

CE394M: Linear Elasticity

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Overview

1 Linear Elasticity

2 Project on FEA of an excavation

3 Slope stability

Isotropic linear elastic stress-strain relations

The linear relationship between the stress and strain tensor is a linear one.
The stress component is a linear combination of the strain tensor:

$$\begin{aligned}\sigma_{ij} = & C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13} + \\& C_{ij21}\varepsilon_{21} + C_{ij22}\varepsilon_{22} + C_{ij23}\varepsilon_{23} + \\& C_{ij31}\varepsilon_{31} + C_{ij32}\varepsilon_{32} + C_{ij33}\varepsilon_{33}\end{aligned}$$

The most general form for *linear* stress-strain relations for a *Cauchy elastic* material is given by:

$$\sigma_{ij} = B_{ij} + C_{ijkl}\varepsilon_{kl}$$

Where B_{ij} is the components of initial stress tensor corresponding to the initial strain free (when all strain components $\varepsilon_{kl} = 0$). C_{ijkl} is the tensor of material *elastic constants*.

If it is assumed that the initial strain free state corresponds to an *initial stress free state*, that is $B_{ij} = 0$, the equations reduces to:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Observation on linear elasticity

- ① $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ is a general expression relating stress to strains for a linear solid.
- ② C_{ijkl} is a 4th order tensor containing 81 terms (we trick using symmetry and reduce order).
- ③ C_{ijkl} material response functions having dimensions F/L^2 .
- ④ Homogeneous: C_{ijkl} independent of position
- ⑤ Isotropic: C_{ijkl} independent of frame of reference.
- ⑥ Because the stress is symmetric: $\sigma_{ij} = \sigma_{ji}$, $C_{ijkl} = C_{jikl}$. Strain is symmetric $\varepsilon_{kl} = \varepsilon_{lk}$ and $C_{ijkl} = C_{ijlk}$. Hence the number of independent variables drop from 81 to 36.
- ⑦ Both the stress and the strain tensor have only 6 independent values, therefore write them as vectors, then the stiffness tensor can be written as a matrix (compromise I can not rotate tensor).

Stress-strain relationship

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

C_{ijkl} is a tensor of material *elastic constants*. However, the above $[\mathbf{C}]$ is not a tensor anymore. So we can not rotate the matrix to another frame of reference. This relationship is useful for isotropic materials, where \mathbf{C} is independent of the frame of reference.

$$\{\sigma\} = [\mathbf{C}] \{\varepsilon\}$$

The inverse of the relationship (Compliance matrix):

$$\{\varepsilon\} = [\mathbf{D}] \{\sigma\} \quad [\mathbf{D}] = [\mathbf{C}]^{-1}$$

Isotropic Linear Elastic Stress-strain relationship

The *isotropic tensor* C_{ijkl} :

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

Where λ , μ , and α are scalar constants. Since C_{ijkl} must satisfy symmetry, $\alpha = 0$.

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

So the stress:

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\varepsilon_{kl}$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

Hence for an isotropic linear elastic material, there are only two independent material constants, λ and μ , which are called *Lame's constants*.

Hooke's law

Empirical observation:

$$\Delta\varepsilon_a = \Delta\sigma_{axial} \cdot \frac{1}{E} \rightarrow \Delta\varepsilon_{11} = \frac{\Delta\sigma_{11}}{E}$$

Where E is defined as the *Young's modulus*.

The lateral strains are defined as:

$$\Delta\varepsilon_{22} = -\nu\Delta\varepsilon_{11}$$

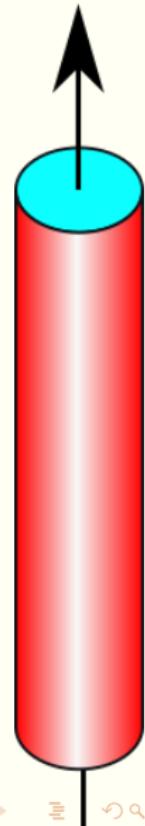
$$\Delta\varepsilon_{33} = -\nu\Delta\varepsilon_{11}$$

Using superposition for principal stresses:

$$\varepsilon_{11} = (1/E) [\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}]$$

$$\varepsilon_{22} = (1/E) [-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}]$$

$$\varepsilon_{33} = (1/E) [-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}]$$



Hooke's law

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{Bmatrix}$$

It is possible to invert the matrix to obtain the generalized Hooke's law:
 $[\sigma] = [\mathbf{C}][\varepsilon]$.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \end{Bmatrix} = \alpha \begin{bmatrix} (1-\mu) & \nu & \nu & 0 & 0 & 0 \\ 0 & (1-\nu) & \nu & 0 & 0 & 0 \\ 0 & 0 & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

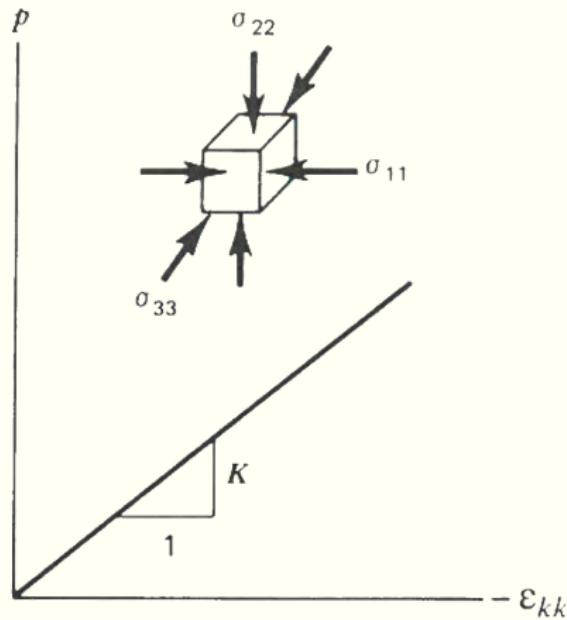
Where $\alpha = E/((1+\mu)(1-2\mu))$. Similarly, we can obtain the inverse matrix.

Hooke's law

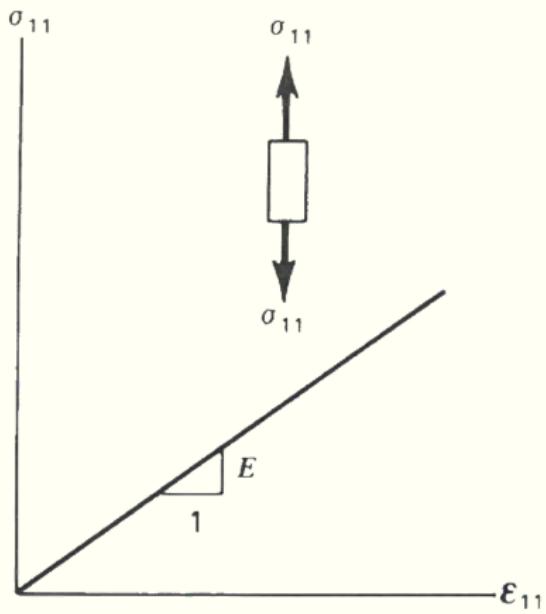
The matrices $[\mathbf{C}]$ and $[\mathbf{D}]$ contains two independent variables E and μ , where $E > 0$ and $-1 \leq \mu \leq 0.5$. The matrix can also be defined in terms of Lame's constants.

$$\begin{cases} \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} & \text{Lame's modulus (wave propagation)} \\ \mu = G = \frac{E}{2(1+\nu)} & \text{Shear modulus (shear behavior)} \\ K = \frac{E}{3(1-2\nu)} & \text{Bulk modulus (volumetric behavior)} \end{cases}$$

Isotropic linear elastic



(a)



(b)

Behavior of isotropic linear elastic material in simple tests: (a) hydrostatic compression test ($\sigma_{11} = \sigma_{22} = \sigma_{33} = p$) and (b) simple tension test (Chen 1994)

Isotropic linear elastic

Hydrostatic compression test The non-zero components of stress:

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p = \sigma_{kk}/3.$$

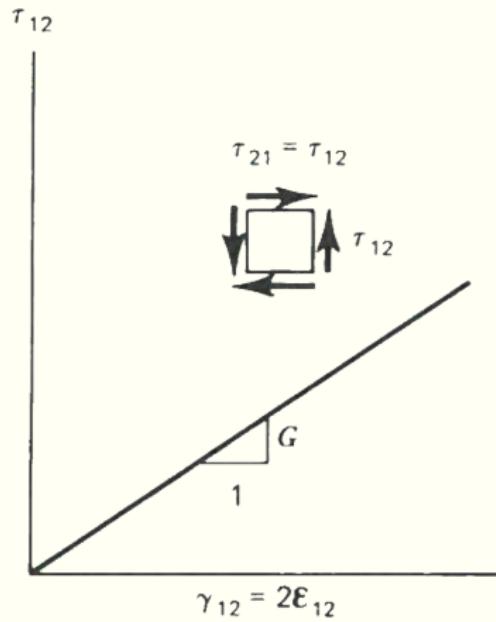
The *Bulk modulus*, K , is defined as the ratio between the *hydrostatic pressure* p and the corresponding volume change $\delta\varepsilon_v = \varepsilon_{kk}$.

$$K = -\frac{p}{\varepsilon_{kk}} = \lambda + \frac{2}{3}\mu$$

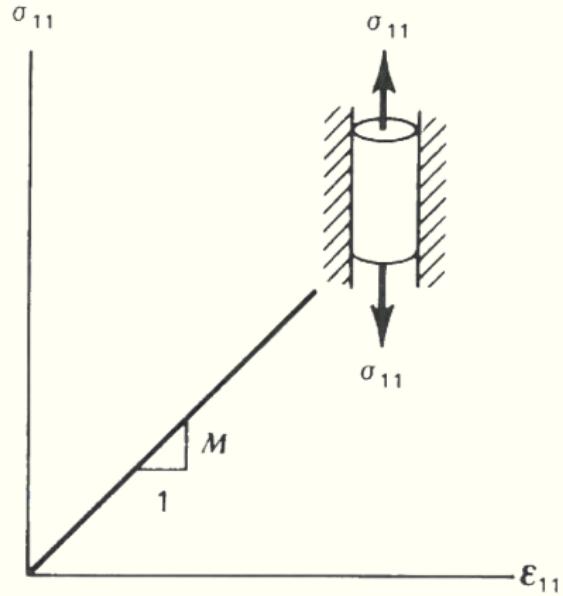
Simple tension test The only non-zero components of stress: $\sigma_{11} = \sigma$
The *Young's modulus*, K , is defined as the ratio between the *hydrostatic pressure* p and the corresponding volume change $\Delta\varepsilon_v = \varepsilon_{kk}$.

$$K = -\frac{p}{\varepsilon_{kk}} = \lambda + \frac{2}{3}\mu$$

Isotropic linear elastic



(c)



(d)

Behavior of isotropic linear elastic material in simple tests: (c) pure shear test, and (d) uniaxial strain test (Chen 1994)

Isotropic linear elastic

Simple shear test

The non-zero components of stress: $\sigma_{12} = \sigma_{21} = \tau_{12} = \tau_{21} = \tau$.

The *Shear modulus, G or μ* , is defined as:

$$G = \mu = \frac{\sigma_{12}}{\gamma_{12}} = \frac{\tau}{2\varepsilon_{12}}$$

Uniaxial strain test The test is carried out by applying a uniaxial stress component σ_{11} in the axial direction of a cylindrical sample, whose lateral surface is *restrained* against lateral movement (Oedometer test). Axial strain ε_{11} is the only nonvanishing component. The *constrained modulus M* or as PLAXIS calls it E_{oed} is defined as the ratio between σ_{11} and ε_{11} .

$$\sigma_{11} = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[(1 - \nu)\varepsilon_{11} + \nu \cancel{\varepsilon_{22}} + \nu \cancel{\varepsilon_{33}} \right] = \frac{E(1 - \nu)\varepsilon_{11}}{(1 + \nu)(1 - 2\nu)}$$

$$M = E_{oed} = \frac{\sigma_{11}}{\varepsilon_{11}} = \frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)} = (\lambda + 2\mu)$$

Plane stress v Plane strain

For a frame with an axis perpendicular to the plane of interest, x_3 or z :

Plane stress $\sigma_{33} = \sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$.

The strain in z is written as:

$$\varepsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy}) = \frac{-\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})$$

The plane stress are commonly used for thin flat plates loaded in the plane of the plate.

Plane strain $\varepsilon_{33} = \varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$.

The stress in z is written as:

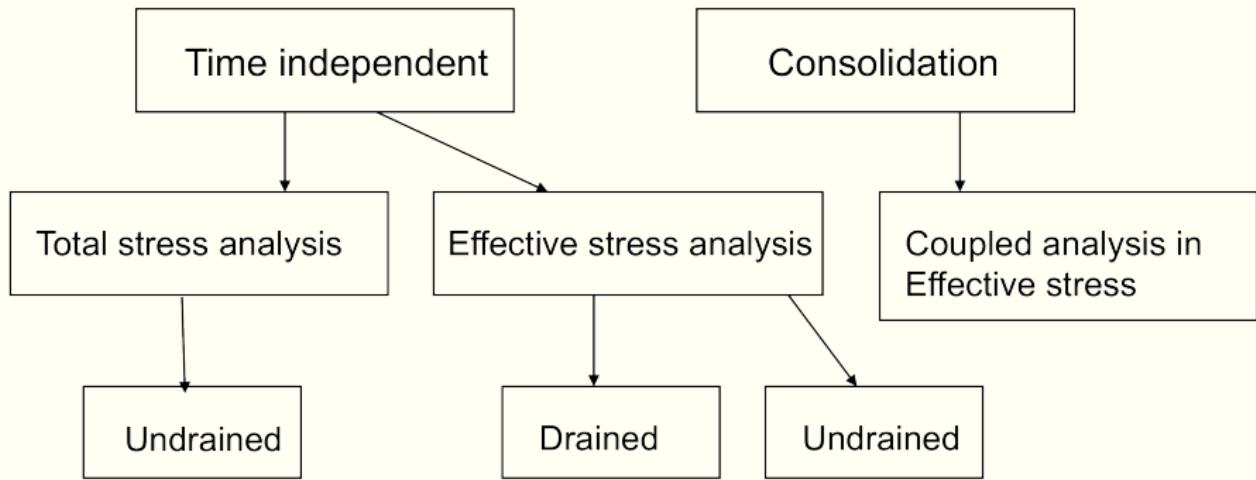
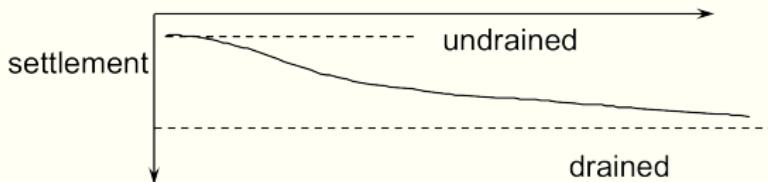
$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

The plane strains are commonly used for elongated bodies of uniform cross sections subjected to uniform loading along the longitudinal axis (tunnels, dams, retaining walls, soil slopes, etc.).

Elastic solutions

- ① *Ease of use* Only two parameters, choose equivalent values representative of strain/stress levels.
- ② *Disadvantage:* No failure criteria.
- ③ Validate code with chart solutions ("Exact solutions"), e.g., Poulos and Davis (1974).
- ④ Useful to get feeling of problem (lo stress levels) not wide distribution of plastic zones.

Pore-pressure analysis in geotechnical engineering



Drained analysis - Effective stress

- ① Need to assign initial effective stresses before the analysis.
- ② Can use any effective stress model: Elastic, Mohr-Coulomb/Drucker Prager and Cam-Clay models.
- ③ If plasticity models are used, need to update the effective stresses at each increment:

$$\sigma'(\text{new}) = \sigma'(\text{old}) + D(\text{soil skeleton})d\varepsilon$$

- ④ Very common.

Undrained analysis - Total stress

- Excess pore pressure cannot be calculated.
- Effective stress state of the soil cannot be examined.
- Elastic model is commonly used for deformation
 - ① Use undrained stiffness E_u and strength parameters s_u
 - ② Poisson's ratio close to 0.5 with a drained simulation
 - ③ Properties can vary with depth. (K_0 varies)
 - ④ Consolidation analysis has no effect and should not be performed.
- Von-Mises model is used for modeling undrained shear strength of clays. (C_u or s_u and undrained friction $\phi_u = 0$)
- Can assign different stiffness and strength at different depths explicitly by assigning different model parameters at different depths.

Undrained analysis - Effective vs Total stress

- Effective stress: E' and ν' .
- Total stress: E_u and ν_u
- No volume change:

$$\nu_u = 0.5; (K_u = E_u/(1 - 2\nu_u)/3 = E_u/0 = \infty)$$

- Pore fluid cannot sustain shear stresses. Soil skeleton carries the shear stresses τ (*orq*).

$$G' = G_u; \quad G' = E'/(1 + \nu')/2 \quad G_u = E_u/(1 + \nu_u)/2$$
$$E'/(1 + \nu')/2 = E_u/(1 + 0.5)/2$$
$$E_u = 1.5E'/(1 + \nu')$$

- In finite element analysis, $\nu_u = 0.5$ cannot be used. Use $\nu_u = 0.49$ or 0.495 . But be careful with *mesh locking* problem.

Undrained analysis - Effective stress

- Need to assign initial effective stresses before the analysis.
- Can use any effective stress model, so the stiffness and strength variation with depth can be modeled implicitly with the one set of model parameters.
- The applied load is carried by the soil skeleton and pore water.
- The contribution of the bulk modulus of water needs to be added:

$$D = D_{(\text{soil skeleton})} + \frac{1}{n} D_{(\text{water})}, \quad \text{where } n \text{ is the porosity}$$

- Effective stress increment can be computed by:

$$d\sigma' = D_{(\text{soil skeleton})} d\varepsilon.$$

- Need to update the effective stresses at each time step.

Effective stress approach or undrained (A)

- Effective stiffness and effective strength parameters are used.
- *Pore pressures are generated*, but may be **inaccurate** depending on the model.
- Undrained shear strength is *not* an input parameter but an outcome of the constitutive model. The resulting shear strength must be checked against known data!
- Consolidation analysis can be performed after the undrained calculation, which *affects the shear strength!*

Equivalent effective stress approach or undrained (B)

- Effective stiffness parameters and *undrained strength parameters* are used.
- *Pore pressures are generated*, but may be highly **inaccurate**.
- Undrained shear strength is an input parameter.
- Consolidation analysis should not be performed after the undrained calculation, s_u must be updated, if consolidation is performed anyway!

Methods of undrained analysis for Mohr-Coulomb clay

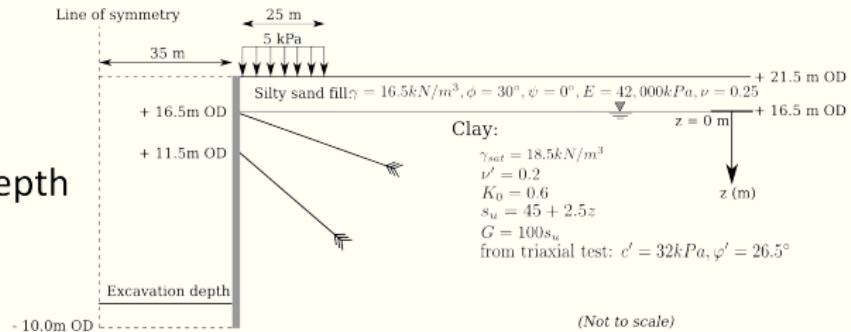
undrained analysis	material type	deformation parameters	strength parameters	initial conditions
Total stress	Non-porous / drained	E_u, ν_u	$c_u, \phi_u = 0$	$K_{0,u}$
Effective stress	Undrained (triaxial parameters)	E', ν'	c', ϕ'	K_0
Equivalent Effective stress	Undrained (strength profile)	E', ν'	c', ϕ'	K_0

Consolidation analysis - Effective stress

- Use Biot's 3D consolidation theory
- Pore pressure and displacement are computed at each time step.
- Need to use effective stress model
- Need permeability
- Lots of computational time
- More realistic. Undrained, partially drained, drained depending on the loading condition, drainage condition, permeability of soil.
- Stress path followed is correct, which should provide a good strain estimate when plasticity models are used.

Total stress evaluating varying K_0

- K_0 Varies with depth

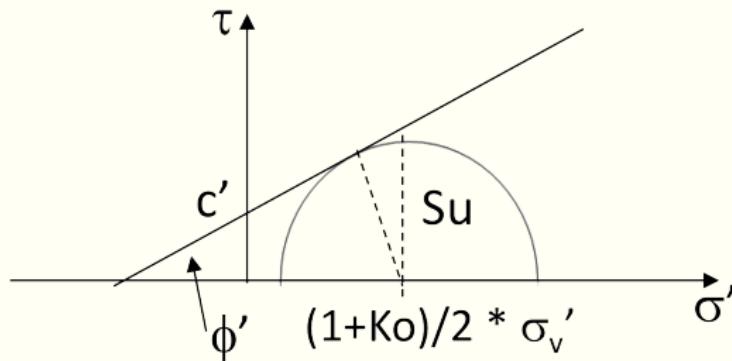


Depth	σ_v	u	σ_v'	$k_0\sigma_v' = \sigma_h'$	σ_h	K_0	K_0 Layer
0	82.5	0	82.5	49.5	49.5	0.6	
6.5	202.75	65	137.75	82.65	147.65	0.728	0.664
11.5	295.25	115	180.25	108.15	223.15	0.756	0.742
16.5	387.75	165	222.75	133.65	298.65	0.77	0.763

Undrained analysis using effective stress method

Effective stress Method A

- Define c' and ϕ' in terms of the real effective stress parameters, assuming zero dilation.
- ν' is the effective Poisson ratio
- E and K_0 should be 'effective stress' based.



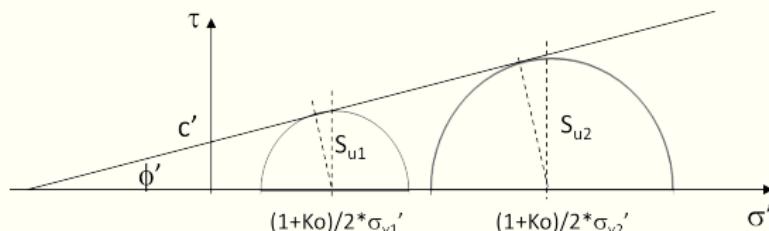
Mohr-Coulomb failure criteria: $\tau = c' + \sigma' \tan \phi'$

$$s_u = c' \cos \phi' + (1 + K_0)/2 \cdot \sin \phi' \cdot \sigma'_{v0}$$

Undrained analysis using equivalent effective stress method

Effective stress Method B

- Define c' and ϕ' in terms of the “*equivalent*” effective stress parameters, with zero dilation. Parameters are defined based on the strength profile with depth.
- ν' is the effective Poisson ratio
- E and K_0 should be ‘*effective stress*’ based.



Mohr-Coulomb failure criteria: $\tau = c' + \sigma' \tan \phi'$

$$s_u = c' \cos \phi' + (1 + K_0)/2 \cdot \sin \phi' \cdot \sigma'_{v0}.$$

2014 Oso landslide



CE394M: Linear Elasticity

└ Slope stability

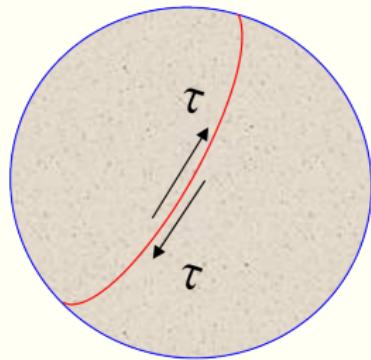
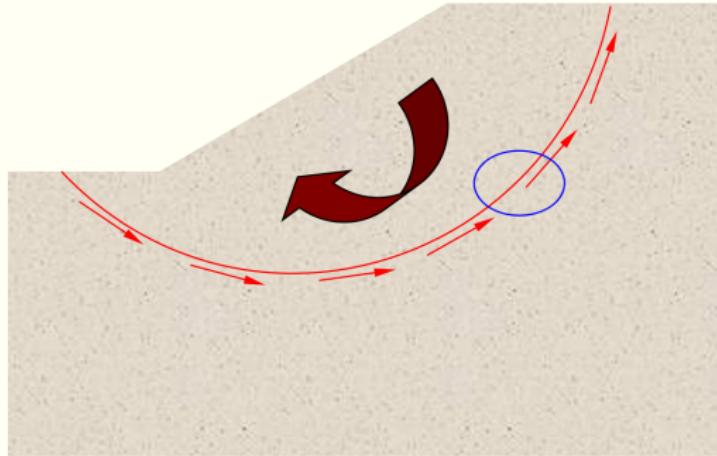
└ 2014 Oso landslide

2014 Oso landslide



- 22nd March 2014 at 10.37 am
- Volume: approx. 8 million m³
- 43 casualties (deadliest landslide in US)
- 1 neighbourhood destroyed
- **Cost unknown** but > USD 150 million + USD 65 million (lawsuit 2017) + indirect costs
- Social tensions (general public & sc. community)
- Indian Snohomish tribe

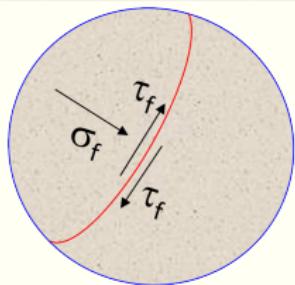
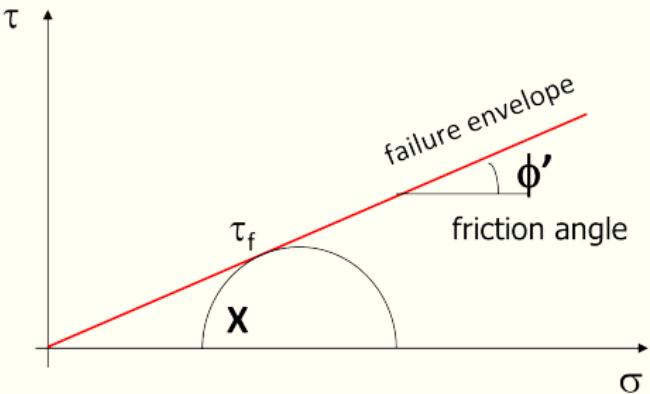
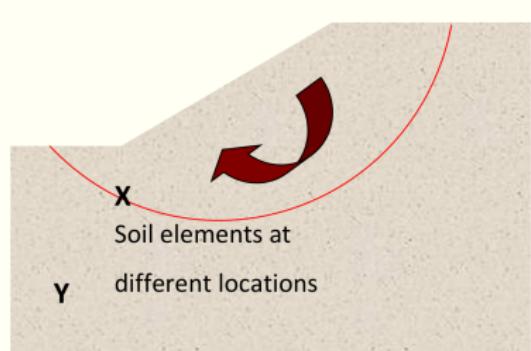
Shear failure plane



At failure, shear stress along the failure surface (τ) reaches the shear strength (τ_f).

Factor of Safety = Resistance (from soil shear strength)/Driving force
(from total stress equilibrium (i.e. weight of the soil))

Dry slope (total stress = effective stress)

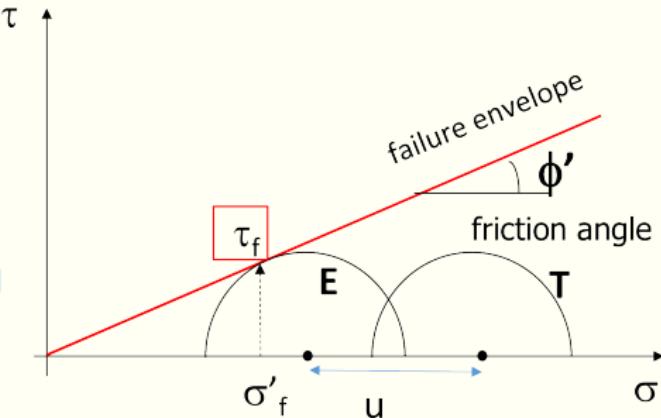
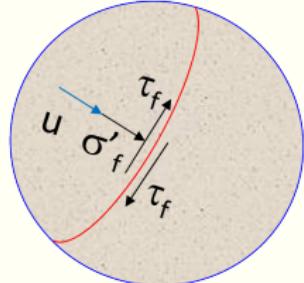
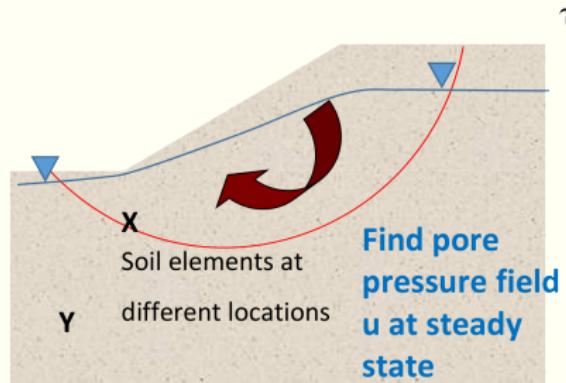


X ~ at failure

Y ~ stable

Saturated slope (total stress = effective stress + pwp)

Drained conditions - need to compute the steady state pore pressure field and then evaluate “effective stress-based” shear strength to find the overall stability (based on total stress equilibrium).

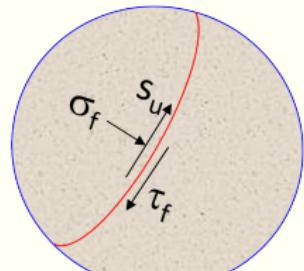
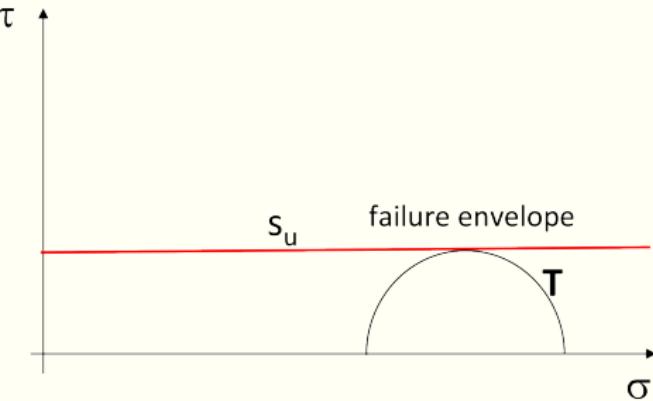
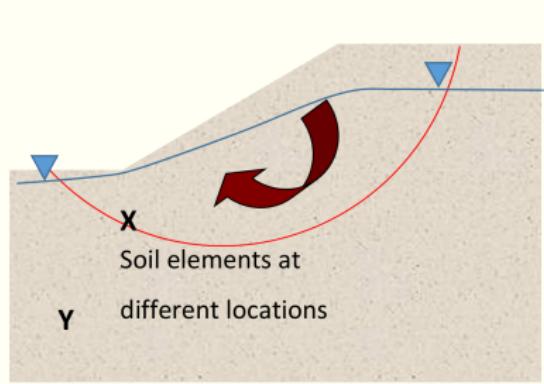


E Effective stress for soil shear resistance estimation

T Total stress for stress equilibrium calculation

Saturated slope (total stress = effective stress + PWP)

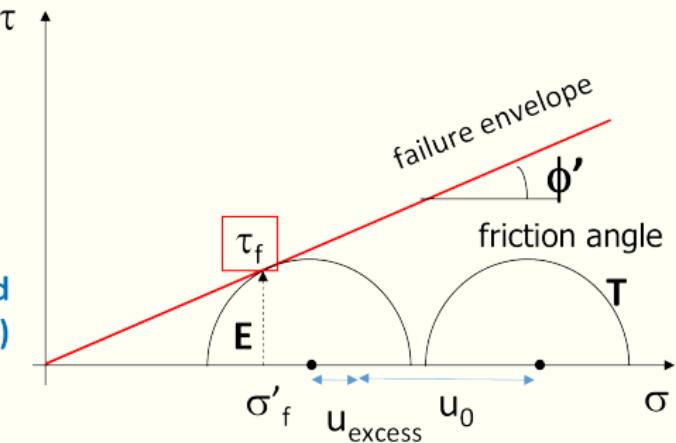
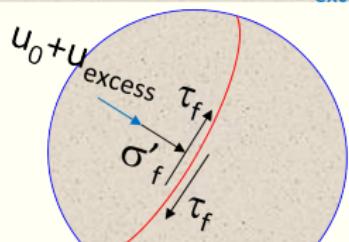
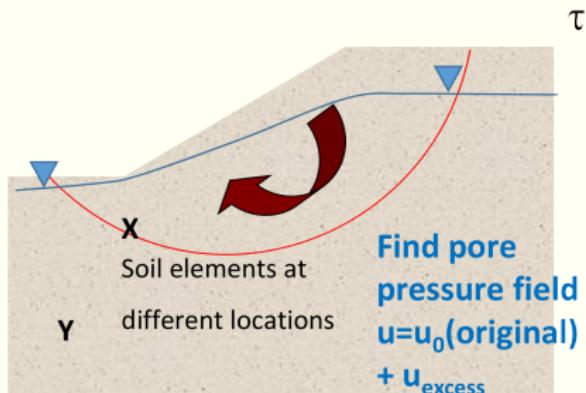
Undrained conditions - (total stress approach) – Use “*total-stress based*” shear strength (s_u) to find the overall stability (based on total stress equilibrium).



T Total stress for stress equilibrium calculation

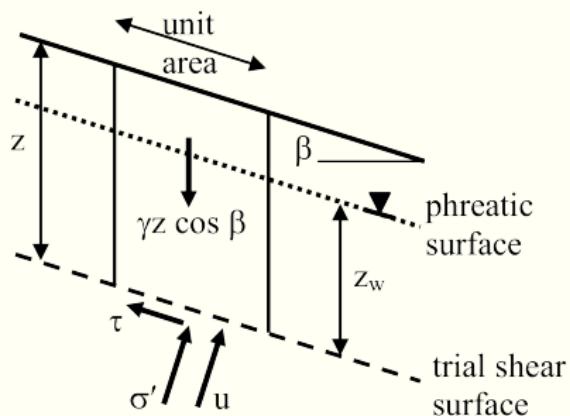
Saturated slope (total stress = effective stress + PWP)

Undrained conditions - (Effective stress approach-not common) - need to compute the pore pressure (including excess pore pressure) field and then evaluate “*effective stress -based*” shear strength to find the overall stability (based on total stress equilibrium).



- E** Effective stress for soil shear resistance estimation
- T** Total stress for stress

Infinite slopes



$$u = \gamma_w z_w \cos^2 \beta$$

$$\sigma = \gamma z \cos^2 \beta$$

$$\sigma' = (\gamma z - \gamma_w z_w) \cos^2 \beta$$

$$\tau = \gamma z \cos \beta \sin \beta$$

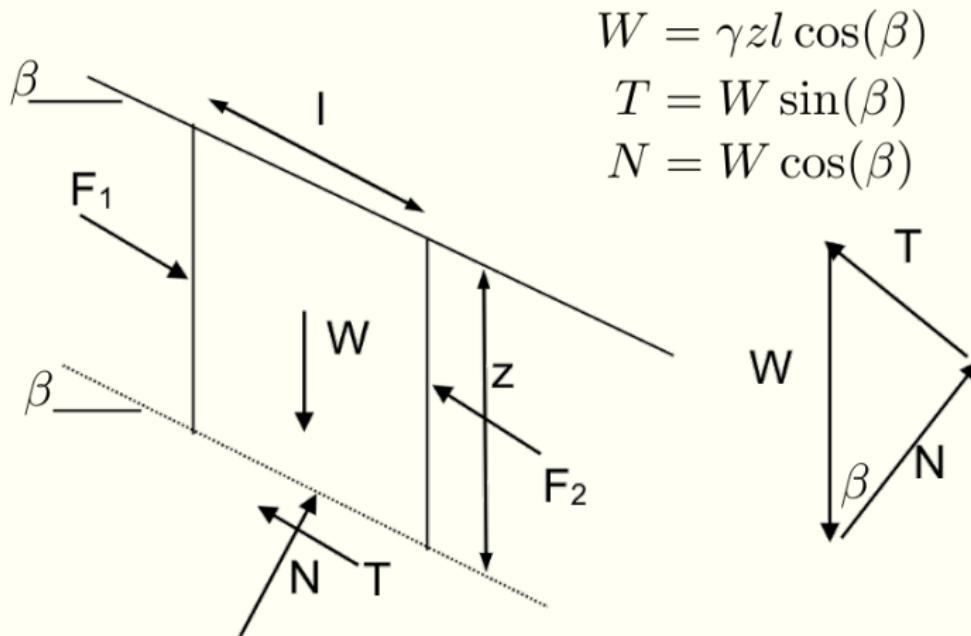
$$\tan \phi_{mob} = \frac{\tau}{\sigma'} = \frac{\tan \beta}{\left(1 - \frac{\gamma_w z_w}{\gamma z}\right)}$$

Soil fails when (dry): $\beta = \phi_{mob}$

Soil fails when (submerged): $\beta = \phi_{mob}$

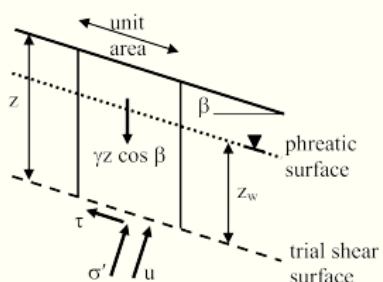
Slope with steady state seepage (drained): $\tan(\beta) = (1 - \gamma_w/\gamma) \tan(\phi_{mob})$

Undrained infinite slope (total stress approach)



But also the shear stress: $T = s_u l$. The slope failure is governed by s_u profile (with depth).

Infinite slope: Summary



$$\begin{aligned} u &= \gamma_w z_w \cos^2 \beta \\ \sigma &= \gamma z \cos^2 \beta \\ \sigma' &= (\gamma z - \gamma_w z_w) \cos^2 \beta \\ \tau &= \gamma z \cos \beta \sin \beta \end{aligned}$$

$$\tan \phi_{\text{mob}} = \frac{\tau}{\sigma'} = \frac{\tan \beta}{\left(1 - \frac{\gamma_w z_w}{\gamma z}\right)}$$

- ① Factor of Safety = resistance / driving
- ② Dry FoS = $\tan(\phi_{\text{mob}})/\tan(\beta)$
- ③ Submerged FoS = $\tan(\phi_{\text{mob}})/\tan(\beta)$
- ④ Undrained FoS = $2s_u/\gamma z \sin(2\beta)$
- ⑤ Steady state seepage FoS = $(1 - \gamma_w/\gamma) \tan(\phi_{\text{mob}})/\tan(\beta)$ where the water table is located at the slope surface