Module 4M12: Partial Differential Equations and Variational Methods
Index Notation

Contents

1 Index notation[1]

1.1 The summation convention

Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices. Write a few common combinations in terms of their components:

- Dot product
$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n=3} x_i y_i$$
- Matrix–vector multiplication
$$[\boldsymbol{A}\boldsymbol{x}]_i = \sum_{j=1}^{n} A_{ij} x_j$$
- Matrix–matrix multiplication
$$[\boldsymbol{A}\boldsymbol{B}]_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Notice the curious thing:

Every sum goes with an index which is repeated *twice*.

Non-repeated indices are not summed.

We can use a simplified notation by adopting the summation convention (due to Einstein),

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Do not write the summation symbol $\sum$. A repeated index implies summation.

(An index may not appear more than twice on one side of an equality.)

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[1] Index notation is also known as 'suffix notation'.

Using the summation convention,

- $\boldsymbol{x} \cdot \boldsymbol{y} = x_i y_i$
- $[\boldsymbol{Ax}]_i = A_{ij} x_j$
- $[\boldsymbol{AB}]_{ij} = A_{ik} B_{kj}$

Summary

If an index occurs *once*, it must occur once in every term of the equation, and the equation is true for each separate value of this index. If an index appears *twice* it is summed over all values. It does not matter what this is called: it is a 'dummy index' whose name can be changed at will. If an index appears three or more times in any given term in an equation, it is wrong!

This may seem a very peculiar trick, with no obvious benefit. However, it will turn out to be surprisingly powerful, and make many calculations involving vector identities and vector differential identities much simpler.

1.2 The Kronecker delta δ_{ij}

Two additional pieces of notation are needed. The first is a way to write the identity matrix $\boldsymbol{I}$,

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We define the 'Kronecker delta' as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

We know that

$$\boldsymbol{I}\boldsymbol{y} = \boldsymbol{y}$$

and

$$\delta_{ij} y_j = y_i$$

In other words 'if one index of δ_{ij} is summed, the effect is to swap this to the other index'.

1.3 The permutation symbol ϵ_{ijk}

Another necessary ingredient is a way to write the cross product of two vectors in index notation,

$$\boldsymbol{x} \times \boldsymbol{y} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{bmatrix} x_2 y_3 - x_3 y_2, & x_3 y_1 - x_1 y_3, & x_1 y_2 - x_2 y_1 \end{bmatrix}$$

where $\boldsymbol{e}_i$ are the basis for the vectors. We have assumed that $\boldsymbol{e}_i$ are the unit vectors for Cartesian coordinates (you may have seen the basis vectors written as $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$). To express the cross product in index notation, we will use the permutation symbol ϵ_{ijk}.

The permutation symbol ϵ_{ijk} is defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

For example,

$$\epsilon_{112} = 0$$

$$\epsilon_{312} = 1$$

$$\epsilon_{132} = -1$$

The permutation symbol is also known as the 'alternating symbol' or the 'Levi-Civita symbol'.

Using the permutation symbol, we can write the cross product of two vectors as:

$$[\boldsymbol{x} \times \boldsymbol{y}]_i = \epsilon_{ijk} x_j y_k$$

To prove this, for each i sum over j and k. The permutation symbol possesses a number of 'symmetries',

$$\begin{aligned} \epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} \quad & \text{(cyclic permutation)} \\ = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \quad & \text{(switch pair } ij\text{, switch pair } ki\text{, switch pair } jk\text{)} \end{aligned}$$

1.4 The $\epsilon_{ijk} - \delta_{ij}$ identity

There is an important identity relating ϵ_{ijk} and δ_{ik}:

$$\underbrace{\epsilon_{ijk}\epsilon_{klm}}_{\text{sum over } k} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

The proof is simple (but somewhat tedious!), just check every case. We sum over k, which leaves four free indices and each index runs $1 \rightarrow 3$, therefore there are $4^3 = 81$ cases. Here are two examples:

- $i = 1,\ j = 2,\ l = 1,\ m = 3$

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{121}\epsilon_{113} + \epsilon_{122}\epsilon_{213} + \epsilon_{123}\epsilon_{313} = 0 + 0 + 0 = 0$$

$$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \delta_{11}\delta_{23} - \delta_{13}\delta_{21} = 0 + 0 = 0$$

- $i = 1,\ j = 2,\ l = 1,\ m = 2$

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{121}\epsilon_{112} + \epsilon_{122}\epsilon_{212} + \epsilon_{123}\epsilon_{312} = 0 + 0 + 1 = 1$$

$$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \delta_{11}\delta_{22} - \delta_{12}\delta_{21} = 1 + 0 = 1$$

Example A vector identity

$$\begin{aligned}
[\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})]_i &= \epsilon_{ijk} a_j (\boldsymbol{b} \times \boldsymbol{c})_k \\
&= \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j b_l c_m \\
&= b_i a_j c_j - c_i a_j b_j \\
&= [(\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c}]_i
\end{aligned}$$

1.5 A trick: symmetry and anti-symmetry

We expect the $\boldsymbol{a} \times \boldsymbol{a} = \mathbf{o}$.

In index notation,

$$[\boldsymbol{a} \times \boldsymbol{a}]_i = \epsilon_{ijk} a_j a_k$$

The term $a_j a_k$ is symmetric in j, k. That is, $a_j a_k = a_k a_j$.

The term ϵ_{ijk} is anti-symmetric in j, k. That is $\epsilon_{ijk} = -\epsilon_{ikj}$.

$$\begin{aligned}
\epsilon_{ijk} a_j a_k &= \frac{1}{2}\left(\epsilon_{ijk} a_j a_k + \epsilon_{ijk} a_j a_k\right) \\
&= \frac{1}{2}\left(\epsilon_{ijk} a_j a_k + \epsilon_{ikj} a_k a_j\right) \\
&= \frac{1}{2}\left(\epsilon_{ijk} a_j a_k - \epsilon_{ijk} a_j a_k\right) \\
&= 0,
\end{aligned}$$

as expected.

Symmetry and anti-symmetry are often useful to simplify expressions. The permutation symbol is anti-symmetric in any of its two indexes. Some other symmetric expressions include

- $a_i a_j$ for any vector $\boldsymbol{a}$
- The Kronecker delta δ_{ij}
- The matrix $\boldsymbol{B}$ when $[\boldsymbol{B}]_{ij} = [\boldsymbol{A}]_{ij} + [\boldsymbol{A}]_{ji}$
- The partial second derivative of a scalar function ϕ, $\dfrac{\partial^2 \phi}{\partial x_i \partial x_j}$

1.6 Vector derivatives

The real power of index notation is revealed when we look at vector differential identities. The vector derivatives known as the gradient, the divergence and the curl can all be written in terms of the operator ∇,

$$\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right],$$

where $[x_1, x_2, x_3]$ are the components of the position vector $\boldsymbol{x}$.

For the scalar function ϕ and a vector function $\boldsymbol{u}$, both of which are a functions of position $\boldsymbol{x}$, we can write

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Gradient

$$\text{grad}\,\phi: \quad [\nabla\phi]_i = \frac{\partial\phi}{\partial x_i} = \left[\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right]$$

Divergence

$$\text{div}\,\boldsymbol{u}: \quad \nabla\cdot\boldsymbol{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Curl (recall that $[\boldsymbol{a}\times\boldsymbol{b}]_i = \epsilon_{ijk}a_j b_k$)

$$\text{curl}\,\boldsymbol{u}: \quad [\nabla\times\boldsymbol{u}]_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

The machinery we have developed for index notation can be used directly to manipulate quantities without having to be constantly thinking about the complicated physical meanings of div or curl, for example.

Example: Product rule for div and curl

Divergence

$$
\begin{aligned}
\nabla \cdot (\phi \boldsymbol{u}) &= \frac{\partial}{\partial x_i}(\phi u_i) \\
&= \frac{\partial \phi}{\partial x_i} u_i + \phi \frac{\partial u_i}{\partial x_i} \qquad \text{(product rule for differentiation)} \\
&= \boldsymbol{u} \cdot \nabla \phi + \phi \nabla \cdot \boldsymbol{u}
\end{aligned}
$$

Curl

$$
\begin{aligned}
[\nabla \times (\phi \boldsymbol{u})]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j}(\phi u_k) \\
&= \epsilon_{ijk} \left(u_k \frac{\partial \phi}{\partial x_j} + \phi \frac{\partial u_k}{\partial x_j} \right) \\
&= [\nabla \phi \times \boldsymbol{u} + \phi \nabla \times \boldsymbol{u}]_i
\end{aligned}
$$

Example: curl (gradϕ)

$$
\begin{aligned}
[\nabla \times (\nabla \phi)]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial \phi}{\partial x_k} \right) \\
&= \epsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \\
&= 0 \qquad \text{(by symmetry and anti-symmetry in } j,\ k\text{)}
\end{aligned}
$$

1.7 Derivatives of the position vector

If the position vector $\boldsymbol{x}$ itself appears in a vector expression which is being manipulated in index notation, then it is useful to notice that

$$
\frac{\partial x_i}{\partial x_j} = \delta_{ij},
$$

which follows directly from the definition of a partial derivative:

$$\frac{\partial x_1}{\partial x_1} = 1, \quad \frac{\partial x_1}{\partial x_2} = 0, \quad \text{etc.}$$

So for example:

- $\nabla \cdot \boldsymbol{x} = \dfrac{\partial x_i}{\partial x_i} = \delta_{ii} = 3$
- $[\nabla \times \boldsymbol{x}]_i = \epsilon_{ijk}\dfrac{\partial x_k}{\partial x_j} = \epsilon_{ijk}\delta_{jk} = 0$ (by symmetry/anti-symmetry in j, k)

1.8 Magnitude of a vector

Sometimes the magnitude (or modulus) of a vector appears in an expression which you wish to differentiate in some way. The trick here is to force it into the form of a scalar product, even though this may make the expression look more complicated than when you started. Index notation allows scalar products to be handled easily, and the normal rules of calculus will allow the calculation to be done without difficulty.

Example: What is $\nabla|\boldsymbol{x}|$?

$$\begin{aligned}[\nabla|\boldsymbol{x}|]_i &= \frac{\partial}{\partial x_i}\left((x_j x_j)^{1/2}\right) \\ &= \frac{1}{2}(x_k x_k)^{-1/2}\, 2x_j \frac{\partial x_j}{\partial x_i}\end{aligned}$$

There is a danger of j appearing four times in the same expression which is not permitted and would indicate an error. Recall that j was a dummy index, summed over all values. So in one of the two expressions, j has been replaced by k (or any other index you like, apart from i or j). Continuing:

$$\begin{aligned}\frac{1}{2}(x_k x_k)^{-1/2}\, 2x_j \frac{\partial x_j}{\partial x_i} &= (x_k x_k)^{-1/2}\, x_j \delta_{ij} \\ &= (x_k x_k)^{-1/2}\, x_i\end{aligned}$$

or in vector notation,

$$\nabla |\boldsymbol{x}| = \frac{\boldsymbol{x}}{|\boldsymbol{x}|},$$

which is just the unit vector in the direction of $\boldsymbol{x}$.

1.9 Integral theorems

Index notation allows the divergence theorem and Stokes's theorem to be written in a way which makes them look familiar. Recall this result for ordinary integration and differentiation of a function $f(x)$ on the interval (a, b):

$$\int_a^b \frac{\partial f}{\partial x}\, dx = f|_a^b = f(b) - f(a)$$

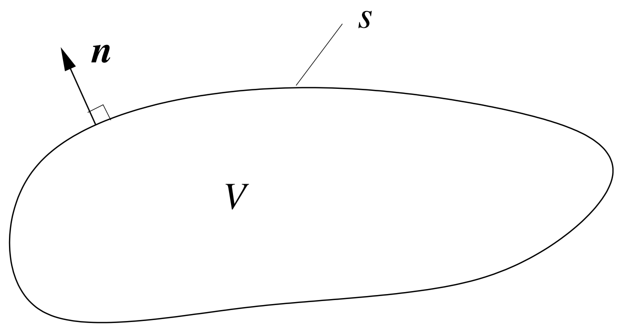
In other words, the integral of the derivative of a function is equal to the function, and can be evaluated from the value of the function on the boundaries of the integration interval. This general description will turn out to apply just as well to volume integrals (via the Divergence Theorem) and to surface integrals (via Stokes's theorem).

1.9.1 Divergence theorem

The divergence theorem states:

$$\int_V \nabla \cdot \boldsymbol{f}\, dV = \int_S \boldsymbol{f} \cdot \boldsymbol{n}\, dS,$$

where $\boldsymbol{f}$ is a vector function of position, V is a volume enclosed by the surface S ($S = \partial V$) and $\boldsymbol{n}$ is the unit outward normal vector on S (it points outwards from V).



We can also write $d\boldsymbol{A} = \boldsymbol{n}\, dS$, or using indexes $dA_i = n_i\, dS$.

For the divergence theorem, we can derive a number of other results (corollaries). For example, what can we say about the gradient of a scalar function, integrated over the volume V? To examine this, consider the divergence theorem using index notation,

$$\int_V \frac{\partial f_i}{\partial x_i}\, dV = \int_S f_i n_i\, dS.$$

Now suppose the we have a scalar function ϕ. We can create a vector function $\boldsymbol{f}$ by multiplying it by a vector $\boldsymbol{a}$,

$$\boldsymbol{f} = \phi \boldsymbol{a},$$

where $\boldsymbol{a}$ is any fixed vector (it does not depend on $\boldsymbol{x}$). Substituting this into the divergence theorem,

$$\int_V \frac{\partial \phi a_i}{\partial x_i}\, dV = \int_S \phi a_i n_i\, dS$$

Since $\boldsymbol{a}$ is fixed it can be taken out of the integral,

$$a_i \left(\int_V \frac{\partial \phi}{\partial x_i}\, dV - \int_S \phi n_i\, dS \right) = 0$$

The vector $\boldsymbol{a}$ is arbitrary, as long as it is fixed . Therefore the above expression must hold for all fixed vectors. This implies that

$$\left(\int_V \frac{\partial \phi}{\partial x_i}\, dV - \int_S \phi n_i\, dS \right) = 0$$

and therefore

$$\int_V \frac{\partial \phi}{\partial x_i}\, dV = \int_S \phi n_i\, dS$$

In vector notation, we have

$$\int_V \nabla\phi \, dV = \int_S \phi \boldsymbol{n} \, dS.$$

A more general result which can be proved using the 'fixed vector' type approach is that

$$\int_V \frac{\partial}{\partial x_i} (\star) \; dV = \int_S (\star) \, \boldsymbol{n} \, dS,$$

where $(\star)$ is any index notation expression. We could, for example, insert

$$\epsilon_{jik} f_k$$

and deduce a 'curl theorem' (check it). We now have a nice generalisation for relating volume integrals of vector derivatives to surface integrals. The integral of any 'regular' function over a volume can be transformed into a surface integral. When in doubt, one can always return to index notation to check and develop the necessary relationships.

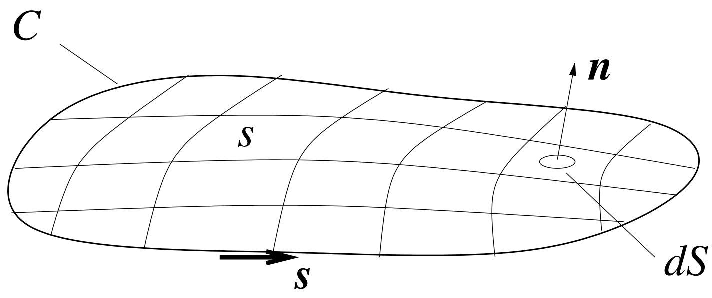
Transforming surface integrals into volume integrals is the key to deriving many equations used in physics and mechanics which come from *conservation laws*. We can start by posing a balance in terms of what is happening on the boundary, and then transform the expression of conservation into a differential equation.

1.9.2 Stokes theorem

Stokes theorem is stated as:

$$\int_S \nabla \times \boldsymbol{f} \, d\boldsymbol{A} = \oint_C \boldsymbol{f} \cdot \, d\boldsymbol{l},$$

where S is a surface (possibly curved), $d\boldsymbol{A} = \boldsymbol{n} \, dS$ is an area vector element and $\boldsymbol{n}$ is the unit normal vector to the surface S, $C = \partial S$ is the curve that bounds the surface and $d\boldsymbol{l} = \boldsymbol{s} \, dC$ is the vector element running along the curve and $\boldsymbol{s}$ is tangential to the boundary. The direction of $\boldsymbol{s}$ is important!



As for the Divergence theorem, we can write Stokes theorem using indices,

$$\int_S \epsilon_{ijk} \frac{\partial f_k}{\partial x_j} n_i \, dS = \oint_C f_i s_i \, dC$$

and we can use the trick with the fixed vector to get

$$\int_S \epsilon_{ijk} \frac{\partial (\star)}{\partial x_j} n_i \, dS = \oint_C (\star) \, s_i \, dC$$

The expression can be manipulated further using another expression which involves another ϵ symbol, and then using the $\epsilon - \delta$ identity to push the ϵ to the other side of the equation. However, it is not possible to eliminate the ϵ entirely, therefore Stokes's theorem always appears to be more complicated that the divergence theorem.

1.9.3 Integration by parts

An important tool, particularly for variational methods, is integration by parts. Starting in one dimension,

$$\int_a^b \frac{d\,(uv)}{dx} \, dx = \int_a^b v \frac{du}{dx} \, dx + \int_a^b u \frac{dv}{dx} \, dx,$$

and then rearranging,

$$\int_a^b v \frac{du}{dx} \, dx = -\int_a^b u \frac{dv}{dx} \, dx + uv|_a^b \, .$$

We want now to generalise this to two and three dimensions. The do this, consider the divergence of a term $\phi \boldsymbol{u}$,

$$\int_V \nabla \cdot (\phi \boldsymbol{u}) \; dV = \int_V \nabla \phi \cdot \boldsymbol{u} \, dV + \int_V \phi \nabla \cdot \boldsymbol{u} \, dV$$

where the product rule for differentiation has been applied. Applying the divergence theorem for the term on the left-hand side,

$$\int_S \phi \boldsymbol{u} \cdot \boldsymbol{n} \, dS = \int_V \nabla \phi \cdot \boldsymbol{u} \, dV + \int_V \phi \nabla \cdot \boldsymbol{u} \, dV$$

The operation can also be performed using index notation,

Product rule for differentiation

$$\int_V \frac{\partial (\phi u_i)}{\partial x_i} \, dV = \int_V \frac{\partial \phi}{\partial x_i} u_i \, dV + \int_V \phi \frac{\partial u_i}{\partial x_i} \, dV$$

Apply divergence theorem to LHS

$$\int_S \phi u_i n_i \, dS = \int_V \frac{\partial \phi}{\partial x_i} u_i \, dV + \int_V \phi \frac{\partial u_i}{\partial x_i} \, dV$$

1.10 Directional derivative

We will make frequent use of the *directional derivative* when working with variational problems. The directional derivative of a function $f(\boldsymbol{u})$ is defined as

$$Df(\boldsymbol{u})[\boldsymbol{v}] = \left. \frac{d f(\boldsymbol{u} + \epsilon \boldsymbol{v})}{d\epsilon} \right|_{\epsilon = 0}$$

It means 'the derivative of f with respect to $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$'. In computing the directional derivative, we first compute the derivative with respect to ϵ, and then set $\epsilon = 0$. For example

For $f = \boldsymbol{u} \cdot \boldsymbol{u}$, where $\boldsymbol{u}$ is some vector function, the directional derivative is

$$
\begin{aligned}
Df(\boldsymbol{u})[\boldsymbol{v}] &= \frac{d}{d\epsilon} \left((\boldsymbol{u} + \epsilon \boldsymbol{v}) \cdot (\boldsymbol{u} + \epsilon \boldsymbol{v}) \right) \Big|_{\epsilon=0} \\
&= 2 \left((\boldsymbol{u} + \epsilon \boldsymbol{v}) \cdot \boldsymbol{v} \right) \big|_{\epsilon=0} \\
&= 2 \boldsymbol{u} \cdot \boldsymbol{v}
\end{aligned}
$$

1.11 Tensors

The manipulation of objects using indices has been presented on the premise of a few basic rules. However, much of what has been presented carries over to the richer field of tensors. Some tensor basic tensor operations will be considered, but only scratching the surface so that we can tackle more problems of practical relevance[2].

We are already familiar with the idea of vectors. However, you might find it hard to give a formal definition of a vector, although you have though been exposed to aspects of the formal definition. For example, vectors are often expressed as

$$
\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k},
$$

or using modern notation as

$$
\boldsymbol{a} = a_i \boldsymbol{e}_i = a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2 + a_3 \boldsymbol{e}_3,
$$

The mathematical definition is in two parts. First, a vector $\boldsymbol{a}$ is quantity which, with respect to a particular Cartesian coordinate system, has three components (a_1, a_2, a_3), or equivalently a_i, $i = 1, 2, 3$. But that is not the whole story. We also need to express the fact that if we choose to rotate our coordinate axes, the vector (the 'arrow' which you might draw in a diagram) remains the same. This means that a vector also has an associated *basis*. In the above example, the basis is $\boldsymbol{e}_i$. We may change the components

[2]Basic consideration of tensors is new in 2011.

and the basis of a vector such that the vector remains the same (it still points in the same direction and still has the same length).

A second-order (or second-rank) tensor is like a matrix but has a basis associated, and can be expressed using the notation

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

Luckily, when working with a fixed orthonormal basis, we can work with A_{ij}, usually manipulating it just as we would a matrix.

You have in fact considered second-order tensors in Part IA without being told (and most likely without the lecturer realising). Recall that 'matrix' can be rotated into a new coordinate system via

$$A'_{ij} = R_{ik}A_{kl}R_{jl}$$

where it turns out that

$$R_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j.$$

Common second-order tensors include stress and strain.

Derivatives and second order tensors

The two operations that we wish to consider are the gradient of a vector and the divergence of a second-order tensor. The gradient of a vector produces a second-order tensor:

$$\begin{aligned}\nabla\boldsymbol{a} &= \nabla \otimes \boldsymbol{a}\\ &= \frac{\partial a_i}{\partial x_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j.\end{aligned}$$

For our purposes, it suffices to just recall $[\nabla\boldsymbol{a}]_{ij} = \partial a_i/\partial x_j$. Just to make things more confusing, in fluid mechanics, the gradient is often defined as the transpose of what is defined here.

Another important operation is the divergence of a second-order tensor. It leads to a vector:

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_{ij}}{\partial x_j}\boldsymbol{e}_i$$

Similar to the gradient, it suffices for our purposes to recall $[\nabla \cdot \boldsymbol{A}]_i = \partial A_{ij}/\partial x_j$. Just as for a vector, the divergence theorem can be applied to a second order tensor:

$$\int_V \nabla \cdot \boldsymbol{A}\, dV = \int_{\partial V} \boldsymbol{A}\boldsymbol{n}\, dS$$

Tensors, in combination balance laws, can be used to systematically derive many differential equations of importance in physics and engineering.

Example: Equilibrium equation of a body with mass density ρ: If the force per unit area acting on a surface is denoted by the vector $\boldsymbol{t}$ (the traction vector), equilibrium (forces sum to zero) requires that

$$\int_{\partial V} \boldsymbol{t} \, dS + \int_{V} \rho \boldsymbol{g} \, dV = \mathbf{o}.$$

The stress (a second order tensor) at a point satisfies by definition $\boldsymbol{t} = \boldsymbol{\sigma} \boldsymbol{n}$, where $\boldsymbol{n}$ is the outward unit normal vector to a surface. Using indices, $t_i = \sigma_{ij} n_j$. Inserting the expression for $\boldsymbol{\sigma}$,

$$\int_{\partial V} \boldsymbol{\sigma} \boldsymbol{n} \, dS + \int_{V} \rho \boldsymbol{g} \, dV = \mathbf{o}.$$

Applying the divergence theorem,

$$\int_{V} \nabla \cdot \boldsymbol{\sigma} \, dV + \int_{V} \rho \boldsymbol{g} \, dV = \mathbf{o}.$$

Since equilibrium must apply to any sub-body of V, the equation can be localised,

$$\nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g} = \mathbf{o}.$$

The is balance of linear momentum. It can be shown that angular momentum balance is satisfied if $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{T}$.