CE394M: An introduction to plasticity

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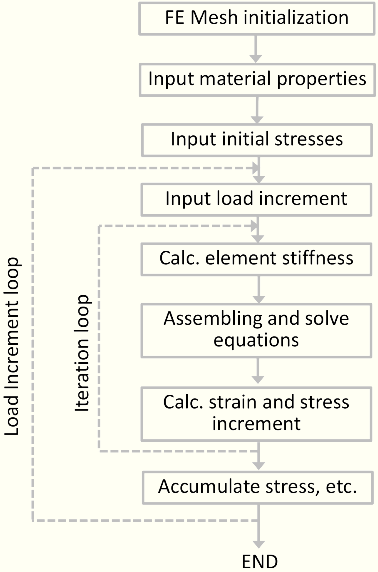
FE workflow



$\boldsymbol{\sigma}$

$\Delta \boldsymbol{f}$

$$K_{ij} \approx \sum_{p}^{n_p} B_i^p D^p B_j^p W_p$$

$\Delta \boldsymbol{u} = \boldsymbol{K}^{-1} \cdot \Delta \boldsymbol{f}$ **Equilibrium eq.**

$\Delta \boldsymbol{\varepsilon} = \boldsymbol{B} \cdot \Delta \boldsymbol{u}$ **Compatibility eq.**

$\Delta \boldsymbol{\sigma} = \boldsymbol{D} \cdot \Delta \boldsymbol{\varepsilon}$ **Constitutive eq.**

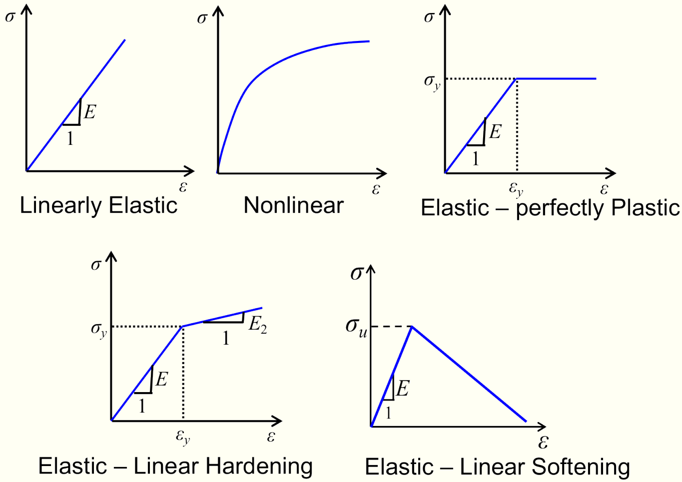
$$\boldsymbol{u} = \boldsymbol{u} + \Delta \boldsymbol{u}$$
$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} + \Delta \boldsymbol{\varepsilon}$$
$$\boldsymbol{\sigma} = \boldsymbol{\sigma} + \Delta \boldsymbol{\sigma}$$

Overview

1. Constitutive modeling
2. Classical plasticity
   - Equations of plasticity

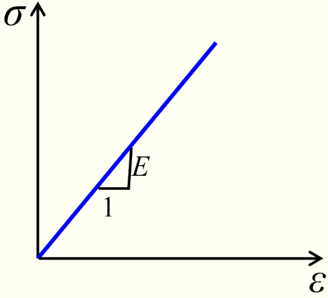
Constitutive law

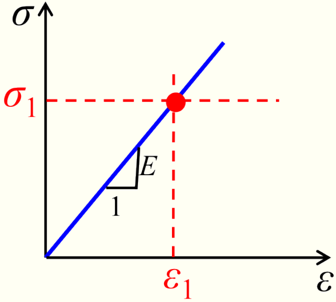
Constitutive law is the stress-strain relationship: $\sigma = f(\varepsilon) \rightarrow \sigma = \mathbf{D} \cdot \varepsilon$



Isotropic linear elasticity

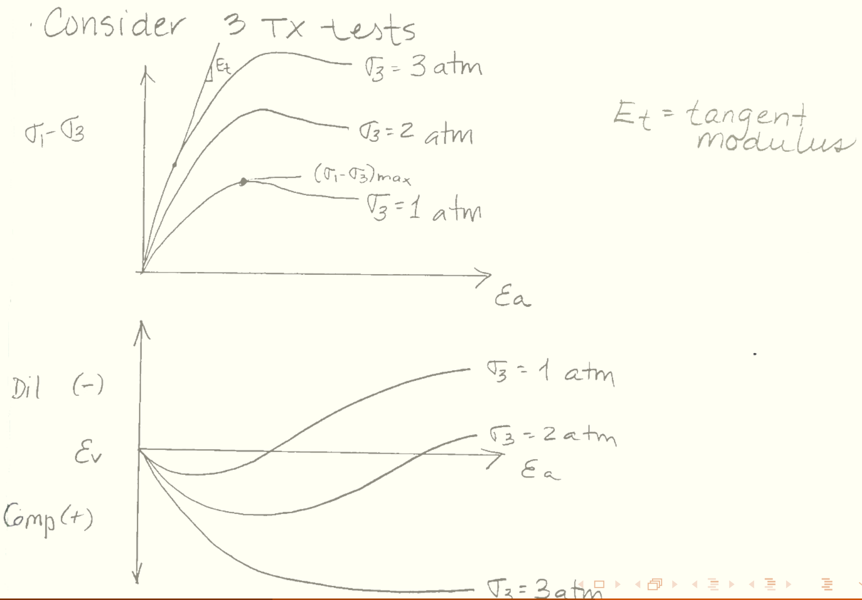
$\sigma = \mathbf{D^{el}} \cdot \varepsilon$ where $\mathbf{D^{el}}$ is the elastic stiffness matrix.

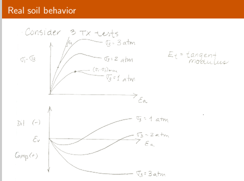




The knowledge of strain alone allows us to obtain the stress value.

Real soil behavior





We would like our model to capture

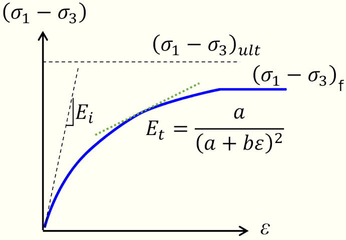
1. $(\sigma_1 - \sigma_3)_{max}$ increases with increasing σ_3.
2. E_t increases with increasing σ_3.
3. E_t decreases with increasing ε_a.
4. ε_v increasses with increasing σ_3.

Could also consider: soil softening, dilation, rate effects... Duncan model captures 1 - 4.

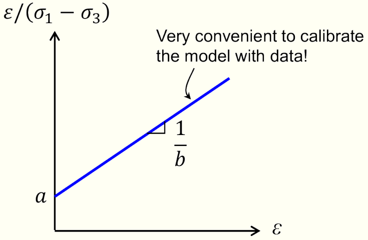
Hyperbolic model (Duncan and Chang., 1970)

$$(\sigma_1 - \sigma_3) = \frac{\varepsilon}{a + b\varepsilon}$$

$$a = \frac{1}{E_i} \quad b = \frac{1}{(\sigma_1 - \sigma_3)_{ult}}$$

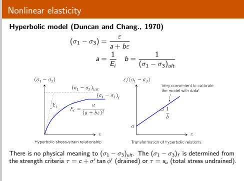


Hyperbolic stress-strain relationship



Transformation of hyperbolic relations

There is no physical meaning to $(\sigma_1 - \sigma_3)_{ult}$. The $(\sigma_1 - \sigma_3)_f$ is determined from the strength criteria $\tau = c + \sigma' \tan\phi'$ (drained) or $\tau = s_u$ (total stress undrained).



R_f is a curve fitting parameter to control ε_f.

$R_f = \frac{(\sigma_1 - \sigma_3)_f}{(\sigma_1 - \sigma_3)_{ULT}}$

As $R_f \to 1$ curve is more hyperbolic, more ductile and ε_f increases.

Lower R_f, more brittle. ε_f decreases.

For most soils, $R_f = 0.6 - 0.95$ with a first guess of 0.8.

Drawbacks of hyperbolic model

Positive aspects

- Relatively simple and easy to use
- Well validated & calibrated
- Based solely on TX testing
- Good for loading conditions (dams)

Original short comings

- Not an effective stress model (no treatment of Δu) - Plaxis can do
- No strain softening (still a problem)
- No dilation - important for dense sand / OC clay (Plaxis can do)
- Empirical model
- Poor performance in excavation problems (where it is linear elastic completely)

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- exceptions: excavations where base-heave is the mode of failure (loading of soil at base).

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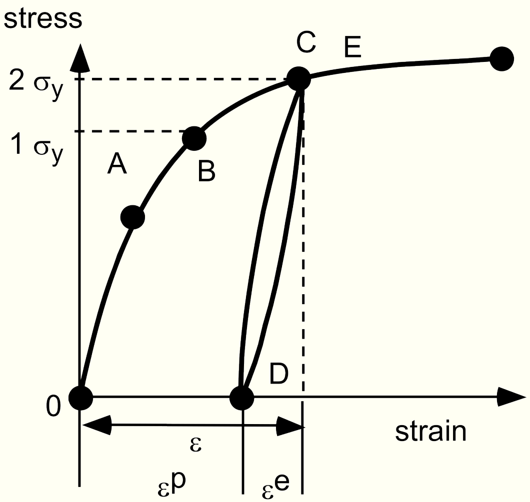
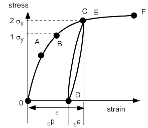
Classical plasticity



Specimen 1 at point 0 same as specimen 2 at point D, but yield stress of specimen 2 is greater than the yield stress of specimen 1 due to **plastic hardening**.

Classical plasticity



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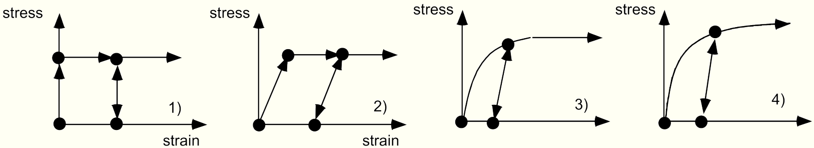
Uniaxial tension test on metal bar:

O→A	Linear elastic, reversible
A→B	Nonlinear elastic, reversible
B	Starts to yield
B→C	Nonlinear elasto-plastic, irreversible
C→D	Elastic with hysteresis
C→F	Nonlinear elasto-plastic, irreversible
F	Peak stress at failure

$\varepsilon = \varepsilon^e + \varepsilon^p$, where e is elastic and p is plastic.

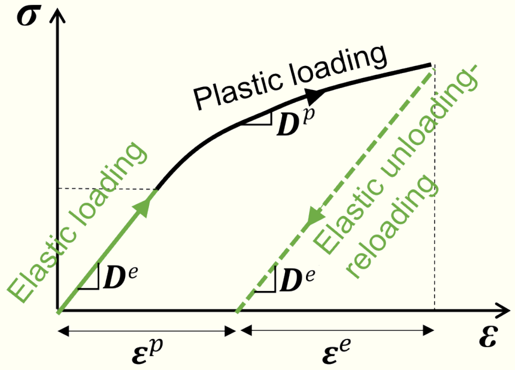
Classical plasticity

- **Plastic behavior**: The direction of plastic strains is governed by the current stress state σ. $d\sigma = f(d\varepsilon) \rightarrow d\sigma = \mathbf{D}^e \cdot d\varepsilon$.
- **Elastic behavior**: The direction of elastic strains is governed by the stress state increment $\delta\sigma$ direction.
- **Plastic models**
    - Rigid - perfectly plastic model - used in static limit equilibrium analysis (no elastic strain and no strain hardening / softening)
    - Linear elastic - perfectly plastic model (Drucker-Prager and Mohr Coulomb models)
    - Hybrid model (nonlinear elastic with perfectly plastic - Duncan and Chang)
    - Work (or strain) hardening plasticity model (Cam-Clay model)



Elasto-plastic materials

Main distinctive feature of elasto-plastic materials: "*irreversibility*" $\rightarrow$ **Plastic deformation** ε^p



$$\varepsilon = \varepsilon^e + \varepsilon^p \quad d\varepsilon = d\varepsilon^e + d\varepsilon^p$$

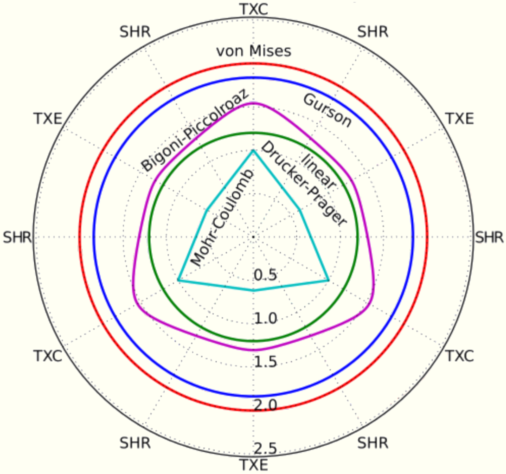
Basic concepts of classical plasticity

To formulate an elasto-plastic constitutive model we need:

1. **Elastic stress-strain relationship:** $\sigma = \mathbf{D}^e \varepsilon^e = \mathbf{D}^e(\varepsilon - \varepsilon^p)$. Describe elastic response.
2. **Yield function**: defines the condition for the onset of plastic strain. Depends on the stress state σ and state parameters (e.g., in MC they are cohesion and friction angle). $F(\sigma', Wp) = 0$.
3. **Plastic potential ($G(\sigma, Wp) = 0$)** defines the direction of plastic strains. Depends on stress state σ and state parameter (for e.g., is dilatancy in MC). Note that the direction of $d\varepsilon^p$ doesn't depend on $d\sigma$ but on the actual stress state σ. **Flow rule** $\varepsilon^p = \lambda(dG/d\sigma)$.
4. **Hardening rule / Hardening law (h)** defines how F changes with plastic strains. Yield function $F = f(\text{stress state}, W_p)$, where W_p is a function of plastic strains. Describes the evolution of state parameters depending on the plastic strain ε^p.

Yield functions

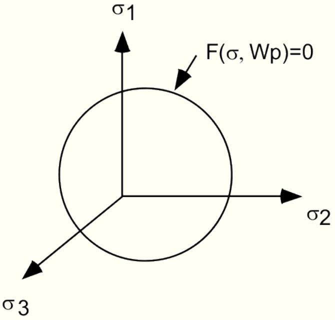
defines when plastic strains occur. If the material is isotropic, we can use the principal stresses to define the stress state.



Wikipedia

Yield functions

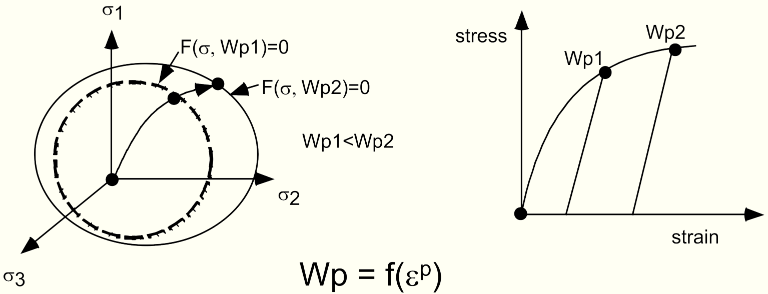
Yield function $F(\sigma, Wp) = 0$.



- if $F = 0$ under loading: yielding and plastic strains and in unloading: elastic strains.
- if $F < 0$ elastic domain.
- if $F > 0$ impossible.

Hardening law

How the threshold of yielding changes with plastic strain or how the yield function changes with plastic strain.

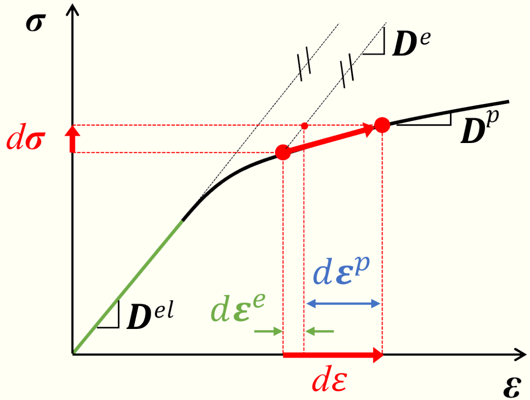


$$Wp = f(\varepsilon^p)$$

Describes the incremental stress-strain relationship.

$$d\sigma = D^e \cdot d\varepsilon^e = D^e \cdot (d\varepsilon - d\varepsilon^p)$$

Where D^e is the elastic stiffness matrix. ^{e} denotes the elastic part.



Equations of elasto-plasticity: 2. Flow rule

- Specifies the direction of plastic strain increments at every yield stress state. It is very important because it controls the ratio of the volumetric and deviatoric components (e.g., dilatancy of the material).
- States that the plastic strain increments ($d\varepsilon^p$) are normal to the plastic potential surface (G).

$$d\varepsilon^p = d\lambda \cdot \vec{P}$$

  Note: typically $\vec{P}$ is **not** a unit vector, so $\vec{P}$ also controls the magnitude of $d\vec{\varepsilon_p}$ (in addition to the direction).
- In many cases, $\vec{P}$ is chosen as the gradient of a function g (if it exists) such that:

$$\vec{P} = \frac{\partial G}{\partial \sigma}$$

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This plastic flow rule was based on the observation by de Saint-Venant (1870) that for metals the principal axes of the plastic strain rate coincide with those of the stress. This is the so-called coaxial assumption, which has been the foundation of almost all the plasticity models used in engineering

Drucker's postulate

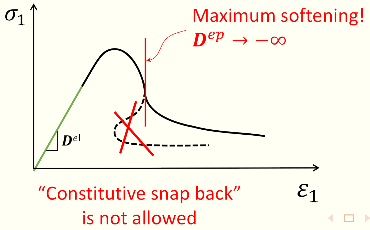
Drucker (1952) established that for a stable inelastic material in a closed stress cycle, a positive work must be done.

$$dW^{plastic} > 0 \rightarrow d\sigma \cdot d\varepsilon^{p} > 0$$

This requirement is satisfied if the normality condition is assumed (but is not the only solution). The gradient of the yield surface (i.e., normal to the surface)

$$\vec{Q} = \frac{\partial F}{\partial \sigma} = \vec{P}$$

It also imposes a constraint that the yield surface must be convex.



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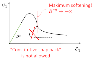
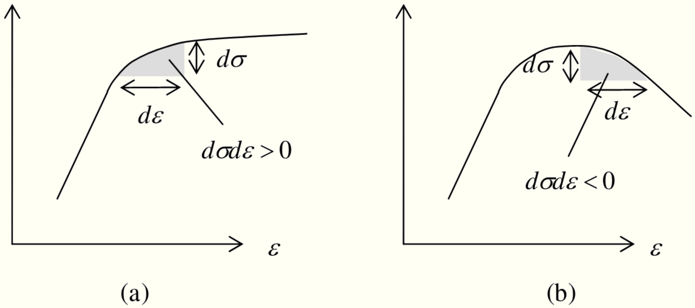


The normality rule has been confirmed by many experiments on metals. However, it is found to be seriously in error for soils and rocks, where, for example, it overestimates plastic volume expansion.

$$Q = \frac{\partial F}{\partial \sigma}$$

$$P = \frac{\partial G}{\partial \sigma}$$

Drucker's postulate of a stable material

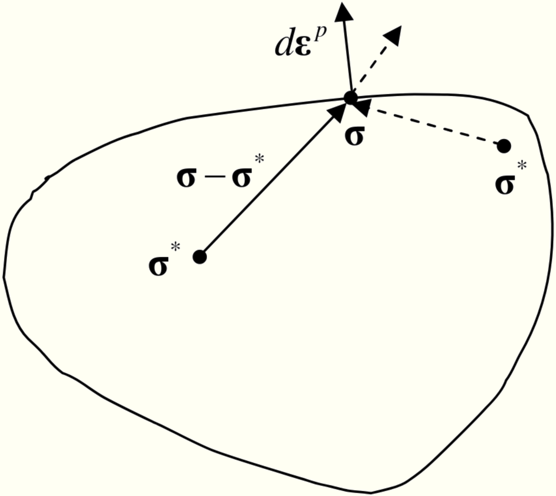


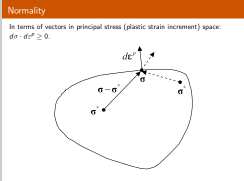
1. Positive work is done by the external agency during the application of the loads
2. The net work performed by the external agency over a stress cycle is nonnegative By this definition, it is clear that a strain hardening material is stable (and satisfies Drucker's postulates).

Note that plastic loading of a softening (or perfectly plastic) material results in a non-positive work

Normality

In terms of vectors in principal stress (plastic strain increment) space: $d\sigma \cdot d\varepsilon^p \geq 0$.



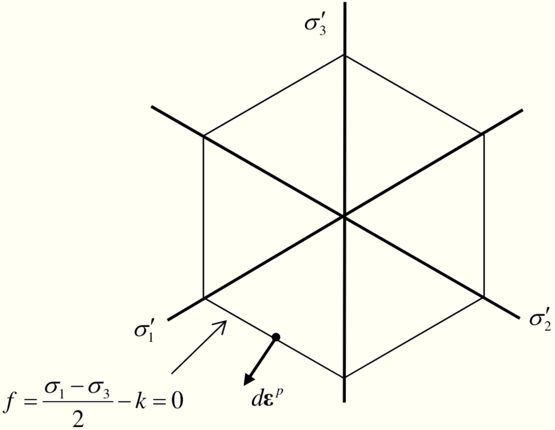


Since the dot product of $d\sigma$ and $d\varepsilon_p$ is non-negative, the angle between the vectors $d\sigma \& d\varepsilon^p$ (with their starting points coincident) must be less than 90^o. This implies that the plastic strain increment vector must be normal to the yield surface since, if it were not, an initial stress state σ^* could be found for which the angle was greater than 90 (as with the dotted vectors in Fig.). Thus a consequence of a material satisfying the stability requirements is that the normality rule holds, i.e. the flow rule is associative

Normality rule: the plastic strain increment vector is normal to the yield surface.

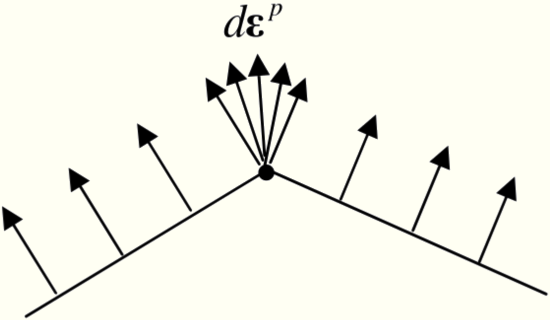
The normality rule has been confirmed by many experiments on metals. However, it is found to be seriously in error for soils and rocks, where, for example, it overestimates plastic volume expansion. For these materials, one must use a non-associative flow-rule.

Normality in Tresca condition



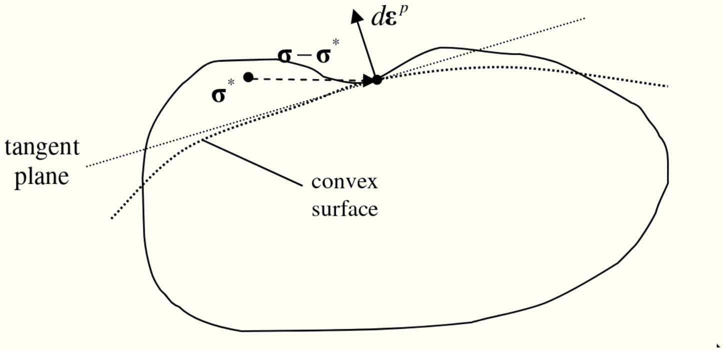
The plastic strain increment vector and the Tresca criterion in the pi-plane (for the associated flow-rule).

Normality in Tresca condition



When the yield surface has sharp corners, as with the Tresca criterion, it can be shown that the plastic strain increment vector must lie within the cone bounded by the normals on either side of the corner

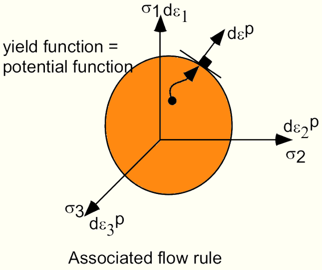
Normality in non-convex surfaces



Using the same arguments, one cannot have a yield surface like the one shown in Fig. In other words, the yield surface is convex: the entire elastic region lies to one side of the tangent plane to the yield surface

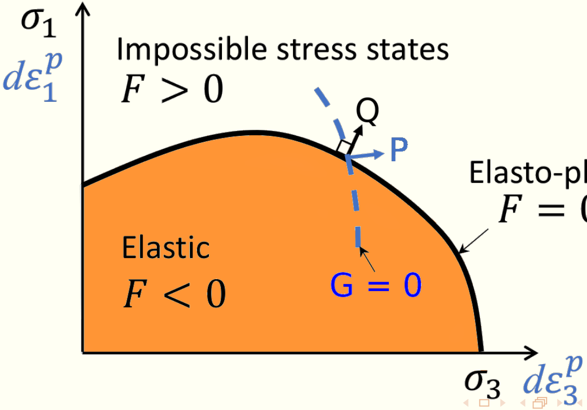
Associated and non-associated plasticity

- **Associated flow rule** - Major principal plastic strain increment ($d\varepsilon_1^p$) is in the direction of the major principal stress (σ_1), and the axes of the principal strain increment and principal stress coincide, i.e., the yield and plastic functions coincide. This results in a symmetric constitutive matrix D^p.
- **Nonassociated flow rule** - The axes of the principal strain increment and principal stress do not coincide.



States that the elastic limit is defined by the yield surface, enforcing points in plastic condition to **remain** on the yield surface.

$$dF = \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \frac{\partial F}{\partial Wp} \cdot dWp = 0$$



Equations of elasto-plasticity: 3. Consistency condition

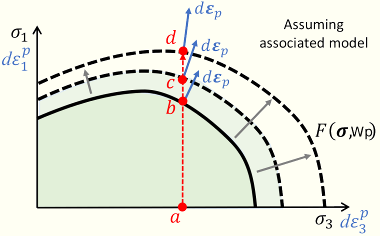
The consistency condition can be written as:

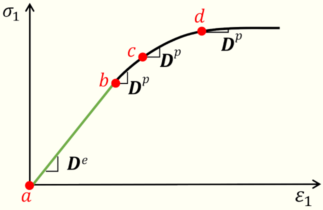
$$
\begin{aligned}
dF &= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \frac{\partial F}{\partial Wp} \cdot dWp && = 0 \\
&= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \left(\frac{\partial F}{\partial Wp}\right) \cdot \left(\frac{\partial Wp}{\partial \varepsilon^p}\right)^T \cdot d\varepsilon^p && = 0 \\
&= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \left(\frac{\partial F}{\partial Wp}\right) \left(\frac{\partial Wp}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma} d\lambda && = 0 \\
dF &= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma - H d\lambda && = 0.
\end{aligned}
$$

if $H > 0$: Hardening, if $H = 0$: perfect plasticity, if $H < 0$: softening.

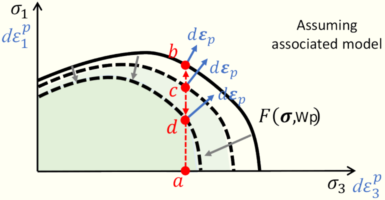
Hardening v Softening

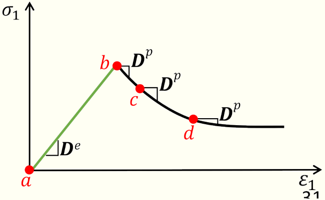
Linear elastic – hardening plastic material $H > 0$





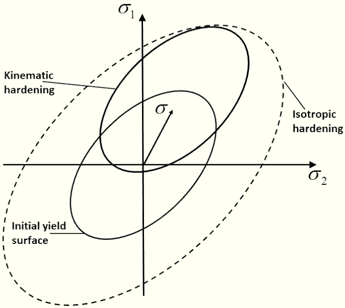
Linear elastic – softening plastic material $H < 0$





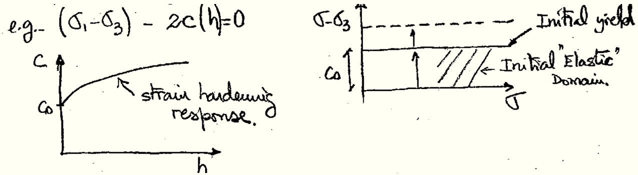
Isotropic v kinematic hardening

Two primary types of hardening laws: density and kinematic hardening.

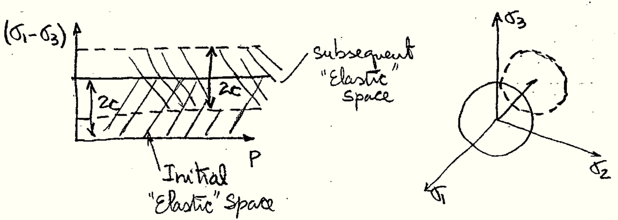


1. **Density hardening**: also referred to as isotropic hardening.
$(\sigma_1 - \sigma_3) - 2c(h) = 0$
2. **Kinematic hardening**: typically describing fabric anisotropy. No change in size of yield surface but translating in stress space.
$F = (\sigma_1 - \sigma_3) - \alpha(h) - 2c = 0$.

Isotropic v kinematic hardening



Isotropic hardening



Kinematic hardening

Basic concepts of elasto-plasticity

The response of the material is:

- **elastic**: as long as the stress state remains **within** the yield surface.

$$F(\sigma, Wp) < 0 \rightarrow d\sigma = D^e \cdot d\varepsilon.$$

- **elasto-plastic**: if the stress state is **on** the yield surface

$$F(\sigma, Wp) = 0 \rightarrow d\sigma = D^{ep} \cdot d\varepsilon.$$

Note: When the response is elastic, we have no problem! Knowing the strain, we directly can calculate the stress ($d\sigma = D^e \cdot d\varepsilon$).
The difficulty arises when the material response is elasto-plastic, we need to determine D^p!

Equations of elasto-plasticity

For an elasto-plastic material:

- Input material: $D^e, \frac{\partial F}{\partial \sigma}, \frac{\partial G}{\partial \sigma}, H$
- Results from FE analysis: $d\varepsilon$
- unknowns: $d\varepsilon^p, d\lambda, d\sigma$
- What we are interested: $d\sigma$

1. Stress-strain: $d\sigma = D^e \cdot (d\varepsilon - d\varepsilon^p)$
2. Consistency-condition: $dF = \frac{\partial F}{\partial \sigma}^T \cdot d\sigma - H d\lambda$
3. Flow rule: $d\varepsilon^p = \frac{\partial G}{\partial \sigma} d\lambda$.

Solution procedure:

1. Eqs 1 and 3 in eq 2 $\rightarrow d\lambda$
2. $d\lambda$ in Eq 3 $\rightarrow d\varepsilon^p$
3. $d\varepsilon^p$ in Eq 1 $\rightarrow d\sigma$

Equations of elasto-plasticity

Eq 1 and 3 (stress-strain & flow rule) in Eq 2 (consistency condition) for $d\lambda$

$$dF = \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot \left(d\varepsilon - \frac{\partial G}{\partial \sigma} d\lambda\right) - H d\lambda = 0$$

$$d\lambda = \frac{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot d\varepsilon}{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \left(\frac{\partial G}{\partial \sigma}\right) + H}$$

$$\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \frac{\partial G}{\partial \sigma} + H > 0$$

$d\lambda$ in Eq 3 $\rightarrow$ $d\varepsilon^p$

$$d\varepsilon^p = \frac{\partial G}{\partial \sigma} d\lambda = \frac{\partial G}{\partial \sigma} \frac{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot d\varepsilon}{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \left(\frac{\partial G}{\partial \sigma}\right) + H}$$

Equations of elasto-plasticity

$d\varepsilon^p$ in Eq 1 $\rightarrow d\sigma$

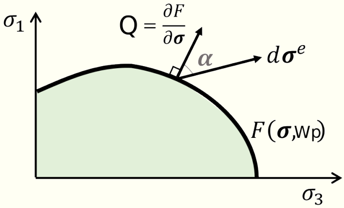
$$d\sigma = D^e \cdot (d\varepsilon - d\varepsilon^p)$$

$$= D^e \cdot \left(d\varepsilon - \frac{\partial G}{\partial \sigma} \frac{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot d\varepsilon}{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \left(\frac{\partial G}{\partial \sigma}\right) + H} \right)$$

$$\boxed{d\sigma' = \left[D^e - \frac{D^e \left(\frac{\partial G}{\partial \sigma'}\right) \left(\frac{\partial F}{\partial \sigma'}\right)^T D^e}{-\left(\frac{\partial F}{\partial W_p}\right) \left(\frac{\partial W_p}{\partial \varepsilon^p}\right)^T \left(\frac{\partial G}{\partial \sigma'}\right) + \left(\frac{\partial F}{\partial \sigma'}\right)^T D^e \left(\frac{\partial G}{\partial \sigma'}\right)} \right] d\varepsilon}$$

Plastic loading v elastic unloading

Depending on α, we might have three different scenarios:



If $\alpha < 90° \Rightarrow \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)^T \cdot \boldsymbol{D}^e \cdot d\boldsymbol{\varepsilon} > 0$

$\Rightarrow d\Lambda > 0 \Rightarrow d\boldsymbol{\varepsilon}^p > 0$

$\Rightarrow$ **Elastoplastic loading**

$\Rightarrow dF = 0$

If $\alpha = 90° \Rightarrow \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)^T \cdot \boldsymbol{D}^e \cdot d\boldsymbol{\varepsilon} = 0$

$\Rightarrow d\Lambda = 0 \Rightarrow d\boldsymbol{\varepsilon}^p = 0$

$\Rightarrow$ **Neutral loading**

$\Rightarrow dF = 0$

If $\alpha > 90° \Rightarrow \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)^T \cdot \boldsymbol{D}^e \cdot d\boldsymbol{\varepsilon} < 0$

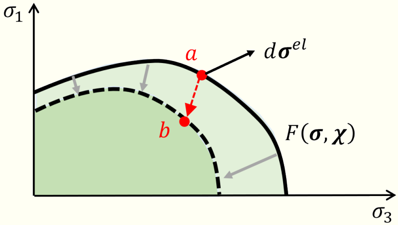
$\Rightarrow \cancel{d\Lambda < 0} \Rightarrow \cancel{d\boldsymbol{\varepsilon}^p = 0}$

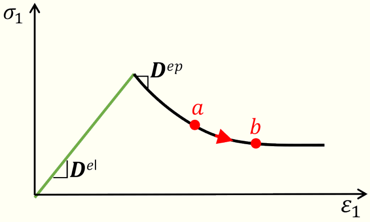
$\Rightarrow$ **Elastic unloading**

$\Rightarrow dF < 0$

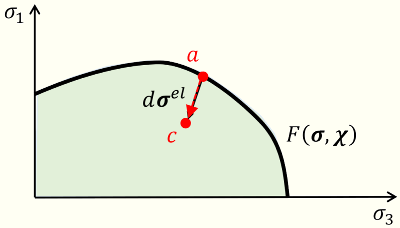
Plastic loading v elastic unloading

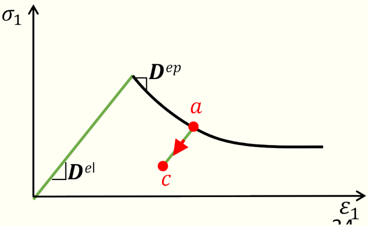
Don't confuse **elastoplastic loading with softenting**...





...with **elastic unloading**



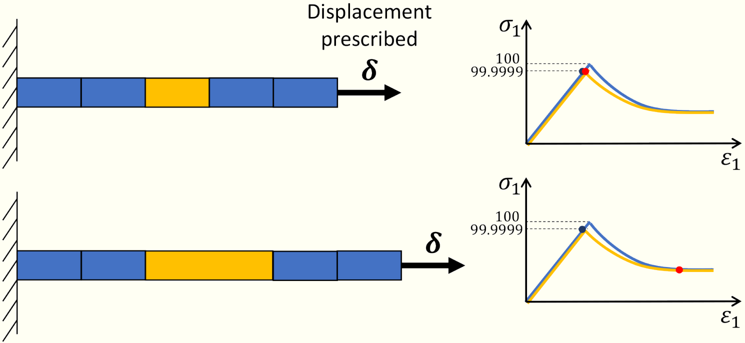


Problems associated to softening

1. The modeling of **strain softening** (i.e. strength decreases from peak to residual conditions) leads to the localization phenomenon.
2. Localization shows up when all deformation is absorbed by one element, while the neighboring elements remain under elastic conditions.
3. The solution depends on the element size, hence the localization is a **mesh dependent numerical problem**.
4. **Solution:** regularization techniques: non-local integration, strain softening models dependent on the element size,...
5. Be careful: Most of the commercial software do not include regularization techniques.

Problems associated to softening

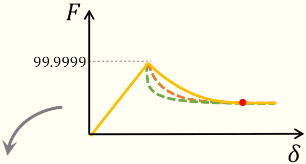
Don't confuse **elastoplastic loading with softenting**...



5 elements → **1 fails**

10 elements → **1 fails**

1,000,000 elements → **1 fails**



Localization =mesh dependent numerical problem