Membranes

Besides conductor paths, membranes are another special type of thin films. Membranes are an important mechanical basic element in microtechnique. They are the microscopic correspondence to macroscopic gaskets, bearings, and springs. They are made of silicon, oxides, nitrides, glasses, polymers, and metals. Their thickness typically is in the range of 0.5–500 µm. Membranes which are thinner than 0.5 µm are very hard to manufacture without holes and are generally not strong enough to withstand usual loads. The upper limit is given by the fact that thicker membranes are no longer a microscopic element. The lateral dimensions of membranes are typically in the range between 100 µm and 10 mm. Again the lower limit is defined by the possibilities of fabrication, while the upper limit is approximately the limit to the macroscopic world. However, all equations discussed here are valid in the macroscopic world also.

There are two types of membranes, which are frequently used in microtechnique: *Thick membranes* and *thin membranes*. A membrane is called thick when its maximum deflection w_0 is much smaller than its thickness d_M and thin when the deflection is larger than the thickness. So, a thick membrane may become a thin one when the pressure drop rises and the deflection is increased. A thick membrane, sometimes, is also called a plate. The shape of the deflection of a thick membrane is determined by the bending moments acting especially at the rim where the membrane is clamped to a frame or housing. Figure 19 shows the characteristic shape of different types of membranes. As usual in microtechnique, the drawings are not to scale; small dimensions are shown larger to make them visible in the vicinity of larger ones. The deflection w of a circular thick membrane with radius R_M is described by the following equation [19]:

$$w(r) = w_0 \left(1 - \frac{r^2}{R_M^2}\right)^2. \tag{23}$$

When the membrane is deflected, its neutral fiber needs to become longer and this strain generates some stress according to Hooke's law. When the deflection becomes much larger than the thickness of the membrane, the effect on the shape of the membrane generated by the stress due to straining gets much larger than the

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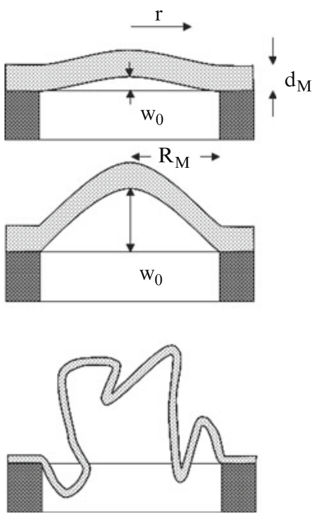
Fig. 19 Thick, thin, and flexible membrane, respectively



effect of the bending moments. In this case, the bending moments may be neglected in calculations. The deflection curve of a circular thin membrane adopts the shape of a parabola and is described by the following equation:

$$w(r) = w_0\left(1 - \frac{r^2}{R_M^2}\right). \tag{24}$$

A third membrane type is the *flexible membrane*, which is shown at the bottom of Fig. 19. It is longer than the distance between the positions where its rim is clamped and it is so thin that bending moments are of minor importance. There is no straining along the neutral fiber and no residual stress. The deflection of such a membrane is not defined. It behaves as a plastic bag which can be deformed with small forces and retains this shape until further forces act on it. A flexible membrane may be used to separate two fluids and to ensure that there is no pressure difference over the membrane. It may also be used to limit the volume of a fluid because much pressure is required when a certain displacement of the membrane shall be overcome.

The *deflection of a circular thin membrane* loaded by a constant pressure drop Δp can be calculated from the equilibrium of forces at the rim. The absolute value of the total force F_p acting on the membrane is the pressure difference times the membrane area πR_M^2 (cf. Fig. 20). This force is balanced by the force F_F of the frame fixing the membrane at the circumference. The lateral components of this force cancel out when summed over the entire rim of the membrane, because there is no lateral movement of the membrane. The vertical components $F_{F,z}$ and $F_{p,z}$ of the force of the frame and the pressure drop, respectively, are in equilibrium:

$$F_{p,z} = \Delta p\ \pi\ R_M^2 = -F_{F,z} = -\sigma_M\ d_M\ 2\ \pi\ R_M \sin(\alpha). \tag{25}$$

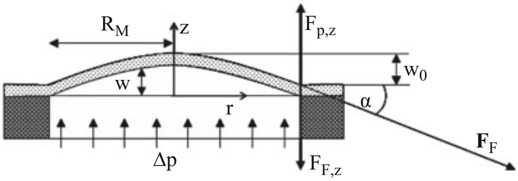


Fig. 20 Cross-section of a circular thin membrane loaded with a pressure drop

The force of the frame in (25) is calculated from the membrane stress σ_M times the cross-section of the membrane around the circumference, which is the membrane thickness d_M times the length of the circumference $2\,\pi\,R_M$. The vertical component of the force of the frame is obtained by multiplying with the sine of the angle at which the membrane touches the frame. For comparatively small angles α, the sine is approximately the same as the tangent which is the slope of the membrane at its rim and can be calculated as the derivative of the deflection curve w(r) at the rim. Equation (25) becomes now:

$$\Delta p\ \pi\ R_M^2 \approx -\sigma_M\ d_M\ 2\ \pi\ R_M \tan(\alpha) = -\sigma_M\ d_M\ 2\ \pi\ R_M \left.\frac{\partial w}{\partial r}\right|_{r=R_M}. \tag{26}$$

The deflection curve of the membrane is the parabola described by (24). Calculating the derivative of this, inserting in (26) and solving for the pressure drop yields:

$$\Delta p\ \pi\ R_M^2 = -\sigma_M\ d_M\ 2\ \pi\ R_M\ w_0 \frac{2}{R_M} = 4\ \pi\ w_0\ d_M\ \sigma_M \Rightarrow \Delta p = \frac{4\ w_0\ d_M}{R_M^2}\sigma_M. \tag{27}$$

The stress σ_M of the membrane consists of two parts, the residual stress σ_0, which is already present when there is no deflection of the membrane, and the stress σ_D due to Hooke's law generated by the deflection of the membrane. The latter part can be calculated from the strains ε_R and ε_T in radial and tangential direction, respectively, generated by the deflection. According to Hooke's law, the radial strain is calculated with the following equation:

$$\varepsilon_R = \frac{\sigma_R}{E_M} + \nu_M \frac{\sigma_T}{E_M} = \frac{1}{E_M}(\sigma_R + \nu_M\ \sigma_T). \tag{28}$$

The first term in (28) is the strain generated by the radial stress σ_R of the membrane and the second term is due to the transverse strain generated by the tangential stress σ_T of the membrane. ν_M and E_M denote Poisson's ratio and Young's modulus of the membrane, respectively. The tangential strain ε_T is calculated accordingly to (28):

$$\varepsilon_T = \frac{\sigma_T}{E_M} + \nu_M \frac{\sigma_R}{E_M} = \frac{1}{E_M}(\sigma_T + \nu_M\ \sigma_R). \tag{29}$$

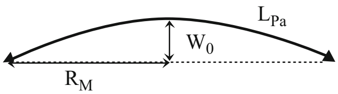


Fig. 21 Length L_{Pa} of a parabola with a peak height w_0 and a base length 2 R_M

Radial strain is assumed to be constant over the entire membrane. This can be adequate for thin membranes only, because bending moments, which result in a strain change over the thickness of the membrane, are comparatively small and may be neglected in thin membranes. Radial membrane strain is estimated by the extension along the neutral fiber of the membrane which is necessary for deflection (cf. Fig. 21). In formularies such as [21], the length L_{Pa} of a parabola with a comparatively small height w_0 at its peak and a base length 2 R_M is found to be:

$$L_{Pa} \approx 2R_M\left(1 + \frac{2}{3}\frac{w_0^2}{R_M^2} - \frac{2}{5}\frac{w_0^4}{R_M^4}\right). \tag{30}$$

Thus, the strain ε_R necessary to extend the base length to the length of the parabola is:

$$\varepsilon_R \approx \frac{2}{3}\frac{w_0^2}{R_M^2}. \tag{31}$$

The tangential strain of the membrane is currently unknown. Only two boundary conditions can be determined:

1. In the center of the membrane, radial and tangential strains are equal because the strain is not a function of direction at this position due to symmetry.
2. At the circumference, tangential strain is zero because the membrane is clamped there and the frame does not allow any movements of the membrane.

In the following, the two ansatz-equations are made that either the first or the second boundary condition would be fulfilled everywhere on the membrane. Later the results obtained this way are compared with each other.

If tangential and radial strains are equal throughout the membrane (31), (28), and (29) result in:

$$\sigma_R = \varepsilon_R \frac{E}{1 - \nu_M}. \tag{32}$$

If tangential strain is assumed to be zero everywhere, we obtain:

$$\sigma_R = \varepsilon_R \frac{E}{1 - \nu_M^2}. \tag{33}$$

For most materials, Poisson's ratio is approximately 0.3. As a consequence, the maximum change in the quantity $(1-\nu_M)$ versus $(1-\nu_M{}^2)$ is 11%. For polymers, Poisson's ratio may approach 0.5 resulting in a ratio of 66%. For a general understanding of the interrelationships, this is an approximation which is still good enough.

Equation (31) is substituted in (32) and the total stress σ_M of the thin circular membrane is obtained from the sum of the residual stress σ_0, and the stress generated by the deflection of the membrane σ_R:

$$\sigma_M = \sigma_0 + \frac{2}{3}\frac{w_0^2}{R_M^2}\frac{E_M}{1 - \nu_M}. \tag{34}$$

This is introduced now to (27) resulting in an interrelationship between the pressure drop Δp over a thin, circular membrane and its radius R_M, thickness d_M, Young's modulus E_M, Poisson's ratio ν_M, residual stress σ_0, and deflection w_0:

$$\Delta p = 4\frac{w_0\, d_M}{R_M^2}\left(\sigma_0 + \frac{2}{3}\frac{w_0^2}{R_M^2}\frac{E_M}{1 - \nu_M}\right). \tag{35}$$

Equation (35) is *Cabrera's equation* which is frequently used to calculate membrane deflections [22]. However, it is based on a rough approximation, and it could have been assumed also that tangential strain is zero throughout the membrane (33). As a consequence of this assumption, (35) would be:

$$\Delta p = 4\frac{w_0\, d_M}{R_M^2}\left(\sigma_0 + \frac{2}{3}\frac{w_0^2}{R_M^2}\frac{E_M}{1 - \nu_M^2}\right). \tag{36}$$

The exact equation for thin, circular membranes was found by using finite element methods (FEM) [23]. An ansatz with variable parameters was made for the pressure drop Δp as a function of Poisson's ratio and the parameters were determined by a fit to the FEM result. The solutions for a circular and a square membrane with an edge length a_M are as follows.

Circular membrane

$$\Delta p = 4\frac{w_0\, d_M}{R_M^2}\left(\sigma_0 + \frac{2}{3}\frac{w_0^2}{R_M^2}\frac{E_M}{1.026 - 0.793\nu_M - 0.233\nu_M^2}\right). \tag{37}$$

Square membrane

$$\Delta p = 13.6\frac{w_0\, d_M}{a_M^2}\left(\sigma_0 + 1.61\frac{w_0^2}{a_M^2}\frac{(1.446 - 0.427\ \nu_M)E_M}{1 - \nu_M}\right). \tag{38}$$

It should be noted that all the calculations above are for thin membranes only, i.e., bending moments are not included. When a membrane is thick, however, bending moments dominate the behavior of the membrane and the stress may be neglected for some cases. In the macroscopic world, the deflection of membranes is very often dominated by bending moments; therefore, solutions for this problem have existed for decades. These are the calculations for the deflection of plates, commonly found in classical mechanics books. After solving these classic macroscopic differential equations, the following equations are found [19].

Circular membrane

$$\Delta p = \frac{16}{3}\frac{d_M^3}{R_M^4}\frac{E_M}{1-\nu_M^2} w_0 \Rightarrow w_0 = \frac{3}{16}\frac{R_M^4}{d_M^3}\frac{1-\nu_M}{E_M}\Delta p. \tag{39}$$

Square membrane

$$\Delta p = 66\frac{d_M^3}{a_M^4}\frac{E_M}{1-\nu_M^2} w_0 \Rightarrow w_0 = \frac{1}{66}\frac{a_M^4}{d_M^3}\frac{1-\nu_M^2}{E_M}\Delta p. \tag{40}$$

In general, however, both bending moments and stress affect membrane deflections. If it is desired to take both into account, the *Ritz method* can be used to find the resultant deflection [24]. In this approach, the potential energy of the membrane is calculated as a function of one or more free parameters, which are obtained by calculating the extremes of the energy function. A local minimum corresponds to a stable equilibrium of the membrane, while a maximum is an unstable one.

In the following, the Ritz method is utilized in a simple example of the deflection of a spring loaded with a mass m_K (cf. Fig. 22). The potential energy V_p of the system is the sum of the energy of the spring and the mass:

$$V_p = -m_K\, g_e\, w_0 + \frac{k}{2} w_0^2. \tag{41}$$

The derivative of the potential energy is the force. Therefore, the extremes of the potential energy correspond to the positions where the sum of the forces is zero, i.e., the equilibrium of forces. The derivative of (41) with respect to the deflection w_0 of the spring is:

$$\frac{\partial V_p}{\partial w_0} = 0 = F_{Ku} + F_k = -m_K\, g_e + k w_0. \tag{42}$$

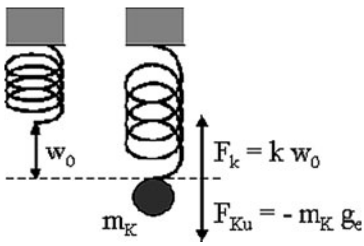


Fig. 22 Equilibrium of forces at a spring loaded with a mass

Solving this equation yields the deflection of the spring:

$$w_0 = \frac{m_K\ g_e}{k}. \tag{43}$$

This result could have also been found by directly calculating the equilibrium of forces, but with a more difficult problem, using the extremes of the potential energy according to the Ritz method is much easier.

The potential energy of a membrane with thickness d_M, radius R_M, Young's modulus E_M, Poisson's ratio ν_M, and deflection w can generally be calculated from a differential equation. If membrane bending moments, residual stress, stress due to straining, and the pressure drop are included, the equation takes the following form in rectangular and polar coordinates, respectively [19]:

$$\begin{aligned} V_p = \iint dx\,dy\Bigg[&\frac{E_M\,d_M^3}{24\,(1-\nu_M^2)}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2+\frac{d_M}{2}\left(\sigma_{x0}\left\{\frac{\partial w}{\partial x}\right\}^2+\sigma_{y0}\left\{\frac{\partial w}{\partial y}\right\}^2\right)\\ &+\frac{d_M}{8}\frac{E_M}{1-\nu_M^2}\left(\left\{\frac{\partial w}{\partial x}\right\}^4+\left\{\frac{\partial w}{\partial y}\right\}^4\right)-w\Delta p\Bigg], \end{aligned} \tag{44}$$

$$\begin{aligned} V_p = \int_0^{2\pi} d\varphi \int_0^{R_M} r\,dr\Bigg[&\frac{E_M\ d_M^3}{24\ (1-\nu_M^2)}\ \left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)^2\\ &+\frac{d_M}{2}\ \left(\sigma_{r0}\left\{\frac{\partial w}{\partial r}\right\}^2+\frac{\sigma_{\varphi 0}}{r^2}\left\{\frac{\partial w}{\partial \varphi}\right\}^2\right)+\frac{d_M}{8}\ \frac{E_M}{1-\nu_M^2}\left(\frac{\partial w}{\partial r}\right)^4-w\ \Delta p\Bigg]. \end{aligned} \tag{45}$$

The first term in the brackets of the two equations above corresponds to the effect of the bending moments, the second to residual membrane stress in x- and y-, (radial and tangential) directions, respectively, the third to the contribution of the stress due to straining of the neutral fiber, and the last to the energy generated by moving the membrane at the pressure difference.

Equations (44) and (45) are differential equations. In principle, the deflection curve w(r) of the membrane would need to be found which yields the minimum potential energy. An ansatz with free parameters was used instead of the unknown deflection curve. This ansatz is chosen such that it describes the expected deflection curve as good as possible. For a circular membrane with radius R_M and center deflection w_0, an ansatz is made with a fourth-order polynomial with the free parameters w_0, a_0, a_1, a_2, a_3, and a_4:

$$w(r) = w_0\left(a_0 + a_1\frac{r}{R_M} + a_2\frac{r^2}{R_M^2} + a_3\frac{r^3}{R_M^3} + a_4\frac{r^4}{R_M^4}\right). \tag{46}$$

Some of the free parameters can be derived from boundary conditions: The slope of the membrane at its center needs to be zero, i.e. the derivative of w(r) at $r = 0$

needs to be zero; therefore $a_1 = 0$. Also, the deflection at the center of the membrane is defined to be w_0, resulting in $a_0 = 1$. Finally, the deflection at the rim and the slope of the membrane at the rim are necessarily zero. Using these parameters, the following equations are created:

$$a_2 = a_4 - 3 \quad \text{and} \quad a_3 = 2 - 2a_4. \tag{47}$$

As a consequence, w_0 and a_4 are the only free parameters remaining, because a_2 and a_3 depend on a_4, and (46) becomes:

$$w(r) = w_0\left(1 + [a_4 - 3]\frac{r^2}{R_M^2} + [2 - 2a_4]\frac{r^3}{R_M^3} + a_4\frac{r^4}{R_M^4}\right), \tag{48}$$

where w_0 is the center deflection of the membrane and a_4 is a measure of the deflection shape of the membrane. The case when $a_4 = 1$ corresponds to a circular, thick membrane [cf. (23) on page 29]. Figure 23 shows the deflection curve as a function of the parameter a_4. A thin membrane [(24) on page 30] would correspond to the parabola shown in the figure. Obviously, a parabola cannot be perfectly modeled by this ansatz, because the slope of a parabola is not zero at the rim. However, the parabola is best approximated by the ansatz when a_4 is between 3 and 5.

The ansatz (48) is now differentiated and the results are inserted into (45). The integral can then be calculated because the resulting function is simply a polynomial. This calculation comprises a lot of simple steps which all bear the risk of mistakes. Therefore, computer codes are used to perform the analytical calculations. The following equation is the result of such a code:

$$\begin{aligned} V_p = 2\pi\Bigg[&\frac{E_M d_M^3}{24(1-\nu_M^2)}\frac{w_0^3}{R_M^2}\left(9+\frac{6}{5}a_4+\frac{7}{15}a_4^2\right)+\frac{d_M}{2}\sigma_{r0}\, w_0^2\left(\frac{3}{5}+\frac{2}{35}a_4+\frac{1}{105}a_4^2\right)\\ &+\frac{E_M d_M}{1-\nu_M^2}\frac{w_0^4}{R_M^2}\left(\frac{9}{70}+\frac{6}{385}a_4+\frac{3}{385}a_4^2+\frac{2}{5{,}005}a_4^3+\frac{1}{30{,}030}a_4^4\right)\\ &-\Delta p\, w_0\, R_M^2\left(\frac{3}{20}+\frac{1}{60}a_4\right)\Bigg]. \end{aligned} \tag{49}$$

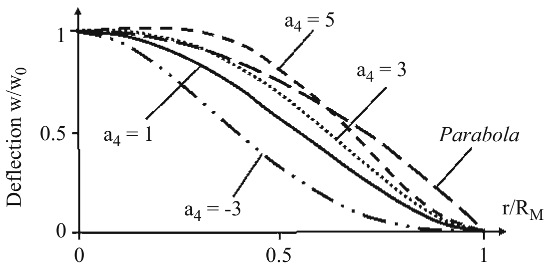


Fig. 23 Deflection curve of a circular membrane calculated with (48) as a function of the parameter a_4

The equation above describes the potential energy of the membrane as a function of the free parameters w_0 and a_4. As before, we are looking for those values of the parameters for which the potential energy is minimal. As usual, the minimum of a function is found by calculating the zero of the derivative with respect to the corresponding parameter, i.e. the following system of two equations needs to be solved for w_0 and a_4:

$$\begin{aligned}0=\frac{\partial V_p}{\partial w_0}=2\,\pi\Bigg[&\frac{E_M\,d_M^3}{24\,(1-v_M^2)}\frac{w_0}{R_M^2}\left(9+\frac{6}{5}a_4+\frac{7}{15}a_4^2\right)+\frac{d_M}{2}\sigma_{r0}\,w_0\left(\frac{3}{5}+\frac{2}{35}a_4+\frac{1}{105}a_4^2\right)\\&+\frac{E_M\,d_M}{1-v_M^2}\frac{w_0^3}{R_M^2}4\left(\frac{9}{70}+\frac{6}{385}a_4+\frac{3}{385}a_4^2+\frac{2}{5{,}005}a_4^3+\frac{1}{30{,}030}a_4^4\right)\\&-\Delta p\,R_M^2\left(\frac{3}{20}+\frac{1}{60}a_4\right)\Bigg],\end{aligned}\tag{50}$$

$$\begin{aligned}0=\frac{\partial V_p}{\partial a_4}=2\,\pi\Bigg[&\frac{E_M\,d_M^3}{24(1-v_M^2)}\frac{w_0^2}{R_M^2}\left(\frac{6}{5}+\frac{14}{15}a_4\right)+d_M\,\sigma_{r0}\,w_0^2\left(\frac{1}{35}a_4+\frac{1}{105}a_4\right)\\&+\frac{E_M\,d_M}{1-v_M^2}\frac{w_0^4}{R_M^2}\left(\frac{6}{385}+\frac{6}{385}a_4+\frac{6}{5{,}005}a_4^2+\frac{2}{15{,}015}a_4^3\right)-\frac{\Delta p\,w_0\,R_M^2}{60}\Bigg].\end{aligned}\tag{51}$$

Unfortunately, this system of equations is too complicated to be solved analytically, but numerically it can be solved easily. In Fig. 24, the interrelationship between pressure drop and both membrane deflection w_0 and the parameter a_4 are shown for two typical cases. On the left, there is the linear relationship between pressure drop and deflection of a 20-µm thick silicon membrane with a radius of 500 µm, no residual stress, and a ratio of Young's modulus E_M to $(1-\nu_M{}^2)$ of 240 GPa. a_4 is approximately one for all pressure values, as expected for a thick membrane.

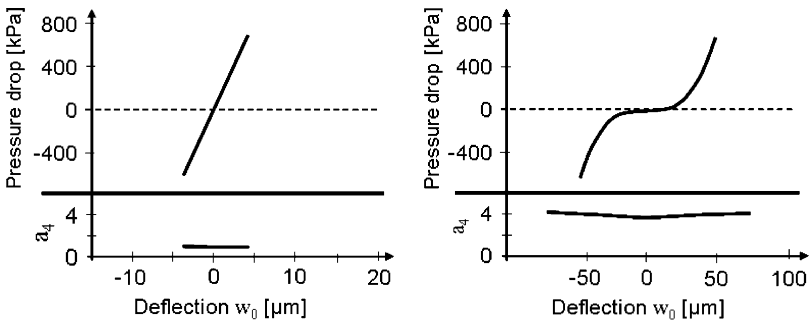


Fig. 24 Interrelationship between the pressure drop over a circular membrane and its center deflection w_0 and the parameter a_4 describing the deflection curve of the membrane both calculated by numerically solving the system of equations (50) and (51). On the left, the result is shown for a thick silicon membrane and on the right for a thin polyimide membrane

On the right of Fig. 24 are the results for a 5-μm thick polyimide membrane also with a radius of 500 μm but with a tensile residual stress of 50 MPa and a ratio of Young's modulus E_M to $(1-\nu_M{}^2)$ of 3.3 GPa. The interrelationship between pressure drop and deflection is not linear in this case. This is in agreement with Cabrera's equation for a thin membrane [(35) on page 33], which contains a nonlinear term due to the straining of the neutral fiber. Accordingly the value of the parameter a_4 is around 4, which approximates a parabola shaped deflection curve of the membrane (cf. Fig. 23).

As the system of (50) and (51) cannot be solved analytically, a simpler ansatz is made calculating the membrane deflection with the Ritz method. It is assumed that the parameter a_4 always equals one. As noted above, this is a good approximation for a thick membrane but does not match well for the case of a thin membrane. Therefore, precision is lost with a more general solution which can be used to study more general cases. The only free parameter remaining in the ansatz is the deflection w_0 of the membrane. At the extremes of the potential energy, the derivative of the potential energy with respect to the deflection is zero:

$$0 = \frac{\partial V_p}{\partial w_0} = \frac{2\,\pi}{3}\left(\frac{8}{3}\,\frac{E_M}{1-\nu_M^2}\,\frac{d_M^3}{R_M^2}\,w_0 + 2\,d_M\,\sigma_0\,w_0 + \frac{128}{105}\,\frac{E_M}{1-\nu_M^2}\,\frac{d_M\,w_0^3}{R_M^2} - \frac{1}{2}\Delta p\,R_M^2\right) = F_M. \tag{52}$$

The derivative of the potential energy with respect to the movement is the force, in general. Therefore, (52) is a description of the force F_M acting on the membrane, and the deflection where the force becomes zero corresponds to the equilibrium of forces as in the simple case of (42) on page 34.

The four terms in the brackets of (52) correspond to the contributions of bending moments, residual stress σ_0, stress due to straining of the neutral fiber, and pressure drop Δp over the membrane, respectively.

Solving (52) for the pressure drop Δp over the membrane yields an equation which describes the relationship between pressure drop and the deflection of a membrane:

$$\Delta p = \frac{4\,d_M\,w_0}{R_M^2}\left(\frac{4}{3}\,\frac{d_M^2}{R_M^2}\,\frac{E_M}{1-\nu_M^2} + \sigma_0 + \frac{64}{105}\,\frac{w_0^2}{R_M^2}\,\frac{E_M}{1-\nu_M^2}\right). \tag{53}$$

This equation can be used to investigate the behavior of membranes. The three terms in the brackets describe the contribution of the bending moments, residual stress σ_0, and stress due to straining of the neutral fiber, respectively. Bending moments and residual stress show a linear interrelationship between pressure drop and membrane deflection, while the stress due to straining contributes with its third power. That is, if the first and the second term of the bracket of (53) are much larger than the third term, the pressure is a linear function of the deflection.

This is important, if a pressure sensor has to be designed which shows a linear characteristic curve. The characteristic curve (53) gets more linear by increasing the

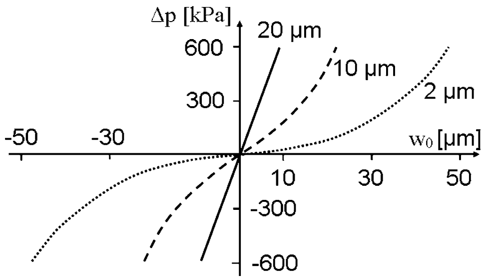


Fig. 25 Pressure drop over a circular membrane as a function of its deflection and different thicknesses calculated with (53)

thickness d_M of the membrane, which enlarges the first term in the brackets compared with the others. In Fig. 25, the pressure drop is shown as a function of the deflection for different thicknesses of a membrane as calculated with (53). Residual stress σ_0, radius R_M, and ratio of Young's modulus E_M to $(1-\nu_M)$ of this membrane are 300 MPa, 600 µm, and 120 GPa, respectively.

As expected, the figure shows that the pressure drop is a more linear function of the deflection for thicker membranes and that this function is linear approximately up to a deflection as large as the thickness of the membrane. So it is an option to design a thicker membrane, if a linear interrelationship between pressure drop and deflection is desired. However, Fig. 25 also shows that the deflection of thicker membranes is less at a certain pressure drop. That is, if a thick membrane is used for a pressure sensor, it is less sensitive. Therefore, there is a compromise between linearity and sensitivity.

The same trend is observed when (53) is investigated with respect to changes in residual stress. A larger residual stress results in a more linear function as well as less sensitivity for a pressure sensor equipped with such a membrane.

If a *compressive residual stress* is assumed for the membrane, negative values need to be used for σ_0 in (53). This is again a rough approximation because it is assumed that the compressive stress of the membrane would be constant throughout the membrane. In general, this is not true, but the negative input value of σ_0 may be considered as a kind of effective residual stress which generates a similar effect as the true distribution of the stress. So a precise description of membrane deflection cannot be expected, but the general trend will be shown.

Figure 26 shows the change which occurs when a compressive residual stress of $\sigma_0 = -600$ MPa is used instead of the tensile stress assumed for Fig. 25. In a certain pressure range, there are two stable states for the membrane: It may buckle up (positive deflection, right side of the graph) or buckle down (negative deflection, left side of the graph). This is called a *bistable membrane*. When there is no pressure drop over the membrane and it is buckling down, it will deflect approximately -55 µm. If the pressure is then raised, the downward deflection becomes increasingly smaller until at approximately -30 µm the membrane snaps over to the opposite side, following the dashed arrow and arriving at an upward deflection of approximately 60 µm. Starting from that point, the pressure can be reversed to about -300 kPa before the membrane snaps back to a downward deflection.

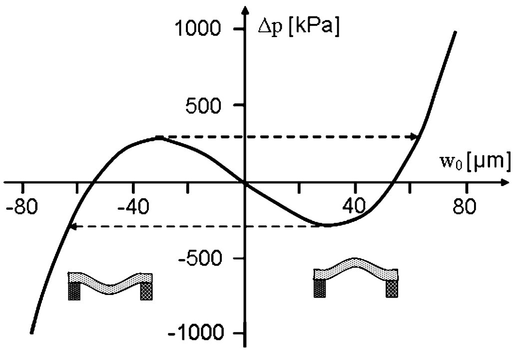


Fig. 26 Pressure drop over a circular membrane as a function of its deflection calculated with (42) as in Fig. 21 but with a compressive residual stress of −600 MPa

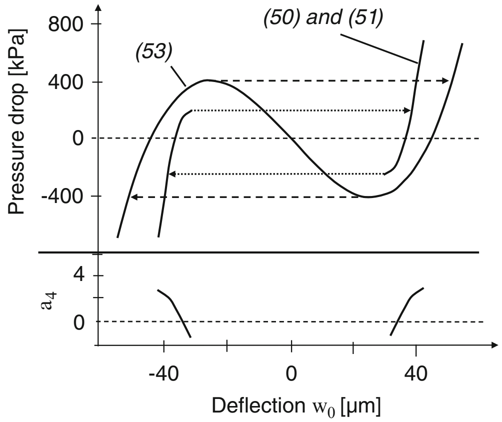


Fig. 27 Interrelationship between the pressure drop over a circular membrane with compressive stress and its center deflection as calculated with (53) and (50) and (51) on page 37

The curve between the two points where the membrane is snapping over cannot be reached by adjusting the pressure difference over the membrane to a certain value. However, it is a real part of the function and can be reached by holding the membrane at a certain deflection by some other means. The pressure drop shown in the graph corresponds to the force per unit area of the membrane necessary to hold the membrane in position and prevent further deflection. The closer the membrane is brought to the non-deflected position, the smaller is the force necessary to hold it at that position. The non-deflected position corresponds to an instable equilibrium of forces of the membrane.

The curves shown in Fig. 26 were also calculated by a numerical solution of (50) and 51 (see page 37) with the same compressive residual stress. In Fig. 27, these curves are compared with each other. The numerical solution is a better approximation because it includes the change of the deflection shape of the membrane. Snapping over is calculated to occur already at a larger deflection and at a smaller pressure difference than calculated with the simpler (53). The calculation also shows that the deflection shape changes from parabolic shape to something beyond the shape of a thick membrane when the snapping over position is approached.

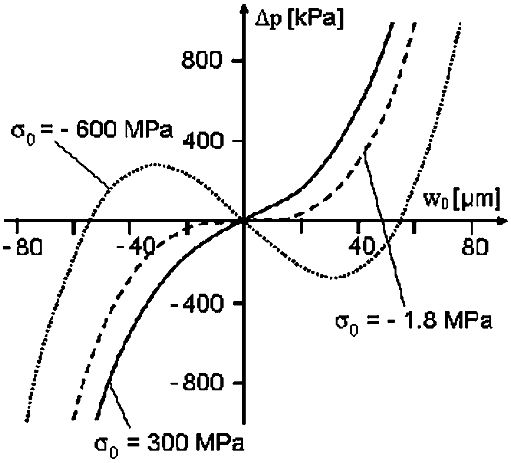


Fig. 28 Interrelationship between the pressure drop over a circular membrane and its center deflection for different residual stresses as calculated with (53)

Figure 28 shows the interrelationship between the pressure drop over a circular membrane and its center deflection for different residual stresses. At a residual stress of -1.8 MPa, there is a transition from a monostable to a bistable membrane. This stress, called the critical stress σ_k of the membrane, occurs where the slope of the function at the origin is zero. Therefore, the critical stress at which the transition takes place can be calculated from the zero of the derivative of (53):

$$0 = \left.\frac{\partial \Delta p}{\partial w_0}\right|_{w_0=0} = \frac{4\,d_M}{R_M^2}\left(\frac{4}{3}\frac{d_M^2}{R_M^2}\frac{E_M}{1-\nu_M^2} + \sigma_0\right) \Rightarrow \sigma_k = -\frac{4}{3}\frac{d_M^2}{R_M^2}\frac{E_M}{1-\nu_M^2}. \tag{54}$$

When the residual stress is smaller (more compressive) than the critical stress, the membrane becomes bistable and buckles if there is no pressure drop. A membrane with larger (more tensile) stress is flat and does not deflect without a pressure drop. Note that the critical stress is as large as the contribution of the bending moments in (53). In other words, buckling takes place when the residual stress overcomes the bending moments of the membrane.

The deflection $w_0(\Delta p = 0)$ of a membrane without pressure drop is calculated now by inserting a pressure drop equal to zero into (53):

$$\Delta p = 0 = \frac{4\,d_M\,w_0}{R_M^2}\left(\frac{4}{3}\frac{d_M^2}{R_M^2}\frac{E_M}{1-\nu_M^2} + \sigma_0 + \frac{64}{105}\frac{w_0^2}{R_M^2}\frac{E_M}{1-\nu_M^2}\right). \tag{55}$$

There are two possibilities to solve this equation:

$$w_0 = 0 \tag{56}$$

and

$$\frac{4}{3}\frac{d_M^2}{R_M^2}\frac{E_M}{1-\nu_M^2}+\sigma_0+\frac{64}{105}\frac{w_0^2}{R_M^2}\frac{E_M}{1-\nu_M^2}=0$$
$$\Rightarrow w_0=\pm\frac{\sqrt{35}}{4}d_M\sqrt{\frac{\sigma_0}{-\frac{4}{3}\left(d_M^2/R_M^2\right)\left(E_M/(1-\nu_M^2)\right)}-1}. \tag{57}$$

The first solution corresponds to a flat membrane, while the second describes the deflection of a bistable membrane with compressive stress. The denominator under the square root is the critical stress (cf. 54), and this solution gives a suitable result only if the residual stress is more compressive than the critical stress (cf. Fig. 29).

Equation (56) is a valid solution for all residual stresses of the membrane. In fact, if the residual stress is more tensile than the critical stress, it is the only solution, because the term under the root is negative. If the residual stress is more compressive than the critical one, the membrane is in an unstable, non-deflected position.

The residual stress of a membrane is changed as a function of the temperature, if the frame or housing it is fixed at shows a different thermal expansion. Equation (57) may be useful to calculate the temperature at which the membrane buckles.

The buckling of a bistable membrane with compressive stress was employed to design the microvalve shown in Fig. 30 [25]. The compressive stress of the membrane results in a force which presses the membrane against the orifice of the inlet and keeps it closed against an outer pressure. When the valve is opened, the heater in the actuator chamber is switched on and the air escapes through the orifice to the environment. After this, the heater is switched off and the decreasing pressure in the actuator chamber pulls the membrane into the downward stable position. For closing, the membrane is pushed up again by activating the heater. The electrical

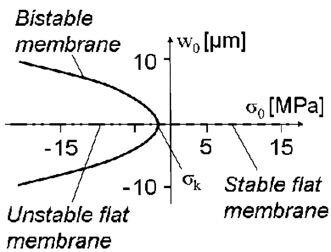


Fig. 29 Deflection of a circular membrane without pressure difference as a function of its residual stress calculated with (57) and (56) (*dashed line*)

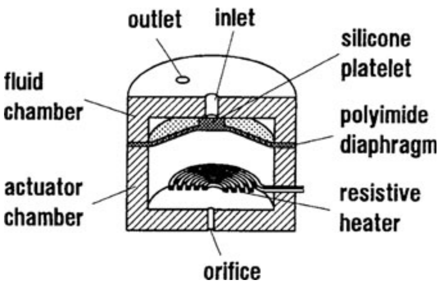


Fig. 30 Microvalve equipped with a bistable membrane [25]

current through the heater is then lowered slowly, allowing the pressure in the actuator chamber to equalize through the orifice and avoiding the pulling down of the membrane.

To design the membrane of a valve of this type its deflection and elastic force need to be calculated. The optimum distance between the undeflected membrane and the inlet orifice with respect to a maximized force on the inlet also needs to be known.

If a force such as the pressure from the inlet of the valve is acting on the center of a circular membrane, the deflection can be approximated with (52) on page 38. Fig. 31 shows the calculated elastic force generated by a 2-µm thick, circular membrane (radius of 600 µm and a ratio $E_M/(1-\nu_M^2)$ of 120 GPa) as a function of membrane deflection w_0 and stress σ_0, if there is no pressure difference over the membrane. The situation is similar to the case of a membrane loaded with a pressure shown in Fig. 28. The only difference is that the membrane is deflected by a force instead of a pressure. When the stress is more compressive than the critical stress, it becomes bistable. If there is no force acting on the membrane and it is buckling downward (negative w_0), its deflection is approximately −55 µm. If it is deflected upward, then it generates a negative elastic force which tries to deflect it down further. The absolute value of this force is increasing until the deflection w_{U-} is reached at a force F_{U-} and the membrane snaps over to the upper side.

The extremes of the deflection are found by calculating the derivative of (52) on page 38 and finding the zeros:

$$\frac{\partial F_M}{\partial w_0} = \frac{2\pi}{3}\left(\frac{8}{3}\frac{E_M\, d_M^3}{1-\nu_M^2}\frac{1}{R_M^2} + 2\, d_M\, \sigma_0 + \frac{128}{35}\frac{E_M\, d_M}{1-\nu_M^2}\frac{w_0^2}{R_M^2}\right) = 0. \tag{58}$$

$$\begin{aligned}\Rightarrow w_U &= \pm\frac{\sqrt{35}}{\sqrt{3}\,4} d_M \sqrt{\frac{\sigma_0}{-\frac{4}{3}\left(d_M^2/R_M^2\right)\left(E_M/1-\nu_M^2\right)} - 1} \\ &= \pm\frac{\sqrt{35}}{\sqrt{3}\,4} d_M \sqrt{\frac{\sigma_0}{\sigma_k} - 1} = \frac{1}{\sqrt{3}} w_0(\Delta p = 0).\end{aligned} \tag{59}$$

The comparison of the above result with (57) shows that the deflection at the snapping over point is just a factor $1/\sqrt{3} \approx 0.58$, smaller than the deflection when

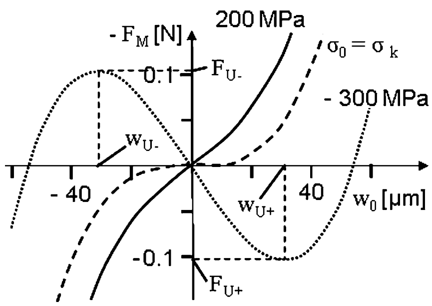


Fig. 31 Elastic force of a circular membrane as a function of its deflection and stress calculated with (52) on page 38

no force or pressure difference is acting on the membrane. The maximum force generated by the membrane is achieved at w_U and is found by inserting (59) into (52):

$$F_U = \pm\frac{\pi}{3}\left(\frac{7}{3}\sqrt{\frac{35}{3}}\, d_M^2\, \sigma_k\left(\frac{\sigma_0}{\sigma_k} - 1\right)^{3/2} - \Delta p\, R_M^2\right). \tag{60}$$

Figure 32 shows the effect of a pressure difference on snapping over and maximum force of a membrane with a residual stress of −300 MPa as calculated with (52).

The pressure drop simply adds a constant to (52). Thus, the deflections w_{U+} and w_{U-} at which the membrane snaps over are not changed. The maximum forces F_{U+} and F_{U-}, however, become asymmetric and the bistable membrane positions without an outer force acting are changed also.

If the stress of a membrane becomes increasingly compressive, more complex bulging may occur which is not described with a simple ansatz such as (48) on page 36. The membrane then contains several folds as seen in Fig. 33, which shows the membrane of a micropump heated and thus being exposed to large compressive stress [26].

When a membrane is designed for a pressure sensor, a microvalve, or a micropump, it is important to know at which *burst pressure* the membrane breaks. The

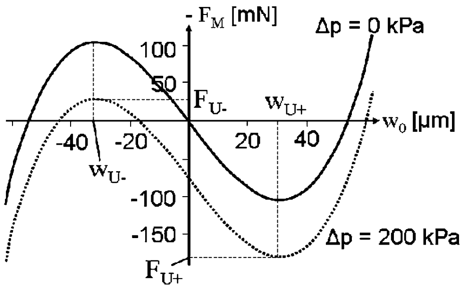


Fig. 32 Elastic force of a circular membrane as a function of its deflection and stress calculated with (52) on page 38

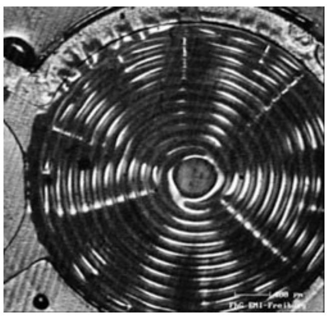


Fig. 33 Membrane of a micropump showing several folds due to large compressive stress generated by heating of the membrane [26].

burst pressure can be calculated from Cabrera's equation or another equation describing the relationship between deflection w_0, pressure drop Δp, and total stress of the membrane σ_M. Here the calculation based on Cabrera's equation [(35) on page 33] assumes that the membrane is a thin membrane. This assumption will be suitable for most applications because the deflection of a membrane will be larger than its thickness when it bursts. However, membranes from brittle materials such as silicon dioxide and ceramics may be exceptions.

The membrane bursts when the total stress σ_M becomes larger than the yield stress σ_y. [The total stress is described by (34) on page 33.] At the burst pressure Δp_y, the membrane will have the burst deflection w_{0y} and (34) and (35) become:

$$\sigma_y = \sigma_0 + \frac{2}{3}\frac{w_{0y}^2}{R_M^2}\frac{E_M}{1-\nu_M}. \tag{61}$$

$$\Delta p_y = 4\frac{w_{0y}\, d_M}{R_M^2}\sigma_y. \tag{62}$$

The burst deflection is calculated from (61)

$$w_{0y} = R_M\sqrt{\frac{3}{2}\left(\sigma_0 - \sigma_y\right)\frac{1-\nu_M}{E_M}} \tag{63}$$

and inserted into (62), resulting in an expression with which the burst pressure can be calculated as follows:

$$\Delta p_y = 4\frac{d_M}{R_M}\sigma_y\sqrt{\frac{3}{2}\left(\sigma_0 - \sigma_y\right)\frac{1-\nu_M}{E_M}}. \tag{64}$$

To calculate the burst pressure, the yield stress of the membrane material needs to be known. The yield stress can be determined by increasing the pressure differential in small increments and measuring the deflection. When the membrane breaks, the last deflection and pressure measured are the burst deflection and burst pressure, respectively, and the yield stress can be calculated by solving (62).

The calculations shown above for a circular membrane can also be performed for square membranes with edge length a_M. A parabolic deflection is assumed in x- and y-direction:

$$w(x,y) = w_0\left(1 - 4\frac{x^2}{a_M^2}\right)^2\left(1 - 4\frac{y^2}{a_M^2}\right)^2. \tag{65}$$

The potential energy is calculated from (44) (page 35) with this ansatz and differentiated with respect to the central deflection w_0 to obtain the force acting on the membrane. It is assumed that the stress σ_x and σ_y of the square membrane may be different in the x- and y-direction, respectively:

$$\frac{\partial V_p}{\partial w_0} = F_M = \frac{64}{75}\left(\frac{2^{10}}{49}\frac{E_M}{1-\nu_M^2}\frac{d_M^3\ w_0}{a_M^2} + \frac{2^{10}}{49\times 9}d_M\ w_0\left(\sigma_x+\sigma_y\right)\right.$$
$$\left. + \frac{2^{28}}{7\times 9\times 11^2\times 13^2\times 17}\frac{d_M\ E_M}{1-\nu_M^2}\frac{w_0^3}{a_M^2} - \frac{1}{3}\Delta p\ a_M^2\right)$$
$$= 17.8\frac{E_M\ d_M^3}{1-\nu_M^2}\frac{w_0}{a_M^2} + 1.98\ d_M\left(\sigma_x+\sigma_y\right)w_0 + 10.46\frac{E_M\ d_M}{1-\nu_M^2}\frac{w_0^3}{a_M^2} - 0.284\ \Delta p\, a_M^2. \tag{66}$$

If the force acting on the membrane is zero, i.e. the membrane is in equilibrium, this equation can be solved for Δp:

$$\Delta p = 3\frac{d_M}{a_M^2}w_0\left(20.9\frac{E_M}{1-\nu_M^2}\frac{d_M^2}{a_M^2} + 2.32\left(\sigma_x+\sigma_y\right) + 12.3\frac{E_M}{1-\nu_M^2}\frac{w_0^2}{a_M^2}\right). \tag{67}$$

From this equation, the critical stress of a square membrane can be calculated. Similar to the case of a circular membrane (cf. page 41), the critical stress σ_k is achieved when the second term in the parenthesis of (67) is more compressive than the first one. In other words, the average of the stress components in the x- and y-directions is more negative than the following:

$$\sigma_k = -\frac{9}{2}\frac{E_M}{1-\nu_M^2}\frac{d_M^2}{a_M^2}. \tag{68}$$

The deflection w_0 ($\Delta p = 0$) of a square membrane with a compressive stress larger than the critical stress σ_k and no pressure difference, the deflection w_U at the snapping over point, and the force F_U needed for snapping over are calculated in the same way as for a circular membrane:

$$w_0(\Delta p = 0) = \pm\frac{11\times 13}{3\times 2^8}\sqrt{\frac{17}{7}}\ d_M\sqrt{\frac{\sigma_x+\sigma_y}{2\ \sigma_k} - 1}$$
$$= \pm 0.29\ d_M\sqrt{\frac{\sigma_x+\sigma_y}{2\ \sigma_K} - 1}. \tag{69}$$

$$w_U = \pm\frac{11\times 13}{3\times 2^8}\sqrt{\frac{17}{3\times 7}}\ d_M\sqrt{\frac{\sigma_x+\sigma_y}{2\,\sigma_k} - 1} = \pm 0.17\, d_M\sqrt{\frac{\sigma_x+\sigma_y}{2\,\sigma_k} - 1}$$
$$= \frac{1}{\sqrt{3}}\,w_0(\Delta p = 0). \tag{70}$$

$$F_U = \pm\frac{64}{9\times 25}\left(\frac{8\times 11\times 13\times 23}{3^5\times 49}\sqrt{\frac{17}{21}}\,d_M^2\ \sigma_k\left(\frac{\sigma_x+\sigma_y}{2\ \sigma_k} - 1\right)^{3/2} - \Delta p\ a_M^2\right)$$
$$= \pm 0.28\left(1.99\ d_M^2\ \sigma_k\left(\frac{\sigma_x+\sigma_y}{2\ \sigma_k} - 1\right)^{3/2} - \Delta p\ a_M^2\right). \tag{71}$$

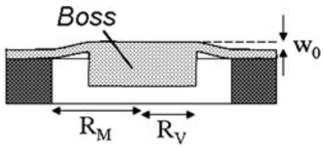


Fig. 34 Membrane with a boss at its center

Figure 25 and the last term of (53) (page 38) show that, in general, the deflection of a membrane is not a linear function of the pressure drop. For several applications, especially pressure sensors, it is desirable to design membranes with a deflection which is a linear function of the pressure difference. Therefore, several designs were developed providing improved linearity.

One possibility is to fabricate a *membrane with a boss* at its center (cf. Fig. 34). The boss stiffens the membrane so much that it may be assumed that only the annular part of the membrane not covered by the boss is strained when a pressure difference is applied to the membrane. As a result, the coefficients in (53) on page 38 need to be altered [27]:

$$\Delta p = \frac{d_M\ w_0}{R_M^2}\left(a_p \frac{d_M^2}{R_M^2}\frac{E_M}{1-\nu_M^2} + 4\ \sigma_0 + b_p \frac{w_0^2}{R_M^2} E_M\right). \tag{72}$$

With

$$a_p = \frac{16}{3}\,\frac{1}{1-\left(R_V^4/R_M^4\right)-4\left(R_V^2/R_M^2\right)\ \ln(R_M/R_V)} \tag{73}$$

and

$$b_p = \frac{(7-\nu_M)\left(1+\left(R_V^2/R_M^2\right)+\left(R_V^4/R_M^4\right)\right)/3+(3-\nu_M)^2/(1+\nu_M)\left(R_V^2/R_M^2\right)}{(1-\nu_M)\left(1-\left(R_V^4/R_M^4\right)\right)\left(1-\left(R_V^2/R_M^2\right)\right)}. \tag{74}$$

In the above equations, R_V denotes the radius of the boss. Equation (72) is an approximation which is valid when the radius of the boss R_V is larger than 15% of the radius of the membrane R_M and the thickness of the boss is at least six times larger than the thickness of the membrane.

The effect of the boss is illustrated in Fig. 35, which shows the coefficients a_p, b_p, and their ratio as a function of the ratio R_V/R_M of the radii of boss and membrane. Poisson's ratio was assumed to be 0.3 for this example calculation. The coefficients are much larger with a sufficiently large boss at the membrane center than without it. (16/3 and $256/(105\ (1-\nu_M^2)) \approx 2.7$, respectively, as shown in (53) on page 38.) This means that the deflection calculated with (72) will be smaller with a boss, resulting in less sensitivity if the membrane is employed in a pressure sensor. On the other hand, the residual stress of the membrane has a comparatively smaller effect on the deflection. This is an advantage because the cross-sensitivity of the sensor to stress changes is reduced. Such stress changes often occur when the

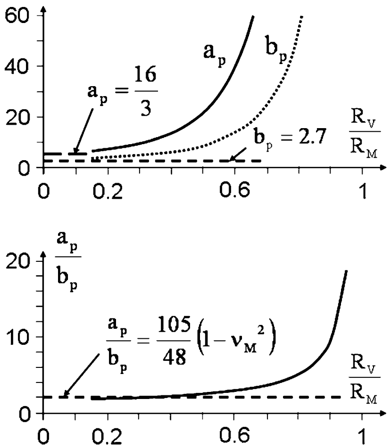


Fig. 35 Coefficients a_p, b_p, and their ratio calculated with (73) and (74)

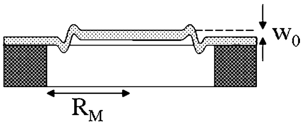


Fig. 36 Membrane with corrugations at its rim

housing of the sensor shows a different thermal expansion than the membrane or when the sensor is mounted somewhere.

The ratio of the coefficients a_p and b_p increases with the radius of the boss. As a consequence, the characteristic curve (membrane deflection as a function of the pressure difference) becomes more linear, because a_p is the coefficient of a linear term and b_p of a nonlinear term in (72). However, a significant effect is achieved, only if the boss covers more than approximately 80% of the membrane (cf. Fig. 35).

Another interesting way to achieve a more linear membrane deflection as a function of the pressure difference is to design a *corrugated membrane* (cf. Fig. 36). Similar to the case of a membrane with a boss, (53) on page 38 needs to be altered to take into account the effect of the corrugations [28]:

$$\Delta p = \frac{d_M\, w_0}{R_M^2}\left(a_p\,\frac{d_M^2}{R_M^2}E_M + 4\sigma_0 + b_p\,\frac{w_0^2}{R_M^2}\,\frac{E_M}{1-\nu_M^2}\right). \tag{75}$$

With

$$a_p = \frac{2}{3}\,\frac{(q+3)(q+1)}{1-(\nu_M^2/q^2)} \tag{76}$$

and

$$b_p = \frac{165(q+1)(q+3)}{q^2(q+4)(q+11)(2\,q+1)(3\,q+5)}. \tag{77}$$

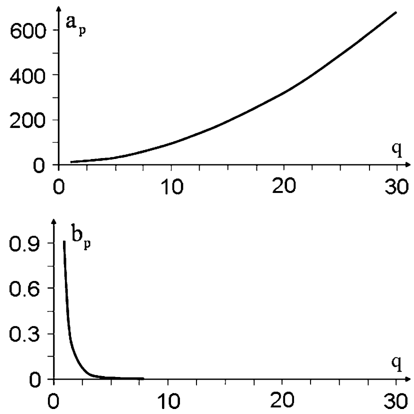


Fig. 37 Coefficients a_p, b_p, and their ratio calculated with (76) and (77)

In these equations, q is the profile factor of the corrugations in the membrane. It is a measure of the size of the corrugations. The calculation is described in [27]. A membrane without any corrugations corresponds to $q = 1$. Typical corrugated membranes correspond to a profile factor between 10 and 30.

Figure 37 shows the coefficients a_p and b_p as a function of q. Only small corrugations are required to reduce the nonlinear third term in the parenthesis of (75) and to allow dominance of the linear first term. Thus, only a few shallow corrugations at the rim of a membrane are enough to achieve a linear displacement as a function of the pressure difference. Since a_p rises quickly with increasing q, the corrugated membrane becomes much stiffer and the effect of residual stress is diminished significantly.

It should be noted that in all calculations described in this chapter, it has been assumed that Young's modulus and Poisson's ratio are isotropic properties of the membrane material. This is true for amorphous and polycrystalline materials. However, for monocrystalline materials such as silicon, which is the material most widely used in microtechnique, these properties are a strong function of crystal orientation. Figure 38 shows Young's modulus and Poisson's ratio as a function of crystal orientation for monocrystalline silicon [29].

Different orientations result in Young's moduli which vary up to 40% and Poisson's ratios can change by as much as a factor 4. Fortunately, in most equations, the ratios $E/(1-\nu)$ or $E/(1-\nu^2)$ occur, which only vary by 23% in the worst case because Young's modulus is largest for the [110] direction where Poisson's ratio is smallest, and the ratio balances these variations slightly.

It is very hard to include the anisotropic properties in analytical calculations. Therefore, effective or average values are often used instead.

The following table provides an overview on the equations which can be used to calculate the interrelationship between the pressure difference over a membrane and its deflection.

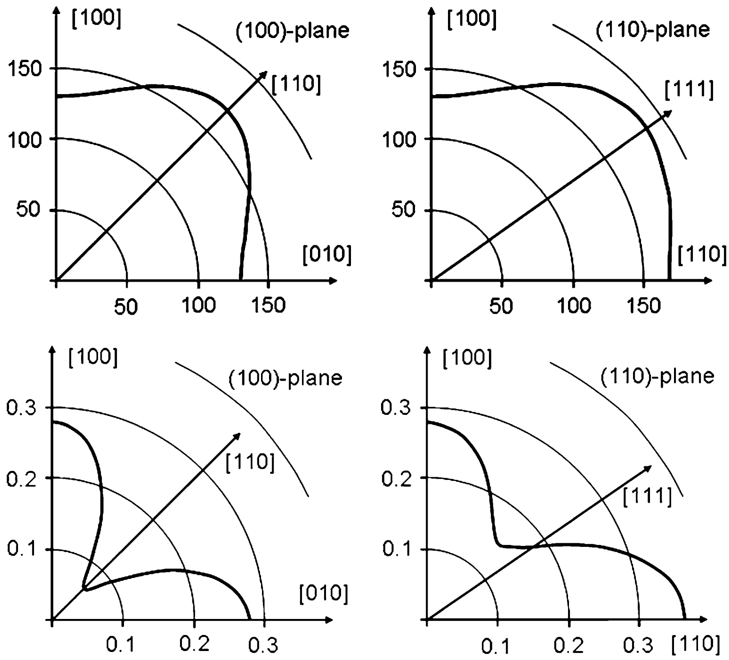


Fig. 38 Young's modulus [GPa] (*upper row*) and Poisson's ratio (*lower row*) of monocrystalline silicon as a function of crystal orientation. Reprinted with permission from [29]. Copyright [1965], American Institute of Physics

Table 2 Equations for the calculation of the interrelationship of pressure difference Δp and deflection w_0 of a membrane

Membrane type	Equation
General solution for circular membrane, rough approximation	$\Delta p = 4\frac{d_M w_0}{R_M^2}\left(\frac{4}{3}\frac{d_M^2}{R_M^2}\frac{E_M}{1-\nu_M^2} + \sigma_0 + \frac{64}{105}\frac{w_0^2}{R_M^2}\frac{E_M}{1-\nu_M^2}\right)$
General solution for square membrane, rough approximation	$\Delta p = 3\frac{d_M w_0}{a_M^2}\left(20.9\frac{E_M}{1-\nu_M^2}\frac{d_M^2}{a_M^2} + 2.32(\sigma_x+\sigma_y) + 12.3\frac{E_M}{1-\nu_M^2}\frac{w_0^2}{a_M^2}\right)$
Thin, circular, without bending moments, exact solution	$\Delta p = 4\frac{d_M w_0}{R_M^2}\left(\sigma_0 + \frac{2}{3}\frac{w_0^2}{R_M^2}\frac{E_M}{1.026-0.793\nu_M-0.233\,\nu_M^2}\right)$
Thin, square, without bending moments, exact solution	$\Delta p = 13.6\frac{d_M w_0}{a_M^2}\left(\sigma_0 + 1.61\frac{w_0^2}{a_M^2}\frac{(1.446-0.427\,\nu_M)E_M}{1-\nu_M}\right)$
Thick, circular, without stress, exact solution	$\Delta p = \frac{16}{3}\frac{d_M^3}{R_M^4}\frac{E_M}{1-\nu_M^2}w_0 \Rightarrow w_0 = \frac{3}{16}\frac{R_M^4}{d_M^3}\frac{1-\nu_M^2}{E_M}\Delta p$
Thick, square, without stress, exact solution	$\Delta p = 66\frac{d_M^3}{a_M^4}\frac{E_M}{1-\nu_M^2}w_0 \Rightarrow w_0 = \frac{1}{66}\frac{a_M^4}{d_M^3}\frac{1-\nu_M^2}{E_M}\Delta p$
Circular membrane, with boss, for a_p and b_p see (73) and (74)	$\Delta p = \frac{d_M w_0}{R_M^2}\left(a_p\frac{d_M^2}{R_M^2}\frac{E_M}{1-\nu_M^2} + 4\,\sigma_0 + b_p\frac{w_0^2}{R_M^2}E_M\right)$
Circular membrane, with corrugations, [a_p, b_p see (76) and (77)]	$\Delta p = \frac{d_M w_0}{R_M^2}\left(a_p\frac{d_M^2}{R_M^2}E_M + 4\,\sigma_0 + b_p\frac{w_0^2}{R_M^2}\frac{E_M}{1-\nu_M^2}\right)$

Exercises

Problem 8

You are asked to design a pressure sensor which fulfills the following specifications:

Measurement range	0–100 kPa
Sensitivity	1 kPa
Maximum deviation of the characteristic curve from linearity	1% of the value measured
Temperature range for the use	−20 to 50°C

Due to the available equipment in your company, you decide to design a circular silicon membrane. The deflection of this membrane shall be ruled by the bending moments. The following material properties are given:

Young's modulus of the membrane	150 GPa
Poisson's ratio	0.3
Thermal strain of the membrane	3×10^{-6}/°C
Thermal strain of the housing	5×10^{-6}/°C

(a) Assume that the sensor membrane is strained directly by the difference in thermal strain of sensor and housing, because it is much thinner than the housing. Which change of the stress of the membrane do you expect over the temperature range specified for the sensor?
(b) The membrane may be deflected up to what portion of the thickness in order to keep the linearity specified? (Neglect the initial stress of the membrane.)
(c) How small may the ratio of thickness to radius of the membrane become to avoid that the thermal strain changes the sensor signal for less than 1%.
(d) How small may the ratio of thickness to radius of the membrane become to avoid that the maximum deflection of the membrane [from (b)] is not exceeded at the upper limit of the measurement range? (Again neglect the initial stress of the membrane.)
(e) What is the smallest deflection of the membrane which needs to be measured to achieve the desired sensitivity? (The diameter of the membrane is 2 mm.)
(f) What thickness is needed for the membrane?

Problem 9

In the lecture, you got to know a membrane with bistable states.

(a) What is the reason for the snapping over of the membrane?

(b) The snapping over can be used to design a bistable valve (cf. Fig. 30 on page 42). In such a bistable valve [30], the membrane consists of polyimide and shows a thickness of $d_M = 25\ \mu m$. The radius of the membrane is $R_M = 1.5$ mm.
(c) Please calculate the critical stress σ_K of this membrane.
(d) What is the deflection of the membrane at the snapping over if the initial stress is $\sigma_0 = -8.5$ MPa?

Young's modulus of polyimide	1.66 GPa	Poisson's ratio of polyimide	0.41

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