CE394M: An introduction to continuum mechanics

Krishna Kumar

University of Texas at Austin

krishnak@utexas.edu

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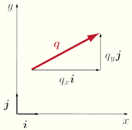
Overview

Vector calculus

A vector is expressed in terms of its components and the unit vectors in the $x-$ and $y-$ directions.



$$\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j}$$

where q_x is the $x-$component and q_y is the $y-$component and i and j are basis vectors (are unit length).

Scalar product:

$$\mathbf{q} \cdot \mathbf{r} = \mathbf{q}^T\mathbf{r} = \begin{bmatrix} q_x & q_y \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix} = q_x r_x + q_y r_y$$

Vector calculus

Grad: If the del operator acts on a scalar field, say temperature $T(x, y)$, it produces a vector that points in the direction of the steepest slope.

$$\nabla \mathbf{T} = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}$$

Divergence: The scalar product of the del operator with a vector field q gives the divergence

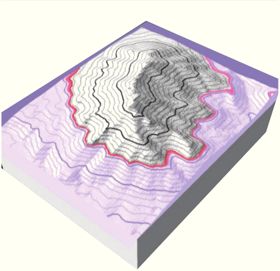
$$div\mathbf{q} = \nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}$$

Notice the divergence of a vector field is a scalar.

Divergence theorem

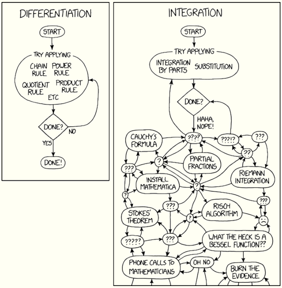
$$\int_\Omega div\mathbf{q} d\Omega = \oint_\Gamma \mathbf{q} \cdot \mathbf{n} d\Gamma$$

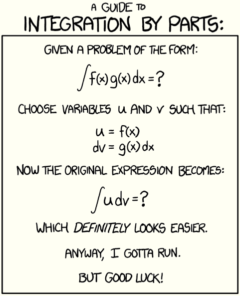
Vector calculus



Contour map for a terrain (left) and the associated three-dimensional model (right). If T is interpreted as the height, the vector ΔT points in the direction of the steepest slope.

Differention and Integration





XKCD - Randall Munroe

The summation convention

Suppose **x** and **y** are vectors, and **A** and **B** are matrices. Write a few common combinations in terms of their components:

- **Dot product:**
- **Matrix-vector product:**
- **Matrix-vector product:**

The summation convention

We can use a simplified notation by adopting the summation convention (due to Einstein), Do not write the summation symbol $\sum$. A repeated index implies summation. (An index may not appear more than twice on one side of an equality.) Using the summation convention

- **Dot product:**
- **Matrix-vector product:**
- **Matrix-vector product:**

This may seem a very peculiar trick, with no obvious benefit. However, it will turn out to be surprisingly powerful, and make many calculations involving vector identities and vector differential identities much simpler.

The Kronecker delta δ_{ij}

The identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We define the '*Kronecker delta*' as:

We know that: $\mathbf{Iy} = \mathbf{y}$

In other words 'if one index of δ_{ij} is summed, the effect is to swap this to the other index'.

The permutation symbol ϵ_{ijk}

Cross product of two vectors:

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{bmatrix} x_2 y_3 - x_3 y_2, & x_3 y_1 - x_1 y_3, & x_1 y_2 - x_2 y_1 \end{bmatrix}$$

where e_i are the basis for the vectors. We have assumed that e_i are the unit vectors for Cartesian coordinates (you may have seen the basis vectors written as i, j and k). To express the cross product in index notation, we will use the permutation symbol ϵ_{ijk}. The permutation symbol ϵ_{ijk} is defined as:

The permutation symbol ϵ_{ijk}

For example:

The permutation symbol is also known as the ‘alternating symbol’ or the ‘Levi-Civita symbol’. Using the permutation symbol, we can write the cross product of two vectors as:

Vector derivatives

The real power of index notation is revealed when we look at vector differential identities. The vector derivatives known as the gradient, the divergence and the curl can all be written in terms of the operator ∇

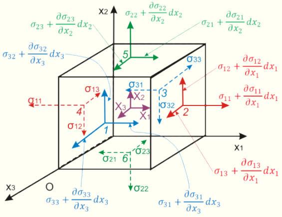
$$\nabla = \left[\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}\right]$$

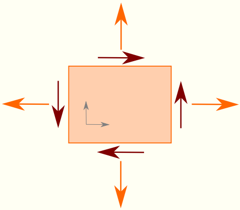
where $[x_1, x_2, x_3]$ are the components of the position vector $\mathbf{x}$.

- **Gradient:**
- **Divergence:**
- **Curl:** $[\mathbf{a} \times \mathbf{b}]_i = \epsilon_{ijk} a_j b_k$

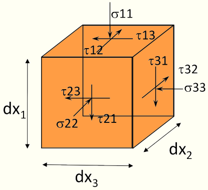
What is a Tensor?

Stresses

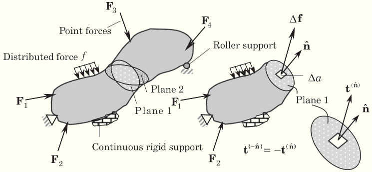




Stresses



Stress vector on a plane

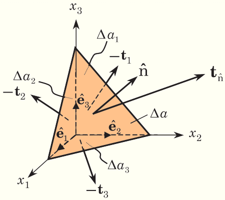


Stress vector on a plane normal to $\hat{\mathbf{n}}$ (Reddy., 2008)

If we denote by $\Delta(\mathbf{f}\hat{\mathbf{n}})$ the force on a small area $\hat{\mathbf{n}}$ located at the position x, the stress vector can be defined:

Cauchy stress tensor

To establish the relationship between $\mathbf{t}$ and $\hat{\mathbf{n}}$ we now set up an infinitesimal tetrahedron in Cartesian coordinates:



If $-\mathbf{t}_1, -\mathbf{t}_2, -\mathbf{t}_3$ and $\mathbf{t}$ denote the stress vectors in the outward directions on the faces of the infinitesimal tetrahedron whose areas are $\Delta a_1, \Delta a_2, \Delta a_3$, and Δa, respectively. Δv is the volume of the tetrahedron, ρ the density, f the body force per unit mass, and $\mathbf{a}$ the acceleration.

Cauchy stress tensor

we have by Newton's second law for the mass inside the tetrahedron:

Since the total vector area of a closed surface is zero (gradient theorem):

The volume Δv can be expressed as:
where Δh is the perpendicular distance from the origin to the slant face.

Cauchy stress tensor

In the limit when the tetrahedron shrinks to a point $\Delta h \to 0$:

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i)\mathbf{t}_i$$

where the summation convention is used.

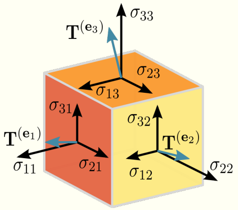
The terms in the parenthesis is the **stress tensor** σ:

$$\sigma \equiv \hat{\mathbf{e}}_1\mathbf{t}_1 + \cdot\hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3$$

The stress tensor is a property of the medium that is independent of the $\hat{\mathbf{n}}$

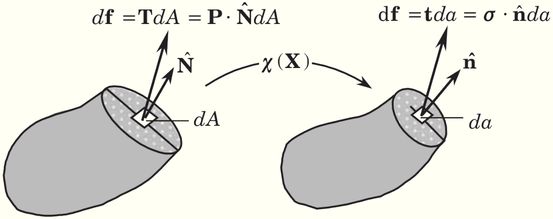
The stress vector $\mathbf{t}$ represents the vectorial stress on a plane whose normal is $\hat{\mathbf{n}}$. σ is the *Cauchy stress tensor* defined to be the *current force per unit deformed area*. In Cartesian component, the Cauchy formula is: $t_i = n_j \sigma_{ji}$.

Cauchy stress tensor



Wikipedia

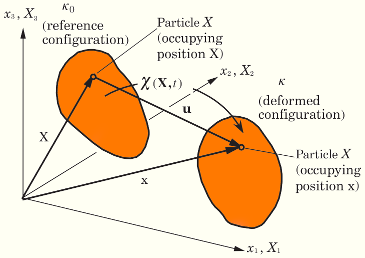
Cauchy stress vs Piola-Kirchoff stress



An introduction to continuum mechanics - J. N. Reddy (2008)

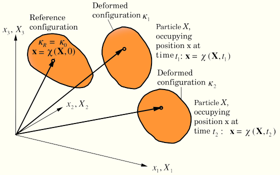
Descriptions of Motion

For a given geometry and loading, the body B will undergo macroscopic geometric changes within the body, which are termed deformation. The geometric changes are accompanied by stresses that are induced in the body.

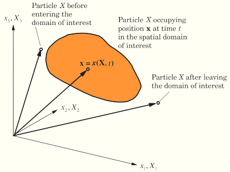


Descriptions of Motion: Displacement field

The displacement of the particle X is given: $\mathbf{u} = \mathbf{x} - \mathbf{X}$.



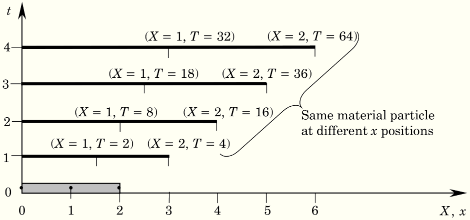
Lagrangian



Eulerian

Displacement field

Consider 1D mapping $\mathbf{x} = \mathbf{X}(1 + 0.5t)$ defining the motion of a rod of initial length two units. The rod experiences a temperature distribution T given by the material description $T = 2\mathbf{X}t^2$ or by the spatial description $T = \mathbf{x}t^2/(1 + 0.5t)$



Material and spatial descriptions of motion (Reddy., 2008)

Deformation Gradient Tensor

deformation gradient F_k of k relative to the reference configuration k_0, which gives the relationship of a material line dX before deformation to the line dx (consisting of the same material as dX) after deformation.

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}$$

Deformation Gradient Tensor

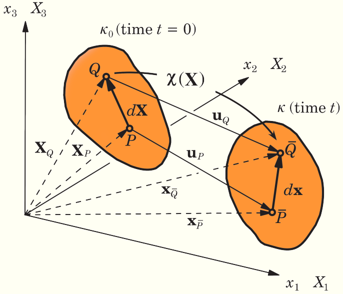
Isotropic compression/extension:

Simple extension:

Simple shear:

It is rather difficult to invert the mapping even for this simple nonhomogeneous deformation.

Cauchy–Green Deformation Tensors



Points P and Q separated by a distance $d\mathbf{X}$ in the undeformed configuration k_0 take up positions $\bar{P}$ and $\bar{Q}$, respectively, in the deformed configuration k, where they are separated by distance $d\mathbf{x}$

Strain tensors

Cauchy-Green deformation tensor The distances between points P and Q and points $\bar{P}$ and $\bar{Q}$ are:

The right *Cauchy-Green deformation tensor*:

Green Strain Tensor The change in the squared lengths that occurs as a body deforms from the reference to the current configuration can be expressed relative to the original length as:

By definition, the Green strain tensor is a symmetric second-order tensor.

Green's strain tensor

If u_1 is displacement in x_1 direction. Then:

$$\varepsilon_{12} = \frac{1}{2}\left\{\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}\right) + \left(\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}\right) + \left(\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right)\right]\right\}$$

Engineering shear strain:

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Typically, we assume small displacements + small strains. Therefore the quadratic terms (higher order) can be ignored. Pile driving or progression of slope failure cannot be modeled as a small strain problem!

Linearization of strain tensor

Ignoring higher order terms is called as the linearization of the strain tensor. This assumption allows for two simplifications:

Alternative, use natural strain approach:

For a 1D deformation of a bar

- $l_0 = 5, l_f = 4.9$
- $l_0 = 5, l_f = 4$

As long as strains are small, small strain formulation is very close - OK! (If you have large strains - a correction / alternatives is to update the **B** matrix every iteration to adhust for finite strains).

Mechanical notation for infinitesimal strains

Ignoring higher order terms is called as the linearization of the strain tensor. This assumption allows for two simplifications:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

i and j refers to directions.
Resulting strain definitions

Strains