THE CRITICAL STATE THEORY AND STRESS DILATANCY

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1. INTRODUCTION

The critical state theory was introduced by **Roscoe et al. (1958)** based on the findings that, under sustained uniform shearing at failure, the magnitude of shear resistance is a function of mean effective pressure and the element volume (void ratio, specific volume, porosity or solid fraction[1]) remains constant (*i.e.* no volume change) once the granular material reaches the ultimate failure (*i.e.* critical state). That is, there is a unique relationship among shear resistance, mean pressure and element volume when a granular material is at failure. This has been a powerful concept in soil mechanics because the relationship allows (i) to evaluate the ultimate shear resistance if the effective mean pressure is given, ii) to quantify the amount of volume change from the initial state of a given set of stresses and element volume, and (iii) to evaluate the stress state at failure if the element volume at failure is given. Although the actual mathematical form of the relationship varies depending on grain characteristics, it is interesting to note that the theory appears to work on a wide range of granular materials from fine grained clayey soils to coarse grained materials such as sands.

2. THE CRITICAL STATE THEORY

2.1 General

Figure 1 illustrates the stress-strain relationship of loosely and densely packed granular systems confined at a given initial confining pressure and then sheared. The densely packed system shows a peak shear resistance and then softens to a residual state (called critical state). The shearing is accompanied by volumetric change. Initially the system contracts (volume or

[1] See Appendix A for the definitions and conversions.

void ratio decreases) but then starts to dilate (volume or void ratio increases). The rate of dilation increases, reaches its maximum and then becomes zero at the critical state. The peak shear resistance often occurs somewhere between the point the dilation starts to happen and the point when the rate of dilation is maximum. Therefore, an increase in pore space essentially contributes to the granular material to start to decrease its shear resistance (i.e. strain-softening). On the other hand, when the loosely packed system is sheared at the same initial confining pressure, the shear resistance monotonically increases and reaches to the critical state. At the same time, the soil volume decreases with increasing shear deformation. The decrease in pore space and the associated denser packing provide the condition for the shear resistance to increase monotonically.

The critical state is marked with a red dot in **Figure 1**. It is important to note that the shear resistance at the critical state of the loosely packed system coincides with that of the densely packed system at a given initial confining pressure. At the same time, the specimen volume (or void ratio) becomes the same. This implies that the initial arrangement of particle packing (dense or loose) is lost by shearing and the granular system at the critical state has its unique packing arrangement when it is sheared continuously. As will be shown later on, this unique packing arrangement depends on the level of confining stress. The critical state void ratio decreases with increasing confining stress.

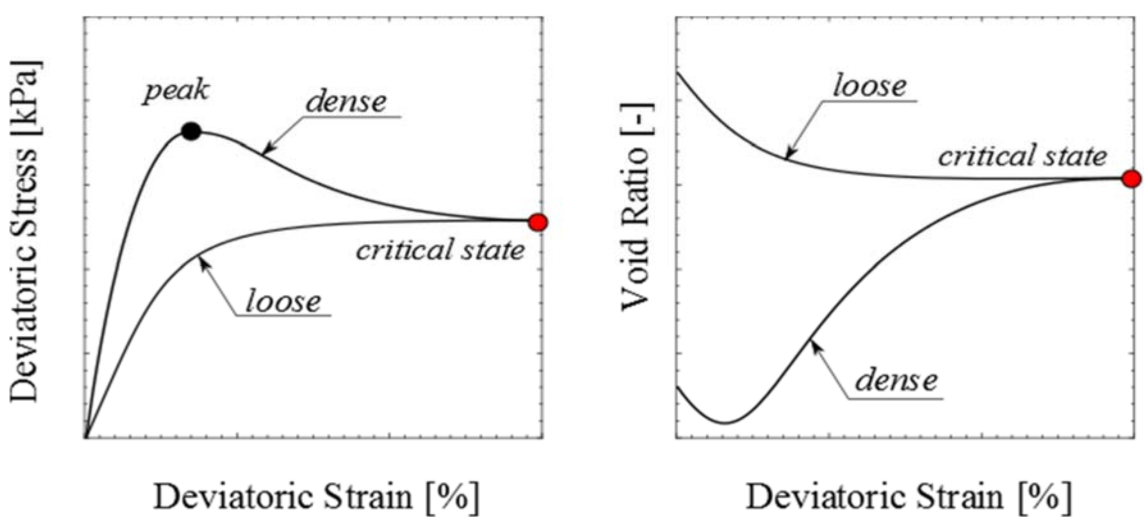

Figure 1: Stress-strain curves of loose and dense soils sheared at the same confining stress

At a microscopic level, a granular system consists of grains. **Figure 2** shows a schematic description of the grain packing at different states. These packets of grains are rearraged when the granular body is sheared.

The densely packed system (**Figure 2-left**) have grains which jam to one another by deformation and these grains have to move apart in order to allow pakets of grains to flow by shear. A simple analalogy is that grains need to override each other, resulting in volume increase. However, discrete element model simulations and microscopic observations through microfocused CT scans show that the grains group together to form chains, which results in larger pore space inside the chains. Hence, the shear-induced dilation is a product of complex granular interactions.

In the loosely packed system (**Figure 2-middle**), the large voids decrease their volumes upon shearing. At the same time the shear resistance increases with better oveall packing arrangement. The strength and density of the granular body then vary until reaching the critical state when the packets of grains can mobilize over each other.

At the critical state (**Figure 2-right**), the shear resistance is constant and the granular structure continously changes by shearing but the average void space remains the same to have the constant void ratio state.

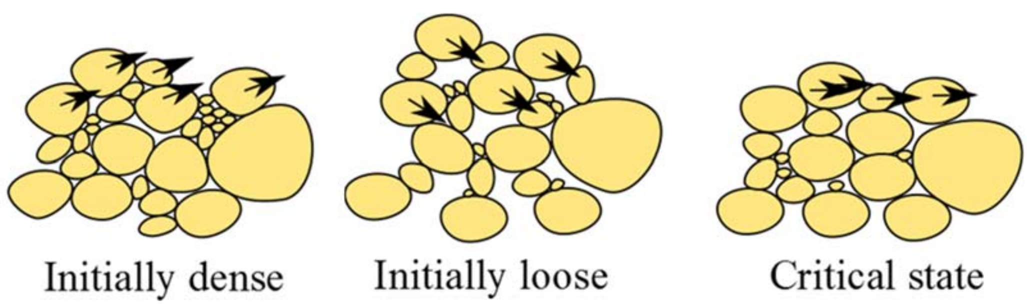

Figure 2: Schematic description of the grain packing at different states

The critical state soil mechanics **(Schofield and Wroth 1968)** uses the following stress and strain invariants.

Mean effective pressure $$p' = tr(\sigma'_{ij})/3$$ Eq. 1

Deviator stress $$q = \sqrt{\frac{3}{2} s_{ij} s_{ij}}$$ Eq. 2

Volumetric strain $\varepsilon_v = tr(\varepsilon_{ij})$ Eq. 3

Deviator strain $\varepsilon_d = \sqrt{\frac{2}{3} e_{ij} e_{ij}}$ Eq. 4

where σ'_{ij} is the effective stress tensor (i,j = 1,3), $s_{ij} = \sigma'_{ij} - p\delta_{ij}$ is the deviator stress tensor, ε_{ij} is the strain tensor, and $e_{ij} = \varepsilon_{ij} - \varepsilon_v \delta_{ij}/3$ is the deviator strain tensor.

At critical state, the granular material will be continuously deformed in shear without any changes in volume or stresses. Such conditions can be expressed mathematically as shown in **Eq. 5**.

$$\frac{d\eta'}{d\varepsilon_d} = \frac{d\varepsilon_v}{d\varepsilon_d} = \frac{d^2\varepsilon_v}{d{\varepsilon_d}^2} = 0 \qquad \text{Eq. 5}$$

where $\eta' = {}^{q}/_{p'}$ is the effective stress ratio with q the deviatoric stress and p' the mean effective stress, ε_v the volumetric strain and ε_d the deviatoric strain, as defined earlier. In this chapter, the dilation rate $\frac{d\varepsilon_v}{d\varepsilon_d}$ is defined as *D*, whereas the rate of dilation rate $\frac{d^2\varepsilon_v}{d{\varepsilon_d}^2}$ is defined as $\frac{dD}{d\varepsilon_d}$.

Roscoe et al. (1958) showed that the stress state and density of any granular body are constant and uniquely defined when the soil is subjected to sufficient shearing. This state is defined by a critical state line (CSL) as shown in **Eq. 6** and **Eq. 7.** Both equations must be fulfilled in order to be at critical state.

$$\eta' = \frac{q_{cs}}{p'_{cs}} = M \qquad \text{Eq. 6}$$

$$e_{cs} = f(p') \qquad \text{Eq. 7}$$

Graphically these conditions are illustrated in **Figure 3** (red lines). When the state of the granular material satisfies both conditions, the material is at critical state failure.

2.2 Critical State of Clay

A normally consolidated clay has never been previously loaded and hence has a loose structure at a given confining stress as shown by point A in **Figure 3 (a-b).** It exhibits contractive behaviour by shearing in drained conditions before reaching the critical state at point B. The initial void ratio is larger than the one at critical state for a given confining stress p' and the specimen must reduce its volume in order to reach the critical state. If a shearing test is conducted in undrained conditions (*i.e.* pore water cannot move in or out of the clay), the constant volume constraint due to the large bulk stiffness of the pore water (i.e. incompressible conditions) will bring the effective stress state of the clay to the position shown in C. A positive excess pore pressure develops and the difference between point A and point C is the magnitude of the excess pore pressure.

An overconsolidated clay undergoes a loading-unloading cycle in terms of confining stress. If the degree of unloading stress relative to the loading stress is large (*i.e.* heavily overconsolidated), the void ratio after unloading can be below the critical state void ratio at a given confining pressure as shown at point A in **Figure 3 (c-d)**. When the clay is sheared in drained conditions, it dilates and increase its volume. During this process, the clay develops a peak strength at Point C. The clay then softens until the stresses reach the critical state stress ratio at point D. When the clay is sheared in undrained conditions, the effective mean stress moves to the right in the ($\ln p', e$) plane and reaches the critical state at point E. The effective mean stress at failure is greater than the effective mean stress prior to shearing. This means that negative excess pore pressure is generated due to dilative tendency of the heavily overconsolidated clay.

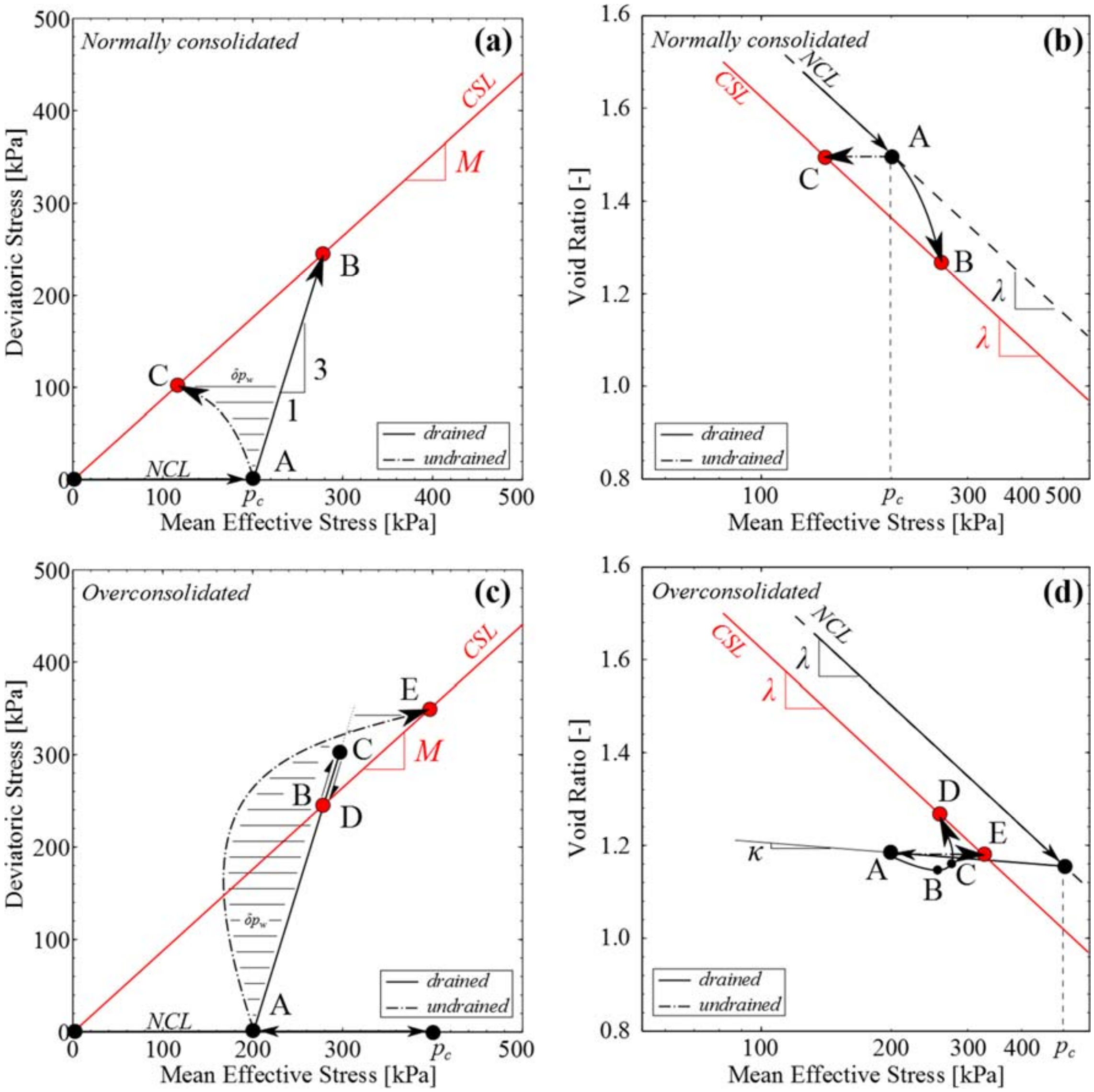

Figure 3: Triaxial compression stress paths in (a-b) drained and (c-d) undrained conditions

There is experimental evidence of the existence of the critical state for clays. **Roscoe et al. (1958)** carried out a series of triaxial compression tests on Weald clay for both normally consolidated and oversconsolidated specimens. **Figure 4** shows the final stress and strain state for all the tests. It can be seen that all tests have reached the same ultimate critical state irrespective of their initial state. **Figure 4 (a)** shows the final stresses in the (p', q) plane and **(b)** in the $(\ln p', e)$ plan for both normally consolidated specimens (white markers) and overconsolidated specimens (black markers). The final stresses are set on a line with a linear relationship between the mean effective stress and the deviatoric stress (i.e. frictional behaviour). The final density of the clay, expressed in terms of specific volume v or void ratio

e, has a logarithmic relation with the mean effective stress p' at critical state **(Eq. 8)**. **Roscoe et al. (1958)** also noticed that the critical state line in the $(\ln p', e)$ plane was parallel to the normal compression line which characterises the changes in volume during isotropic compressions.

$$q_{cs} = M\, p'_{cs} \qquad \text{Eq. 8}$$

$$v_{cs} = 1 + e_{cs} = \Gamma - \lambda \cdot \ln p'_{cs} \qquad \text{Eq. 9}$$

where $v_{cs} = 1 + e_{cs}$ is the critical state specific volume, e_{cs} the critical state void ratio, Γ the critical state specific volume at p'= 1 kPa and λ the slope of the critical state line. M, Γ and λ are material properties determined from experiments. Whether a clay is sheared in drained or undrained conditions, the stress and void ratio state at critical state must satisfy both of these equations.

These two empirical relationships are used for the development of the Cam-Clay models, which are discussed in **Chapter 2**.

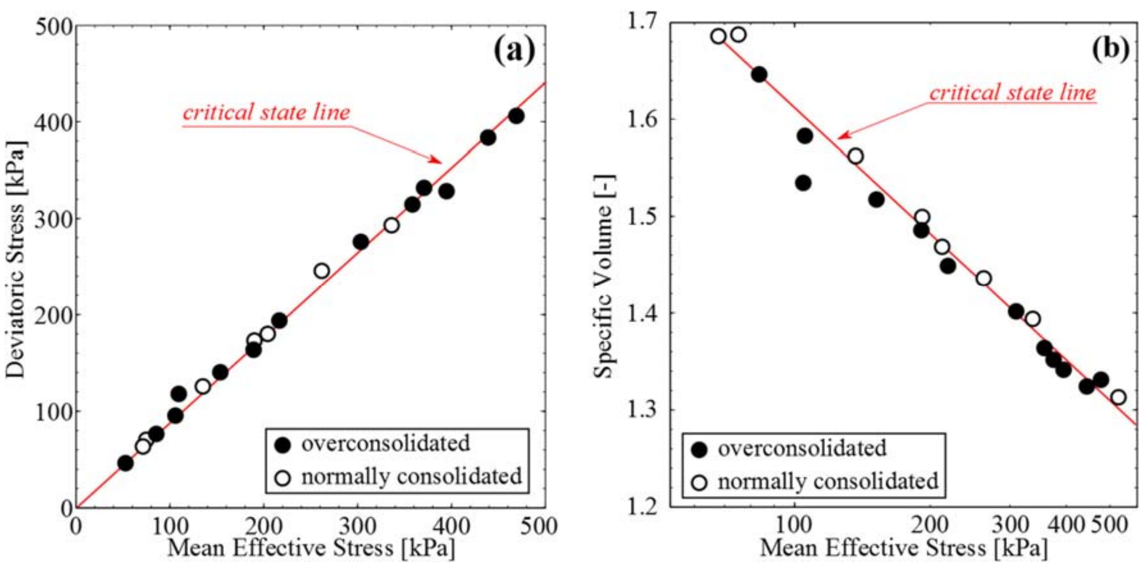

Figure 4: Critical state line (a) in the (*p'*, *q*) plan and (b) in the (ln *p'*, *v*) plan for drained triaxial compression tests on Weald clay (data: Roscoe et al. 1958)

2.3 Critical State of Sand

Similar to what is observed in clays, there are experimental evidences of the existence of the critical state in sands. **Figure 5** shows the results of triaxial compression tests on Toyoura Sand with different sample preparation to create different initial void ratios at different confining stresses **(Verdugo and Ishihara 1996)**. The left figure shows the final strength of the tests along with the critical state line, giving a linear relationship. The right figure shows the initial (black markers) states prior to shearing and the final (white markers) states at critical state. It shows that all tests, no matter whether the specimens are dense or loose, reach a non-linear critical state line in the $(\ln p', e)$ plane.

The following empirical relationships can be used to fit the data at critical state.

$$q_{cs} = M\, p'_{cs} \qquad \text{Eq. 10}$$

$$e_{cs} = e_{max} - \frac{e_{max} - e_{min}}{\ln Q - \ln p'_{cs}} \qquad \text{Eq. 11}$$

where e_{min} is the minimum void ratio and e_{min} is the maximum void ratio evaluated by soil mechanics standards. M and Q are the fitting material parameters. The derivation of **Eq. 11** is given in **Section 3.3**.

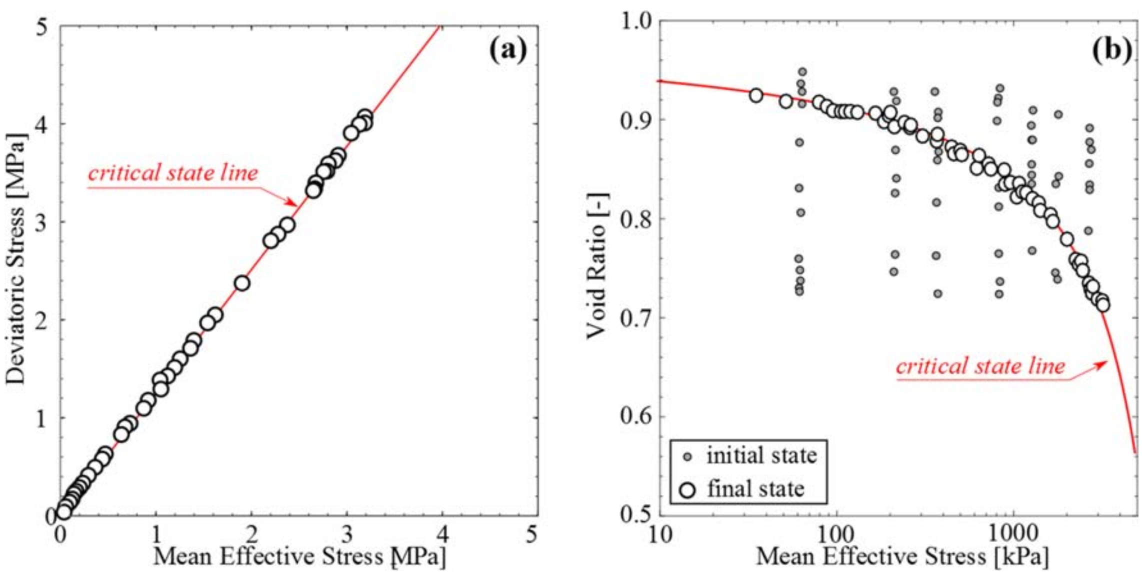

Figure 5: Critical state lines in the (a) (*p'*,*q*) plan and (b) (*e*, *p'*) plan for Toyoura Sand (data: Verdugo & Ishihara 1996)

2.3 Critical State Stress Ratio *M* in the Generalised Stress Space

In soil mechanics, effective friction angle φ'_{cs} (with zero cohesion) is often used as a failure criterion for granular materials with no bonding between grains.

$$\frac{(\sigma'_1 - \sigma'_3)}{(\sigma'_1 + \sigma'_3)} = \sin \varphi'_{cs} \qquad \text{Eq. 12}$$

where σ'_1 is the major principal stress and σ'_3 is the minor principal stress. This means that the classical Mohr-Coulomb model assumes that the magnitude of intermediate principal stress σ'_2 does not contribute to its failure.

In the classical critical soil mechanics, the failure criterion is defined as

$$q_{cs} = Mp'_{cs} \qquad \text{Eq. 13}$$

where *q* and *p'* are defined by Eq. **1** and Eq. **2** and they are a function of intermediate principal stress. This means that the above two failure criteria are not the same.

Experimental data indicate that the *M* value measured when a specimen is sheared in triaxial compression ($\sigma'_1 > \sigma'_2 = \sigma'_3$) is different from that in triaxial extension ($\sigma'_1 = \sigma'_2 > \sigma'_3$). If it is assumed that the friction angle φ'_{cs} is the same for both triaxial compression and extension, the following relationships between *M* and φ'_{cs} can be obtained for the two conditions.

Triaxial compression ($\sigma'_1 > \sigma'_2 = \sigma'_3$):

$$M_{tc} = \frac{6 \sin \varphi'_{cs}}{3 - \sin \varphi'_{cs}} \qquad \text{Eq. 14}$$

Triaxial extension ($\sigma'_1 = \sigma'_2 > \sigma'_3$):

$$M_{te} = \frac{6 \sin \varphi'_{cs}}{3 + \sin \varphi'_{cs}} \qquad \text{Eq. 15}$$

Figure 6 (a) shows the critical state line in triaxial compression and extension, when $q = \sigma'_a - \sigma'_r$, where σ'_a is the axial stress and σ'_r is the radial stress in a triaxial testing apparatus.

For more general principal stress case ($\sigma'_1 > \sigma'_2 > \sigma'_3$), it is common to use an invariant called Lode angle θ and relate *M* as a function of θ.

$$\tan \theta = \sqrt{3} \left[\frac{(\sigma'_2 - \sigma'_3)}{(\sigma'_1 - \sigma'_2) + (\sigma'_1 - \sigma'_3)} \right] \qquad 0° \leq \theta \leq 60° \qquad \text{Eq. 16}$$

The Lode angle is 0° in triaxial compression and 60° in triaxial extension.

In order to use the critical state theory in wider geotechnical problems, it is necessary to extend the prediction of the critical state stress ratio M_θ to any stress state. Since *M* value is often evaluated from a triaxial compression test, M_{tc} can be used as a basis and then extended to other Lode angle values. Three alternatives are presented in this chapter. The most common is to estimate M_θ with Mohr-Coulomb model as given by **Eq. 17**. The second alternative is to use the **Matsuoka and Nakai (1974)** criterion as given by **Eq. 18**. However, it is an implicit function which requires an iterative process to solve. The third alternative is proposed by **Jefferies & Shuttle (2011)** (**Eq. 19**) which approximates the Matsuoka-Nakai criterion but with an explicit function which becomes computationally less expensive.

Mohr-Coulomb
$$M_\theta = \frac{3\sqrt{3}}{\cos\theta\left(1+\frac{6}{M_{tc}}\right)-\sqrt{3}\sin\theta} \quad \text{Eq. 17}$$

Matsuoka-Nakai (1974)
$$\frac{27-3\,M_\theta^2}{3-M_\theta^2+\frac{8}{9}M_\theta^3\sin\theta\cdot\left(\frac{3}{4}-\sin^2\theta\right)} = \frac{27-3\,M_{tc}^2}{3-M_{tc}^2+\frac{2}{9}M_{tc}^3} \quad \text{Eq. 18}$$

Jefferies & Shuttle (2011)
$$M_\theta = M_{tc} - \frac{M_{tc}^2}{3+M_{tc}}\cos\left(\frac{3\theta}{2}\right) \quad \text{Eq. 19}$$

Figure 13 (b) shows the variation in M when M_{tc} = 1.25 for the three different failure criteria.

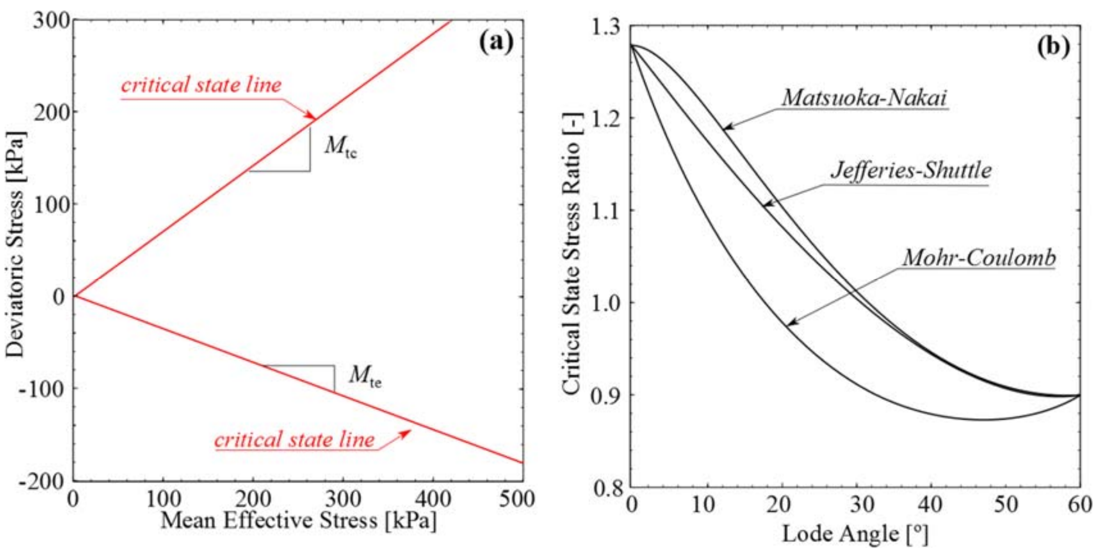

Figure 6 (a) triaxial compression and extension critical state lines and (b) the critical state stress ratio as a function of the Lode angle with M_{tc} = 1.25

3 STATE INDEX – CONTRACTIVE OR DILATIVE

The critical state soil mechanics can be used to evaluate whether the granular material will contract or dilate upon shearing, depending on the stress and void ratio state. If the state is above the critical state line in the (*e-p'*) space, then the material has a contractive tendency during shearing as it reaches to the critical state. If the state is below the critical state line in the (e-p') space, then the material has a dilative tendency during shearing as it reaches to the critical state. Hence, the distance between the current state and the critical state shows the degree of contraction or dilation. Various normalized state indices are available and these state indices can be used to develop a constitutive model.

3.1 State Index for Clay – the Equivalent Liquidity Index by Schofield (1980)

For saturated clay, the critical state specific volume or void ratio can be expressed in terms of gravimetric water content w using the following equation $w = e / G_s$, where G_s is the specific gravity (see Appendix A). It is therefore possible to express the critical state specific volume in terms of critical state water content (**Eq. 20**). Two points of this curve are known – the liquid limit (w_{LL}, p'_{LL}) and plastic limits (w_{PL}, p'_{PL}) which can be quantified by laboratory tests. **Figure 7 (a)** illustrates the critical state line in terms of water content.

$$w_{cs} = \Gamma_{\mathrm{w}} - \lambda \cdot \ln p'_{cs} \qquad \text{Eq. 20}$$

where w_{cs} is the critical state water content and Γ_{w} the critical state water content at p' = 1 kPa.

The liquidity index LI is a normalised index which indicates the state of the clay.

$$LI = (w - w_{PL})/(w_{LL} - w_{PL}) \qquad \text{Eq. 21}$$

The liquidity index LI is 1 at the liquidity limit w_{LL} and nil at the plastic limit w_{PL}. As these two points are known as well as the critical state line, it is possible to obtain the liquidity index at critical state LI_{cs} as shown in **Eq. 22** and illustrated in **Figure 7 (b).**

$$LI_{cs} = \frac{w_{cs} - w_{PL}}{w_{LL} - w_{PL}} = \frac{\ln\left(\frac{p'_{PL}}{p'}\right)}{\ln\left(\frac{p'_{PL}}{p'_{LL}}\right)} \qquad \text{Eq. 22}$$

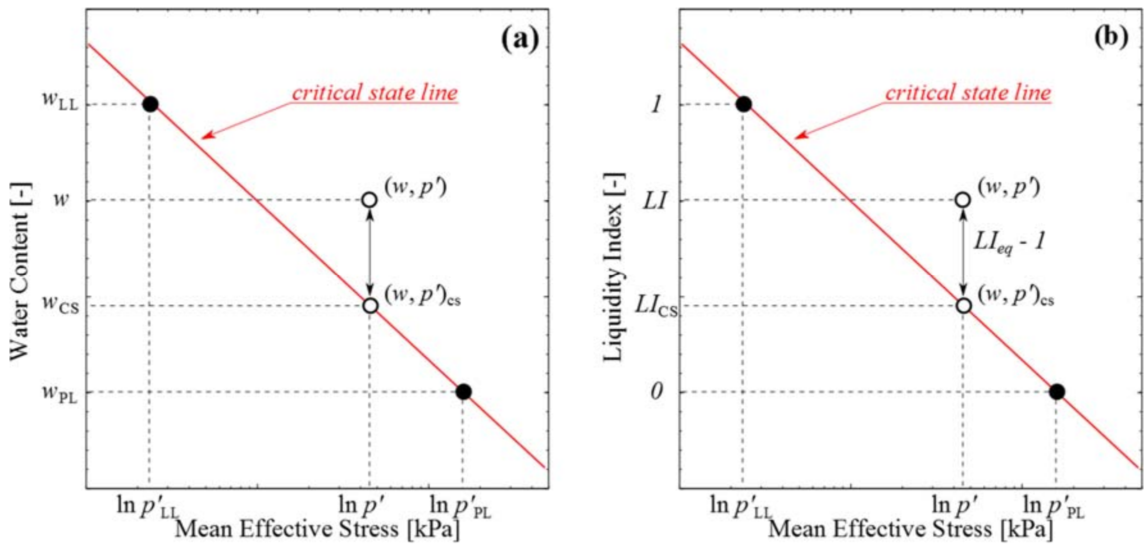

Figure 7: Critical state line expressed in terms (a) water content and (b) liquidity index

The experimental data suggests that the shear strength $q_{LL} \approx$ 4 kPa at the liquid limit and $q_{PL} \approx$ 400 kPa at the plastic limit (Mitchell 1976; Muir Wood 1990). For a given M, then the mean pressures at critical state are given by the following equations.

$$p'_{LL} = \frac{q_{LL}}{M} \quad \text{Eq. 23}$$

$$p'_{PL} = \frac{q_{PL}}{M} \quad \text{Eq. 24}$$

Schofield (1980) suggested the equivalent liquidity index LI_{eq} which indicates the normalised distance from the current state to the critical state (**Eq. 25**).

$$LI_{eq} = LI - LI_{cs} + 1 \quad \text{Eq. 25}$$

The equivalent liquidity index is a measurement of how much a clay must contract or dilate to achieve the critical state and its value is an indicator of the magnitude of volume change. When the equivalent liquidity index LI_{eq} is large than 1, then the clay must contract to achieve to the critical state. When the equivalent liquidity index LI_{eq} is lower than 1, soil must dilate in order to achieve the critical state line. The equivalent liquidity index LI_{eq} is equal to 1 at critical state.

$$LI_{eq} > 1 \rightarrow contracts$$

$$LI_{eq} < 1 \rightarrow dilates$$

$$LI_{eq} = 1 \rightarrow critical\ state$$

3.2 State Index for Sand – the State Parameter by Bean and Jefferies (1985)

Been & Jefferies (1985) suggested the state parameter Ψ as an indicator of the mechanical behaviour of sand by comparing the current state to the equivalent critical state which acts as a reference state. It is an equivalent index for sand as the equivalent liquidity index for clay by **Schofield (1980). Roscoe (1970)** commented on the necessity of establishing such index in order to further develop a constitutive model for sand.

The state parameter measures the distance between the current void ratio and the critical state void ratio (**Eq. 26**) for a given mean effective stress p' and is illustrated in **Figure 8**. Note that the state parameter is denoted as a capital Ψ in order to avoid confusion with the dilatancy angle ψ.

$$\Psi = e - e_{cs} \qquad \text{Eq. 26}$$

where e_{cs} can be evaluated from Eq. 10.

The state parameter Ψ is a theoretical index which finds its roots in the critical state theory. The state parameter is an indicator of the direction and magnitude of the volumetric behaviour of the sand upon shearing. When it is positive, the sand will contract. When it is negative, it will dilate. When it is nil, the sand is at the critical state density and no changes in volume will take place while shearing is taking place.

$$\Psi > 0 \rightarrow contractes$$

$$\Psi < 0 \rightarrow dilates$$

$$\Psi = 0 \rightarrow critical\ state$$

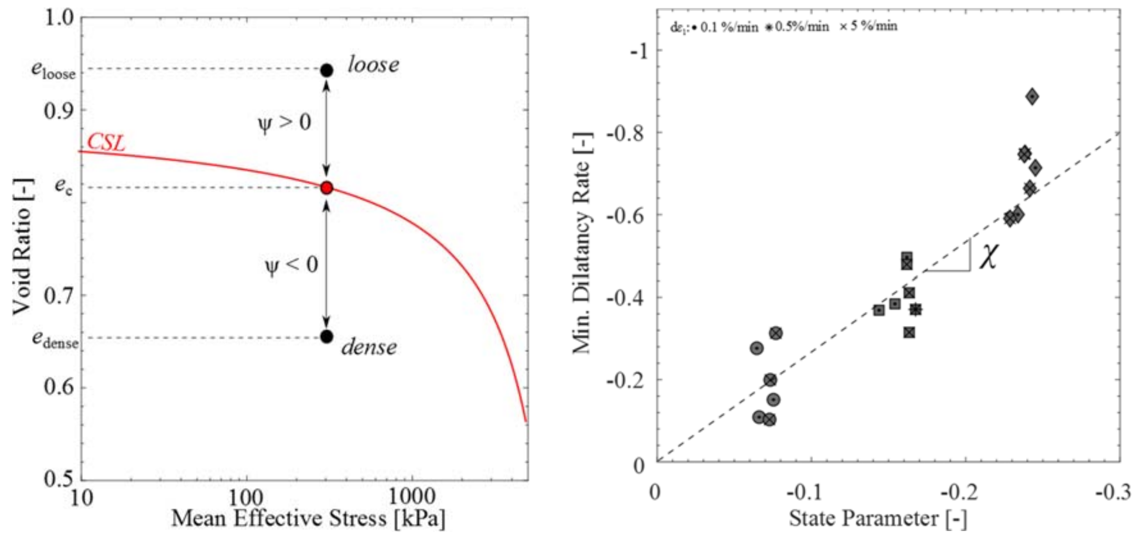

Figure 8: (a) definition of the state parameter, (b) corroletion between the minimum dilatancy rate and the state parameter from Been & Jefferies (1986)

3.3 State Index for Sand – the Relative Dilatancy Index by Bolton (1986)

Bolton (1986) suggested an index called the relative dilatancy index I_R to characterise the dilatancy characteristic of sand, which is influenced by density and pressure (**Eq. 27**).

$$I_R = I_D \cdot I_C - 1 \qquad \text{Eq. 27}$$

When the relative dilatancy index is positive, the sand is dense and has to dilate in order to reach the critical state. When the index is negative, the sand is loose and has to contract in order to reach the critical state. Bolton (1986) showed that the magnitude of the index is proportional to changes in volume required to reach the critical state.

$$I_R > 0 \rightarrow dilates$$

$$I_R < 0 \rightarrow contractes$$

$$I_R = 0 \rightarrow critical\ state$$

The relative dilatancy index I_R is a function of the relative density index I_D and the relative crushability index I_C. The relative density index I_D (**Eq. 28**) defines the density of the packing by comparing the current void ration state with the 'loosest' and 'densest' states possible which are characterised by the maximum void ratio e_{min} and the minimum void ratio e_{max}. These values are material properties and are obtained by laboratory experiments **(Kolbuszewski 1948).**

$$I_D = \frac{e_{max} - e}{e_{max} - e_{min}} \qquad \text{Eq. 28}$$

The relative crushability I_C (**Eq. 29**) defines the influence of pressure p' and the crushing pressure Q on the ability of a sand to dilate. If a sand is subjected to a high confining pressure, it will be more difficult to dilate. This pressure is normalised with the crushing pressure Q which is a function of the mineralogy of the sand, typically 10 MPa for silica sand and 20 MPa for silica silt.

$$I_C = \ln(\frac{Q}{p'}) \qquad \text{Eq. 29}$$

No change in volume is observed at critical state and hence the dilatancy rate must be nil. Therefore, the relative dilatancy index must be nil at critical state. Boulanger (2003) and Mitchell and Soga (2005) suggested to express the critical state line by setting the relative index I_R to nil. Therefore, the relative dilatancy index I_D at critical state can be expressed as shown in **Eq. 30** and in terms of void ratio as shown in **Eq. 31**. This dilatancy-based critical state line is non-linear and takes into account the crushability of the sand grains. **Figure 9** shows the data from triaxial compression tests on Toyoura sand with the dilatancy-based critical state line with a crushing pressure Q = 12 MPa.

$$I_R = 0$$

$$\rightarrow \quad I_{D,cs} = \frac{e_{max} - e_{cs}}{e_{max} - e_{min}} = \frac{1}{\ln Q - \ln p'} \qquad \text{Eq. 30}$$

$$\rightarrow \; e_{cs} = e_{max} - \frac{e_{max} - e_{min}}{\ln Q - \ln p'} \qquad \text{Eq. 31}$$

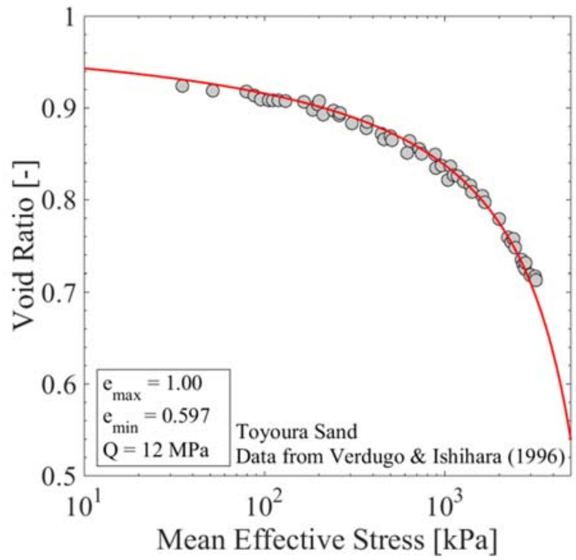

Figure 9: Theoretical and experimental critical state line of Toyoura sand (data from Verdugo & Ishihara 1996)

4. STRESS-DILATANCY RELATIONS

A loosely packed granular material contracts upon shearing, whereas a densely packed granular material dilates upon shearing. As shown in the previous sections, the degree of contraction or dilation depends not only on void ratio (or water content) but also on confining stress. In this section, models that predicts the magnitude of dilation or contraction are presented.

4.1 Stress-dilatancy theory by Taylor (1948) and Roscoe and Schofield (1963)

Taylor (1948) carried out a series of direct shear tests on dry Ottawa Sand and noticed that the development of strength changed with initial density and that this change in behaviour was associated with the volume change of the sand during shearing. A loose sand contracted and did not reach a peak strength whilst a dense sand dilated and exhibited a peak strength. He explained this phenomenon as a consequence of grains interlocking. **Taylor (1948)** proposed a stress-dilatancy theory based on the work balance equation (**Eq. 32**). The external work corresponds to the product of the measured displacements and forces (assuming that the elastic deformation is negligible). The internal work corresponds to the frictional force. **Figure 10** illustrated the direct shear tests.

$$\tau \, \mathrm{d}x - \sigma_n' \mathrm{d}y = \mu \, \sigma_n' \mathrm{d}x \qquad \text{Eq. 32}$$

where τ is the shear stress, σ_n' the normal effective stress, μ the friction coefficient, dx and dy are the incremental displacements in the horizontal and vertical planes, respectively.

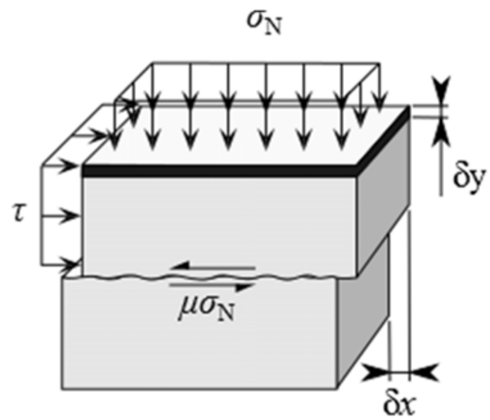

Figure 10: Schematic description of the direct shear test

The work balance equation (**Eq. 32**) can be rearranged to express the strength as a function of the friction coefficient and the incremental displacements characterising the interlocking (Eq. 33).

$$\frac{\tau}{\sigma_n'} = \mu + \frac{dy}{dx} \qquad \text{Eq. 33}$$

The strength can then be express as the mobilised strength ($\tan\varphi_m' = \frac{\tau}{\sigma_n'}$), the friction coefficient as the critical state strength ($\mu = \tan\varphi_{cs}$) and interlocking as the dilatancy rate ($\tan\psi = \frac{dy}{dx}$).

The work balance equation then yields to Taylor's stress-dilatancy theory (**Eq. 34**) in which the mobilised strength corresponds to the critical state strength when the dilatancy is nil.

$$\tan\varphi_m' = \tan\varphi_{cs} + \tan\psi \qquad \text{Eq. 34}$$

Roscoe and Schofield (1963) followed the same approach with their simple shear test results for which the internal work equation can be expressed as shown in **Eq. 35** and illustrated in **Figure 11**.

$$\sigma_n'\delta\varepsilon_v + \tau\delta\gamma = \mu_{cs}\,\sigma_n'\,\delta\gamma \qquad \text{Eq. 35}$$

The total strains can be decomposed into a reversible (elastic) and an irreversible (plastic) part. The work balance for the irreversible and plastic part gives **Eq. 36** which known as the Cam-Clay work equation.

$$\sigma_n' \, \delta\varepsilon_v^p + \tau \, \delta\gamma^p = \mu_{cs} \, \sigma_n' \, \delta\gamma^p \qquad \text{Eq. 36}$$

where the superscript p stands for plastic.

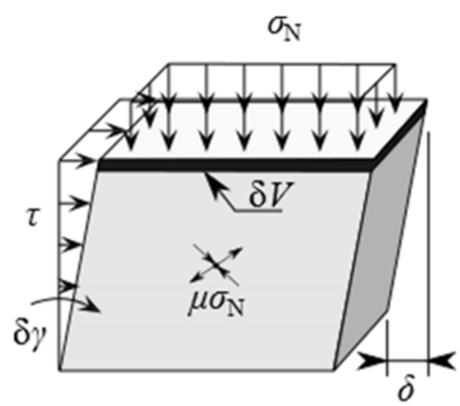

Figure 11: Schematic description of the simple shear test

Eq. 8 can be rearranged in order to express the strength as a function of the critical state strength and dilatancy (**Eq. 37**).

$$\frac{\tau}{\sigma_n'} = \mu_{cs} + \frac{\delta\varepsilon_v^p}{\delta\gamma^p} \qquad \text{Eq. 37}$$

It is possible to express **Eq. 37** with the stress and strain variables defined earlier. Stresses are expressed in terms of effective stress ratio $\eta'(= q/p')$, the critical state stresses in terms of the critical state effective stress ratio M and the volumetric behaviour in terms of dilatancy rates D as shown in **Eq. 38**.

$$\eta' = M + D \qquad \text{Eq. 38}$$

The critical state is defined by an absence of volume change or, in other words, a nil dilatancy conditions. Therefore, at critical state, the stress-dilatancy rule (**Eq. 38**) yields to the critical state condition (**Eq. 39**).

$$D = 0 \rightarrow \eta' = M \qquad \text{Eq. 39}$$

Eq. 38 is the basis for the development of the original Cam-clay model, which will be presented in the next Chapter.

4.2 Stress-dilatancy relationships of sand

Newland and Allely (1957) analysed the dilative behaviour of a granular system at a grain-size scale. They presented the theory for perfectly spherical particles but pointed out that the theory was independent of particle shapes. The stress-dilatancy theory would then becomes **Eq. 40**.

$$\varphi = \varphi_{cs} + \psi \qquad \text{Eq. 40}$$

Rowe (1962) realised that dilatancy would be the consequence of grains sliding and rotating and that the principal stress directions would play a major role and established a new stress-dilatancy relationship. **Bolton (1986)** stated that the strength by Rowe's theory was overestimtaed by 20% in plane strain shear conditions and the following corrected expression was suggested (**Eq. 41**).

$$\varphi = \varphi_{cs} + 0.8\psi \qquad \text{Eq. 41}$$

Following **Rowe (1962)** and **Roscoe and Schofield (1963)**, **Nova (1982)** suggested an expression to express the influence of dilatancy on mobilized strength of sand.

$$\eta' = M + (N-1)D \qquad \text{Eq. 42}$$

where N is the dilatancy parameter which is often 0.2 **(Jefferies and Been 2006).**

Eq. 42 is the basis for the development of the Nor-Sand model, which will be presented in the next Chapter.

There have been many other suggestions for the stress-dilatancy relationship (*i.e.* **Muir Wood et al. 1994; Nova and Muir Wood 1979)**. However, they all have in common the expression of the stress and strain variables as invariants and include the idea that the development of strength is the consequence of grains interlocking. These equations form the base to develop Cambridge-type of constitutive models and will be developed in the next Chapter.

4.3 Dilatancy as function of state indices

Using experimental data, it is possible to develop empirical relationships between dilation rate and state indices such as the relative dilatancy index I_R and state parameter Ψ presented earlier.

Bolton (1986) related the relative dilatancy index I_R to dilatancy rate by using a dilatancy coefficient α as shown in **Eq. 43.**

$$D_{1,max} = \max\left(-\frac{d\varepsilon_v}{d\varepsilon_1}\right) = \alpha \cdot I_R \qquad \text{Eq. 43}$$

where α is the dilatancy coefficient with a default value of 0.3. It is important to note that the definition of dilation by Bolton is different from the dilation defined in this chapter.

Bolton (1986) suggested a dilatancy coefficient α of 0.3. However, **Tatsuoka (1987)** pointed out that this dilatancy coefficient α would also be a function of the soil.

Similarly, it is possible to relate the state parameter Ψ to dilatancy rate with a dilatancy coefficient χ **(Jefferies 1993; Jefferies and Shuttle 2002).**

$$D = \chi \cdot \frac{M_\theta}{M_{tc}} \cdot \Psi \qquad \text{Eq. 44}$$

As for the relative dilatancy index, the conversion is stress state and soil fabric dependent and may somewhat differ from the suggested value **(Jefferies and Been 2006).**

Schofield (1980) presented **Figure 12**, which illustrates the type of plastic behaviour for a given stress ratio and water content. When $LI_{eq} > 1$ (the state above the critical state line), the clay will exhibit uniform contractive behaviour when it is in the plastic state. When $LI_{eq} < 1$ (the state is below the critical state line), the clay exhibits localized dilatant rupture at the stress ratio for a given water content. The clay will fracture when the stress ratio reaches the tensile limit of $\sigma'_3 = 0$ (*i.e.* the clay does not have any cohesion). The tensile limit gives the yield envelope of *q/p'* = 3 for triaxial compression and -1.5 for triaxial extension as shown in the figure.

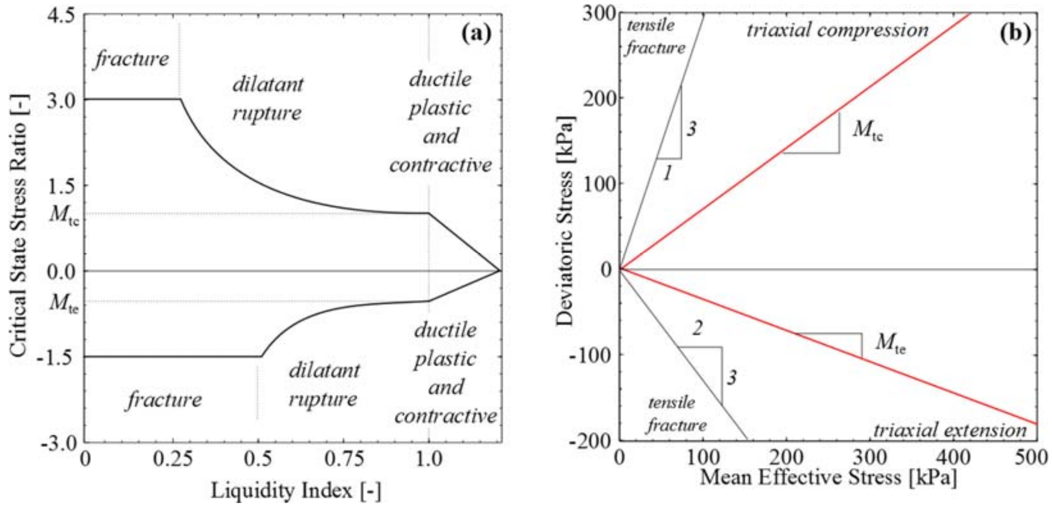

Figure 12: Plastic state of clay in relation to normalized liquidity index: (a) stress ratio when plastic state initiates for a given LI_{eq} and (b) definition of stress ratios used in (a) (after Schofield, 1980)

4.4 A note on strain localization

Direct shear tests induce a predefined failure surface within a specimen. This failure surface is also called a slip surface, and is by essence a localised failure. However, strain localisation is not specific to direct shear tests. **Roscoe (1970)** presented the work of **Coumoulos (1967)** on the micro-mechanical investigation of soil in simple shear tests, and in which the inhomogeneity of the specimen was investigated with γ-ray imagining. **Coumoulos (1967)** showed that the local void ratio in direct shear tests obtained by **Stroud (1971)** were underestimated and **Roscoe (1970)** suggested that this void ratio would transpire to be the critical state void ratio. **Figure 13** shows the results of a simple shear tests on dense Buzzard Leighton sand for which the local and global void ratio are shown.

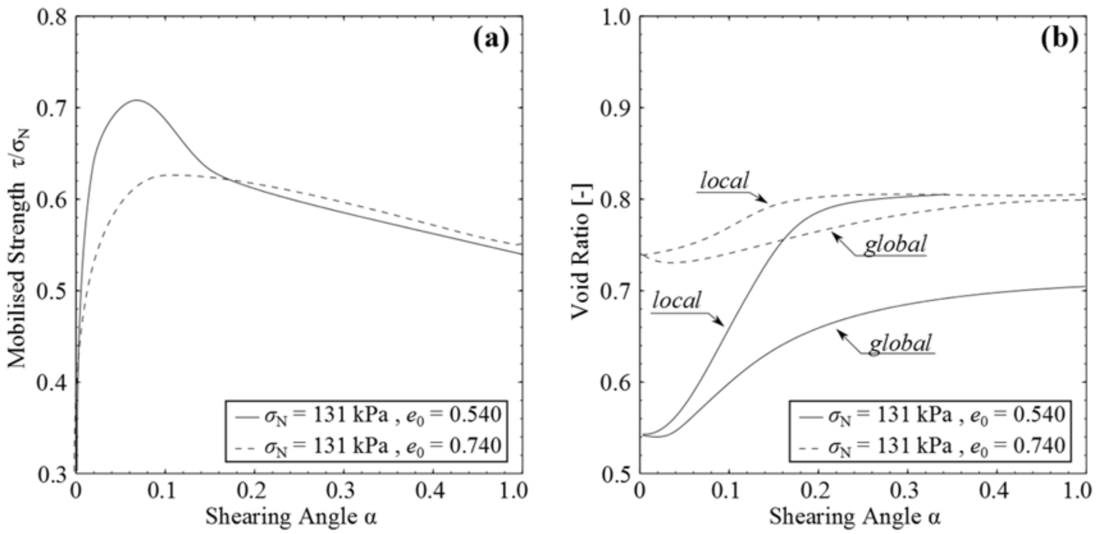

Figure 13: Local and global stress-strain curves of Leighton Buzzard sand in simple shear test modified Roscoe (1970)

Roscoe (1970) said: '*While in the condition between peak and critical sate the sand is unstable and will tend to fail on the thinnest possible zone, band or surface, which is approximately ten grains thick. Once these rupture bands have formed, the strains become concentrated within them and the neighbouring material on either side behaves as a rigid body.'*. This is exactly the type of failure which was investigated by **Coulomb (1776)** and the one reproduced in direct shear tests. Furthermore, Roscoe's view that these local failures or slips reach a critical state whilst the rest of the soil behaves rigidly bridges the gap between Terzghi's Soil Mechanics **(Terzaghi 1943)** and the Critical State Soil Mechanics **(Schofield and Wroth 1968).**

Desrues et al. (1996) carried out a series of drained triaxial compression tests on dense Hostun sand, and post-testing computed tomography (CT) imaging was carried out. **Figure 14** shows the evolution of the void ratio of seven drained triaxial compression tests. **Desrues et al. (1996)** suggested that the local void ratio would be at critical state as the local void ratio reached a steady state of e = 0.85±0.02. It corresponded to the critical state of loose specimen, which did not produce distinctive shear bands. However, it is possible to calculate this value with Eq. 31. The maximum and minimum densities were reported to be 15.99 and 13.24, respectively, and the specific gravity is 2.65. Therefore, the maximum and minimum void ratios are 1.001 and 0.657, respectively. Based on previous analyses of Hostun sand, the crushing pressure was estimated at 10 MPa, a typical value for silica sand. The mean effective stress at failure was 100 kPa. Hence, the critical state void ratio is 0.857 (**Eq. 45**). It can be concluded that the local void ratio estimated from the CT images potentially reached the critical state value.

$$e_{cs} = e_{max} - \frac{e_{max} - e_{min}}{\ln Q - \ln p'} = 1.001 - \frac{1.001 - 0.657}{\ln 10{,}000 - \ln 100} = 0.857 \qquad \text{Eq. 45}$$

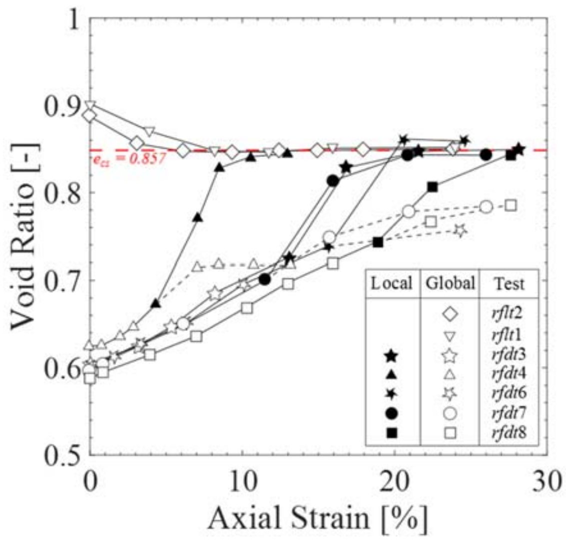

Figure 14: Global and shear band void ratio in Huston sand in triaxial compression tests modified from Desrues et al. (1996)

5. CLOSURE

The critical state theory **(Roscoe et al. 1958)** is a simple theory which predicts with acceptable accuracy the ultimate state of any granular material sheared sufficiently. The critical state is a uniquely defined line in the (p', q, e) space.

$$\frac{q_{cs}}{p'_{cs}} = M(\theta)$$

$$e_{cs} = f(p'_{cs})$$

When a granular material is at critical state failure, the stress and void ratio have to be on this line. They provide a reference state from which it is possible to understand and predict the behaviour of the granular body in any condition or state. It is also possible to formulate state indices in which the critical state is used as a reference (*e.g.* the equivalent liquidity index for clay or the state parameter for sand) and a normalised state is defined as a quantified distance to the critical state. These state indices are of interest for constitutive modelling. Because of these features, the critical state theory has become the most widely used theory in soil mechanics and is a corner stone in assessing the behaviour of any soil.

The critical state theory is more than just a set of mathematical expressions which characterise the mechanical behaviour of granular material. It is a theory based on experimental observations and provides a physical explanation and relationship between the development of strength with respect to the development of strains. It allows us to make reliable prediction on the ultimate strength of any soil subjected to shearing **(Roscoe 1970).** It therefore provides a unique reference state from which all critical state constitutive models are derived.

As shown in the next chapter, in conjunction with the critical state framework **(Schofield and Wroth 1968)**, it is possible to develop powerful constitutive models for both clay and sand. The success of the theory was driven by the establishment of a constitutive model called Cam-Clay **(Roscoe and Burland 1968; Roscoe and Schofield 1963)**. The name Cam-Clay was later introduced by **Schofield & Wroth (1968)** for teaching purposes. The name 'Cam' refers to river Cam in Cambridge, UK; there is no real clay called "Cam-Clay". In fact, the city of Cambridge sits on a very heavily overconsolidated clay called "Gault Clay"!

Roscoe (1970) commented on the difficulties in obtaining an equivalent model for sand as it needed some 'invariant strain parameter related to the critical state' to capture the dilatancy rate but was then unavailable. This issue was overcome by **Bolton (1985)** using relative dilatancy index. **Jefferies (1993)** used the state parameter **(Been and Jefferies 1985)** as the strain invariant related to the critical state theory in a model called Nor-Sand.

Both Cam-Clay and Nor-Sand and their extensions are introduced in the next chapter.

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APPENDIX A: VOLUME AND MASS FRACTIONS

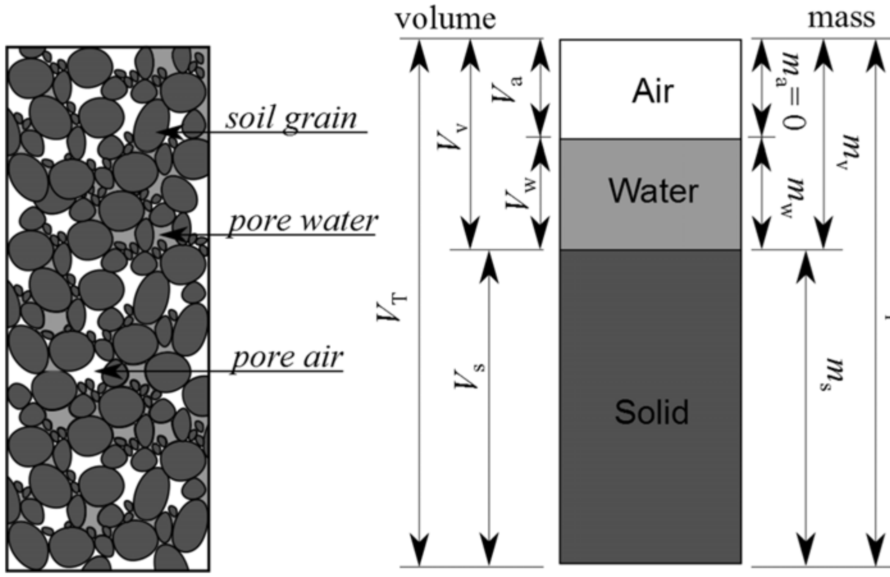

Figure 15: Volume and mass fractions

Void ratio $e = \frac{V_v}{V_s} = \frac{V_v}{V_T - V_v}$

Specific volume $v = \frac{V_T}{V_s} = 1 + e$

Porosity $n = \frac{V_v}{V_T} = \frac{V_v}{V_v + V_s} \quad \rightarrow n = \frac{e}{1+e} = 1 - \frac{\rho_d}{G_s}$

Solid fraction $S_f = \frac{V_s}{V_T} = 1 - n$

Density $\rho = \frac{m_T}{V_t}$

Specific Gravity $G_s = \left(\frac{m_s}{v_s}\right)/\rho_w$; ρ_w is water density

Dry Density $\rho_d = \frac{m_s}{V_t} = \frac{\rho}{1+w}$

Degree of saturation $S_w = \frac{V_w}{V_v}$

Water content $w = \frac{m_w}{m_s} = \frac{eS_w}{G_s}$