

CE394M: An introduction to continuum mechanics

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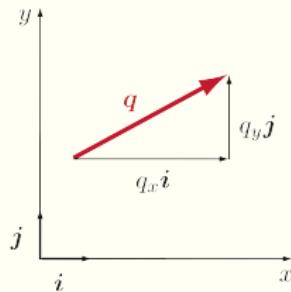
February 8, 2020

Overview

- 1 Review of vector calculus
- 2 Index notation
- 3 Tensors
- 4 Definition of stress and strain tensors
 - Descriptions of Motion
 - Strain measures

Vector calculus

A vector is expressed in terms of its components and the unit vectors in the x - and y - directions.



$$\mathbf{q} = q_x \mathbf{i} + q_y \mathbf{j}$$

where q_x is the x -component and q_y is the y -component and i and j are basis vectors (are unit length).

Scalar product:

$$\mathbf{q} \cdot \mathbf{r} = \mathbf{q}^T \mathbf{r} = \begin{bmatrix} q_x & q_y \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix} = q_x r_x + q_y r_y$$

Vector calculus

Grad: If the del operator acts on a scalar field, say temperature $T(x, y)$, it produces a vector that points in the direction of the steepest slope.

$$\nabla T = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}$$

Divergence: The scalar product of the del operator with a vector field \mathbf{q} gives the divergence

$$div \mathbf{q} = \nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}$$

Notice the divergence of a vector field is a scalar.

Divergence theorem

$$\int_{\Omega} div \mathbf{q} d\Omega = \oint_{\Gamma} \mathbf{q} \cdot \mathbf{n} d\Gamma$$

CE394M: Continuum mechanics

└ Review of vector calculus

└ Vector calculus

Vector calculus

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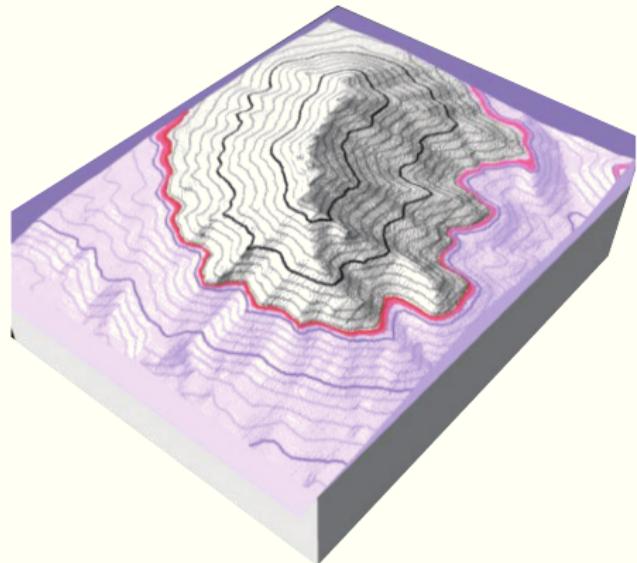
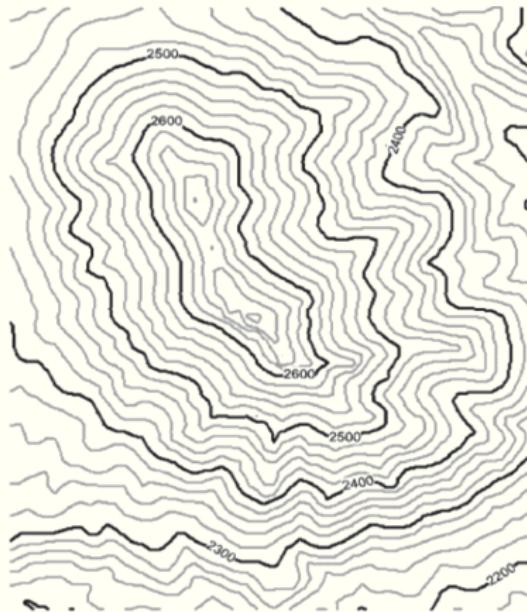
Divergence theorem

$$\int_{\Omega} \operatorname{div} \mathbf{q} d\Omega = \oint_{\Gamma} \mathbf{q} \cdot \mathbf{n} d\Gamma$$

Divergence is the volume density of the outward flux.

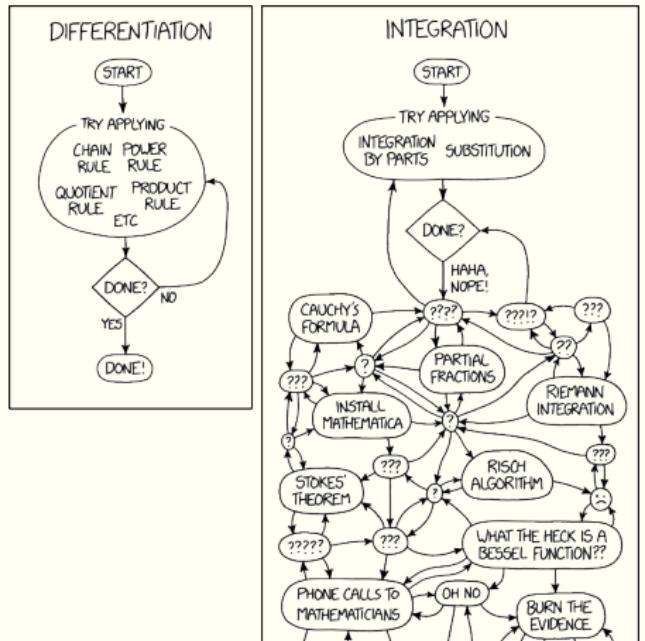
Gauss's theorem or Divergence theorem is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the tensor field inside the surface.

Vector calculus



Contour map for a terrain (left) and the associated three-dimensional model (right). If T is interpreted as the height, the vector ΔT points in the direction of the steepest slope.

Differentiation and Integration



A GUIDE TO INTEGRATION BY PARTS:

GIVEN A PROBLEM OF THE FORM:

$$\int f(x) g(x) dx = ?$$

CHOOSE VARIABLES u AND v SUCH THAT:

$$u = f(x)$$
$$dv = g(x) dx$$

NOW THE ORIGINAL EXPRESSION BECOMES:

$$\int u dv = ?$$

WHICH DEFINITELY LOOKS EASIER.

ANYWAY, I GOTTA RUN.

BUT GOOD LUCK!

XKCD - Randall Munroe

The summation convention

Suppose \mathbf{x} and \mathbf{y} are vectors, and \mathbf{A} and \mathbf{B} are matrices. Write a few common combinations in terms of their components:

- Dot product:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n=3} x_i y_i$$

- Matrix-vector product:

$$[\mathbf{Ax}]_i = \sum_{j=1}^n A_{ij} x_j$$

- Matrix-vector product:

$$[\mathbf{AB}]_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Every sum goes with an index which is repeated twice. Non-repeated indices are not summed.

The summation convention

We can use a simplified notation by adopting the summation convention (due to Einstein), Do not write the summation symbol \sum . A repeated index implies summation. (An index may not appear more than twice on one side of an equality.) Using the summation convention

- Dot product:

$$\mathbf{x} \cdot \mathbf{y} = x_i y_i$$

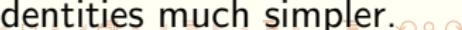
- Matrix-vector product:

$$[\mathbf{Ax}]_i = A_{ij}x_j$$

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$$[\mathbf{AB}]_{ij} = A_{ik}B_{kj}$$

This may seem a very peculiar trick, with no obvious benefit. However, it will turn out to be surprisingly powerful, and make many calculations involving vector identities and vector differential identities much simpler.



The Kronecker delta δ_{ij}

The identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We define the '*Kronecker delta*' as:

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j \end{cases}$$

We know that: $\mathbf{I}\mathbf{y} = \mathbf{y}$

$$\delta_{ij}y_j = y_i$$

In other words 'if one index of δ_{ij} is summed, the effect is to swap this to the other index'.

The permutation symbol ϵ_{ijk}

Cross product of two vectors:

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = [x_2y_3 - x_3y_2, \quad x_3y_1 - x_1y_3, \quad x_1y_2 - x_2y_1]$$

where e_i are the basis for the vectors. We have assumed that e_i are the unit vectors for Cartesian coordinates (you may have seen the basis vectors written as i , j and k). To express the cross product in index notation, we will use the permutation symbol ϵ_{ijk} . The permutation symbol ϵ_{ijk} is defined as:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{otherwise} \end{cases}$$

The permutation symbol ϵ_{ijk}

For example:

$$\epsilon_{112} = 0$$

$$\epsilon_{312} = 1$$

$$\epsilon_{132} = -1$$

The permutation symbol is also known as the ‘alternating symbol’ or the ‘Levi-Civita symbol’. Using the permutation symbol, we can write the cross product of two vectors as:

$$[\mathbf{x} \times \mathbf{y}] = \epsilon_{ijk} x_j y_k$$

To prove this, for each i sum over j and k . The permutation symbol possesses a number of ‘symmetries’,

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} \text{ (cyclic permutation)}$$

$$= -\epsilon_{ikj} = -\epsilon_{ikj} = -\epsilon_{kji} \text{ (switch pair } ij, \text{ switch pair } ki, \text{ switch pair } jk)$$

Vector derivatives

The real power of index notation is revealed when we look at vector differential identities. The vector derivatives known as the gradient, the divergence and the curl can all be written in terms of the operator ∇

$$\nabla = \left[\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3} \right]$$

where $[x_1, x_2, x_3]$ are the components of the position vector \mathbf{x} .

- **Gradient:**

$$grad\phi : [\nabla\phi]_i = \frac{\partial\phi}{\partial x_i} = \left[\frac{\partial\phi}{\partial x_1}, \quad \frac{\partial\phi}{\partial x_2}, \quad \frac{\partial\phi}{\partial x_3} \right]$$

- **Divergence:**

$$div\mathbf{u} : \nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

- **Curl:** $[\mathbf{a} \times \mathbf{b}]_i = \epsilon_{ijk} a_j b_k$

$$curl\mathbf{u} : [\nabla \times \mathbf{u}]_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

What is a Tensor?

- A tensor is a geometric object that maps in a multi-linear manner geometric vectors, scalars, and other tensors to a resulting tensor. Vectors and scalars which are often used in elementary physics and engineering applications, are considered as the simplest tensors.
- Tensors, defined mathematically, are simply arrays of numbers, or functions, that transform according to certain rules under a change of coordinates. Tensors characterize the properties of a physical system.
- An elementary example of mapping, describable as a tensor, is the dot product, which maps two vectors to a scalar.
- A tensor may consist of a single number, in which case it is referred to as a tensor of order zero, or simply a scalar. An example of a scalar field would be the density of a fluid as a function of position.
- A second-order tensor is a vector.

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Assuming a basis of a real vector space, e.g., a coordinate frame in the ambient space, a tensor can be represented as an organized multidimensional array of numerical values with respect to this specific basis. Changing the basis transforms the values in the array in a characteristic way that allows to define tensors as objects adhering to this transformational behavior. For example, there are invariants of tensors that must be preserved under any change of the basis, thereby making only certain multidimensional arrays of numbers a tensor. Compare this to the array representing ε_{ijk} not being a tensor, for the sign change under transformations changing the orientation.

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'a tensor quantity is a physical quantity which has no specified direction but different values in different directions with an associated transformation between different coordinates.'

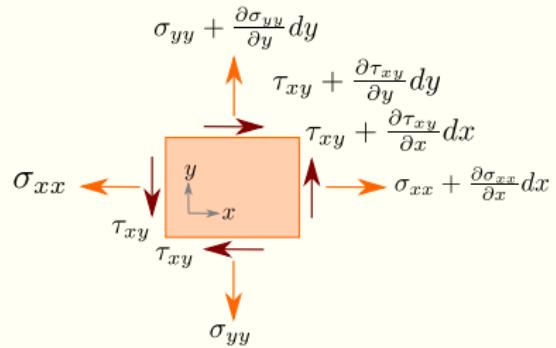
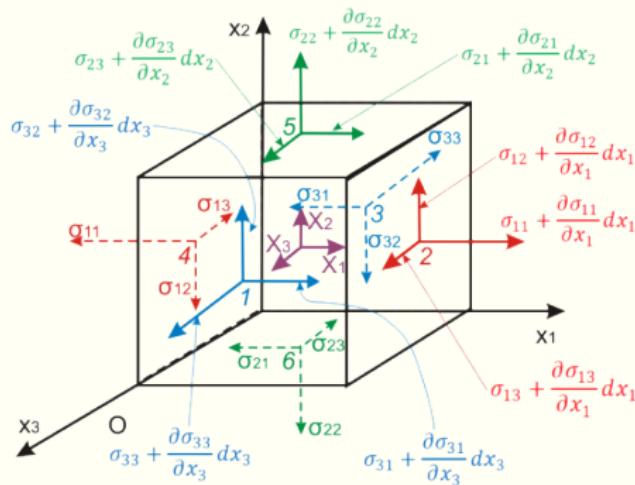
Suppose a car has some velocity represented by $(1, 1, 0)$ m/s, that definition is pretty useless, since there is no information about the coordinate system. However, if we define three orthonormal unit vectors and give the velocity in terms of those vectors, then we have all the information we need. A tensor works is just a generalization of the vector concept (in fact a vector is a rank 1 tensor, and even a scalar can be considered a rank 0 tensor!).

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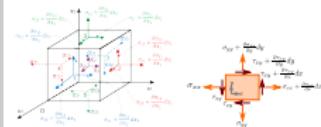
Suppose I write down a matrix on a piece of paper and tell you that it represents the moment of inertia of some object. Then I tell you to transform it into a more convenient coordinate system. (Suppose the object is a torus and it would simplify things dramatically if it were represented in a toroidal coordinate system or something). You can't do it! You have no information about the coordinate system that I used when I calculated the matrix. A moment of inertia tensor is basically a matrix where each element is associated with two unit vectors in some given coordinate system. This is all the info you need to transform into other systems, and the mathematics of such transformations is extremely elegant in tensor notation. So, in short, a tensor is basically a "matrix-like" quantity that is independent of any coordinate system, and can be readily expressed in any coordinate system.

<https://www.physicsforums.com/threads/what-is-tensor-quantity.345535/>

Stresses



- Stress state at a point is defined by σ_{ij} in a frame of reference.
- Equilibrium of moments require $\sigma_{xz} = \sigma_{zx}, \dots$ etc.,

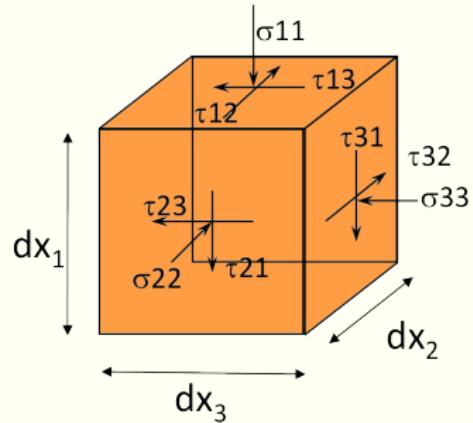


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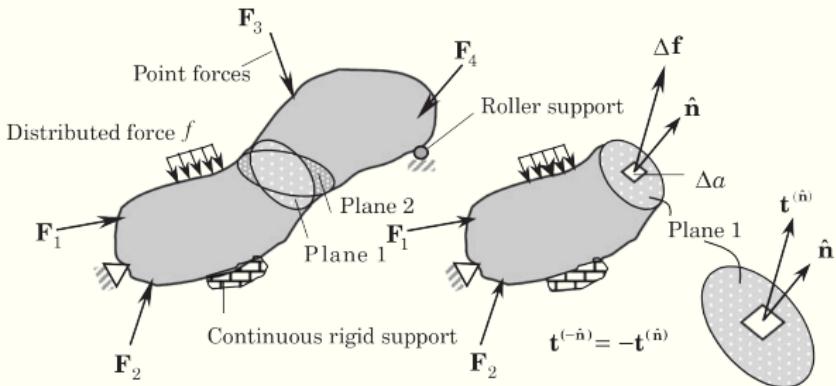
- σ_{xz} stress acting on plane perpendicular to axis x and in the direction of z
- σ_{xx} stress acting on plane perpendicular to axis x and in the direction of x

Stresses

- 9 components of the stress tensor.
- 6 stresses: $\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{12}, \tau_{23}, \tau_{31}$.
- $\tau_{21} = -\tau_{12}, \tau_{32} = -\tau_{23}, \tau_{13} = -\tau_{31}$
- Compression is positive
- Shear stress, anti-clockwise is positive
- In order to write the components in a more concise way we can use the indices notation: σ_{ij} (use $i = 1, 2, 3$ and $j = 1, 2, 3$)
- Correspondence from x, y, z to $1, 2, 3$ (e.g., $\sigma_{11} = \sigma_{xx}, \sigma_{12} = \sigma_{xy}$)



Stress vector on a plane



Stress vector on a plane normal to \hat{n} (Reddy., 2008)

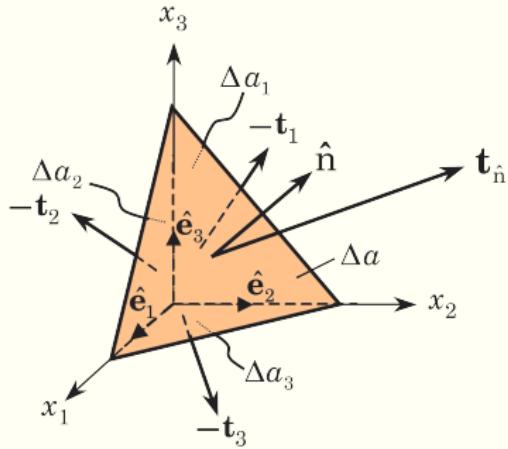
If we denote by $\Delta f(\hat{n})$ the force on a small area \hat{n} located at the position x , the stress vector can be defined:

$$t(\hat{n}) = \lim_{\Delta a \rightarrow 0} \frac{\Delta f(\hat{n})}{\Delta a}$$

Cauchy stress is the true stress, that is, stress in the deformed configuration.

Cauchy stress tensor

To establish the relationship between \mathbf{t} and $\hat{\mathbf{n}}$ we now set up an infinitesimal tetrahedron in Cartesian coordinates:



If $-\mathbf{t}_1, -\mathbf{t}_2, -\mathbf{t}_3$ and \mathbf{t} denote the stress vectors in the outward directions on the faces of the infinitesimal tetrahedron whose areas are $\Delta a_1, \Delta a_2, \Delta a_3$, and Δa , respectively. Δv is the volume of the tetrahedron, ρ the density, f the body force per unit mass, and \mathbf{a} the acceleration.

Cauchy stress tensor

we have by Newton's second law for the mass inside the tetrahedron:

$$\mathbf{t}\Delta a - \mathbf{t}_1\Delta a_1 - \mathbf{t}_2\Delta a_2 - \mathbf{t}_3\Delta a_3 + \rho\Delta v\mathbf{f} = \rho\Delta v\mathbf{a}$$

Since the total vector area of a closed surface is zero (gradient theorem):

$$\Delta a\hat{\mathbf{n}} - \Delta a_1\hat{\mathbf{e}}_1 - \Delta a_2\hat{\mathbf{e}}_2 - \Delta a_3\hat{\mathbf{e}}_3 = \mathbf{0}$$

$$\Delta a_1 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\Delta a, \quad \Delta a_2 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\Delta a, \quad \Delta a_3 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\Delta a.$$

The volume Δv can be expressed as: $\Delta v = (\Delta h/3)\Delta a$

where Δh is the perpendicular distance from the origin to the slant face.

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3 + \rho \frac{\Delta h}{3}(\mathbf{a} - \mathbf{f})$$

Cauchy stress tensor

In the limit when the tetrahedron shrinks to a point $\Delta h \rightarrow 0$:

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i)\mathbf{t}_i$$

where the summation convention is used.

$$\mathbf{t} = \hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_1\mathbf{t}_1 + \hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3).$$

The terms in the parenthesis is the **stress tensor** σ :

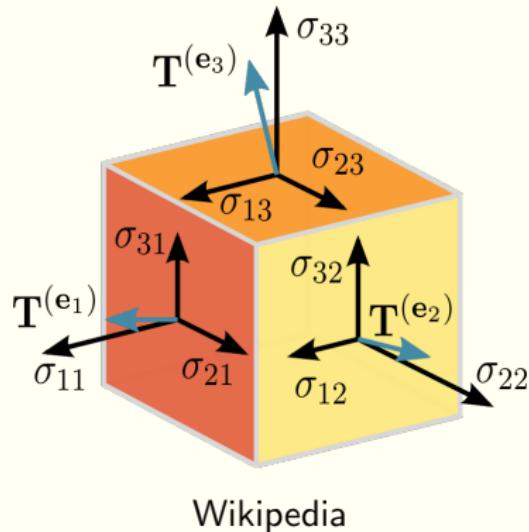
$$\sigma \equiv \hat{\mathbf{e}}_1\mathbf{t}_1 + \hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3$$

The stress tensor is a property of the medium that is independent of the $\hat{\mathbf{n}}$

$$\mathbf{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \sigma = \sigma^T \hat{\mathbf{n}}$$

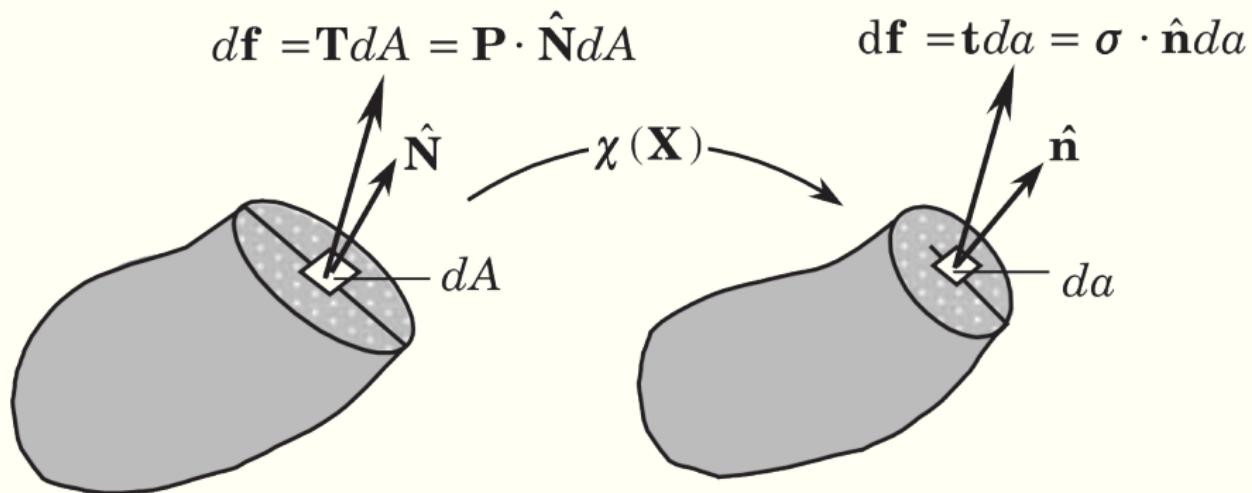
The stress vector \mathbf{t} represents the vectorial stress on a plane whose normal is $\hat{\mathbf{n}}$. σ is the *Cauchy stress tensor* defined to be the *current force per unit deformed area*. In Cartesian component, the Cauchy formula is: $t_i = n_j \sigma_{ji}$.

Cauchy stress tensor



The Cauchy stress tensor σ , which takes a directional unit vector e as input and maps it to the stress vector $T(e)$, which is the force (per unit area) exerted by material on the negative side of the plane orthogonal to e against the material on the positive side of the plane, thus expressing a relationship between these two vectors

Cauchy stress vs Piola-Kirchoff stress

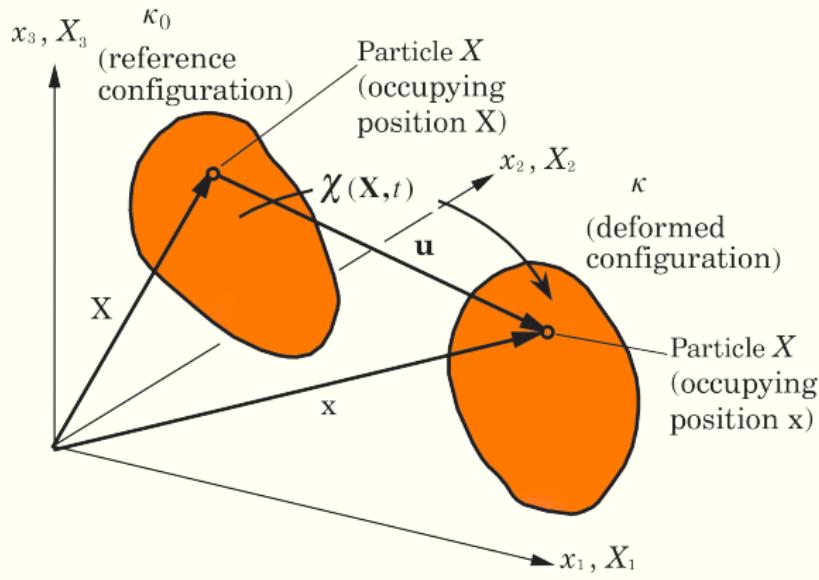


An introduction to continuum mechanics - J. N. Reddy (2008)

- The first Piola–Kirchhoff stress tensor, also referred to as the *nominal stress tensor*, or *Lagrangian stress tensor*, gives the current force per unit undeformed area.

Descriptions of Motion

For a given geometry and loading, the body B will undergo macroscopic geometric changes within the body, which are termed deformation. The geometric changes are accompanied by stresses that are induced in the body.

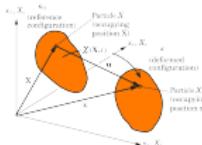


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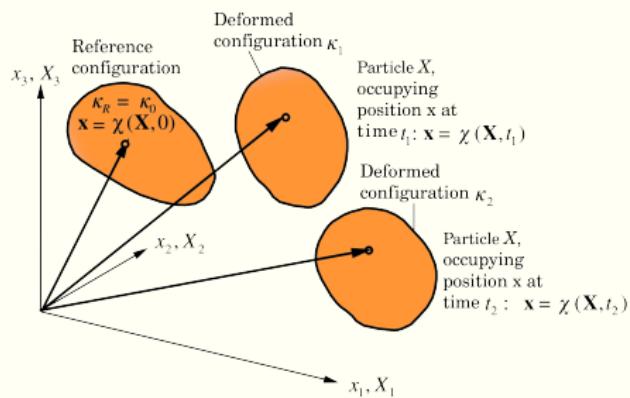
The mathematical description of the deformation of a continuous body follows one of the two approaches: (1) the material description and (2) spatial description. The material description is also known as the Lagrangian description, and the spatial description is known as the Eulerian description.

In the material description, the motion of the body is referred to a reference configuration k_R , which is often chosen to be the undeformed configuration, $k_R = k_0$. Thus, in the Lagrangian description, the current coordinates ($x \in k$) are expressed in terms of the reference coordinates ($X \in k_0$).

In the spatial description, the motion is referred to the current configuration k occupied by the body B , and ϕ is described with respect to the current position ($x \in k$) in space, currently occupied by material particle X .

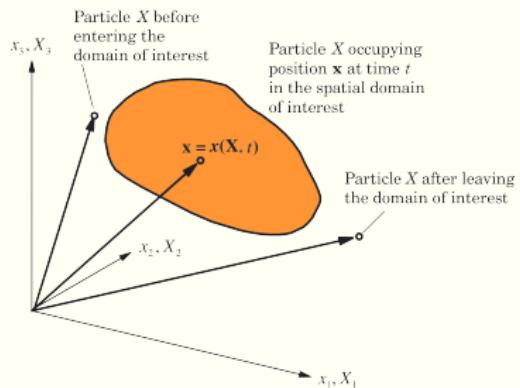
Descriptions of Motion: Displacement field

The displacement of the particle X is given: $\mathbf{u} = \mathbf{x} - \mathbf{X}$.



Lagrangian

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$$



Eulerian

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t)$$

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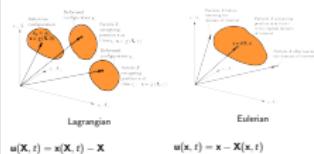
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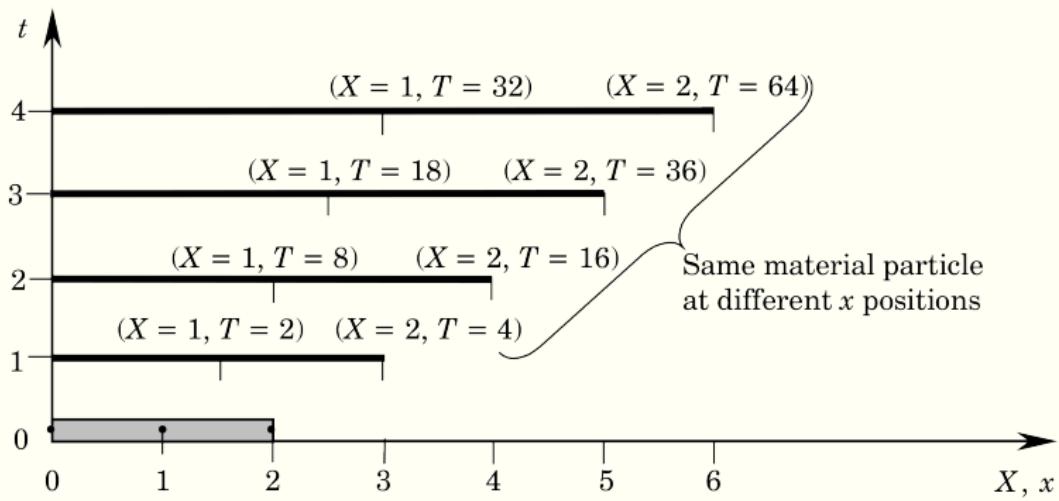
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For a given geometry and loading, the body B will undergo macroscopic geometric changes within the body, which are termed deformation. The geometric changes are accompanied by stresses that are induced in the body. In the Lagrangian description, the displacements are expressed in terms of the material coordinates X_i : $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$.

Displacement field

Consider 1D mapping $\mathbf{x} = \mathbf{X}(1 + 0.5t)$ defining the motion of a rod of initial length two units. The rod experiences a temperature distribution T given by the material description $T = 2\mathbf{X}t^2$ or by the spatial description $T = 2\mathbf{x}t^2/(1 + 0.5t)$



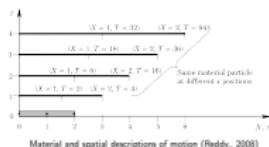
Material and spatial descriptions of motion (Reddy., 2008)

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Displacement field

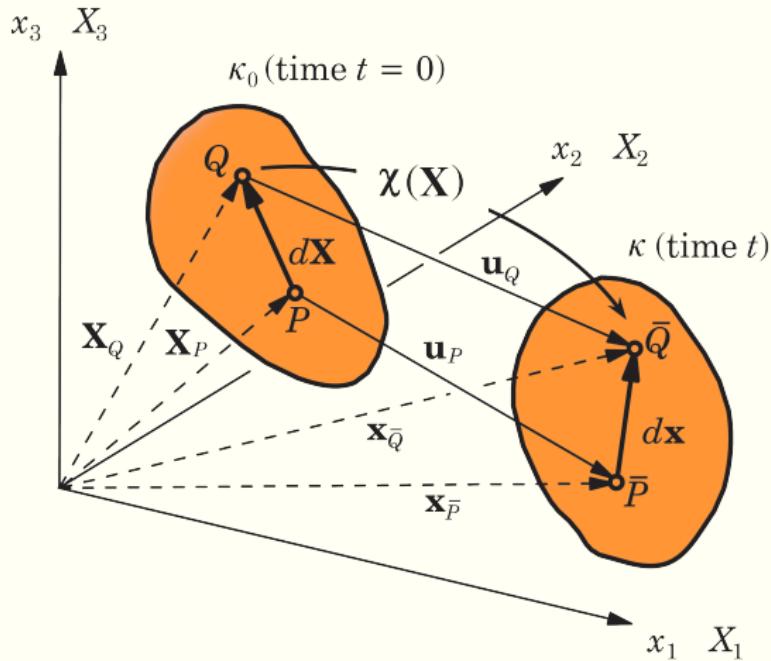
Consider 1D mapping $\mathbf{x} = \mathbf{X}(1 + 0.5t)$ defining the motion of a rod of initial length two units. The rod experiences a temperature distribution T given by the material description $T = 2\mathbf{X}t^2$ or by the spatial description $T = 2\mathbf{x}t^2/(1 + 0.5t)$



From the figure, we see that the particle's material coordinate (label) X remains associated with the particle while its spatial position x changes. The temperature at a given time can be found in one of the two ways: for example, at time $t = 3$, the temperature of the particle labeled $X = 2$ is $T = 2 \times 2 \times (3)^2 = 36$; alternatively, the temperature of the same particle which at $t = 3$ is at a spatial position $x = 2(1 + 0.5 \times 3) = 5$ is $T = 2 \times 5(3)^2 / (1 + 0.5 \times 3) = 36$. The displacement of a material point occupying position X in k_0 is:

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} = X(1 + 0.5t) - X = 0.5Xt.$$

Deformation Gradient Tensor

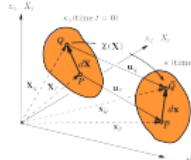


Points P and Q separated by a distance $d\mathbf{X}$ in the undeformed configuration k_0 take up positions \bar{P} and \bar{Q} , respectively, in the deformed configuration k , where they are separated by distance $d\mathbf{x}$

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Deformation Gradient Tensor



Points P and Q separated by a distance dX in the undeformed configuration k_0 take up positions \tilde{P} and \tilde{Q} , respectively, in the deformed configuration k , where they are separated by distance ds .

The deformation gradient is a tensor that quantifies both the 3D and 2D shape change as well as overall material rotation, making it superior to strain as an all-encompassing measure of deformation of material elements.

Deformation Gradient Tensor

deformation gradient F_k of k relative to the reference configuration k_0 , which gives the relationship of a material line dX before deformation to the line dx (consisting of the same material as dX) after deformation.

$$dx = \mathbf{F} \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^T$$

$$\mathbf{F} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right)^T$$

$$d\mathbf{X} = \mathbf{F}^{-1} \cdot dx = dx \cdot \mathbf{F}^{-T}$$

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}$$

Deformation Gradient Tensor

Isotropic compression/extension:

$$[\mathbf{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Isochoric (constant volume) is a special case where $\lambda = 1$.

Simple extension:

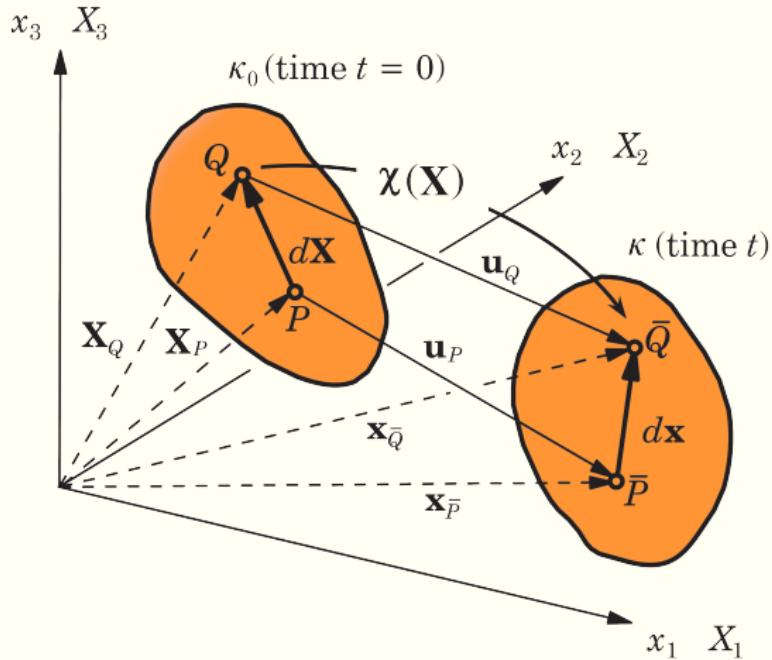
$$[\mathbf{F}] = \begin{bmatrix} 1 + \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Simple shear:

$$[\mathbf{F}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is rather difficult to invert the mapping even for this simple nonhomogeneous deformation.

Cauchy–Green Deformation Tensors



Points P and Q separated by a distance $d\mathbf{X}$ in the undeformed configuration k_0 take up positions \bar{P} and \bar{Q} , respectively, in the deformed configuration k , where they are separated by distance $d\mathbf{x}$

Strain tensors

Cauchy-Green deformation tensor The distances between points P and Q and points \bar{P} and \bar{Q} are:

$$(dS)^2 = d\mathbf{X} \cdot d\mathbf{X},$$

$$(ds)^2 = d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot (\mathbf{F}^T \cdot \mathbf{F}) \cdot d\mathbf{X} \equiv d\mathbf{X} \cdot \mathbf{C} \cdot d\mathbf{X}$$

The right *Cauchy-Green deformation tensor*: $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$

Green Strain Tensor The change in the squared lengths that occurs as a body deforms from the reference to the current configuration can be expressed relative to the original length as:

$$(ds)^2 - (dS)^2 = 2d\mathbf{X} \cdot \mathbf{E} \cdot d\mathbf{X},$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

By definition, the Green strain tensor is a symmetric second-order tensor.

Green's strain tensor

If u_1 is displacement in x_1 direction. Then:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial x_1} \right)^2 + \left(\frac{\partial u_2}{\partial x_2} \right)^2 + \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right]$$

$$\varepsilon_{12} = \frac{1}{2} \left\{ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial x_1} \frac{\partial u_1}{\partial x_2} \right) + \left(\frac{\partial u_2}{\partial x_1} \frac{\partial u_2}{\partial x_2} \right) + \left(\frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_2} \right) \right] \right\}$$

Engineering shear strain: $\gamma_{12} = 2\varepsilon_{12}$.

Typically, we assume small displacements + small strains. Therefore the quadratic terms (higher order) can be ignored. Pile driving or progression of slope failure cannot be modeled as a small strain problem!

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

Linearization of strain tensor

Ignoring higher order terms is called as the linearization of the strain tensor. This assumption allows for two simplifications:

- ① Calculate strains based on undeformed/original geometry
- ② Calculate matrix \mathbf{B} based on original / undeformed element geometry.
(\mathbf{B} is the strain-displacement matrix)

Alternative, use natural strain approach:

$$\varepsilon_n = \int_{l_{initial}}^{l_{final}} \frac{dl}{l} = -\ln\left(\frac{l_{final}}{l_{initial}}\right) = -\ln\left(1 - \frac{\Delta l}{l_0}\right) = -\ln(1 - \varepsilon_{engg})$$

For a 1D deformation of a bar

- $l_0 = 5, l_f = 4.9 \quad \varepsilon = 0.1/5.0 = 2\% \quad \varepsilon_n = \ln(5/4.9) = 2.02\%$
- $l_0 = 5, l_f = 4 \quad \varepsilon = 1/5.0 = 20\% \quad \varepsilon_n = \ln(5/4) = 22.3\%$

As long as strains are small, small strain formulation is very close - OK! (If you have large strains - a correction / alternatives is to update the \mathbf{B} matrix every iteration to adjust for finite strains).

Mechanical notation for infinitesimal strains

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} (u_{i,j} + u_{j,i})$$

i and *j* refers to directions.

Resulting strain definitions

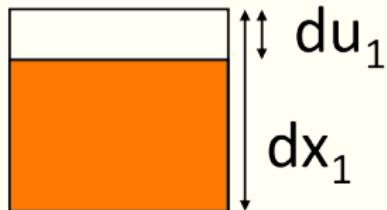
$$\varepsilon_{11} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) = \frac{\partial u_1}{\partial x_1}$$

$$\varepsilon_{22} = \frac{1}{2} \left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \right) = \frac{\partial u_2}{\partial x_2}$$

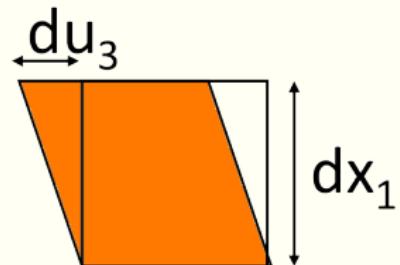
$$\varepsilon_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)$$

Strains

Normal strain



Shear strain



$$\varepsilon_{11} = du_1/dx_1$$

$$\gamma_{13} = du_3/dx_1$$

- 6 strains: $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{12}, \gamma_{23}, \gamma_{31}$.
- $\gamma_{21} = -\gamma_{12}, \gamma_{32} = -\gamma_{23}, \gamma_{13} = -\gamma_{31}$
- Compression is positive
- anti-clockwise is positive