CE394M: Critical State and Cam-Clay

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Overview

1. Critical State Soil Mechanics
2. Cam-Clay
3. Modified Cam-Clay
4. Cam-Clay material properties determination

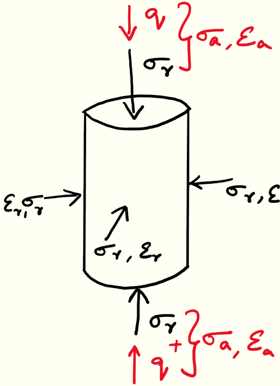
Critical State Soil Mechanics

Roscoe et al., (1958), Schofield & Worth (1968), Wood (1990):

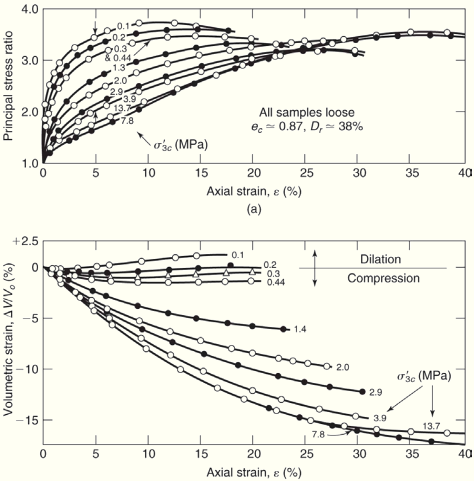
- Provides a conceptual framework in which to interpret stress-strain-strength-volumetric strain response of soil.
- Started as a qualitative, rather than a mathematical model
- A unified framework of known or observed soil responses: drained / undrained / etc

Critical state variables

- Mean stress: $p' = \frac{\sigma'_a + 2\sigma'_r}{3} = p - u.$
- Deviatoric stress: $q = \sigma'_a - \sigma'_r = \sigma_a - \sigma_r$
- Specific volume: $v = \frac{V_T}{V_s} = \frac{V_s + V_v}{V_s} = 1 + e.$

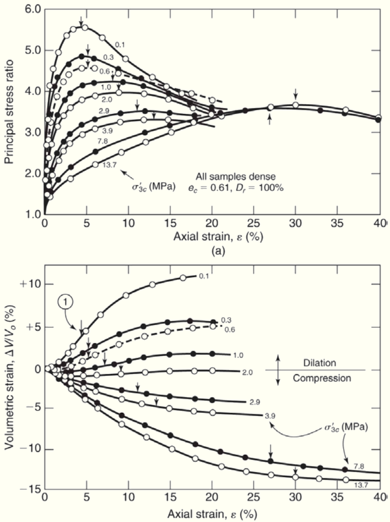


Triaxial compression drained: loose



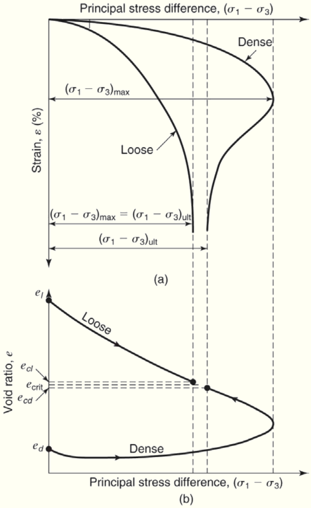
Loose Sacramento River sand: (a) principal stress ratio versus axial strain; (b) volumetric strain versus axial strain (Lee, 1965).

Triaxial compression drained: dense



Dense Sacramento River sand: (a) principal stress ratio versus axial strain; (b) volumetric strain versus axial strain (Lee, 1965).

Triaxial compression drained: loose v dense

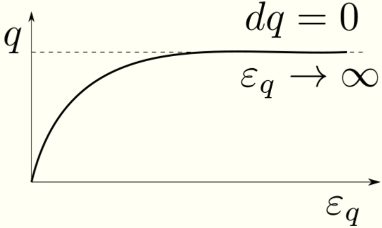


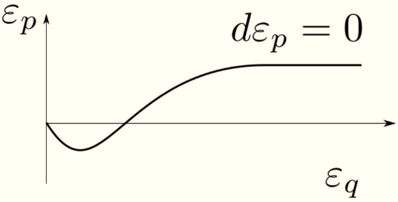
Triaxial tests on "loose" and "dense" specimens of a typical sand: (a) stress-strain curves; (b) void ratio changes during shear (Hirschfeld, 1963).

Critical State Hypothesis: I

Roscoe, Schofield & Worth (1958): **At shear-failure, soil exists at a unique state**

- $d\varepsilon_s >> 0$ unlimited shear strain potential.
- $dp' = dq = d\varepsilon_v = 0$ no change in p', q, ε_v.
- Critical state stress ratio: $\eta = q/p' = const = M$ at failure $q = Mp'$.





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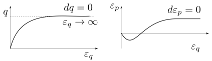
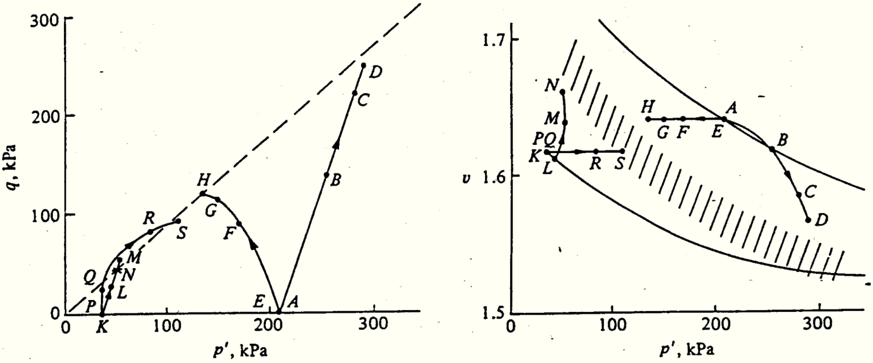


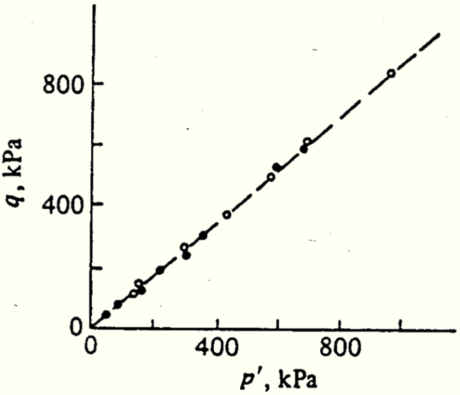
Soil is sheared to a point where stresses are stationary ($dq = dp' = 0$) with no futher change in volume ($d\varepsilon_v = 0$), unlimited shear strains ($d\varepsilon_s >> 0$) and q/p' has a fixed value: **critical state**.

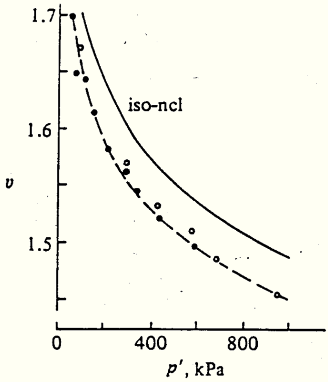
M can be related to phi': $M = \frac{6 \sin \phi'}{3 - \sin \phi'}$.

Conventional drained - undrained TXC

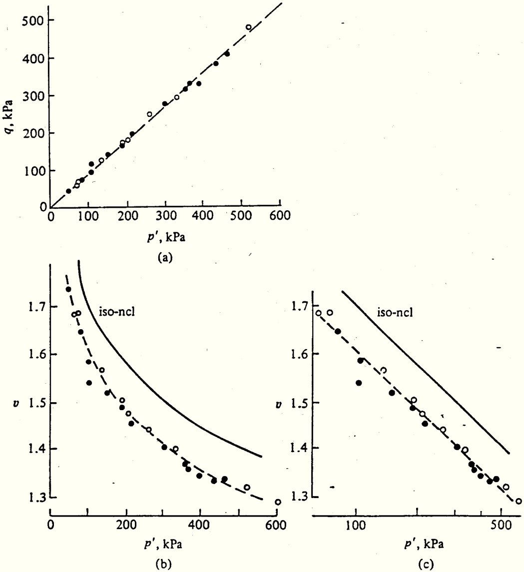


Conventional Drained TXC

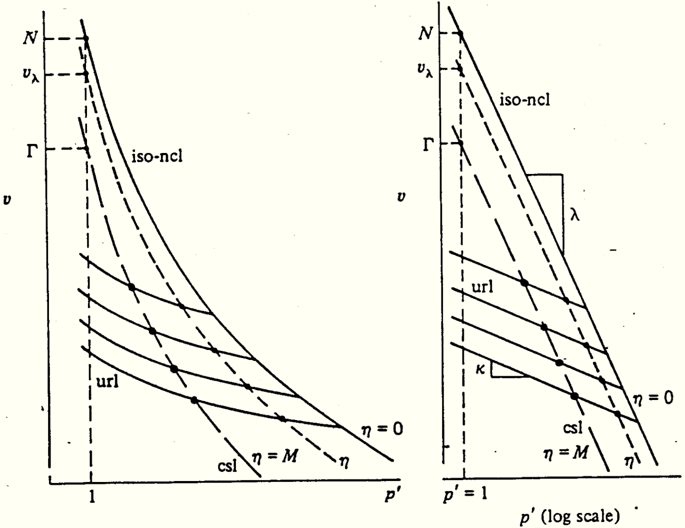




Conventional Undrained TXC

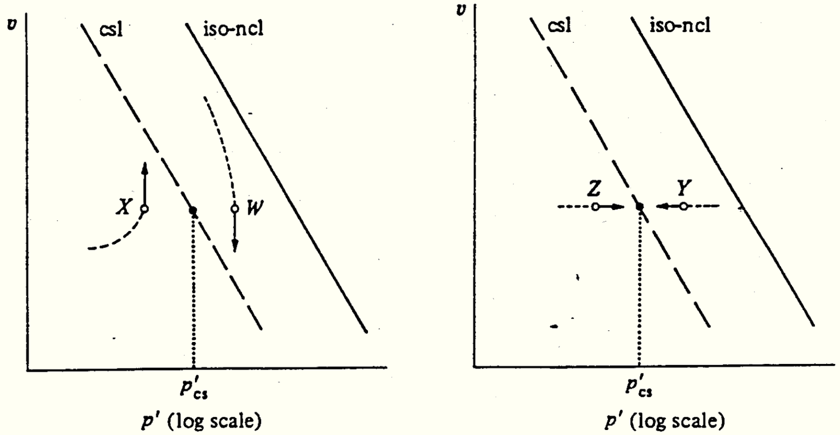


v - ln p



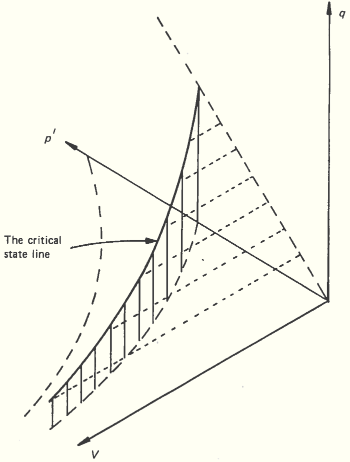
N
v_λ
Γ
v
iso-ncl
url
csl
$\eta = M$
η
$\eta = 0$
1
p'
N
v_λ
Γ
v
iso-ncl
λ
url
κ
csl
$\eta = M$
η
$\eta = 0$
$p' = 1$
p' (log scale)

Conventional drained - undrained TXC



Critical State Hypothesis: II

Critical state is a function of q, p', v.



The CSL (p', v, q) space is given by the intersection of two planes: $q = Mp'$ and a cruved vertical plane $v = \Gamma - \lambda \ln p'$

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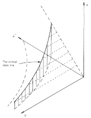

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Critical state curve connecting critical state points:

- Crticial state line
- Defined in 3D but we'll look at projections into $q - p'$ and $v - p'$ space

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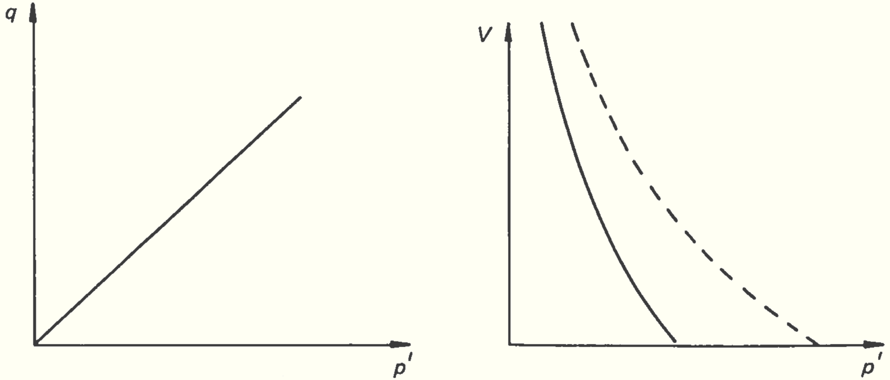

The CSL in (a) (p', q) plot and (b) (p', v) plot (isotropic normal compression line is shown in dashed)

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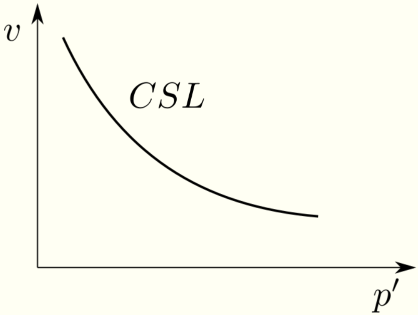

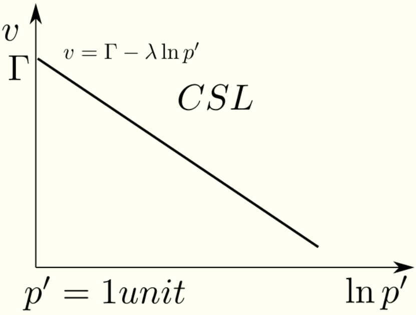

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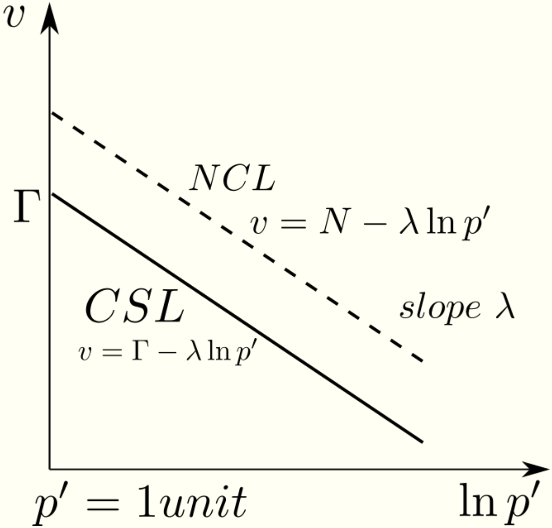
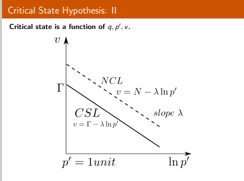
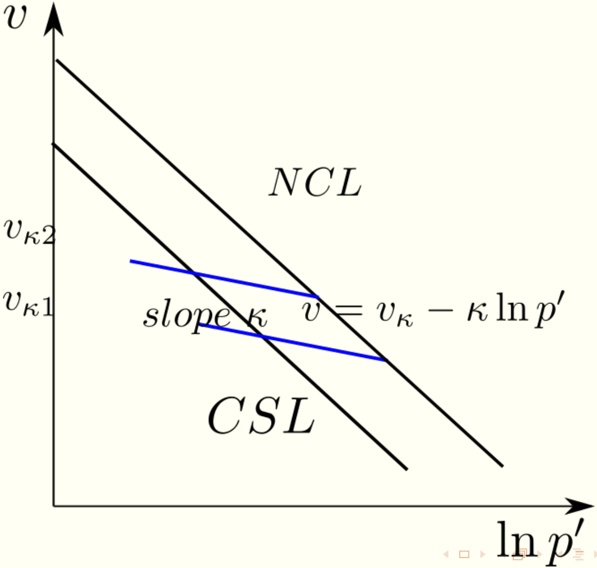
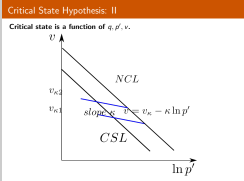




Isotropic virgin compression line (VCL) $\eta = 0$. NCL is parallel to CSL. VCL is $\eta = 0$, while CSL $\eta = M$. Oedometer falls between VCL and CSL at a constant η: $0 < \eta < M$.

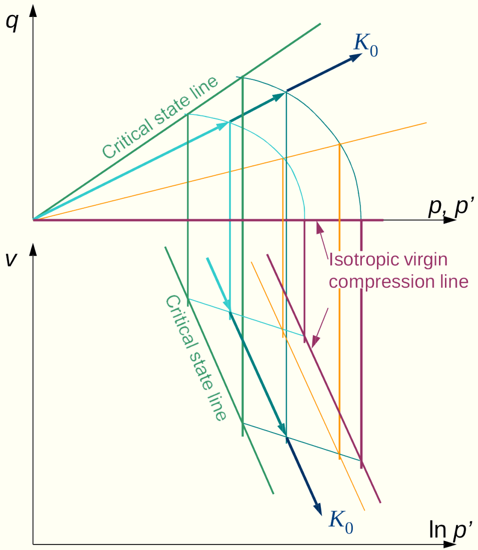
Critical state is a function of q, p', v.



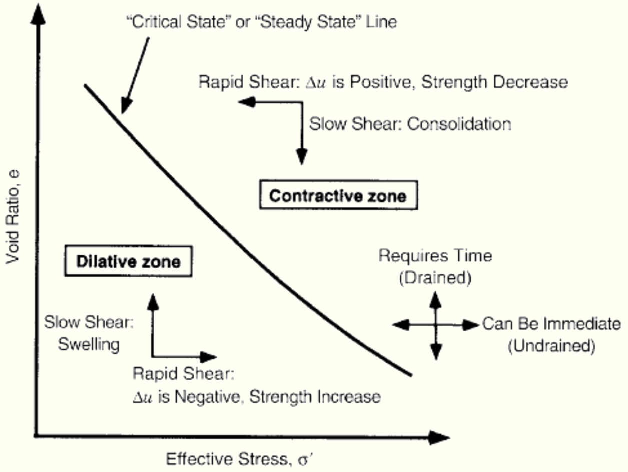


v_κ depends on which κ line you are on. $\kappa \neq c_r$ and $\lambda \neq C_c$

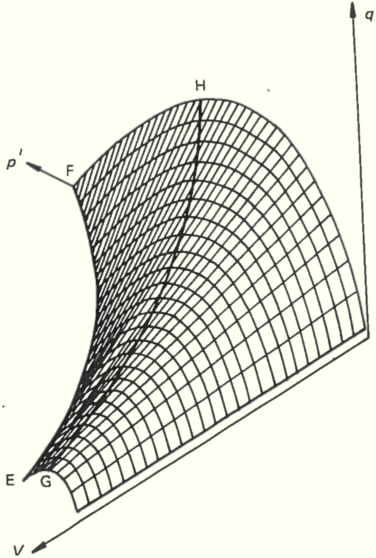
Stress paths $\sigma_3'/\sigma_1' = K_c = const$



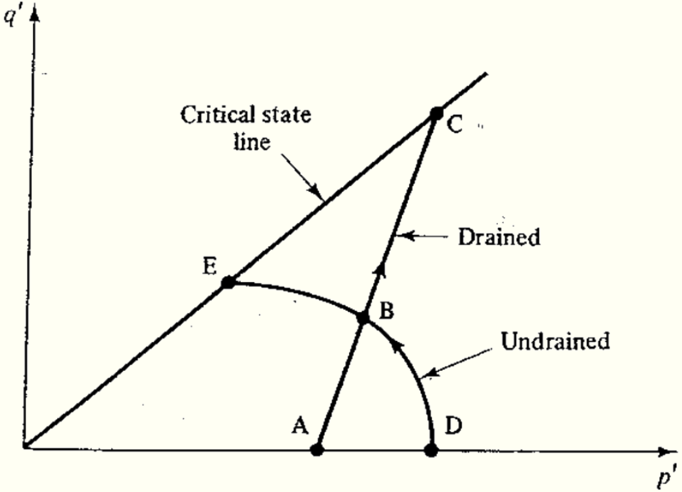
Clay behavior



Critical state boundary surface

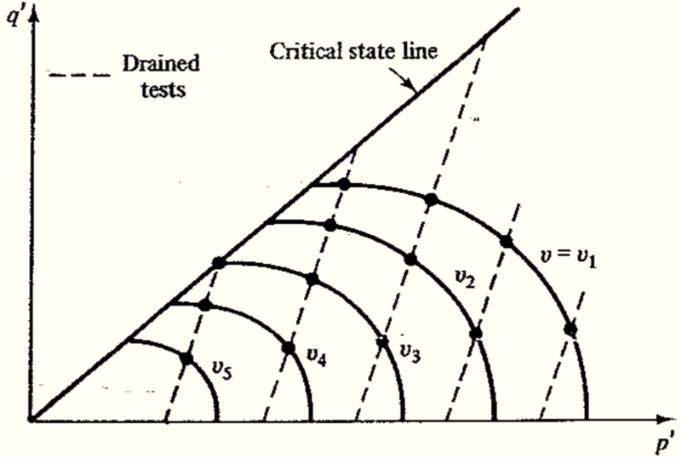


Critical state boundary surface: Drained vs undrained



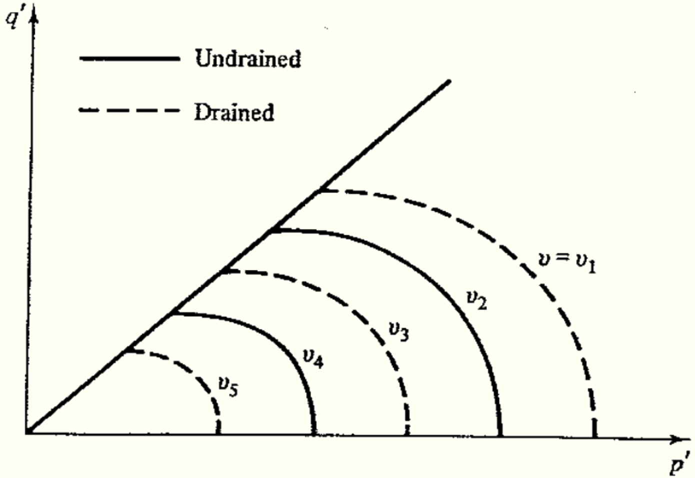
Drained and undrained paths in $q - p'$

Critical state boundary surface



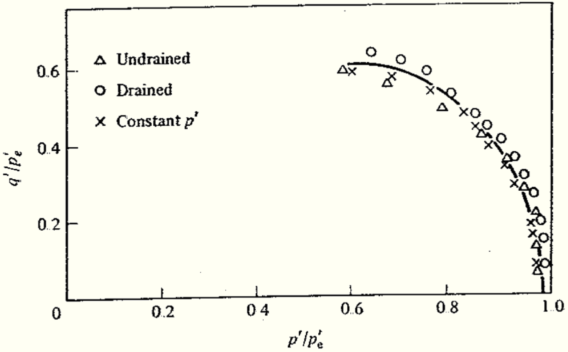
Contours of constant v from drained test

Critical state boundary surface



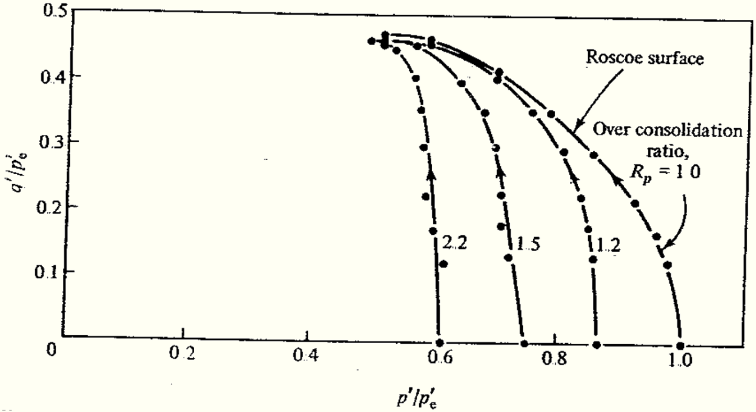
Contours of constant v from drained and undrained test

Critical state boundary surface



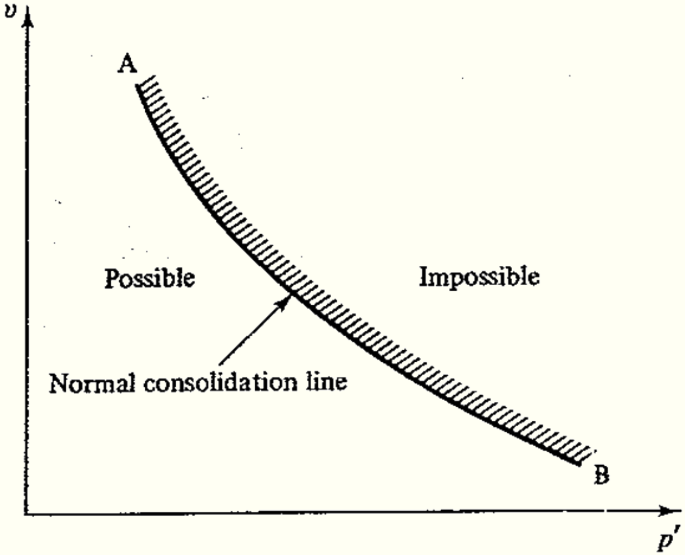
Test paths in $q/p'_e and p'/p'_e$ for a drained, undrained and a test at a constant p' on NC Kaolin clay (Balasubramanian., 1969).

p'_e is the mean normal effective stress on the NCL at that specific volume.

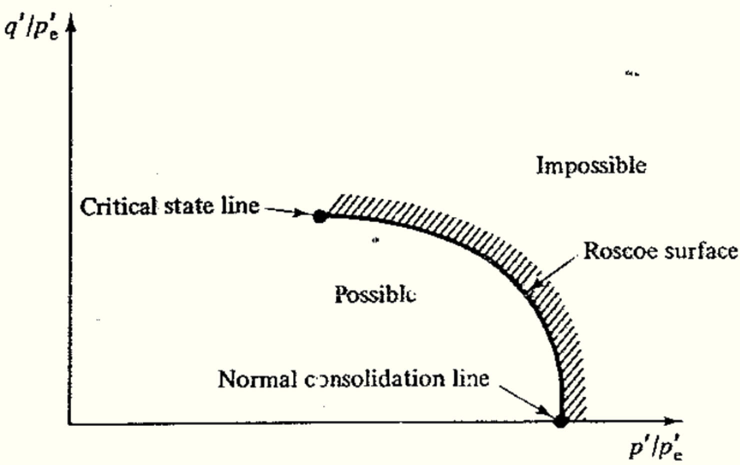


Paths in $q/p'_e and p'/p'_e$ for undrained tests on LOC samples of Kaolin clay (Loudon., 1967).

Roscoe surface

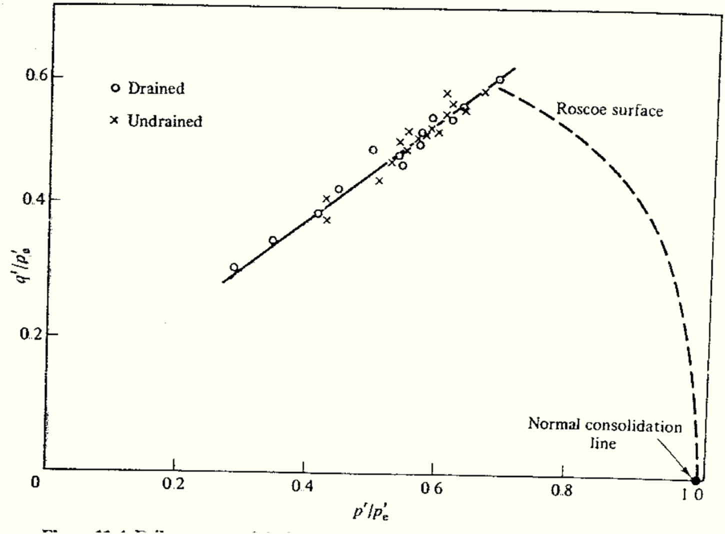


Roscoe surface



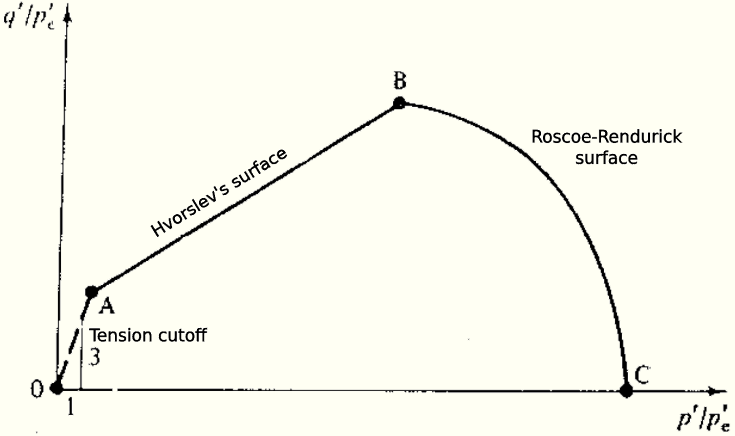
The Roscoe - Rendulic (1936) surface as a state boundary surface.

Hvorslev-Roscoe surface



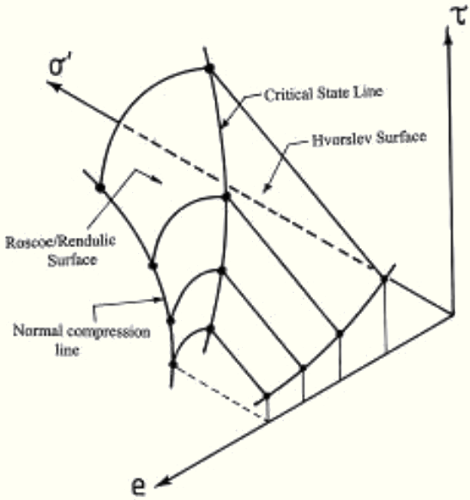
Failure states of drained and undrained tests on OC samples of Weald clay (Parry 1960)

Critical state boundary surface



The complete state boundary surface $q/p'_e andp'/p'_e$

Critical state boundary surface



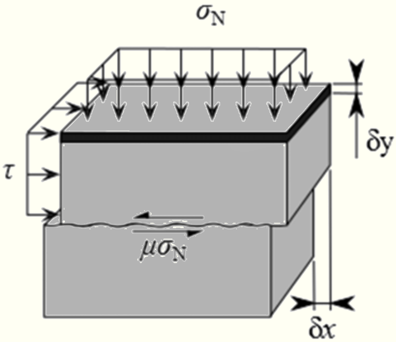
The complete state boundary surface $q - p' - v$ space

Summary of critical state behavior

- Can only traverse NCL in one direction
- Can traverse RCL (κ-line) in both directions
- To move from one κ-line to another must move along NCL. Hence, plastic volumetric strains must occur.
- Critical state line is **NOT** a yield surface. It's where it's going but a lot of plastic straining is needed to get there. (if $CSL = F = 0$) then with associative flow rule $d\varepsilon_v^p \neq 0$ at critical state. Real F is horizontal at critical state.

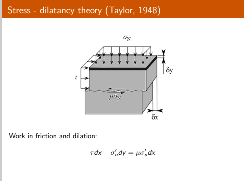
Overview

Stress - dilatancy theory (Taylor, 1948)

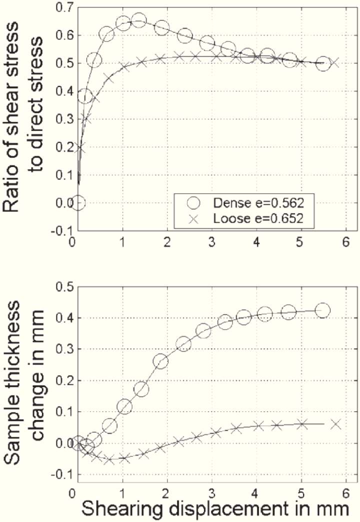


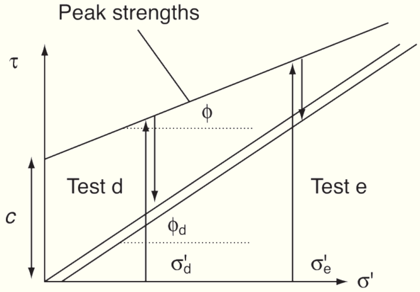
Work in friction and dilation:

$$\tau dx - \sigma'_n dy = \mu \sigma'_n dx$$



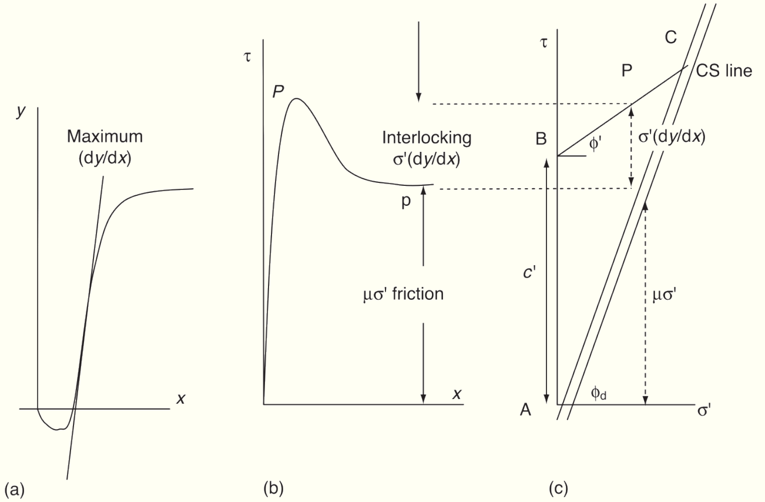
Taylor (1948) proposed a stress-dilatancy theory based on the work balance equation: The external work corresponds to the product of the measured displacements and forces (assuming that the elastic deformation is negligible). The internal work corresponds to the frictional force.

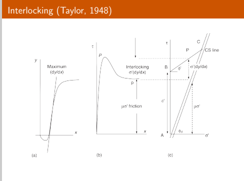




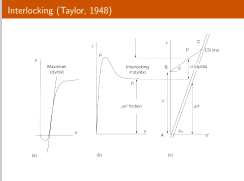
Work in Fig. 1(a) is $\tau\, dx - \sigma'\, dy = \mu\sigma'\, dx$
This gives strengths $\tau/\sigma' = \mu + (dy/dx)$

Interlocking (Taylor, 1948)





Terzaghi's review of Taylor's book "I am convinced that the theories of soil mechanics and the results of laboratory tests serve only to guide the engineer toward a recognition of the factors which may affect the design and construction of a real project" from review sent to Wiley by R B Peck July 31 1944 quoted from page 213 "Karl Terzaghi; the Engineer as Artist" R E Goodman (1999)

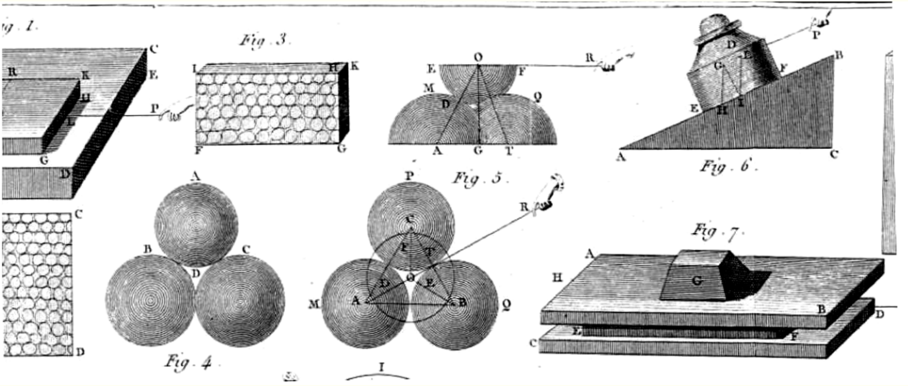


John Wiley & Sons reply to Terzaghi

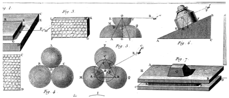
... (Taylor's book) will be published by one of our competitors if we do not take it. Under the circumstances, we see nothing to do but publish it. However, as I said in the first paragraph of this letter, we believe that each book will be judged on its own merits, and certainly we have no fears for the success of (Terzaghi & Peck).

- E P Hamilton (President) December 17, 1946

Belidor's friction hypothesis, 1737



Belidor's friction hypothesis, 1737

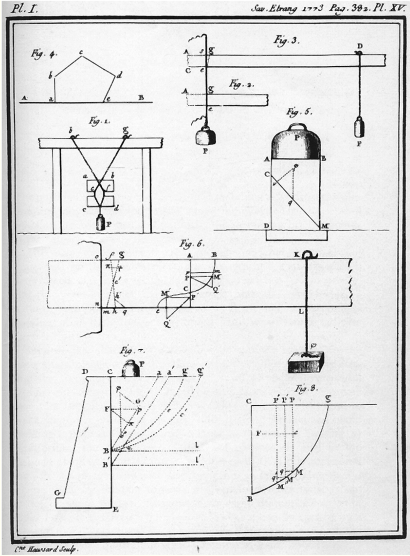


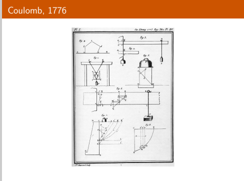
Belidor attributed sliding friction coefficients of 1/3 to the hemispherical geometry of roughness Navier (1819) called Belidor's theory très-fautive (very faulty) but he offered no alternative to it.

Coulomb, 1776

Coulomb, at school in Mezieres, learned friction theory from a text book written by Belidor in 1737 (reprinted with notes by Navier in 1819) and a Dutch concept of (cohesion) = (adhesion). In his 1773 paper he reported new rock strength data Terzaghi (1936), in "A fundamental fallacy in earth pressure computation", rejected Rankine's theory of limiting statics of granular media, (Sokolovski), for lacking consideration of strains

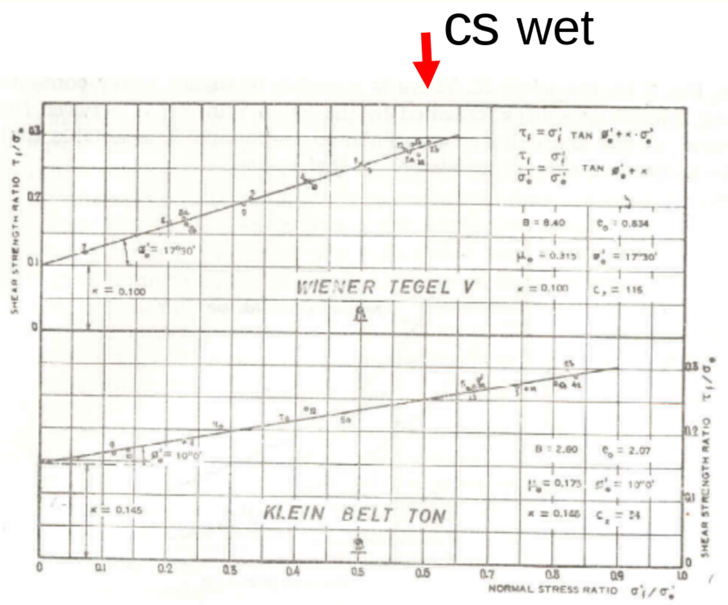
Coulomb, 1776





Coulomb defined soil internal friction as the angle of repose ϕ_d of drained slopes. In Coulomb's rock tests, cohesion in shear was slightly greater than adhesion in tension, so he considered it safe to design with tension data His wall design assumed that newly compacted soil has zero cohesion

Terzaghi interprets Hvorslev's (1937) shear box tests



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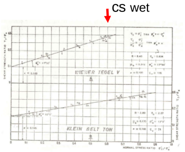
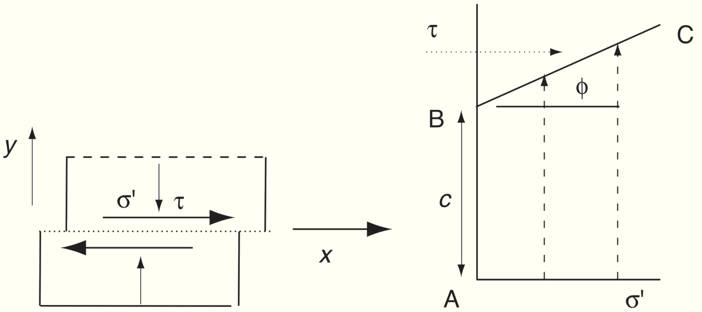


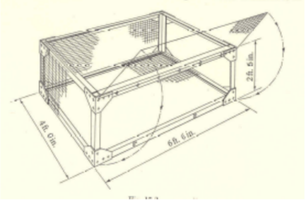
Terzaghi fitted "true" cohesion and friction to peak strengths found by Hvorslev in shear box tests, normalısing them by equıvalent pressure.

A point Terzaghi missed in interpreting test data: Hvorslev's data ended at a critical state point. Terzaghi should have asked Hvorslev why he put equations in space where there were no peak strengths. Filling the space meant that he asked no questions about the wet side of critical states $v_\lambda = v + \lambda \ln p' > \Gamma$.

Terzaghi misinterprets Hvorslev's (1937) shear box tests



Plastic design of a steel frame, Baker (1948)



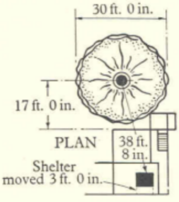
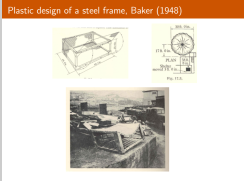


Fig. 17.3.



Cambridge text book example plastic design of shelter to resist floor load 20 lb/sq.ft falling 9 ft in bombed house; Mother and 3 children survived WW II 250kg bomb in Falmouth, UK.

A paste of soil saturated with water is plastic, (from the Greek word $\pi\lambda\alpha\sigma\sigma\epsilon l\nu$ *plassein* to mould, as in moulding pottery from clay).

An aggregate of separate hard grains in a critical state behaves as a ductile plastic continuum. Plastic design guides us to select construction materials and methods; soil is not plastic and ductile if over compacted to high peak strength

Calladine's associated plastic flow (1963)

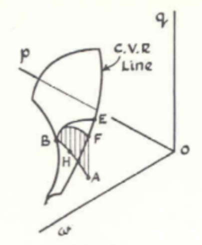


Fig. 4.

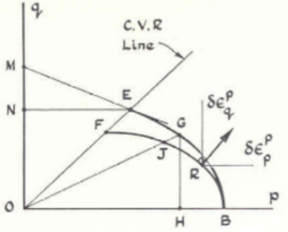


Fig. 5.

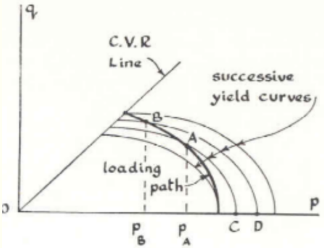


Fig. 6.

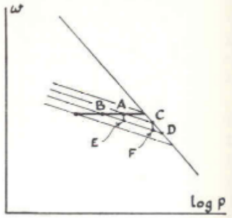
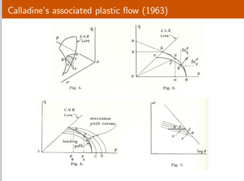


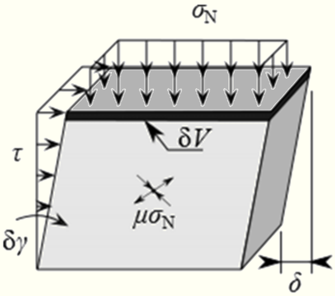
Fig. 7.



Yield loci for paste with $v_\kappa =$ (const) on wet side of Critical-states, satisfy the associated flow rule $dp'dv + dqd\varepsilon = 0$. The Original Cam-clay locus was based on this plus Thurairajah's dissipation function.

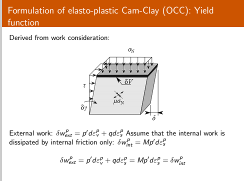
Formulation of elasto-plastic Cam-Clay (OCC): Yield function

Derived from work consideration:



External work: $\delta w^p_{ext} = p' d\varepsilon^p_v + q d\varepsilon^p_s$ Assume that the internal work is dissipated by internal friction only: $\delta w^p_{int} = M p' d\varepsilon^p_s$

$$\delta w^p_{ext} = p' d\varepsilon^p_v + q d\varepsilon^p_s = M p' d\varepsilon^p_s = \delta w^p_{int}$$



This dissipation function can be regarded simply as generalisation of Taylor's equation. It should be noted that both Taylor's equation and Cam-Clay dissipation function equation assume that when there is some combination of volumechange (dy or $\partial\varepsilon_v$) and of shear distortion (dx or $\partial\varepsilon_s$) it is the shear strain that determines the dissipation rate. The dilation or volume change is a geometrical consequence of interlocking, and does not appear explicitly in the dissipation function.

Cam-Clay (OCC): Stress dilatancy relation

$$p' d\varepsilon_v^p + q d\varepsilon_s^p = M p' d\varepsilon_s^p$$

Rearranging the terms (divide by $p' d\varepsilon_s^p$):

$$\frac{d\varepsilon_v^p}{d\varepsilon_s^p} = M - \frac{q}{p'} = M - \eta$$

Where $\eta = q/p'$ is defined as the stress-ratio. This equation is known as the dilatancy expression and expresses the ratio in plastic volumetric and deviatoric components.

$$q/p < M: \quad \frac{d\varepsilon_v^p}{d\varepsilon_q^p} > 0 \rightarrow \quad d\varepsilon_v^p > 0 \quad \text{Contractive response}$$

$$q/p > M: \quad \frac{d\varepsilon_v^p}{d\varepsilon_q^p} < 0 \rightarrow \quad d\varepsilon_v^p < 0 \quad \text{Dilative response}$$

$$q/p = M: \quad d\varepsilon_v^p = 0 \quad \text{No volume change}$$

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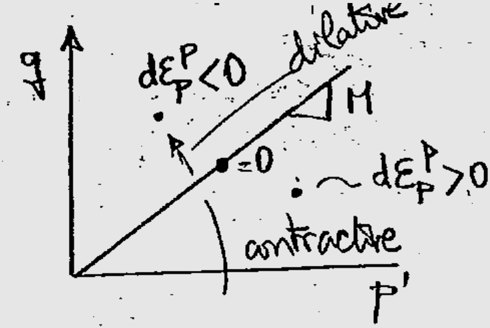
The critical state is defined by an absence of volume change or, in other words, a nil dilatancy conditions. Therefore, at critical state, the stress-dilatancy rule yields to the critical state condition $\eta = M$.



The original idea was very simple. The yield locus must be such that each associated flow rule $(d\varepsilon_v, d\varepsilon_s)$ would be orthogonal to the tangent to the yield locus.

$$\frac{d\varepsilon_v}{d\varepsilon_s} = -\frac{dq}{dp'}$$

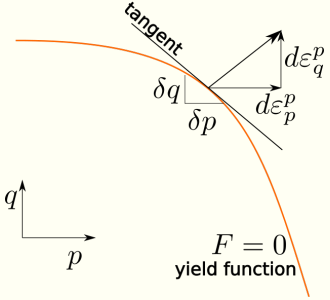
From stress dilation condition:

$$\frac{dq}{dp'} = -(M - \eta) = -M + \eta$$

Integrating we obtain:

$$q = Mp' \ln\left(\frac{p'_c}{p'}\right)$$

Where p'_c is the value of p' at $q = 0$.



$$\eta = q/p \quad \rightarrow d\eta = \frac{\partial \eta}{\partial q} dq + \frac{\partial \eta}{\partial p'} dp'$$

Which gives:

$$d\eta = \frac{dq}{p} - \frac{q}{p^2} dp \rightarrow \quad dq = p d\eta + \eta dp$$

We know from flow rule and orthogonality: $dq = dp(-M + \eta)$
Equating the above 2 equations:

$$dp = p d\eta + \eta dp = dp(-M + \eta)$$

$$p d\eta = -M dp \rightarrow \quad d\eta = -M \frac{dp}{p}$$

Integrating this expression we obtain:

$$\eta = -M \ln p + C \tag{1}$$

Original Cam-Clay integration

$$\eta = -M \ln p + C \tag{2}$$

To find the constants, for $\eta = 0$, we get $p = p_c$:

$$0 = -M \ln p_c + C \quad C = M \ln p_c$$

Which gives:

$$\eta = M \ln p_c - M \ln p$$
$$q/p = M \ln (p_c/p)$$

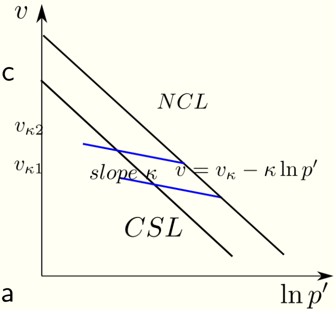
Yield function:

$$F = q - Mp' \ln(p'_c/p') = 0$$

Cam-Clay (OCC): Elastic properties

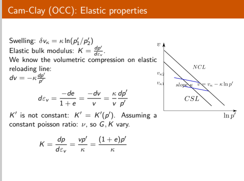
Swelling: $\delta v_\kappa = \kappa \ln(p'_1/p'_2)$
Elastic bulk modulus: $K = \frac{dp'}{d\varepsilon_v}$.
We know the volumetric compression on elastic reloading line:
$dv = -\kappa \frac{dp'}{p'}$

$$d\varepsilon_v = \frac{-de}{1+e} = \frac{-dv}{v} = \frac{\kappa}{v}\frac{dp'}{p'}$$



K' is not constant: $K' = K'(p')$. Assuming a constant poisson ratio: ν, so G, K vary.

$$K = \frac{dp}{d\varepsilon_v} = \frac{vp'}{\kappa} = \frac{(1+e)p'}{\kappa}$$



Observation:

- Stiffness K increases with p': correct.
- Stiffness increases with void ratio (not right)!

Note: The original derivation assumed that there were no recoverable (elastic) shear strains so $G = \infty$. We can find the stress-strain relationships for a single element in this case, but for a finite element forumulation we need to have a finite G^e. So there are two options:

- Define $G = f(e, p')$.
- Use a constant "elastic" Poisson ratio. Ratio between the shear and bulk modulus is constant. $2G/K = const$.

The first alternative has the shortcoming that depending on the choice of G we may have unreasonable values of the "elastic" Poisson's ratio. I prefer the second choice.

Cam-Clay (OCC): Hardening law

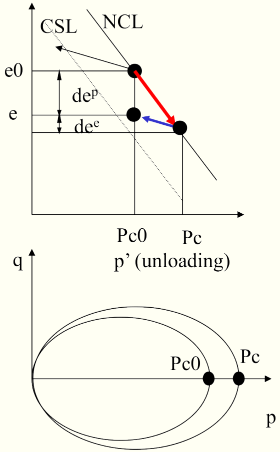
We need to define how the yield surface hardens as plastic work is being performed. Only "*memory*" parameter in our yield surface is the size: p'_c. From the isotropic NCL:

$$d\varepsilon_v = \frac{-dv}{v} = \frac{-de}{1+e} = \frac{+\lambda}{v}\frac{dp'_c}{p'_c}$$

But the increment in elastic volumetric strain is:

$$d\varepsilon_v^e = \left(\frac{-dv}{v}\right)^{elastic} = +\frac{\kappa}{v}\left(\frac{dp'_c}{p'_c}\right)$$

$$
\begin{aligned}
d\varepsilon_{vol} &= -\frac{de}{(1+e)} \\
&= \frac{\kappa}{(1+e)}\frac{dp'}{p'} + \frac{\lambda-\kappa}{(1+e)}\frac{dp'_c}{p'_c} \\
&= elastic + plastic \\
&= d\varepsilon^e_{vol} + d\varepsilon^p_{vol}
\end{aligned}
$$



Therefore the increment of p_c can be related to the increment of plastic volumetric strain:

$$d\varepsilon^p_v = d\varepsilon_v - d\varepsilon^e_v = (\lambda - \kappa)\left(\frac{dp'_c}{p'_c}\right)$$

$$dp'_c = \left(\frac{v \cdot p'_c}{(\lambda-\kappa)}\right) \cdot d\varepsilon^p_v$$

Cam-Clay (OCC): Hardening law

We have seen that the hardening law:

$$H = -\left(\frac{\partial F}{\partial Wp}\right)\left(\frac{\partial Wp}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma}$$

W_p is the vector of memory parameters. In our case, the CC model has only one parameter: p'_c and it's variation is only a function of the plastic volumetric strain. So:

$$H = -\left(\frac{\partial F}{\partial p'_c}\right)\left(\frac{\partial p'_c}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma}$$

We know:

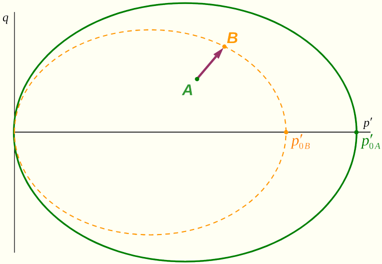
$$\frac{\partial F}{\partial p'_c} = -Mp'/p'_c$$

$$\frac{\partial p'_c}{\partial \varepsilon^p} = \frac{v}{(\lambda - \kappa)} p'_c$$

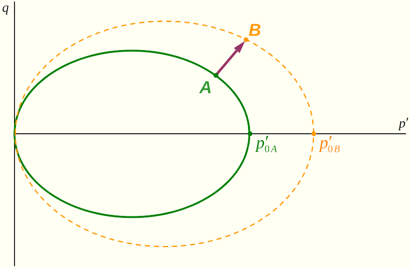
$$\frac{\partial G}{\partial \sigma} = P_p = Q_p = M - \eta$$

$$\boxed{H = -\left(\frac{\partial F}{\partial p'_c}\right)\left(\frac{\partial p'_c}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma} = M\frac{(M-\eta)}{(\lambda-\kappa)} \cdot (1+e) \cdot p'}$$

Deformations under an applied stress path

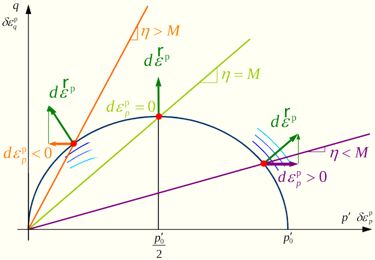


$P'_{0B} > P'_{0A}$ Elastic



$P'_{0B} < P'_{0A}$ Elasto-plastic

Hardening law



$$d\varepsilon_p^p = \frac{\lambda - \kappa}{v p_0'} dp_0'$$

- $\eta < M \rightarrow d\varepsilon_p^p > 0 \quad dp' > 0$
  Yield surface "*expands*"
- $\eta > M \rightarrow d\varepsilon_p^p < 0 \quad dp' < 0$
  Yield surface "*contracts*"
- $\eta = M \rightarrow d\varepsilon_p^p = 0 \quad dp' = 0$
  Yield surface "*constant*"

Stress-strain relation in (p', q) and $(\varepsilon_v, \varepsilon_s)$ drained TX

1. Give strain and/or stress increments
2. Check if the current stress state is inside the yield surface or outside the yield surface $q/p' = M \ln(p_c/p')$
    1. If Elastic (stress inside yield surface):

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_s \end{bmatrix} = \begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

    2. If Elastic-plastic (stress on the yield surface)

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_s \end{bmatrix} = D^{ep} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$$D^{ep} = \left[\begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} + \frac{1}{Mp'} \frac{(\lambda - \kappa)}{(1 + e_0)} \begin{bmatrix} M - (q/p') & 1 \\ 1 & 1/(M - (q/p')) \end{bmatrix} \right]$$

3. Compute the unknown stress or strain increments and update the stress and strains
4. If plastic deformation, update p_c to satisfy Cam-Clay yield surface
5. Go back to step 1

Stress-strain relation in (p', q) and $(\varepsilon_v, \varepsilon_s)$ undrained TX

$d\varepsilon_v = d\varepsilon_a + 2d\varepsilon_r = 0$ (constant volume)
$d\varepsilon_s = (2/3)(d\varepsilon_a - d\varepsilon_r) = (2/3)(d\varepsilon_a - (-0.5d\varepsilon_a)) = d\varepsilon_a$

1. Give axial strain increment $d\varepsilon_a$ or dq
2. Check if the current stress state is inside the yield surface or outside the yield surface $q/p' = M \ln(p_c/p')$
   - If Elastic (stress inside yield surface):

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_s \end{bmatrix} = \begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$d\varepsilon_v =$ from the first equation gives $dp' = 0$. This means that the effective stress path is fixed to go in the vertical direction in $p' - q$ space irrespective of any total stress path.
The second equation gives $d\varepsilon_s$ for a given dq or dq for a $d\varepsilon_s$.

Stress-strain relation in (p', q) and $(\varepsilon_v, \varepsilon_s)$ undrained TX

4. - If Elastic-plastic (stress on the yield surface)

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_s \end{bmatrix} = D^{ep} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$$D^{ep} = \left[\begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} + \frac{1}{Mp'} \frac{(\lambda - \kappa)}{(1 + e_0)} \begin{bmatrix} M - (q/p') & 1 \\ 1 & 1/(M - (q/p')) \end{bmatrix} \right]$$

   $d\varepsilon_v = 0$ and $d\varepsilon_s = d\varepsilon_a$ give dp' and dq
   or
   $d\varepsilon_v = 0$ and dq gives $d\varepsilon_s (= d\varepsilon_a)$ and dp'.

5. Update the stress and strain. The difference between the total mean pressure p and the effective mean pressure will give the pore pressure.
6. If plastic deformation, update p_c to satisfy Cam-Clay yield surface.
7. Go back to step 1

Limitations of original Cam-Clay

1. For an isotropically normally consolidated (saturated clay) specimen in TXC: Overpredicts the excess pore-pressure at failure.
2. Yield surface / plastic potential function produces too much shearing at low stress-ratios. At low stress ratio you would expect mostly plastic volumetric strains rather than deviatoric stress.
3. Yield surface is discontinous at the hydrostatic axis.
4. Overpredicts K_0 for a normally consolidated clay under 1D loading. For low $\phi_c s$ we get K_0 larger than 1 (unrealistic).
5. Other modes of shearing?
6. Anisotropy?

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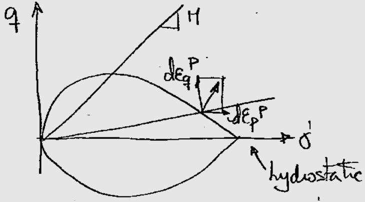

Overview

What can we change?

1. Yield function (yes)
2. Elastic constants (not really) controlled by the compression model.
3. Flow rule (for associated models it is tied to the yield function).
4. Hardening laws (constrained already by the compression model)

MCC: Yield function

Derived from work considerations (Burland 1965, Roscoe and Burland 1968):

$$dW^p_{int} = p\sqrt{(d\varepsilon^p_v)^2 + (Md\varepsilon^p_s)^2}$$

This is the new equation describing the energy dissipated by the soil. Following similar arguments to CC:

$$dW^p_{ext} = pd\varepsilon^p_v + qd\varepsilon^p_s = p\sqrt{(d\varepsilon^p_v)^2 + (Md\varepsilon^p_s)^2} = dW^p_{int}$$

Squaring and re-arranging the terms:

$$\frac{d\varepsilon^p_v}{d\varepsilon^p_s} = \frac{M^2 - \eta^2}{2\eta} = -\frac{dq}{dp'}$$

Assuming an associative flow rule gives an elliptic yield surface:

$$\boxed{F(q,p) = q^2 - M^2 p^2 \left(\frac{p'_c}{p'} - 1\right) = 0}$$

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There appears to be a measure of agreement over the belief that irrecoverable deformation (both volumetric and distortional) of soil can be attributed primarily to grain slip. It occurred to the Writer (Burland) that the work dissipated during grain slip might be related to some generalized measure of total irrecoverable strain which does not differentiate between the components of deformation.

Modified Cam-Clay integration

Employing Cambridge parameters: p, q, v, the increment of work dissipated per unit bulk volume of an isotropic continuum during deformation can be written as:

$$dW = p' d\varepsilon_p + q d\varepsilon_s$$

For particular case of wet clay, the basic assumption is it behaves *macroscopically* as a homogenous isotropic continuum. Under isotropic stress ($q = 0$) there is no distortion $d\varepsilon_s = 0$. So:

$$(dW)_{q=0} = p' d\varepsilon_p$$

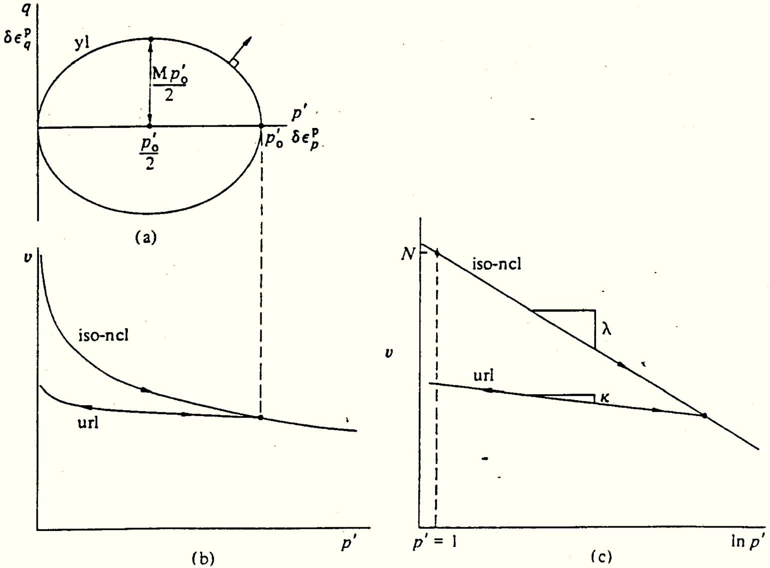
At CS: $q/p = M$ and $d\varepsilon_p = 0$ so:

$$(dW)_{q=Mp'} = p' M d\varepsilon_s$$

A general equation would be:

$$dW = p' \sqrt{d\varepsilon_p^2 + (M d\varepsilon_s)^2}$$

Modified Cam-Clay integration



Modified Cam-Clay integration

A convenient way of writing the equation of the ellipse in Fig is:

$$\frac{p'}{p'_c} = \frac{M^2}{M^2 + \eta^2}$$

In differential form:

$$\frac{\delta p'}{p'} + \frac{2\eta d\eta}{M^2 + \eta^2} - \frac{dp'_c}{p'_c} = 0$$

Incorporating the particular form in the general framework:

$$f = q^2 - M^2[p'(p'_c - p')] = 0$$

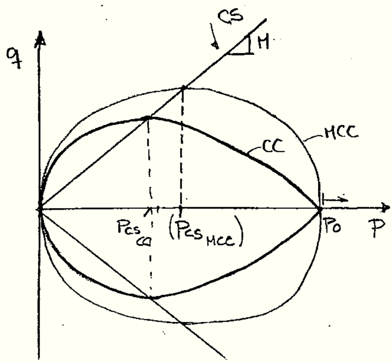
Modified Cam-Clay integration

$$\frac{d\varepsilon_v^p}{d\varepsilon_s^p} = \frac{\partial g/\partial p'}{\partial g/\partial q} = \frac{M^2(2p' - p'_c)}{2q} = \frac{M^2 - \eta^2}{2\eta}$$

$$\frac{d\varepsilon_v^p}{d\varepsilon_s^p} = \frac{M^2 - \eta^2}{2\eta}$$

This is referred to as the dilatancy expression. In contrast to the original Cam-Clay, this equation predicts only plastic volumetric strain at $\eta = 0$ (isotropic state).

Modified Cam-Clay integration

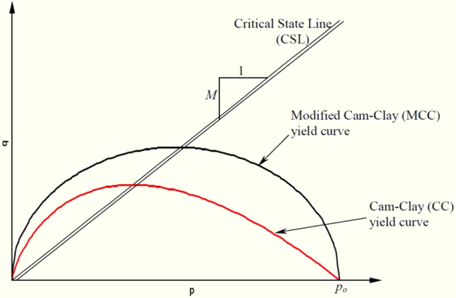


For the MCC we can find the value of p_{cs} the stress corresponding to the critical stress at CS: $q_{cs} = Mp'_{cs}$ and should be on the yield surface:

$$(Mp'_{cs})^2 = M^2(p'_{cs})^2\left(\frac{p'_c}{p'_{cs}} - 1\right) \qquad \frac{p'_c}{p'_{cs}} = 2 \rightarrow \qquad p'_{cs} = p'_c/2$$

OCC v MCC

	Original Cam-clay model (Schofield and Wroth, 1968)	Modified Cam-clay model (Roscoe and Burland, 1968)
Dissipative work	$p' d\varepsilon_v^p + q d\varepsilon_s^p = M p' d\varepsilon_s^p$	$p' d\varepsilon_v^p + q d\varepsilon_s^p = p' \sqrt{(d\varepsilon_v^p)^2 + (M d\varepsilon_s^p)^2}$
Associated flow rule	$(d\varepsilon_s^p / d\varepsilon_v^p)(dq/dp') = -1$	$(d\varepsilon_s^p / d\varepsilon_v^p)(dq/dp') = -1$
Yielding of Cam-clay	$q = M p' \ln(p_c / p')$	$q^2 + M^2 p'^2 = M^2 p' p_c$
N and Γ	$N = \Gamma + \lambda - \kappa$	$N = \Gamma + (\lambda - \kappa) \ln 2$



MCC: Stress-strain relationship

$$\begin{bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ d\tau_{12} \\ d\tau_{23} \\ d\tau_{31} \end{bmatrix} = \left[D(6\times 6)\right] \begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ d\gamma_{12} \\ d\gamma_{23} \\ d\gamma_{31} \end{bmatrix}$$

$$d\sigma' = \left[\boxed{D_e} - \frac{\boxed{D_e}(\partial F / \partial \sigma')(\partial F / \partial \sigma')^T \boxed{D_e}}{-(\partial F / \partial p_c)(dp_c / d\varepsilon_v^p)(\partial F / \partial p') + (\partial F / \partial \sigma')^T \boxed{D_e}(\partial F / \partial \sigma')} \right] d\varepsilon$$

Elastic Stiffness

$$D_e = \begin{bmatrix} K+4/3G & K-2/3G & K-2/3G & 0 & 0 & 0 \\ K-2/3G & K+4/3G & K-2/3G & 0 & 0 & 0 \\ K-2/3G & K-2/3G & K+4/3G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \qquad K = \frac{\nu p'}{\kappa}, G = \frac{3K(1-2\nu)}{2(1+\nu)}$$

$$d\sigma' = \left[D_e - \frac{D_e \boxed{(\partial F/\partial\sigma')} \boxed{(\partial F/\partial\sigma')^T} D_e}{-(\partial F/\partial p_c)(dp_c/d\varepsilon_v^p)(\partial F/\partial p') + \boxed{(\partial F/\partial\sigma')^T} D_e \boxed{(\partial F/\partial\sigma')}} \right] d\varepsilon$$

(A) Calculation of $\partial F/\partial\sigma'$

$$F = \frac{q^2}{M^2} - p' p_c + p^2 = 0$$

$$\frac{\partial F}{\partial\sigma'} = \frac{\partial F}{\partial p'}\frac{\partial p'}{\partial\sigma'} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial\sigma'}$$

$$\partial F/\partial p' = 2p - p_c$$

$$\partial F/\partial q = 2q/M^2$$

$$\partial p'/\partial\sigma' = \begin{Bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \partial q/\partial\sigma' = (3/2q)\begin{Bmatrix} \sigma_{xx} - p \\ \sigma_{yy} - p \\ \sigma_{zz} - p \\ 2\sigma_{xy} \\ 2\sigma_{yz} \\ 2\sigma_{zx} \end{Bmatrix}$$

$$\frac{\partial F}{\partial\sigma'} = \begin{Bmatrix} (2p - p_c)/3 + 3(\sigma_{xx} - p)/M^2 \\ (2p - p_c)/3 + 3(\sigma_{yy} - p)/M^2 \\ (2p - p_c)/3 + 3(\sigma_{zz} - p)/M^2 \\ 6\sigma_{xy}/M^2 \\ 6\sigma_{yz}/M^2 \\ 6\sigma_{zx}/M^2 \end{Bmatrix}$$

$$d\sigma' = \left[D_e - \frac{D_e (\partial F / \partial \sigma')(\partial F / \partial \sigma')^T D_e}{-\boxed{(\partial F / \partial p_c)(dp_c / d\varepsilon_v^p)(\partial F / \partial p')} + (\partial F / \partial \sigma')^T D_e (\partial F / \partial \sigma')} \right] d\varepsilon$$

(B) Calculation of $(\partial F / \partial p_c)(dp_c / d\varepsilon_v^p)(\partial F / \partial p)$

$$\frac{\partial F}{\partial p_c} = -p \qquad \frac{dp_c}{d\varepsilon_v^P} = \frac{v p_c}{(\lambda - \kappa)} \qquad \frac{\partial F}{\partial p} = 2p - p_c$$

$$(\partial F / \partial p_c)(dp_c / d\varepsilon_v^p)(\partial F / \partial p) = -p \frac{v p_c}{(\lambda - \kappa)} (2p - p_c)$$

(C) Assemble [6x6] matrix

$$d\sigma' = \left[D_e - \frac{D_e (\partial F / \partial \sigma')(\partial F / \partial \sigma')^T D_e}{-(\partial F / \partial p_c)(dp_c / d\varepsilon_v^p)(\partial F / \partial p') + (\partial F / \partial \sigma')^T D_e (\partial F / \partial \sigma')} \right] d\varepsilon$$

$$[6\text{x}1] = \left[[6\text{x}6] - \frac{[6\text{x}6][6\text{x}1][1\text{x}6][6\text{x}6]}{-[1\text{x}1][1\text{x}1][1\text{x}1] + [1\text{x}6][6\text{x}6][6\text{x}1]} \right] [6\text{x}1]$$

Overview

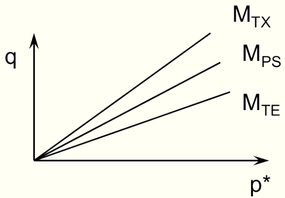
Cam-Clay material properties determination: I

Test required

1. Slow drained or undrained test with pore pressure measurement at different preconsolidation pressure. Test must be taken to very large strains to reach to the critical state.
2. Isotropic consolidateion test or incrementental loading consolidation test.

Slope of failure line: M **If** ϕ_{cs} **is assumed to be constant, then** M **is not constant.**

- In TXC: $M_{TXC} = \frac{6 \sin \phi_{cs}}{3 - \sin \phi_{cs}}$
- In TXE: $M_{TXE} = \frac{6 \sin \phi_{cs}}{3 + \sin \phi_{cs}}$
- In PS: $M_{PS} = 2 \sin \phi_{cs}$ (assuming $\sigma_2 \approx \frac{\sigma_1 + \sigma_3}{2}$)

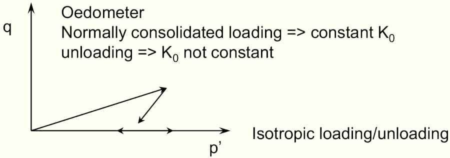


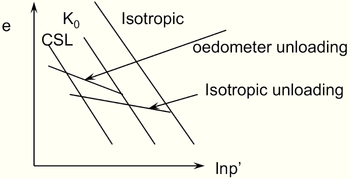
λ and κ: Compression index and swelling index

1. Use isotropic consolidation. Plot in $v - \ln p'$ plane. Measure λ & κ.
2. From odeometer test, plot void ratio e vs $\log \sigma'_v$:

$$\lambda = 0.4343 C_c$$
$$\kappa = 0.4343(1.4 C_{r/s}) = 0.63 C_{r/s} = (0.2\ 0.33)\lambda$$

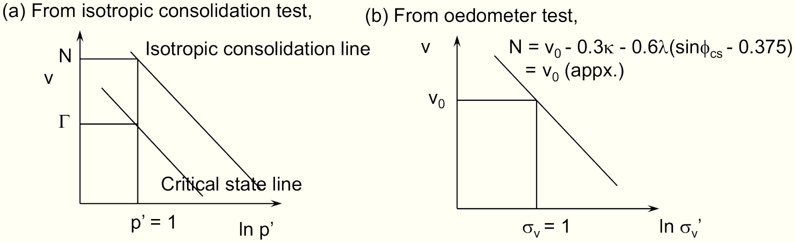
1. swelling and recompression lines are highly nonlinear in $e - \ln p'$ space.
2. κ is usually not less than 0.1λ or greater than 0.5λ.
3. $\kappa$is about 0.03 to 0.06 for many medium plasticity clays.
4. modify κ to try to fit data. λ is usually easier to accurately calculate.





Γ **or** N

1. Critical state line: $v = \Gamma - \lambda \ln p'$
2. Isotropic normally consolidated line: $v = N - \lambda \ln p'$
3. Original Cam-Clay: $N = \Gamma + (\lambda - \kappa)$
4. Modified Cam-Clay: $N = \Gamma + (\lambda - \kappa) \ln 2$



Cam-Clay material properties determination: IV

Elastic properties: G **or** ν

1. Elastic bulk modulus $K_e = dp'/d\varepsilon_v = vp'/\kappa$
2. Choosing Poisson's ratio ν constant gives G_e which varies as K_e.
3. Typically $\nu = 0.2 - 0.4$.

Preconsolidation pressure p_c

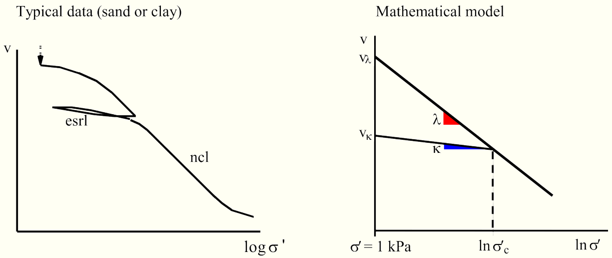
1. Estimate OCR
2. $\sigma_{v,max} = OCR\sigma_{v,current}$
3. $\sigma_{h,max} = K_{0,nc}\sigma_{v,max}$
4. $q_{max} = \sigma_{v,max} - \sigma_{h,max}, p_{max} = (1/3)(\sigma_{v,max} + 2\sigma_{h,max})$
5. Define p_c:
   - Cam-Clay: $q_{max} = Mp_{max}\ln(p_c/p_{max})$
   - MCC: $q_{max}^2 = M^2 p_{max}(p_c - p_m ax)$

Cam-Clay material properties determination: V

Cam-clay parameters : Correlation For clayey soils, attempts have been made to obtain the cam-clay parameters from index tests, especially plasticity index. It should be remembered that most of the soils tested for correlation were remoulded and direct application to natural clays requires caution. The users should be fully aware that limitations exist and these correlations should be regarded as a first approximation.

- Atkinson (1993) $\lambda = PIG_s/460 \quad \Gamma = 1.25 + \lambda \ln 10,000$
- Nakase, Kamei and Kusakabe (1988)- study based on Japanese clays: $\lambda = 0.02 + 0.0045PI \qquad \kappa = 0.00084(PI - 4.6) \qquad N = 1.52 + 0.19PI$
- Nakase et al. (1988) and Frydman (1990) observed that M was independent of PI. Atkinson (1993) finds that M tends to increase with decreasing PI.

Critical state parameters: 1D compression

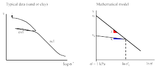


Plastic compression stress σ'_c is taken as the larger of the initial aggregate crushing stress and the historic maximum effective vertical stress. Clay muds are taken to begin with $\sigma'_c = 1$kPa.

Plastic compression (normal compression line): $v = v_\lambda - \lambda \ln \sigma'$ for $\sigma' = \sigma'_c$.

Elastic swelling and recompression line ($\kappa-$line): $v = v_c + \kappa(\ln \sigma'_c - \ln \sigma'_v)$.

Critical state parameters: 1D compression



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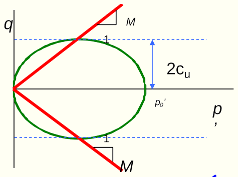
Equivalent parameters for log 10 stress scale:
Terzaghi's compression index: $C_c = \lambda \log 10 = \lambda \times 2.3026$
Terzaghi's swelling index: $C_s/C_r = \kappa \log 10 = \lambda \times 2.3026$

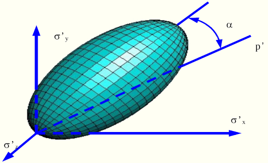
Drawbacks of MCC: Computational problems

- In undrained triaxial test on a heavily overconsolidated soil, after the stress point reaches the yield surface (above M line), due to the negative direction of volumetric plastic strain vector, the yield surface contracts.
- This phenomenon is referred to as **strain softening**.
- Even though the constitutive model is perfectly able to model this aspect of mechanical behaviour, strain softening may lead to problems in a finite element analysis: e.g. mesh dependency and problems with convergence.
- That can be overcome with good coding & algorithms, but many leading codes still struggle and diverge or give erroneous results!

Drawbacks of MCC: Strength prediction in undrained conditions

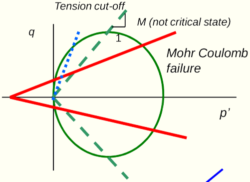
- MCC model assumes Drucker-Prager failure condition, which overestimates undrained strength in triaxial extension.
- Better predictions if Mohr Coulomb failure or Lode angle dependency is introduced
- Real soils are anisotropic and both the shape and size of the yield surface would need to change (see e.g., Wheeler et al. 2003, Can Geotech J for S-Clay1 model)

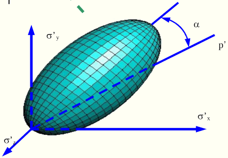




Drawbacks of MCC: K_0 prediction

- Given MCC assumes an associated flow rule, the model predicts unrealistically high K_0 values in normally consolidated range
- This has been fixed e.g. in the Soft Soil model by de-coupling the volumetric yield surface (cap) from the failure line
- Consequently, in the Soft Soil model, M as become a "*shape*" coefficient and no longer corresponds to the critical state line
- Alternatively, the anisotropic S-CLAY1 model also gives good K0 prediction
- Non-associated flow rule is also an option which will help with that issue





Advantages and Limitations of Cam-Clay models

Advantages:

- Unified framework of elastic - plastic behaviour, deals with volumetric & shearing strains, drained & undrained, prefailure and failure,
- Uses a small number of parameters $(M, \lambda, \kappa, \Gamma, \nu)$
- fairly well established,
- Qualitative prediction of soil behaviour - replicates many behaviour not possible to replicate in e.g. Mohr-Coulomb model.

Limitations:

1. Largely elastic for heavily OC clays (and dense sands).
2. Soils do not always reach the critical state or necessarily stay there, such as by breakdown in fabric strain softening.
3. No anisotropy.
4. Stress reversals within elastic domain are purely elastic after initial yielding.
5. No creep.
6. Model predicts conservative K_0 value.

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- Overestimates K_0 loading (initial conditions)
- Wet of CS is well described but description of Over-consolidated soils is more difficult - requires Hvorslev correction
- Not used for sands in its current state
- Not very good elastic model
- Real data shows yield surface should be rotated about K_0 line not hydrostatic axis (anisotropic stres-history)
- over predicts significantly elastic region in extension test.

Critical state concept

Interchangeable parameters for stress at yield and $d\varepsilon^p$.

System	Effective normal stress	Plastic normal strain	Effective shear stress	Plastic shear strain	Critical stress ratio	Plastic normal stress	Critical normal stress
General	σ^*	ε^*	τ^*	γ^*	μ^*_{crit}	σ^*_c	σ^*_{crit}
SSA	σ'	ε	τ	γ	$\tan \phi_{crit}$	σ'_c	σ'_{crit}
BA-PS	s'	ε_v	t	ε_γ	$\sin \phi_{crit}$	s'_c	s'_{crit}
TA-AS	p'	ε_v	q	ε_s	M	p'_c	p'_{crit}

Plastic work and dissipation: $\sigma^* d\varepsilon^* + \tau^* d\gamma^* = \mu^*_{crit} \sigma^* d\gamma^*$.

General yield surface: $\frac{\tau^*}{\sigma^*} = \mu^* = \mu^*_{crit} \ln \left[\frac{\sigma^*_c}{\sigma} \right]$