CE394M Advanced Analysis in Geotechnical Engineering: Introduction

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Overview

1. Geotechnical modeling
   - Complexity in Geotechnical modeling
   - Classical vs advanced analysis
2. Numerical methods for differential equations
   - Direct method: Matrix analysis of structures
   - Numerical analysis of engineering problems
3. Governing equations in stress-deformation analysis
   - Stress equilibrium
   - Compatibility condition
   - Stress-strain relationship

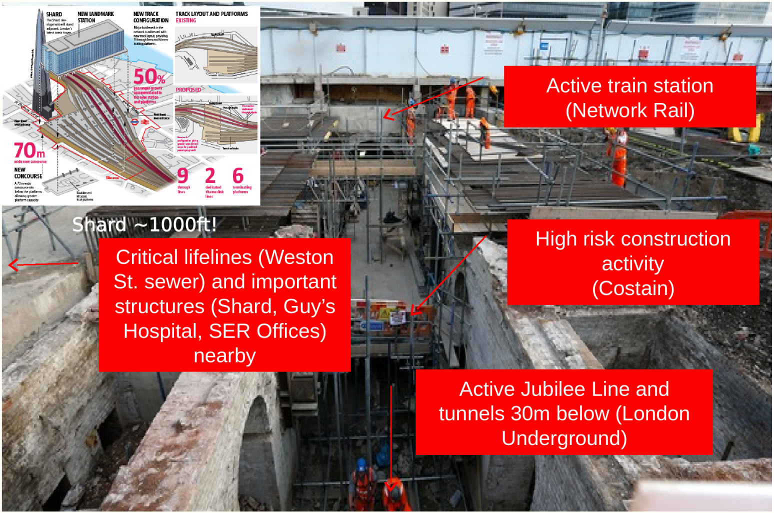


Fig. London Bridge Station, London, UK

Geotechnical modeling of the complex world

Fig. London Victoria station upgrade, London, UK

Geotechnical modeling of the complex world

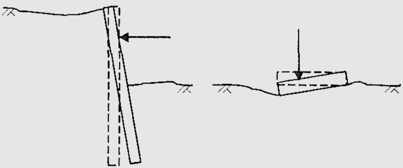
Fig. London Victoria station upgrade, London, UK

Movements must be estimated, both of the structure and of the ground. This is particularly important if there are adjacent buildings and for sensitive services. For example, if an excavation is to be made in an urban area close to existing services and buildings, one of the key design constraints is the effect that the excavation has on the adjacent structures and services. It may be necessary to predict any structural forces induced in these existing structures and/or services.

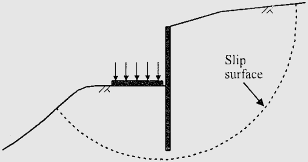
Geotechnical modeling of the complex world

Fig. London Victoria station upgrade, London, UK

When designing any geotechnical structure, the engineer must ensure that it is stable. Stability can take several forms. Firstly, the structure and support system must be stable as a whole. There must be no danger of rotational, vertical or translational failure (**local stability**). Secondly, **overall stability** must be established. For example, if a retaining structure supports sloping ground, the possibility of the construction promoting an overall slope failure should be investigated.

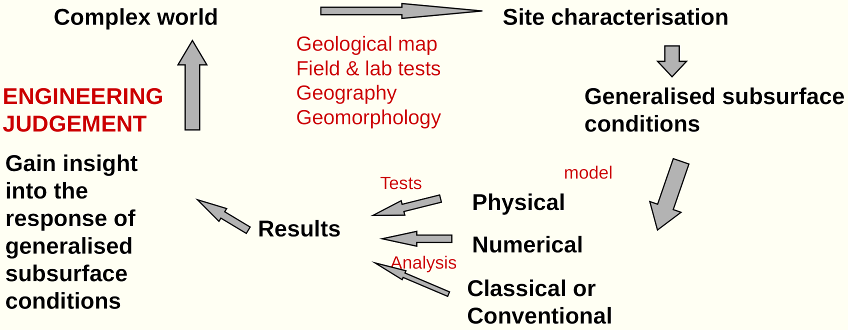


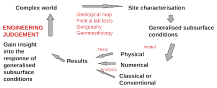
Local stability



Overall stability

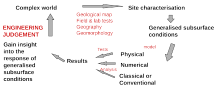
Geotechnical modeling





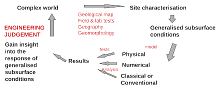
Design requirements: Before the design process can begin, a considerable amount of information must be assembled. The basic geometry and loading conditions must be established. These are usually defined by the nature of the engineering project.

A geotechnical site investigation is then required to establish the ground conditions. Both the soil stratigraphy and soil properties should be determined. In this respect it will be necessary to determine the strength of the soil and, if ground movements are important, to evaluate its stiffness too. The position of the ground water table and whether or not there is underdrainage or artesian conditions must also be established. The possibility of any changes to these water conditions should be investigated. For example, in many major cities around the world the ground water level is rising.



The site investigation should also establish the location of any services (gas, water, electricity, telecommunications, sewers andlor tunnels) that are in the vicinity of the proposed construction. The type (strip, raft andlor piled) and depth of the foundations of any adjacent buildings should also be determined. The allowable movements of these services and foundations should then be established.

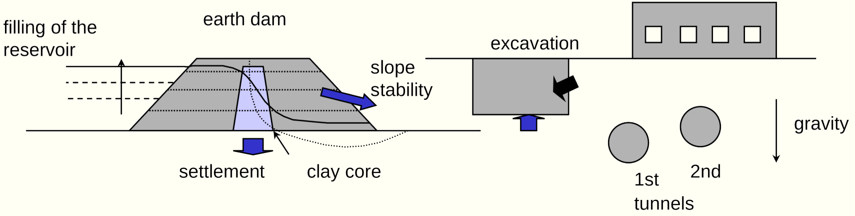
Any restrictions on the performance of the new geotechnical structure must be identified. Such restrictions can take many different forms. For example, due to the close proximity of adjacent services and structures there may be restrictions imposed on ground movements.



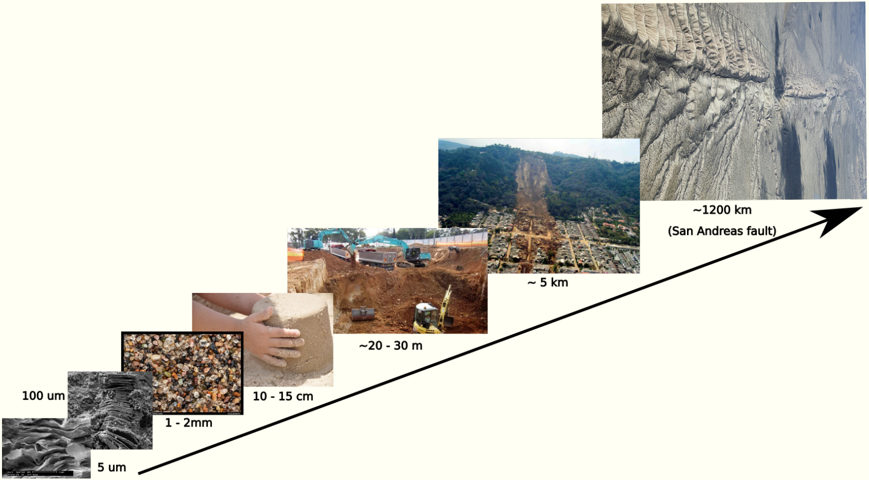
Once the above information has been collected, the design constraints on the geotechnical structure can be established. These should cover the construction period and the design life of the structure. This process also implicitly identifies which types of structure are and are not appropriate. For example, when designing an excavation, if there is a restriction on the movement of the retained ground, propped or anchored embedded retaining walls are likely to be more appropriate than gravity or reinforced earth walls. The design constraints also determine the type of design analysis that needs to be undertaken.

Geotechnical modeling: What should be modeled?

- Self weight effect of soils (This is why soil moves)
- Construction sequence (Complex geometry)
- Water movement (undrained, consolidation, drained)
- Insitu stresses (stiffness/strength depends on current stresses and stress history)
- Predict the ability of a design to withstand extreme loading conditions (you only have one chance)



Scales of modeling in geotechnical engineering



Soil behavior

- nonhomogeneous,
- anisotropic,
- non-linear,
- initial stress conditions,
- stress history
- Geometry - very complex

Soil Mechanics in practice - largely empirical

Advanced analysis in geotechnical engineering

Geotechnical design:

- Assess applied forces
- evaluate "performance" (stability & movements) under working and ultimate loads

Analysis:

- Mathematical framework to perform calculations for these quantities
- Requires idealization of: geometry, soil properties, and loading conditions
- Analysis is a tool in design, but design involves more: acceptable movements, constraints, site characterization, etc.

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As part of the design process, it is necessary for an engineer to perform calculations to provide estimates of the above quantities. Analysis provides the mathematical framework for such calculations. A good analysis, which simulates real behaviour, allows the engineer to understand problems better. While an important part of the design process, analysis only provides the engineer with a tool to quantify effects once material properties and loading conditions have been set. The design process involves considerably more than analysis.

Classical vs advanced analysis

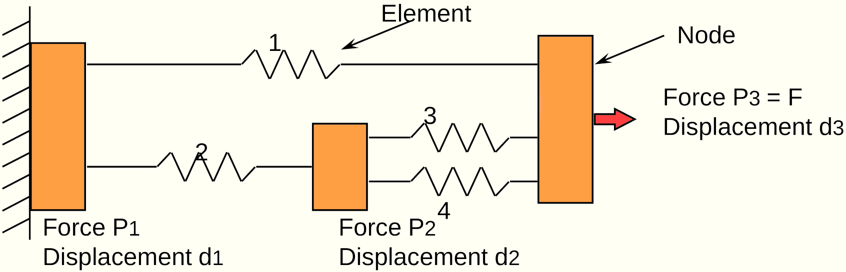
Classical approach:

- Failure estimates
  - rigid perfectly-plastic stress-strain assumptions
  - calculate factor of safety (What value do you pick?)
- Deformation estimates
  - Elastic analysis
  - Use average elastic properties (like what??)

Advanced analysis:

- Failure and deformation are obtained from the same analysis
- Handle complex geometry
- More difficult to perform, more computational requirements and more info on soil behavior $\sigma - \epsilon$
- Need to know how to do it right!

Matrix analysis of structures



- What are the known variables? $d_1 = 0, P_2 = 0, P_3 = F$(constant)
- What are the unknowns? P_1, d_2, d_3
- What do we know? Force or distortion relations at an element level.

Matrix analysis of structures: Equilibrium

A structure is considered to be in equilibrium if, initially at rest, it remains at rest when subjected to a system of forces and couples. If a structure is in equilibrium, then all of its members and joints must also be in equilibrium.

- $P_1 = -S_1 - S_2$
- $P_2 = S_2 - S_3 - S_4$
- $P_3 = S_1 + S_3 + S_4$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ s_4 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{A^T S}$$

Matrix analysis of structures: Compatibility

- compatibility relates the deformations of a structure so that its various parts (members, joints, and supports) fit together without any gaps or overlaps.
- ensure that the deformed shape of the structure is continuous (except at the locations of any internal hinges or rollers), and is consistent with the support conditions.

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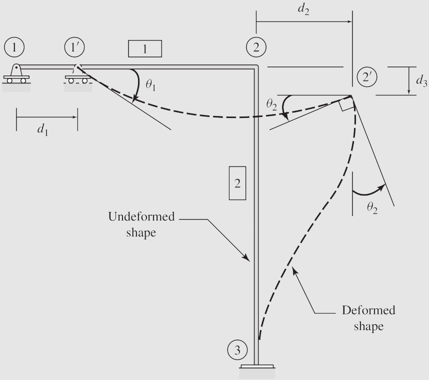
Deflection profile shown for a hinge is incompatible, because it can't be differentiated (the slope cannot be evaluated) at the hinge. Hence incompatible.



Matrix analysis of structures: Compatibility

v = internal spring distortion d = nodal displacement

- $v_1 = d_3 - d_1$
- $v_2 = d_2 - d_1$
- $v_3 = d_3 - d_2$
- $v_4 = d_3 - d_2$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\mathbf{v} = \mathbf{A}\mathbf{d}$$

Matrix analysis of structures: Physical condition

Force-distance relationship: spring constant

spring #	1	2	3	4
stiffness ($F.L^{-1}$)	3	2	1	2

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$\mathbf{s} = \mathbf{D}\mathbf{v}$$

Combine all the equations: $\mathbf{P} = \mathbf{A^T S} = \mathbf{A^T D v} = \mathbf{A^T D A d} = \mathbf{K d}$
where $\mathbf{K} = A^T DA$ (Global stiffness matrix)

$$K = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & -3 \\ -2 & 5 & -3 \\ -3 & -3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -3 \\ -2 & 5 & -3 \\ -3 & -3 & 6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Apply Boundary conditions $d_1 = 0$, $P_2 = 0$ and $P_3 = F$ and solve P_1, d_2 and d_3

Matrix analysis of structures

known displacements and forces

↓

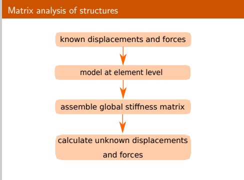
model at element level

↓

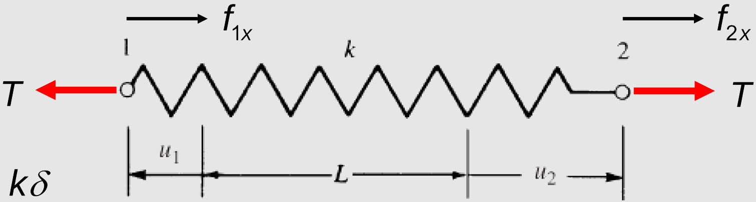
assemble global stiffness matrix

↓

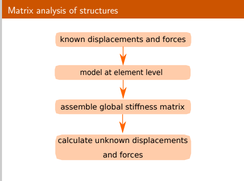
calculate unknown displacements and forces



Stiffness method: Tensile forces produce a total elongation (deformation) δ of the spring. For linear springs, the force T and the displacement u are related by Hooke's law. where deformation of the spring δ is given as: $\delta = u(L) - u(0) = u_2 - u_1$. Forces $f_{1x} = -T$ and $f_{2x} = T$.



$$T = k\delta$$



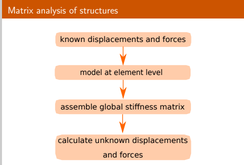
We can now derive the spring element stiffness matrix as follows. Rewrite the forces in terms of the nodal displacements:

$$T = -f_{1x} = k(u_2 - u_1) \rightarrow f_{1x} = k(u_1 - u_2)$$

$$T = f_{2x} = k(u_2 - u_1) \rightarrow f_{2x} = k(-u_1 + u_2)$$

We can write the last two force-displacement relationships in matrix form as:

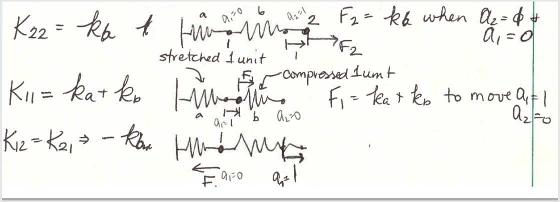
$$\begin{bmatrix} f_{1x} \\ f_{2x} \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



Stiffness method Force $F = K * a$. Where, K represents force at DOF given unit displacement at DOF.

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

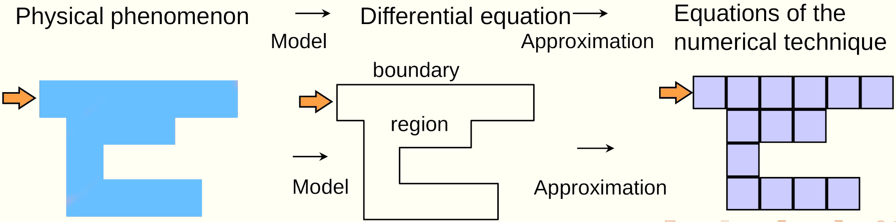
K_{ij} is force at DOF i given unit displacement at DOF j with all other dof held constant.



$$K = \begin{bmatrix} k_a + k_b & -k_b \\ -k_b & k_b \end{bmatrix}$$

Numerical analysis of engineering problems

- Conceptualize the system
  - Geometry
  - Properties
  - Processes
- Describe it mathematically
  - Select the relevant differential equations
- Solve the equations (numerically)
  - Discretize the system
  - Settle for approximations (numerical techniques)
- Interpret the results



Boundary value problems

Differential equations coupled with boundary conditions

- Steady state (time-independent)
  - Static load-deformation problems: $\partial \sigma / \partial x = 0$ (force + disp. B.C)
  - Steady seepage state, flow problems: $\partial q / \partial x = 0$ (head + flow B.C)
- Transient (time-dependent)
  - Consolidation/pore-fluid migration/multiphase flows
  - Dynamic loading (earthquakes, wave actions)
  - Contaminant transport processes

Numerical solutions to differential equations

- Finite differences: Approximate derivatives with expansion into a Taylor series
- Finite elements
- Boundary element method (BEM)
- Meshless methods
- Discrete/discontinuous element methods
- Others...

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Differential equation is a continous problem, a computer can only solve a finite problem, a discrete problem. So the goal is to take this continous differential equation, like the stresses in the soil during an excavation, and solve it as a finite problem. One way is to take derivatives and replace by finite differences. So the derivative, the slope of a function, in calculus is the exact slope at a point, while the finite differences uses the slope between one point and the next point.

Numerical solutions to differential equations

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- Others...

Galerkin is a different one, a different starting point, he took some trial functions, functions that he hoped whose combinations could be close the right answer and the problem then is to find how much of each function goes into the good answer. We are talking about approximations, not exact solutions, geotechnical engineering involves approximations, and is quite OK. 100 years later because of the computers it was possible to work with a hundred thousand functions. Galerkin worked with 2 or 3 trial functions, so he had to make a very close guess of the solution, but if we have a hundred thousand functions they can be just maybe litlle hat functions, just up and down again simple functions and their combinations, if we have many many can give us close to the correct answer.

Numerical solutions to differential equations

- Finite differences: Approximate derivatives with expansion into a Taylor series
- Finite elements
- Boundary element method (BEM)
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- Others...

The whole idea of the FEM is a combination of the Galerkin idea of test functions with the idea of simple functions, where the physics is imple and the equations stay simple, but many many functions and that's what a computer is happy with. So instead of a differential equation, we have a big system to solve.

Our unknowns are how much of each hat function goes into the final solution, it's the coefficients we want, the number that multiplies each of the hat functions, when we add them together that we get a big system of equations for those numbers to multiply the hat functions and the system is very well organized for mathematics, so the question is how close is the approximation let's say if I approximate a function like e^x by a thousand hat functions and take a combination of the thousand hat functions to be close to e^x, if I take two-thousand hat functions how much closer do I get?

Governing equations in stress-deformation analysis

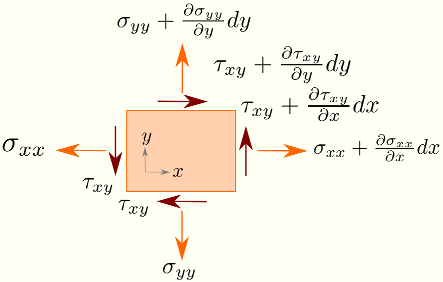
In stress-deformation analysis, we need to consider:

- **Equilibrium - static conditions**
  - forces and stress must agree across the region of interest. (geometric problem)
- **Compatibility-kinematic conditions**
  - geometry, displacement and strains must agree across the region of interest. (geometric problem)
- **Stress-strain relationship on physical conditions**
  - material dependent relationship between stress and strain must be specified. (element level)

Governing equations in stress-deformation analysis

The governing differential equation for equilibrium expresses: $\sum \mathbf{F} = m\mathbf{a}$

- σ_{xx} acting on face dy in the $-x$ direction
- τ_{xy} acting on face dx in the $-x$ direction
- $\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx$ acting on face dy in the $+x$ direction
- $\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy$ acting on face dx in the $+x$ direction
- Plus "body forces" due to gravity: $\rho f_x dxdy$ where f_x is body force per unit mass



Equilibrium equations

Summing all this in the x-direction gives:

$$
\begin{aligned}
-\sigma_{xx}dy - \tau_{xy}dx + \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)dy \\
+ \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}dy\right)dx + \rho f_x dxdy = \rho dxdya_x
\end{aligned}
$$

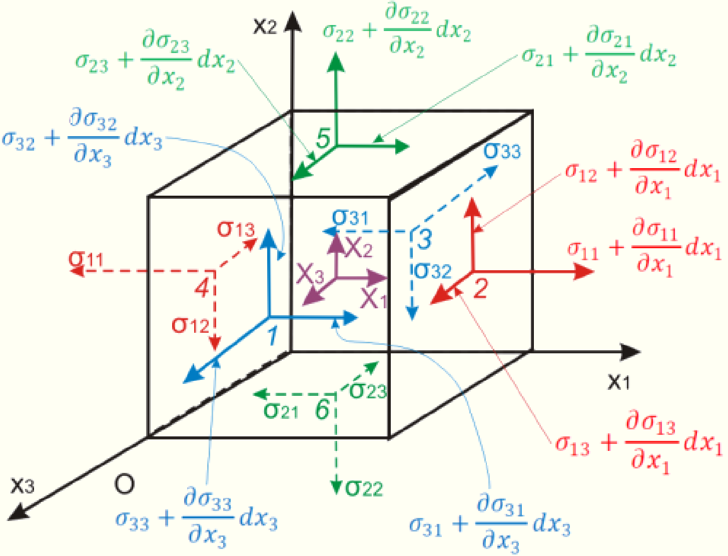
Cleaning up terms that cancel, and dividing through by $dxdy$ gives

$$
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho f_x = \rho a_x
$$

And summing forces in the y-direction leads to:

$$
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho f_y = \rho a_y
$$

Equilibrium in 3D



Equilibrium in 3D

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x = \rho a_x$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y = \rho a_y$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z = \rho a_z$$

The governing differential equation for equilibrium expresses $\sum \mathbf{F} = m\mathbf{a}$ in terms of derivatives of the stress tensor as: $\nabla \boldsymbol{\sigma} + \rho \mathbf{f} = \rho \mathbf{a}$

$\boldsymbol{\sigma}$ is the stress tensor,

ρ is density,

$\mathbf{f}$ is the body force vector per unit mass and

$\mathbf{a}$ is the acceleration vector.

Stress equilibrium

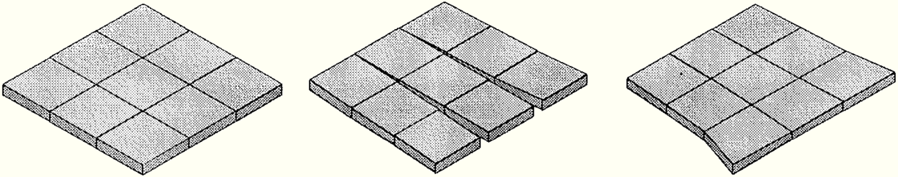
If the object is in equilibrium, then $\mathbf{a} = 0$ and $\sum \mathbf{F} = 0$.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

Stresses in Voigt notation: $[\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}]^T$.
Equilibrium equation: $\nabla^T \boldsymbol{\sigma} + \mathbf{b} = 0$
Then:

$$\nabla^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$

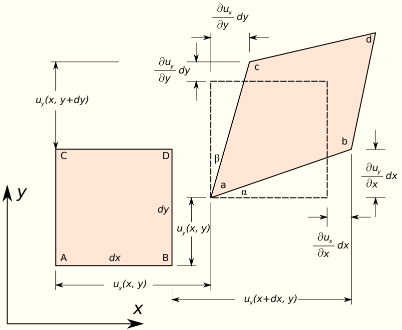
Governing equations: Compatibility



(a) original (b) non-compatible (c) compatible

Displacement - strain relationship: $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$
Where,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$
$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$
$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$
$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$
$$\gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$



$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Equilibrium and compatibility conditions

Combining the Equilibrium and Compatibility conditions gives:

- Unknowns: 6 stresses + 6 strains + 3 displacements = 15
- Equations: 3 equilibrium + 6 compatibility = 9

To obtain a solution therefore requires 6 more equations. These come from the constitutive relationships

Governing equations: Stress-strain relationship

Stress - strain relationship: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} D_{xxxx} & D_{xxyy} & D_{xxzz} & D_{xxxy} & D_{xxyz} & D_{xxzx} \\ D_{yyxx} & D_{yyyy} & D_{yyzz} & D_{yyxy} & D_{yyyz} & D_{yyzx} \\ D_{zzxx} & D_{zzyy} & D_{zzzz} & D_{zzxy} & D_{zzyz} & D_{zzzx} \\ D_{xyxx} & D_{xyyy} & D_{xyzz} & D_{xyxy} & D_{xyyz} & D_{xyzx} \\ D_{yzxx} & D_{yzyy} & D_{yzzz} & D_{yzxy} & D_{yzyz} & D_{yzzx} \\ D_{zxxx} & D_{zxyy} & D_{zxzz} & D_{zxxy} & D_{zxyz} & D_{zxzx} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

Governing equations in stress-deformation analysis

What are the variables used in the governing equations?

1. displacements $\mathbf{u}$ in the body
2. strains ε in the body or within the elements
3. stresses $\boldsymbol{\sigma}$ in the body or within the elements

Advanced analysis involves:

1. Equilibrium: External forces + internal stresses agree
2. Compatibility: Displacements fields agree (no gaps) + strains (derivatives)
3. Stress-strain relationship (constitutive behaviour)