CE394M: Critical State and Cam-Clay

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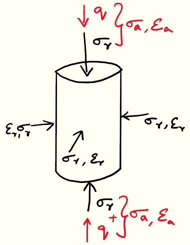
Overview

Critical State Soil Mechanics

Roscoe et al., (1958), Schofield & Worth (1968), Wood (1990):

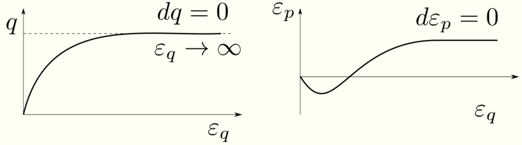
Critical state variables

- Mean stress: $p' = \frac{\sigma'_a + 2\sigma'_r}{3} = p - u$.
- Deviatoric stress: $q = \sigma'_a - \sigma'_r = \sigma_a - \sigma_r$



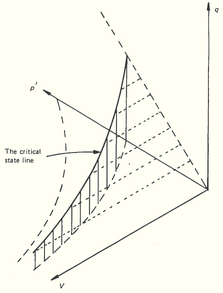
Critical State Hypothesis: I

Roscoe, Schofield & Worth (1958): **At shear-failure, soil exists at a unique state**



Critical State Hypothesis: II

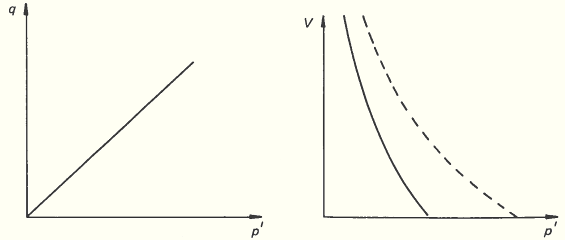
Critical state is a function of q, p', v**.**



The CSL (p', v, q) space is given by the intersection of two planes: $q = Mp'$ and a cruved vertical plane $v = \Gamma - \lambda \ln p'$

Critical State Hypothesis: II

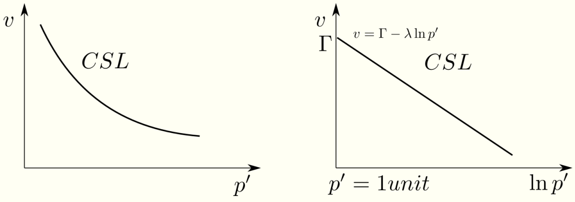
Critical state is a function of q, p', v.



The CSL in (a) (p', q) plot and (b) (p', v) plot (isotropic normal compression line is shown in dashed)

Critical State Hypothesis: II

Critical state is a function of q, p', v.



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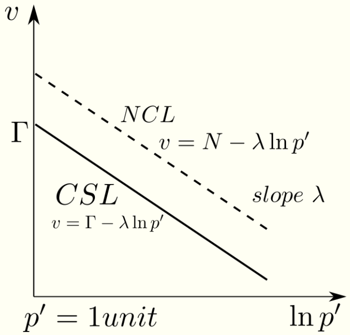

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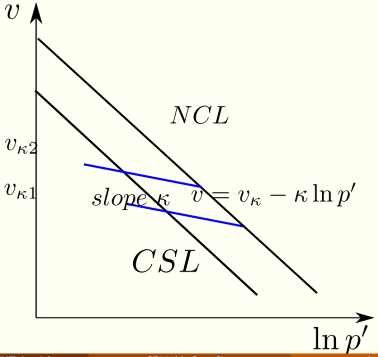
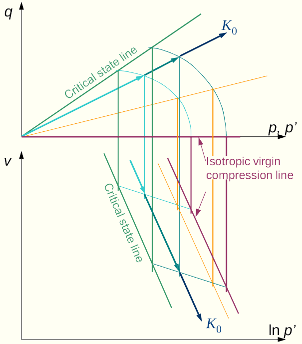
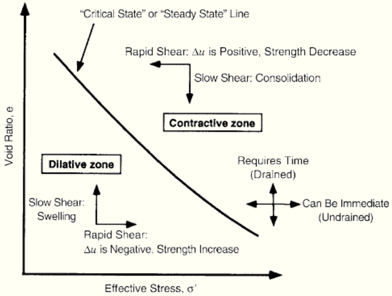


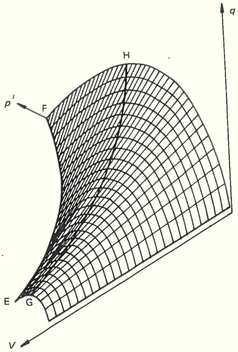
Stress paths $\sigma'_3/\sigma'_1 = K_c = const$



Clay behavior

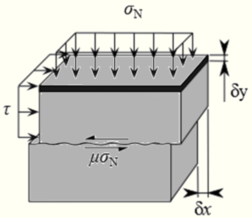


Critical state boundary surface



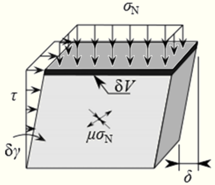
Summary of critical state behavior

Stress - dilatancy theory (Taylor, 1948)



Formulation of elasto-plastic Cam-Clay (OCC): Yield function

Derived from work consideration:



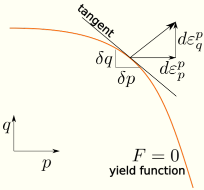
Cam-Clay (OCC): Stress dilatancy relation

$$p' d\varepsilon_v^p + q d\varepsilon_s^p = M p' d\varepsilon_s^p$$

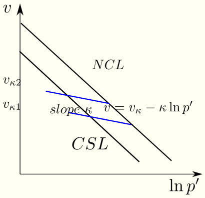
Rearranging the terms (divide by $p' d\varepsilon_s^p$):

Cam-Clay (OCC): flow-rule

The original idea was very simple. The yield locus must be such that each associated flow rule $(\delta\varepsilon_v, \delta\varepsilon_s)$ would be orthogonal to the tangent to the yield locus.



Cam-Clay (OCC): Elastic properties



Cam-Clay (OCC): Hardening law

We need to define how the yield surface hardens as plastic work is being performed. Only "*memory*" parameter in our yield surface is the size: p'_c. From the isotropic NCL:

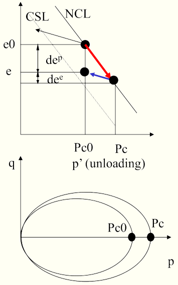
$$d\varepsilon_v = \frac{-dv}{v} = \frac{-de}{1+e} = \frac{+\lambda}{v}\frac{dp'_c}{p'_c}$$

But the increment in elastic volumetric strain is:

$$d\varepsilon_v^e = \left(\frac{-dv}{v}\right)^{elastic} = +\frac{\kappa}{v}\left(\frac{dp'_c}{p'_c}\right)$$

Cam-Clay (OCC): Hardening law

$$
\begin{aligned}
d\varepsilon_{vol} &= -\frac{de}{(1+e)} \\
&= \frac{\kappa}{(1+e)}\frac{dp'}{p'} + \frac{\lambda-\kappa}{(1+e)}\frac{dp'_c}{p'_c} \\
&= elastic + plastic \\
&= d\varepsilon^e_{vol} + d\varepsilon^p_{vol}
\end{aligned}
$$



Therefore the increment of p_c can be related to the increment of plastic volumetric strain:

$$
d\varepsilon^p_v = d\varepsilon_v - d\varepsilon^e_v = (\lambda-\kappa)\left(\frac{dp'_c}{p'_c}\right)
$$

$$
dp'_c = \left(\frac{v\cdot p'_c}{(\lambda-\kappa)}\right)\cdot d\varepsilon^p_v
$$

Cam-Clay (OCC): Hardening law

We have seen that the hardening law:

$$
H = -\left(\frac{\partial F}{\partial Wp}\right)\left(\frac{\partial Wp}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma}
$$

W_p is the vector of memory parameters. In our case, the CC model has only one parameter:

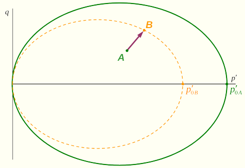
Cam-Clay (OCC): Hardening law

We know:

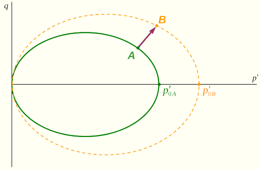
$$\frac{\partial F}{\partial p'_c} = -Mp'/p'_c$$
$$\frac{\partial p'_c}{\partial \varepsilon^p} = \frac{v}{(\lambda - \kappa)} p'_c$$
$$\frac{\partial G}{\partial \sigma} = P_p = Q_p = M - \eta$$

$$\boxed{H = -\left(\frac{\partial F}{\partial p'_c}\right)\left(\frac{\partial p'_c}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma} = M\frac{(M-\eta)}{(\lambda - \kappa)} \cdot (1+e) \cdot p'}$$

Deformations under an applied stress path

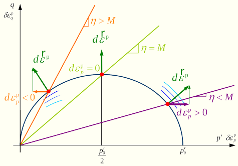


$P'_{0B} > P'_{0A}$ Elastic



$P'_{0B} < P'_{0A}$ Elasto-plastic

Hardening law



$$d\varepsilon_p^p = \frac{\lambda - \kappa}{v p_0'} dp_0'$$

Stress-strain relation in (p', q) and $(\varepsilon_v, \varepsilon_s)$ drained TX

1. Give strain and/or stress increments
2. Check if the current stress state is inside the yield surface or outside the yield surface $q/p' = M \ln(p_c/p')$
    1. If Elastic (stress inside yield surface):

    $$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_s \end{bmatrix} = \begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

    2. If Elastic-plastic (stress on the yield surface)

    $$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_s \end{bmatrix} = D^{ep} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$$D^{ep} = \left[\begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} + \frac{1}{Mp'} \frac{(\lambda - \kappa)}{(1 + e_0)} \begin{bmatrix} M - (q/p') & 1 \\ 1 & 1/(M - (q/p')) \end{bmatrix} \right]$$

3. Compute the unknown stress or strain increments and update the stress and strains
4. If plastic deformation, update p_c to satisfy Cam-Clay yield surface
5. Go back to step 1

Stress-strain relation in (p', q) and $(\varepsilon_v, \varepsilon_s)$ undrained TX

$d\varepsilon_v = d\varepsilon_a + 2d\varepsilon_r = 0$ (constant volume)
$d\varepsilon_s = (2/3)(d\varepsilon_a - d\varepsilon_r) = (2/3)(d\varepsilon_a - (-0.5d\varepsilon_a)) = d\varepsilon_a$

1. Give axial strain increment $d\varepsilon_a$ or dq
2. Check if the current stress state is inside the yield surface or outside the yield surface $q/p' = M\ln(p_c/p')$
   - If Elastic (stress inside yield surface):

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_s \end{bmatrix} = \begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$d\varepsilon_v =$ from the first equation gives $dp' = 0$. This means that the effective stress path is fixed to go in the vertical direction in $p' - q$ space irrespective of any total stress path.
The second equation gives $d\varepsilon_s$ for a given dq or dq for a $d\varepsilon_s$.

Stress-strain relation in (p', q) and $(\varepsilon_v, \varepsilon_s)$ undrained TX

4. - If Elastic-plastic (stress on the yield surface)

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_s \end{bmatrix} = D^{ep} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$$D^{ep} = \left[\begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} + \frac{1}{Mp'} \frac{(\lambda - \kappa)}{(1 + e_0)} \begin{bmatrix} M - (q/p') & 1 \\ 1 & 1/(M - (q/p')) \end{bmatrix} \right]$$

$d\varepsilon_v = 0$ and $d\varepsilon_s = d\varepsilon_a$ give dp' and dq
or
$d\varepsilon_v = 0$ and dq gives $d\varepsilon_s (= d\varepsilon_a)$ and dp'.

5. Update the stress and strain. The difference between the total mean pressure p and the effective mean pressure will give the pore pressure.
6. If plastic deformation, update p_c to satisfy Cam-Clay yield surface.
7. Go back to step 1

Limitations of original Cam-Clay

What can we change?

MCC: Yield function

Derived from work considerations (Burland 1965, Roscoe and Burland 1968):

$$dW^p_{int} = p\sqrt{(d\varepsilon^p_v)^2 + (Md\varepsilon^p_s)^2}$$

This is the new equation describing the energy dissipated by the soil. Following similar arguments to CC:

$$dW^p_{ext} = pd\varepsilon^p_v + qd\varepsilon^p_s = p\sqrt{(d\varepsilon^p_v)^2 + (Md\varepsilon^p_s)^2} = dW^p_{int}$$

Squaring and re-arranging the terms:

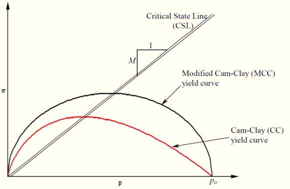
$$\frac{d\varepsilon^p_v}{d\varepsilon^p_s} = \frac{M^2 - \eta^2}{2\eta} = -\frac{dq}{dp'}$$

Assuming an associative flow rule gives an elliptic yield surface:

$$\boxed{F(q,p) = q^2 - M^2 p^2 \left(\frac{p'_c}{p'} - 1\right) = 0}$$

OCC v MCC

	Original Cam-clay model (Schofield and Wroth, 1968)	Modified Cam-clay model (Roscoe and Burland, 1968)
Dissipative work	$p'd\varepsilon_v^p + qd\varepsilon_s^p = Mp'd\varepsilon_s^p$	$p'd\varepsilon_v^p + qd\varepsilon_s^p = p'\sqrt{(d\varepsilon_v^p)^2 + (Md\varepsilon_s^p)^2}$
Associated flow rule	$(d\varepsilon_s^p/d\varepsilon_v^p)(dq/dp') = -1$	$(d\varepsilon_s^p/d\varepsilon_v^p)(dq/dp') = -1$
Yielding of Cam-clay	$q = Mp' \ln(p_c/p')$	$q^2 + M^2p'^2 = M^2p'p_c$
N and Γ	$N = \Gamma + \lambda - \kappa$	$N = \Gamma + (\lambda - \kappa)\ln 2$



MCC: Stress-strain relationship

$$\begin{bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ d\tau_{12} \\ d\tau_{23} \\ d\tau_{31} \end{bmatrix} = \left[D(6\times 6)\right] \begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ d\gamma_{12} \\ d\gamma_{23} \\ d\gamma_{31} \end{bmatrix}$$

$$d\sigma' = \left[\boxed{D_e} - \frac{\boxed{D_e}(\partial F/\partial\sigma')(\partial F/\partial\sigma')^T \boxed{D_e}}{-(\partial F/\partial p_c)(dp_c/d\varepsilon_v^p)(\partial F/\partial p') + (\partial F/\partial\sigma')^T \boxed{D_e}(\partial F/\partial\sigma')}\right] d\varepsilon$$

Elastic Stiffness

$$D_e = \begin{bmatrix} K+4/3G & K-2/3G & K-2/3G & 0 & 0 & 0 \\ K-2/3G & K+4/3G & K-2/3G & 0 & 0 & 0 \\ K-2/3G & K-2/3G & K+4/3G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \quad K = \frac{\nu p'}{\kappa}, G = \frac{3K(1-2\nu)}{2(1+\nu)}$$

MCC: I

$$d\sigma' = \left[D_e - \frac{D_e\boxed{(\partial F/\partial\sigma')}\boxed{(\partial F/\partial\sigma')^T} D_e}{-(\partial F/\partial p_c)(dp_c/d\varepsilon_v^p)(\partial F/\partial p') + \boxed{(\partial F/\partial\sigma')^T} D_e\boxed{(\partial F/\partial\sigma')}}\right] d\varepsilon$$

(A) Calculation of $\partial F/\partial\sigma'$

$$F = \frac{q^2}{M^2} - p' p_c + p^2 = 0$$

$$\frac{\partial F}{\partial\sigma'} = \frac{\partial F}{\partial p'}\frac{\partial p'}{\partial\sigma'} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial\sigma'}$$

$$\partial F/\partial p' = 2p - p_c$$

$$\partial F/\partial q = 2q/M^2$$

$$\partial p'/\partial\sigma' = \begin{Bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \partial q/\partial\sigma' = (3/2q)\begin{Bmatrix} \sigma_{xx} - p \\ \sigma_{yy} - p \\ \sigma_{zz} - p \\ 2\sigma_{xy} \\ 2\sigma_{yz} \\ 2\sigma_{zx} \end{Bmatrix}$$

$$\frac{\partial F}{\partial\sigma'} = \begin{Bmatrix} (2p-p_c)/3 + 3(\sigma_{xx}-p)/M^2 \\ (2p-p_c)/3 + 3(\sigma_{yy}-p)/M^2 \\ (2p-p_c)/3 + 3(\sigma_{zz}-p)/M^2 \\ 6\sigma_{xy}/M^2 \\ 6\sigma_{yz}/M^2 \\ 6\sigma_{zx}/M^2 \end{Bmatrix}$$

MCC: II

$$d\sigma' = \left[D_e - \frac{D_e(\partial F/\partial \sigma')(\partial F/\partial \sigma')^T D_e}{-\boxed{(\partial F/\partial p_c)(dp_c/d\varepsilon_v^p)(\partial F/\partial p')} + (\partial F/\partial \sigma')^T D_e(\partial F/\partial \sigma')} \right] d\varepsilon$$

(B) Calculation of $(\partial F/\partial p_c)(dp_c/d\varepsilon_v^p)(\partial F/\partial p)$

$$\frac{\partial F}{\partial p_c} = -p \qquad \frac{dp_c}{d\varepsilon_v^p} = \frac{vp_c}{(\lambda-\kappa)} \qquad \frac{\partial F}{\partial p} = 2p - p_c$$

$$(\partial F/\partial p_c)(dp_c/d\varepsilon_v^p)(\partial F/\partial p) = -p\frac{vp_c}{(\lambda-\kappa)}(2p-p_c)$$

(C) Assemble [6x6] matrix

$$d\sigma' = \left[D_e - \frac{D_e(\partial F/\partial \sigma')(\partial F/\partial \sigma')^T D_e}{-(\partial F/\partial p_c)(dp_c/d\varepsilon_v^p)(\partial F/\partial p') + (\partial F/\partial \sigma')^T D_e(\partial F/\partial \sigma')} \right] d\varepsilon$$

$$[6\text{x}1] = \left[[6\text{x}6] - \frac{[6\text{x}6][6\text{x}1][1\text{x}6][6\text{x}6]}{-[1\text{x}1][1\text{x}1][1\text{x}1] + [1\text{x}6][6\text{x}6][6\text{x}1]} \right] [6\text{x}1]$$

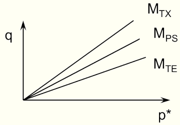
Cam-Clay material properties determination: I

Test required

1. Slow drained or undrained test with pore pressure measurement at different preconsolidation pressure. Test must be taken to very large strains to reach to the critical state.
2. Isotropic consolidateion test or incrementental loading consolidation test.

Slope of failure line: M If ϕ_{cs} is assumed to be constant, then M is not constant.

- In TXC: $M_{TXC} = \frac{6\sin\phi_{cs}}{3-\sin\phi_{cs}}$
- In TXE: $M_{TXE} = \frac{6\sin\phi_{cs}}{3+\sin\phi_{cs}}$
- In PS: $M_{PS} = 2\sin\phi_{cs}$ (assuming $\sigma_2 \approx \frac{\sigma_1+\sigma_3}{2}$)



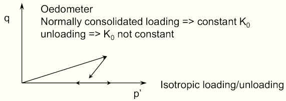
Cam-Clay material properties determination: II

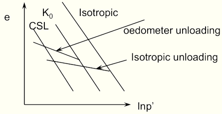
λ and κ: Compression index and swelling index

1. Use isotropic consolidation. Plot in $v - \ln p'$ plane. Measure λ & κ.
2. From odeometer test, plot void ratio e vs $\log \sigma'_v$:

$$\lambda = 0.4343 C_c$$
$$\kappa = 0.4343(1.4 C_{r/s}) = 0.63 C_{r/s} = (0.2\ 0.33)\lambda$$

1. swelling and recompression lines are highly nonlinear in $e - \ln p'$ space.
2. κ is usually not less than 0.1λ or greater than 0.5λ.
3. $\kappa$is about 0.03 to 0.06 for many medium plasticity clays.
4. modify κ to try to fit data. λ is usually easier to accurately calculate.



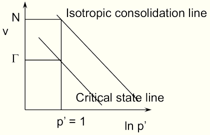


Cam-Clay material properties determination: III

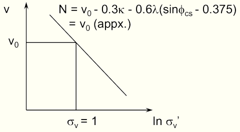
Γ or N

1. Critical state line: $v = \Gamma - \lambda \ln p'$
2. Isotropic normally consolidated line: $v = N - \lambda \ln p'$
3. Original Cam-Clay: $N = \Gamma + (\lambda - \kappa)$
4. Modified Cam-Clay: $N = \Gamma + (\lambda - \kappa) \ln 2$

(a) From isotropic consolidation test,



(b) From oedometer test,



Cam-Clay material properties determination: IV

Elastic properties: G or ν

1. Elastic bulk modulus $K_e = dp'/d\varepsilon_v = vp^p rime/\kappa$
2. Choosing Poisson's ratio ν constant gives G_e which varies as K_e.
3. Typically $\nu = 0.2 - 0.4$.

Preconsolidation pressure p_c

1. Estimate OCR
2. $\sigma_{v,max} = OCR\sigma_{v,current}$
3. $\sigma_{h,max} = K_{0,nc}\sigma_{v,max}$
4. $q_{max} = \sigma_{v,max} - \sigma_{h,max}, p_{max} = (1/3)(\sigma_{v,max} + 2\sigma_{h,max})$
5. Define p_c:
   - Cam-Clay: $q_{max} = Mp_{max}\ln(p_c/p_{max})$
   - MCC: $q_{max}^2 = M^2 p_{max}(p_c - p_m ax)$

Cam-Clay material properties determination: V

Cam-clay parameters : Correlation For clayey soils, attempts have been made to obtain the cam-clay parameters from index tests, especially plasticity index. It should be remembered that most of the soils tested for correlation were remoulded and direct application to natural clays requires caution. The users should be fully aware that limitations exist and these correlations should be regarded as a first approximation.

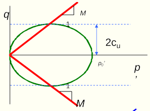
- Atkinson (1993) $\lambda = PIG_s/460 \quad \Gamma = 1.25 + \lambda \ln 10,000$
- Nakase, Kamei and Kusakabe (1988)- study based on Japanese clays: $\lambda = 0.02+0.0045PI \qquad \kappa = 0.00084(PI-4.6) \qquad N = 1.52+0.19PI$
- Nakase et al. (1988) and Frydman (1990) observed that M was independent of PI. Atkinson (1993) finds that M tends to increase with decreasing PI.

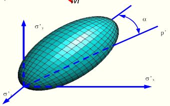
Drawbacks of MCC: Computational problems

- In undrained triaxial test on a heavily overconsolidated soil, after the stress point reaches the yield surface (above M line), due to the negative direction of volumetric plastic strain vector, the yield surface contracts.
- This phenomenon is referred to as **strain softening**.
- Even though the constitutive model is perfectly able to model this aspect of mechanical behaviour, strain softening may lead to problems in a finite element analysis: e.g. mesh dependency and problems with convergence.
- That can be overcome with good coding & algorithms, but many leading codes still struggle and diverge or give erroneous results!

Drawbacks of MCC: Strength prediction in undrained conditions

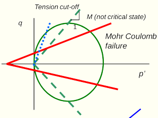
- MCC model assumes Drucker-Prager failure condition, which overestimates undrained strength in triaxial extension.
- Better predictions if Mohr Coulomb failure or Lode angle dependency is introduced
- Real soils are anisotropic and both the shape and size of the yield surface would need to change (see e.g., Wheeler et al. 2003, Can Geotech J for S-Clay1 model)

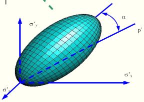




Drawbacks of MCC: K 0 prediction

- Given MCC assumes an associated flow rule, the model predicts unrealistically high K_0 values in normally consolidated range
- This has been fixed e.g. in the Soft Soil model by de-coupling the volumetric yield surface (cap) from the failure line
- Consequently, in the Soft Soil model, M as become a "*shape*" coefficient and no longer corresponds to the critical state line
- Alternatively, the anisotropic S-CLAY1 model also gives good K0 prediction
- Non-associated flow rule is also an option which will help with that issue





Advantages and Limitations of Cam-Clay models

Advantages:

- Unified framework of elastic - plastic behaviour, deals with volumetric & shearing strains, drained & undrained, prefailure and failure,
- Uses a small number of parameters $(M, \lambda, \kappa, \Gamma, \nu)$
- fairly well established,
- Qualitative prediction of soil behaviour - replicates many behaviour not possible to replicate in e.g. Mohr-Coulomb model.

Limitations:

1. Largely elastic for heavily OC clays (and dense sands).
2. Soils do not always reach the critical state or necessarily stay there, such as by breakdown in fabric strain softening.
3. No anisotropy.
4. Stress reversals within elastic domain are purely elastic after initial yielding.
5. No creep.
6. Model predicts conservative K_0 value.