

# CE394M: Stress paths and invariants

Krishna Kumar

University of Texas at Austin

*krishnak@utexas.edu*

April 4, 2020

# Overview

- 1 Stresses / strains in typical geotechnical lab tests
- 2 Friction
- 3 Stress invariants

# Stresses / strains

## 1D consolidation / simple shear

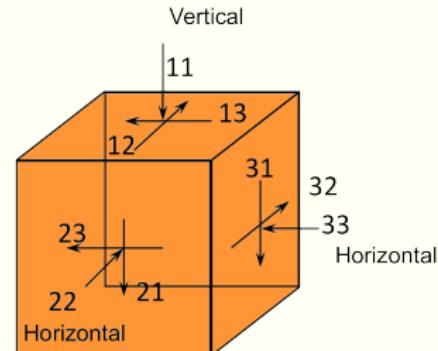
- Zero lateral strain ( $\varepsilon_{22} = \varepsilon_{33} = 0$ )
- Stresses:  $\sigma$  and  $\tau$
- Strains:  $\varepsilon_{11} = \varepsilon_v$  and  $\gamma$

## 2D plane strain

- Zero lateral strain ( $\varepsilon_{22} = \gamma_{12} = \gamma_{23} = 0$ )
- Stresses:  $s$  and  $t$
- Strains:  $\varepsilon_v$  and  $\varepsilon_\gamma$

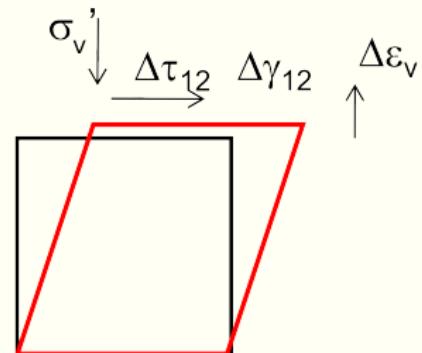
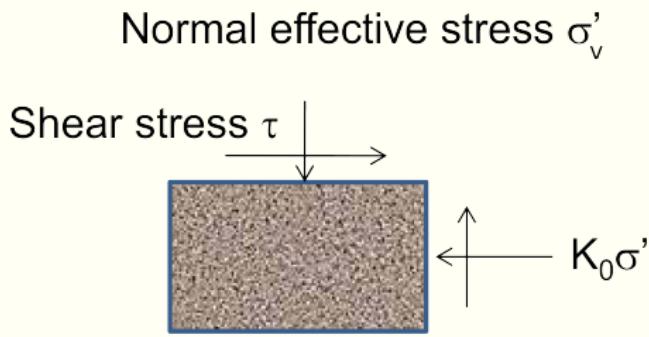
## 3D general (axi-symmetric as a special case)

- Stresses:  $p$  and  $q$
- Strains:  $\varepsilon_v$  and  $\varepsilon_s$



# 1D simple shear

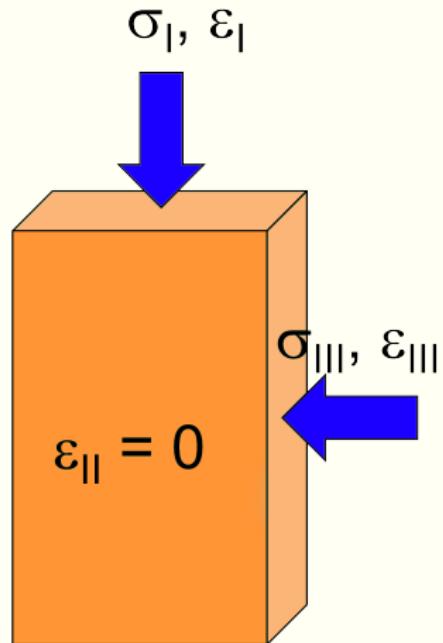
- ① No lateral strain
- ② Constant normal effective stress  $\sigma'_v$
- ③ Increasing shear strain  $\gamma$
- ④ Measure shear resistance  $\tau$
- ⑤ Measure volumetric strain  $\varepsilon_v$  or void ratio  $e = e_0 - (1 + e_0)\varepsilon_v$
- ⑥ **No information for the lateral direction**



# 2D plane strain / Mohr-Coulomb model

## Stresses and strains: independent components

- ① Mean stress:  $s = (\sigma_I + \sigma_{III})/2$  and  $s' = s - u$
- ② Volumetric strain:  $\varepsilon_v = (\varepsilon_I + \varepsilon_{III})$
- ③ Deviatoric / shear stress:  $t = (\sigma_I - \sigma_{III})/2$  and  $t' = t$ .
- ④ Deviatoric / shear strain:  $\varepsilon_\gamma = (\varepsilon_I - \varepsilon_{III})$
- ⑤ Work increment:  
$$\delta W = \sigma'_I \delta \varepsilon_I + \sigma'_{III} \delta \varepsilon_{III} = s' \delta \varepsilon_v + t \delta \varepsilon_\gamma$$
- ⑥  $s$  and  $t$  are often used to derive parameters for Mohr-Coulomb model because it only considers  $\sigma_I$  and  $\sigma_{III}$  and not  $\sigma_{II}$ .



## CE394M: Stresses - paths &amp; invariants

└ Stresses / strains in typical geotechnical lab tests

└ 2D plane strain / Mohr-Coulomb model

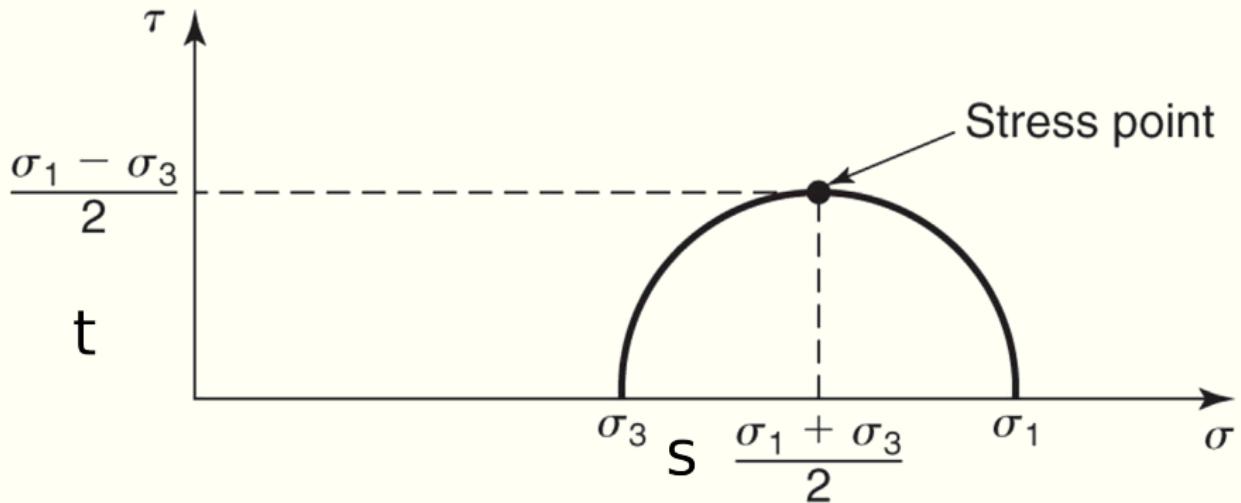
## Stresses and strains: independent components

- Mean stress:  $s = (\sigma_I + \sigma_{II})/2$  and  $s' = s - u$
  - Volumetric strain:  $\varepsilon_v = (\varepsilon_I + \varepsilon_{II})$
  - Deviatoric / shear stress:  $t = (\sigma_I - \sigma_{II})/2$  and  $t' = t$ .
  - Deviatoric / shear strain:  $\varepsilon_t = (\varepsilon_I - \varepsilon_{II})$
  - Work increment:
- $$\delta W = \sigma'_I \delta \varepsilon_I + \sigma'_{II} \delta \varepsilon_{II} = s' \delta \varepsilon_v + t \delta \varepsilon_t$$
- $\varepsilon_{II} = 0$        $\sigma'_{II}, \varepsilon_{II}$
- $G' = G - u$

Principal stresses  $\sigma_I > \sigma_{II} > \sigma_{III}$ . This equation holds good and is the definition of  $\sigma$  in principal stress notations, i.e.,  $I > II > III$ .

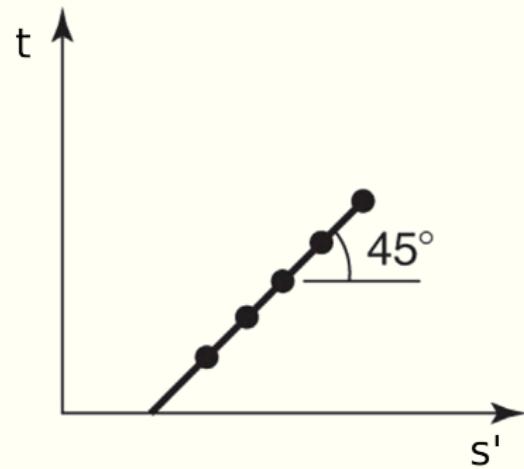
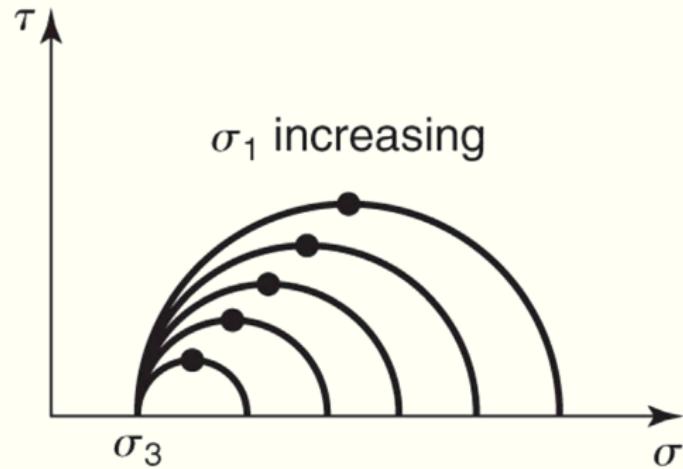
No shear stresses on these planes.

## 2D Mohr circle



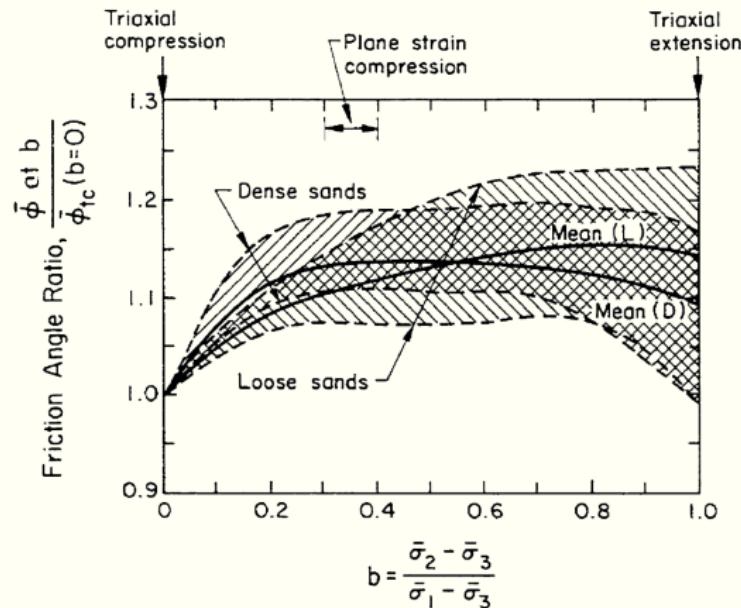
# Mohr circle to 2D stress components

Successive Mohr circles and stress path for constant  $\sigma_3$  and increasing  $\sigma_1$ :



# Effect of $\sigma_{II}$

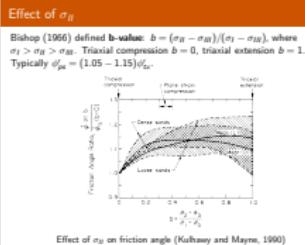
Bishop (1966) defined **b-value**:  $b = (\sigma_{II} - \sigma_{III}) / (\sigma_I - \sigma_{III})$ , where  $\sigma_I > \sigma_{II} > \sigma_{III}$ . Triaxial compression  $b = 0$ , triaxial extension  $b = 1$ . Typically  $\phi'_{ps} = (1.05 - 1.15)\phi'_{tx}$ .



Effect of  $\sigma_{II}$  on friction angle (Kulhawy and Mayne, 1990)

## CE394M: Stresses - paths &amp; invariants

## └ Stresses / strains in typical geotechnical lab tests

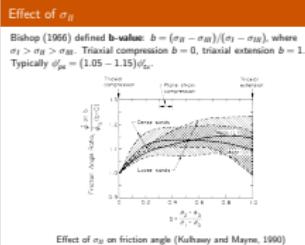
└ Effect of  $\sigma_{II}$ 

$\sigma_{II}$  do have an effect on soil behavior. For example, the friction angle depends on the loading condition: triaxial compression, plane-strain, triaxial extension and others ...., The effect of  $\sigma_{II}$  can be measured using true triaxial apparatus or hollow cylinder torsional shear apparatus. In general, the peak friction angle increases 10 to 15 percent from  $b = 0$  (triaxial compression) to  $b = 0.3$  to  $0.4$  (plane strain), and it stays constant or slightly decreases as  $b$  reaches  $1$  (tri-axial extension).

The variation of measured friction angle with changes in  $\sigma_{II}$  can be attributed to the effects of different mean stress and stress anisotropy on the dilatancy and particle rearrangement contributions to the total strength. For given maximum and minimum principal stresses, the TXE conditions have the largest mean effective stress, whereas the triaxial compression conditions have the smallest mean effective stress. The higher confinement for triaxial extension and plane strain conditions contributes to the increasing friction angle for these conditions. (Mitchell and Soga., 2005)

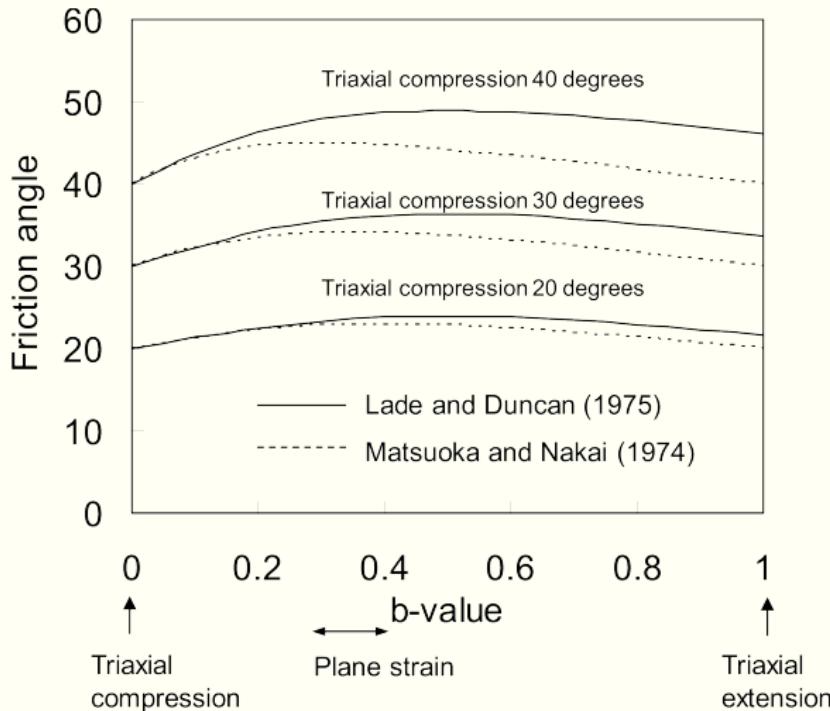
## CE394M: Stresses - paths &amp; invariants

## └ Stresses / strains in typical geotechnical lab tests

└ Effect of  $\sigma_{II}$ 

In TXC major principal stress,  $\sigma_1$  acts vertically, compared to triaxial extension mode of loading, where  $\sigma_1$  acts horizontally. The effect of the intermediate principal stress is comparatively smaller, and is often assessed using the parameter  $b = (\sigma_{II} - \sigma_{III})/(\sigma_1 - \sigma_{III})$ , which is zero in triaxial compression, and one in triaxial extension. In TXE  $\sigma_1 = \sigma_{II}$  (radial), while  $\sigma_{III}$  is the vertical stress.

# Effect of $\sigma_{II}$ on friction



Chapter 11., Mitchell and Soga, 2005

## Effect of $\sigma_{II}$ on friction

Various models fit the experimental data showing the intermediate stress effect. Lade and Duncan (1975)

$$\frac{I_1^3}{I_3} = \text{const}$$

Matsuoka and Nakai (1974)

$$\frac{I_1 I_2}{I_3} = \text{const}$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

Matsuoka and Nakai's model gives the same friction angle for compression and extension, whereas Lade and Duncan's model gives the ratio of the TXE friction angle ( $\phi'_{TXE}$ ) to the TXC friction angle ( $\phi'_{TXC}$ ) to be 1.08 at  $\phi'_{TXC} = 20^\circ$  to 1.15 at  $\phi'_{TXC} = 40^\circ$ .

## CE394M: Stresses - paths &amp; invariants

## └ Stresses / strains in typical geotechnical lab tests

└ Effect of  $\sigma_{II}$  on frictionEffect of  $\sigma_R$  on friction

Various models fit the experimental data showing the intermediate stress effect. Lade and Duncan (1975)

$$\frac{I_2^2}{I_3} = \text{const}$$

Matsuoka and Nakai (1974)

$$\frac{I_2 I_3}{I_1} = \text{const}$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1$$

$$I_3 = \sigma_1 \sigma_2 \sigma_3$$

Matsuoka and Nakai's model gives the same friction angle for compression and extension, whereas Lade and Duncan's model gives the ratio of the TXE friction angle ( $\phi'_{TXE}$ ) to the TXC friction angle ( $\phi'_{TXC}$ ) to be 1.08 at  $\phi'_{TXC} = 20^\circ$  to 1.15 at  $\phi'_{TXC} = 40^\circ$ .

Given the large scatter in the published experimental data, it is not possible to conclude that one model is better than the other.

# Triaxial stresses and strains: independent components

- We split the stress system into:

- purely volumetric deformation 'p'
- purely distortional deformation 'q'

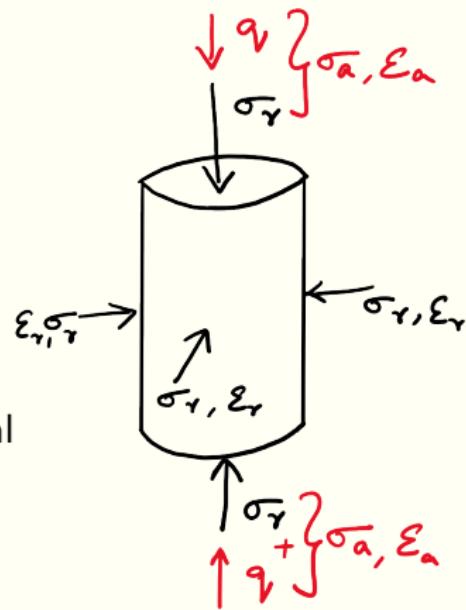
- Mean stress:

$$p = (\sigma_a + 2\sigma_r)/3 \quad p' = p - u$$

- Volumetric strain:  $\varepsilon_v = (\varepsilon_a + 2\varepsilon_r)$
- Deviatoric / shear stress (purely distortional deformation):

$$q = (\sigma_a - \sigma_r) \quad q' = q$$

- Deviatoric / shear strain:  $\varepsilon_s = 2/3(\varepsilon_a - \varepsilon_r)$



# Triaxial deviatoric strain

- Deviatoric / shear strain:  $\varepsilon_s = 2/3(\varepsilon_a - \varepsilon_r)$
- Why 2/3?:
- Work equation:  $\delta W = p'\delta\varepsilon_v + q'\delta\varepsilon_s$

$$\delta W = \sigma'_1\delta\varepsilon_1 + \sigma'_2\delta\varepsilon_2 + \sigma'_3\delta\varepsilon_3 = \sigma'_a\delta\varepsilon_a + 2\sigma'_r\delta\varepsilon_r$$

$$\begin{aligned}\delta W &= (1/3\sigma'_a + 2/3\sigma'_r)(\delta\varepsilon_a + 2\delta\varepsilon_r) + (\sigma'_a - \sigma'_r)(2/3)(\delta\varepsilon_a - \delta\varepsilon_r) \\ &= \sigma'_a\delta\varepsilon_a + 2\sigma'_r\delta\varepsilon_r\end{aligned}$$

# Triaxial stress - strain relationship

Relationship in terms of axial and radial direction:

$$\begin{bmatrix} \varepsilon_a \\ \varepsilon_r \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -2\nu \\ -\nu & (1-\nu) \end{bmatrix} \begin{bmatrix} \sigma_a \\ \sigma_r \end{bmatrix} \quad (1)$$

Relationship in terms of 'p' and 'q':

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_a \\ \sigma_r \end{bmatrix} \quad \text{invert} \rightarrow \begin{bmatrix} \sigma_a \\ \sigma_r \end{bmatrix} = \begin{bmatrix} 1 & 2/3 \\ 1 & -1/3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \quad (2)$$

Strain:

$$\begin{bmatrix} \varepsilon_p \\ \varepsilon_q \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} \varepsilon_a \\ \varepsilon_r \end{bmatrix} \quad (3)$$

# Volume - shear coupling

Equation (1) and eq. (2) in eq. (3) to relate  $\varepsilon_p, \varepsilon_q$  to  $p, q$ :

$$\begin{bmatrix} \varepsilon_p \\ \varepsilon_q \end{bmatrix} = \begin{bmatrix} 3(1-2\nu)/E & 0 \\ 0 & 2(1+\nu)/3E \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

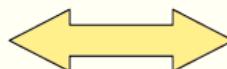
$$\begin{bmatrix} \varepsilon_p \\ \varepsilon_q \end{bmatrix} = \begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

- ① Off-diagonal zeros indicate no coupling between volumetric and distortional effects.
- ② Applying pure shear cannot induce volume changes - **not true for soils!**
- ③ Plasticity theory will deal with this.

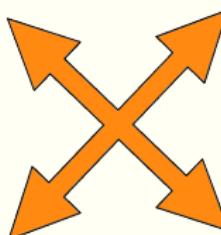
# Volume shear coupling

Usually volume change and shear deformation are decoupled (i.e. bulk modulus and shear modulus), but in soils they are coupled:

Change in  
confining stress



Volumetric strain  
increment



If particulate matter

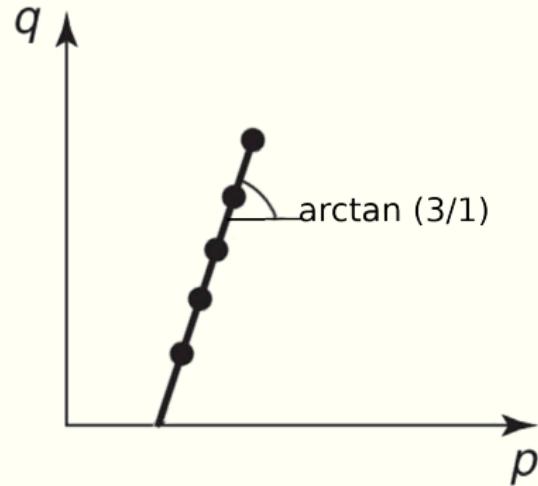
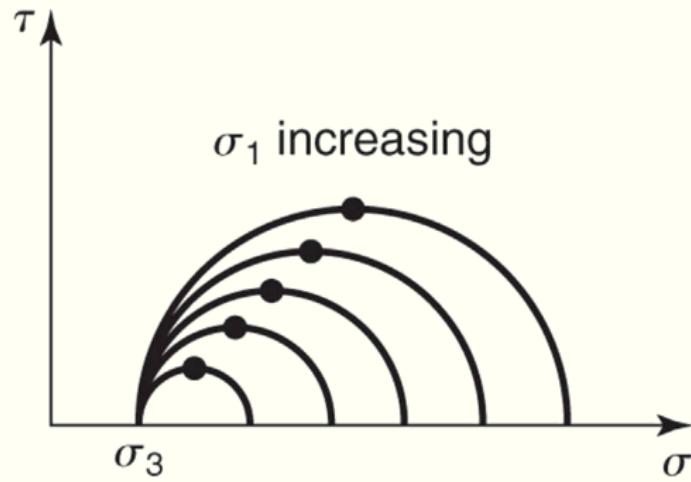
Change in shear  
stress



Deviator strain  
increment

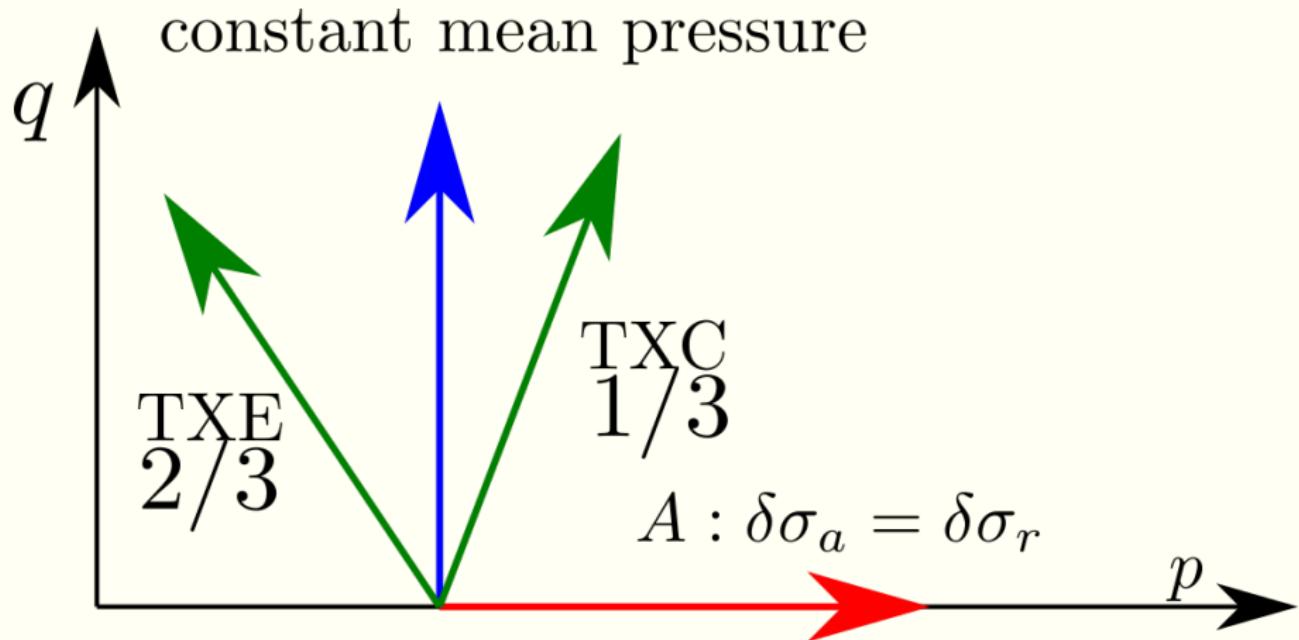
# Mohr circle to 3D stress components

Successive Mohr circles and stress path for constant  $\sigma_3$  and increasing  $\sigma_1$ :



## Stress paths p-q

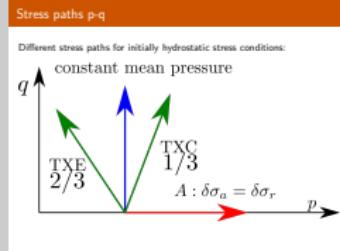
Different stress paths for initially hydrostatic stress conditions:



## CE394M: Stresses - paths &amp; invariants

- └ Stresses / strains in typical geotechnical lab tests

- └ Stress paths p-q



A mean total stress  $p = \frac{\sigma_a + 2\sigma_r}{3}$  and deviatoric stress  $q = \sigma_a - \sigma_r$ . In triaxial compression the cell pressure is held constant,  $\delta\sigma_r = 0$  and axial load is increased.

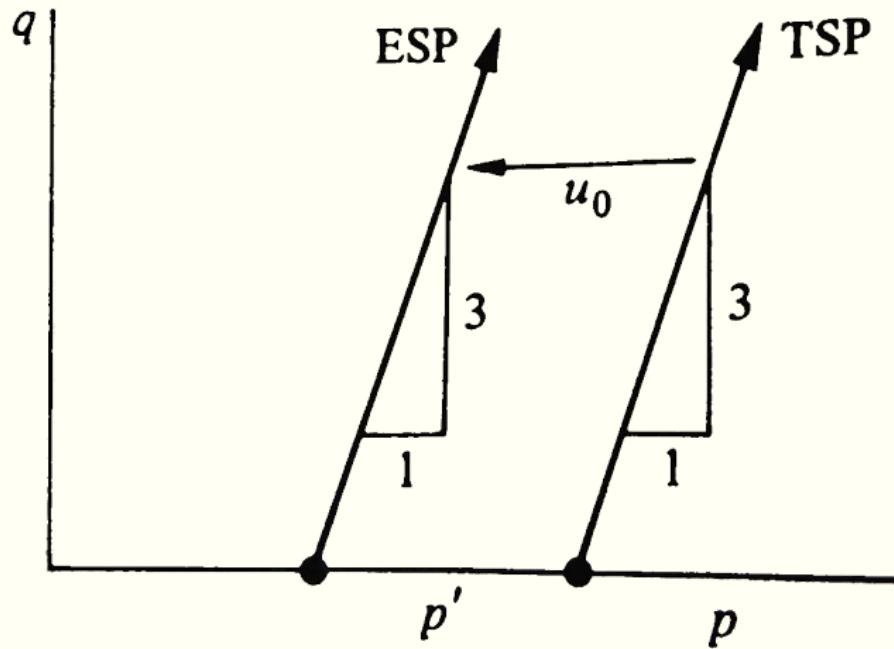
$$\delta p = \frac{\delta q}{3}$$

In triaxial extension, to keep the axial stress constant while changing the cell pressure requires that the ram force (or deviatoric stress) and cell pressure must be changed simultaneously. For  $\delta\sigma_a = 0, \delta\sigma_r = -\delta q$ . The differential form is:

$$\delta p = \frac{2\delta\sigma_r}{3} = \frac{-2\delta q}{3}$$

# Triaxial compression

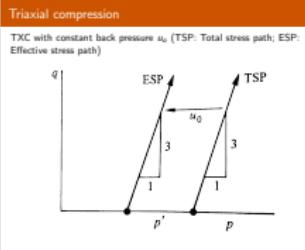
TXC with constant back pressure  $u_0$  (TSP: Total stress path; ESP: Effective stress path)



## CE394M: Stresses - paths &amp; invariants

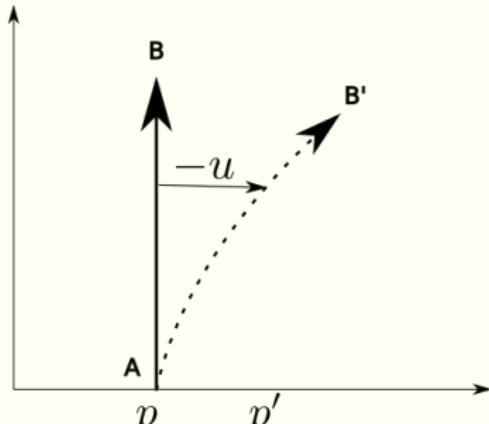
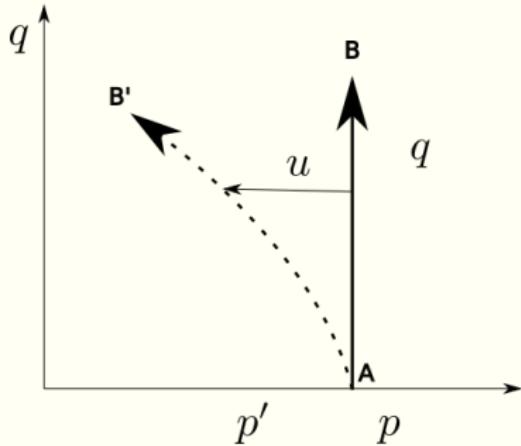
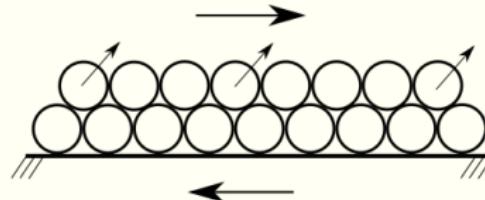
## └ Stresses / strains in typical geotechnical lab tests

## └ Triaxial compression



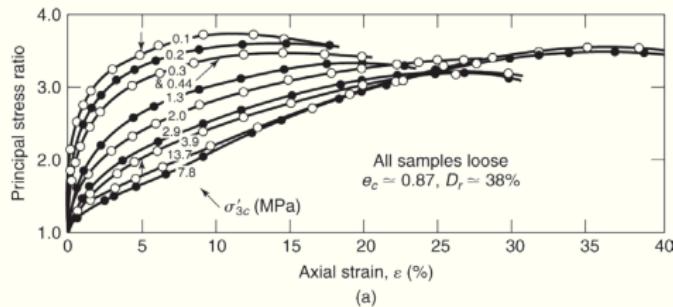
If drainage can occur freely from the soil sample to atmospheric pressure, then the pore pressure will be zero, and total and effective stresses will be identical. The measurement of volume changes of triaxial samples require the measurement of volume of pore fluid flowing into or out of the sample. If part of this pore fluid is emerging as gas (air), then part of the volume change of the sample will not be observed as a change in the level of liquid in a burette. It is often desirable to ensure that the pore fluid is indeed saturated by subjecting the whole pore fluid system to a pressure, called a *back pressure*, which is held constant at a value of  $u_0$  during the test in order to ensure that the gaseous phase remains in solution. Drainage can occur freely, but against this back pressure. In such a test, the total and effective stress paths will be separated by a constant distance  $u_0$ .

## Triaxial compression undrained: loose v dense

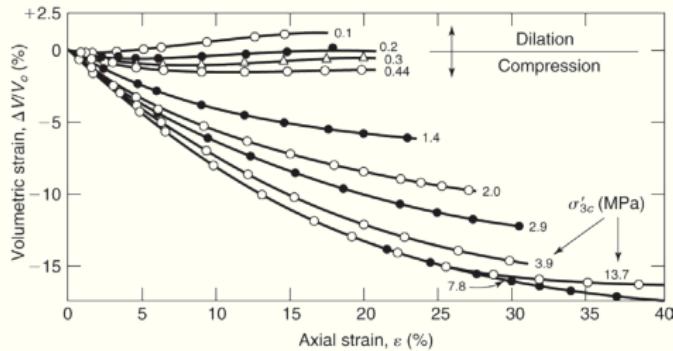


Total and effective stress paths for undrained triaxial test: (a) on soil that wishes to contract as it is sheared, and (b) on soil that wishes to expand as it is sheared.

# Triaxial compression drained: loose

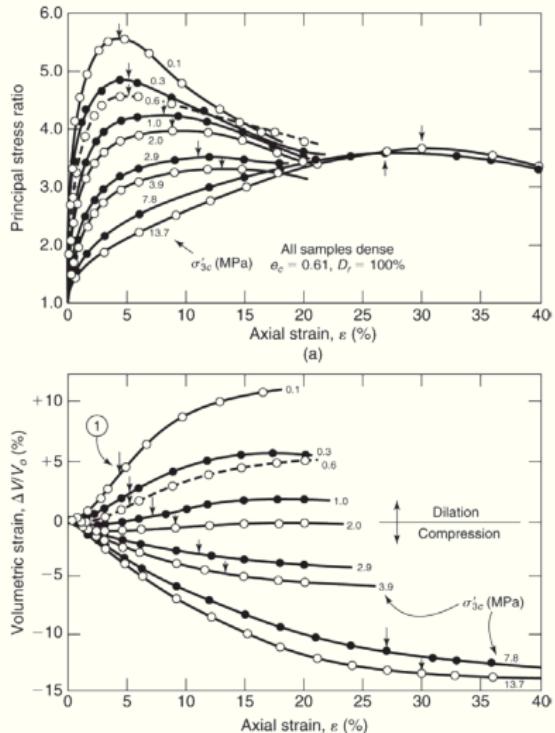


(a)



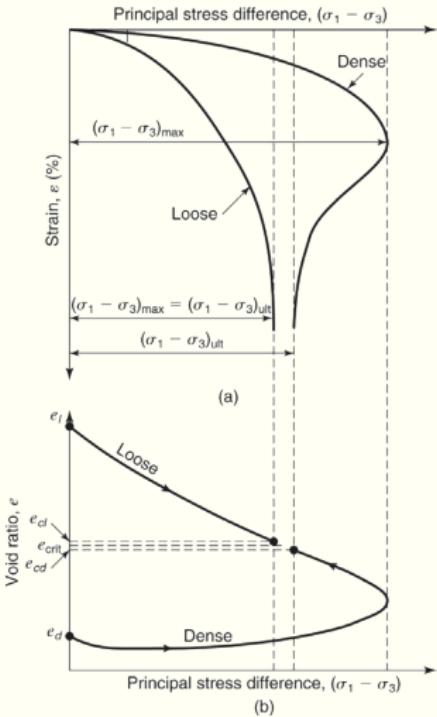
Loose Sacramento River sand: (a) principal stress ratio versus axial strain; (b) volumetric strain versus axial strain (Lee, 1965).

# Triaxial compression drained: dense



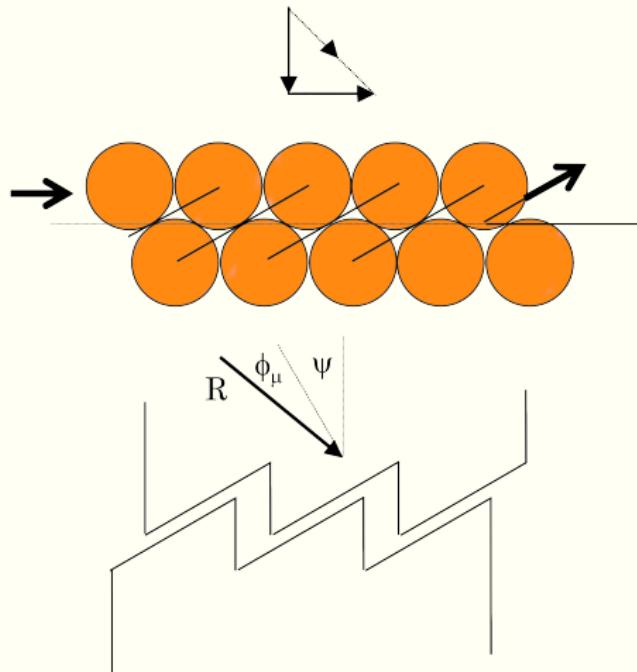
Dense Sacramento River sand: (a) principal stress ratio versus axial strain; (b) volumetric strain versus axial strain (Lee, 1965).

# Triaxial compression drained: loose v dense



Triaxial tests on “loose” and “dense” specimens of a typical sand: (a) stress-strain curves; (b) void ratio changes during shear (Hirschfeld, 1963).

# Friction: Is this correct?



$$\phi_{ss} = \phi_\mu + \psi_{ss}$$

# Discrete Element Method

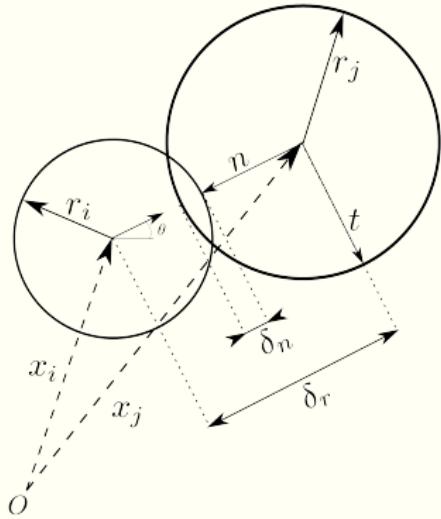
- ① Particle level interaction based on Newton's equation of motion
- ② The contact normal force is computed as:

$$F_n = \begin{cases} 0, & \delta_n > 0 \\ -k_n \delta_n - \gamma_n \frac{d\delta_n}{dt}, & \delta_n < 0 \end{cases}$$

- ③ The contact tangential force is computed in a similar way, but has a frictional limit.

$$F_t \leq \mu F_n$$

- ④ Solve Newton's second law and the angular momentum equation (including rotational resistance).

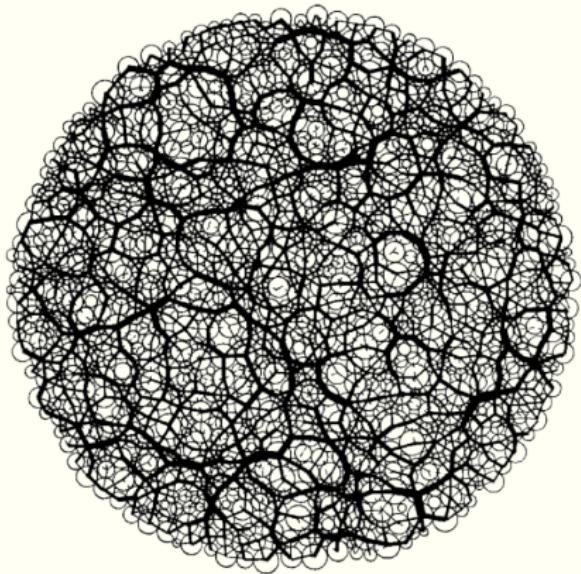


# Interparticle friction angles

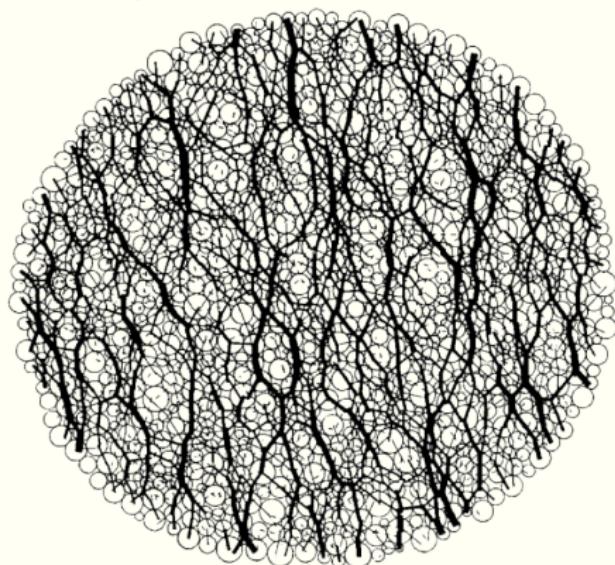
- For Quartz Sands: 26 degrees
- For Sheet Minerals (muscovite, phlogopite, biotite and chlorite): 7 - 13 degrees
  - Water acts as a lubricant
- Clay minerals: Probably 7 - 13 degrees
  - Similar to reported residual friction angles.
  - Sodium Montmorillonite: 4 degrees

# Strong force network vs weak clusters

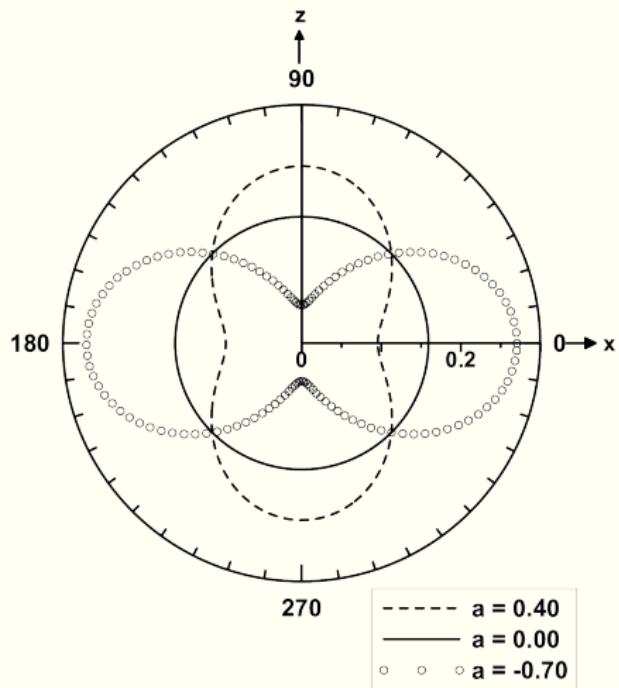
Isotropic Loading



Biaxial Loading

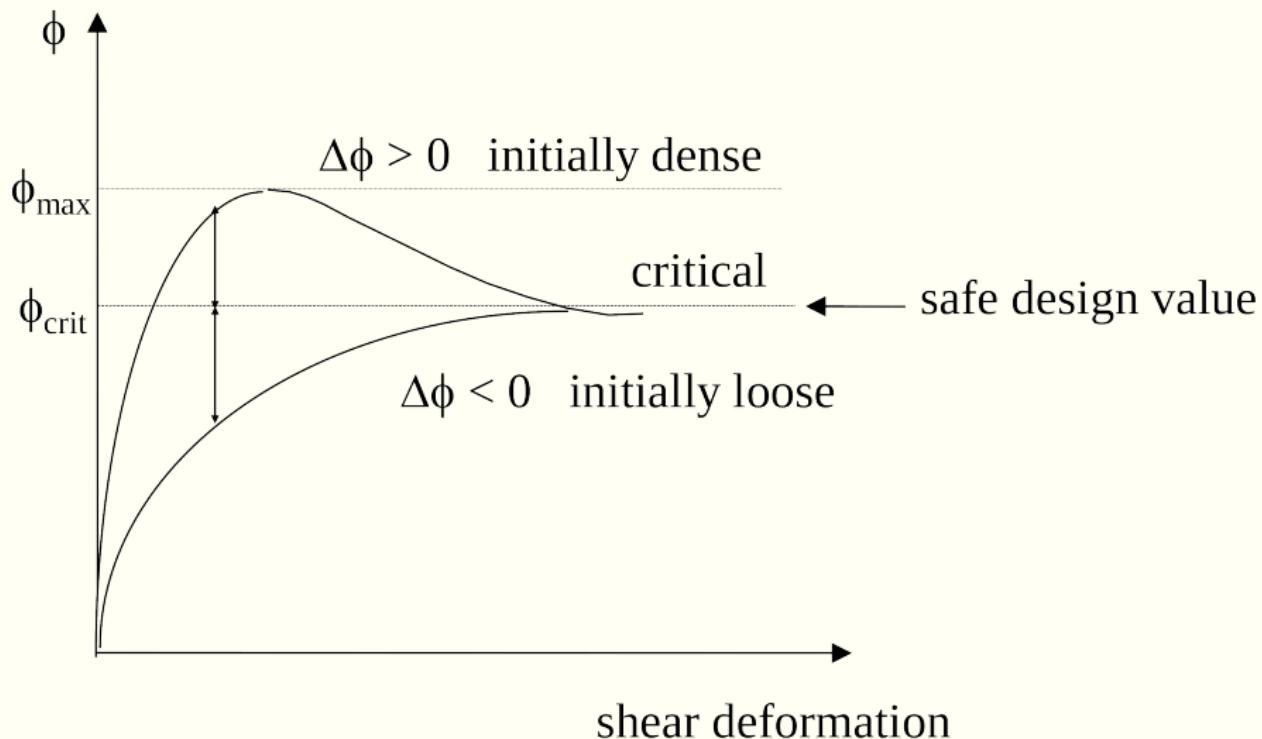


# Fabric anisotropy



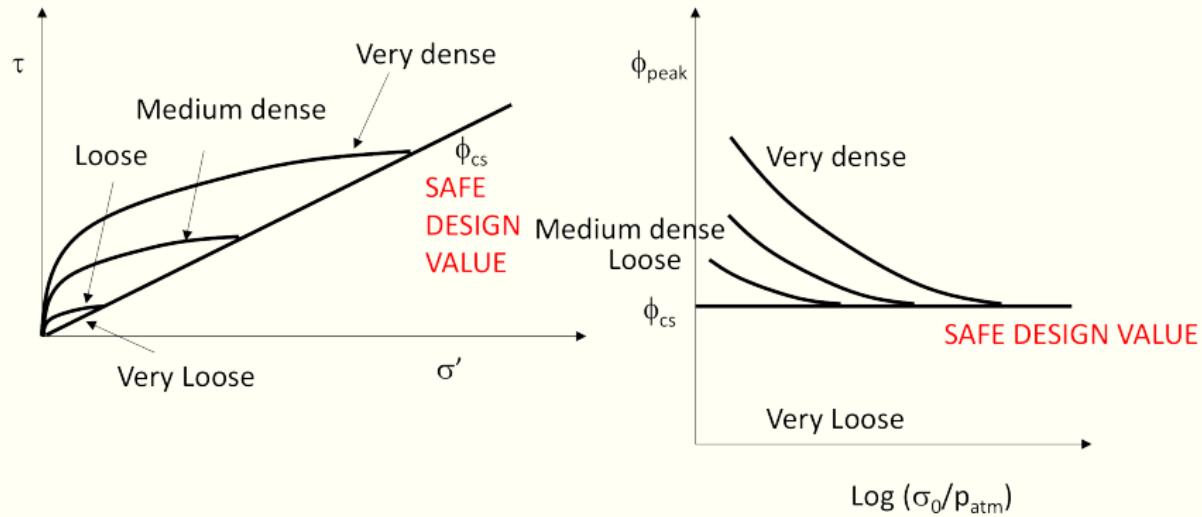
Fabric anisotropy under different stress conditions

Macroscopically, as soil aggregates...



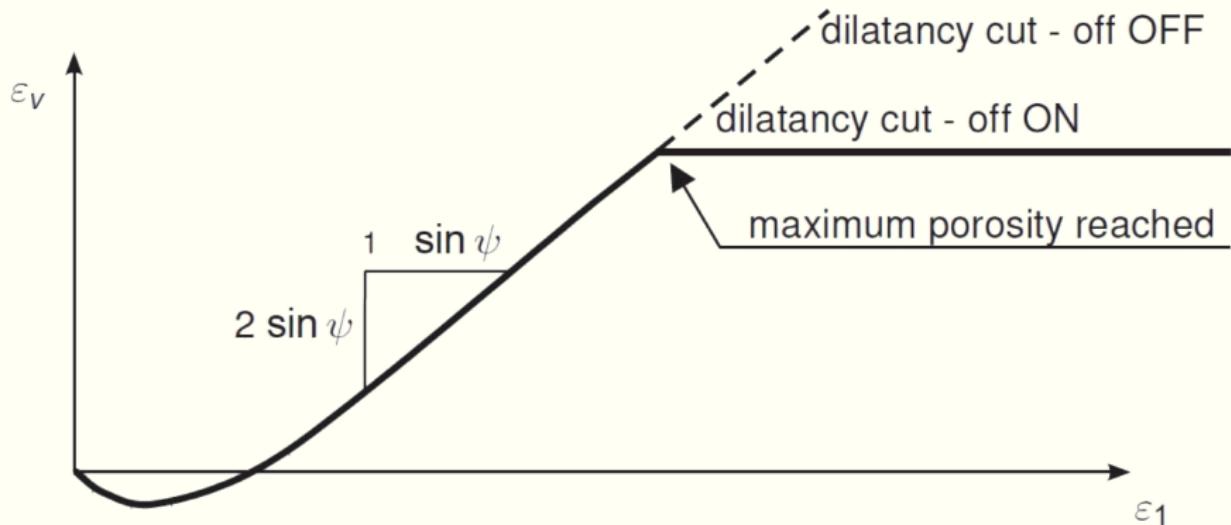
$$\phi = \phi_{cs} + \Delta\phi_{dilatancy}$$

# Critical state friction angle and density



$$\phi_{peak} = \phi_{cs} + \Delta\phi \text{ (confining stress, initial void ratio)}$$

# Dilation angle $\psi$



- ① Simulate CD or CU test and check the volume change behavior (for CD) and excess pore pressure behavior (for CU)!
- ② For CU, if the computed undrained shear strength is greater than the actual value, use Undrained (B).

# Friction angle (Bolton., 1986)

## Friction angle

$$\begin{aligned}\phi_{peak} - \phi_{cs} &= 3I_R \quad \text{for triaxial compression} \\ &= 5I_R \quad \text{for plane strain}\end{aligned}$$

## Relative density index

$$I_R = I_D \cdot I_c - 1$$

$$\text{Relative crushability } I_c = \ln \left( \frac{\sigma_c}{p'} \right)$$

$$\sigma_c \text{ for quartz} = 20 \text{ GPa}$$

$$\text{Relative density } I_D = \frac{e_{max} - e}{e_{max} - e_{min}}$$

## Dilation angle

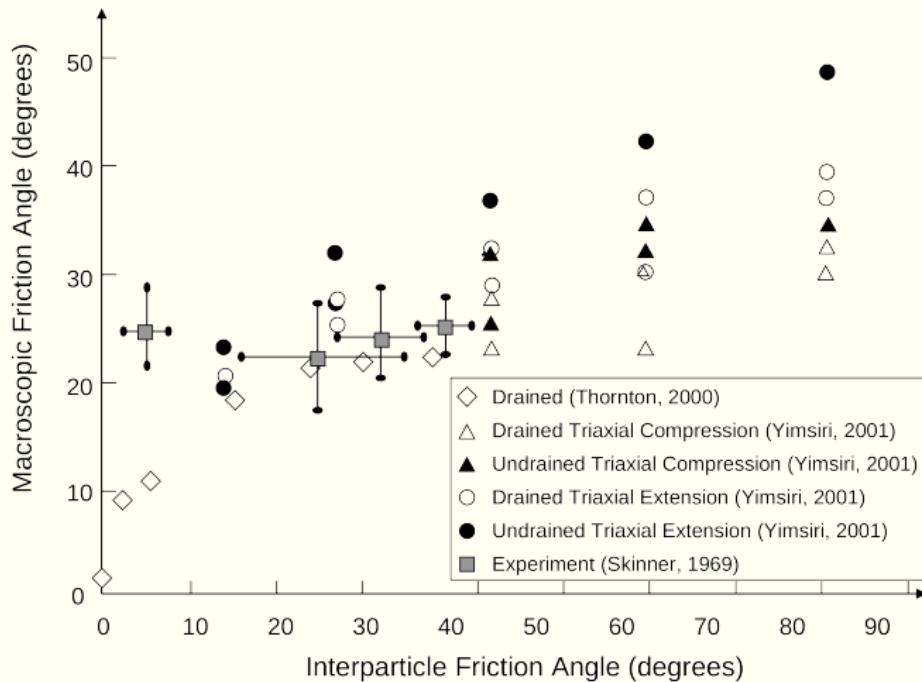
$$\psi \approx 1.25(\phi_{peak} - \phi_{cs})$$

$$\psi = 0 \text{ when } \phi_{peak} = \phi_{cs}$$

# Macroscopic friction angle

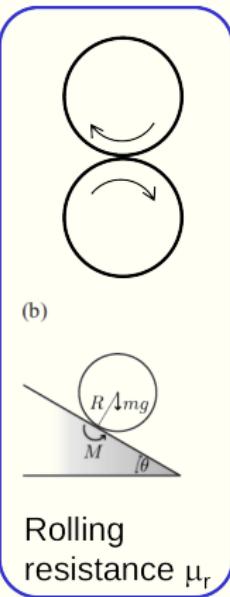
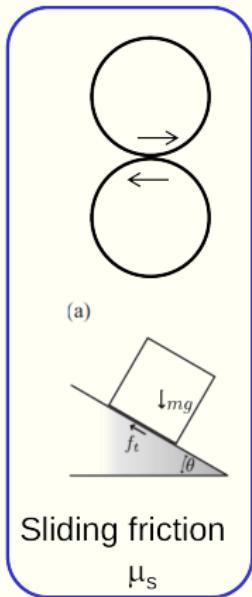
- $\phi_{crit}$  is the angle of friction measured at constant volume of a soil aggregate, and  $\Delta\phi$  dilatancy is the extra dilatant contribution to friction angle  $\phi$ . Typical values are:
- Critical state friction  $\phi_{crit}$ :
  - clay:  $22^\circ$
  - uniform rounded sand:  $32^\circ$
  - well-graded angular sandy gravel:  $38^\circ$
- peak strength of pre-compressed or uncrushable grains, densely compacted, and: shearing in plane strain  $\Delta\phi$ :
  - shearing in plane strain:  $\Delta\phi_{max} = 20^\circ$
  - shearing in axial symmetry:  $\Delta\phi_{max} = 12^\circ$

# Micro to Macroscopic friction angle



Relationship between macroscopic friction angle and interparticle friction angle (no rolling resistance) - Yimisir and Soga (2001)

# Micro to Macroscopic friction angle: Rolling resistance

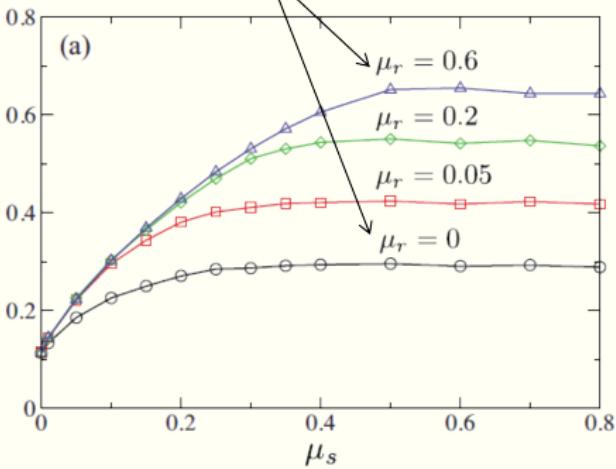


Sliding friction  
 $\mu_s$

Rolling  
resistance  $\mu_r$

Macroscopic friction angle  
 $\mu^* = \tan \phi_{\text{crit}}$

Different rolling resistances

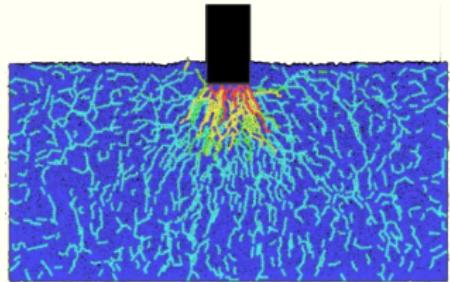


Microscopic sliding friction at  
particle contacts     $\mu_s = \tan \phi_\mu$

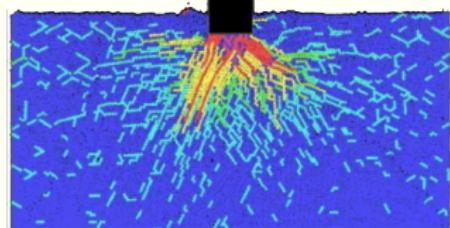
Estrada et al., (2001)

# Interparticle friction angles

- The interparticle friction acts as a kinematic constraint of the strong force network and not as the direct source of macroscopic resistance to shear.
- Increased friction at the contacts increases the stability of the system (development of anisotropic fabric) and reduces the number of contacts required to achieve a stable condition.
- As long as the strong force network can be formed, the magnitude of the interparticle friction becomes of secondary importance.



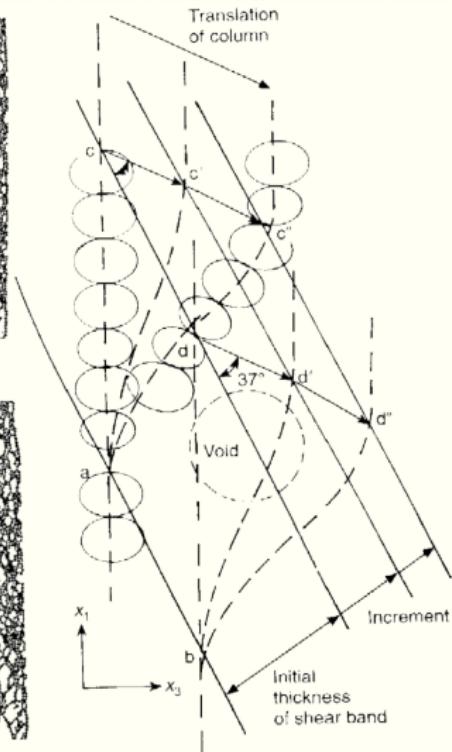
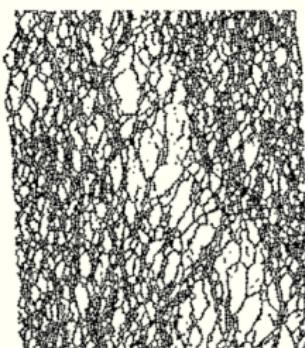
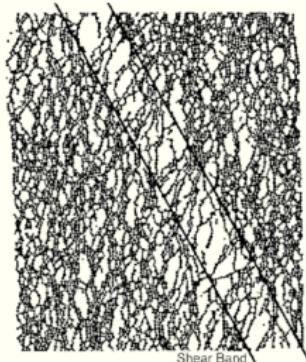
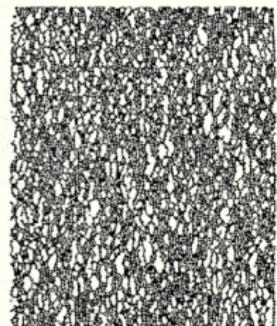
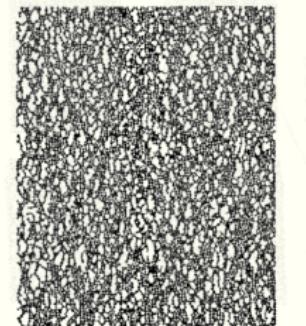
Loose



Dense

Muthuswamy and  
Tordesillas (2006)

# What is dilation? shear band



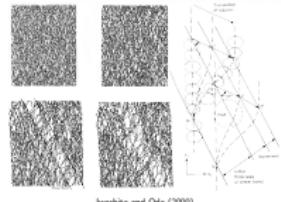
Iwashita and Oda (2000)

## CE394M: Stresses - paths &amp; invariants

## └ Friction

## └ What is dilation? shear band

What is dilation? shear band



Kuhn (1999) reports that their thicknesses are  $1.5D_{50}$  to  $2.5D_{50}$  in the early stages of shearing and increase to between  $1.5D_{50}$  and  $4D_{50}$  as deformation proceeds.

# Fabric evolution at critical state

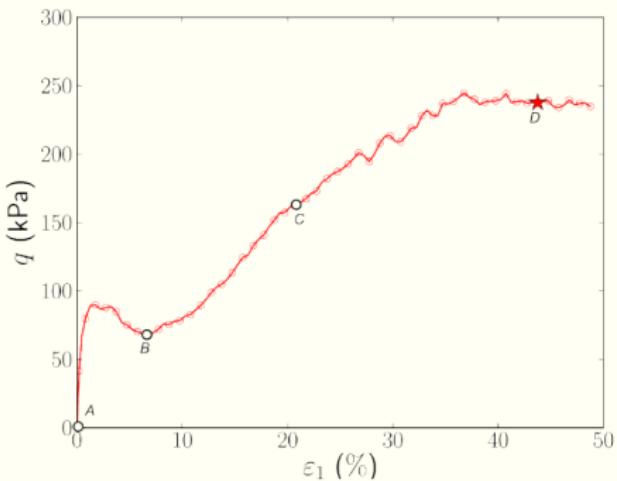


Fig.1 Undrained shear response of a medium dense sand and four stress states selected for examination of internal structure.

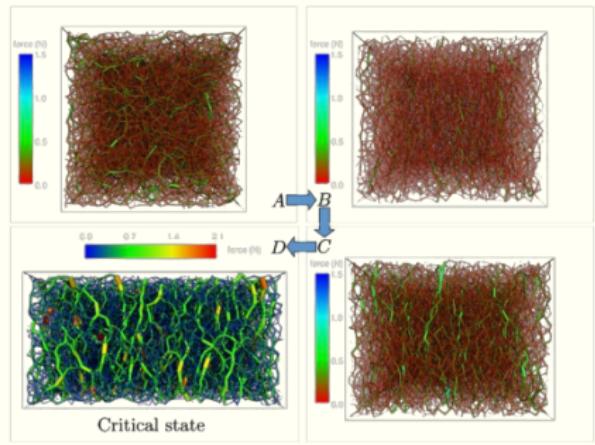


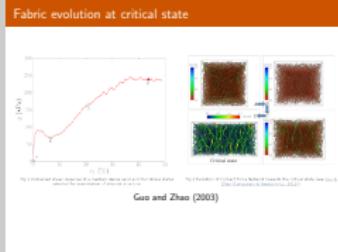
Fig.2 Evolution of Contact Force Network towards the critical state (see [Guo & Zhao \[Computers & Geotechnics, 2013\]](#)).

Guo and Zhao (2003)

## CE394M: Stresses - paths &amp; invariants

## └ Friction

## └ Fabric evolution at critical state



As deformation progresses,

- The number of particles in the strong force network decreases.
- With fewer particles sharing the increased loads.
- Anisotropic fabric develops, showing the formation of strong force network. Fabric of particles associated with strong forces is different from that associated with weak clusters.
- At critical state, force chain forms and buckles continuously. Likely to buckle when a force chain has 8 particles

# Stress and strain invariants in 3D

## 6 stresses and strains

- Mean pressure:  $p' = (\sigma'_{11} + \sigma'_{22} + \sigma'_{33})/3$
- Deviator stress:  $q = \sqrt{3/2} \sqrt{s_{11}^2 + s_{22}^2 + s_{33}^2 + 2\tau_{12}^2 + 2\tau_{23}^2 + 2\tau_{31}^2}$
- where  $s_{11} = \sigma'_{11} - p'$ ,  $s_{22} = \sigma'_{22} - p'$ ,  $s_{33} = \sigma'_{33} - p'$
- Volumetric strain:  $\varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$
- Deviatoric strain:  
$$\varepsilon_s = \sqrt{2/3} \sqrt{e_{11}^2 + e_{22}^2 + e_{33}^2 + \gamma_{12}^2/2 + \gamma_{23}^2/2 + \gamma_{31}^2/2}$$
- where  $e_{11} = \varepsilon_{11} - \varepsilon_v/3$ ,  $e_{22} = \varepsilon_{22} - \varepsilon_v/3$ ,  $e_{33} = \varepsilon_{33} - \varepsilon_v/3$

# Stress and strain invariants in 3D

## 3 Principal stresses and strains

- Mean pressure:  $p' = (\sigma'_I + \sigma'_{II} + \sigma'_{III})/3$
- Deviator stress:  $q = \sqrt{1/2} \sqrt{(\sigma'_I - \sigma'_{II})^2 + (\sigma'_{II} - \sigma'_{III})^2 + (\sigma'_{III} - \sigma'_I)^2}$
- Volumetric strain:  $\varepsilon_v = \varepsilon_I + \varepsilon_{II} + \varepsilon_{III}$
- Deviatoric strain:  $\varepsilon_s = \frac{\sqrt{2}}{3} \sqrt{(\varepsilon'_I - \varepsilon'_{II})^2 + (\varepsilon'_{II} - \varepsilon'_{III})^2 + (\varepsilon'_{III} - \varepsilon'_I)^2}$

### In triaxial condition (principal stresses/strains)

$$p' = (\sigma_I + 2\sigma_{III})/3 \quad q = \sigma_I - \sigma_{III} \quad \varepsilon_v = \varepsilon_I + 2\varepsilon_{III} \quad \varepsilon_s = 2(\varepsilon_I - \varepsilon_{III})/3$$

## CE394M: Stresses - paths &amp; invariants

## └ Stress invariants

## └ Stress and strain invariants in 3D

## 3 Principal stresses and strains

- Mean pressure:  $p' = (\sigma'_x + \sigma'_y + \sigma'_z)/3$

- Deviator stress:  $q = \sqrt{1/2} \sqrt{(\sigma'_x - \sigma'_y)^2 + (\sigma'_y - \sigma'_z)^2 + (\sigma'_z - \sigma'_x)^2}$

- Volumetric strain:  $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$

- Deviatoric strain:  $\varepsilon_d = \frac{1}{2} \sqrt{(\varepsilon'_x - \varepsilon'_y)^2 + (\varepsilon'_y - \varepsilon'_z)^2 + (\varepsilon'_z - \varepsilon'_x)^2}$

In triaxial condition (principal stresses/strains)

$$\rho' = (\sigma_1 + 2\sigma_{23})/3 \quad q = \sigma_1 - \sigma_{23} \quad \varepsilon_v = \varepsilon_1 + 2\varepsilon_{23} \quad \varepsilon_d = 2(\varepsilon_1 - \varepsilon_{23})/3$$

The magnitudes of the components of the stress vector (i.e.  $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}$ ) depend on the chosen direction of the coordinate axes. The principal stresses ( $\sigma_I, \sigma_{II}, \text{ and } \sigma_{III}$ ) however, always act on the same planes and have the same magnitude, no matter which direction is chosen for the coordinate axes. They are therefore invariant to the choice of axes. Consequently, the state of stress can be fully defined by either specifying the six component values for a fixed direction of the coordinate axis, or by specifying the magnitude of the principal stresses and the directions of the three planes on which these principal stresses act. In either case six independent pieces of information are required.

# The missing stress invariant

- **General stresses:**  $\sigma_{ij}$  ( $i, j = 1, 2, 3$ )     $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}$



- **Principal stresses:**  $\sigma_I, \sigma_{II}, \sigma_{III}$  + 3 angles



Two stress invariant (mean pressure  $p$  or  $s$ , deviator stress  $q$  or  $t$ ). One more stress invariant is needed to go back to 3 principal stresses: **Lode angle!**

# Stress invariant

Principal stresses (3 components)  $\Leftrightarrow$  Invariants (3 components)

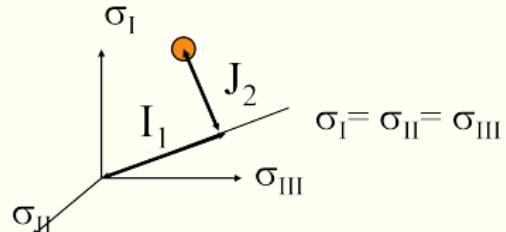
- $I_1 = p' = \sigma'_{ii}/3$

- $s_{ij} = \sigma'_{ij} - \delta_{ij}p'$

- $J_2 = \frac{1}{2}s_{ij}s_{ij}$

- $J_2 = \frac{1}{2}s_{ij}s_{ij}$

- $q = \sqrt{3/2}(s_{ij}s_{ij})^{0.5} = \sqrt{3}J_2$



Lode angle:

- $\sin 3\theta = \frac{-3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$ .

- $\theta = \tan^{-1} \left[ \frac{1}{\sqrt{3}} \left( 2 \frac{(\sigma_2 - \sigma_3)}{(\sigma_1 - \sigma_3)} - 1 \right) \right]$

- $TXC = \pi/6 \quad TXE = -\pi/6$

- $SHR = 0$ , when

$$\sigma_2 = (\sigma_1 + \sigma_3)/2 \text{ (note } SHR \neq PS\text{)}$$

## CE394M: Stresses - paths &amp; invariants

## └ Stress invariants

## └ Stress invariant

## Stress invariant

Principal stresses (3 components)  $\Leftrightarrow$  Invariants (3 components)

- $k_1 = p' = \sigma_{zz}'/3$
- $k_2 = \sigma_{yy}' - \delta_{yz}\sigma_z'$
- $J_3 = \frac{1}{2}\sigma_y\sigma_z$
- $J_2 = \frac{1}{2}\sigma_y^2 - \frac{1}{2}\sigma_z^2$
- $q = \sqrt{3/2}(\lambda/\lambda_0)^{0.5} = \sqrt{3}J_2$

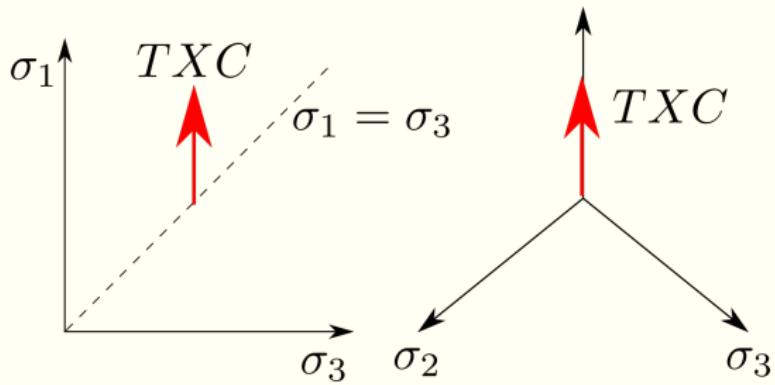
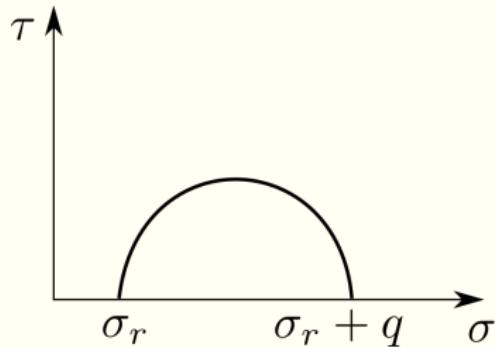
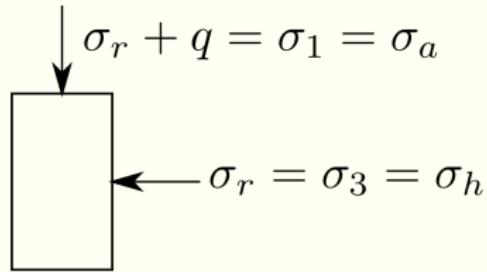


## Lode angle:

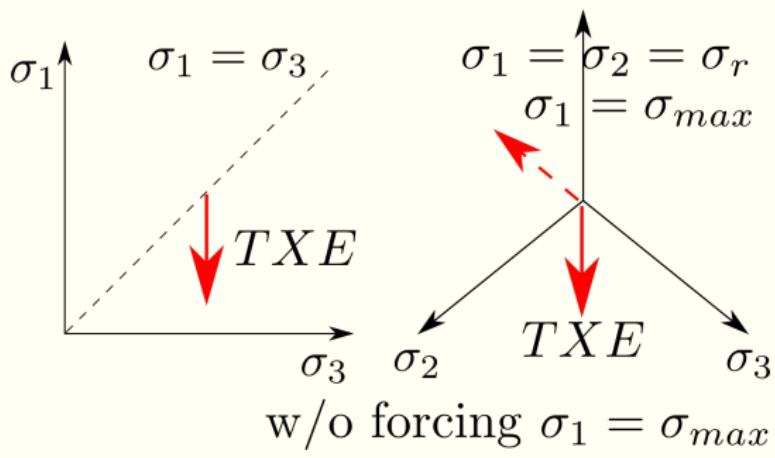
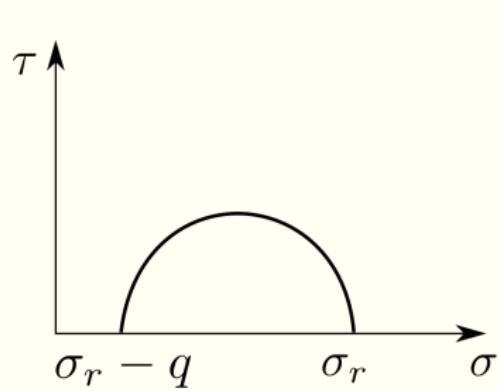
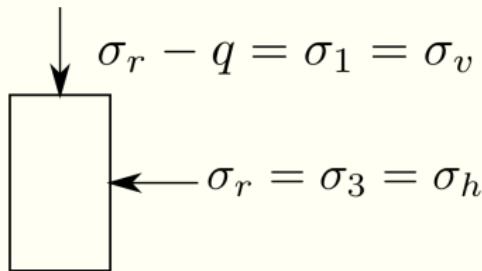
- $\sin 3\theta = \frac{-\lambda\sqrt{3}}{J_2} \frac{\partial \sigma_z}{\partial \sigma_y}$
- $\theta = \tan^{-1} \left[ \frac{1}{\sqrt{3}} \left( 2 \frac{(\sigma_y - \sigma_z)}{(\sigma_y + \sigma_z)} - 1 \right) \right]$
- $TXC = \pi/6$ ,  $TXE = -\pi/6$
- $SHR = 0$ , when  $\sigma_2 = (\sigma_1 + \sigma_3)/2$  (note  $SHR \neq PS$ )

If one is interested in the overall magnitude of the stress, all three principal stresses would be needed, but not the directions of the planes on which they act. In geotechnical engineering it is often convenient to work with alternative invariant quantities which are combinations of the principal effective stresses.

# $\pi$ plane: Triaxial compression



## $\pi$ plane: Triaxial extension



# $\pi$ plane: Random stress paths

