CE394M Advanced Analysis in Geotechnical Engineering: Introduction

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Overview

1. Geotechnical modeling
   - Complexity in Geotechnical modeling
   - Classical vs advanced analysis
2. Numerical methods for differential equations
   - Direct method: Matrix analysis of structures
   - Numerical analysis of engineering problems
3. Governing equations in stress-deformation analysis
   - Stress equilibrium
   - Compatibility condition
   - Stress-strain relationship

Geotechnical modeling of the complex world

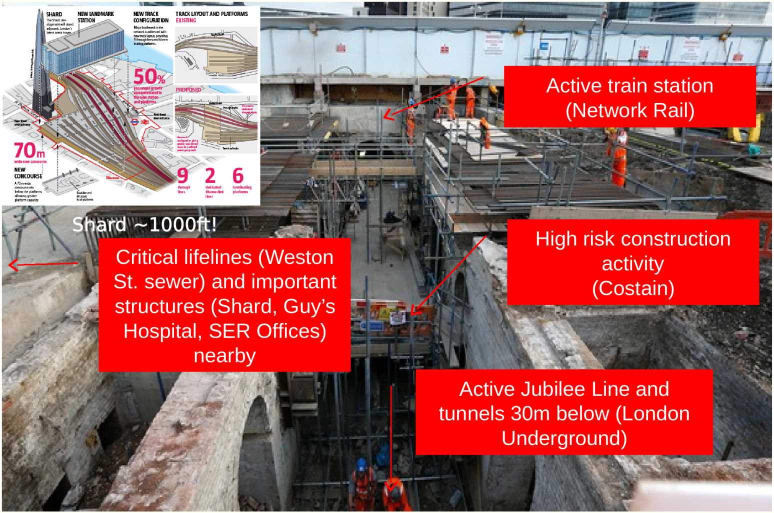
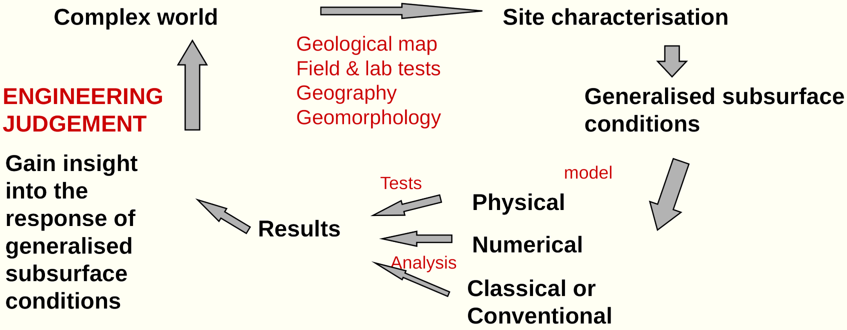


Fig. London Bridge Station, London, UK

Geotechnical modeling of the complex world

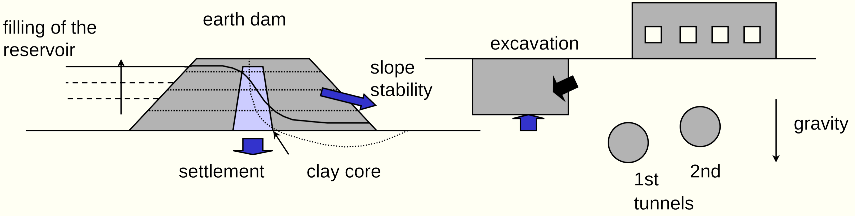
Fig. London Victoria station upgrade, London, UK

Geotechnical modeling

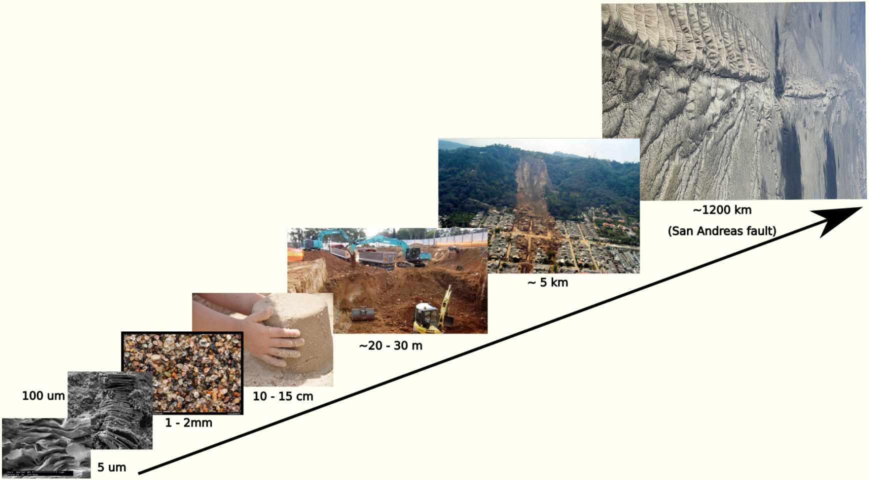


Geotechnical modeling: What should be modeled?

- Self weight effect of soils (This is why soil moves)
- Construction sequence (Complex geometry)
- Water movement (undrained, consolidation, drained)
- Insitu stresses (stiffness/strength depends on current stresses and stress history)
- Predict the ability of a design to withstand extreme loading conditions (you only have one chance)



Scales of modeling in geotechnical engineering



Soil behavior

- nonhomogeneous,
- anisotropic,
- non-linear,
- initial stress conditions,
- stress history
- Geometry - very complex

Soil Mechanics in practice - largely empirical

Advanced analysis in geotechnical engineering

Geotechnical design:

- Assess applied forces
- evaluate "performance" (stability & movements) under working and ultimate loads

Analysis:

- Mathematical framework to perform calculations for these quantities
- Requires idealization of: geometry, soil properties, and loading conditions
- Analysis is a tool in design, but design involves more: acceptable movements, constraints, site characterization, etc.

Classical vs advanced analysis

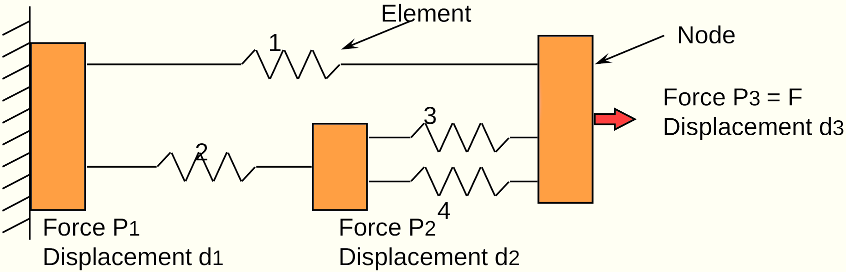
Classical approach:

- Failure estimates
  - rigid perfectly-plastic stress-strain assumptions
  - calculate factor of safety (What value do you pick?)
- Deformation estimates
  - Elastic analysis
  - Use average elastic properties (like what??)

Advanced analysis:

- Failure and deformation are obtained from the same analysis
- Handle complex geometry
- More difficult to perform, more computational requirements and more info on soil behavior $\sigma - \epsilon$
- Need to know how to do it right!

Matrix analysis of structures



- What are the known variables? $d_1 = 0, P_2 = 0, P_3 = F(\text{constant})$
- What are the unknowns? P_1, d_2, d_3
- What do we know? Force or distortion relations at an element level.

Matrix analysis of structures: Equilibrium

A structure is considered to be in equilibrium if, initially at rest, it remains at rest when subjected to a system of forces and couples. If a structure is in equilibrium, then all of its members and joints must also be in equilibrium.

- $P_1 =-S_1 - S_2$
- What are the unknowns? P_1, d_2, d_3
- What do we know? Force or distortion relations at an element level.

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ s_4 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{A^T S}$$

Matrix analysis of structures: Compatibility

- compatibility relates the deformations of a structure so that its various parts (members, joints, and supports) fit together without any gaps or overlaps.
- ensure that the deformed shape of the structure is continuous (except at the locations of any internal hinges or rollers), and is consistent with the support conditions.

Matrix analysis of structures: Compatibility

v = internal spring distortion δ = nodal displacement

- $v_1 = d_3 - d_1$
- $v_2 = d_2 - d_1$
- $v_3 = d_3 - d_2$
- $v_4 = d_3 - d_2$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\mathbf{v} = \mathbf{A}\mathbf{d}$$

Matrix analysis of structures: Physical condition

Force-distance relationship: spring constant

spring #	1	2	3	4
stiffness $(F.L^{-1})$	3	2	1	2

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$\mathbf{s} = \mathbf{D}\mathbf{v}$$

Matrix analysis of structures: Direct Method

Combine all the equations: $\mathbf{P} = \mathbf{A^T S} = \mathbf{A^T D v} = \mathbf{A^T D A d} = \mathbf{K d}$
where $\mathbf{K} = A^T DA$ (Global stiffness matrix)

$$K = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & -3 \\ -2 & 5 & -3 \\ -3 & -3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -3 \\ -2 & 5 & -3 \\ -3 & -3 & 6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Apply Boundary conditions $d_1 = 0$, $P_2 = 0$ and $P_3 = F$ and solve P_1, d_2 and d_3

Matrix analysis of structures

known displacements and forces

↓

model at element level

↓

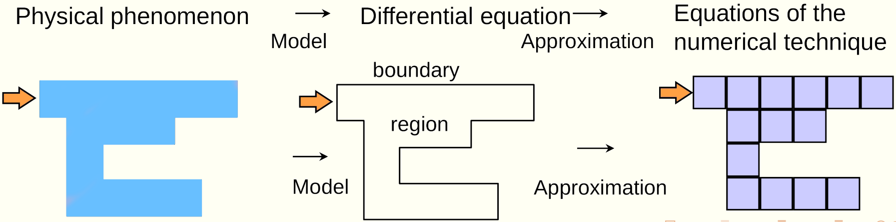
assemble global stiffness matrix

↓

calculate unknown displacements and forces

Numerical analysis of engineering problems

- Conceptualize the system
  - Geometry
  - Properties
  - Processes
- Describe it mathematically
  - Select the relevant differential equations
- Solve the equations (numerically)
  - Discretize the system
  - Settle for approximations (numerical techniques)
- Interpret the results



Boundary value problems

Differential equations coupled with boundary conditions

- Steady state (time-independent)
  - Static load-deformation problems: $\partial\sigma/\partial x = 0$ (force + disp. B.C)
  - Steady seepage state, flow problems: $\partial q/\partial x = 0$ (head + flow B.C)
- Transient (time-dependent)
  - Consolidation/pore-fluid migration/multiphase flows
  - Dynamic loading (earthquakes, wave actions)
  - Contaminant transport processes

Numerical solutions to differential equations

- Finite differences: Approximate derivatives with expansion into a Taylor series
- Finite elements
- Boundary element method (BEM)
- Meshless methods
- Discrete/discontinuous element methods
- Others...

Governing equations in stress-deformation analysis

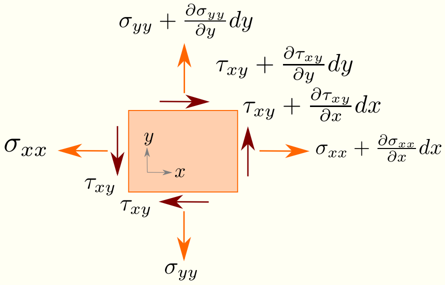
In stress-deformation analysis, we need to consider:

- **Equilibrium - static conditions**
  - forces and stress must agree across the region of interest. (geometric problem)
- **Compatibility-kinematic conditions**
  - geometry, displacement and strains must agree across the region of interest. (geometric problem)
- **Stress-strain relationship on physical conditions**
  - material dependent relationship between stress and strain must be specified. (element level)

Governing equations in stress-deformation analysis

The governing differential equation for equilibrium expresses: $\sum \mathbf{F} = m\mathbf{a}$

- σ_{xx} acting on face dy in the $-x$ direction
- τ_{xy} acting on face dx in the $-x$ direction
- $\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx$ acting on face dy in the $+x$ direction
- $\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy$ acting on face dx in the $+x$ direction
- Plus "body forces" due to gravity: $\rho f_x dx dy$ where f_x is body force per unit mass



Equilibrium equations

Summing all this in the x-direction gives:

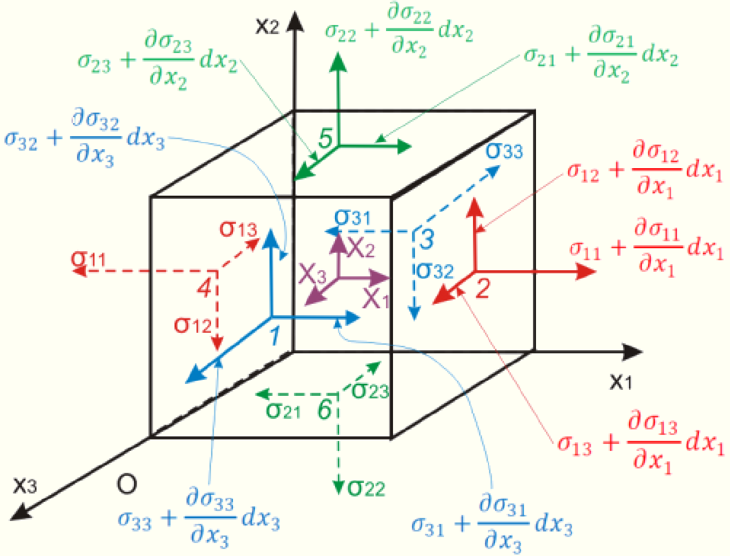
$$
\begin{aligned}
-\sigma_{xx}dy - \tau_{xy}dx + \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)dy \\
+ \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}dy\right)dx + \rho f_x dxdy = \rho dxdya_x
\end{aligned}
$$

Cleaning up terms that cancel, and dividing through by *dxdy* gives

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho f_x = \rho a_x$$

And summing forces in the y-direction leads to:

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho f_y = \rho a_y$$



X2
X1
X3
O
$\sigma_{23} + \frac{\partial \sigma_{23}}{\partial x_2} dx_2$
$\sigma_{22} + \frac{\partial \sigma_{22}}{\partial x_2} dx_2$
$\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2$
$\sigma_{32} + \frac{\partial \sigma_{32}}{\partial x_3} dx_3$
$\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1$
$\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1$
$\sigma_{13} + \frac{\partial \sigma_{13}}{\partial x_1} dx_1$
$\sigma_{33} + \frac{\partial \sigma_{33}}{\partial x_3} dx_3$
$\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3$
σ33
σ31
σ32
σ13
σ11
σ12
σ21
σ23
σ22
1
2
3
4
5
6
X2
X3
X1

Equilibrium in 3D

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x = \rho a_x$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y = \rho a_y$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z = \rho a_z$$

The governing differential equation for equilibrium expresses $\sum \mathbf{F} = m\mathbf{a}$ in terms of derivatives of the stress tensor as: $\nabla \boldsymbol{\sigma} + \rho \mathbf{f} = \rho \mathbf{a}$

$\boldsymbol{\sigma}$ is the stress tensor,

ρ is density,

$\mathbf{f}$ is the body force vector per unit mass and

$\mathbf{a}$ is the acceleration vector.

Stress equilibrium

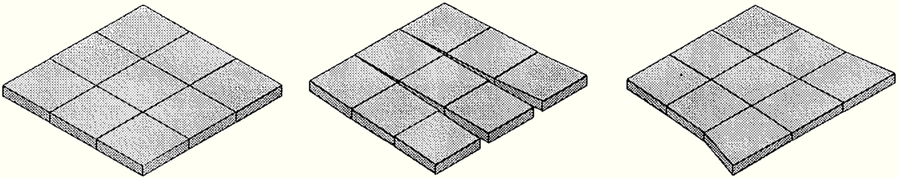
If the object is in equilibrium, then $\mathbf{a} = 0$ and $\sum \mathbf{F} = 0$.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

Stresses in Voigt notation: $[\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}]^T$.
Equilibrium equation: $\nabla^T \boldsymbol{\sigma} + \mathbf{b} = 0$
Then:

$$\nabla^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$

Governing equations: Compatibility



(a) original (b) non-compatible (c) compatible

Displacement - strain relationship: $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$
Where,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

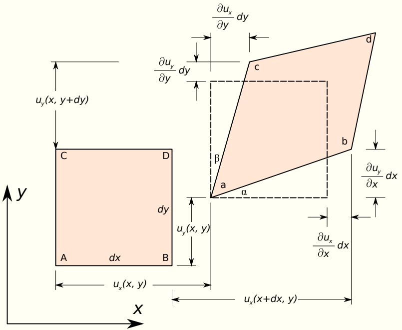
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

$$\gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$



$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Equilibrium and compatibility conditions

Combining the Equilibrium and Compatibility conditions gives:

- Unknowns: 6 stresses + 6 strains + 3 displacements = 15
- Equations: 3 equilibrium + 6 compatibility = 9

To obtain a solution therefore requires 6 more equations. These come from the constitutive relationships

Stress - strain relationship: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} D_{xxxx} & D_{xxyy} & D_{xxzz} & D_{xxxy} & D_{xxyz} & D_{xxzx} \\ D_{yyxx} & D_{yyyy} & D_{yyzz} & D_{yyxy} & D_{yyyz} & D_{yyzx} \\ D_{zzxx} & D_{zzyy} & D_{zzzz} & D_{zzxy} & D_{zzyz} & D_{zzzx} \\ D_{xyxx} & D_{xyyy} & D_{xyzz} & D_{xyxy} & D_{xyyz} & D_{xyzx} \\ D_{yzxx} & D_{yzyy} & D_{yzzz} & D_{yzxy} & D_{yzyz} & D_{yzzx} \\ D_{zxxx} & D_{zxyy} & D_{zxzz} & D_{zxxy} & D_{zxyz} & D_{zxzx} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

Governing equations in stress-deformation analysis

What are the variables used in the governing equations?

1. displacements $\mathbf{u}$ in the body
2. strains $\boldsymbol{\varepsilon}$ in the body or within the elements
3. stresses $\boldsymbol{\sigma}$ in the body or within the elements

Advanced analysis involves:

1. Equilibrium: External forces + internal stresses agree
2. Compatibility: Displacements fields agree (no gaps) + strains (derivatives)
3. Stress-strain relationship (constitutive behaviour)