIS adds the stiffness of water to the stiffness matrix in order to distinguish between effective stresses and (excess) pore pressures (= effective stress analysis). The advantage of using effective strength parameters in undrained case is that after consolidation a qualitative increase in the shear strength can be obtained, although this increased shear strength could be quantitatively wrong.
Figure 1 illustrates an example using the Mohr-Coulomb model. When the Drainage type is set to Undrained (A), the model will follow an effective stress path where the mean effective stress, p', remains constant all the way up to failure (1). It is known that especially soft soils, like normally

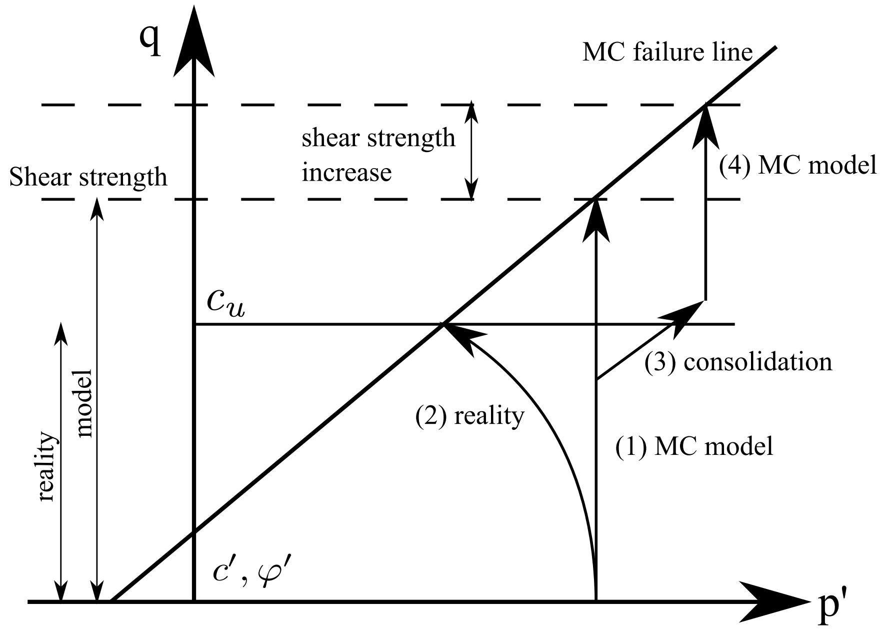

Figure 1: Stress paths in Mohr-Coulomb model

consolidated clays and peat, will follow an effective stress path in undrained loading where p' reduces significantly as a result of shear induced pore pressure (2). As a result, the maximum deviatoric stress that can be reached in the model is over-estimated in the Mohr-Coulomb model. In other words, the mobilized shear strength in the model supersedes the available undrained shear strength. If, at some stress state, the soil is consolidated, the mean effective stress will increase (3). Upon further undrained loading with the Mohr-Coulomb model, the observed shear strength will be increased (4) compared to the previous shear strength, but this increased shear strength may again be unrealistic, especially for soft soils.
On the other hand, advanced models do include, to some extent, the reduction of mean effective stress in undrained loading, but even when using advanced models it is generally advised to check the mobilised shear strength in the Output program against the available (undrained) shear strength when this approach is followed. Note that whenever the Drainage type parameter is set to Undrained (A), effective values must be entered for the stiffness parameters (Young's modulus E' and Poisson ratio φ') in case of the Mohr-Coulomb model or the respective stiffness parameters in the advanced models).

Undrained Effective Stress Analysis with Undrained Strength Parameters

Undrained analysis for soil layers with a known undrained shear strength profiles, can be performed by setting the drainage type as Undrained (B) with direct input of undrained shear strength, i.,e. setting the friction angle to zero and the cohesion equal to undrained shear strength ($\varphi = \varphi_u = 0°; c = s_u$). Also in this case, distinction is made between pore pressures and effective stresses. Although the pore pressures and effective stress path may not be fully correct, the resulting undrained shear strength is not affected, since it is directly specified as an input parameter.
Source: PLAXIS 2D 2011 Material Models Manual