CE394M: Stress paths and invariants

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Overview

1D consolidation / simple shear

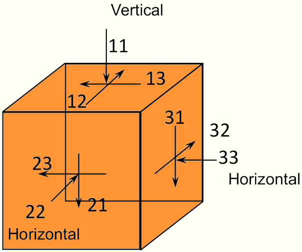
- Zero lateral strain ($\varepsilon_{22} = \varepsilon_{33} = 0$)
- Stresses: σ and τ
- Strains: $\varepsilon_{11} = \varepsilon_v$ and γ

2D plane strain

- Zero lateral strain ($\varepsilon_{22} = \gamma_{12} = \gamma_{23} = 0$)
- Stresses: s and t
- Strains: ε_v and ε_γ

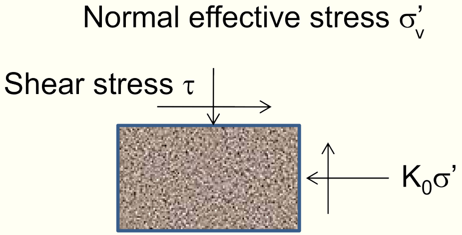
3D general (axi-symmetric as a special case)

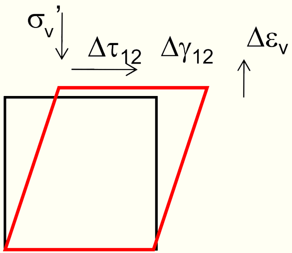
- Stresses: p and q
- Strains: ε_v and ε_s



1D simple shear

1. No lateral strain
2. Constant normal effective stress σ'_v
3. Increasing shear strain γ
4. Measure shear resistance τ
5. Measure volumetric strain ε_v or void ratio $e = e_0 - (1 + e_0)\varepsilon_v$
6. **No information for the lateral direction**



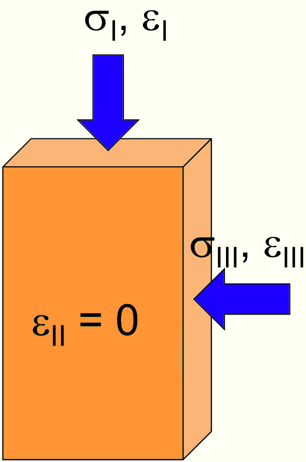


Stresses and strains: independent components

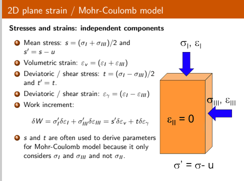
1. Mean stress: $s = (\sigma_I + \sigma_{III})/2$ and $s' = s - u$
2. Volumetric strain: $\varepsilon_v = (\varepsilon_I + \varepsilon_{III})$
3. Deviatoric / shear stress: $t = (\sigma_I - \sigma_{III})/2$ and $t' = t$.
4. Deviatoric / shear strain: $\varepsilon_\gamma = (\varepsilon_I - \varepsilon_{III})$
5. Work increment:

$$\delta W = \sigma'_I \delta\varepsilon_I + \sigma'_{III}\delta\varepsilon_{III} = s'\delta\varepsilon_v + t\delta\varepsilon_\gamma$$

6. s and t are often used to derive parameters for Mohr-Coulomb model because it only considers σ_I and σ_{III} and not σ_{II}.



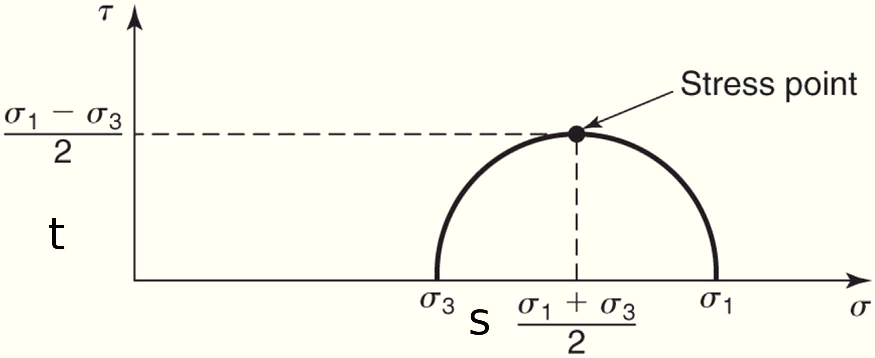
$\sigma' = \sigma - u$



Principal stresses $\sigma_I > \sigma_{II} > \sigma_{III}$. This equation holds good and is the definition of σ in principal stress notations, i.e., $I > II > III$.

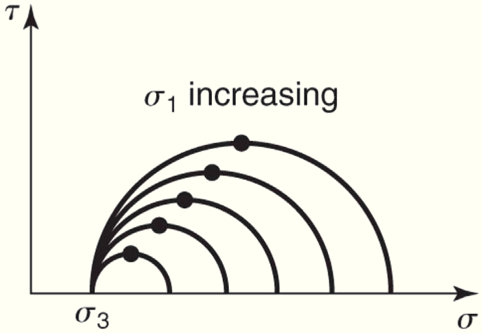
No shear stresses on these planes.

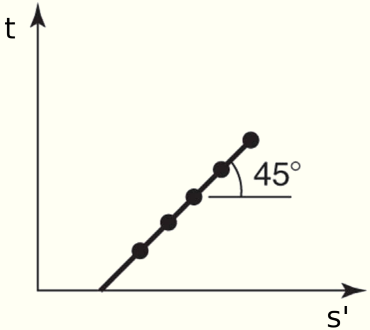
2D Mohr circle



Mohr circle to 2D stress components

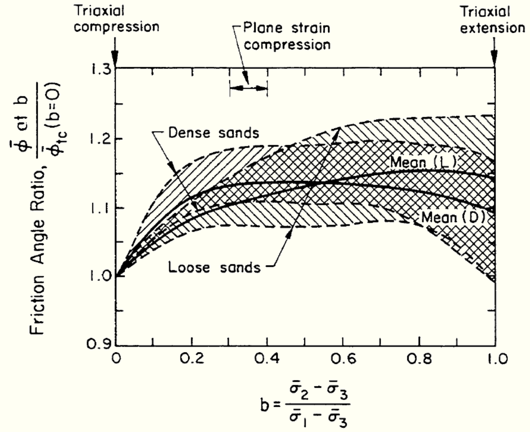
Successive Mohr circles and stress path for constant σ_3 and increasing σ_1:





Effect of σ_{II}

Bishop (1966) defined **b-value**: $b = (\sigma_{II} - \sigma_{III})/(\sigma_I - \sigma_{III})$, where $\sigma_I > \sigma_{II} > \sigma_{III}$. Triaxial compression $b = 0$, triaxial extension $b = 1$. Typically $\phi'_{ps} = (1.05 - 1.15)\phi'_{tx}$.



Effect of σ_{II} on friction angle (Kulhawy and Mayne, 1990)

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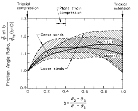

Effect of σ_{II} on friction angle (Kulhawy and Mayne, 1990)

σ_{II} do have an effect on soil behavior. For example, the friction angle depends on the loading condition: triaxial compression, plane-strain, triaxial extension and others ...., The effect of σ_{II} can be measured using true triaxial apparatus or hollow cylinder torsional shear apparatus. In general, the peak friction angle increases 10 to 15 percent from b = 0 (triaxial compression) to b = 0.3 to 0.4 (plane strain), and it stays constant or slightly decreases as b reaches 1 (tri-axial extension).

The variation of measured friction angle with changes in σ_{II} can be attributed to the effects of different mean stress and stress anisotropy on the dilatancy and particle rearrangement contributions to the total strength. For given maximum and minimum principal stresses, the TXE conditions have the largest mean effective stress, whereas the triaxial compression conditions have the smallest mean effective stress. The higher confinement for triaxial extension and plane strain conditions contributes to the increasing friction angle for these conditions. (Mitchell and Soga., 2005)

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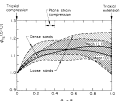
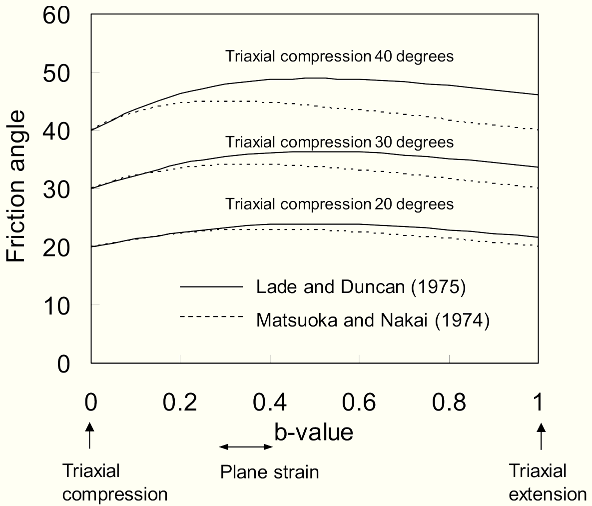


Effect of σ_{II} on friction angle (Kulhawy and Mayne, 1990)

In TXC major principal stress, σ_1 acts vertically, compared to triaxial extension mode of loading, where σ_1 acts horizontally. The effect of the intermediate principal stress is comparatively smaller, and is often assessed using the parameter $b = (\sigma_{II} - \sigma_{III})/(\sigma_I - \sigma_{III})$, which is zero in triaxial compression, and one in triaxial extension. In TXE $\sigma_I = \sigma_{II}$ (radial), while σ_{III} is the vertical stress.

Effect of σ_{II} on friction



Chapter 11., Mitchell and Soga, 2005

Effect of σ_{II} on friction

Various models fit the experimental data showing the intermediate stress effect. Lade and Duncan (1975)

$$\frac{I_1^3}{I_3} = const$$

Matsuoka and Nakai (1974)

$$\frac{I_1 I_2}{I_3} = const$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

Matsuoka and Nakai's model gives the same friction angle for compression and extension, whereas Lade and Duncan's model gives the ratio of the TXE friction angle (ϕ'_{TXE}) to the TXC friction angle (ϕ'_{TXC}) to be 1.08 at $\phi'_{TXC} = 20^o$ to 1.15 at $\phi'_{TXC} = 40^o$.

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$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

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Given the large scatter in the published experimental data, it is not possible to conclude that one model is better than the ohter.

Triaxial stresses and strains: independent components

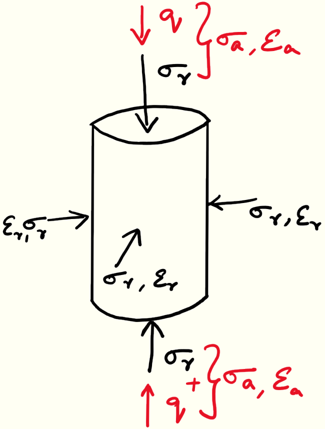
- We split the stress system into:
  - purely volumetric deformation 'p'
  - purely distortional deformation 'q'
- Mean stress:

$$p = (\sigma_a + 2\sigma_r)/3 \quad p' = p - u$$

- Volumetric strain: $\varepsilon_v = (\varepsilon_a + 2\varepsilon_r)$
- Deviatoric / shear stress (purely distortional deformation):

$$q = (\sigma_a - \sigma_r) \quad q' = q$$

- Deviatoric / shear strain: $\varepsilon_s = 2/3(\varepsilon_a - \varepsilon_r)$



Triaxial deviatoric strain

- Deviatoric / shear strain: $\varepsilon_s = 2/3(\varepsilon_a - \varepsilon_r)$
- Why 2/3?:
- Work equation: $\delta W = p'\delta\varepsilon_v + q'\delta\varepsilon_s$

$$\begin{aligned}
\delta W &= \sigma'_1\delta\varepsilon_1 + \sigma'_2\delta\varepsilon_2 + \sigma'_3\delta\varepsilon_3 = \sigma'_a\delta\varepsilon_a + 2\sigma'_r\delta\varepsilon_r \\
\delta W &= (1/3\sigma'_a + 2/3\sigma'_r)(\delta\varepsilon_a + 2\delta\varepsilon_r) + (\sigma'_a - \sigma'_r)(2/3)(\delta\varepsilon_a - \delta\varepsilon_r) \\
&= \sigma'_a\delta\varepsilon_a + 2\sigma'_r\delta\varepsilon_r
\end{aligned}$$

Triaxial stress - strain relationship

Relationship in terms of axial and radial direction:

$$\begin{bmatrix} \varepsilon_a \\ \varepsilon_r \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -2\nu \\ -\nu & (1-\nu) \end{bmatrix} \begin{bmatrix} \sigma_a \\ \sigma_r \end{bmatrix} \tag{1}$$

Relationship in terms of 'p' and 'q':

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_a \\ \sigma_r \end{bmatrix} \quad \text{invert} \rightarrow \begin{bmatrix} \sigma_a \\ \sigma_r \end{bmatrix} \begin{bmatrix} 1 & 2/3 \\ 1 & -1/3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{2}$$

Strain:

$$\begin{bmatrix} \varepsilon_v \\ \varepsilon_s \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} \varepsilon_a \\ \varepsilon_r \end{bmatrix} \tag{3}$$

Volume - shear coupling

Equation (1) and eq. (2) in eq. (3) to relate $\varepsilon_v, \varepsilon_s$ to p, q:

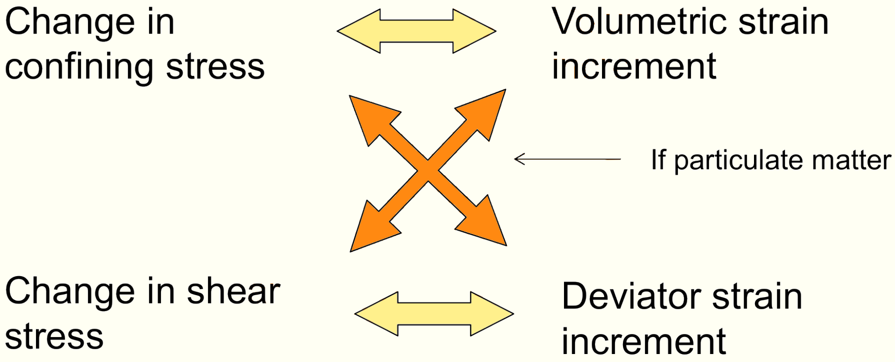
$$\begin{bmatrix} \varepsilon_v \\ \varepsilon_s \end{bmatrix} = \begin{bmatrix} 3(1-2\nu)/E & 0 \\ 0 & 2(1+\nu)/3E \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_v \\ \varepsilon_s \end{bmatrix} = \begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

1. Off-diagonal zeros indicate no coupling between volumetric and distortional effects.
2. Applying pure shear cannot induce volume changes - **not true for soils!**
3. Plasticity theory will deal with this.

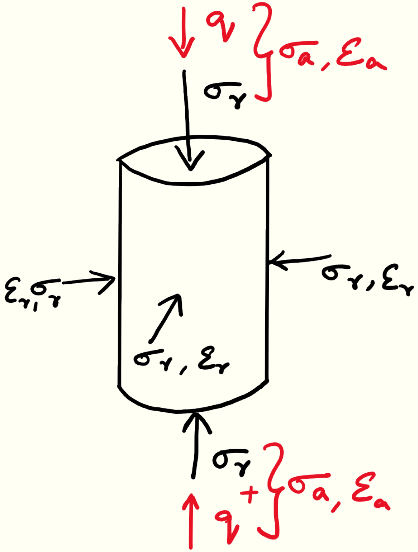
Volume shear coupling

Usually volume change and shear deformation are decoupled (i.e. bulk modulus and shear modulus), but in soils they are coupled:



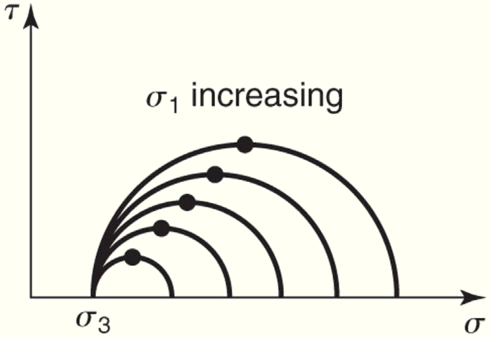
Overview

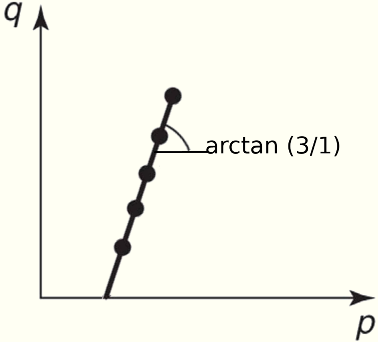
Triaxial test



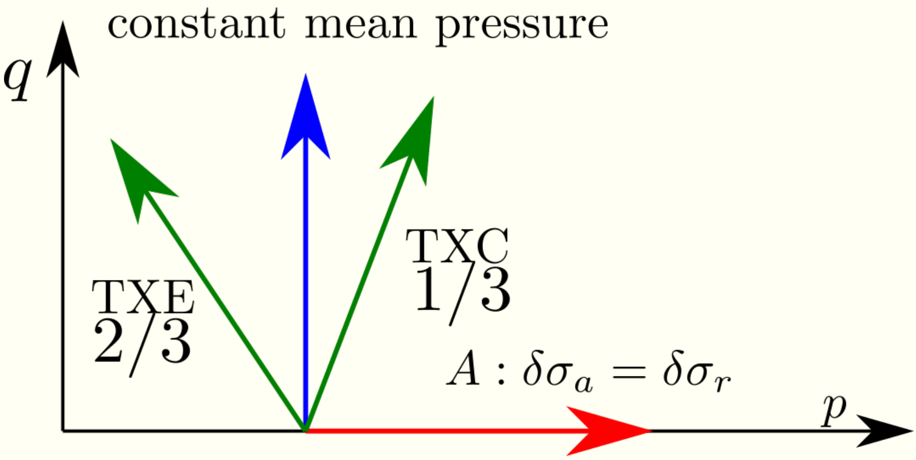
Mohr circle to 3D stress components

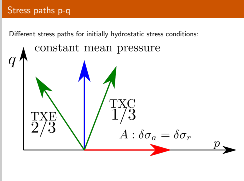
Successive Mohr circles and stress path for constant σ_3 and increasing σ_1:





Different stress paths for initially hydrostatic stress conditions:





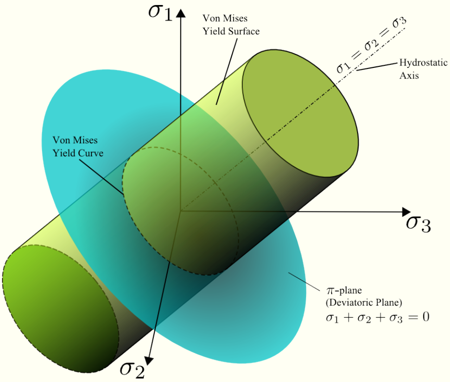
A mean total stress $p = \frac{\sigma_a + 2\sigma_r}{3}$ and deviatoric stress $q = \sigma_a - \sigma_r$. In triaxial compression the cell pressure is held constant, $\delta\sigma_r = 0$ and axial load is increased.

$$\delta p = \frac{\delta q}{3}$$

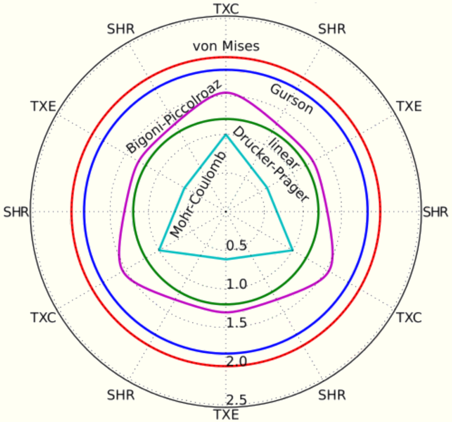
In triaxial extension, to keep the axial stress constant while changing the cell pressure requires that the ram force (or deviatoric stress) and cell pressure must be changed simultaneously. For $\delta\sigma_a = 0, \delta\sigma_r = -\delta q$. The the differential form is:

$$\delta p = \frac{2\delta\sigma_r}{3} = \frac{-2\delta q}{3}$$

Pi-plane

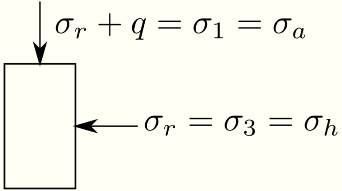


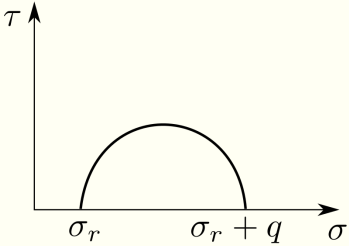
σ_1
Von Mises
Yield Surface
$\sigma_1 = \sigma_2 = \sigma_3$
Hydrostatic
Axis
Von Mises
Yield Curve
σ_3
π-plane
(Deviatoric Plane)
$\sigma_1 + \sigma_2 + \sigma_3 = 0$
σ_2

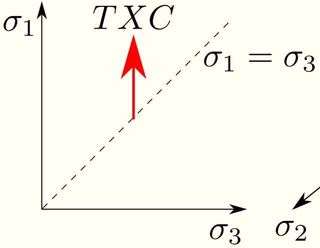


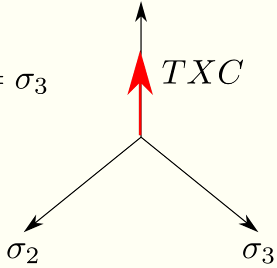
TXC
SHR
SHR
von Mises
TXE
TXE
Gurson
Bigoni-Piccolroaz
linear
Drucker-Prager
Mohr-Coulomb
SHR
SHR
0.5
1.0
1.5
2.0
2.5
TXC
TXC
SHR
SHR
TXE

π plane: Triaxial compression

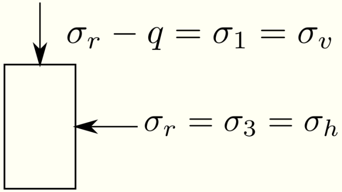


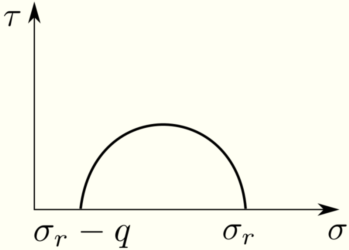


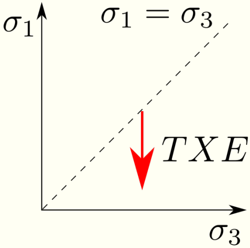


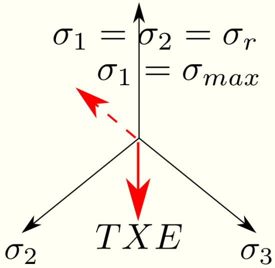


π plane: Triaxial extension



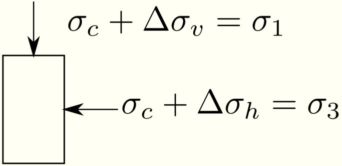


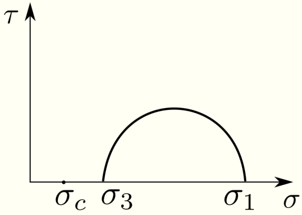


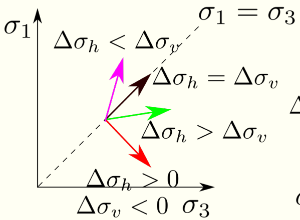


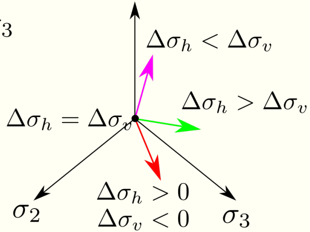
w/o forcing $\sigma_1 = \sigma_{max}$

π plane: Random stress paths









Overview

Stress and strain invariants in 3D

6 stresses and strains

- Mean pressure: $p' = (\sigma'_{11} + \sigma'_{22} + \sigma'_{33})/3$
- Deviator stress: $q = \sqrt{3/2}\sqrt{s_{11}^2 + s_{22}^2 + s_{33}^2 + 2\tau_{12}^2 + 2\tau_{23}^2 + 2\tau_{31}^2}$
- where $s_{11} = \sigma'_{11} - p', s_{22} = \sigma'_{22} - p', s_{33} = \sigma'_{33} - p'$
- Volumetric strain: $\varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$
- Deviatoric strain: $\varepsilon_s = \sqrt{2/3}\sqrt{e_{11}^2 + e_{22}^2 + e_{33}^2 + \gamma_{12}^2/2 + \gamma_{23}^2/2 + \gamma_{31}^2/2}$
- where $e_{11} = \varepsilon_{11} - \varepsilon_v/3, e_{22} = \varepsilon_{22} - \varepsilon_v/3, e_{33} = \varepsilon_{33} - \varepsilon_v/3$

3 Principal stresses and strains

- Mean pressure: $p' = (\sigma'_I + \sigma'_{II} + \sigma'_{III})/3$
- Deviator stress: $q = \sqrt{1/2}\sqrt{(\sigma'_I - \sigma'_{II})^2 + (\sigma'_{II} - \sigma'_{III})^2 + (\sigma'_{III} - \sigma'_I)^2}$
- Volumetric strain: $\varepsilon_v = \varepsilon_I + \varepsilon_{II} + \varepsilon_{III}$
- Deviatoric strain: $\varepsilon_s = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon'_I - \varepsilon'_{II})^2 + (\varepsilon'_{II} - \varepsilon'_{III})^2 + (\varepsilon'_{III} - \varepsilon'_I)^2}$

In triaxial condition (principal stresses/strains)

$$p' = (\sigma_I + 2\sigma_{III})/3 \quad q = \sigma_I - \sigma_{III} \quad \varepsilon_v = \varepsilon_I + 2\varepsilon_{III} \quad \varepsilon_s = 2(\varepsilon_I - \varepsilon_{III})/3$$

Stress and strain invariants in 3D

3 Principal stresses and strains

- Mean pressure: $p' = (\sigma'_I + \sigma'_{II} + \sigma'_{III})/3$
- Deviator stress: $q = \sqrt{1/2}\sqrt{(\sigma'_I - \sigma'_{II})^2 + (\sigma'_{II} - \sigma'_{III})^2 + (\sigma'_{III} - \sigma'_I)^2}$
- Volumetric strain: $\varepsilon_v = \varepsilon_I + \varepsilon_{II} + \varepsilon_{III}$
- Deviatoric strain: $\varepsilon_s = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon'_I - \varepsilon'_{II})^2 + (\varepsilon'_{II} - \varepsilon'_{III})^2 + (\varepsilon'_{III} - \varepsilon'_I)^2}$

In triaxial condition (principal stresses/strains)

$p' = (\sigma_I + 2\sigma_{III})/3 \quad q = \sigma_I - \sigma_{III} \quad \varepsilon_v = \varepsilon_I + 2\varepsilon_{III} \quad \varepsilon_s = 2(\varepsilon_I - \varepsilon_{III})/3$

The magnitudes of the components of the stress vector (i.e. $\sigma_{\sigma_{xx},\sigma_{yy},\sigma_{zz},\tau_{xy},\tau_{xz},\tau_{yz}}$) depend on the chosen direction of the coordinate axes. The principal stresses $(\sigma_I, \sigma_{II}, and \sigma_{III})$ however, always act on the same planes and have the same magnitude, no matter which direction is chosen for the coordinate axes. They are therefore invariant to the choice of axes. Consequently, the state of stress can be fully defined by either specifying the six component values for a fixed direction of the coordinate axis, or by specifying the magnitude of the principal stresses and the directions of the three planes on which these principal stresses act. In either case six independent pieces of information are required.

The missing stress invariant

- **General stresses:** $\sigma_{ij}(i,j=1,2,3) \quad \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}$

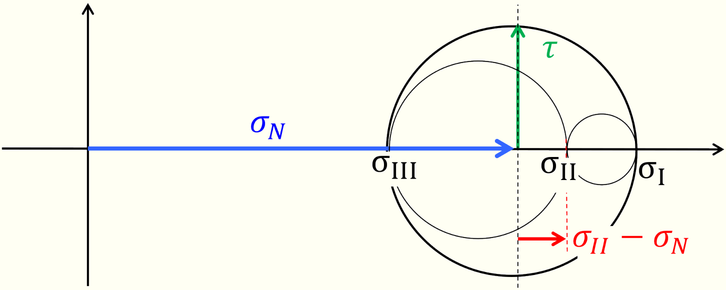
$$\Updownarrow$$

- **Principal stresses:** $\sigma_I, \sigma_{II}, \sigma_{III} + 3 angles$

$$\Downarrow \quad \not\Uparrow$$

Two stress invariant (mean pressure p or s, deviator stress q or t). One more stress invariant is needed to go back to 3 principal stresses: **Lode angle!**

The Lode parameter

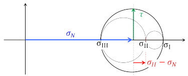


Maximum shear stress: $\tau = \frac{\sigma_I - \sigma_{III}}{2}$

Normal stress: $\sigma_N = \frac{\sigma_I + \sigma_{III}}{2}$

Position of intermediate principal stress: $L = \frac{\sigma_{II} - \sigma_N}{\tau}$

The Lode parameter



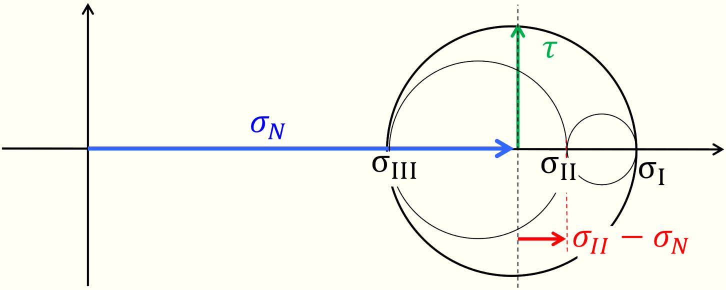
Maximum shear stress: $\tau = \frac{\sigma_I - \sigma_{III}}{2}$

Normal stress: $\sigma_N = \frac{\sigma_I + \sigma_{III}}{2}$

Position of intermediate principal stress: $L = \frac{\sigma_{II} - \sigma_N}{\tau}$

If one is interested in the overall magnitude of the stress, all three principal stresses would be needed, but not the directions of the planes on which they act. In geotechnical engineering it is often convenient to work with alternative invariant quantities which are combinations of the principal effective stresses.

The Lode parameter



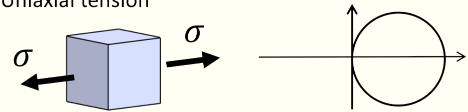
Maximum shear stress: $\tau = \frac{\sigma_I - \sigma_{III}}{2}$

Normal stress: $\sigma_N = \frac{\sigma_I + \sigma_{III}}{2}$

Position of intermediate principal stress: $L = \frac{\sigma_{II} - \sigma_N}{\tau}$

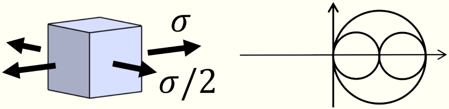
Which states have the same Lode parameter?

- Uniaxial tension



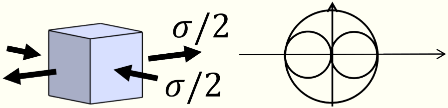
$$\left.\begin{aligned}\sigma_m &= \sigma/3\\ \bar{\sigma} &= \sigma\end{aligned}\right\}$$

- Plane strain tension

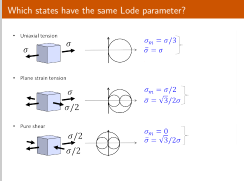


$$\left.\begin{aligned}\sigma_m &= \sigma/2\\ \bar{\sigma} &= \sqrt{3}/2\sigma\end{aligned}\right\}$$

- Pure shear



$$\left.\begin{aligned}\sigma_m &= 0\\ \bar{\sigma} &= \sqrt{3}/2\sigma\end{aligned}\right\}$$



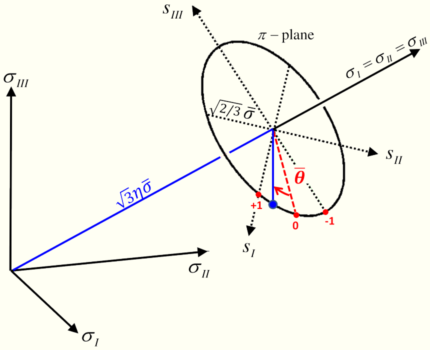
Smallest angle between the line of pure shear and the projection of the stress tensor on the deviatoric plane

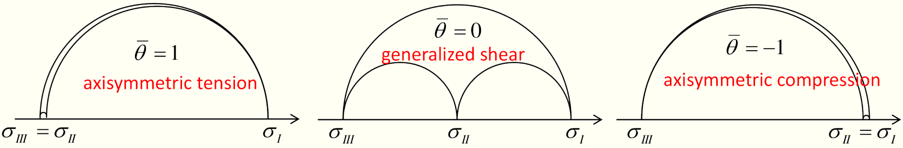
Lode angle parameter

Lode angle parameter (Lode, 1926)

$$L = \frac{2\sigma_{II} - \sigma_I - \sigma_{III}}{\sigma_I - \sigma_{III}}$$

Lode angle $\bar{\theta} \approx -L$



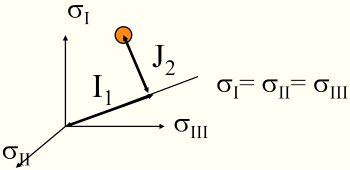


Overview

Stress invariant

Principal stresses (3 components) $\Leftrightarrow$ Invariants (3 components)

- $I_1 = p' = \sigma'_{ii}/3$
- $s_{ij} = \sigma'_{ij} - \delta_{ij} p'$
- $J_2 = \frac{1}{2} s_{ij} s_{ij}$
- $J_2 = \frac{1}{2} s_{ij} s_{ij}$
- $q = \sqrt{3/2}(s_{ij} s_{ij})^{0.5} = \sqrt{3} J_2$



Lode angle:

- $\sin 3\theta = \frac{-3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$.
- $\theta = \tan^{-1}\left[\frac{1}{\sqrt{3}}\left(2\frac{(\sigma_2 - \sigma_3)}{(\sigma_1 - \sigma_3)} - 1\right)\right]$
- $TXC = \pi/6 \quad TXE = -\pi/6$
- $SHR = 0$, when $\sigma_2 = (\sigma_1 + \sigma_3)/2$ (note $SHR \neq PS$)