

CE394M Advanced Analysis in Geotechnical Engineering: Introduction

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# Overview

## 1 Geotechnical modeling

- Complexity in Geotechnical modeling
- Classical vs advanced analysis

## 2 Numerical methods for differential equations

- Direct method: Matrix analysis of structures
- Numerical analysis of engineering problems

## 3 Governing equations in stress-deformation analysis

- Equilibrium

# Geotechnical modeling of the complex world



Fig. London Bridge Station, London, UK

# Geotechnical modeling of the complex world

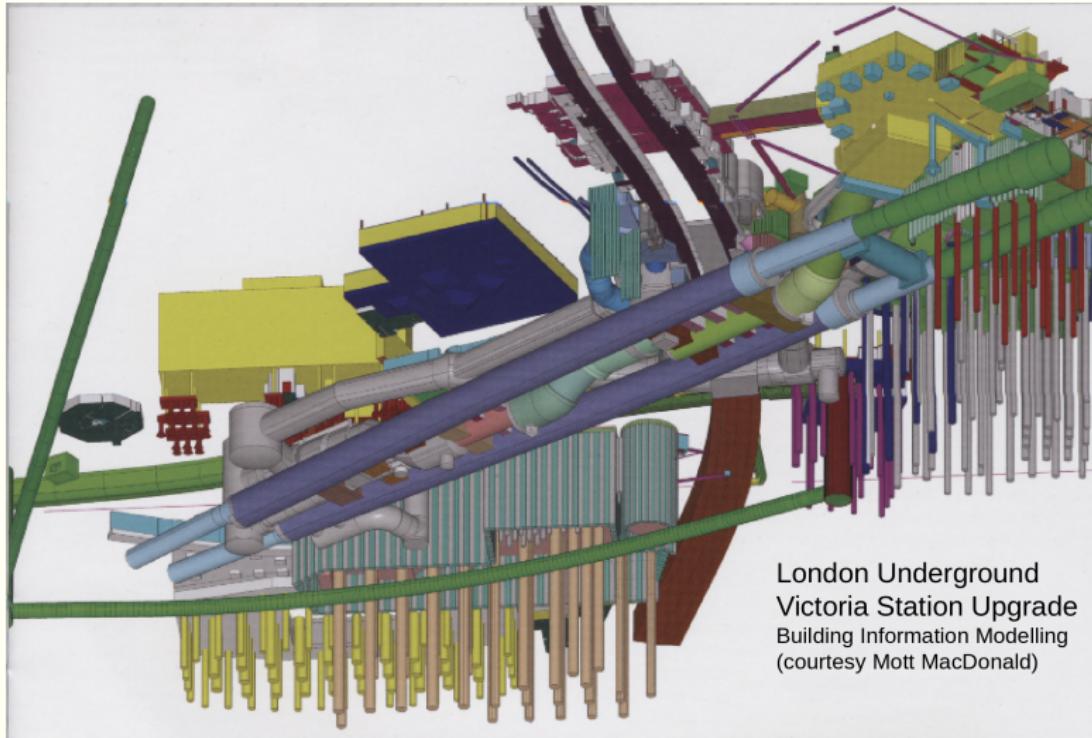
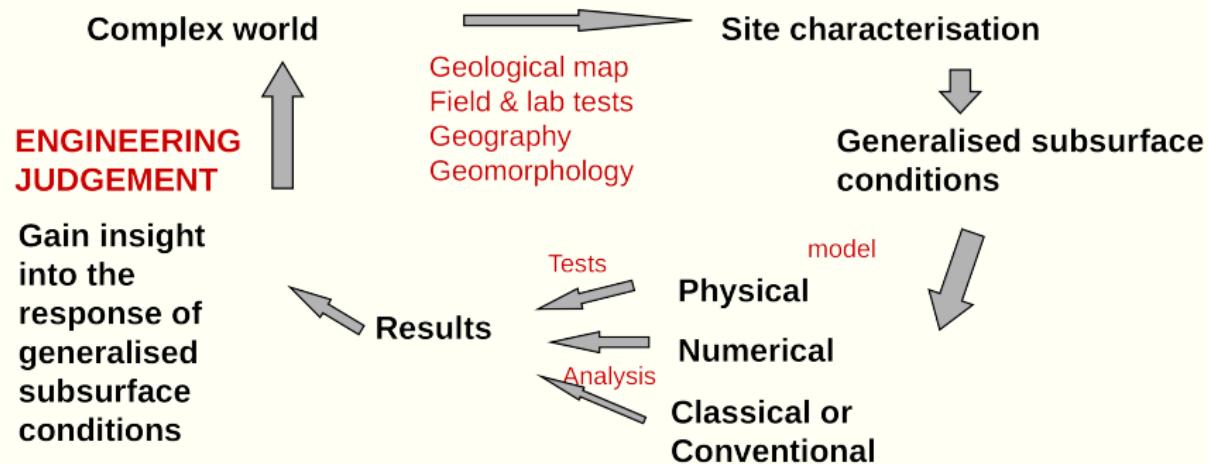


Fig. London Victoria station upgrade, London, UK

# Geotechnical modeling



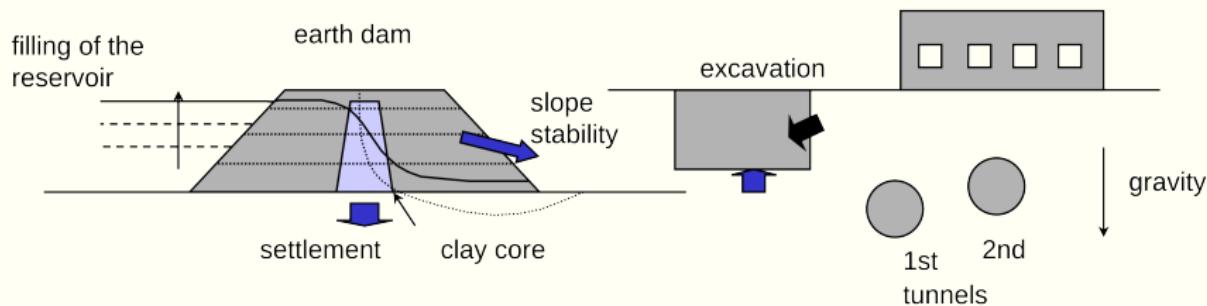
# Soil behavior

- nonhomogeneous,
- anisotropic,
- non-linear,
- initial stress conditions,
- stress history
- Geometry - very complex

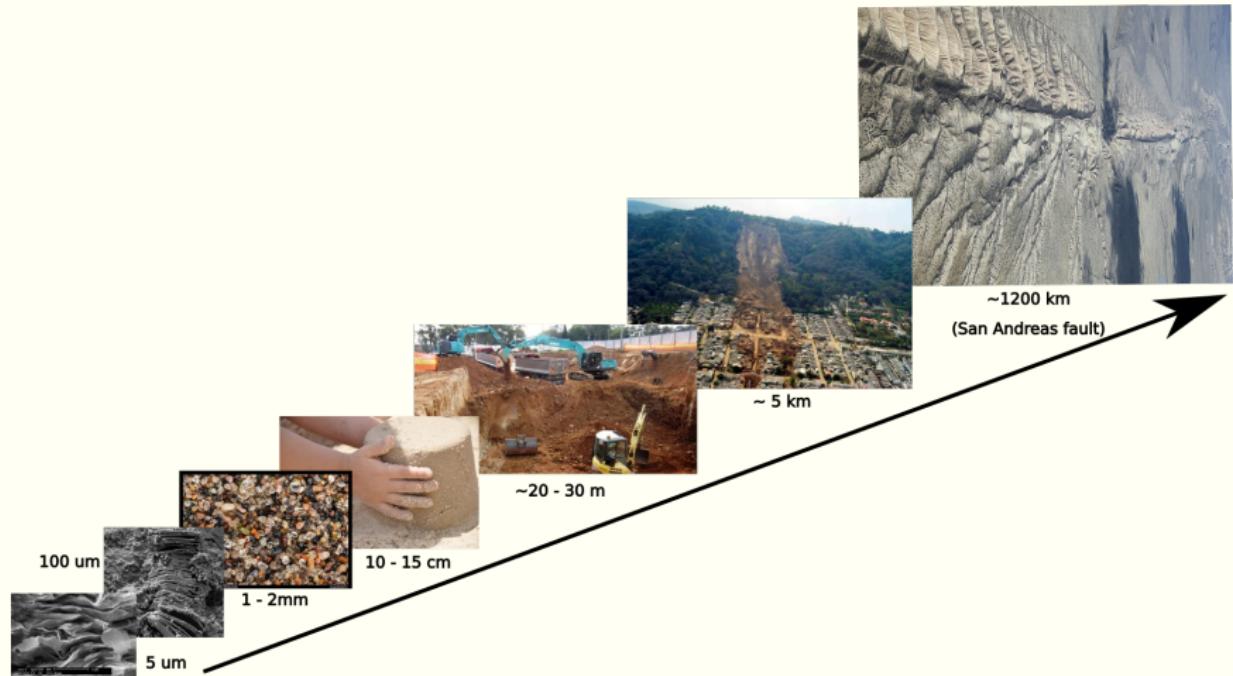
**Soil Mechanics in practice - largely empirical**

# Geotechnical modeling: What should be modeled?

- Self weight effect of soils (This is why soil moves)
- Construction sequence (Complex geometry)
- Water movement (undrained, consolidation, drained)
- Insitu stresses (stiffness/strength depends on current stresses and stress history)
- Predict the ability of a design to withstand extreme loading conditions (you only have one chance)



# Scales of modeling in geotechnical engineering



# Advanced analysis in geotechnical engineering

## Geotechnical design:

- Assess applied forces
- evaluate “performance” (stability & movements) under working and ultimate loads

## Analysis:

- Mathematical framework to perform calculations for these quantities
- Requires idealization of: geometry, soil properties, and loading conditions
- Analysis is a tool in design, but design involves more: acceptable movements, constraints, site characterization, etc.

# Classical vs advanced analysis

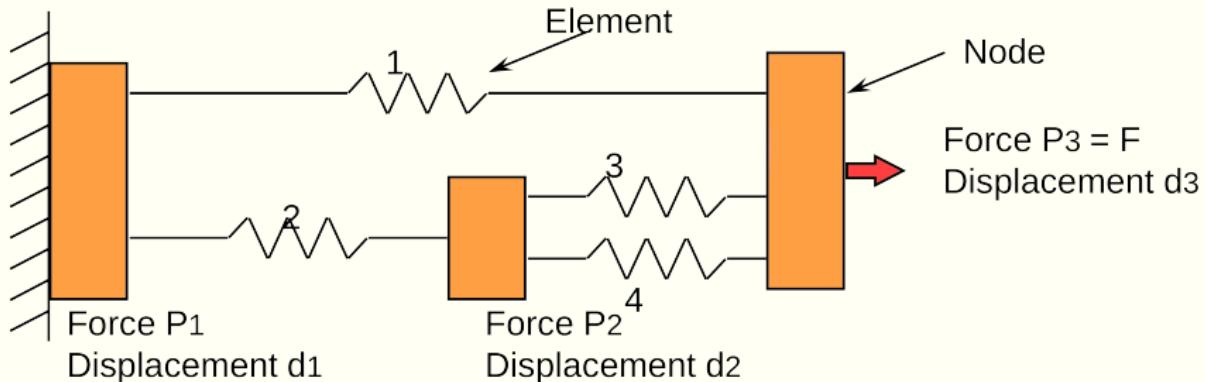
## Classical approach:

- Failure estimates
  - rigid perfectly-plastic stress-strain assumptions
  - calculate factor of safety (What value do you pick?)
- Deformation estimates
  - Elastic analysis
  - Use average elastic properties (like what??)

## Advanced analysis:

- Failure and deformation are obtained from the same analysis
- Handle complex geometry
- More difficult to perform, more computational requirements and more info on soil behavior  $\sigma - \epsilon$
- Need to know how to do it right!

# Matrix analysis of structures



- What are the known variables?  $d_1 = 0, P_2 = 0, P_3 = F(\text{constant})$
- What are the unknowns?  $P_1, d_2, d_3$
- What do we know? Force or distortion relations at an element level.

# Matrix analysis of structures: Equilibrium

A structure is considered to be in equilibrium if, initially at rest, it remains at rest when subjected to a system of forces and couples. If a structure is in equilibrium, then all of its members and joints must also be in equilibrium.

- $P_1 = -S_1 - S_2$
- What are the unknowns?  $P_1, d_2, d_3$
- What do we know? Force or distortion relations at an element level.

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{A}^T \mathbf{S}$$

# Matrix analysis of structures: Compatibility

- compatibility relates the deformations of a structure so that its various parts (members, joints, and supports) fit together without any gaps or overlaps.
- ensure that the deformed shape of the structure is continuous (except at the locations of any internal hinges or rollers), and is consistent with the support conditions.

# Matrix analysis of structures: Compatibility

$v$  = internal spring distortion  $\delta$  = nodal displacement

- $v_1 = d_3 - d_1$
- $v_2 = d_2 - d_1$
- $v_3 = d_3 - d_2$
- $v_4 = d_3 - d_2$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\mathbf{v} = \mathbf{A}\mathbf{d}$$

# Matrix analysis of structures: Physical condition

Force-distance relationship: spring constant

spring #	1	2	3	4
stiffness ( $F.L^{-1}$ )	3	2	1	2

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$\mathbf{s} = \mathbf{Dv}$$

## Matrix analysis of structures: Direct Method

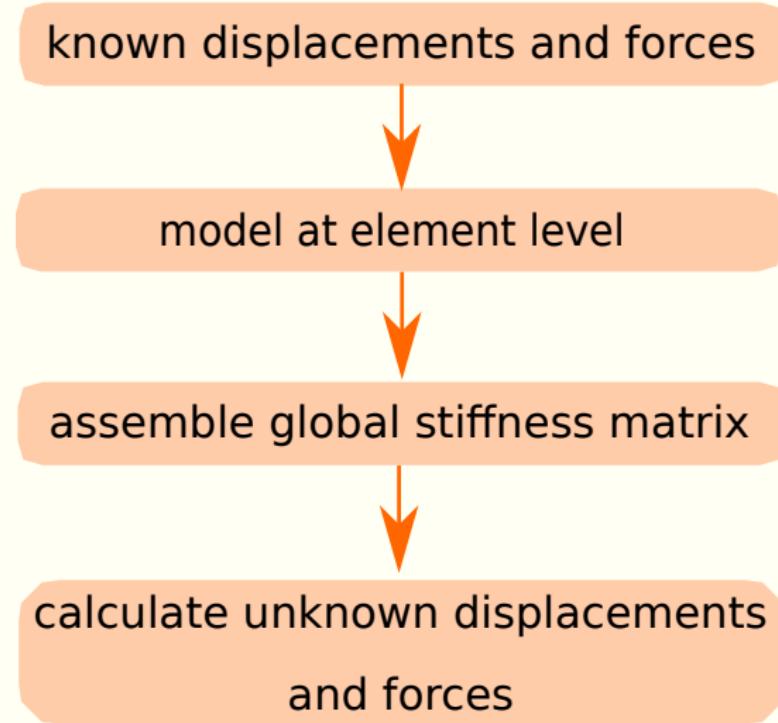
Combine all the equations:  $\mathbf{P} = \mathbf{A}^T \mathbf{S} = \mathbf{A}^T \mathbf{D} \mathbf{v} = \mathbf{A}^T \mathbf{D} \mathbf{A} \mathbf{d} = \mathbf{K} \mathbf{d}$   
where  $\mathbf{K} = \mathbf{A}^T \mathbf{D} \mathbf{A}$  (Global stiffness matrix)

$$\begin{aligned}\mathbf{K} &= \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -2 & -3 \\ -2 & 5 & -3 \\ -3 & -3 & 6 \end{bmatrix}\end{aligned}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -3 \\ -2 & 5 & -3 \\ -3 & -3 & 6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

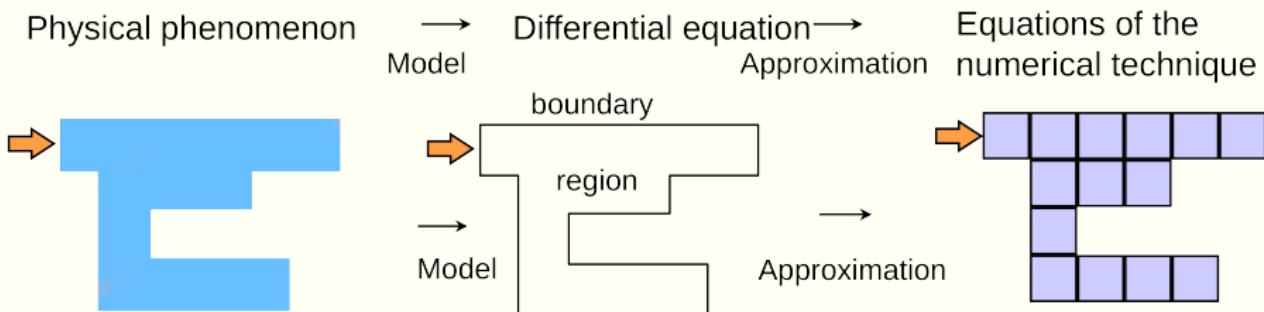
Apply Boundary conditions  $d_1 = 0$ ,  $P_2 = 0$  and  $P_3 = F$  and solve  $P_1$ ,  $d_2$  and  $d_3$

# Matrix analysis of structures



# Numerical analysis of engineering problems

- Conceptualize the system
  - Geometry
  - Properties
  - Processes
- Describe it mathematically
  - Select the relevant differential equations
- Solve the equations (numerically)
  - Discretize the system
  - Settle for approximations (numerical techniques)
- Interpret the results



# Boundary value problems

Differential equations coupled with boundary conditions

- Steady state (time-independent)
  - Static load-deformation problems:  $\partial\sigma/\partial x = 0$  (force + disp. B.C)
  - Steady seepage state, flow problems:  $\partial q/\partial x = 0$  (head + flow B.C)
- Transient (time-dependent)
  - Consolidation/pore-fluid migration/multiphase flows
  - Dynamic loading (earthquakes, wave actions)
  - Contaminant transport processes

# Numerical solutions to differential equations

- Finite differences: Approximate derivatives with expansion into a Taylor series
- Finite elements
- Boundary element method (BEM)
- Meshless methods
- Discrete/discontinuous element methods
- Others...

# Governing equations in stress-deformation analysis

In stress-deformation analysis, we need to consider:

- **Equilibrium - static conditions**

- forces and stress must agree across the region of interest. (geometric problem)

- **Compatibility-kinematic conditions**

- geometry, displacement and strains must agree across the region of interest. (geometric problem)

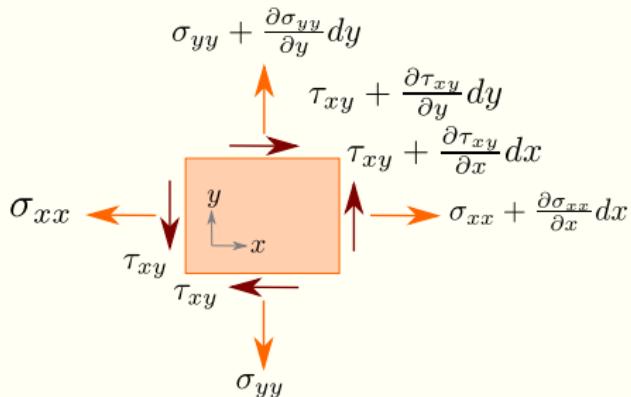
- **Stress-strain relationship on physical conditions**

- material dependent relationship between stress and strain must be specified. (element level)

# Governing equations in stress-deformation analysis

The governing differential equation for equilibrium expresses:  $\sum \mathbf{F} = ma$

- $\sigma_{xx}$  acting on face  $dy$  in the  $-x$  direction
- $\tau_{xy}$  acting on face  $dx$  in the  $-x$  direction
- $\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx$  acting on face  $dy$  in the  $+x$  direction
- $\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy$  acting on face  $dx$  in the  $+x$  direction
- Plus “body forces” due to gravity:  $\rho f_x dx dy$  where  $f_x$  is body force per unit mass



# Equilibrium equations

Summing all this in the x-direction gives:

$$-\sigma_{xx}dy - \tau_{xy}dx + \left( \sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}dx \right) dy + \left( \tau_{xy} + \frac{\partial\tau_{xy}}{\partial y}dy \right) dx + \rho f_x dxdy = \rho dxdy a_x$$

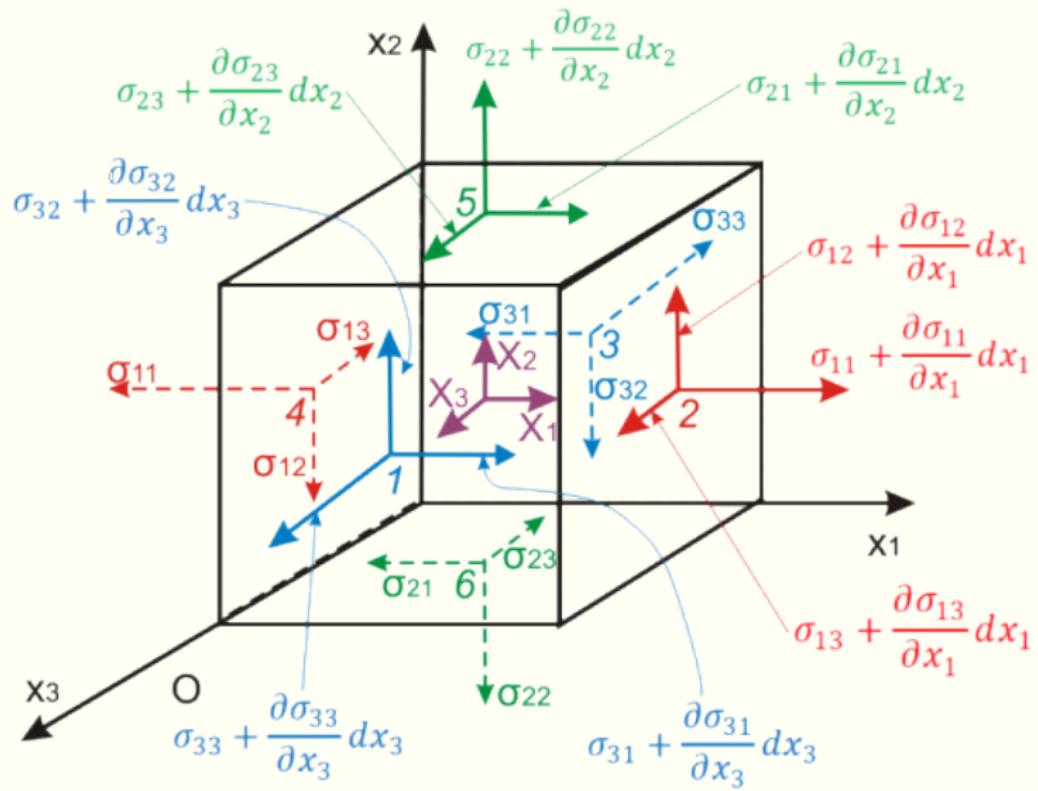
Cleaning up terms that cancel, and dividing through by  $dxdy$  gives

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \rho f_x = \rho a_x$$

And summing forces in the y-direction leads to:

$$\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \rho f_y = \rho a_y$$

# Equilibrium in 3D



# Equilibrium in 3D

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x = \rho a_x$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y = \rho a_y$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z = \rho a_z$$

The governing differential equation for equilibrium expresses  $\sum \mathbf{F} = m\mathbf{a}$  in terms of derivatives of the stress tensor as:  $\nabla \boldsymbol{\sigma} + \rho \mathbf{f} = \rho \mathbf{a}$

$\boldsymbol{\sigma}$  is the stress tensor,  
 $\rho$  is density,  
 $\mathbf{f}$  is the body force vector per unit mass and  
 $\mathbf{a}$  is the acceleration vector.

# Stress equilibrium

If the object is in equilibrium, then  $\mathbf{a} = 0$  and  $\sum \mathbf{F} = 0$ .

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

Stresses in Voigt notation:  $[\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}]^T$ .

Equilibrium equation:  $\nabla^T \boldsymbol{\sigma} + \mathbf{b} = 0$

Then:

$$\nabla^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$