CE394M: Isoparametric elements and Gauss integration

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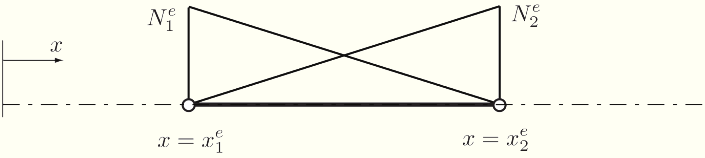
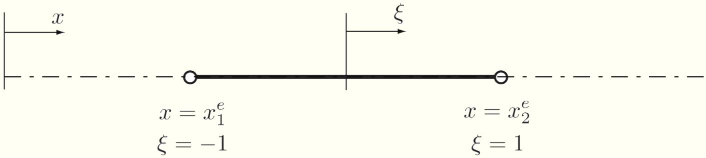
Overview

Isoparametric mapping in 1D



Shape functions for a two-noded element.

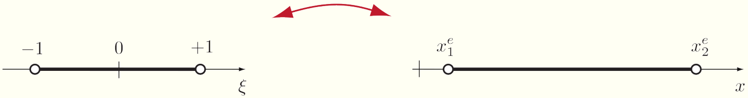
$$N_1(x) = 1 - \frac{x - x_1^e}{x_2^e - x_1^e} \qquad N_2(x) = \frac{x - x_1^e}{x_2^e - x_1^e}$$



Global coordinate x and local coordinate ξ

Isoparametric mapping in 1D

consider a coordinate transformation which transforms (maps) the coordinate x into a local (element specific) coordinate ξ:



Mapping of the parent element onto the physical element.

The coordinate ξ fulfills the relationships:

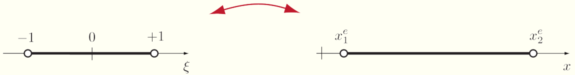
"stretch transformation" of $x(\xi)$:

$$
\begin{aligned}
x(\xi) &= x_1^e + \frac{1}{2}(x_2^e - x_1^e)(1+\xi) \\
&= \frac{x_1^e + x_2^e}{2} + \frac{x_2^e - x_1^e}{2}\xi
\end{aligned}
$$

Isoparametric mapping in 1D

The shape functions $N_1^e = (1 - x/l)$ and $N_2^e = (x/l)$ can also be expressed using ξ:

The key idea of the isoparametric concept is to use these shape functions for writing the coordinate transformation between x and ξ



Mapping of the parent element onto the physical element.

Isoparametric mapping in 1D

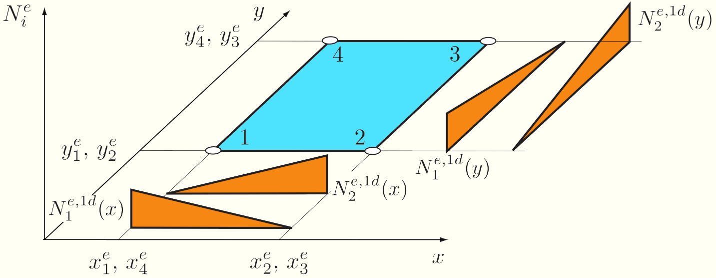
- The coordinate ξ is usually called the **natural coordinate** and always lies by definition between -1 and $+1$.
- The parent element is solely for numerical purposes.
- The finite element analysis is still performed over the physical domain.

In an isoparametric element the field variable, like displacement, is approximated with the same set of shape functions as those used for the coordinate transformation:

To compute the derivatives which appear in the weak form the chain rule is used:

4-noded rectangular element

An alternative and more elegant approach is to construct the shape functions by the **tensor product method.** This is based on taking products of one-dimensional shape functions.



Construction of two dimensional shape functions.

$$N_2^e = N_2^{e,1d}(x) \times N_1^{e,1d}(y)$$

4-noded rectangular element

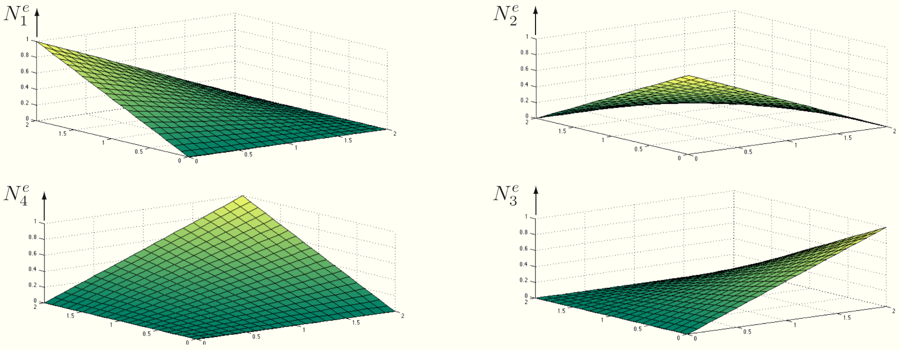
The four shape functions, also called **bilinear shape functions**, for the quadrilateral element are:

$$
\begin{aligned}
N_1^e(x,y) &= \frac{x - x_2^e}{x_1^e - x_2^e}\frac{y - y_4^e}{y_1^e - y_4^e} = \frac{1}{A^e}(x - x_2^e)(y - y_4^e) \\
N_2^e(x,y) &= \frac{x - x_1^e}{x_2^e - x_1^e}\frac{y - y_4^e}{y_1^e - y_4^e} = -\frac{1}{A^e}(x - x_1^e)(y - y_4^e) \\
N_3^e(x,y) &= \frac{x - x_1^e}{x_2^e - x_1^e}\frac{y - y_1^e}{y_4^e - y_1^e} = \frac{1}{A^e}(x - x_1^e)(y - y_1^e) \\
N_4^e(x,y) &= \frac{x - x_2^e}{x_1^e - x_2^e}\frac{y - y_1^e}{y_4^e - y_1^e} = -\frac{1}{A^e}(x - x_2^e)(y - y_1^e)
\end{aligned}
$$

where A^e is the area of the element.

4-noded rectangular element

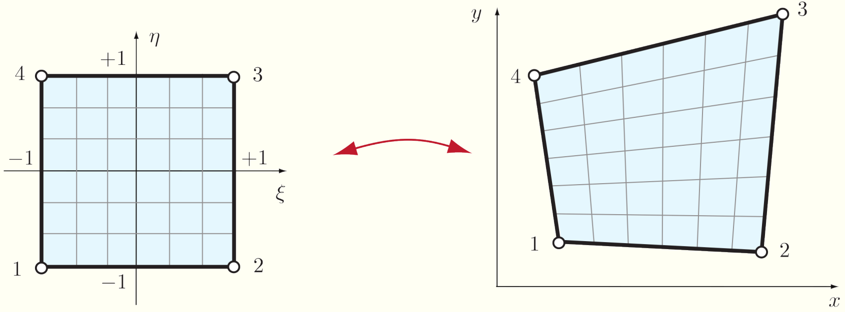
The four shape functions are plotted in the following figure:



Four shape functions of the rectangular element (on $[0,2] \times [0,2]$).

Isoparametric mapping of a quadrilateral element

The idea of isoparametric mapping is used for deriving shape functions for arbitrary quadrilateral elements:



Mapping of the bi-unit parent element onto the quadrilateral element in the physical space.

Isoparametric mapping of a quadrilateral element

The bi-unit square is the parent domain and ξ and η are its natural coordinates.
To map points from the parent domain onto the quadrilateral in the physical domain the four nodal shape functions are used:

where $N^{4Q}(\xi, \eta)$ are the four-node element shape functions in the natural coordinates and x^e and y^e are the vectors of the element coordinates:

$$\mathrm{x}^e = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix} \quad \mathrm{y}^e = \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

Isoparametric mapping of a quadrilateral element

As the parent element is a bi-unit square its shape functions are identical to those of the rectangular element expressed in ξ and η coordinates.

$$
\begin{aligned}
N_1^{4Q}(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta) \\
N_2^{4Q}(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta) \\
N_3^{4Q}(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta) \\
N_4^{4Q}(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta)
\end{aligned}
$$

Isoparametric shape functions

The pwp will be approximated with the same shape functions:

The element is called *isoparametric* because the pwp approximation and the mapping of the geometry is accomplished with the same shapefn.
The displacement will be approximated as:

$$u = \begin{bmatrix} N_1^{4Q} & 0 & N_2^{4Q} & 0 & N_3^{4Q} & 0 & N_4^{4Q} & 0 \\ 0 & N_1^{4Q} & 0 & N_2^{4Q} & 0 & N_3^{4Q} & 0 & N_4^{4Q} \end{bmatrix} \begin{bmatrix} a_{1x}^e \\ a_{1y}^e \\ a_{2x}^e \\ a_{2y}^e \\ a_{3x}^e \\ a_{3y}^e \\ a_{4x}^e \\ a_{4y}^e \end{bmatrix}$$

Derivatives isoparametric shape functions

The gradient of displacement for the four-node (isoparametric) quadrilateral element is:
Strain: $\epsilon = \mathrm{B}^e \mathrm{a}^e$

$$\begin{bmatrix} \epsilon^e_{xx} \\ \epsilon^e_{yy} \\ 2\epsilon^e_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^{4Q}}{\partial x} & 0 & \frac{\partial N_2^{4Q}}{\partial x} & 0 & \frac{\partial N_3^{4Q}}{\partial x} & 0 & \frac{\partial N_4^{4Q}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{4Q}}{\partial y} & 0 & \frac{\partial N_2^{4Q}}{\partial y} & 0 & \frac{\partial N_3^{4Q}}{\partial y} & 0 & \frac{\partial N_4^{4Q}}{\partial y} \\ \frac{\partial N_1^{4Q}}{\partial y} & \frac{\partial N_1^{4Q}}{\partial x} & \frac{\partial N_2^{4Q}}{\partial y} & \frac{\partial N_2^{4Q}}{\partial x} & \frac{\partial N_3^{4Q}}{\partial y} & \frac{\partial N_3^{4Q}}{\partial x} & \frac{\partial N_4^{4Q}}{\partial y} & \frac{\partial N_4^{4Q}}{\partial x} \end{bmatrix} \begin{bmatrix} u^e_{1x} \\ u^e_{1y} \\ u^e_{2x} \\ u^e_{2y} \\ u^e_{3x} \\ u^e_{3y} \\ u^e_{4x} \\ u^e_{4y} \end{bmatrix}$$

Derivatives isoparametric shape functions

To compute shape function derivatives the chain rule will be used:

written as matrices and vectors this becomes:
J^e is the Jacobian which contains the derivatives of the physical coordinates with respect to the natural coordinates.

Derivatives isoparametric shape functions

The derivatives required for the weak form are computed by inverting the above expression:

$$\begin{bmatrix} \frac{\partial N_I^{4Q}}{\partial x} \\ \frac{\partial N_I^{4Q}}{\partial y} \end{bmatrix} = (\mathsf{J}^e)^{-1} \begin{bmatrix} \frac{\partial N_I^{4Q}}{\partial \xi} \\ \frac{\partial N_I^{4Q}}{\partial \eta} \end{bmatrix}$$

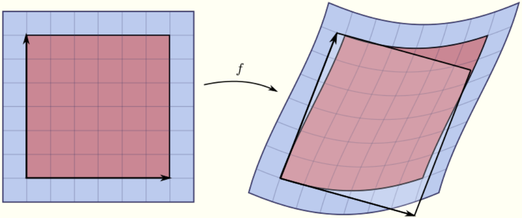
The inverse of J^e is:

$$\mathsf{J}^{e-1} = \frac{1}{|\mathsf{J}^e|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

Where $|\mathsf{J}^e|$ is the determinant of the Jacobian, which represents the ratio of an area element in the physical domain to the corresponding area element in the parent domain.

Jacobian

The Jacobian can also be thought of as describing the amount of "stretching", "rotating" or "transforming" that a transformation imposes locally. For example, if $(x', y') = f(x, y)$ is used to transform an image, the Jacobian $\mathrm{J}f(x, y)$, describes how the image in the neighborhood of (x, y) is transformed.



The Jacobian at a point gives the best linear approximation of the distorted parallelogram near that point (in translucent white on the right-side), and the Jacobian determinant gives the ratio of the area of the approximating parallelogram to that of the original square.

Derivatives isoparametric shape functions

The isoparametric mapping is used to compute the Jacobian:

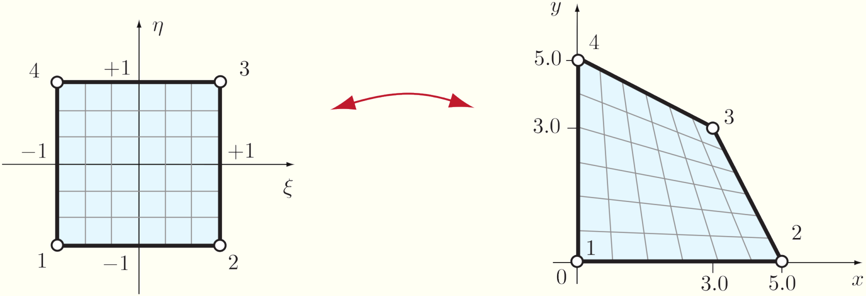
$$x(\xi,\eta) = \mathsf{N}^{4Q}(\xi,\eta)\mathsf{x}^e \quad y(\xi,\eta) = \mathsf{N}^{4Q}(\xi,\eta)\mathsf{y}^e$$

which leads to:

$$\mathsf{J}^e = \begin{bmatrix} \frac{\partial N_1^{4Q}}{\partial \xi} & \frac{\partial N_2^{4Q}}{\partial \xi} & \frac{\partial N_3^{4Q}}{\partial \xi} & \frac{\partial N_4^{4Q}}{\partial \xi} \\ \frac{\partial N_1^{4Q}}{\partial \eta} & \frac{\partial N_2^{4Q}}{\partial \eta} & \frac{\partial N_3^{4Q}}{\partial \eta} & \frac{\partial N_4^{4Q}}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix}$$

Effect of element shape on isoparametric mapping

Consider the following isoparametric mapping for a four-node quadrilateral element:



Effect of element shape on isoparametric mapping

The Jacobian of this mapping:

$$\mathrm{J}^e = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

is computed from

Effect of element shape on isoparametric mapping

After some algebra we obtain:

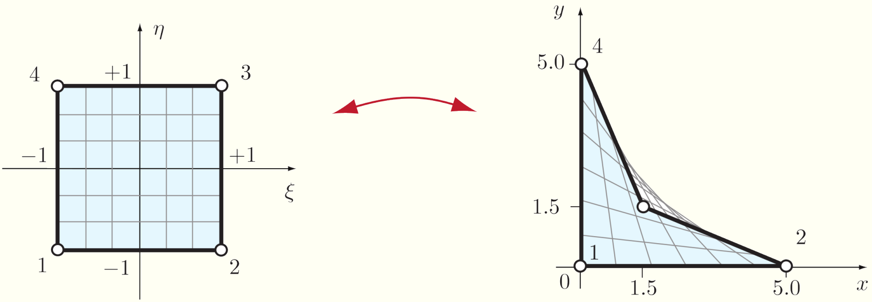
$$\mathsf{J}^e = \begin{bmatrix} 2 - \frac{\eta}{2} & -\frac{1}{2} - \frac{1}{2}\eta \\ -\frac{1}{2} - \frac{1}{2}\xi & 2 - \frac{1}{2}\xi \end{bmatrix}$$

The Jacobian has to be invertible for computing the derivatives of the shape functions with respect to the physical coordinates.
For the mapping to be invertible, the determinant of the Jacobian has to be larger than zero over the entire element:

$$det\mathsf{J}^e = \frac{5}{4}(3 - \xi - \eta)$$

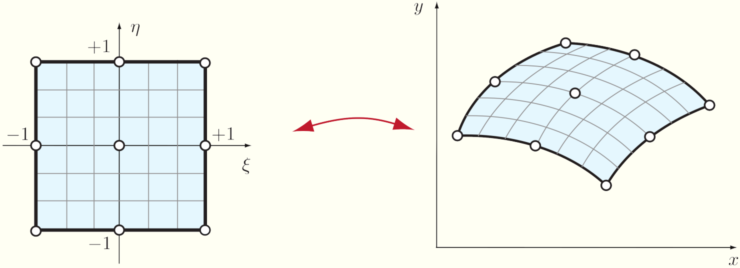
which is the case for this mapping.

In contrast to the previous mapping, it can be for the following mapping shown:



that the determinant of the Jacobian is zero or negative close to the non-convex corner.

Higher order quadrilateral element



Nine-node ispoarametric quadrilateral in parameter (left) and physical space (right).

Numerical integration (Quadrature)

- During the finite element solution procedure, it is necessary to integrate various quantities, for instance the stiffness matrix or force vectors.
- For isoparametric elements these quantities must in general be integrated numerically due to the presence of the Jacobian.
- Although there are many different numerical integration techniques, in finite elements Gauss integration is preferred.

Review of Basic Integration Rules

Consider the following integral which is to be integrated numerically:

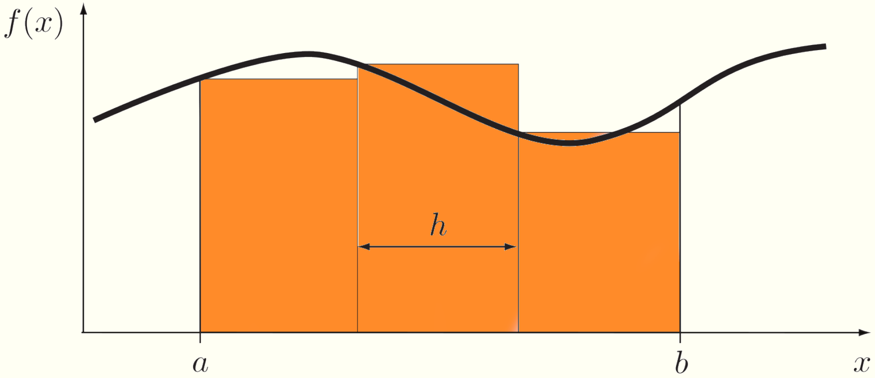
$$\int_a^b f(x)dx$$

The integration domain $[a, b]$ is first split into N equidistant subintervals:

with:

There are various schemes for approximating the integral using the function values at N positions

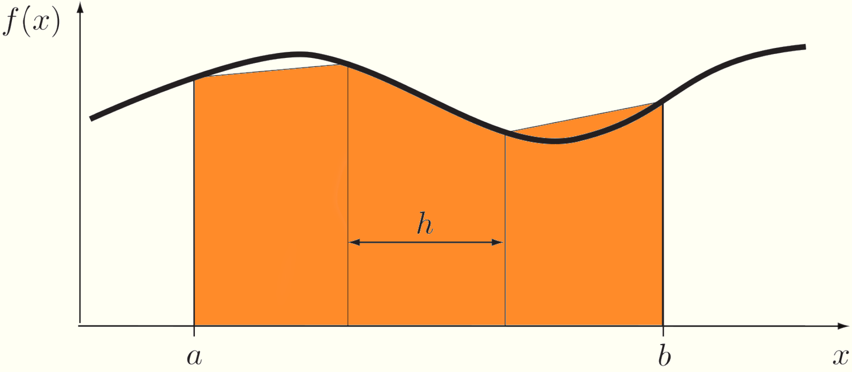
Rectangle rule (constant approximation)



Integration of $f(x)$ with the rectangle rule between $x = a$ and $x = b$.

$$\int_a^b f(x)dx \approx h(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{N-1}))$$

Trapezoidal rule (linear approximation)



Integration of $f(x)$ with the trapezoidal rule between $x = a$ and $x = b$.

$$\int_a^b f(x)dx \approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N))$$

Integration rule

Simpson's rule (quadratic approximation)

$$\int_a^b f(x)dx \approx \frac{1}{3}h(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{N-1})+f(x_N))$$

Notice that all these integration rules can be written in the following form:

$$\int_a^b f(x)dx \approx \sum_{I=0}^{N} w_I f(x_I)$$

where w_I are the integration weights and x_I are the integration points.

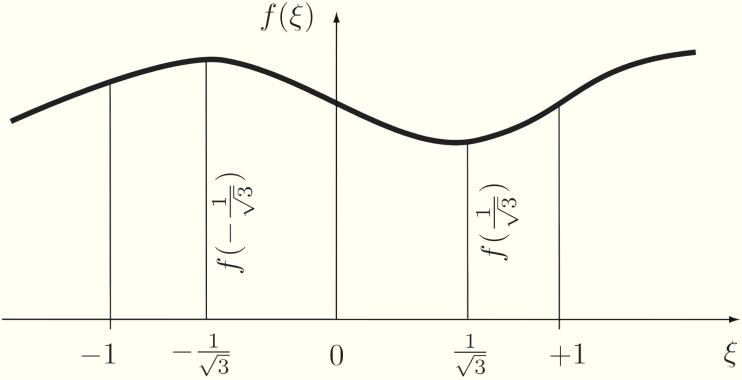
Gauss integration in 1D

Gauss integration formulas are always given over the parent domain $[-1, 1]$:

1. a very efficient technique for integrating functions that are (almost) polynomials, like finite element shape functions.
2. needs fewer subintervals to provide the same accuracy as the other integration rules.
3. Number of integration points is of great importance for practical computations since the fewer the integration points the faster the finite element analysis will be.

Two-point Gauss integration in 1D

Integration of $f(\xi)$ between $\xi = -1$ and $\xi = 1$ using two Gauss points:



Gauss integration weights and position in 1D

N	location ξ_I	weight w_I
1	0	2
2	$-1/\sqrt{3}$	1
	$1/\sqrt{3}$	1
3	$-\sqrt{0.6}$	$5/9$
	0	$8/9$
	$\sqrt{0.6}$	$5/9$

With N Gauss points the integration is exact up to polynomial order $(2N - 1)$.

Gauss integration

$$\int_{-1}^{1} (3\xi^2 + \xi) d\xi$$

The quadrature approximation is:

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{I=1}^{N} w_I f(\xi_I)$$

where GP is the number of Gauss points.
one-point Gaussian integration:

Gauss integration

The quadrature approximation is:

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{I=1}^{N} w_I f(\xi_I)$$

Two-point Gaussian integration:

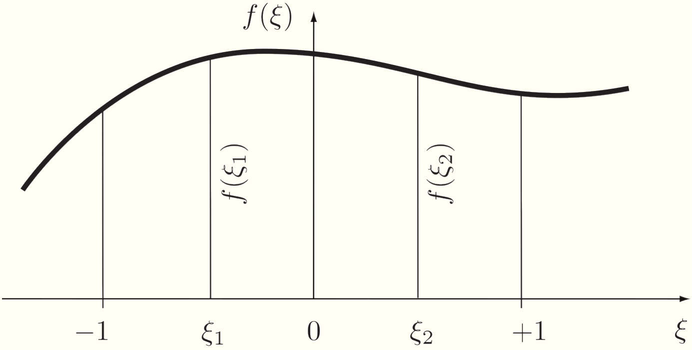
Three-point Gaussian integration:

Gauss integration

Assume that we want to integrate an arbitrary cubic polynomial over the parent domain $[-1, 1]$:

$$f(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are constants. The integral will be approximated with two integration points:



Gauss integration

Integration of $f(\xi)$ between $\xi = -1$ and $\xi = 1$ using two Gauss points:

$$\begin{aligned}\int_{-1}^{+1} f(\xi) d\xi &= w_1 f(\xi_1) + w_2 f(\xi_2) \\ &= w_1(\alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3) \\ &+ w_2(\alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3)\end{aligned}$$

Alternatively, the integration can be performed analytically:

$$\begin{aligned}\int_{-1}^{+1} f(\xi) d\xi &= \int_{-1}^{+1} (\alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3) dx \\ &= \left[\alpha_1\xi + \alpha_2\frac{\xi^2}{2} + \alpha_3\frac{\xi^3}{3} + \alpha_4\frac{\xi^4}{4}\right]_{-1}^{+1} \\ &= 2\alpha_1 + 0 + \frac{2}{3}\alpha_3 + 0\end{aligned}$$

Gauss integration

Comparing the coefficients of the constants in equations:

$$
\begin{aligned}
w_1 + w_2 &= 2 \\
w_1\xi_1 + w_2\xi_2 &= 0 \\
w_1\xi_1^2 + w_2\xi_2^2 &= \frac{2}{3} \\
w_1\xi_1^3 + w_3\xi_2^2 &= 0
\end{aligned}
$$

This is a set of four nonlinear equations for determining the values of w_1, w_2, ξ_1 and ξ_2. Symmetry considerations require that

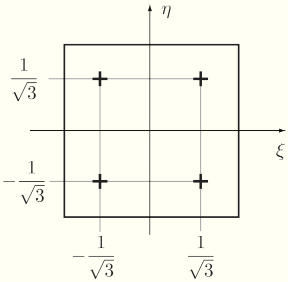
$$
w_1 = w_2 = 1
$$

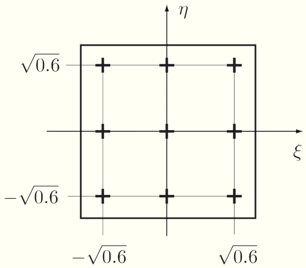
$$
\xi_1 = -\frac{1}{\sqrt{3}} \quad \text{and} \quad \xi_2 = \frac{1}{\sqrt{3}}
$$

Integration over quadrilateral elements

For the integration over the bi-unit parent element we have:

$$\int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)d\xi d\eta = \int_{-1}^{+1}\left(\int_{-1}^{+1} f(\xi,\eta)d\xi\right)d\eta$$
$$= \sum_{J=1}^{M}\left(\sum_{I=1}^{N} f(\xi_I,\eta_J)w_I\right)w_K$$





Four-point (left) and nine-point (right) integration over bi-unit square.

Evaluation of FE integrals

The integration domain for finite element matrices and vectors is the physical element domain. Therefore, we need to consider the isoparametric mapping from the the parent domain ($[-1,+1] \times [-1,+1]$) to the physical element domain Ω_e:

$$\begin{aligned}
\mathsf{K}^e &= \int_{\Omega_e} \mathsf{B}^{e^T}(x,y)\mathsf{D}\mathsf{B}^e(x,y)d\Omega \\
&= \int_{\Omega_e} \mathsf{B}^{e^T}(x,y)\mathsf{D}\mathsf{B}^e(x,y)dxdy \\
&= \int_{-1}^{+1}\int_{-1}^{+1} \mathsf{B}^{e^T}(\xi,\eta)\mathsf{D}\mathsf{B}^e(\xi,\eta)\left|\mathsf{J}^e(\xi,\eta)\right| d\xi d\eta
\end{aligned}$$

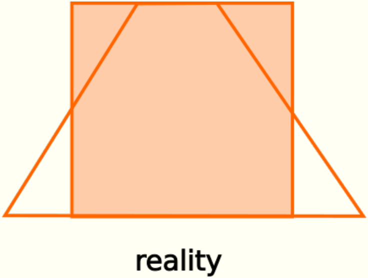
where $|\mathsf{J}^e|$ is the Jacobian of the isoparametric mapping and takes care of the mapping of infinitesimal area elements from parent to the physical domain.

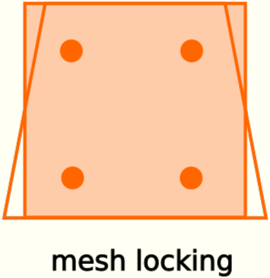
$$dxdy = \left|\mathsf{J}^e\right| d\xi d\eta$$

Number of integration points to use

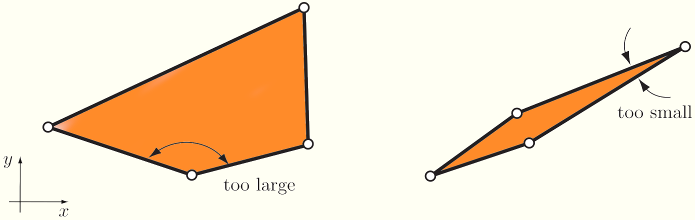
Mesh locking

Volume mesh locking Conventional FE formulations produce very stiff behavior of an element when modeling nearly incompressible materials. The volume at each integration point is fixed and this condition puts sever constraints on the kinematically admissible displacement fields at the nodes. The stiff behavior is called "**mesh locking**". For soils, incompressible conditions (no volume change) occurs when it is in undrained condition.





Modeling considerations: Element geometries



Quadrilateral element geometries to be avoided.

Modeling considerations: Higher-order element

Notice for higher order elements, like the nine-noded one, the position of the mid-nodes contribute to the element distortion. Therefore, they must lie at a certain distance from the corner nodes.