# CE394M: An introduction to continuum mechanics

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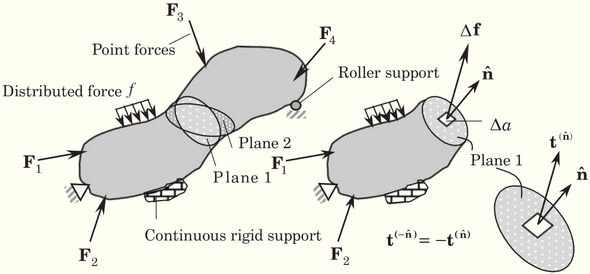
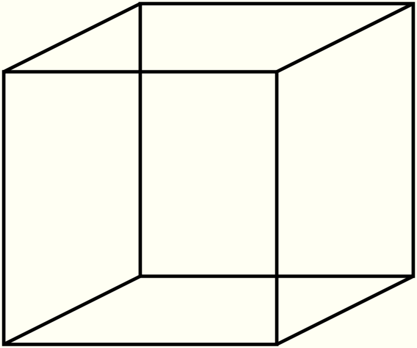
# Stress vector on a plane



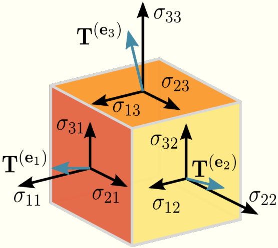

Stress vector on a plane normal to $\hat{\mathbf{n}}$ (Reddy., 2008)

If we denote by $\Delta\mathbf{f}(\hat{\mathbf{n}})$ the force on a small area $\hat{\mathbf{n}}$ located at the position $x$, the stress vector can be defined:

# Stress vector on a plane



The stress vector in the direction of $\hat{\mathbf{n}}$ is:



# Cauchy Stress theorem

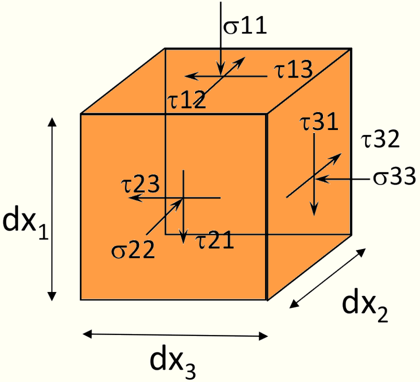
The stress vector **t** is now written as a 3-component column vector. The equation tells how to write three equations for the components of **t**, by multiplying the stress matrix by the unit vector **n**.

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$
$$t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$
$$t_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

# Stress tensor

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

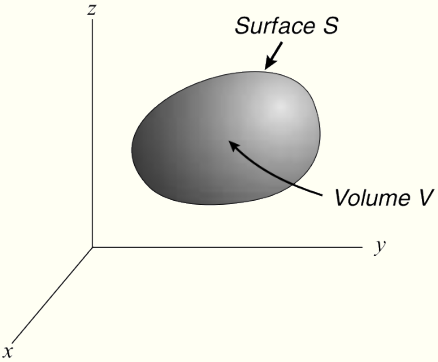


# What is a Tensor?

# Divergence theorem

The total contact force acting on $V$ is:

Divergence theorem converts an integral over $S$ into an integral over the volume $V$ enclosed by $S$.



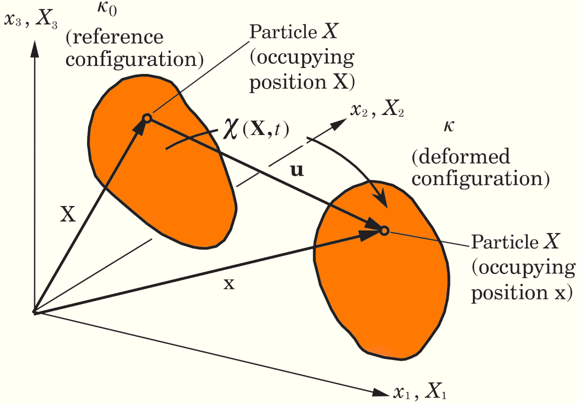
# Equilibrium

If we assume that the body is in static equilibrium, we must set the contact forces and body forces equal.

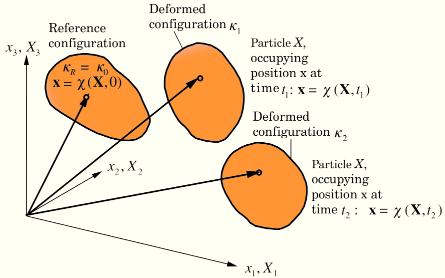
$$
\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} - b_x = 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} - b_y = 0 \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - b_z = 0
\end{aligned}
$$

# Descriptions of Motion

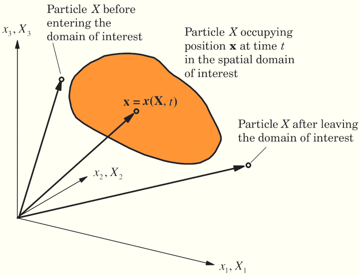
For a given geometry and loading, the body $B$ will undergo macroscopic geometric changes within the body, which are termed deformation. The geometric changes are accompanied by stresses that are induced in the body.



The displacement of the particle X is given: $\mathbf{u} = \mathbf{x} - \mathbf{X}$.

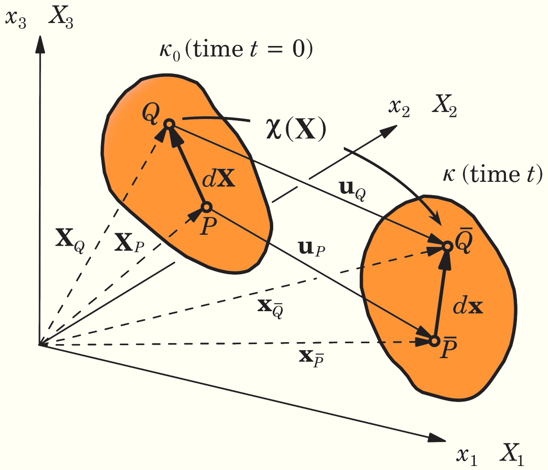



Lagrangian




Eulerian

# Deformation Gradient Tensor




Points $P$ and $Q$ separated by a distance $d\mathbf{X}$ in the undeformed configuration $k_0$ take up positions $\bar{P}$ and $\bar{Q}$, respectively, in the deformed configuration $k$, where they are separated by distance $d\mathbf{x}$

# Deformation Gradient Tensor

deformation gradient $F_k$ of $k$ relative to the reference configuration $k_0$, which gives the relationship of a material line $dX$ before deformation to the line $dx$ (consisting of the same material as $dX$) after deformation.

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}$$

# Deformation Gradient Tensor

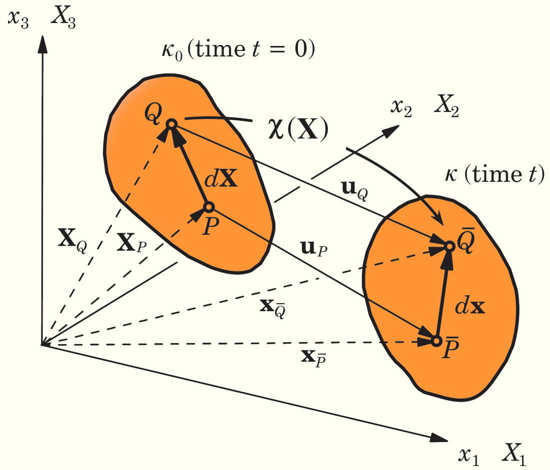
Isotropic compression/extension:

Simple extension:

Simple shear:

It is rather difficult to invert the mapping even for this simple
nonhomogeneous deformation

# Deformation Tensors




Points $P$ and $Q$ separated by a distance $d\mathbf{X}$ in the undeformed configuration $k_0$ take up positions $\bar{P}$ and $\bar{Q}$, respectively, in the deformed configuration $k$, where they are separated by distance $d\mathbf{x}$

# Eulerian Strain Tensor

The change in the squared lengths that occurs as a body deforms from the reference to the current configuration can be expressed relative to the original length as:

$$(ds)^2 - (dS)^2 = 2d\mathbf{x} \cdot \mathbf{e} \cdot d\mathbf{x},$$

$$\varepsilon = \frac{1}{2}(\mathbf{I} - \mathbf{F}^T \cdot \mathbf{F}^{-1})$$

The Eulerian strain tensor is written as:

# Eulerian strain tensor

If $u_1$ is displacement in $x_1$ direction. Then:

$$\varepsilon_{12} = \frac{1}{2}\left\{\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}\right) + \left(\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}\right) + \left(\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right)\right]\right\}$$

Engineering shear strain:
Typically, we assume small displacements + small strains. Therefore the quadratic terms (higher order) can be ignored. Pile driving or progression of slope failure cannot be modeled as a small strain problem!

# Linearization of strain tensor

Ignoring higher order terms is called as the linearization of the strain tensor. This assumption allows for two simplifications:

Alternative, use natural strain approach:

For a 1D deformation of a bar

- $l_0 = 5, l_f = 4.9$
- $l_0 = 5, l_f = 4$

As long as strains are small, small strain formulation is very close - OK! (If you have large strains - a correction / alternatives is to update the **B** matrix every iteration to adjust for finite strains).

# Mechanical notation for infinitesimal strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$i$ and $j$ refers to directions.

Resulting strain definitions

# Mechanical notation for infinitesimal strains

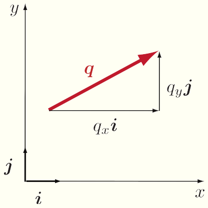
$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

# Reference

Below sections are for reference only.

# Vector calculus

A vector is expressed in terms of its components and the unit vectors in the $x-$ and $y-$ directions.




$$\mathbf{q} = q_x \mathbf{i} + q_y \mathbf{j}$$

where $q_x$ is the $x-$component and $q_y$ is the $y-$component and $i$ and $j$ are basis vectors (are unit length).

**Scalar product:**

$$\mathbf{q} \cdot \mathbf{r} = \mathbf{q}^T \mathbf{r} = \begin{bmatrix} q_x & q_y \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix} = q_x r_x + q_y r_y$$

# Vector calculus

**Grad:** If the del operator acts on a scalar field, say temperature $T(x, y)$, it produces a vector that points in the direction of the steepest slope.

$$\nabla \mathbf{T} = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}$$

**Divergence:** The scalar product of the del operator with a vector field q gives the divergence

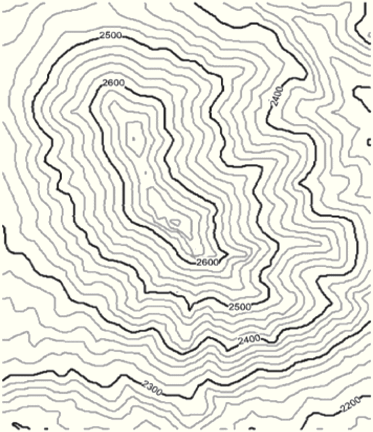
$$div\mathbf{q} = \nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}$$

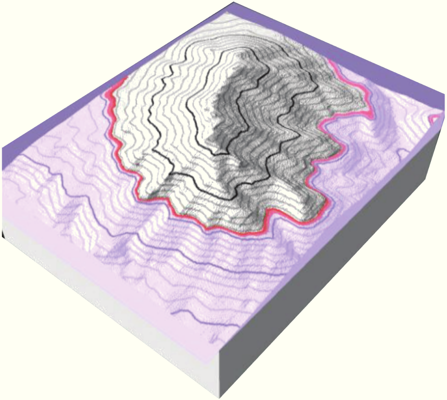
Notice the divergence of a vector field is a scalar.

**Divergence theorem**

$$\int_\Omega div\mathbf{q} d\Omega = \oint_\Gamma \mathbf{q} \cdot \mathbf{n} d\Gamma$$

# Vector calculus



Contour map for a terrain (left) and the associated three-dimensional model (right). If $T$ is interpreted as the height, the vector $\Delta T$ points in the direction of the steepest slope.

# The summation convention

Suppose $\mathbf{x}$ and $\mathbf{y}$ are vectors, and $\mathbf{A}$ and $\mathbf{B}$ are matrices. Write a few common combinations in terms of their components:

- **Dot product:**
- **Matrix-vector product:**
- **Matrix-vector product:**

# The summation convention

We can use a simplified notation by adopting the summation convention (due to Einstein), Do not write the summation symbol $\sum$. A repeated index implies summation. (An index may not appear more than twice on one side of an equality.) Using the summation convention

- **Dot product:**
- **Matrix-vector product:**
- **Matrix-vector product:**

This may seem a very peculiar trick, with no obvious benefit. However, it will turn out to be surprisingly powerful, and make many calculations involving vector identities and vector differential identities much simpler.

# The Kronecker delta $\delta_{ij}$

The identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We define the '*Kronecker delta*' as:

We know that: $\mathbf{Iy} = \mathbf{y}$

In other words 'if one index of $\delta_{ij}$ is summed, the effect is to swap this to the other index'.

# The permutation symbol $\epsilon_{ijk}$

Cross product of two vectors:

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{bmatrix} x_2y_3 - x_3y_2, & x_3y_1 - x_1y_3, & x_1y_2 - x_2y_1 \end{bmatrix}$$

where $e_i$ are the basis for the vectors. We have assumed that $e_i$ are the unit vectors for Cartesian coordinates (you may have seen the basis vectors written as $i$, $j$ and $k$). To express the cross product in index notation, we will use the permutation symbol $\epsilon_{ijk}$. The permutation symbol $\epsilon_{ijk}$ is defined as:

# The permutation symbol $\epsilon_{ijk}$

For example:

The permutation symbol is also known as the ‘alternating symbol’ or the ‘Levi-Civita symbol’. Using the permutation symbol, we can write the cross product of two vectors as:

# Vector derivatives

The real power of index notation is revealed when we look at vector differential identities. The vector derivatives known as the gradient, the divergence and the curl can all be written in terms of the operator $\nabla$

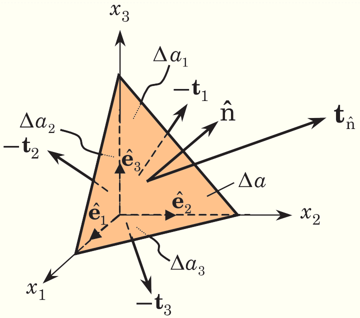
$$\nabla = \left[ \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3} \right]$$

where $[x_1, x_2, x_3]$ are the components of the position vector $\mathbf{x}$.

- **Gradient:**
- **Divergence:**
- **Curl:** $[\mathbf{a} \times \mathbf{b}]_i = \epsilon_{ijk} a_j b_k$

# Cauchy stress tensor

To establish the relationship between **t** and **n̂** we now set up an infinitesimal tetrahedron in Cartesian coordinates:




If $-\mathbf{t}_1, -\mathbf{t}_2, -\mathbf{t}_3$ and **t** denote the stress vectors in the outward directions on the faces of the infinitesimal tetrahedron whose areas are $\Delta a_1, \Delta a_2, \Delta a_3$, and $\Delta a$, respectively. $\Delta v$ is the volume of the tetrahedron, $\rho$ the density, $f$ the body force per unit mass, and **a** the acceleration.

# Cauchy stress tensor

we have by Newton's second law for the mass inside the tetrahedron:

Since the total vector area of a closed surface is zero (gradient theorem):

The volume $\Delta v$ can be expressed as:
where $\Delta h$ is the perpendicular distance from the origin to the slant face.

# Cauchy stress tensor

In the limit when the tetrahedron shrinks to a point $\Delta h \rightarrow 0$:

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i)\mathbf{t}_i$$

where the summation convention is used.

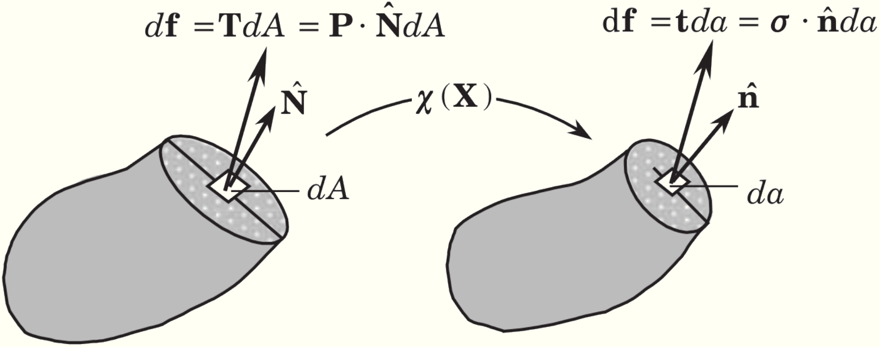
The terms in the parenthesis is the **stress tensor** $\sigma$:

$$\sigma \equiv \hat{\mathbf{e}}_1\mathbf{t}_1 + \cdot\hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3$$

The stress tensor is a property of the medium that is independent of the $\hat{\mathbf{n}}$

The stress vector $\mathbf{t}$ represents the vectorial stress on a plane whose normal is $\hat{\mathbf{n}}$. $\sigma$ is the *Cauchy stress tensor* defined to be the *current force per unit deformed area*. In Cartesian component, the Cauchy formula is: $t_i = n_j \sigma_{ji}$.

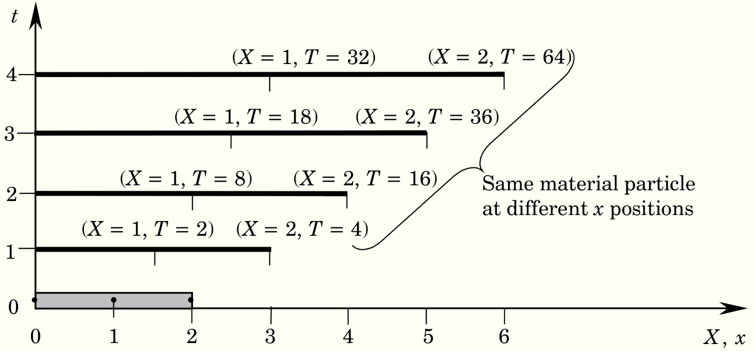
# Cauchy stress vs Piola-Kirchoff stress




An introduction to continuum mechanics - J. N. Reddy (2008)

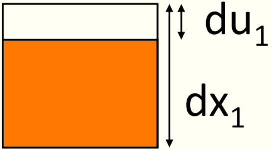
# Displacement field

Consider 1D mapping $\mathbf{x} = \mathbf{X}(1 + 0.5t)$ defining the motion of a rod of initial length two units. The rod experiences a temperature distribution $T$ given by the material description $T = 2\mathbf{X}t^2$ or by the spatial description $T = 2\mathbf{x}t^2/(1 + 0.5t)$



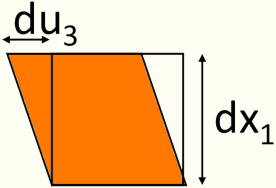

Material and spatial descriptions of motion (Reddy., 2008)

Normal strain




$\varepsilon_{11} = du_1/dx_1$

Shear strain




$\gamma_{13} = du_3/dx_1$