In general, the behavior of real materials is very complicated. Drastic idealizations and simplifications are essential in order to model mathematically and approximately the real material behavior for the solution of a practical problem. For example, the material behavior may be drastically idealized as *time independent*, such as *elastic* and *elastic–plastic* materials, where time effects are neglected. For an ideal elastic material model the behavior is further idealized as reversible and load path independent, whereas for a plastic model, it is irreversible and load path dependent. On the other hand, in the *time-dependent* material idealization, such as viscoelastic and viscoplastic models, time effects are considered, and they are generally capable of describing rate- and history-dependent behavior. Thus constitutive relations corresponding to each of these idealized material models can describe only a limited number of actual physical phenomena of a real material.

It must be emphasized here that the previous idealizations and subsequent classification of material models are only for mathematical convenience in describing the mechanical behavior of real materials. Nothing can compel real materials to behave according to these idealized models. Indeed, a commonly used engineering material such as a structural steel exhibits the behavior of most of these models under certain conditions of stresses, temperatures, vibrations, and strain rates. Therefore, in the solution of a practical problem, it is essential that we determine the range and conditions under which the material of the body can sensibly be assumed to exhibit the dominant characteristics of a particular type of the idealized models. Furthermore, since any idealized model has its own shortcomings, all the results obtained must be interpreted carefully in terms of these shortcomings.

4.2 BASIC ASSUMPTIONS (HYPOTHESES)

In the majority of engineering applications of structural and geological materials such as metals, concrete, soils, rocks, and rubber under short-term loading, time independence of the stress–strain relation is a reasonable approximation and is made most often. The physical behavior of these materials can vary very widely when subjected to extreme conditions. In this chapter, we limit our discussions to a few typical time-independent constitutive models for study. These include the linear elastic and nonlinear elastic models for materials that reversibly load and unload along the same curve, and the hypoelastic and the deformation theory of plasticity for the irreversible plastic range. In outline form, the constitutive models or stress–strain relations to be described in the forthcoming are based on the following two assumptions:

1. Material behavior is *time independent*. Therefore, rate sensitivity, creep, and relaxation are not included in such behavior. Also, *time does not appear explicitly* as a variable in the constitutive equations for such material.

2. Interaction between mechanical and thermal processes is neglected. Thus only material bodies under *isothermal conditions* are considered. Moreover, effect of temperature on constitutive equations is *not* considered.

4.3 NEED FOR ELASTIC MATERIAL MODELS

In this chapter, various elastic constitutive relationships are formulated to model the mechanical behavior of the class of real materials that can be described as *elastic*. In practice, the study of these elastic models is needed for two important reasons:

1. By themselves, elastic models describe well the behavior of many engineering materials at the *working load* levels. For example, the linear elastic model has been used successfully to describe the behavior of metal materials at stress levels below the *elastic limit*. Therefore, elastic constitutive relations are the basis for the *theory of elasticity*, which has found many applications in different engineering problems.
2. The elastic constitutive models are needed in the *theory of plasticity*, which can be considered as a generalization of the theory of elasticity. For instance, elastic–plastic models are used extensively for metal materials at *overload stages*, when the stress levels exceed the elastic limit and yielding of material occurs.

4.4 DEFINITIONS

Elastic Material

A material body is deformed when subjected to applied forces. If upon the release of the applied forces the body recovers its original shape and size, then the material body is called *elastic*. For such a material, the current state of stress depends *only* on the current state of deformation; that is, the stress is a function of strain. Mathematically, the constitutive equations for this material are given by

$$\sigma_{ij} = F_{ij}(\varepsilon_{kl}) \tag{4.3}$$

where the function F_{ij} is the *elastic response function*. Thus the *elastic* behavior described by Eq. (4.3) is both *reversible* and *path independent* in the sense that strains are uniquely determined from the current state of stress or vice versa. There is no dependence of the behavior on the stress or strain histories followed to reach the current state of stress or strain. The elastic material defined above is usually termed *Cauchy elastic material* (e.g., Eringen, 1962;

and Malvern, 1969). It can be shown that Cauchy elastic material may generate energy under certain loading–unloading cycles. Clearly, in such cases it violates the laws of thermodynamics. Therefore, the term *hyperelastic* or *Green elastic* material is used (e.g., Fung, 1965; Eringen, 1962; Green and Zerna, 1954; and Malvern, 1969) to indicate that the elastic response function in Eq. (4.3) is further restricted by the existence of an *elastic strain energy* function W, which is in general a function of strain components ε_{ij}, such that

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \tag{4.4}$$

This ensures that no energy can be generated through load cycles and thermodynamic laws are *always* satisfied.

Sometimes the term *hypoelastic* model is used to describe the *incremental* elastic constitutive relations (e.g., Malvern, 1969; and Truesdell, 1955). These models are often used to describe the behavior of a class of materials in which the state of stress is generally a function of the *current state* of strain as well as of the *stress path* followed to reach that state. Thus for a hypoelastic material the constitutive equations are generally expressed as

$$\dot{\sigma}_{ij} = F_{ij}(\dot{\varepsilon}_{kl}, \sigma_{mn}) \tag{4.5}$$

where $\dot{\sigma}_{ij}$ = stress rate (or increment) tensor
$\dot{\varepsilon}_{kl}$ = strain rate (or increment) tensor
$F_{ij}(\dot{\varepsilon}_{kl}, \sigma_{mn})$ = elastic response function

Material Symmetry Properties

If the mechanical behavior of the material is the *same* in certain directions, then the material is said to have material symmetry with respect to these directions. Material symmetry is exhibited by the *form invariance* of the constitutive relations under a group of transformations of the coordinate axes. If there is *no* material symmetry at all, the material is termed *anisotropic*. In the following, three types of material symmetry are described:

1. Orthotropic Material: An orthotropic material has three orthogonal planes of material symmetry. Wood materials are often treated as orthotropic. Fig. 4.2*a* shows a small wood material element with three planes of material symmetry. These symmetry planes are taken as normal, tangential (parallel), and radial with respect to the grains of wood, as shown in Fig. 4.2*a*. Thus material symmetry is exhibited under any 180° rotation of the axes x_1, x_2, and x_3 shown.

2. Transversely Isotropic Material: This material has *rotational symmetry* with respect to one of the coordinate axes. As an example, a small material element of a rolled plate is shown in Fig. 4.2*b*. The material properties are the

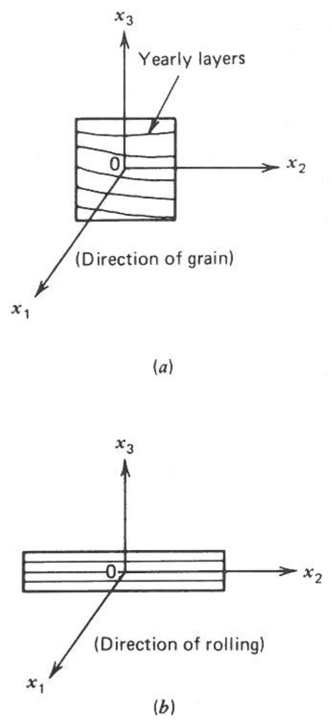

FIGURE 4.2 Material symmetry for orthotropic and transversely isotropic materials. (*a*) Orthotropic material (wood). (*b*) Transversely isotropic material (rolled plate).

same in any direction of the plane x_1–x_2 parallel to the direction of rolling. However, the properties in the direction normal to the direction of rolling are generally different. Material symmetry is exhibited under any rotation of the coordinate axes about the axis normal to the rolling direction (axis x_3 in Fig. 4.2*b*). Plane x_1–x_2 parallel to the direction of rolling is called the *plane of isotropy*.

3. Isotropic Materials: In an isotropic material, the mechanical behavior is identical in *all* directions. Any plane is a plane of material symmetry, and any axis is an axis of rotational symmetry. Isotropic material models are used extensively in engineering applications for elastic materials. As an example, polycrystalline metals, composed of randomly oriented grains, are considered *isotropic*. It is important to note that this initial isotropy is destroyed after plastic deformations occur, since plastic deformations are physically anisotropic.

In the subsequent sections, various elastic constitutive models are described. The discussion is taken up in the following sequence. First we derive the well-known isotropic linear elastic relations. Then we extend it to some simple anisotropic linear elastic relations. Nonlinearity of isotropic material is then introduced through an obvious modification of the linear isotropic forms. Fundamental questions arise which require much additional thought. In the sections that follow, considerations of strain energy and complementary energy density for nonlinear elastic materials suggest general normality and convexity relations for our idealized materials. Uniqueness of a solution and stability of a material are seen to be closely related to the forms of stress–strain relations used. Explicit stress–strain relations in matrix forms are written for several commonly used material models. A general discussion of the incremental approach suitable for materials that are load path dependent is given near the end of this chapter. Finally, a variable moduli model for isotropic materials based on this approach is presented. Some remarks on the suitability of this model are then made.

4.5 ISOTROPIC LINEAR ELASTIC STRESS–STRAIN RELATIONS (GENERALIZED HOOKE'S LAW)

The most general form for *linear* stress–strain relations for a *Cauchy elastic* material is given by

$$\sigma_{ij} = B_{ij} + C_{ijkl}\varepsilon_{kl} \tag{4.6}$$

where B_{ij} = components of initial stress tensor corresponding to the *initial strain free* state (when all strain components $\varepsilon_{kl} = 0$)
C_{ijkl} = tensor of material *elastic constants*

If it is assumed that the initial strain free state corresponds to an *initial stress free* state, that is, $B_{ij} = 0$, then Eq. (4.6) reduces to

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{4.7}$$

It may also be remarked that Eq. (4.7) is the simplest generalization of the linear dependence of stress on strain observed in the familiar Hooke's experiment in simple tension test and consequently Eq. (4.7) is often referred to as the generalized Hooke's law.

Since both σ_{ij} and ε_{kl} are second-order tensors, it follows that C_{ijkl} is a fourth-order tensor (quotient rule of Chapter 1). In general, there are $(3)^4 = 81$ constants for such a tensor C_{ijkl}. However, since σ_{ij} and ε_{kl} are both *symmetrical*, one has the following symmetry conditions:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} \tag{4.8}$$

Hence the maximum number of independent constants is reduced to 36.

For a *Green elastic* material, it is shown later that the four subscripts of the elastic constants can be considered as pairs $C_{(ij)(kl)}$, and the order of the pairs can be interchanged, $C_{(ij)(kl)} = C_{(kl)(ij)}$. As a result, the number of independent constants needed is reduced from 36 to 21. That is, if we know these 21 constants, we know all 81 constants. If, in addition, we have a plane of elastic symmetry, the number of elastic constants is reduced further from 21 to 13. If there is a second plane of elastic symmetry orthogonal to the first, the number of elastic constants is reduced still further. The second plane of symmetry implies also the symmetry about the third orthogonal plane (*orthotropic symmetry*) and the number of elastic constants is reduced to 9. For a *transversely isotropic* material, the number is reduced to 5. Further, if we specify *cubic symmetry*, that is, the properties along the x-, y-, z-directions are identical, then we can not distinguish between directions x, y, and z. It follows that it takes only three independent constants to describe the elastic behavior of such a material. Finally, if we have a solid whose elastic properties are not a function of direction at all, then we need only two independent elastic constants to describe its behavior. Details of this are given below.

4.5.1 Isotropic Linear Elastic Stress–Strain Relations

For an *isotropic* material, the elastic constants in Eq. (4.7) must be the same for all directions. Thus tensor C_{ijkl} must be an isotropic fourth-order tensor. It can be shown that the most general form for the *isotropic tensor* C_{ijkl} is given by (Sec. 1.14)

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{4.9}$$

where λ, μ, and α are scalar constants. Now, since C_{ijkl} must satisfy the symmetry conditions in Eqs. (4.8), we have $\alpha = 0$ in Eq. (4.9). Thus Eq. (4.9) must take the form

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4.10}$$

From Eqs. (4.7) and (4.10), we get

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\varepsilon_{kl}$$

or

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{4.11}$$

Hence for an isotropic linear elastic material, there are only *two* independent material constants, λ and μ, which are called *Lame's constants*.

Conversely, strains ε_{ij} can be expressed in terms of stresses in the constitutive law of Eq. (4.11). From Eq. (4.11), one has

$$\sigma_{kk} = (3\lambda + 2\mu)\varepsilon_{kk}$$

or

$$\varepsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu} \tag{4.12}$$

Substituting this value of ε_{kk} into Eq. (4.11) and solving for ε_{ij}, we get

$$\varepsilon_{ij} = \frac{-\lambda\delta_{ij}}{2\mu(3\lambda+2\mu)}\sigma_{kk} + \frac{1}{2\mu}\sigma_{ij} \tag{4.13}$$

Equations (4.11) and (4.13) are the general forms of the constitutive law for an isotropic linear elastic material. An important consequence of these equations is that for an isotropic material, principal directions of stress and strain tensors *coincide*. The constants λ and μ are determined from *experimental test results* for some simple states of stresses and strains. In the following, some of these simple tests are described. Also, alternative definitions of elastic constants, which are used frequently in engineering applications, are given.

1. Hydrostatic Compression Test (Fig. 4.3a): In this case, $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p = \sigma_{kk}/3$ are the only nonzero components of stress. *Bulk modulus*, K, is defined for this case as the ratio between the hydrostatic pressure p and the corresponding volume change ε_{kk}. Thus, from Eq. (4.12),

$$K = -\frac{p}{\varepsilon_{kk}} = \lambda + \tfrac{2}{3}\mu \tag{4.14}$$

2. Simple Tension Test (Fig. 4.3b): The only nonzero stress component is $\sigma_{11} = \sigma$. Defining the *Young's modulus*, E, and *Poisson's ratio*, ν, as

$$E = \frac{\sigma_{11}}{\varepsilon_{11}}, \qquad \nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}} \tag{4.15}$$

we have from Eqs. (4.11) and (4.13)

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}, \qquad \nu = \frac{\lambda}{2(\lambda+\mu)} \tag{4.16}$$

3. Simple Shear Test (Fig. 4.3c): Here $\sigma_{12} = \sigma_{21} = \tau_{12} = \tau_{21} = \tau$, and all other stress components are zero. The *shear modulus*, G, is defined as

$$G = \frac{\sigma_{12}}{\gamma_{12}} = \frac{\tau}{2\varepsilon_{12}}$$

From Eq. (4.11), we have

$$G = \mu \tag{4.17}$$

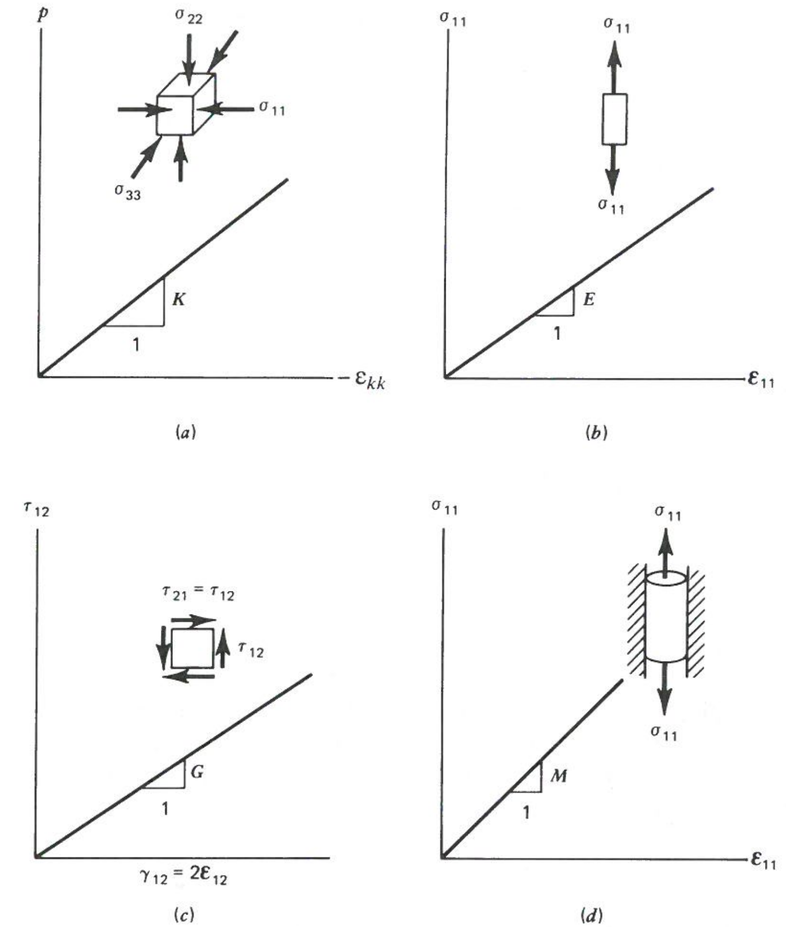

FIGURE 4.3 Behavior of isotropic linear elastic material in simple tests. (*a*) Hydrostatic compression test ($\sigma_{11} = \sigma_{22} = \sigma_{33} = p$). (*b*) Simple tension test. (*c*) Pure shear test. (*d*) Uniaxial strain test.

4. Uniaxial Strain Test (Fig. 4.3d): This test is carried out by applying a uniaxial stress component σ_{11} in the axial direction of a cylindrical test sample whose lateral surface is *restrained* against lateral movement. Thus axial strain ε_{11} is the only nonvanishing component in this case. The *constrained modulus*, M, is defined as the ratio between σ_{11} and ε_{11}. Substituting in Eq. (4.11) for $\varepsilon_{kk} = \varepsilon_{11}$, the stress σ_{11} is given by

$$\sigma_{11} = \lambda\varepsilon_{11} + 2\mu\varepsilon_{11}$$

or

$$M = \frac{\sigma_{11}}{\varepsilon_{11}} = (\lambda + 2\mu) \tag{4.18}$$

The important relations among different elastic moduli are summarized in Table 4.1, which is very helpful in solving practical problems. Figure 4.3 illustrates the stress–strain relations describing the model behavior under the simple test conditions outlined above.

The constitutive relations given in Eqs. (4.11) and (4.13) can be written in different forms by utilizing the relations between the elastic moduli given in Table 4.1. In particular, the following forms are used frequently in practice:

$$\sigma_{ij}=\frac{E}{(1+\nu)}\varepsilon_{ij}+\frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij} \tag{4.19}$$

$$\varepsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}\delta_{ij} \tag{4.20}$$

$$\sigma_{ij}=2G\varepsilon_{ij}+\frac{3\nu K}{(1+\nu)}\varepsilon_{kk}\delta_{ij} \tag{4.21}$$

$$\varepsilon_{ij}=\frac{1}{2G}\sigma_{ij}-\frac{\nu}{3K(1-2\nu)}\sigma_{kk}\delta_{ij} \tag{4.22}$$

Decomposition of Stress–Strain Relations

A neat and logical separation exists between the mean (hydrostatic or volumetric) and the deviatoric (shear) response components in the isotropic linear elastic model. The hydrostatic response can be derived directly from Eq. (4.11) upon contraction ($i=j$, $\sigma_{ii}=I_1=3p$, $\delta_{ii}=3$); that is,

$$3p=(3\lambda+2\mu)\varepsilon_{kk}=(3\lambda+2G)\varepsilon_{kk} \tag{4.23}$$

Substituting for $3\lambda+2G=3K$, from Table 4.1, we have

$$p=K\varepsilon_{kk} \tag{4.24}$$

To derive the deviatoric response relations, we use the relation $s_{ij}=\sigma_{ij}-p\delta_{ij}$, and substitute for σ_{ij} and p from Eqs. (4.11) and (4.23), respectively. This leads to

$$s_{ij}=(\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij})-\tfrac{1}{3}(3\lambda+2G)\varepsilon_{kk}\delta_{ij}$$

which upon substitution for $\varepsilon_{ij}=e_{ij}+\frac{1}{3}\varepsilon_{kk}\delta_{ij}$ and $\mu=G$ becomes

$$s_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2G\left(e_{ij}+\tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right)-\tfrac{1}{3}(3\lambda+2G)\varepsilon_{kk}\delta_{ij}$$

Simplifying, we have the relation

$$s_{ij}=2Ge_{ij} \tag{4.25}$$

TABLE 4.1 Relationships Among Elastic Moduli E, G, K, ν, λ, M

	Shear Modulus, G	Young's Modulus, E	Constrained Modulus, M	Bulk Modulus, K	Lame Parameter, λ	Poisson's Ratio, ν
G, E	G	E	$\frac{G(4G-E)}{3G-E}$	$\frac{GE}{9G-3E}$	$\frac{G(E-2G)}{3G-E}$	$\frac{E-2G}{2G}$
G, M	G	$\frac{G(3M-4G)}{M-G}$	M	$M-\frac{4}{3}G$	$M-2G$	$\frac{M-2G}{2(M-G)}$
G, K	G	$\frac{9GK}{3K+G}$	$K+\frac{4}{3}G$	K	$K-\frac{2G}{3}$	$\frac{3K-2G}{2(3K+G)}$
G, λ	G	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\lambda+2G$	$\lambda+\frac{2G}{3}$	λ	$\frac{\lambda}{2(\lambda+G)}$
G, ν	G	$2G(1+\nu)$	$\frac{2G(1-\nu)}{1-2\nu}$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$\frac{2G\nu}{1-2\nu}$	ν
E, K	$\frac{3KE}{9K-E}$	E	$\frac{K(9K+3E)}{9K-E}$	K	$\frac{K(9K-3E)}{9K-E}$	$\frac{3K-E}{6K}$
E, ν	$\frac{E}{2(1+\nu)}$	E	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$\frac{E}{3(1-2\nu)}$	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	ν
K, λ	$\frac{3(K-\lambda)}{2}$	$\frac{9K(K-\lambda)}{3K-\lambda}$	$3K-2\lambda$	K	λ	$\frac{\lambda}{3K-\lambda}$
K, M	$\frac{3(M-K)}{4}$	$\frac{9K(M-K)}{3K+M}$	M	K	$\frac{3K-M}{2}$	$\frac{3K/M-1}{3K/M+1}$
K, ν	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$3K(1-2\nu)$	$\frac{3K(1-\nu)}{1+\nu}$	K	$\frac{3K\nu}{1+\nu}$	ν

Equations (4.24) and (4.25) give the required separation of the hydrostatic and deviatoric relations. Combining these two equations, we can write the total elastic strains ε_{ij} in terms of the hydrostatic and deviatoric stresses as

$$\varepsilon_{ij}=\tfrac{1}{3}\varepsilon_{kk}\delta_{ij}+e_{ij}=\frac{1}{3K}p\delta_{ij}+\frac{1}{2G}s_{ij} \tag{4.26}$$

or

$$\varepsilon_{ij}=\frac{1}{9K}I_1\delta_{ij}+\frac{1}{2G}s_{ij} \tag{4.27}$$

Similarly, σ_{ij} can be expressed in terms of the volumetric and deviatoric strains in the following form:

$$\sigma_{ij}=K\varepsilon_{kk}\delta_{ij}+2Ge_{ij} \tag{4.28}$$

Restrictions Imposed on Engineering Elastic Constants
(Experimental Facts)

Experimental results for *real* elastic materials have shown that the constants E, G, and K are always positive, that is,

$$E>0, \qquad G>0, \qquad K>0 \tag{4.29}$$

These conditions state that a body must allow loading to do work on it. Thus, as expected, a uniaxial tensile stress in a certain direction causes an extension of the material in the same direction. Similarly, a shear strain caused by a simple shearing stress has the direction of this stress. Finally, a volume decrease is produced by a hydrostatic pressure. Using inequalities (4.29) and the relations in Table 4.1, it can be shown that

$$-1 \leq \nu \leq \tfrac{1}{2} \tag{4.30}$$

We have no practical experience with any existing material that will exhibit a negative value of ν. Thus actual values of ν for most materials in practice are positive. The value $\nu=0.5$ implies that $G=E/3$ and $1/K=0$, or *elastic incompressibility*. Some rubberlike materials are almost incompressible and have a value of $\nu \approx 0.48$.

Typical experimental values for E are

10.6×10^6 psi (73×10^6 kN/m^2) for aluminum
30×10^6 psi (207×10^6 kN/m^2) for steel
4×10^6 psi (27.6×10^6 kN/m^2) for concrete
0.2×10^6 psi (1.38×10^6 kN/m^2) for cellulose acetate.

For the shear modulus G, typical values are

4×10^6 psi (27.6×10^6 kN/m^2) for aluminum
12×10^6 psi (82.8×10^6kN/m^2) for steel
0.17×10^6 psi (1.173×10^6 kN/m^2) for polystyrene.

Typical values for ν are

0.29 for steel
0.19 for concrete
0.33 for aluminum
0.45 for lead.

Finally, for bulk modulus, K, typical values are

24×10^6 psi (165.6×10^6 kN/m^2) for steel
10×10^6 psi (69×10^6 kN/m^2) for aluminum
17.9×10^6 psi (122.9×10^6 kN/m^2) for copper.

It is to be emphasized that only two of these engineering elastic constants are independent for isotropic linear elastic materials.

4.5.2 Isotropic Linear Elastic Stress–Strain Relations in Matrix Form

The stress–strain relationships discussed above can be conveniently expressed in matrix forms. These forms are suitable for use in solutions by numerical methods (e.g., finite element method). In the following, matrix forms are given for various cases.

1. Three-Dimensional Case: The stress and strain components are defined by the two vectors $\{\sigma\}$ and $\{\varepsilon\}$, respectively, which are given by

$$\{\sigma\}=\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}, \qquad \{\varepsilon\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix} \tag{4.31}$$

Now Eq. (4.19) can be written in matrix form as

$$\{\sigma\}=[C]\{\varepsilon\} \tag{4.32}$$

where matrix $[C]$ is called the *elastic constitutive* or *elastic moduli matrix*, which is given by

$$[C]=\frac{E}{(1+\nu)(1-2\nu)} \times \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0\\ \nu & (1-\nu) & \nu & 0 & 0 & 0\\ \nu & \nu & (1-\nu) & 0 & 0 & 0\\ 0 & 0 & 0 & \dfrac{(1-2\nu)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \dfrac{(1-2\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \dfrac{(1-2\nu)}{2}\end{bmatrix} \tag{4.33a}$$

or alternatively, substituting for ν and E in terms of K and G from Table 4.1 gives

$$[C]=\begin{bmatrix} (K+\frac{4}{3}G) & (K-\frac{2}{3}G) & (K-\frac{2}{3}G) & 0 & 0 & 0 \\ (K-\frac{2}{3}G) & (K+\frac{4}{3}G) & (K-\frac{2}{3}G) & 0 & 0 & 0 \\ (K-\frac{2}{3}G) & (K-\frac{2}{3}G) & (K+\frac{4}{3}G) & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{4.33b}$$

Also, Eq. (4.20) can be written in matrix form as

$$\{\varepsilon\}=[C]^{-1}\{\sigma\}=[D]\{\sigma\} \tag{4.34}$$

where the *elastic compliance* matrix, $[D]$, is given by the inverse of matrix $[C]$:

$$[D]=\frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{4.35}$$

2. Plane Stress Case: It can be shown that Eqs. (4.32) and (4.34), when reduced to the two-dimensional plane stress case ($\sigma_z=\tau_{yz}=\tau_{zx}=0$), take the following simple forms:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}=\frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{4.36}$$

and

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}=\frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{4.37}$$

It is to be noted that in the plane stress case, the strain component ε_z is nonzero, while shear strain components γ_{yz} and γ_{zx} are zero. The component ε_z

has the value

$$\varepsilon_z=\frac{-\nu}{E}(\sigma_x+\sigma_y)=\frac{-\nu}{1-\nu}(\varepsilon_x+\varepsilon_y) \tag{4.38}$$

That is, ε_z is a linear function of ε_x and ε_y.

The plane stress relations given above are commonly used in many practical applications. For instance, the analysis of thin flat plates loaded in the plane of the plate ($x-y$ plane) are often treated as plane stress problems.

3. Plane Strain Case: The plane strain conditions ($\varepsilon_z=\gamma_{yz}=\gamma_{zx}=0$) are normally found in the elongated bodies of uniform cross sections subjected to uniform loading along their longitudinal axis (z-axis), such as in the case of tunnels, soil slopes, and retaining walls. Under the conditions of plane strain, Eqs. (4.32) and (4.34) can be reduced to the simple form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{4.39}$$

and

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}=\frac{(1+\nu)}{E}\begin{bmatrix} (1-\nu) & -\nu & 0 \\ -\nu & (1-\nu) & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{4.40}$$

For this case the stress components $\tau_{yz}=\tau_{zx}=0$, and the stress component σ_z has the value

$$\sigma_z=\nu(\sigma_x+\sigma_y) \tag{4.41}$$

4. Axisymmetric Case: Analysis of bodies of revolution under axisymmetric loading is similar to those of plane stress and plane strain conditions since it is also two-dimensional. With reference to Fig. 4.4, the nonzero stress components in the axisymmetric case are σ_r, σ_z, σ_θ, and τ_{rz}, and the corresponding strains are ε_r, ε_z, ε_θ, and γ_{rz}. Equations (4.32) and (4.34) can be reduced to the forms ($\tau_{z\theta}=\tau_{\theta r}=\gamma_{z\theta}=\gamma_{\theta r}=0$):

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & \nu & 0 \\ \nu & (1-\nu) & \nu & 0 \\ \nu & \nu & (1-\nu) & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{zr} \end{Bmatrix} \tag{4.42}$$

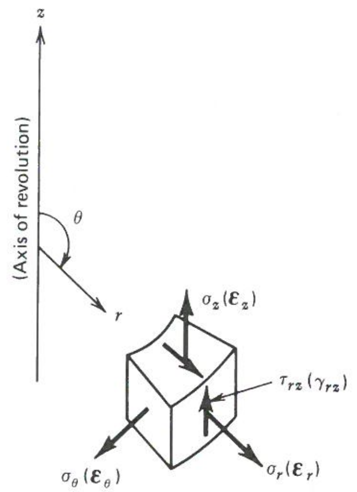

FIGURE 4.4 Stress and strain components in an axisymmetric case.

and

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 \\ -\nu & 1 & -\nu & 0 \\ -\nu & -\nu & 1 & 0 \\ 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} \tag{4.43}$$

Example 4.1: Show that the *principal axes of stress and strain coincide* for an isotropic linear elastic material.

Proof: Referring to the principal axes of strain, the strain tensor is given by (ε_1, ε_2, and ε_3 are the principal strains)

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

If these values are substituted into Eq. (4.11), the resulting shear stress components are all zero; that is,

$$\sigma_{ij} = 0 \qquad \text{for } i \neq j$$

Thus the principal stresses and the principal strains are coaxial.

Example 4.2: The elastic *complementary energy density*, Ω, is defined as (see Sec. 4.7)

$$\Omega = \int_0^{\sigma_{ij}} \varepsilon_{ij}\, d\sigma_{ij} \tag{4.44}$$

Find an expression for Ω in terms of the stress invariants I_1 and J_2 for an isotropic linear elastic material.

Solution: Substituting for $\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}$ in Eq. (4.20), we have

$$\varepsilon_{ij} = \frac{1+\nu}{E} s_{ij} + \frac{1-2\nu}{3E} I_1 \delta_{ij} \tag{4.45}$$

where $I_1 = \sigma_{kk}$. Substituting for ε_{ij} from Eq. (4.45), we can write Ω as

$$\Omega = \frac{1+\nu}{E}\int_0^{\sigma_{ij}} s_{ij}\, d\sigma_{ij} + \frac{1-2\nu}{3E}\int_0^{\sigma_{ij}} I_1 \delta_{ij}\, d\sigma_{ij}$$

which can be reduced to ($J_2 = \frac{1}{2}s_{ij}s_{ij}$, $dJ_2 = s_{ij}\,ds_{ij} = s_{ij}\,d\sigma_{ij}$, $dI_1 = \delta_{ij}\,d\sigma_{ij}$)

$$\Omega = \frac{1+\nu}{E}\int_0^{J_2} dJ_2 + \frac{1-2\nu}{3E}\int_0^{I_1} I_1\, dI_1 \tag{4.46}$$

$$= \frac{1+\nu}{E} J_2 + \frac{1-2\nu}{6E} I_1^2 \tag{4.47a}$$

or, in terms of G, K, we have

$$\Omega = \frac{J_2}{2G} + \frac{I_1^2}{18K} \tag{4.47b}$$

For positive values of bulk modulus K and shear modulus G the complementary energy density Ω in Eq. (4.47b) is a *positive definite quadratic* form in the components of stress (since both I_1^2 and J_2 are always positive and can not be zero unless $\sigma_{ij}=0$). For an isotropic linear elastic material, Ω is found explicitly in terms of the existing components of stress (current values of I_1 and J_2) irrespective of the loading (stress) path followed to reach these current stress components; that is, Ω in this case is *path independent*. However, in general this is not true for Cauchy elastic materials, whether linear or nonlinear. This is illustrated further in the following example for a linear Cauchy elastic model.

Example 4.3: In the two-dimensional principal space (σ_1, σ_2, ε_1, and ε_2), the behavior of a *linear Cauchy elastic* material is described by the stress–strain relations:

$$\begin{aligned} \varepsilon_1 &= a_{11}\sigma_1 + a_{12}\sigma_2 \\ \varepsilon_2 &= a_{21}\sigma_1 + a_{22}\sigma_2 \end{aligned} \tag{4.48}$$

where a_{11}, a_{12}, a_{21}, and a_{22} are material constants and $a_{12} \neq a_{21}$. Consider two different stress paths 1 and 2 as shown in Fig. 4.5a. Path 1 is from (0,0) to