CE394M: Linear Elasticity

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Overview

Isotropic linear elastic stress-strain relations

The linear relationship between the stress and strain tensor is a linear one. The stress component is a linear combination of the strain tensor:

$$\begin{aligned}\sigma_{ij} = C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13} + \\ C_{ij21}\varepsilon_{21} + C_{ij22}\varepsilon_{22} + C_{ij23}\varepsilon_{23} + \\ C_{ij31}\varepsilon_{31} + C_{ij32}\varepsilon_{32} + C_{ij33}\varepsilon_{33}\end{aligned}$$

The most general form for *linear* stress-strain relations for a *Cauchy elastic* material is given by:

$$\sigma_{ij} = B_{ij} + C_{ijkl}\varepsilon_{kl}$$

Where B_{ij} is the components of initial stress tensor corresponding to the initial strain free (when all strain components $\varepsilon_{kl} = 0$). C_{ijkl} is the tensor of material *elastic constants*.

If it is assumed that the initial strain free state corresponds to an *initial stress free state*, that is $B_{ij} = 0$, the equations reduces to:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Observation on linear elasticity

1. $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ is a general expression relating stress to strains for a linear solid.
2. C_{ijkl} is a 4th order tensor containing 81 terms (we trick using symmetry and reduce order).
3. C_{ijkl} material response functions having dimensions F/L^2.
4. Homogeneous: C_{ijkl} independent of position
5. Isotropic: C_{ijkl} independent of frame of reference.
6. Because the stress is symmetric: $\sigma_{ij} = \sigma_{ji}$, $C_{ijkl} = C_{jikl}$. Strain is symmetric $\varepsilon_{kl} = \varepsilon_{lk}$ and $C_{ijkl} = C_{ijlk}$. Hence the number of independent variables drop from 81 to 36.
7. Both the stress and the strain tensor have only 6 independent values, therefor write them as vectors, then the stiffness tensor can be written as a matrix (compromise I can not rotate tensor).

Stress-strain relationship

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{23}\\ \varepsilon_{31}\end{bmatrix}$$

C_{ijkl} is a tensor of material *elastic constants*. However, the above [**C**] is not a tensor anymore. So we can not rotate the matrix to another frame of reference. This relationship is useful for isotropic materials, where **C** is independent of the frame of reference.

$$\{\sigma\} = [\mathbf{C}]\,\{\varepsilon\}$$

The inverse of the relationship (Compliance matrix):

$$\{\varepsilon\} = [\mathbf{D}]\,\{\sigma\} \qquad [\mathbf{D}] = [\mathbf{C}]^{-1}$$

Isotropic Linear Elastic Stress-strain relationship

The *isotropic tensor* C_{ijkl}:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

Where $\lambda, \mu,$ and $, \alpha$ are scalar constants. Since C_{ijkl} must satisfy symmetry, $\alpha = 0$.

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

So the stress:

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\varepsilon_{kl}$$
$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

Hence for an isotropic linear elastic material, there are only two independent material constants, λ and μ, which are called *Lame's constants*.

Hooke's law

Empirical observation:

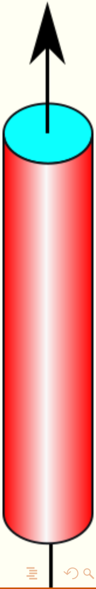
$$\Delta\varepsilon_a = \Delta\sigma_{axial} \cdot \frac{1}{E} \rightarrow \Delta\varepsilon_{11} = \frac{\Delta\sigma_{11}}{E}$$

Where E is defined as the *Young's modulus*.
The lateral strains are defined as:

$$\Delta\varepsilon_{22} = -\nu\Delta\varepsilon_{11}$$
$$\Delta\varepsilon_{33} = -\nu\Delta\varepsilon_{11}$$

Using superposition for principal stresses:

$$\varepsilon_{11} = (1/E)\left[\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}\right]$$
$$\varepsilon_{22} = (1/E)\left[-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}\right]$$
$$\varepsilon_{33} = (1/E)\left[-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}\right]$$



$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{Bmatrix}$$

It is possible to invert the matrix to obtain the generalized Hooke's law: $[\sigma] = [\mathbf{C}][\varepsilon]$.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \alpha \begin{bmatrix} (1-\mu) & \nu & \nu & 0 & 0 & 0 \\ & (1-\nu) & \nu & 0 & 0 & 0 \\ & & (1-\nu) & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & & & & \frac{(1-2\nu)}{2} & 0 \\ & & & & & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

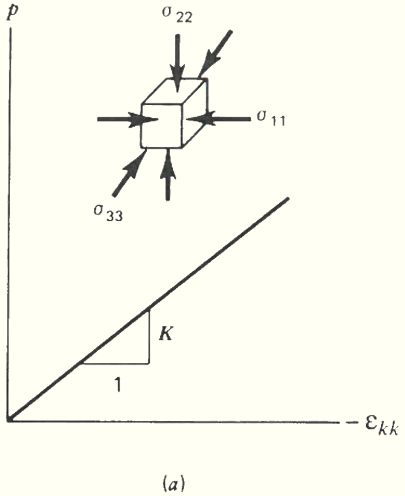
Where $\alpha = E/((1+\mu)(1-2\mu))$. Similarly, we can obtain the inverse matrix.

Hooke's law

The matrices [**C**] and [**D**] contains two indepdent variables E and μ, where $E > 0$ and $-1 \leq \mu \leq 0.5$. The matrix can also be defined in terms of Lame's constants.

$$\begin{cases} \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} & \text{Lame's modulus (wave propagation)} \\ \mu = G = \frac{E}{2(1+\nu)} & \text{Shear modulus (shear behavior)} \\ K = \frac{E}{3(1-2\nu)} & \text{Bulk modulus (volumetric behavior)} \end{cases}$$

Isotropic linear elastic



Behavior of isotropic linear elastic material in simple tests: (a) hydrostatic compression test ($\sigma_{11} = \sigma_{22} = \sigma_{33} = p$) and (b) simple tension test (Chen 1994)

Isotropic linear elastic

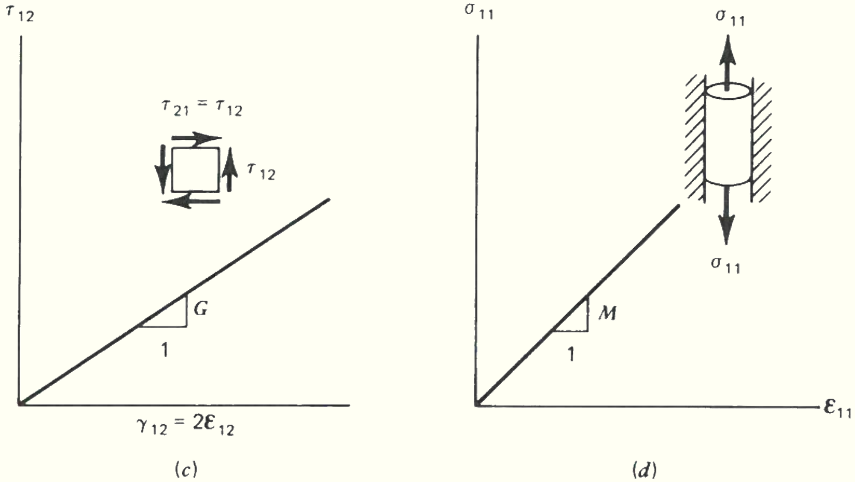
Hydrostatic compression test The non-zero components of stress: $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p = \sigma_{kk}/3$.
The *Bulk modulus, K,* is defined as the ratio between the *hydrostatic pressure p* nd the corresponding volume change $\delta\varepsilon_v = \varepsilon_{kk}$.

$$K = -\frac{p}{\varepsilon_{kk}} = \lambda + \frac{2}{3}\mu$$

Simple tension test The only non-zero components of stress: $\sigma_{11} = \sigma$
The *Young's modulus, E,* and *Poisson's ratio,* ν as.

$$E = \frac{\sigma_{11}}{\varepsilon_{11}} \quad \nu = \frac{-\varepsilon_{22}}{\varepsilon_{11}} = \frac{-\varepsilon_{33}}{\varepsilon_{11}}$$

Isotropic linear elastic



Behavior of isotropic linear elastic material in simple tests: (c) pure shear test, and (d) uniaxial strain test (Chen 1994)

Isotropic linear elastic

Simple shear test

The non-zero components of stress: $\sigma_{12} = \sigma_{21} = \tau_{12} = \tau_{21} = \tau$.
The *Shear modulus, G or μ*, is defined as:

$$G = \mu = \frac{\sigma_{12}}{\gamma_{12}} = \frac{\tau}{2\varepsilon_{12}}$$

Uniaxial strain test The test is carried out by applying a uniaxial stress component σ_{11} in the axial direction of a cylindrical sample, whose lateral surface is *restrained* against lateral movement (Oedometer test). Axial strain ε_{11} is the only nonvanishing component. The *constrained modulus* M or as PLAXIS calls it E_{oed} is defined as the ratio between σ_{11} and $\varepsilon_1 1$.

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{11} + \nu\cancelto{0}{\varepsilon_{22}} + \nu\cancelto{0}{\varepsilon_{33}}\right] = \frac{E(1-\nu)\varepsilon_{11}}{(1+\nu)(1-2\nu)}$$

$$M = E_{oed} = \frac{\sigma_{11}}{\varepsilon_{11}} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} = (\lambda + 2\mu)$$

Plane stress v Plane strain

For a frame with an axis perpendicular to the plane of interest, x_3 or z:
Plane stress $\sigma_{33} = \sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$.
The strain in z is written as:

$$\varepsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy}) = \frac{-\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})$$

The plane stress are commonly used for thin flat plates loaded in the plane of the plate.
Plane strain $\varepsilon_{33} = \varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$.
The stress in z is written as:

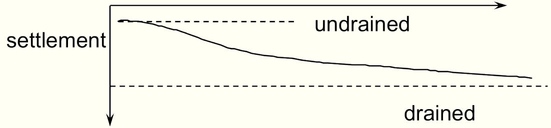
$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

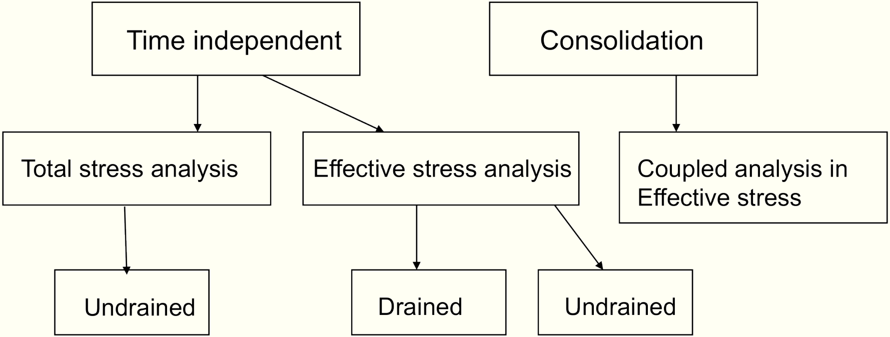
The plane strains are commonly used for elongated bodies of uniform cross sections subjected to uniform loading along the longitudinal axis (tunnels, dams, retaining walls, soil slopes, etc.).

Elastic solutions

1. *Ease of use* Only two parameters, choose equivalent values representative of stran/stress levels.
2. *Disadvantage*: No failure criteria.
3. Validate code with chart solutions ("*Exact solutions*")., e.g., Poulos and Davis (1974).
4. Useful to get feeling of problem (lo stress levels) not wide distribution of plastic zones.

Pore-pressure analysis in geotechnical engineering





Drained analysis - Effective stress

1. Need to assign initial effective stresses before the analysis.
2. Can use any effective stress model: Elastic, Mohr-Coulomb/Drucker Prager and Cam-Clay models.
3. If plasticity models are used, need to update the effective stresses at each increment:

$$\sigma'(new) = \sigma'(old) + D(\text{soil skeleton})d\varepsilon$$

4. Very common.

Undrained analysis - Total stress

- Excess pore pressure cannot be calculated.
- Effective stress state of the soil cannot be examined.
- Elastic model is commonly used for deformation
  1. Use undrained stiffness E_u and strength parameters s_u
  2. Poisson's ratio close to 0.5 with a drained simulation
  3. Properties can vary with depth. (K_0 varies)
  4. Consolidation analysis has no effect and should not be performed.
- Von-Mises model is used for modeling undrained shear strength of clays. (C_u or s_u and undrained friction $\phi_u = 0$)
- Can assign different stiffness and strength at different depths explicity by assigning different model parameters at different depths.

Undrained analysis - Effective vs Total stress

- Effective stress: E' and ν'.
- Total stress: E_u and ν_u
- No volume change:

$$\nu_u = 0.5; (K_u = E_u/(1 - 2\nu_u)/3 = E_u/0 = \infty)$$

- Pore fluid cannot sustain shear stresses. Soil skeleton carries the shear stresses $\tau(orq)$.

$$G' = G_u; \quad G' = E'/(1+\nu')/2 \quad G_u = E_u/(1+\nu_u)/2$$
$$E'/(1+\nu')/2 = E_u/(1+0.5)/2$$
$$E_u = 1.5E'/(1+\nu')$$

- In finite element analysis, $\nu_u = 0.5$ cannot be used. Use $\nu_u = 0.49$ or 0.495. But be careful with *mesh locking* problem.

Undrained analysis - Effective stress

- Need to assign initial effective stresses before the analysis.
- Can use any effective stress model, so the stiffness and strength variation with depth can be modeled implicitly with the one set of model parameters.
- The applied load is carried by the soil skeleton and pore water.
- The contribution of the bulk modulus of water needs to be added:

$$D = D_{(\text{soil skeleton})} + \frac{1}{n} D_{(water)}, \quad \text{where } n \text{ is the porosity}$$

- Effective stress increment can be computed by:

$$d\sigma' = D_{(\text{soil skeleton})} d\varepsilon.$$

- Need to update the effective stresses at each time step.

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- Need to update the effective stresses at each time step.

$$u = K_w \cdot \varepsilon_{v,water}$$

Where, K_w is the Bulk modulus of water $\approx 2 \times 10^6 kN/m^2$.

$$\begin{aligned}
\varepsilon_{v,water} &= \Delta V_w / V_w \\
\varepsilon_{v,soil} &= \Delta V_T / V_T = \Delta V_w / V_T \quad \text{(assuming solid is incompressible)} \\
\frac{\varepsilon_{v,soil}}{\varepsilon_{v,water}} &= \frac{\Delta V_w / V_T}{\Delta V_w / V_w} = \frac{V_w}{V_T} = n \quad \text{(porosity for s = 100\%)}
\end{aligned}$$

Therefore,

$$u = K_w \cdot \frac{\varepsilon_{v,soil}}{n} = (K_w / n) \cdot \varepsilon_{v,soil}$$

Undrained analysis - Effective stress

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- Effective stress increment can be computed by:

$$d\sigma' = D_{(\text{soil skeleton})} d\varepsilon.$$

- Need to update the effective stresses at each time step.

Consider equivalent (K_w/n) in soil model. Separate total and effective stresses:

$$
\begin{aligned}
\sigma_m &= K_u \cdot \varepsilon_{v,soil} \quad K_u \text{undrained bulk modulus of soil} \\
\sigma'_m &= K' \cdot \varepsilon_v \\
u &= \sigma_m - \sigma' = (K_u - K') \cdot \varepsilon_v \\
K_w/n &= K_u - K'
\end{aligned}
$$

Effective stress approach or undrained (A)

- Effective stiffness and effective strength parameters are used.
- *Pore pressures are generated*, but may be **inaccurate** depending on the model.
- Undrained shear strength is *not* an input parameter but an outcome of the constitutive model. The resulting shear strength must be checked against known data!
- Consolidation analysis can be performed after the undrained calculation, which *affects the shear strength*!

Equivalent effective stress approach or undrained (B)

- Effective stiffness parameters and *undrained strength parameters* are used.
- *Pore pressures are generated*, but may be hightly **inaccurate**.
- Undrained shear strength is an input parameter.
- Consolidation analysis should not be performed after the undrained calculation, s_u must be updated, if consolidation is performed anyway!

Methods of undrained analysis for Mohr-Coulomb clay

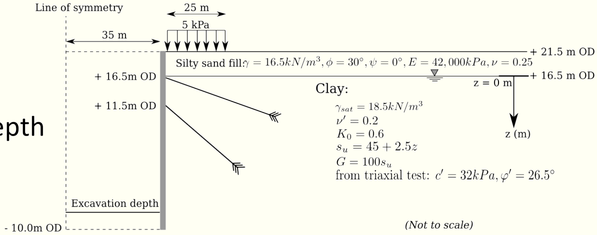
undrained analysis	material type	deformation parameters	strength parameters	initial conditions
Total stress	Non-porous / drained	E_u, ν_u	$c_u, \phi_u = 0$	$K_{0,u}$
Effective stress	Undrained (triaxial parameters)	E', ν'	c', ϕ'	K_0
Equivalent Effective stress	Undrained (strength profile)	E', ν'	c', ϕ'	K_0

Consolidation analysis - Effective stress

- Use Biot's 3D consolidation theory
- Pore pressure and displacement are computed at each time step.
- Need to use effective stress model
- Need permeability
- Lots of computational time
- More realistic. Undrained, partially drained, drained depending on the loading condition, drainage condition, permeability of soil.
- Stress path followed is correct, which should provide a good strain estimate when plasticity models are used.

Total stress evaluating varying K_0

- K_0 Varies with depth

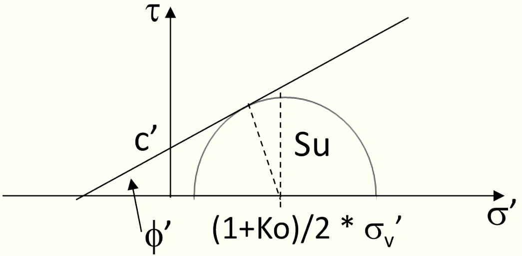


Depth	σ_v	u	σ_v'	$k_0\sigma_v'=\sigma_h'$	σ_h	K_0	K_0 Layer
0	82.5	0	82.5	49.5	49.5	0.6	
6.5	202.75	65	137.75	82.65	147.65	0.728	0.664
11.5	295.25	115	180.25	108.15	223.15	0.756	0.742
16.5	387.75	165	222.75	133.65	298.65	0.77	0.763

Undrained analysis using effective stress method

Effective stress Method A

- Define c' and ϕ' in terms of the real effective stress parameters, assuming zero dilation.
- ν' is the effective Poisson ratio
- E and K_0 should be '*effective stress*' based.



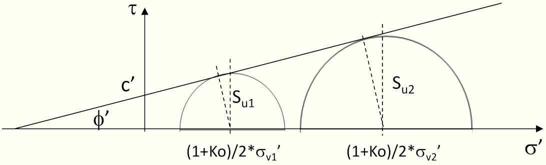
Mohr-Coulomb failure criteria: $\tau = c' + \sigma' \tan \phi'$

$$s_u = c' \cos \phi' + (1 + K_0)/2 \cdot \sin \phi' \cdot \sigma'_{v0}.$$

Undrained analysis using equivalent effective stress method

Effective stress Method B

- Define c' and ϕ' in terms of the "*equivalent*" effective stress parameters, with zero dilation. Parameters are defined based on the strength profile with depth.
- ν' is the effective Poisson ratio
- E and K_0 should be '*effective stress*' based.



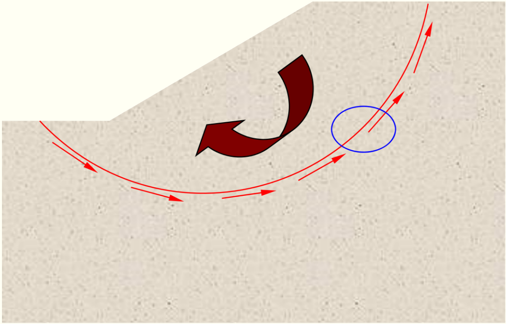
Mohr-Coulomb failure criteria: $\tau = c' + \sigma' \tan\phi'$

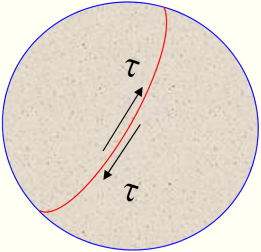
$$s_u = c' \cos\phi' + (1 + K_0)/2 \cdot \sin\phi' \cdot \sigma'_{v0}.$$

2014 Oso landslide

- 22nd March 2014 at 10.37 am
- Volume: approx. 8 million m3
- 43 casualties (deadliest landslide in US)
- 1 neighboured destroyed
- **Cost unknown** but $>$ USD 150 million $+$ USD 65 million (lawsuit 2017) $+$ indirect costs
- Social tensions (general public & sc. community)
- Indian Snohomish tribe

Shear failure plane

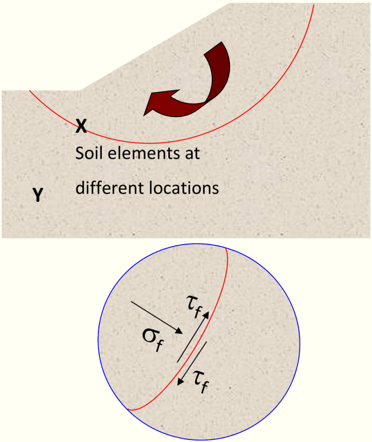


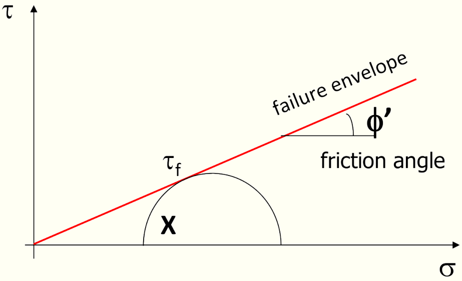


At failure, shear stress along the failure surface (τ) reaches the shear strength (τ_f).

Factor of Safety = Resistance (from soil shear strength)/Driving force (from total stress equilibrium (i.e. weight of the soil))

Dry slope (total stress = effective stress)



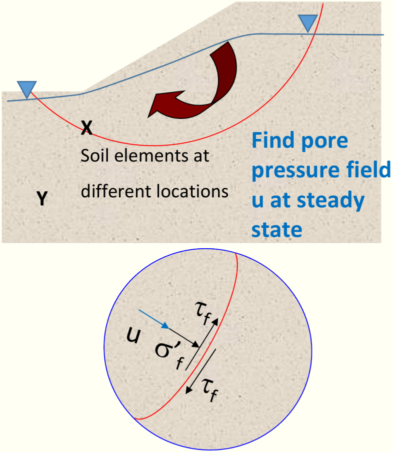


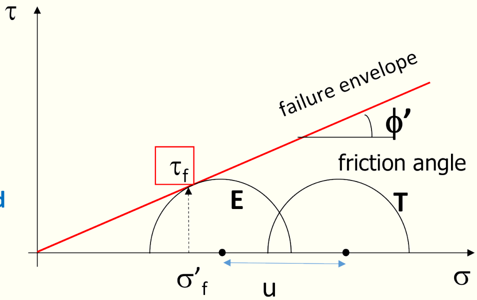
X ~ at failure

Y ~ stable

Saturated slope (total stress = effective stress + pwp)

Drained conditions - need to compute the steady state pore pressure field and then evaluate "effective stress-based" shear strength to find the overall stability (based on total stress equilibrium).



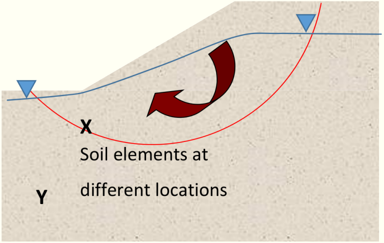


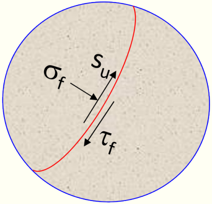
E Effective stress for soil shear resistance estimation

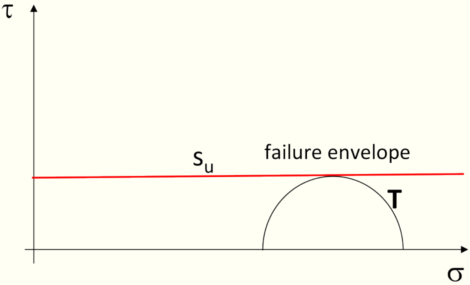
T Total stress for stress equilibrium calculation

Saturated slope (total stress = effective stress + PWP)

Undrained conditions - (total stress approach) – Use "*total-stress based*" shear strength (s_u) to find the overall stability (based on total stress equilibrium).



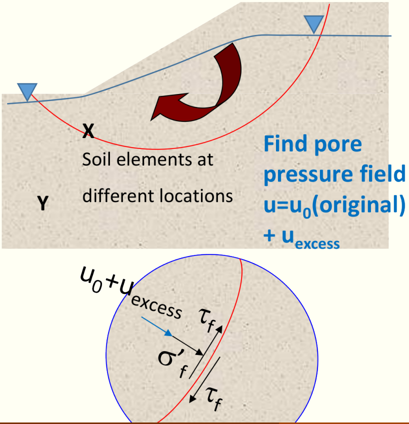


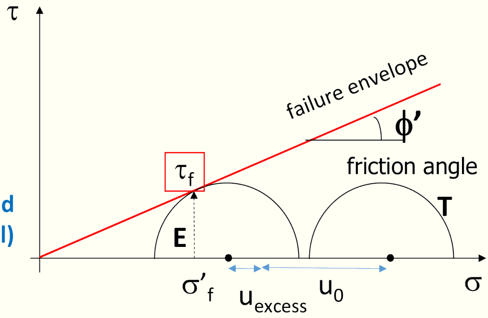


T Total stress for stress equilibrium calculation

Saturated slope (total stress = effective stress + PWP)

Undrained conditions - (Effective stress approach-not common) - need to compute the pore pressure (including excess pore pressure) field and then evaluate "*effective stress -based*" shear strength to find the overall stability (based on total stress equilibrium).

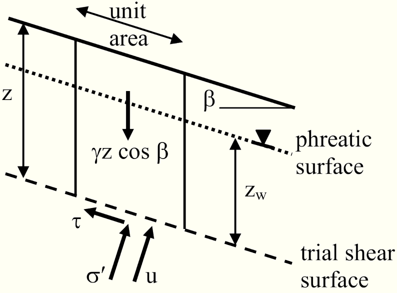




E Effective stress for soil shear resistance estimation

T Total stress for stress

Infinite slopes

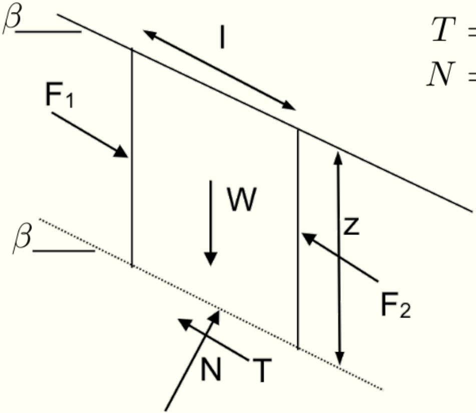


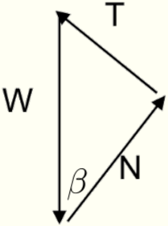
$$u = \gamma_w z_w \cos^2\beta$$
$$\sigma = \gamma z \cos^2\beta$$
$$\sigma' = (\gamma z - \gamma_w z_w) \cos^2\beta$$
$$\tau = \gamma z \cos\beta \sin\beta$$

$$\tan\phi_{mob} = \frac{\tau}{\sigma'} = \frac{\tan\beta}{\left(1 - \frac{\gamma_w z_w}{\gamma z}\right)}$$

Soil fails when (dry): $\beta = \phi_{\text{mob}}$
Soil fails when (submerged): $\beta = \phi_{\text{mob}}$
Slope with steady state seepage (drained): $\tan(\beta) = (1 - \gamma_w/\gamma)\tan(\phi_{\text{mob}})$

Undrained infinite slope (total stress approach)

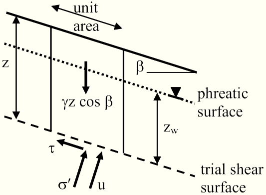




$$W = \gamma z l \cos(\beta)$$
$$T = W \sin(\beta)$$
$$N = W \cos(\beta)$$

But also the shear stress: $T = s_u l$. The slope failure is governed by s_u profile (with depth).

Infinite slope: Summary



$$u = \gamma_w z_w \cos^2\beta$$
$$\sigma = \gamma z \cos^2\beta$$
$$\sigma' = (\gamma z - \gamma_w z_w) \cos^2\beta$$
$$\tau = \gamma z \cos\beta \sin\beta$$

$$\tan\phi_{mob} = \frac{\tau}{\sigma'} = \frac{\tan\beta}{\left(1 - \frac{\gamma_w z_w}{\gamma z}\right)}$$

1. Factor of Safety = resistance / driving
2. Dry FoS $= \tan(\phi_{\text{mob}})/\tan(\beta)$
3. Submerged FoS $= \tan(\phi_{\text{mob}})/\tan(\beta)$
4. Undrained FoS $= 2s_u/\gamma z \sin(2\beta)$
5. Steady state seepage FoS $= (1 - \gamma_w/\gamma)\tan(\phi_{\text{mob}})/\tan(\beta)$ where the water table is located at the slope surface