CE394M: Stress paths and invariants

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Overview

1. Stresses / strains in typical geotechnical lab tests
2. Friction
3. Stress invariants

1D consolidation / simple shear

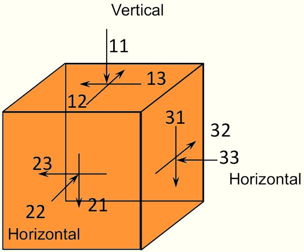
- Zero lateral strain ($\varepsilon_{22} = \varepsilon_{33} = 0$)
- Stresses: σ and τ
- Strains: $\varepsilon_{11} = \varepsilon_v$ and γ

2D plane strain

- Zero lateral strain ($\varepsilon_{22} = \gamma_{12} = \gamma_{23} = 0$)
- Stresses: s and t
- Strains: ε_v and ε_γ

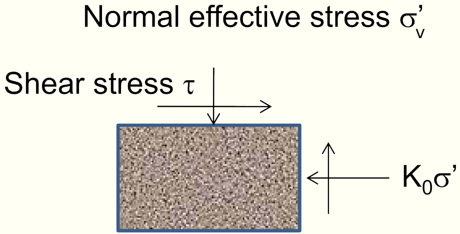
3D general (axi-symmetric as a special case)

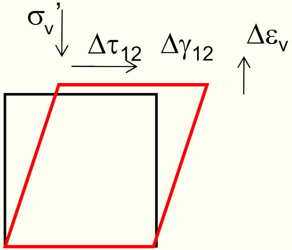
- Stresses: p and q
- Strains: ε_v and ε_s



1D simple shear

1. No lateral strain
2. Constant normal effective stress σ'_v
3. Increasing shear strain γ
4. Measure shear resistance τ
5. Measure volumetric strain ε_v or void ratio $e = e_0 - (1 + e_0)\varepsilon_v$
6. **No information for the lateral direction**



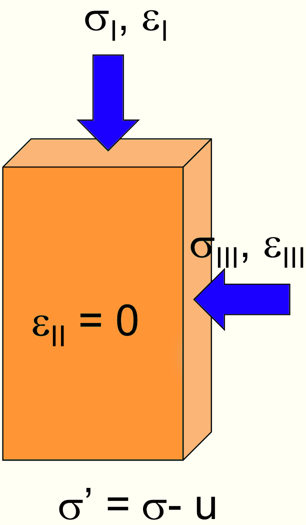


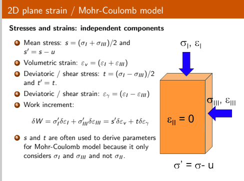
Stresses and strains: independent components

1. Mean stress: $s = (\sigma_I + \sigma_{III})/2$ and $s' = s - u$
2. Volumetric strain: $\varepsilon_v = (\varepsilon_I + \varepsilon_{III})$
3. Deviatoric / shear stress: $t = (\sigma_I - \sigma_{III})/2$ and $t' = t$.
4. Deviatoric / shear strain: $\varepsilon_\gamma = (\varepsilon_I - \varepsilon_{III})$
5. Work increment:

$$\delta W = \sigma_I' \delta\varepsilon_I + \sigma_{III}' \delta\varepsilon_{III} = s' \delta\varepsilon_v + t \delta\varepsilon_\gamma$$

6. s and t are often used to derive parameters for Mohr-Coulomb model because it only considers σ_I and σ_{III} and not σ_{II}.

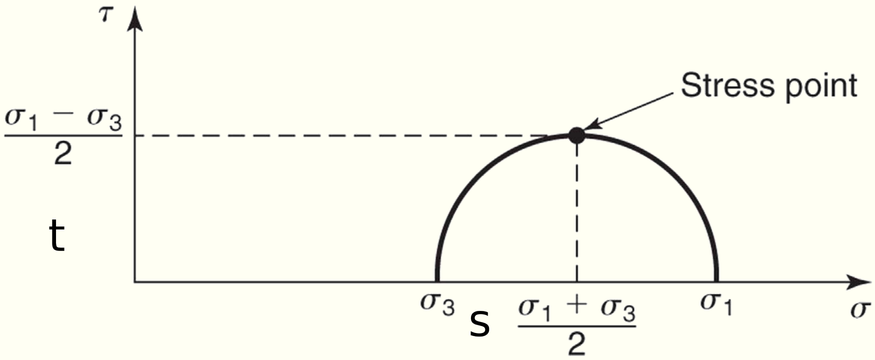




Principal stresses $\sigma_I > \sigma_{II} > \sigma_{III}$. This equation holds good and is the definition of σ in principal stress notations, i.e., $I > II > III$.

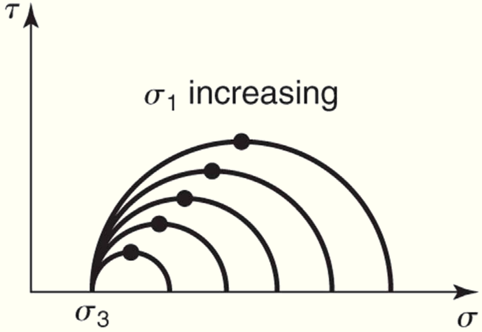
No shear stresses on these planes.

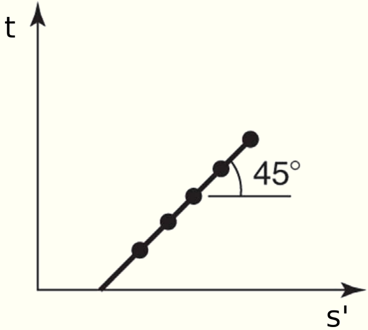
2D Mohr circle



Mohr circle to 2D stress components

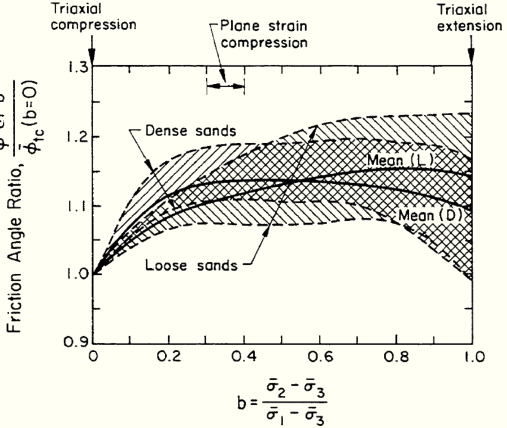
Successive Mohr circles and stress path for constant σ_3 and increasing σ_1:





Effect of σ_{II}

Bishop (1966) defined **b-value**: $b = (\sigma_{II} - \sigma_{III})/(\sigma_I - \sigma_{III})$, where $\sigma_I > \sigma_{II} > \sigma_{III}$. Triaxial compression $b = 0$, triaxial extension $b = 1$. Typically $\phi'_{ps} = (1.05 - 1.15)\phi'_{tx}$.



Effect of σ_{II} on friction angle (Kulhawy and Mayne, 1990)

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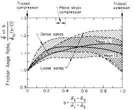
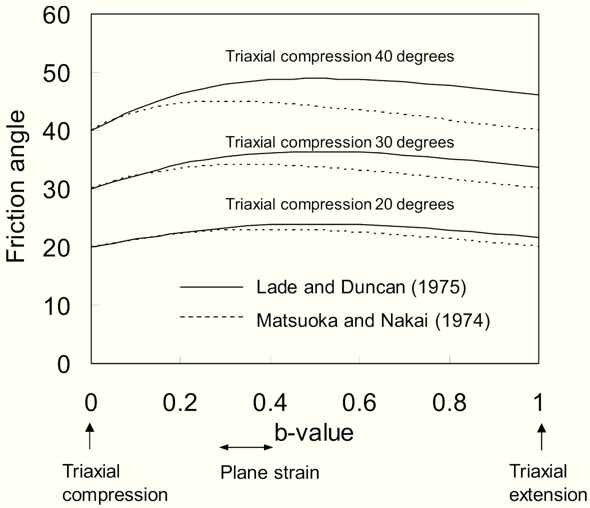


Effect of σ_{II} on friction angle (Kulhawy and Mayne, 1990)

σ_{II} do have an effect on soil behavior. For example, the friction angle depends on the loading condition: triaxial compression, plane-strain, triaxial extension and others . . . ., The effect of σ_{II} can be measured using true triaxial apparatus or hollow cylinder torsional shear apparatus. In general, the peak friction angle increases 10 to 15 percent from b = 0 (triaxial compression) to b = 0.3 to 0.4 (plane strain), and it stays constant or slightly decreases as b reaches 1 (tri-axial extension).

The variation of measured friction angle with changes in σ_{II} can be attributed to the effects of different mean stress and stress anisotropy on the dilatancy and particle rearrangement contributions to the total strength. For given maximum and minimum principal stresses, the TXE conditions have the largest mean effective stress, whereas the triaxial compression conditions have the smallest mean effective stress. The higher confinement for triaxial extension and plane strain conditions contributes to the increasing friction angle for these conditions. (Mitchell and Soga., 2005)

Effect of σ_{II} on friction



Chapter 11., Mitchell and Soga, 2005

Effect of σ_{II} on friction

Various models fit the experimental data showing the intermediate stress effect. Lade and Duncan (1975)

$$\frac{I_1^3}{I_3} = const$$

Matsuoka and Nakai (1974)

$$\frac{I_1 I_2}{I_3} = const$$

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned}$$

Matsuoka and Nakai's model gives the same friction angle for compression and extension, whereas Lade and Duncan's model gives the ratio of the TXE friction angle (ϕ'_{TXE}) to the TXC friction angle (ϕ'_{TXC}) to be 1.08 at $\phi'_{TXC} = 20^o$ to 1.15 at $\phi'_{TXC} = 40^o$.

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$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

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Given the large scatter in the published experimental data, it is not possible to conclude that one model is better than the ohter.

Triaxial stresses and strains: independent components

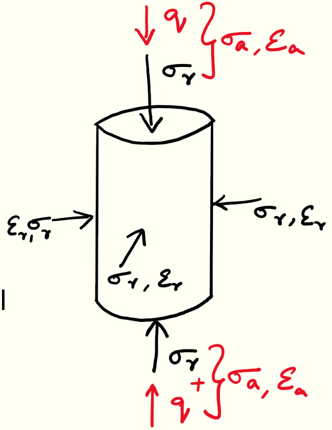
- We split the stress system into:
  - purely volumetric deformation 'p'
  - purely distortional deformation 'q'
- Mean stress:

$$p = (\sigma_a + 2\sigma_r)/3 \quad p' = p - u$$

- Volumetric strain: $\varepsilon_v = (\varepsilon_a + 2\varepsilon_r)$
- Deviatoric / shear stress (purely distortional deformation):

$$q = (\sigma_a - \sigma_r) \quad q' = q$$

- Deviatoric / shear strain: $\varepsilon_s = 2/3(\varepsilon_a - \varepsilon_r)$



Triaxial deviatoric strain

- Deviatoric / shear strain: $\varepsilon_s = 2/3(\varepsilon_a - \varepsilon_r)$
- Why 2/3?:
- Work equation: $\delta W = p'\delta\varepsilon_v + q'\delta\varepsilon_s$

$$\begin{aligned}
\delta W &= \sigma'_1\delta\varepsilon_1 + \sigma'_2\Delta\varepsilon_2 + \sigma'_3\Delta\varepsilon_3 = \sigma'_a\delta\varepsilon_a + 2\sigma'_r\delta\varepsilon_r \\
\delta W &= (1/3\sigma'_a + 2/3\sigma'_r)(\varepsilon_a + 2\varepsilon_r) + (\sigma'_a - \sigma'_r)(2/3)(\varepsilon_a - \varepsilon_r) \\
&= \sigma'_a\delta\varepsilon_a + 2\sigma'_r\delta\varepsilon_r
\end{aligned}$$

Triaxial stress - strain relationship

Relationship in terms of axial and radial direction:

$$\begin{bmatrix} \varepsilon_a \\ \varepsilon_r \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -2\nu \\ -\nu & (1-\nu) \end{bmatrix} \begin{bmatrix} \sigma_a \\ \sigma_r \end{bmatrix} \tag{1}$$

Relationship in terms of 'p' and 'q':

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_a \\ \sigma_r \end{bmatrix} \quad \text{invert} \ \rightarrow \begin{bmatrix} \sigma_a \\ \sigma_r \end{bmatrix} \begin{bmatrix} 1 & 2/3 \\ 1 & -1/3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{2}$$

Strain:

$$\begin{bmatrix} \varepsilon_p \\ \varepsilon_q \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} \varepsilon_a \\ \varepsilon_r \end{bmatrix} \tag{3}$$

Volume - shear coupling

Equation (1) and eq. (2) in eq. (3) to relate $\varepsilon_p, \varepsilon_q$ to p, q:

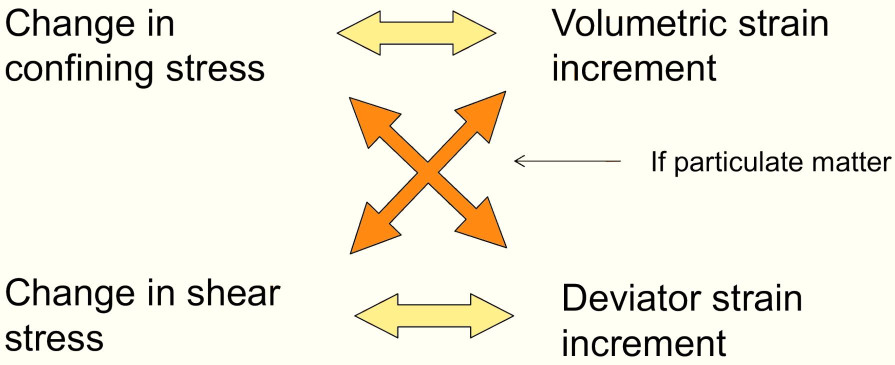
$$\begin{bmatrix} \varepsilon_p \\ \varepsilon_q \end{bmatrix} = \begin{bmatrix} 3(1-2\nu)/E & 0 \\ 0 & 2(1+\nu)/3E \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_p \\ \varepsilon_q \end{bmatrix} = \begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

1. Off-diagonal zeros indicate no coupling between volumetric and distortional effects.
2. Applying pure shear cannot induce volume changes - **not true for soils!**
3. Plasticity theory will deal with this.

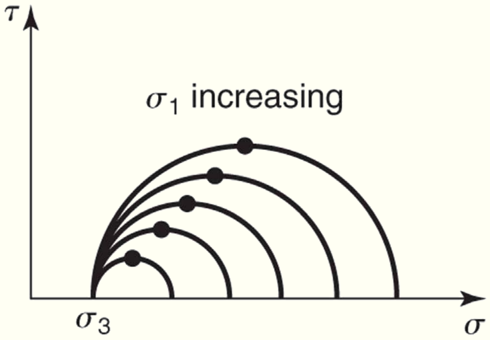
Volume shear coupling

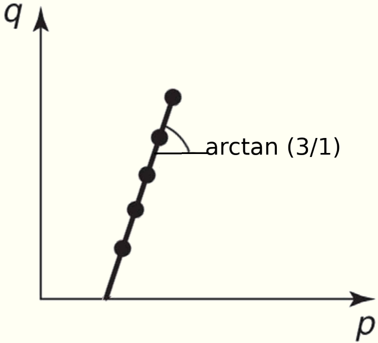
Usually volume change and shear deformation are decoupled (i.e. bulk modulus and shear modulus), but in soils they are coupled:



Mohr circle to 3D stress components

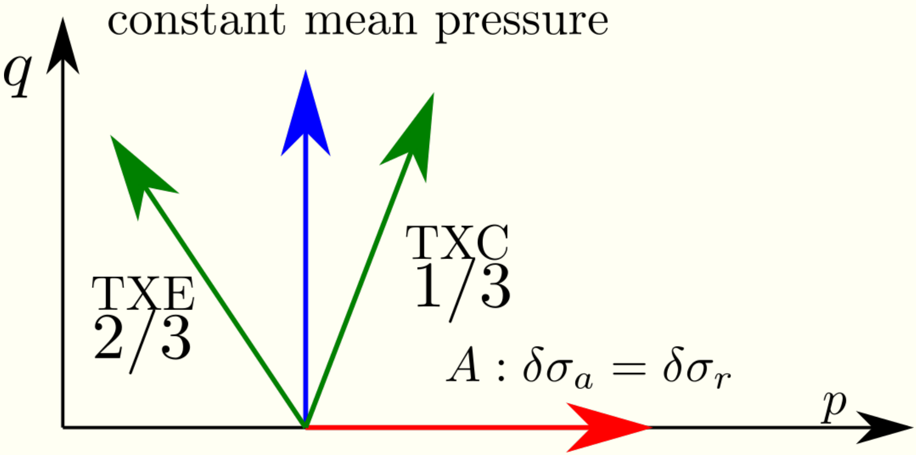
Successive Mohr circles and stress path for constant σ_3 and increasing σ_1:



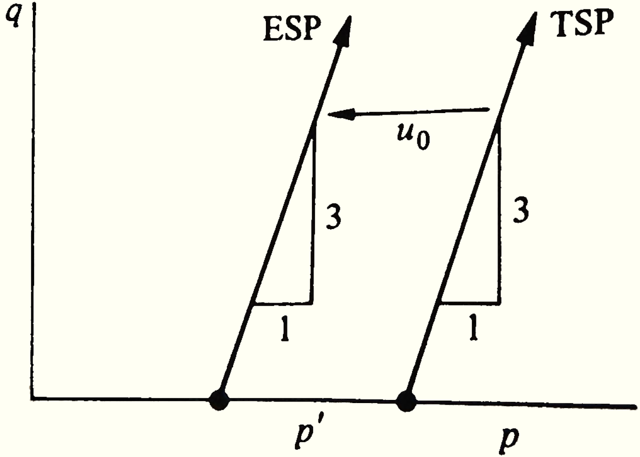


Stress paths p-q

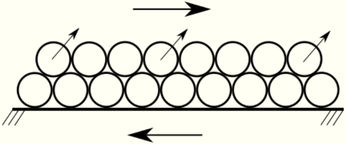
Different stress paths for initially hydrostatic stress conditions:

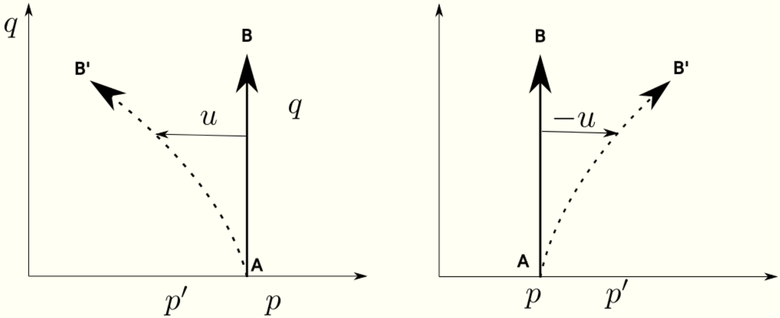


TXC with constant back pressure u_o (TSP: Total stress path; ESP: Effective stress path)



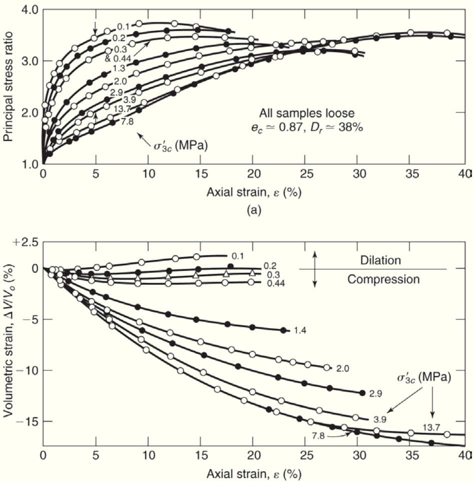
Triaxial compression undrained: loose v dense





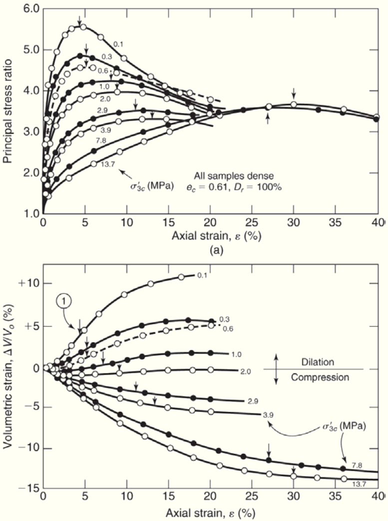
Total and effective stress paths for undrained triaxial test: (a) on soil that wishes to contract as it is sheared, and (b) on soil that wishes to expand as it is sheared.

Triaxial compression drained: loose



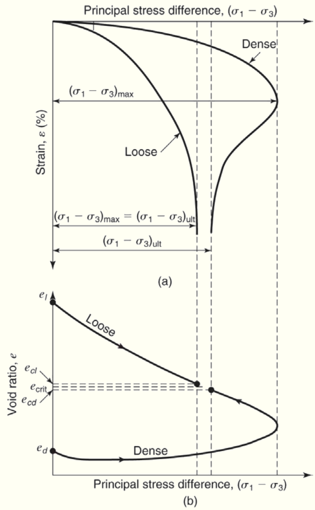
Loose Sacramento River sand: (a) principal stress ratio versus axial strain; (b) volumetric strain versus axial strain (Lee, 1965).

Triaxial compression drained: dense



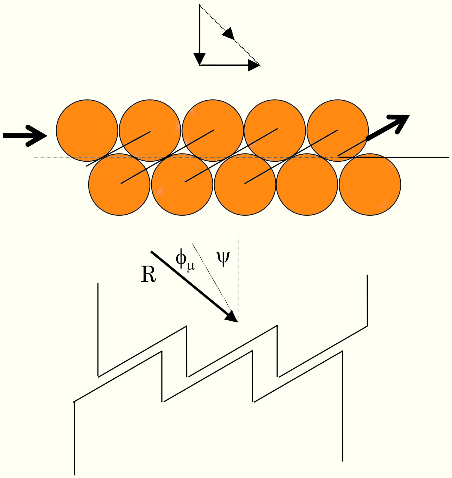
Dense Sacramento River sand: (a) principal stress ratio versus axial strain; (b) volumetric strain versus axial strain (Lee, 1965).

Triaxial compression drained: loose v dense



Triaxial tests on “loose” and “dense” specimens of a typical sand: (a) stress-strain curves; (b) void ratio changes during shear (Hirschfeld, 1963).

Friction: Is this correct?



$$\phi_{ss} = \phi_\mu + \psi_{ss}$$

Discrete Element Method

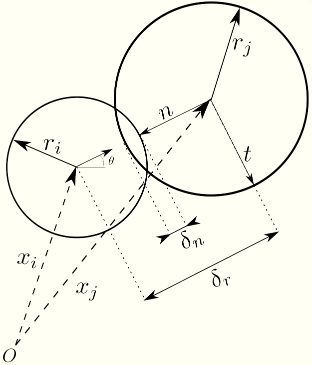
1. Particle level interaction based on Newton's equation of motion
2. The contact normal force is computed as:

$$F_n = \begin{cases} 0, & \delta_n > 0 \\ -k_n\delta_n - \gamma_n \frac{d\delta_n}{dt}, & \delta_n < 0 \end{cases}$$

3. The contact tangential force is computed in a similar way, but has a frictional limit.

$$F_t \leq \mu F_n$$

4. Solve Newton's second law and the angular momentum equation (including rotational resistance).

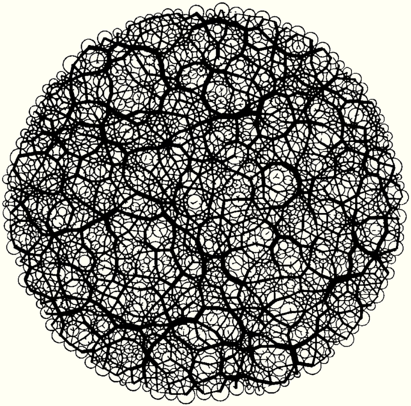


Interparticle friction angles

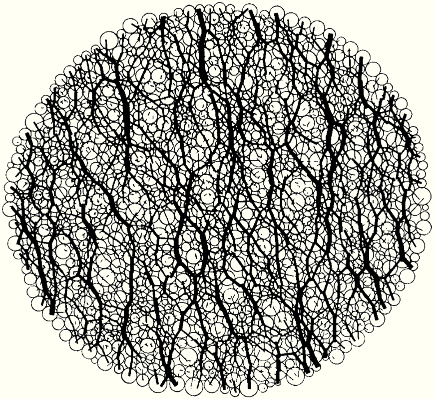
- For Quartz Sands: 26 degrees
- For Sheet Minerals (muscovite, phlogopite, biotite and chlorite): 7 - 13 degrees
  - Water acts as a lubricant
- Clay minerals: Probably 7 - 13 degrees
  - Similar to reported residual friction angles.
  - Sodium Montmorillonite: 4 degrees

Strong force network vs weak clusters

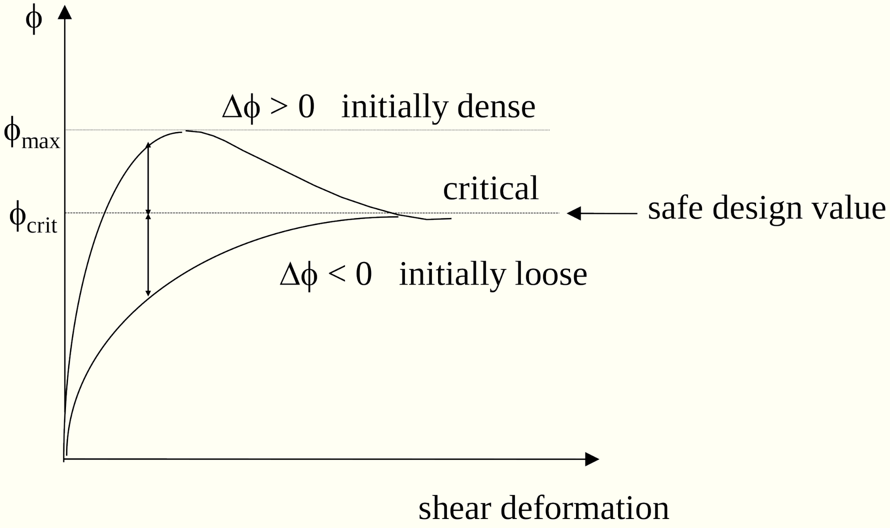
Isotropic Loading



Biaxial Loading



Macroscopically, as soil aggregates...

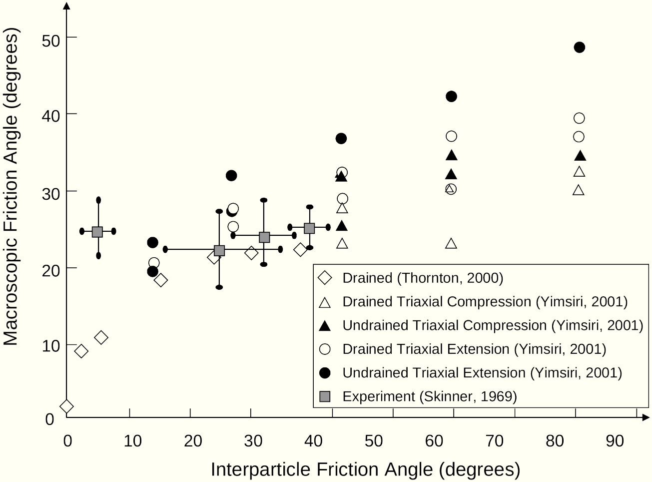


$$\phi = \phi_{crit} + \Delta\phi_{dilatancy}$$

Macroscopic friction angle

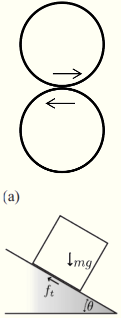
- ϕ_{crit} is the angle of friction measured at constant volume of a soil aggregate, and $\Delta\phi$ dilatancy is the extra dilatant contribution to friction angle ϕ. Typical values are:
- Critical state friction ϕ_{crit}:
  - clay: 22°
  - uniform rounded sand: 32°
  - well-graded angular sandy gravel: 38°
- peak strength of pre-compressed or uncrushable grains, densely compacted, and: shearing in plane strain $\Delta\phi$:
  - shearing in plane strain: $\Delta\phi_{max} = 20^\circ$
  - shearing in axial symmetry: $\Delta\phi_{max} = 12^\circ$

Micro to Macroscopic friction angle

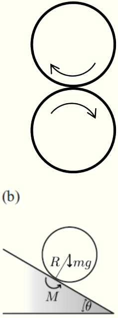


Relationship between macroscopic friction angle and interparticle friction angle (no rolling resistance) - Yimisir and Soga (2001)

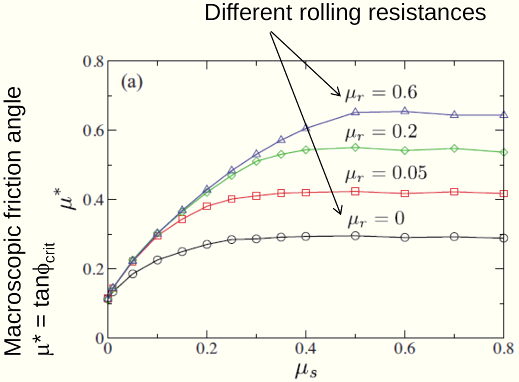
Micro to Macroscopic friction angle: Rolling resistance



Sliding friction μ_s



Rolling resistance μ_r

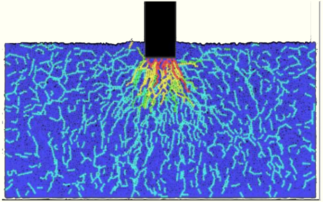


Microscopic sliding friction at particle contacts $\mu_s = \tan\phi_\mu$

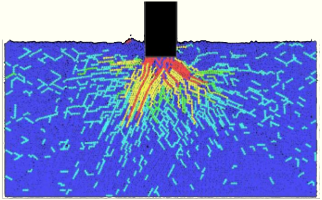
Estrada et al., (2001)

Interparticle friction angles

- The interparticle friction acts as a kinematic constraint of the strong force network and not as the direct source of macroscopic resistance to shear.
- Increased friction at the contacts increases the stability of the system (development of anisotropic fabric) and reduces the number of contacts required to achieve a stable condition.
- As long as the strong force network can be formed, the magnitude of the interparticle friction becomes of secondary importance.



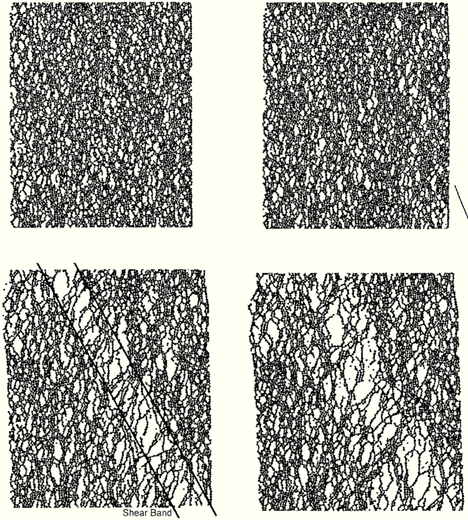
Loose

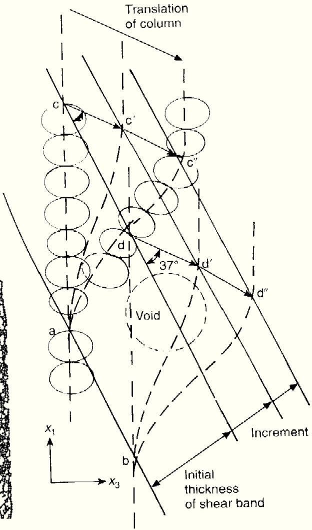


Dense

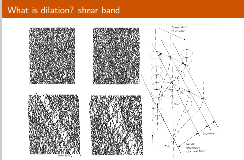
Muthuswamy and Tordesillas (2006)

What is dilation? shear band





Iwashita and Oda (2000)



Iwashita and Oda (2000)

Kuhn (1999) reports that their thicknesses are $1.5D_{50}$ to $2.5D_{50}$ in the early stages of shearing and increase to between $1.5D_{50}$ and $4D_{50}$ as deformation proceeds.

Fabric evolution at critical state

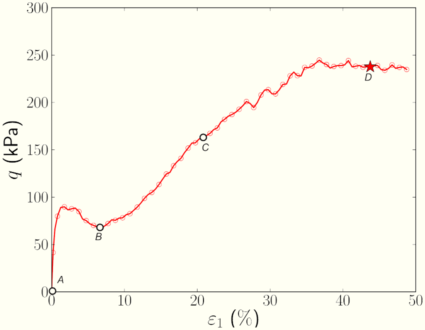


Fig.1 Undrained shear response of a medium dense sand and four stress states selected for examination of internal structure.

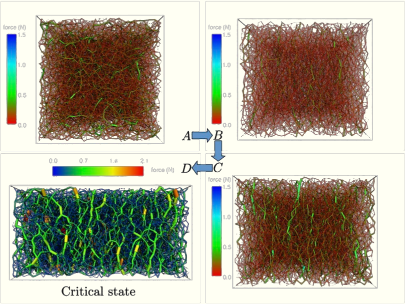
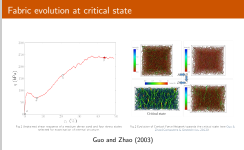


Fig.2 Evolution of Contact Force Network towards the critical state (see Guo & Zhao [Computers & Geotechnics, 2013]).

Guo and Zhao (2003)



As deformation progresses,

- The number of particles in the strong force network decreases.
- With fewer particles sharing the increased loads.
- Anisotropic fabric develops, showing the formation of strong force network. Fabric of particles associated with strong forces is different from that associated with weak clusters.
- At critical state, force chain forms and buckles continuously. Likely to buckle when a force chain has 8 particles

Stress and strain invariants in 3D

6 stresses and strains

- Mean pressure: $p' = (\sigma'_{11} + \sigma'_{22} + \sigma'_{33})/3$
- Deviator stress: $q = \sqrt{3/2}\sqrt{s_{11}^2 + s_{22}^2 + s_{33}^2 + 2\tau_{12}^2 + 2\tau_{23}^2 + 2\tau_{31}^2}$
- where $s_{11} = \sigma'_{11} - p', s_{22} = \sigma'_{22} - p', s_{33} = \sigma'_{33} - p'$
- Volumetric strain: $\varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$
- Deviatoric strain:
  $\varepsilon_s = \sqrt{2/3}\sqrt{e_{11}^2 + e_{22}^2 + e_{33}^2 + \gamma_{12}^2/2 + \gamma_{23}^2/2 + \gamma_{31}^2/2}$
- where $e_{11} = \varepsilon_{11} - \varepsilon_v/3, e_{22} = \varepsilon_{22} - \varepsilon_v/3, e_{33} = \varepsilon_{33} - \varepsilon_v/3$

3 Principal stresses and strains

- Mean pressure: $p' = (\sigma'_I + \sigma'_{II} + \sigma'_{III})/3$
- Deviator stress: $q = \sqrt{1/2}\sqrt{(\sigma'_I - \sigma'_{II})^2 + (\sigma'_{II} - \sigma'_{III})^2 + (\sigma'_{III} - \sigma'_I)^2}$
- Volumetric strain: $\varepsilon_v = \varepsilon_I + \varepsilon_{II} + \varepsilon_{III}$
- Deviatoric strain: $\varepsilon_s = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon'_I - \varepsilon'_{II})^2 + (\varepsilon'_{II} - \varepsilon'_{III})^2 + (\varepsilon'_{III} - \varepsilon'_I)^2}$

In triaxial condition (principal stresses/strains)

$$p' = (\sigma_I + 2\sigma_{III})/3 \quad q = \sigma_I - \sigma_{III} \quad \varepsilon_v = \varepsilon_I + 2\varepsilon_{III} \quad \varepsilon_s = 2(\varepsilon_I - \varepsilon_{III})/3$$

Stress and strain invariants in 3D

3 Principal stresses and strains

- Mean pressure: $p' = (\sigma'_I + \sigma'_{II} + \sigma'_{III})/3$
- Deviator stress: $q = \sqrt{1/2}\sqrt{(\sigma'_I - \sigma'_{II})^2 + (\sigma'_{II} - \sigma'_{III})^2 + (\sigma'_{III} - \sigma'_I)^2}$
- Volumetric strain: $\varepsilon_v = \varepsilon_I + \varepsilon_{II} + \varepsilon_{III}$
- Deviatoric strain: $\varepsilon_s = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon'_I - \varepsilon'_{II})^2 + (\varepsilon'_{II} - \varepsilon'_{III})^2 + (\varepsilon'_{III} - \varepsilon'_I)^2}$

In triaxial condition (principal stresses/strains)

$p' = (\sigma_I + 2\sigma_{III})/3 \quad q = \sigma_I - \sigma_{III} \quad \varepsilon_v = \varepsilon_I + 2\varepsilon_{III} \quad \varepsilon_s = 2(\varepsilon_I - \varepsilon_{III})/3$

The magnitudes of the components of the stress vector (i.e. $\sigma_{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}}$) depend on the chosen direction of the coordinate axes. The principal stresses $(\sigma_I, \sigma_{II}, and \sigma_{III})$ however, always act on the same planes and have the same magnitude, no matter which direction is chosen for the coordinate axes. They are therefore invariant to the choice of axes. Consequently, the state of stress can be fully defined by either specifying the six component values for a fixed direction of the coordinate axis, or by specifying the magnitude of the principal stresses and the directions of the three planes on which these principal stresses act. In either case six independent pieces of information are required.

The missing stress invariant

- **General stresses:** $\sigma_{ij}(i,j=1,2,3) \quad \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}$

$$\Updownarrow$$

- **Principal stresses:** $\sigma_I, \sigma_{II}, \sigma_{III} + 3 angles$

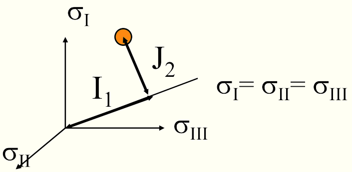
$$\Downarrow \quad \not\Uparrow$$

Two stress invariant (mean pressure p or s, deviator stress q or t). One more stress invariant is needed to go back to 3 principal stresses: **Lode angle!**

Stress invariant

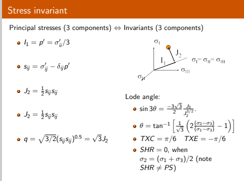
Principal stresses (3 components) $\Leftrightarrow$ Invariants (3 components)

- $I_1 = p' = \sigma'_{ii}/3$
- $s_{ij} = \sigma'_{ij} - \delta_{ij} p'$
- $J_2 = \frac{1}{2} s_{ij} s_{ij}$
- $J_2 = \frac{1}{2} s_{ij} s_{ij}$
- $q = \sqrt{3/2}(s_{ij} s_{ij})^{0.5} = \sqrt{3} J_2$



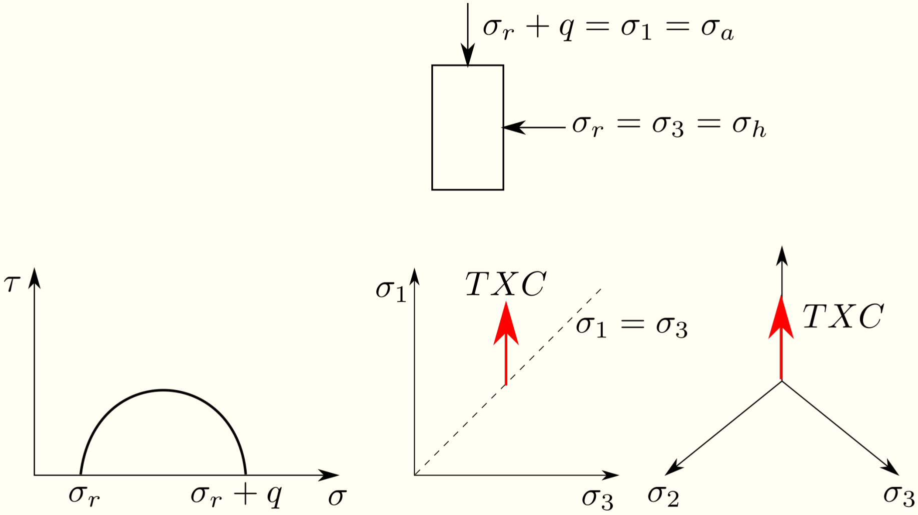
Lode angle:

- $\sin 3\theta = \frac{-3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$.
- $\theta = \tan^{-1}\left[\frac{1}{\sqrt{3}}\left(2\frac{(\sigma_2-\sigma_3)}{(\sigma_1-\sigma_3)} - 1\right)\right]$
- $TXC = \pi/6 \quad TXE = -\pi/6$
- $SHR = 0$, when $\sigma_2 = (\sigma_1 + \sigma_3)/2$ (note $SHR \neq PS$)

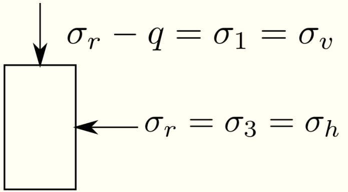


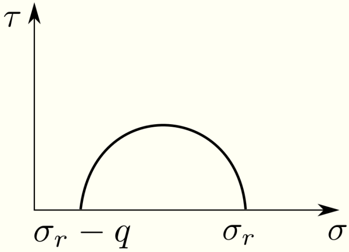
If one is interested in the overall magnitude of the stress, all three principal stresses would be needed, but not the directions of the planes on which they act. In geotechnical engineering it is often convenient to work with alternative invariant quantities which are combinations of the principal effective stresses.

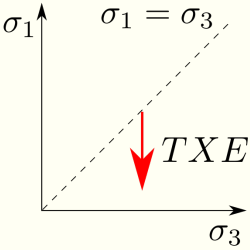
π plane: Triaxial compression

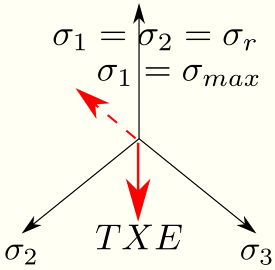


π plane: Triaxial extension









w/o forcing $\sigma_1 = \sigma_{max}$

π plane: Random stress paths