

# CE394M: 1D-Finite Element Method

Krishna Kumar

University of Texas at Austin

*krishnak@utexas.edu*

February 8, 2020

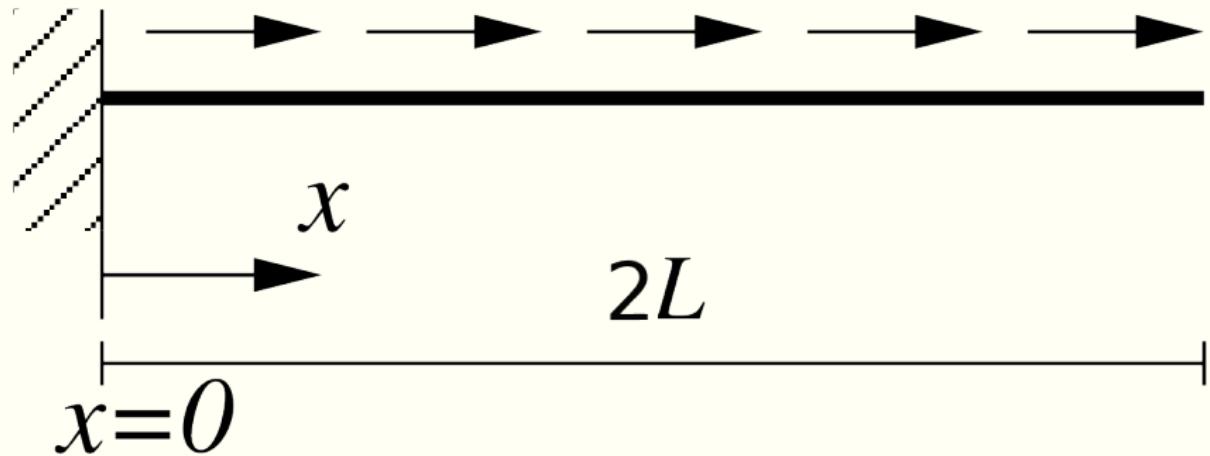
# Overview

- 1 FEM workflow
- 2 1D FEM
- 3 Assembly
- 4 Errors in the FEM
- 5 A case-study of a FE failure
- 6 2D truss analysis

# Finite Element Analysis

FEM is a systematic procedure for approximating differential equations.  
For any problem in any spatial dimension it follows the same steps:

# 1D Finite Element Analysis of a cantilever beam

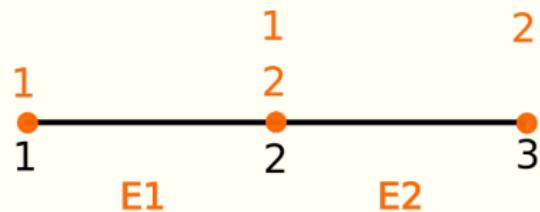


1D cantilever beam

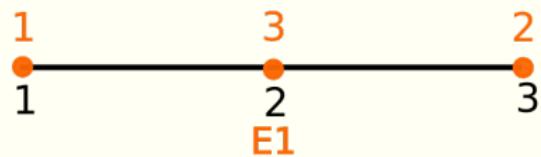
Assume  $L$  as unit length  $L = 1$ . Unit force  $f = 1$ .

# 1D Finite Element Analysis of a cantilever beam

What element should be used?



Linear elements



Quadrilateral element

1D discretization of a cantilever beam

# 1D FEM: Shape functions and derivatives

Shape function  $\mathbf{N}$ :

$\mathbf{B}$  is the derivatives of the shape functions:

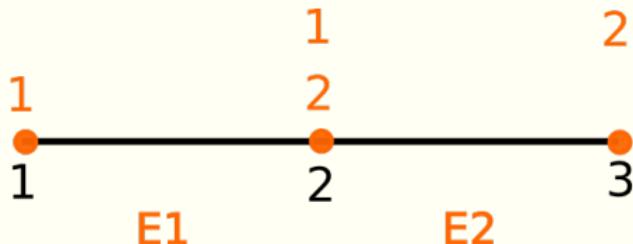
In matrix format:

# 1D FEM: Stiffness and force

Element stiffness  $k_e$ :

Right-hand side vector  $b_e$  is:

# 1D FEM: Assembly

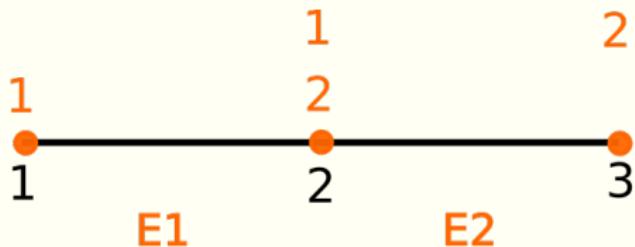


Element stiffness  $k_e$ :

$$k_e = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$= EA \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

# 1D FEM: Assembly

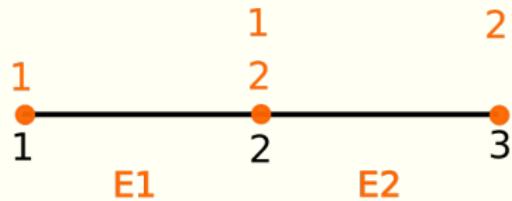


---

element 1	element 2
Local dof	
Global dof	

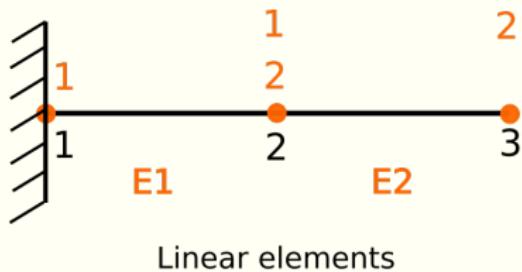
---

# 1D FEM: Global stiffness matrix



Global stiffness  $\mathbf{K}$ :

# Boundary conditions



Global stiffness  $\mathbf{K}$ :

# Applying boundary conditions

Consider a global system that has already been assembled by adding the contribution of each element:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} \\ 0 & 0 & 0 & K_{54} & K_{55} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

To apply a displacement of  $g$  at node one ( $a_1 = g$ ):

# Global system of equations

Assemble the global system of equations:

$$\mathbf{K} = \begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

# Applying boundary conditions

Assemble the global system of equations:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}$$

We need to apply the boundary condition  $u(0) = 0$ , which requires that  $a_1 = 0$ . The simplest way to impose this condition is to delete the first row and column of the stiffness matrix:

Solving this system we have:

# Analytical solution of a 1D cantilever beam

The Euler–Bernoulli equation describes the relationship between the beam's deflection and the applied load:

$$-EA \frac{d^2u}{dx^2} = 1$$

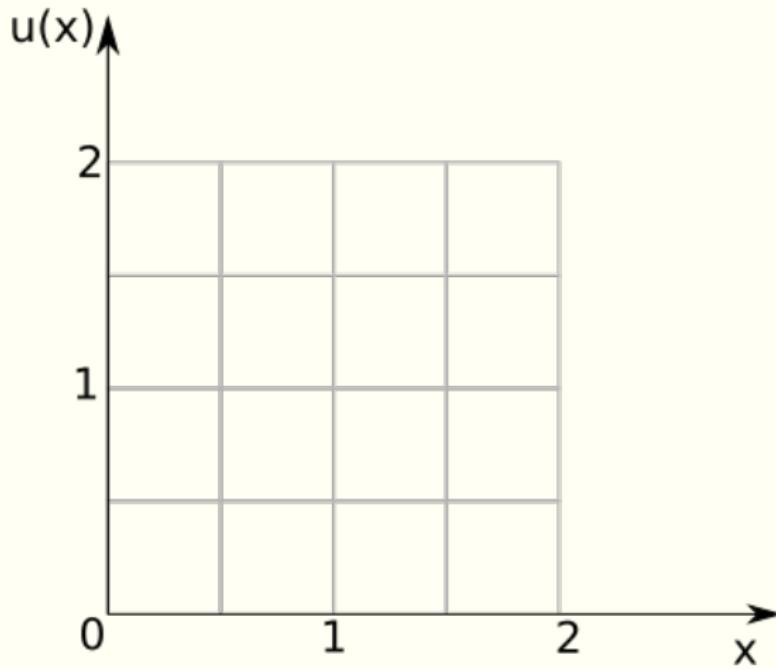
The exact solution is:

$$u = \frac{1}{EA} \left( \frac{-x^2}{2} + Cx + D \right)$$

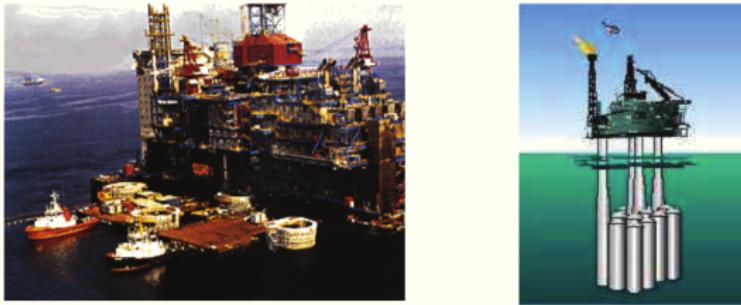
Using the boundary conditions:  $u(0) = 0$  and  $\frac{du(2)}{dx} = 0$ .

# Error in the Finite Element Methods

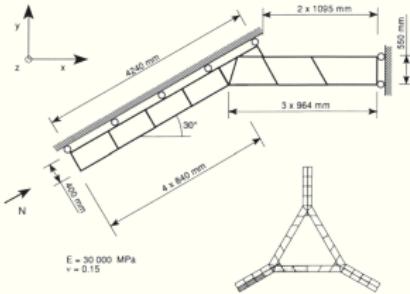
This nodal ‘exactness’ means that looking at error in the displacement at the nodes does not tell us about the error. Displacement error norm:



# Sleipner A offshore platform sprung leak

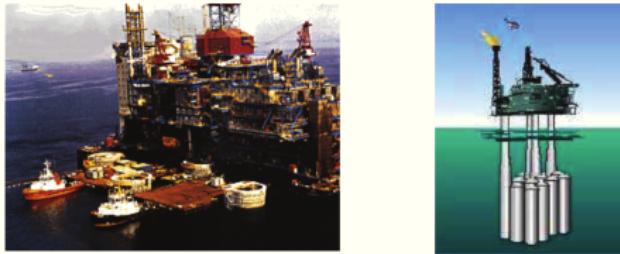


Platform with a reinforced concrete base structure.

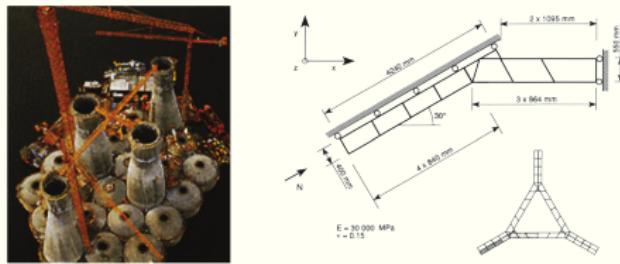


Concrete substructure during manufacturing on shore. Finite element mesh of a tricell detail. A tricell is a triangular concrete frame placed where three cells meet.

# Sleipner A offshore platform sprung leak



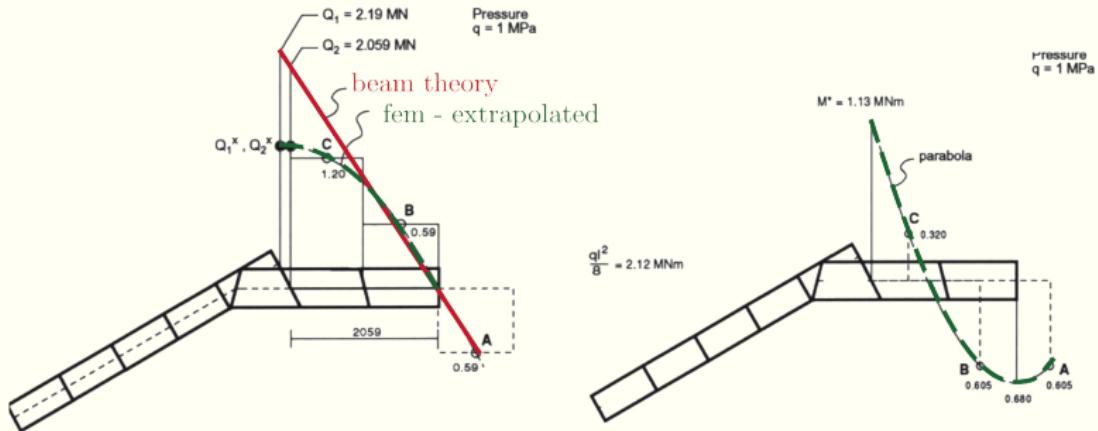
*Platform with a reinforced concrete base structure.*



*Concrete substructure during manufacturing on shore. Finite element mesh of a tricell detail. A tricell is a triangular concrete frame placed where three cells meet.*

Using the FE program NASTRAN, the shear stresses in the tricells were under-estimated by 47%. First, the chosen finite element mesh was exceedingly coarse so that the finite element shear stress was significantly too small.

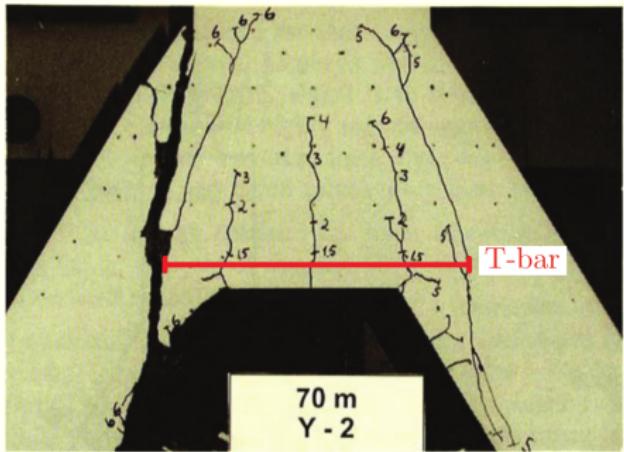
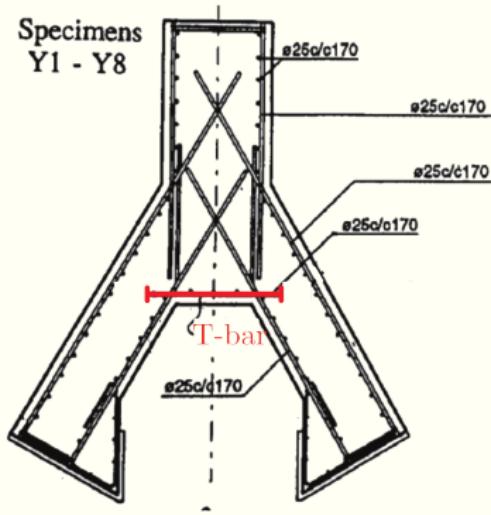
# Sleipner A offshore platform sprung leak



Comparison of the finite element shear forces (left) and moments (right) with the beam theory solution.

Second, the shear stresses at the boundary have been quadratically extrapolated using the shear forces at points A, B and C. We know however from beam theory that the shear force distribution is linear so that the shear force at the boundary is underestimated by  $\approx 40\%$

# Sleipner A offshore platform sprung leak

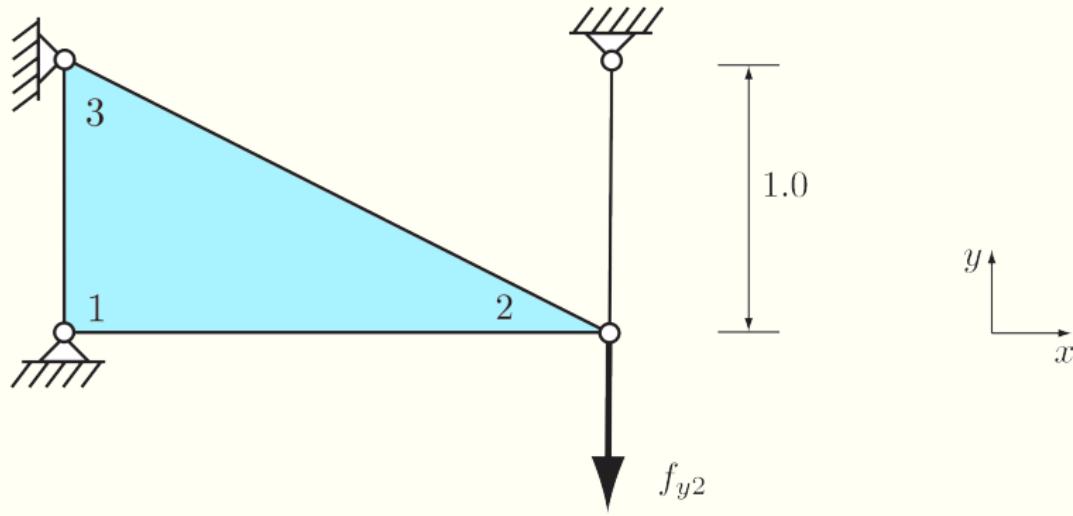


*Experimental investigation of the tricell failure. As can be concluded from the failure mode the T-bar length is too short.*

To make matters worse, the necessary reinforcement was automatically dimensioned based on the FE results without any checking by an engineer.

# 2D truss analysis

Consider the following structural system consisting of a three-noded triangle and a cable element (i.e. two-noded one-dimensional element):



## 2D truss analysis

The triangle element is fixed at the (local) nodes 1 and 3 and its stiffness matrix for the unconstrained degrees of freedom at node 2 is

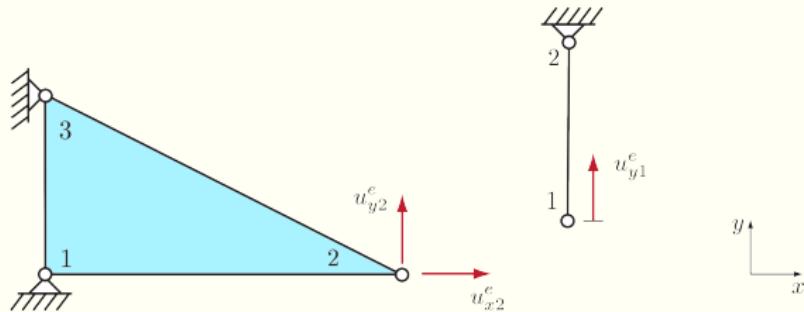
$$\mathbf{K} = 10^9 \begin{bmatrix} 1.97 & 0 \\ 0 & 0.66 \end{bmatrix} \begin{bmatrix} u_{x2}^e \\ u_{y2}^e \end{bmatrix}$$

For the cable, the product of the Young's modulus and cross-sectional area is  $EA = 1.0 \cdot 10^9$ . Further, the system is loaded with a nodal force of  $f_{y2} = -10000$ .

Inserting  $EA$  and the length  $h = 1.0$  gives the stiffness matrix for a two-noded one-dimensional element as:

## 2D truss analysis

To compose the global stiffness matrix we consider the correspondence between the element specific (local) degrees of freedom of the triangle and the cable



In the global stiffness matrix, the element stiffness matrix components corresponding to  $u_{y2}^e$  of the triangle and  $u_{y1}^e$  of the cable are add up:

# 2D truss analysis

Hence, the final equation system for determining the nodal displacements is: