Learning What the Higgs Is Mixed With

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(with H. E. Logan, R. Killick - arXiv:1305.7236)

OUTLINE

- Motivation for measuring *hhVV*
- Measuring Higgs Couplings - LHC, ILC
- Parametrization of Couplings
- 3 Benchmark Models (BMs)
- *hhVV* and *hhh* from di-Higgs rates at ILC
- M_{hh} as kinematic discriminant
- Caveats / Viability of BMs
- Conclusions

Motivation

- We’ve entered the stage of measuring the properties of the 125 GeV resonance more precisely
- Through a long-term experimental program we hope to find out more about the nature of this particle and EWSB
- Many extensions of the SM involve the Higgs mixing with another scalar
- This scenario can be tested by measuring coupling deviations or by direct searches
- We shall focus on the scenario where no new particles are observed

Motivation

With no new particles seen measuring coupling deviations is crucial

$$\mathcal{L} \supset M_V^2 V_\mu^* V^\mu \left[1 + a_V \frac{2h}{v} + b_V \frac{h^2}{v^2}\right] - m_f \bar{f} f \left[1 + c_f \frac{h}{v}\right]$$
$$- \frac{1}{2} M_h^2 h^2 \left[1 + d_3 \frac{h}{v} + d_4 \frac{h^2}{4v^2}\right]$$

- We assume the new scalar doesn't contribute to EWSB (has zero or tiny vev) and so doesn't couple singly to vector bosons
- We assume the new scalar doesn't couple singly to quarks or charged leptons to avoid stringent constraints from FCNCs

Motivation

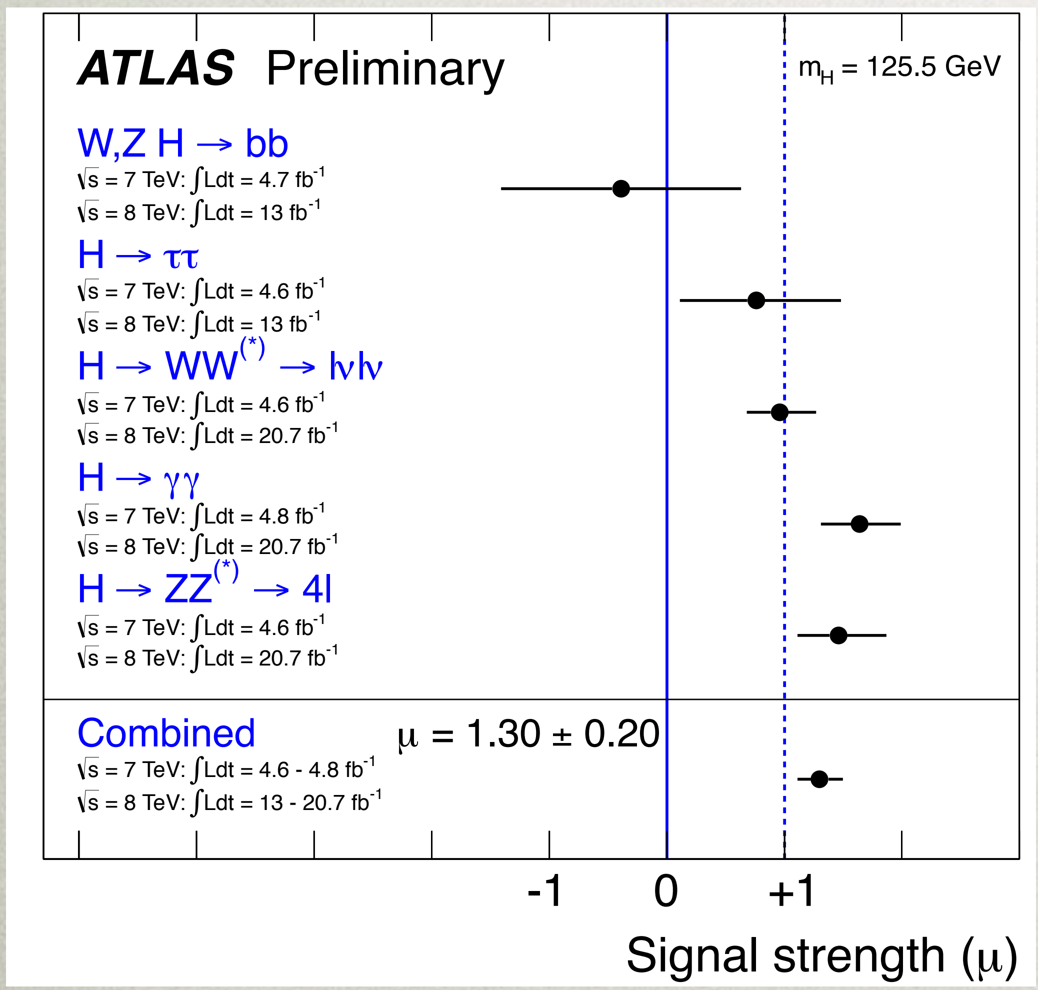
- Modifies *hVV* and *hff* couplings by a common multiplicative factor
- The factor depends on the mixing angle
- But measuring the mixing angle alone doesn't give us information about the EW quantum numbers of the new scalar
- Measuring the *hhVV* and *hhh* coupling can help to distinguish between models where the Higgs mixes with scalars with different EW quantum numbers

Motivation

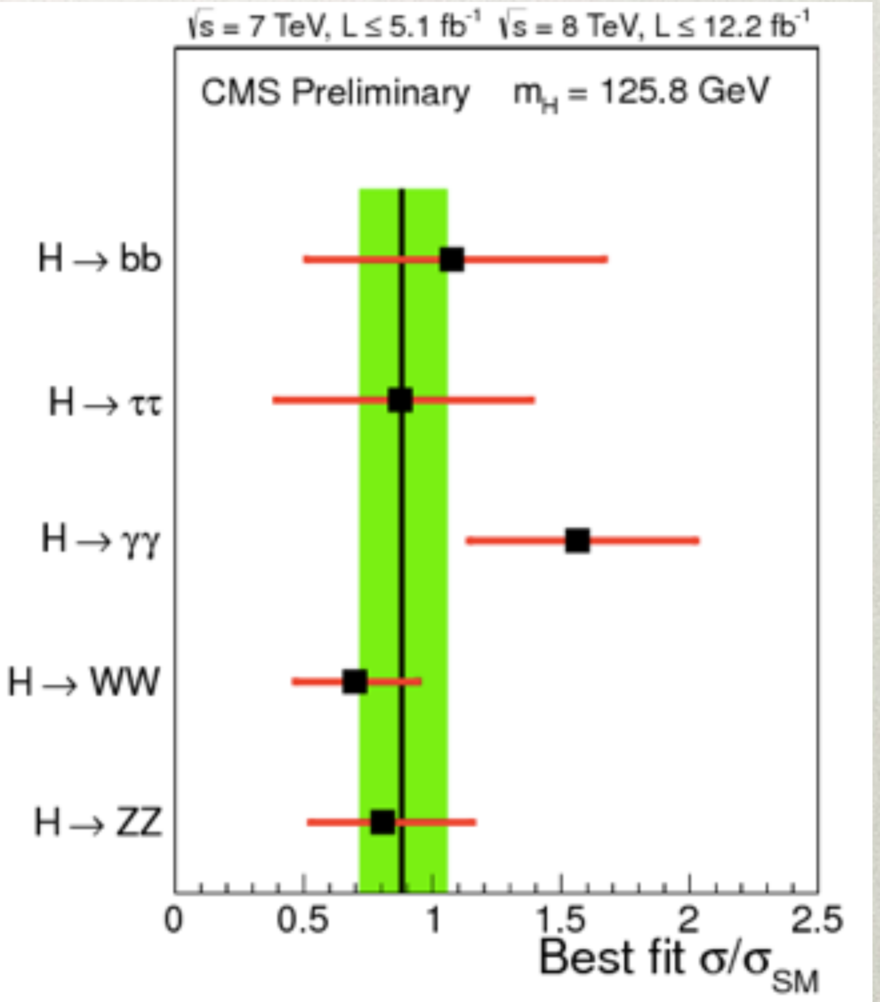
- Measuring these at the LHC is extremely hard (details to follow)
- We propose to extract them from cross sections of $e^+e^- \to Zhh$ and $e^+e^- \to \nu\bar{\nu}hh$ at the proposed ILC
- These processes have been looked at in the past only to extract the Higgs self coupling
- So the main goal of this work is to make a case for doing this hard measurement at the ILC in order to find out more about the EW quantum numbers of the scalar the Higgs mixes with

Measuring Higgs Couplings - LHC

- At the LHC what we measure are signal strengths (production x Branching Ratio)



Moriond 2013

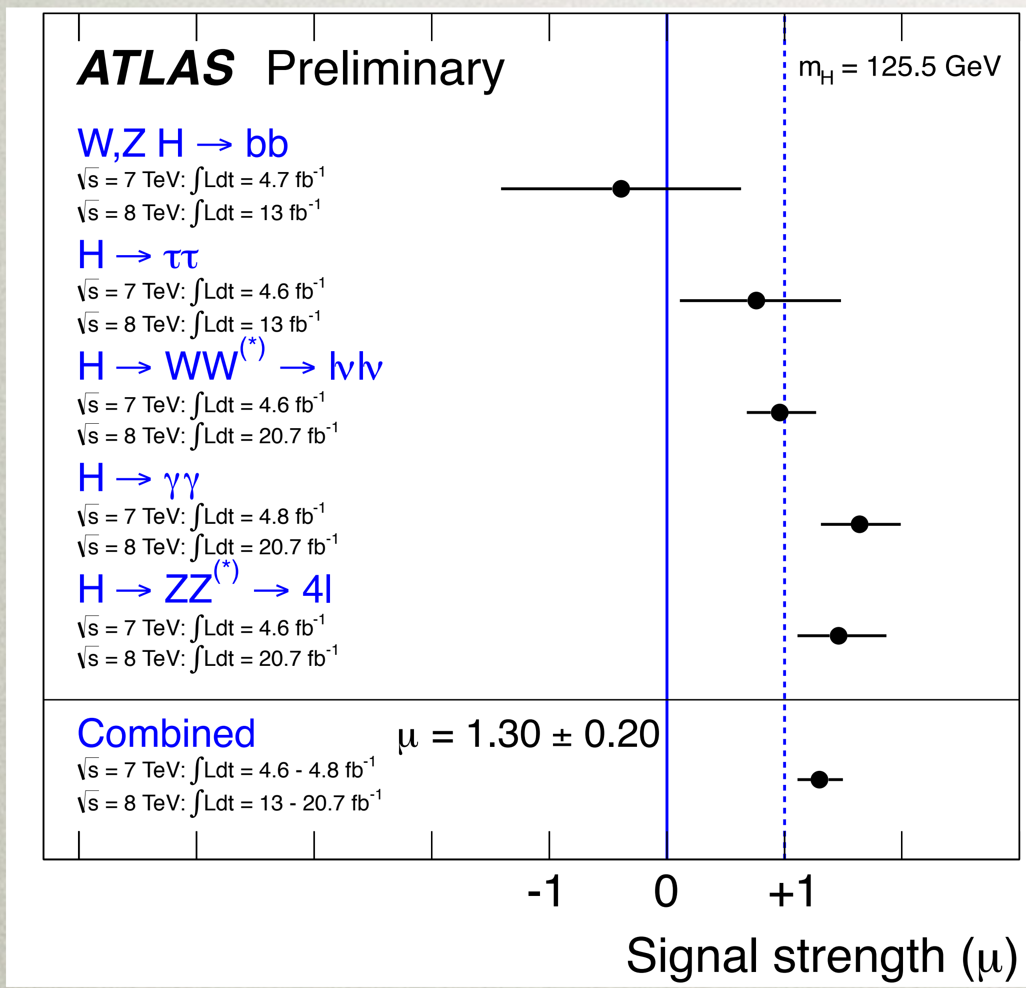


$$\hat{\mu} = 0.88 \pm 0.21$$

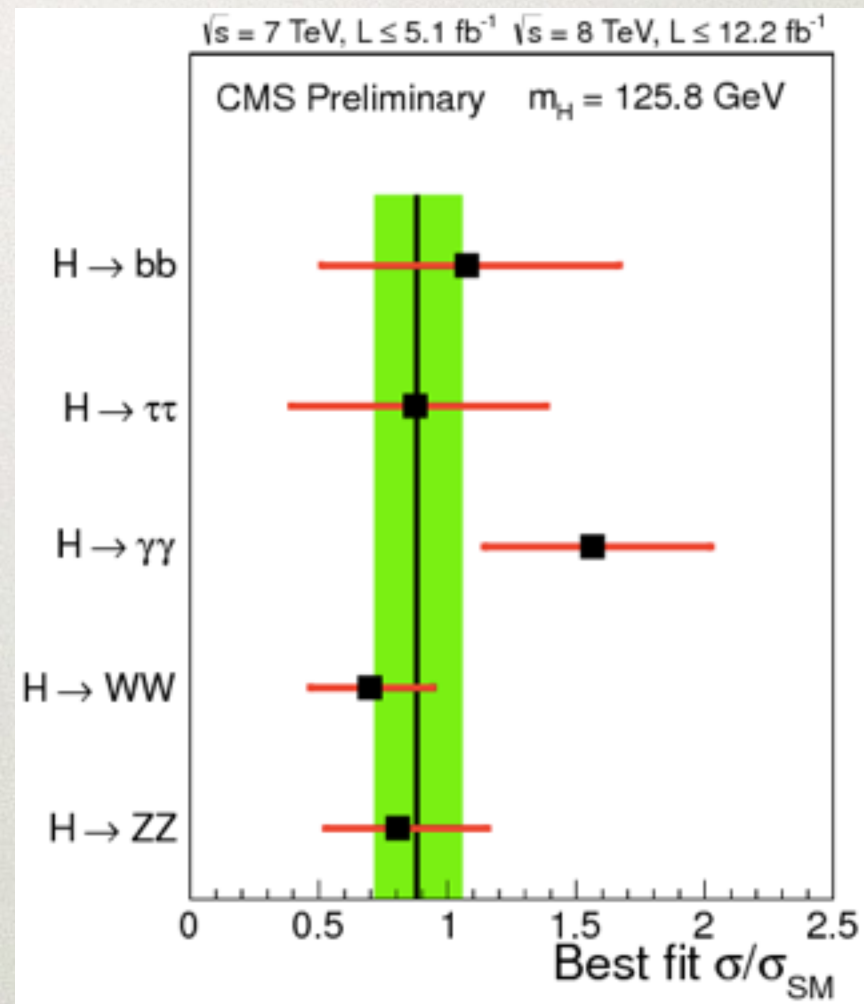
HCP 2012

Measuring Higgs Couplings - LHC

- Extracting Higgs couplings in a model independent way from the signal strength require global fits (too many parameters vs model dependence)



Moriond 2013

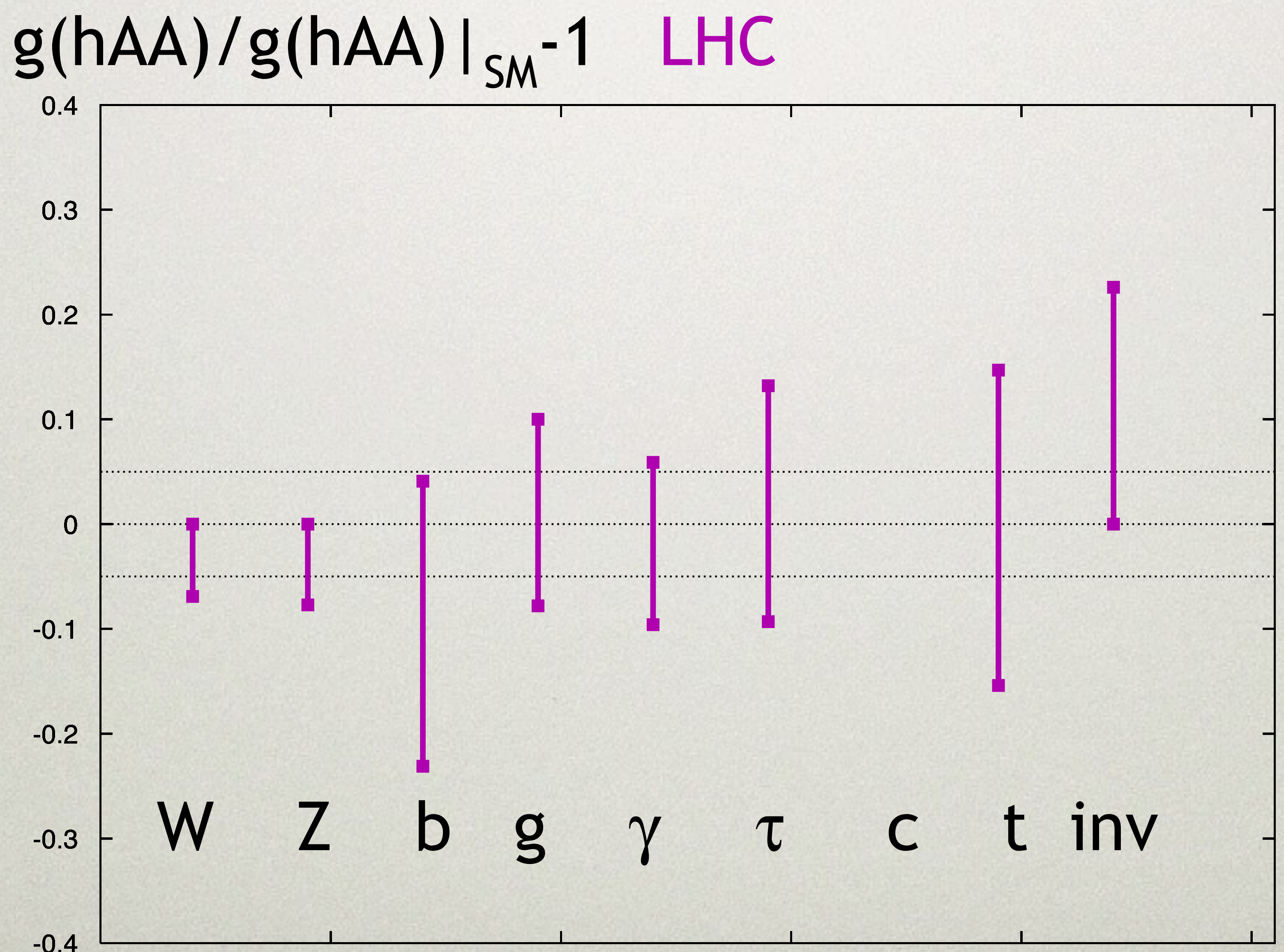


$\hat{\mu} = 0.88 \pm 0.21$

HCP 2012

Measuring Higgs Couplings - LHC

- Estimates for accuracy in Higgs coupling measurements with 300 inv fb of data (end of this decade)



M. Peskin, arXiv : 1208.5152

Measuring Higgs Couplings - LHC

- *hhVV* and *hhh* couplings are hard to measure because the cross sections for di-Higgs production are small
- At 14 TeV LHC
  pp -->h ~ 50 pb (gluon fusion)
  pp --> h h ~ 20 fb (gluon fusion)
  pp --> h h ~ 2 fb (VBF)

A. Djouadi, W. Kilian, M. Muhlleitner and P. Zerwas, Eur. Phys. J. C10 (1999) 45

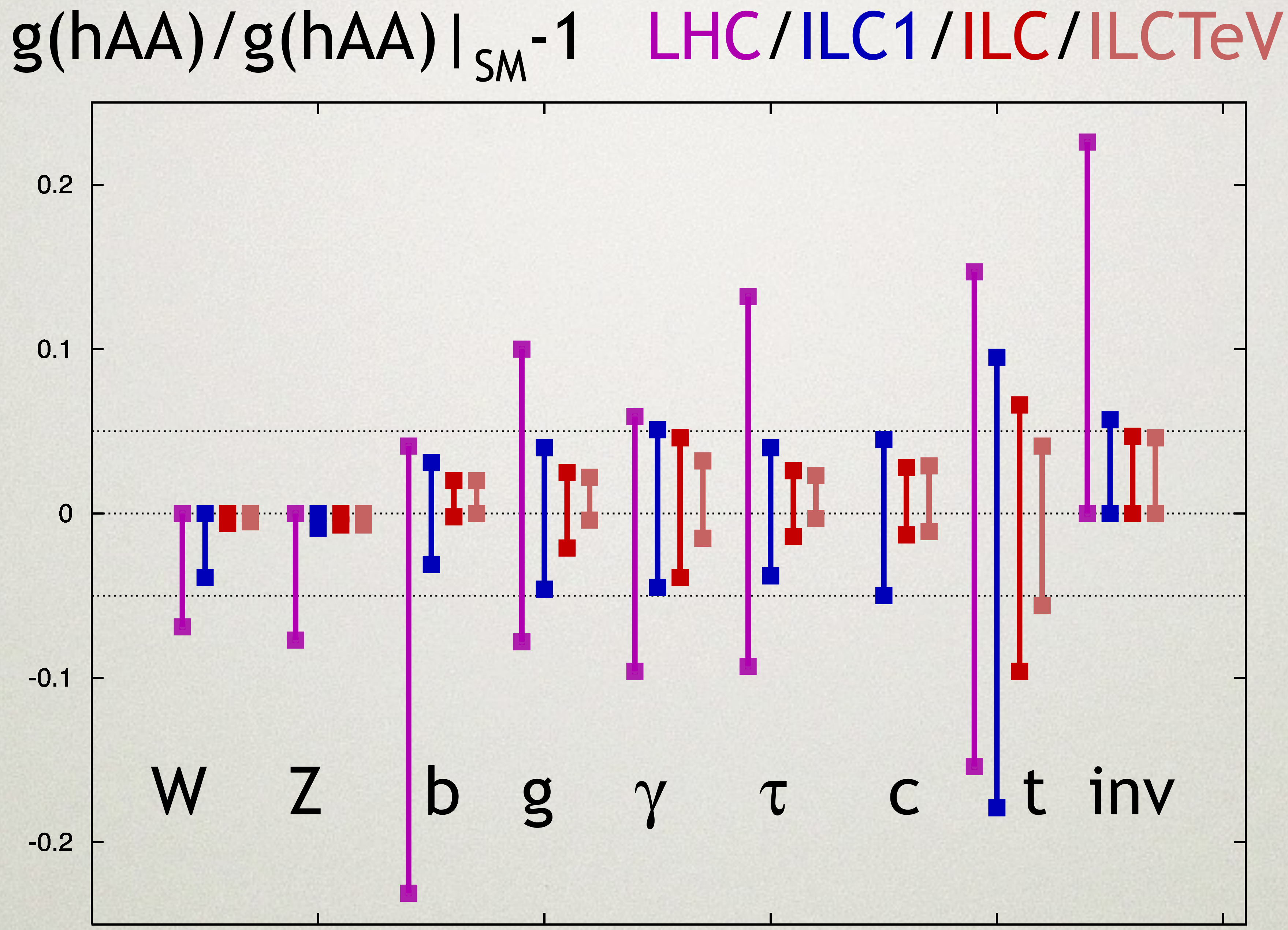
Building the ILC

- There are good reasons to do better even in channels that the LHC measures to ~10% precision
- A number of NP scenarios with a light Higgs and other particles (heavier than a TeV) can cause deviations smaller than that in one or more of the Higgs couplings (decoupling limit)
- In the absence of any other particles being discovered at the LHC, measuring the Higgs couplings more precisely is crucial
- Measuring these couplings more precisely is one of the main physics reasons to build a linear collider like the proposed ILC

Advantages

- e+ e- collisions have much smaller total cross sections (~100 nb as compared to ~100 mb)
- No pile up or hadrons from underlying event
- Z and W bosons are recognized easily even in hadronic decay modes
- Absolute branching ratios of the Higgs can be measured as the Higgs can be tagged when it recoils against the Z boson in $e^+e^- \to Zh$ at 250 GeV
- Combined with $\sigma(e^+e^- \to \nu\bar{\nu}h \to b\bar{b})$ at 500 GeV gives the Higgs width to 6% accuracy

LHC-ILC comparison



LHC - 14 TeV , 300 inv. fb

ILC1 - 250 GeV, 250 inv. fb

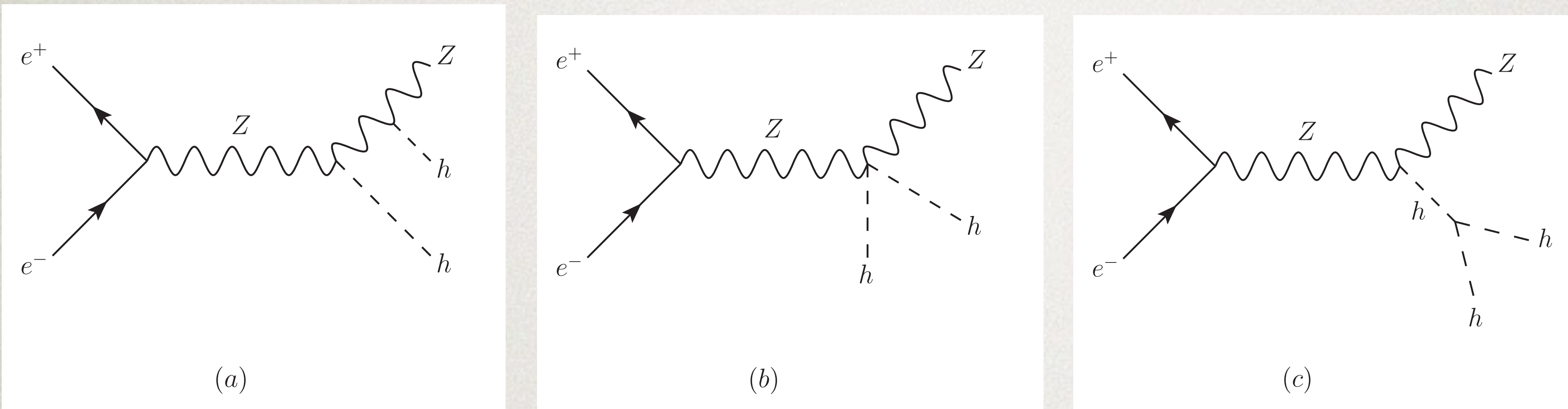
ILC - 500 GeV, 500 inv. fb

ILCTeV - 1 TeV, 1000 inv. fb

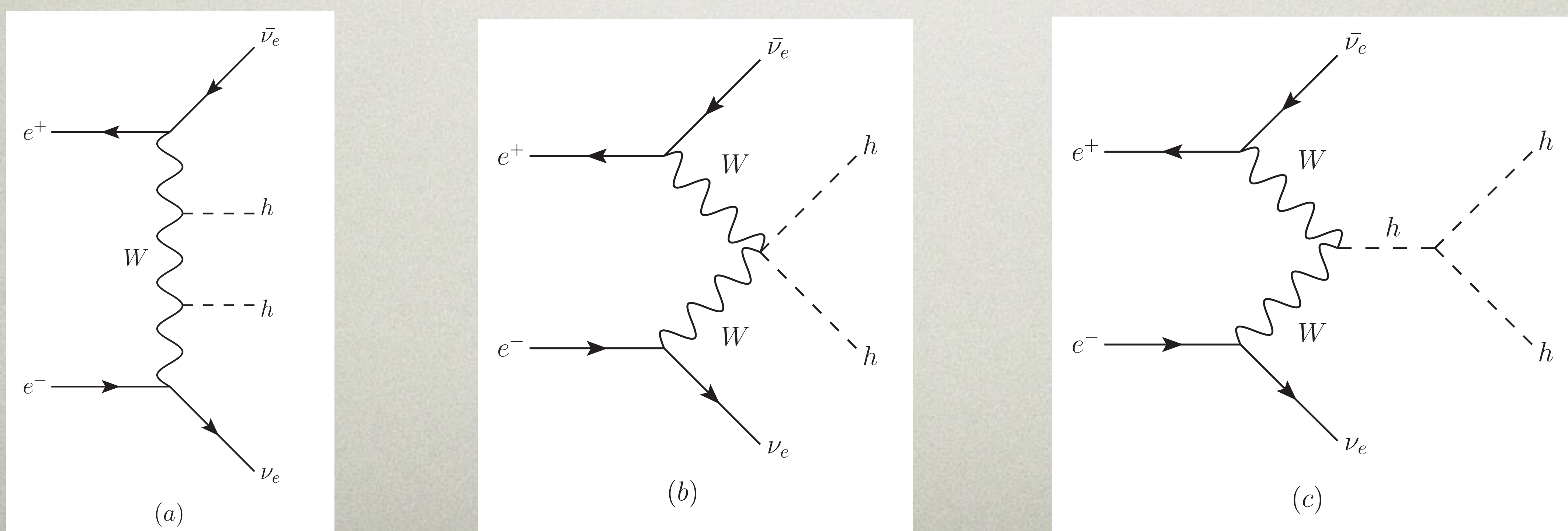
M. Peskin, arXiv : 1208.5152

Double Higgs production at ILC

$$e^+e^- \to Zhh$$



$$e^+e^- \to \nu\bar{\nu}hh$$

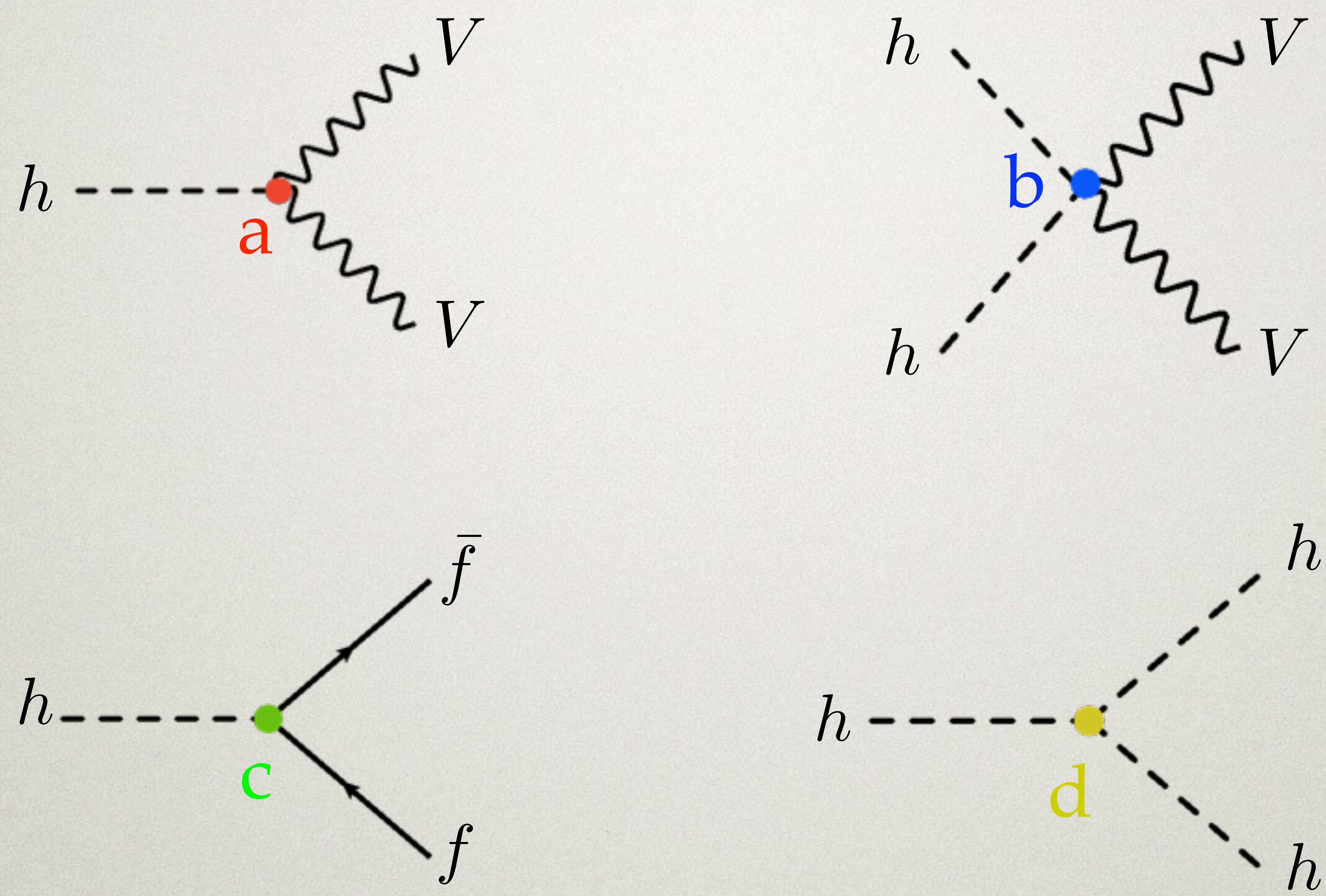


PARAMETRIZATION OF COUPLINGS

$$\mathcal{L} \supset M_V^2 V_\mu^* V^\mu \left[1 + a_V \frac{2h}{v} + b_V \frac{h^2}{v^2}\right] - m_f \bar{f} f \left[1 + c_f \frac{h}{v}\right]$$
$$-\frac{1}{2} M_h^2 h^2 \left[1 + d_3 \frac{h}{v} + d_4 \frac{h^2}{4v^2}\right]$$

- In the SM, $a_i, b_i, c_i, d_i = 1$
- c_f can be different for each fermion
- $(a_W, b_W) \neq (a_Z, b_Z)$ in models where custodial SU(2) is violated

Nomenclature



a,b,c,d are multiplicative factors by which the SM couplings are modified and V denotes the W or Z boson

COUPLING MODIFICATIONS

$$h = \phi\cos\theta - \chi\sin\theta.$$

- χ - real neutral component of general electroweak multiplet X
- doesn't couple to fermions and since it doesn't acquire a vev it doesn't couple singly to VV.

$$a = \cos\theta, \qquad c = \cos\theta$$

- $hhVV$ couplings are modified by $\chi\chi VV$ couplings

$$\chi\chi W_\mu^+ W_\nu^- : ig^2\left[T(T+1) - \frac{Y^2}{4}\right] g_{\mu\nu}$$

$$\chi\chi Z_\mu Z_\nu : i\frac{g^2}{2c_W^2} Y^2 g_{\mu\nu}$$

COUPLING MODIFICATIONS

- After mixing,

$$b_W = \cos^2\theta + 2\left[T(T+1) - \frac{Y^2}{4}\right]\sin^2\theta,$$
$$b_Z = \cos^2\theta + Y^2\sin^2\theta.$$

- Models that preserve a custodial SU(2) have $b_W = b_Z$
- $b_V > 1$ is possible when T > 1/2

BENCHMARK MODELS

- I : SM + Real Singlet Scalar

$$h = \phi \cos\theta - s \sin\theta$$

$$b_W = b_Z = \cos^2\theta = a^2$$

- II : SM + Inert Doublet

Consider a Type 1 2HDM where the doublet Φ_2 has small vev (fine-tuned)

$$h = \phi_1 \cos\theta - \phi_2 \sin\theta$$

$$b_W = b_Z = \cos^2\theta + \sin^2\theta = 1$$

BENCHMARK MODELS

- III : Georgi-Machacek model
- Contains the SM doublet along with a complex triplet (Y=2) and a real triplet (Y=0)
- Note that each of those triplets taken individually would violate custodial SU(2)
- Together they can be arranged so as to preserve it

$$\Phi = \begin{pmatrix} \phi^{0*} & \phi^{+} \\ -\phi^{+*} & \phi^{0} \end{pmatrix} \qquad \chi = \begin{pmatrix} \chi^{0*} & \xi^{+} & \chi^{++} \\ -\chi^{+*} & \xi^{0} & \chi^{+} \\ \chi^{++*} & \xi^{-} & \chi^{0} \end{pmatrix}$$

BENCHMARK MODELS

- The model contains two custodial SU(2) singlets that can mix to produce the observed resonance

$$H_1^0 = \phi \qquad\qquad H_1^{0\prime} = \sqrt{\frac{2}{3}}\chi^{0,r} + \frac{1}{\sqrt{3}}\xi^0$$

$$h = H_1^0 \cos\theta - H_1^{0\prime}\sin\theta$$

- The vevs of the triplets can be chosen to be ~ zero resulting in the following coupling modifications

$$b_W = b_Z = \cos^2\theta + \frac{8}{3}\sin^2\theta = a^2 + \frac{8}{3}(1 - a^2)$$

Measuring the $hhVV$ Coupling

- Scenario : LHC + 250 GeV ILC data point to the Higgs mixing with another scalar that doesn't couple singly to gauge bosons or fermions

$$a = c = 0.9$$

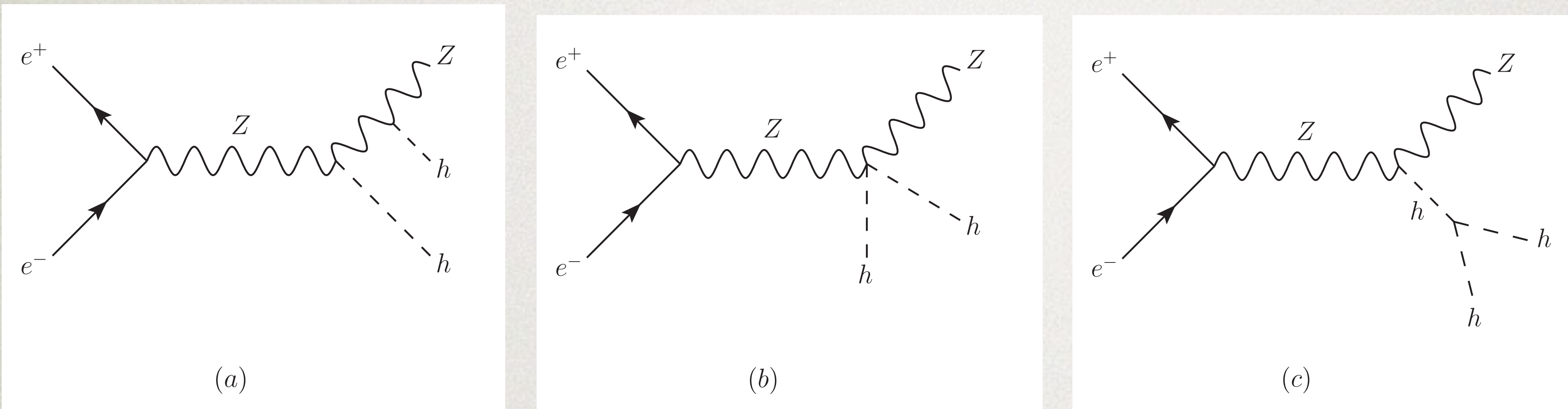
- The production rate would be scaled by a factor of 0.81 but all the BRs would stay the same
- The 250 GeV ILC measurement of $e^+e^- \to Zh$

  would yield *a* to a precision of $\Delta a/a = 1.3\%$

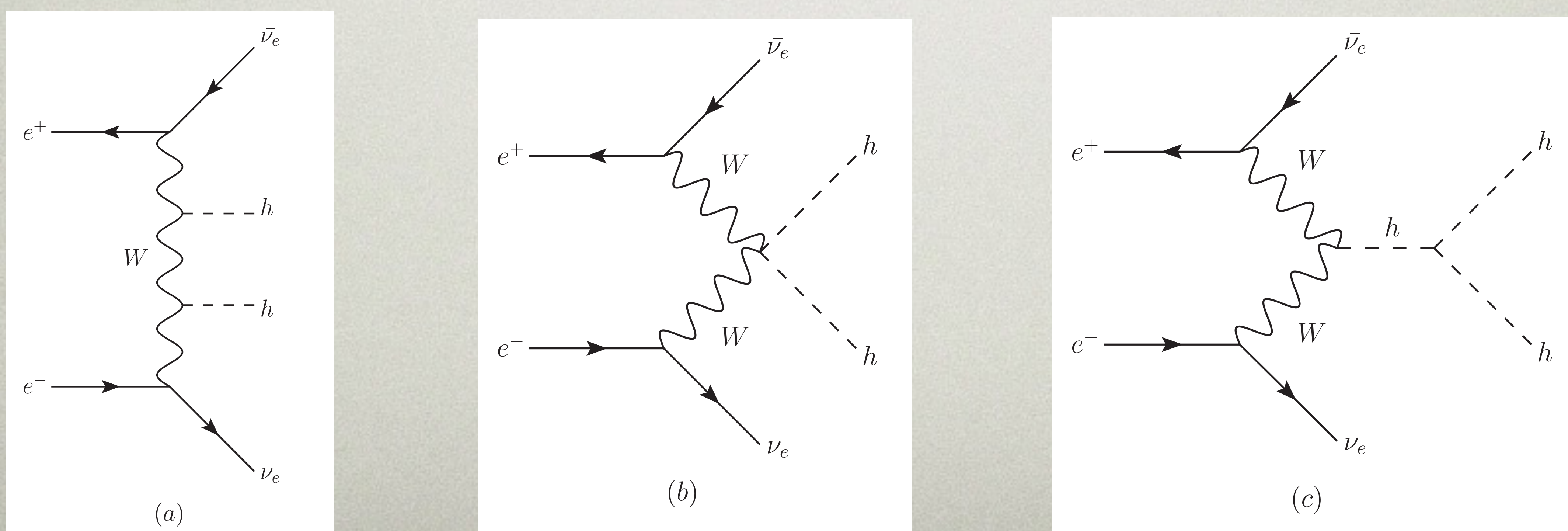
  with 250 inv. fb of data (~ 3 yrs)

Double Higgs production at ILC

$$e^+e^- \to Zhh$$



$$e^+e^- \to \nu\bar{\nu}hh$$



Double Higgs production at ILC

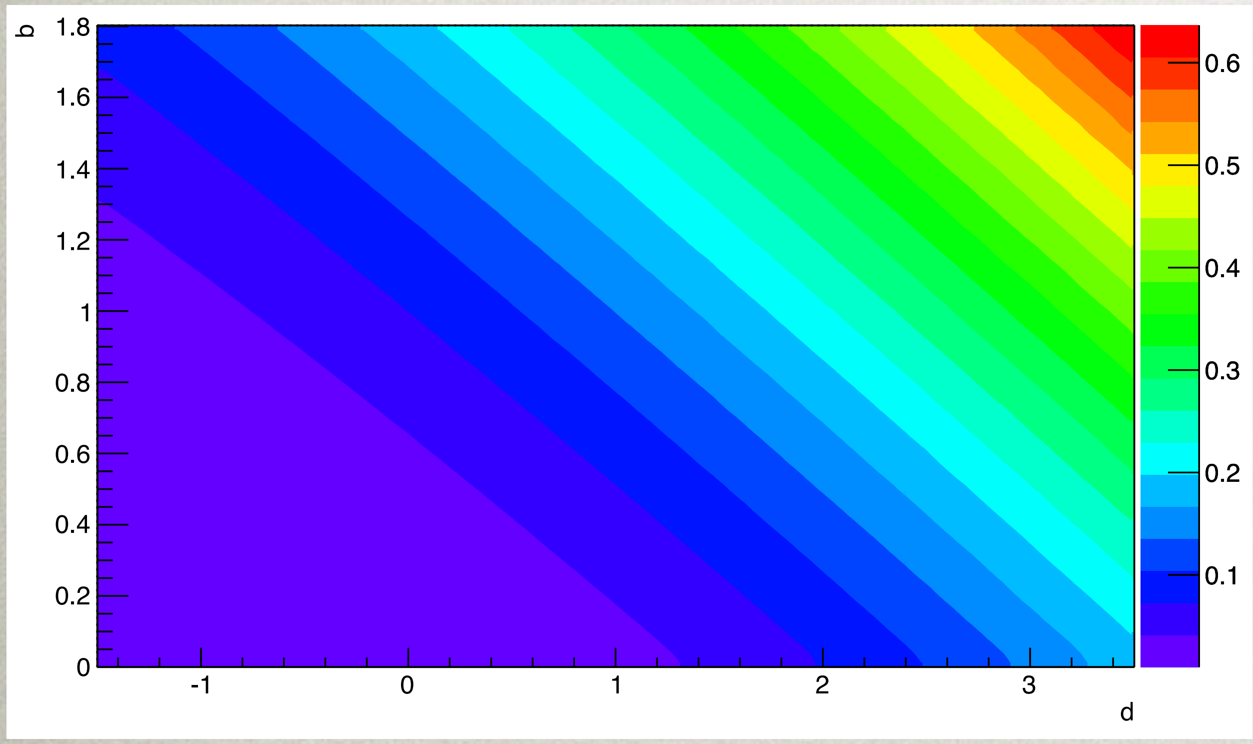
- We Z boson fusion because the cross section is much smaller than WBF
- We calculate the cross sections using CalcHEP and MG5 for di-higgs production for the SM and the three benchmark models assuming a = c = 0.9 and d = 1

Model	b	$\sigma^{500}(Zhh)$	$\sigma^{1000}(Zhh)$	$\sigma^{1000}(\mathrm{WBF})$
Singlet	0.81	0.109 fb	0.0815 fb	0.0411 fb
Doublet	1	0.136 fb	0.113 fb	0.0273 fb
GM	1.32	0.188 fb	0.183 fb	0.0901 fb
SM	1	0.157 fb	0.119 fb	0.0712 fb

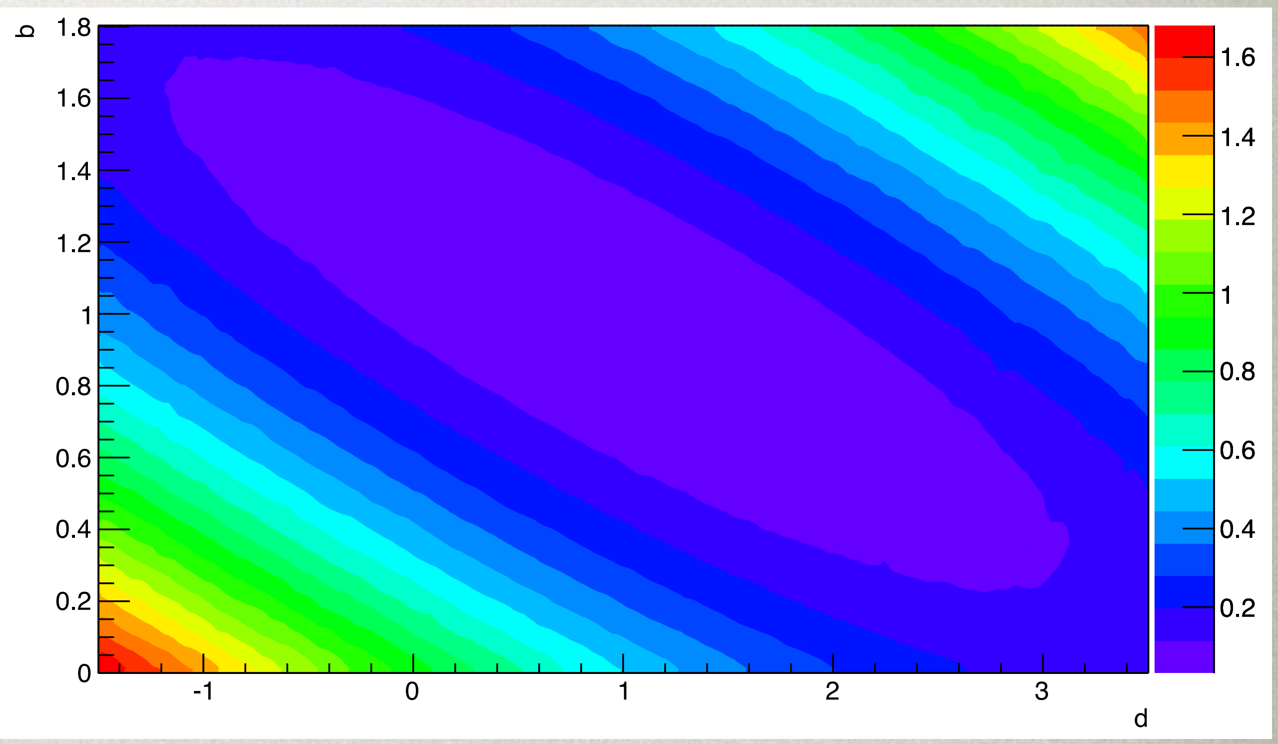
Extracting b and d

- The cross sections for the two processes depend differently on b and d
- This dependence also varies with the CoM energy

$e^+e^- \to Zhh$ (500 GeV)



$e^+e^- \to \nu\bar{\nu}hh$ (1 TeV)



Contour Plots of Cross sections (in fb)

Extracting b and d

- Measurements of $e^+e^- \to Zhh$ at 500 GeV and $e^+e^- \to \nu\bar{\nu}hh$ at 1 TeV can be used to fit for b and d
- We can compute the two cross sections in terms of our effective lagrangian for $a = 0.9$ while varying b and d
- Next we can plot 68% and 95% CL chi sq plots for each Benchmark Model

$$\chi^2(b,d) = \sum_{i=1,2} \frac{(\sigma_i(b,d) - \sigma_{BM,i})^2}{\Delta\sigma^2_{BM,i}}$$

Extracting b and d

$$\chi^2(b,d) = \sum_{i=1,2} \frac{(\sigma_i(b,d) - \sigma_{BM,i})^2}{\Delta\sigma^2_{BM,i}}$$

- We use cross section uncertainties from the ILC Large Detector Study for the ILC Detailed Baseline Design (DBD) Report
- These uncertainties are scaled appropriately for the Benchmark Models as the cross section

Measuring *b* and *d*

- Account for different selection efficiencies for (Z--> v v) hh and WBF at 1 TeV by scaling the Zhh process to get the relative efficiency of 11%

Model	b	$\Delta\sigma/\sigma(Zhh, 500\text{ GeV})$	$\Delta\sigma/\sigma(\nu\nu hh, 1\text{ TeV})$
Singlet	0.81	38%	32%
Doublet	1	32%	42%
GM	1.32	24%	18%
SM	1	27%	23%

- Beam Polarisation : P(e-,e+) : (-0.8, +0.3) at 500 GeV

  and (-0.8,0.2) at 1 TeV , Int. Lum. = 2000 inv. fb

- Change in relative contribution due to kinematic cuts is beyond the scope of this work

Measuring b and d

- Relative uncertainty increases for Singlet and Doublet Benchmark model and decreases for GM as one would expect from the table of cross sections

Model	b	$\Delta\sigma/\sigma(Zhh, 500\ \mathrm{GeV})$	$\Delta\sigma/\sigma(\nu\nu hh, 1\ \mathrm{TeV})$
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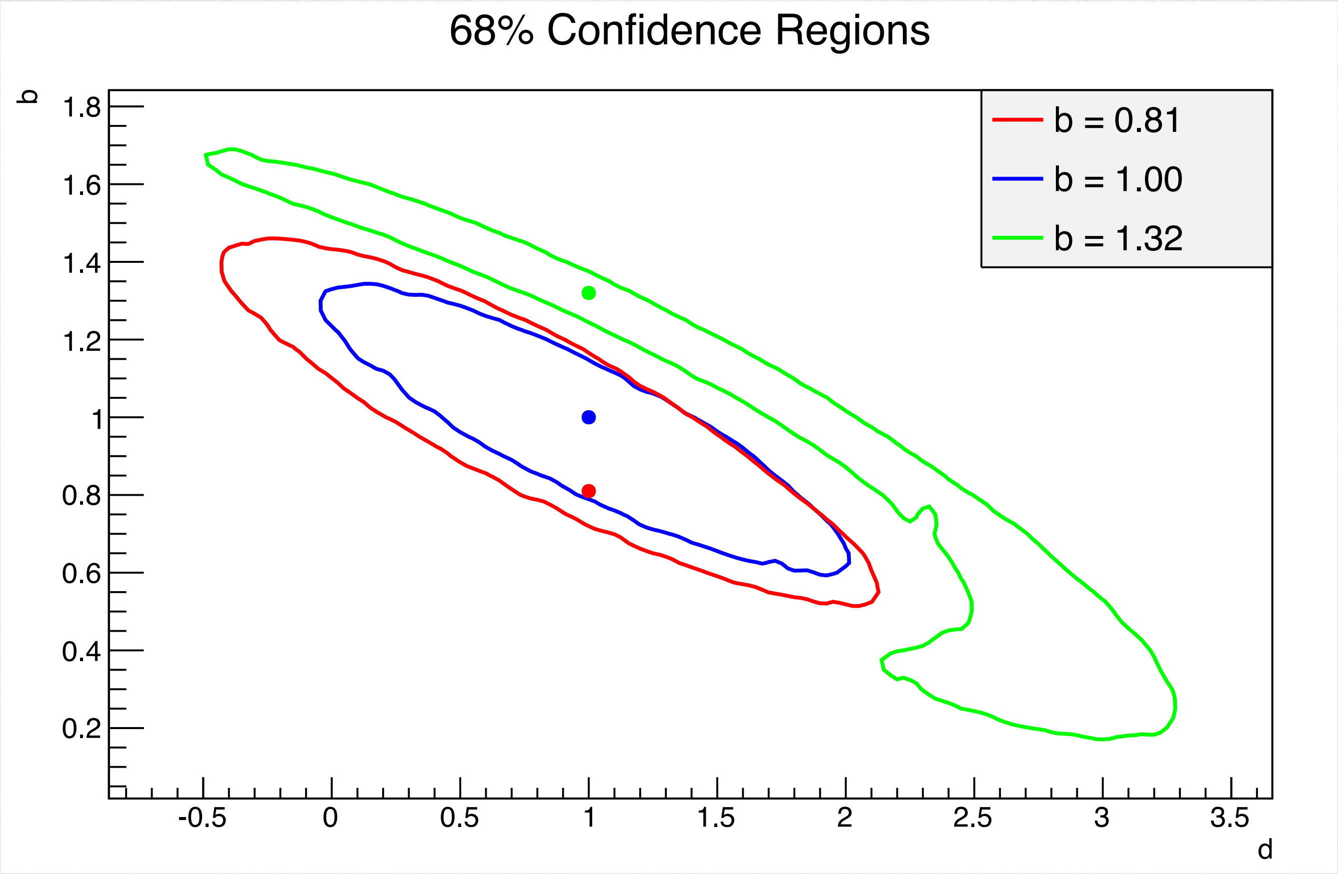
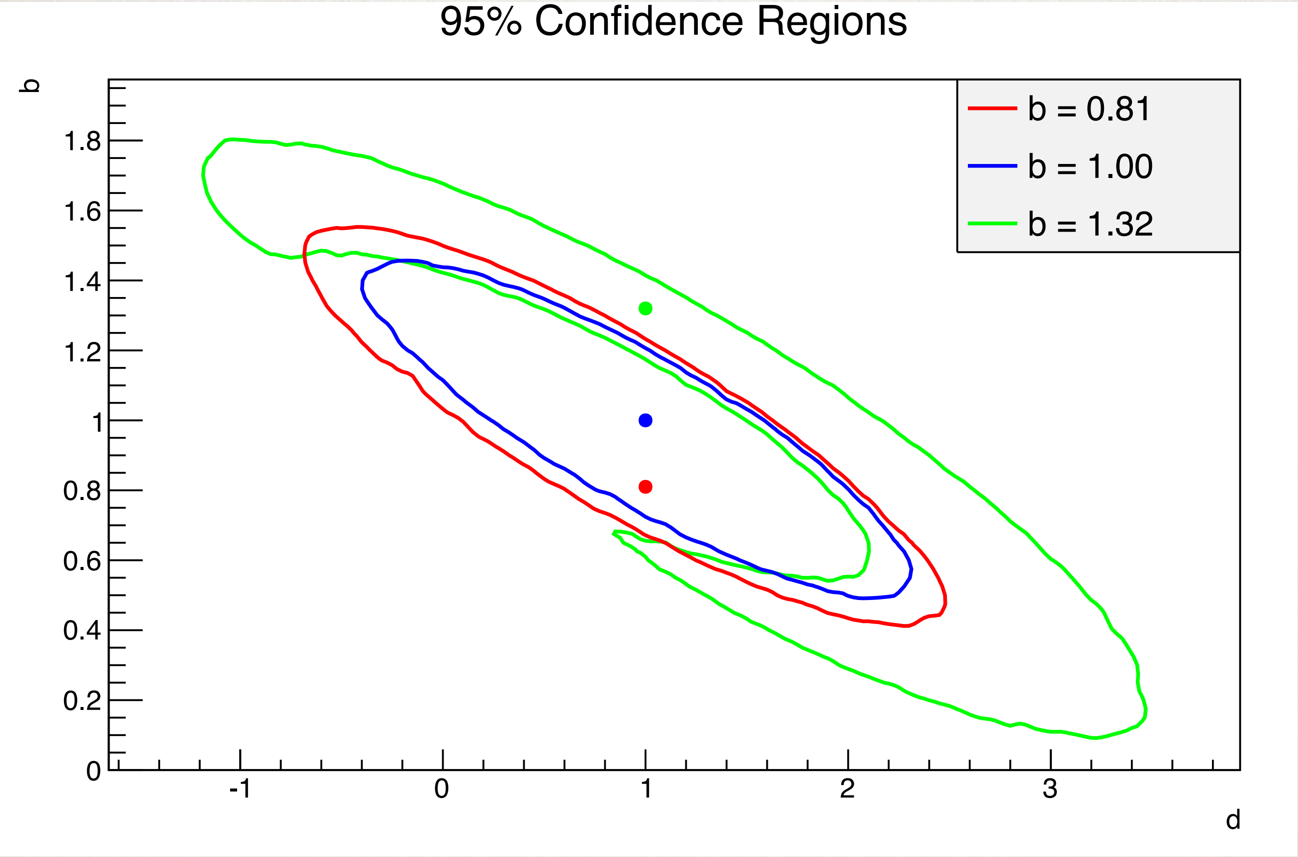
Fit Results

- GM model can be distinguished from singlet and doublet benchmarks at 68% CL (chi sq. = 2.28)



Fit Results

- Overlap at 98% CL is minimal (chi sq. = 5.99)

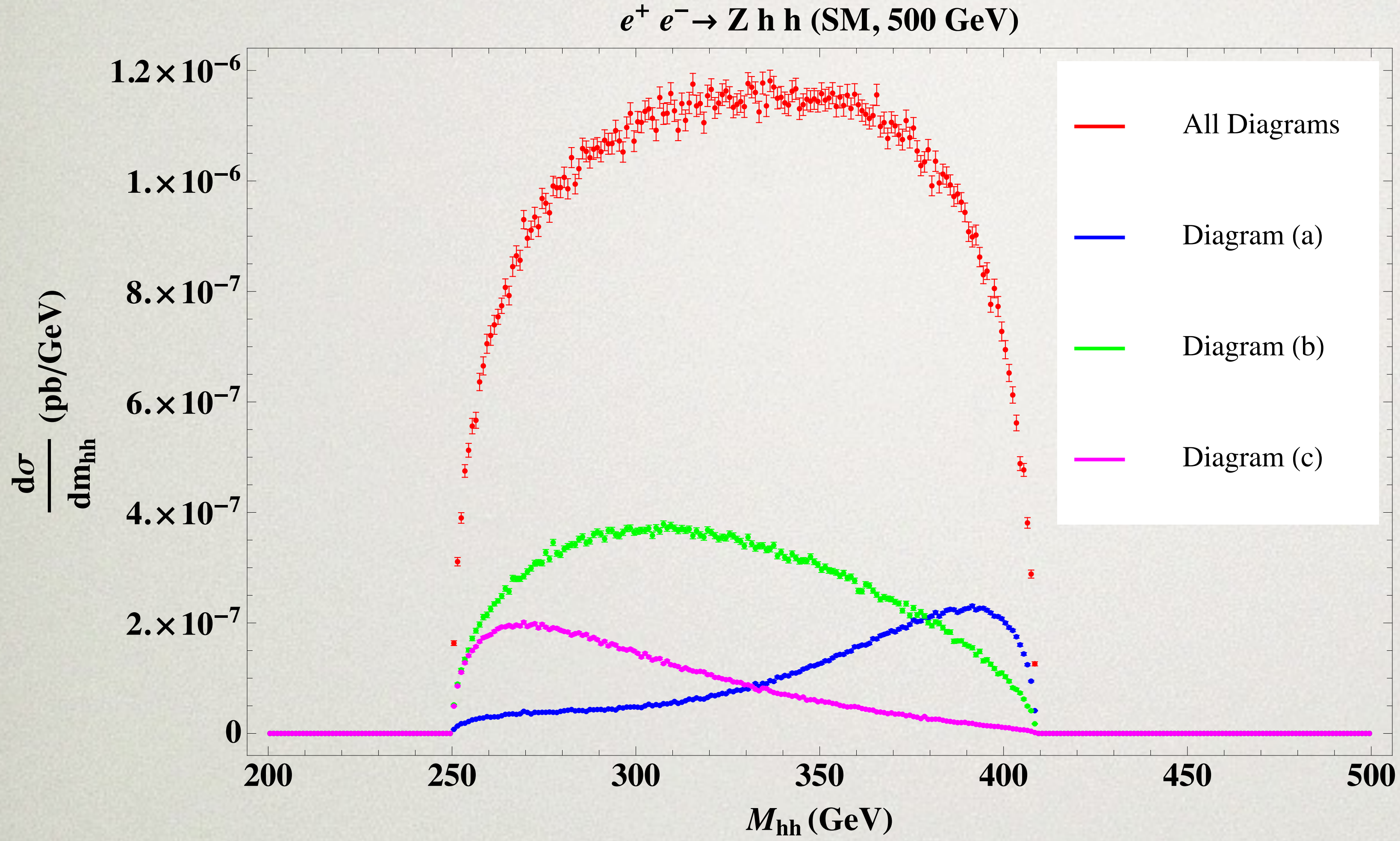


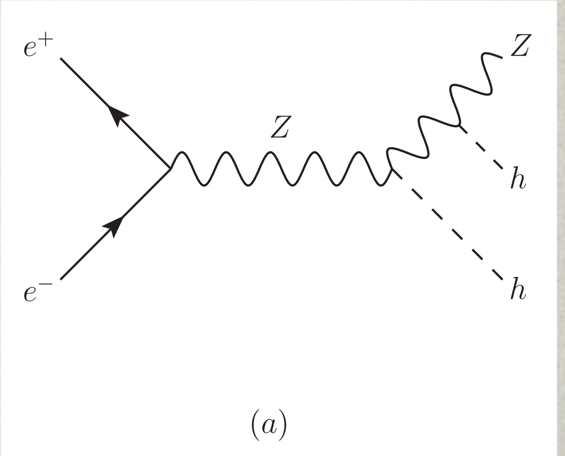
- Crescent shape due to WBF cross section not being monotonic in *b*

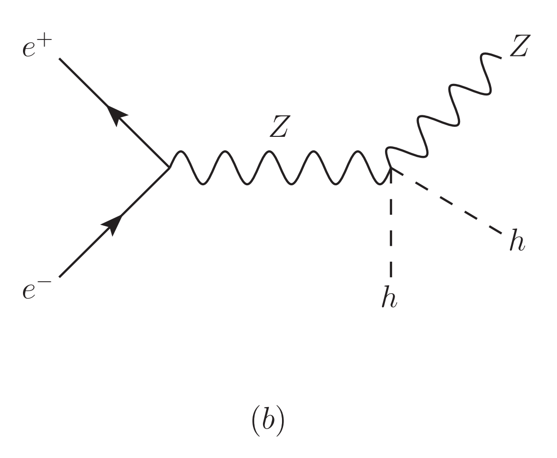
Measuring *b* and *d*

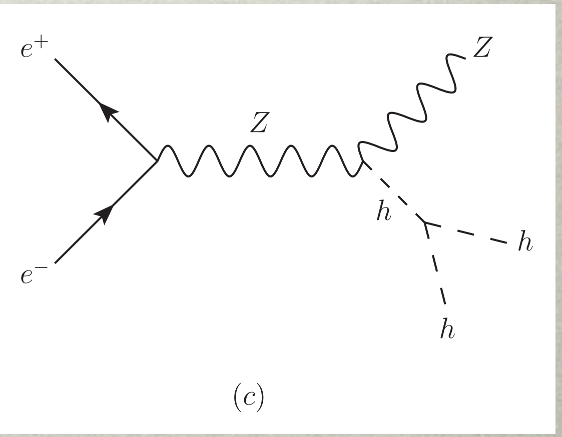
- The DBD report assumed a Higgs mass of 120 GeV and considered the channel where higgs decays to bottom quark pairs
- At 125 GeV this would reduce the cross sections by about 20%
- The lost precision can be regained by including the h h --> W+W- b b

M_{hh} as Kinematic Discriminant





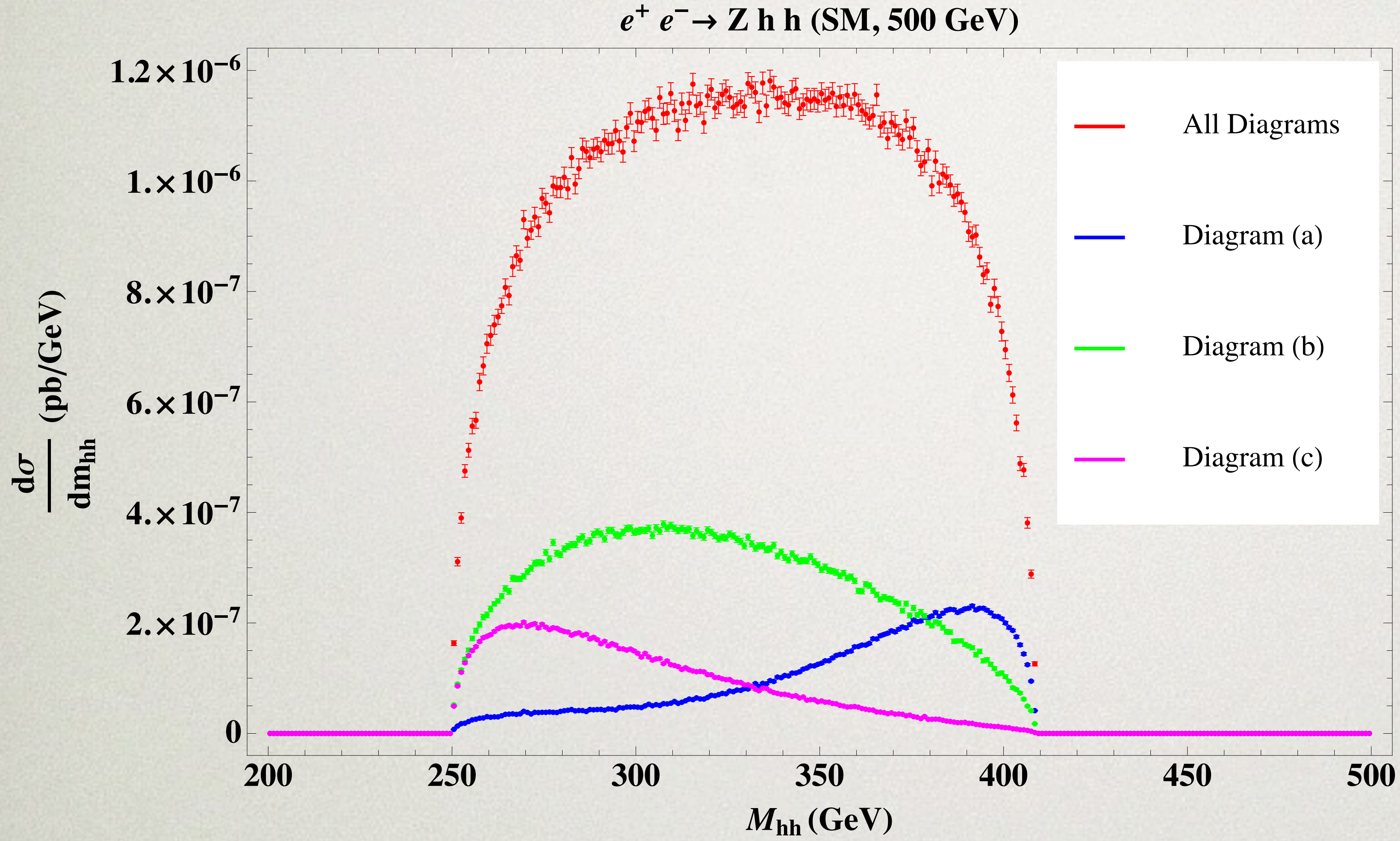


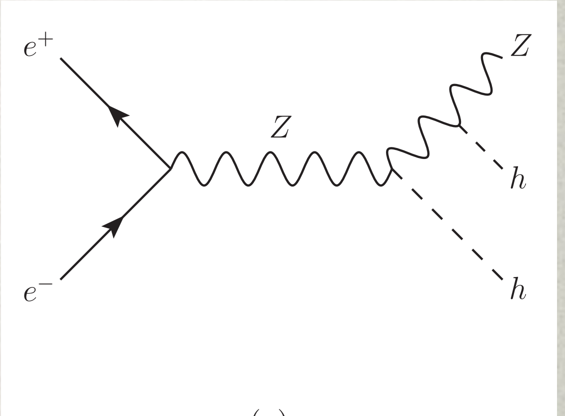


- ILD collaboration is exploring improving sensitivity to d by weighting events based on M_{hh}

J.Tian, talk at LCWS2012, http://www.uta.edu/physics/lcws12/

M_{hh} as Kinematic Discriminant





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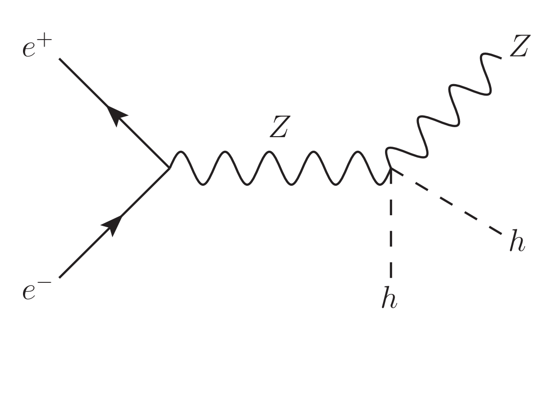

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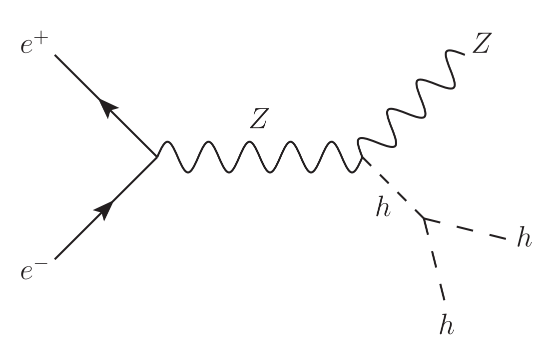
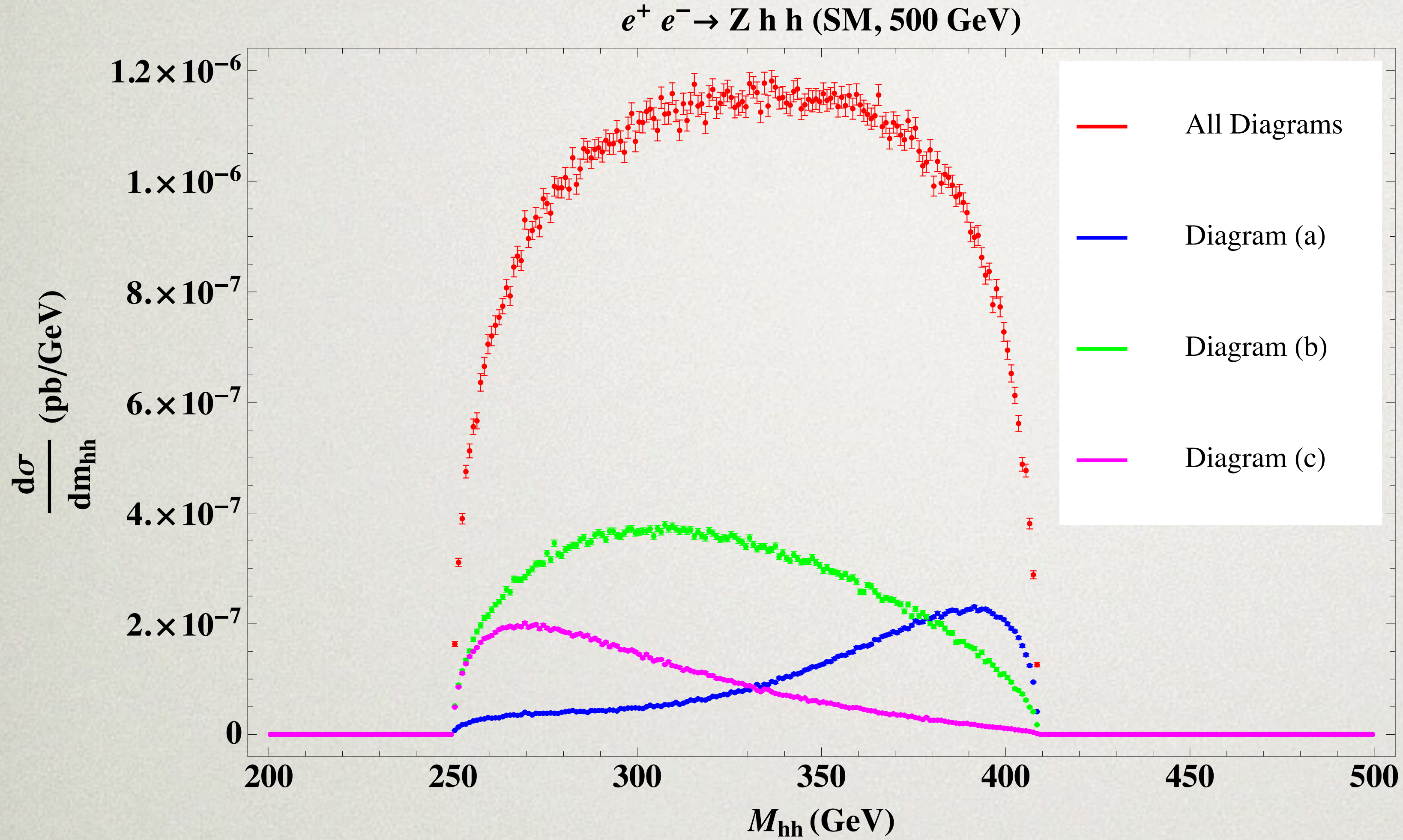


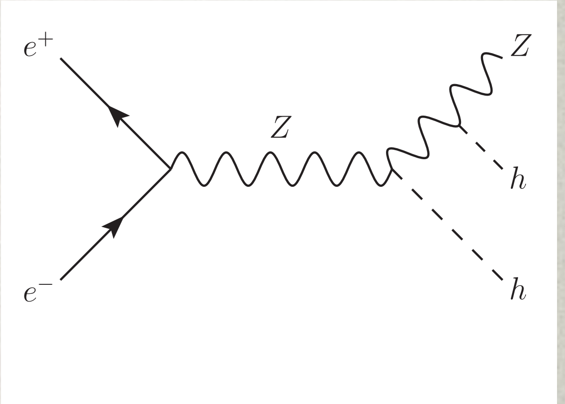
(c)

- Improves precision on d by 10% at 500 GeV and 1 TeV for the SM case ($a = b = 1$)

K. Fujii, Talk at Higgs Snowmass Workshop 2013, http://physics.princeton.edu/snowmass

M_{hh} as Kinematic Discriminant





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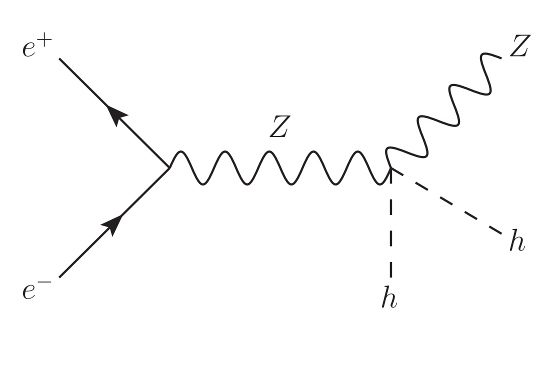

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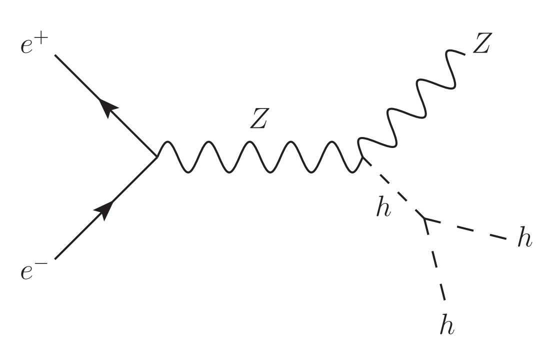
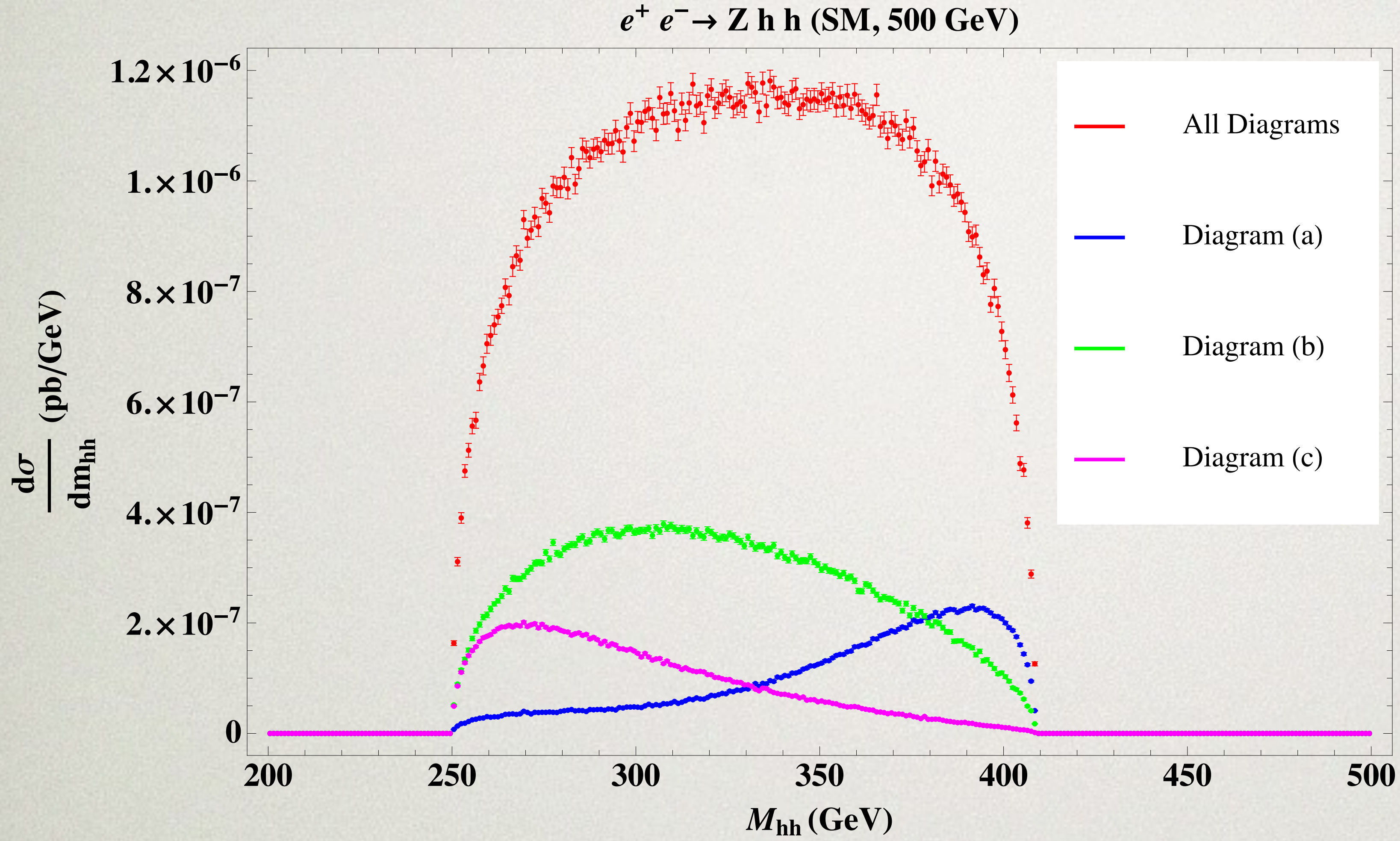
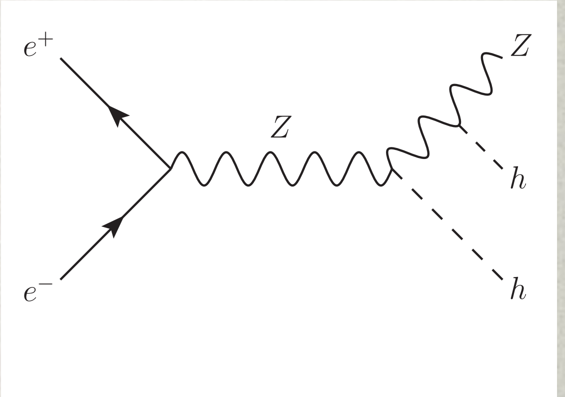


(c)

- Method can be adapted to improve extraction of b and d
- Significant contribution from interference
- Contribution from (c) is higher at lower M_{hh} and from (b) has a broader M_{hh} dependence

M_{hh} as Kinematic Discriminant





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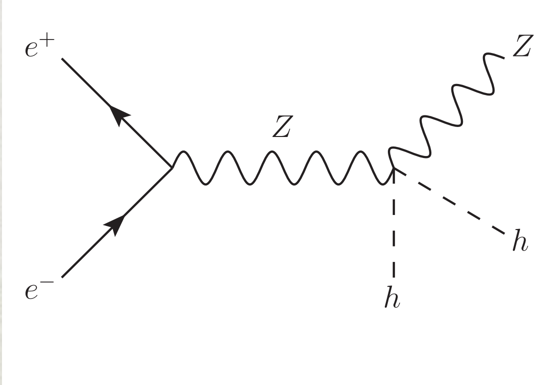

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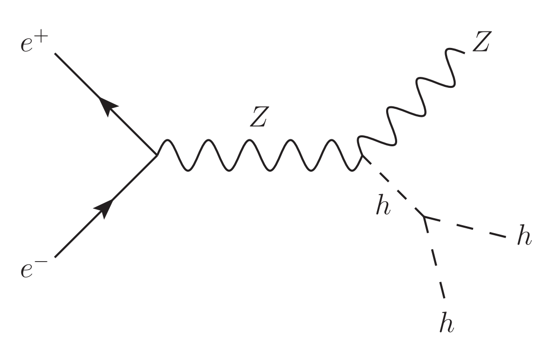
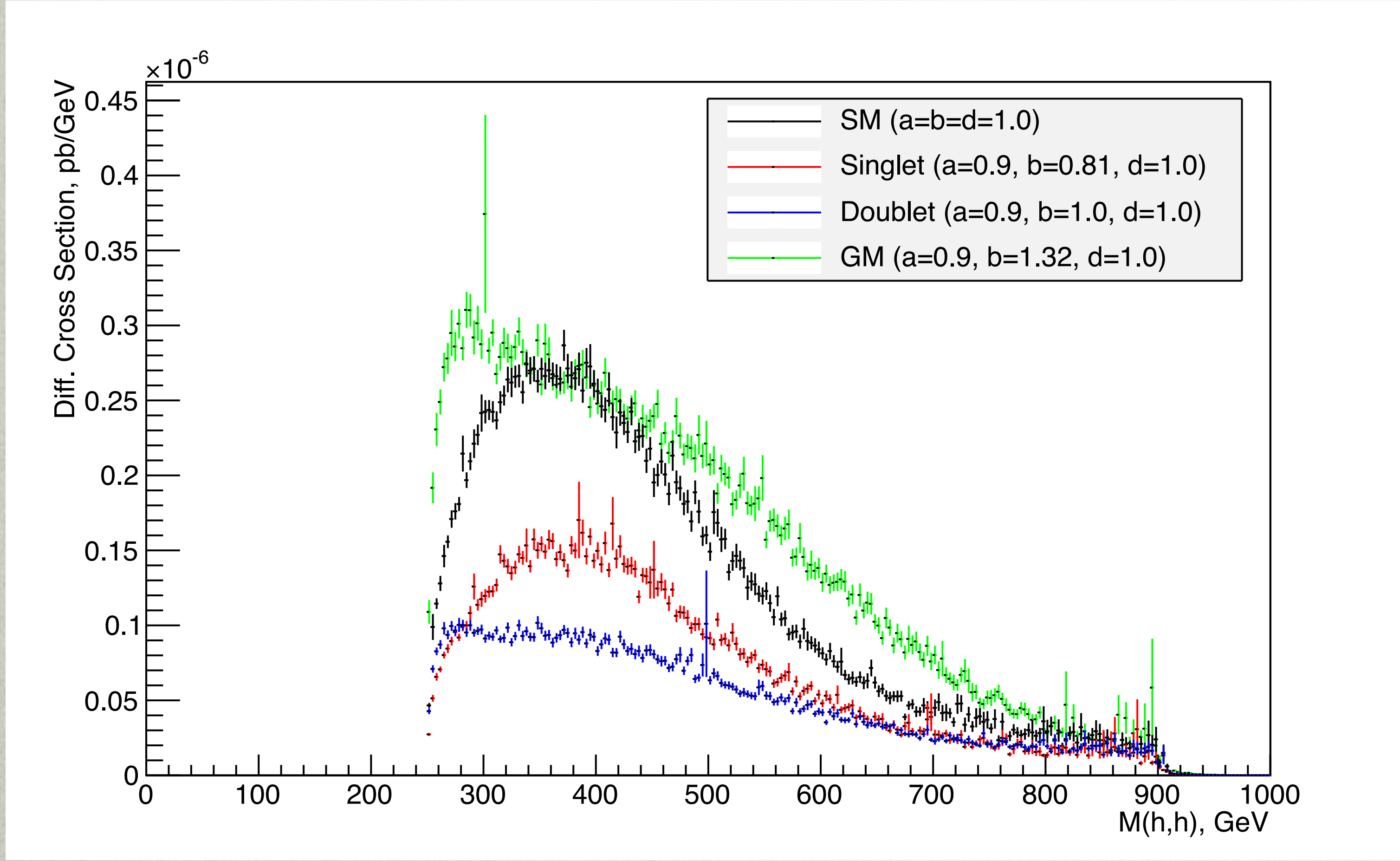
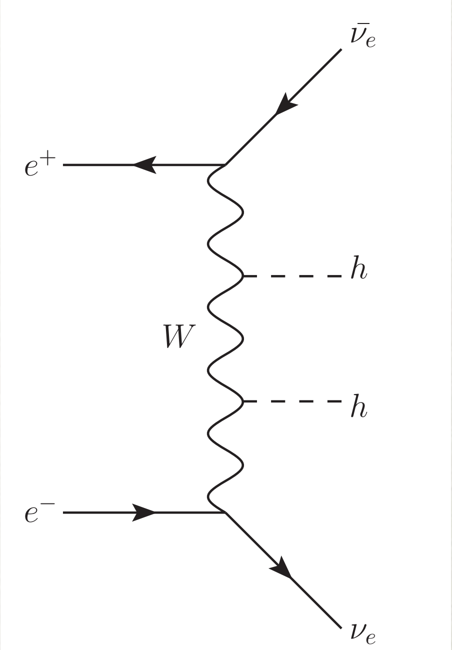


(c)

- Contribution from d is highest at low M_{hh} and from b has a broader M_{hh} dependence

WBF at 1 TeV





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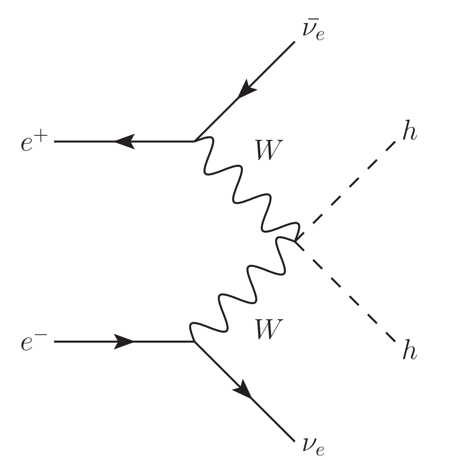

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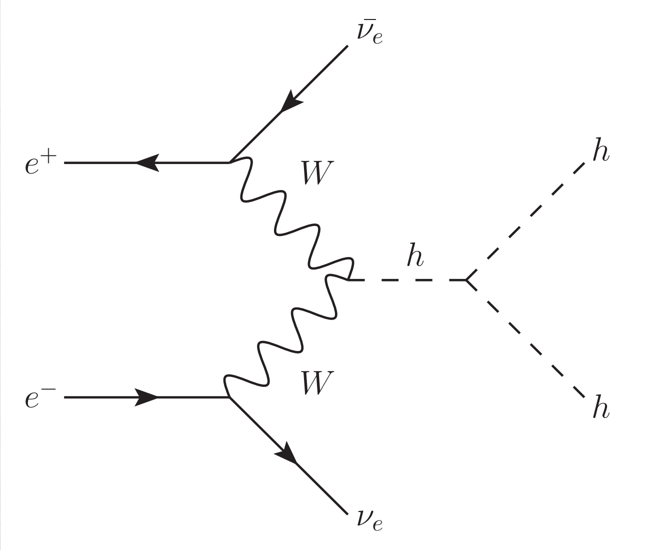
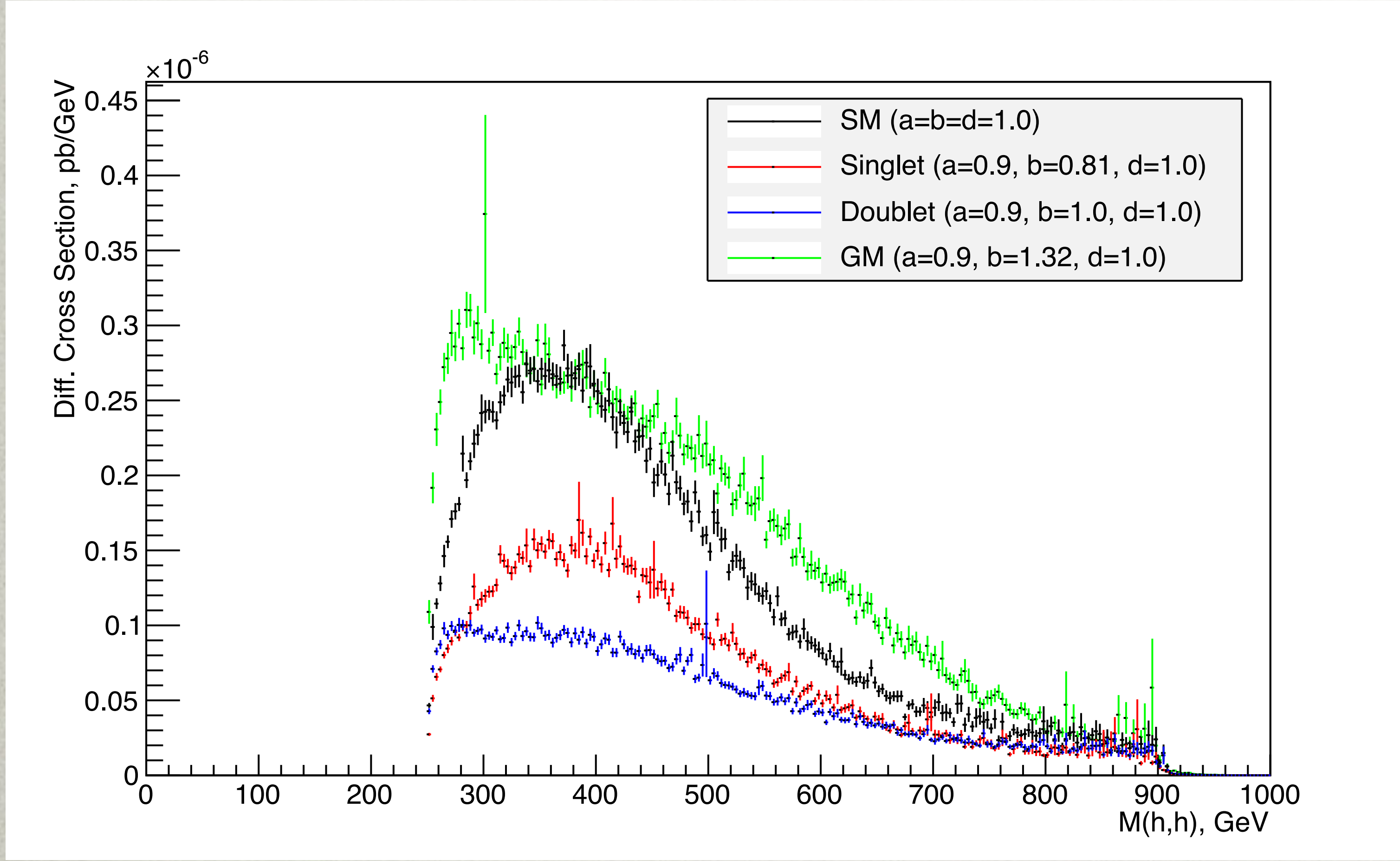
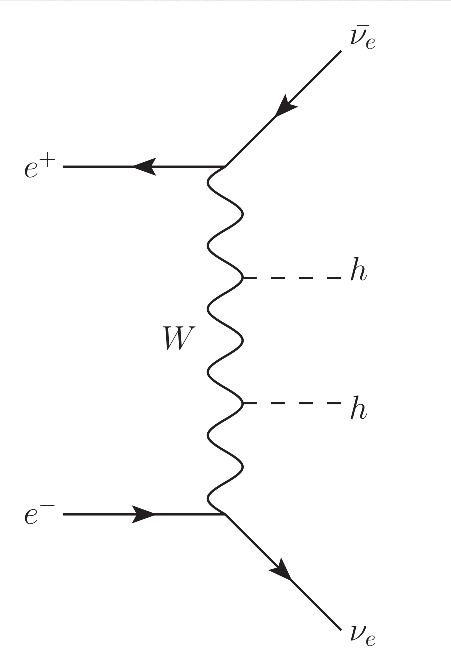


(c)

- Benchmark Models differ just in value of *b*
- Dig. (b) and (c) interfere constructively leading to enhancement at lower M_{hh} for larger *b* values (GM or Doublet model vs Singlet)

WBF at 1 TeV





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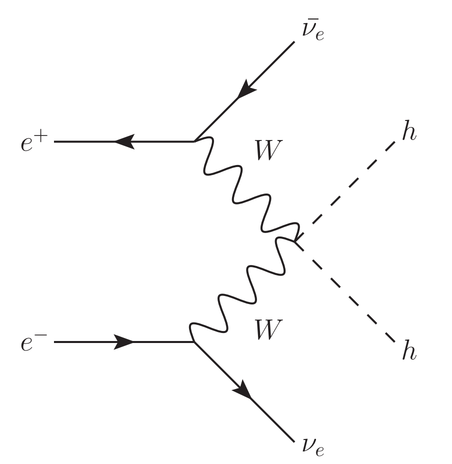

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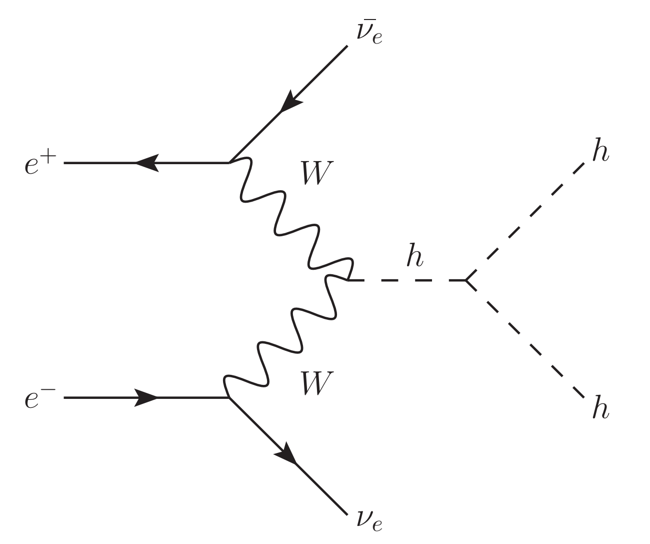
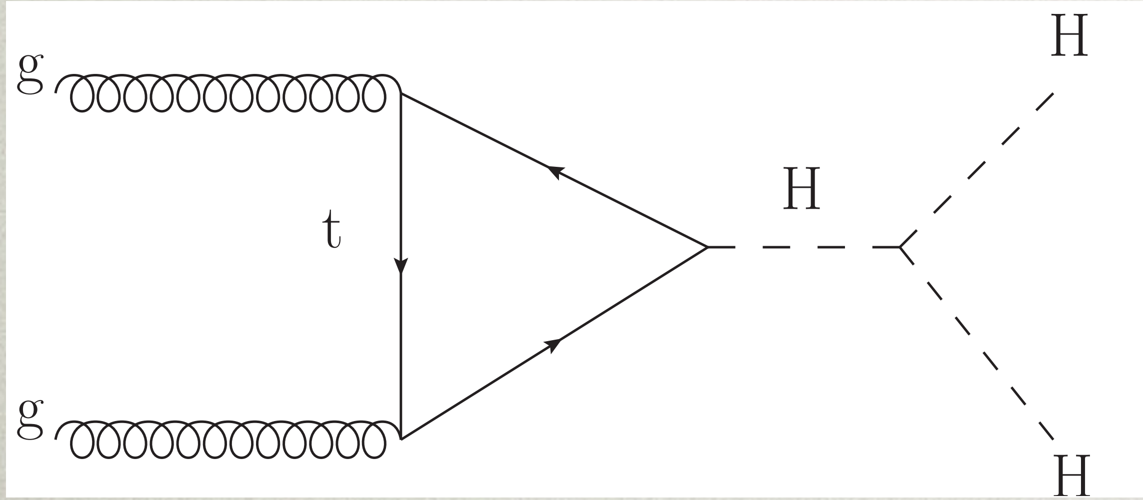


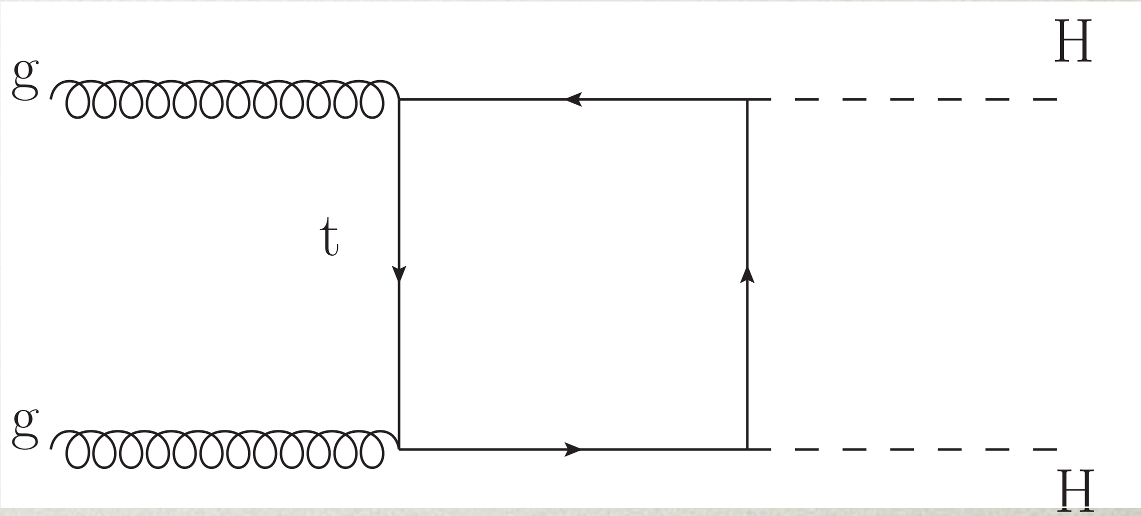
(c)

- Dig. (a) and (b) interfere destructively
- Leads to a flatter spectrum at intermediate M_{hh} for large b values (Doublet vs Singlet model)

Constraints on *d* from LHC

- Accessed via di-Higgs production through gluon fusion
- Depends on top Yukawa coupling as well as new particles in the gluon-fusion loop





figs. from F. Goertz, A. Papaefstathiou, L.L.Yang, J. Zurita [arXiv : 1301.3492]

Constraints on *d* from LHC

- di-Higgs production at LHC is not very sensitive to *b* (the hhVV coupling modification)
- *d* can be constrained to be +ve at 96% CL using 600 inv. fb at 14 TeV LHC F. Goertz et al. [arXiv : 1301.3492]
- With 3000 inv. fb the 1 sigma uncertainty is reduced to +30% and -20%
- The study assumed $c = 1$ and no new particles in the loop
- A joint analysis from LHC and ILC data can thus be used to constrain *b*, *d* and new colored particles

Viability of Benchmark Models

Double Higgs production at ILC

- The approach we used to calculate di-Higgs rates doesn't account for contribution from t- and u-channel exchange of SU(2) triplet states in the doublet and GM model
- Doesn't include H -> h h where H is the heavier custodial singlet
- We assume these states are heavy enough to be kinematically forbidden at the 1 TeV ILC

$$M_H \gtrsim 910 \text{ GeV} \qquad e^+e^- \to Z(H \to hh)$$

$$M_{H^\pm, A^0} \gtrsim 875 \text{ GeV} \qquad e^+e^- \to h(A^0 \to Zh)$$

For the Inert Doublet case including these states increases the di-Higgs cross section by less than 3% if the states are heavier than 1 TeV

Unitarity Constraints on Heavy States

- We cannot assume the heavier states to be arbitrarily heavy
- This is because in the presence of Higgs coupling deviations we need contributions from NP to ensure perturbative unitarity
- $V_L V_L \to hh$
- $V_L V_L \to V_L V_L$

Unitarity limits

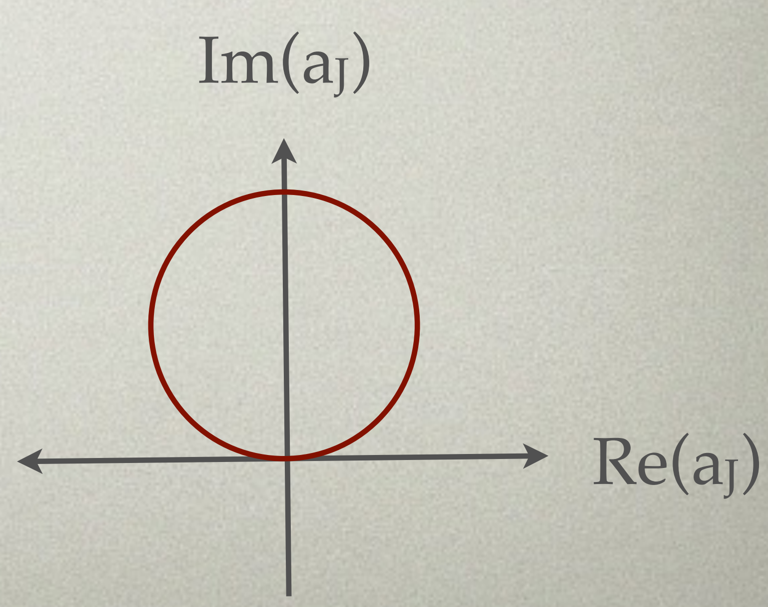
The amplitude for a scattering process can be written in terms of partial wave amplitudes of definite angular momentum

$$\mathcal{M} = 16\pi \sum_J (2J+1) a_J P_J(\cos\theta)$$

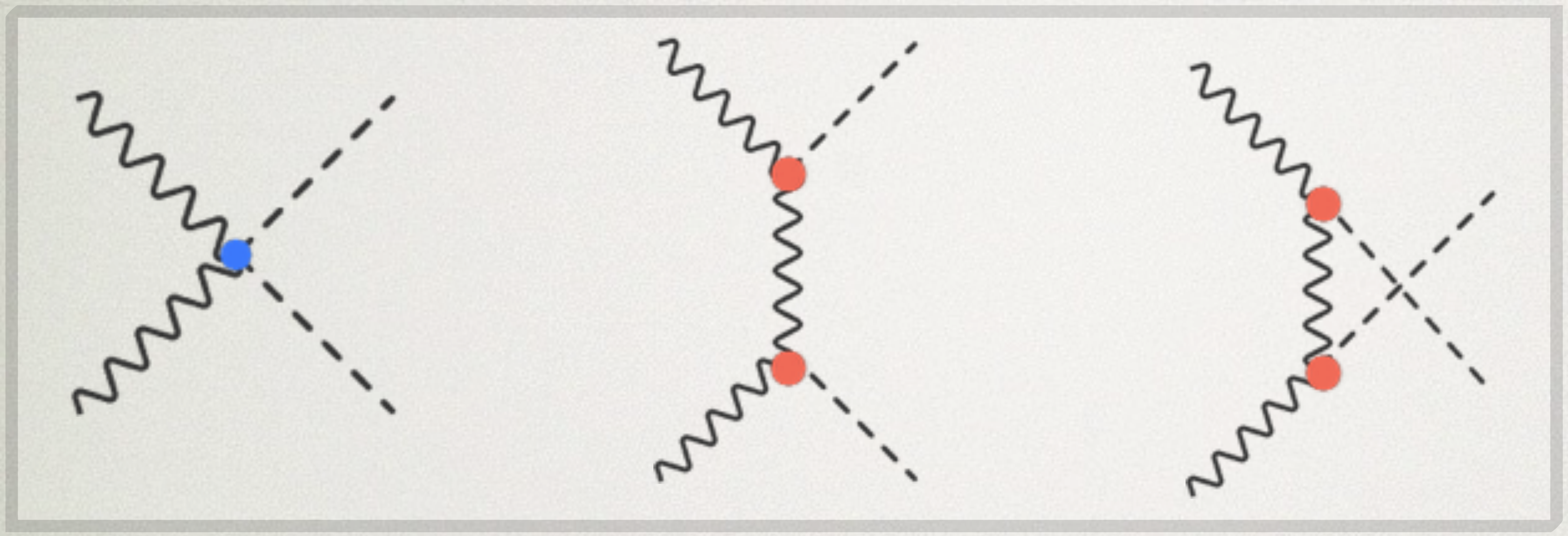
The cross section in each partial wave is limited

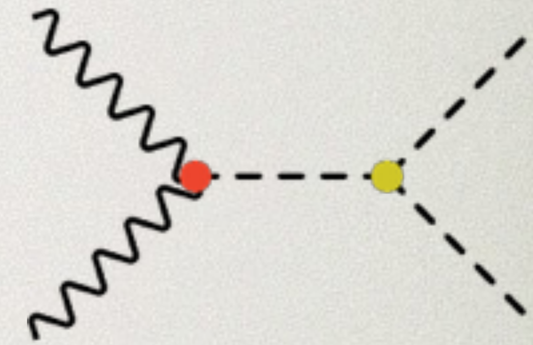
Since this is a result of the unitarity of the S-matrix the bounds on the partial wave amplitudes are called unitarity limits

$$|\mathrm{Re}(a_J)| \leq \frac{1}{2}$$



$V_L V_L \to hh$



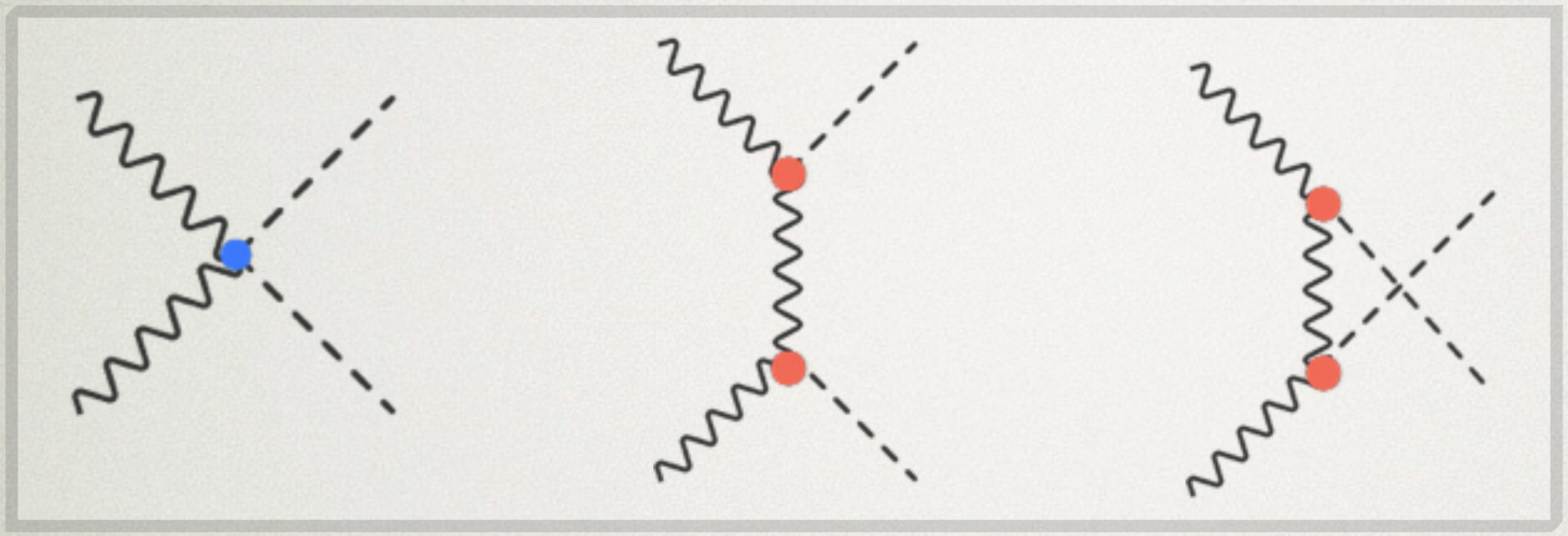


$\mathcal{O}\left(E^0\right)$

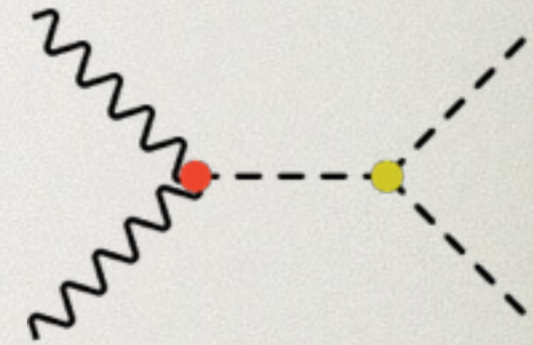
$\mathcal{O}\left(\frac{E^2}{v^2}\right)$

- Clearly the t- and u-channel exchange is required to restore unitarity when $b - a^2 \neq 0$

$V_L V_L \to hh$



$\mathcal{O}\left(\frac{E^2}{v^2}\right)$



$\mathcal{O}\left(E^0\right)$

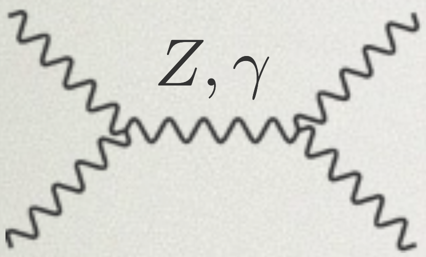
$$\text{Doublet}: \quad m^2_{H^\pm, A^0} \lesssim \frac{8\pi v^2}{\sqrt{3}(1-a^2)} \simeq (2150\ \text{GeV})^2$$

$\boxed{a=0.9}$

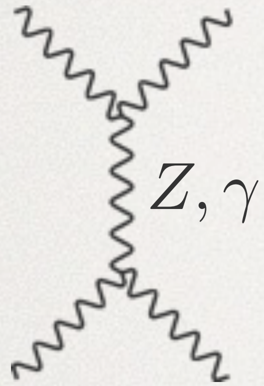
$$\text{GM}: \quad m^2_{H_3^\pm, A_3^0} \lesssim \frac{3\pi v^2}{\sqrt{3}(1-a^2)} \simeq (1320\ \text{GeV})^2$$

Assuming $4\pi v^2 \gg m_h^2, m_W^2$

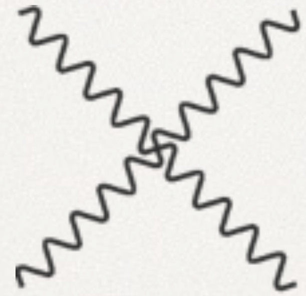
$V_L V_L \to V_L V_L$



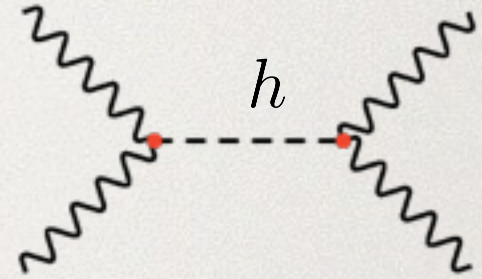
$\mathcal{O}\left(\frac{E^4}{v^4}\right)$



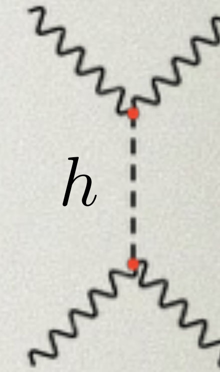
$\mathcal{O}\left(\frac{E^4}{v^4}\right)$



$\mathcal{O}\left(\frac{E^4}{v^4}\right)$



$\mathcal{O}\left(\frac{E^2}{v^2}\right)$



$\mathcal{O}\left(\frac{E^2}{v^2}\right)$

$$\mathcal{O}\left(\frac{E^2}{v^2}\right)$$

$$m_H^2 \lesssim \frac{8\pi v^2}{3(1-a^2)} \simeq (1630 \text{ GeV})^2 \longleftarrow \boxed{a=0.9}$$

- Thus perturbative unitarity constraints do not prevent us from assuming H--> hh contributions are beyond the kinematic reach of the 1 TeV ILC

$a \neq c$

- Benchmark Models assumed Higgs mixing with a scalar that doesn't participate in EWSB or break custodial SU(2)
- There are well motivated models for which these assumptions do not hold
- If new scalar participates in EWSB then $a \neq c$
- This can be determined from the high precision measurements of single Higgs couplings at the ILC

$a \neq c$

- Considering the G-M model. We can still solve for the mixing angle.

$$c = \frac{\cos\theta}{v_\phi / v_{SM}}$$

$$v_\phi^2 + 8v_\chi^2 = v_{SM}^2$$

$$a = \cos\theta \frac{v_\phi}{v_{SM}} - \sin\theta \frac{8}{\sqrt{3}} \frac{v_\chi}{v_{SM}}$$

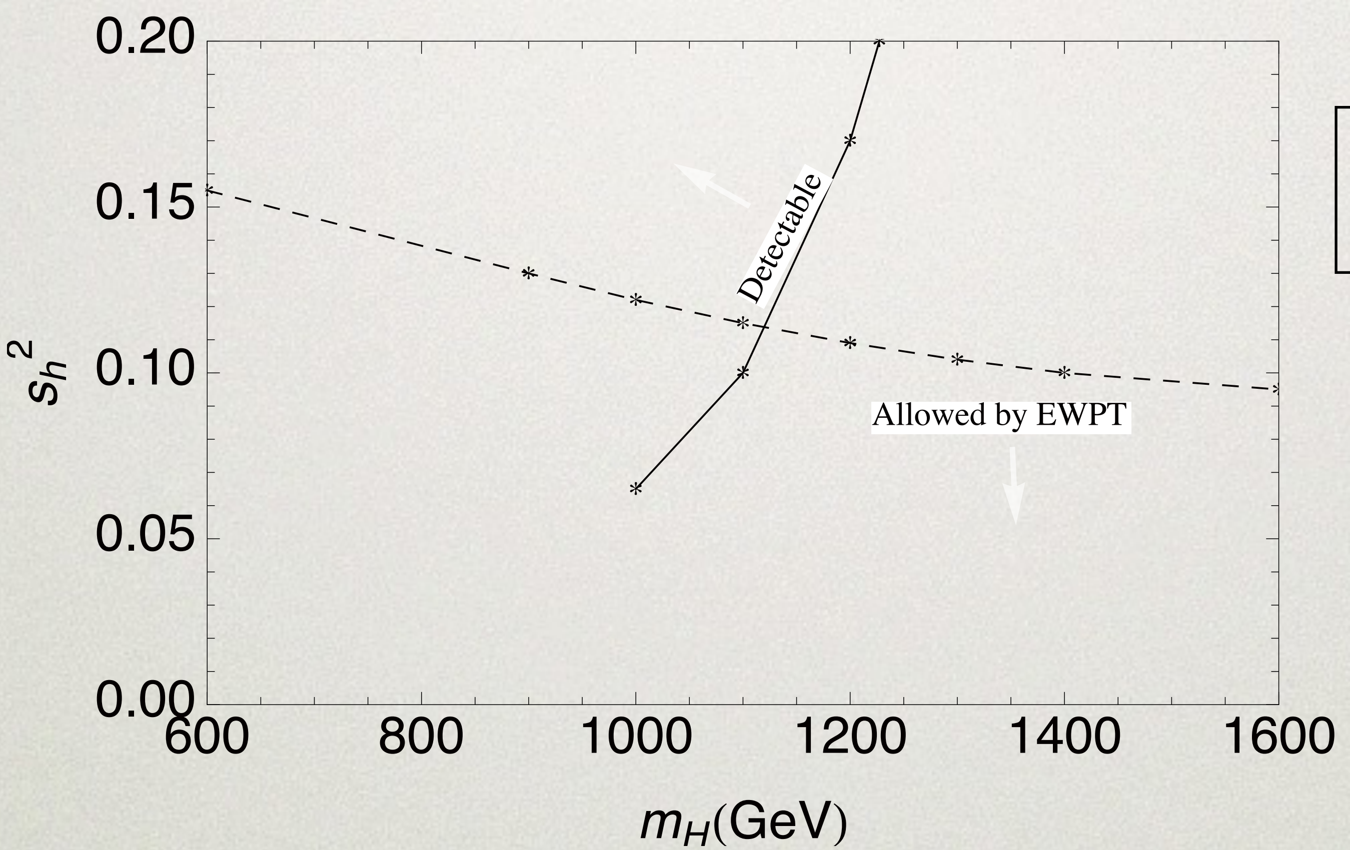
$$b = \cos^2\theta + \frac{8}{3}\sin^2\theta$$

Broken Custodial SU(2)

- If custodial SU(2) is broken then $b_W \neq b_Z$
- eg: complex triplet in type-2 seesaw mechanism for neutrino masses
- Two measurements will not be sufficient to measure b_W, b_Z and d
- Additional information from kinematic discriminants or the LHC will be needed

Electroweak Precision Constraints

- For a mixed-in scalar



R. Gupta, J.Wells, H. Rhezhak, arXiv : 1206.3560

- Additional new physics could compensate for a = 0.9
- At the very least a = 0.95 is safe for the mH range we are interested in

Conclusions

- If Higgs couplings show deviations from the SM expectation a direct measurement of the $hhVV$ would be important to determine EW quantum numbers of the other scalar
- This measurement is extremely difficult at the LHC
- At the ILC it is accessible via di-Higgs production
- di-Higgs production has mainly been studied as a handle on the triple-Higgs coupling

Conclusions

- The *hhVV* can be separated from the *hhh* coupling with rate measurements at two different centre of mass energies
- In addition LHC measurements can constrain the *hhh* independently

BACKUP SLIDES

di-Higgs from VBF

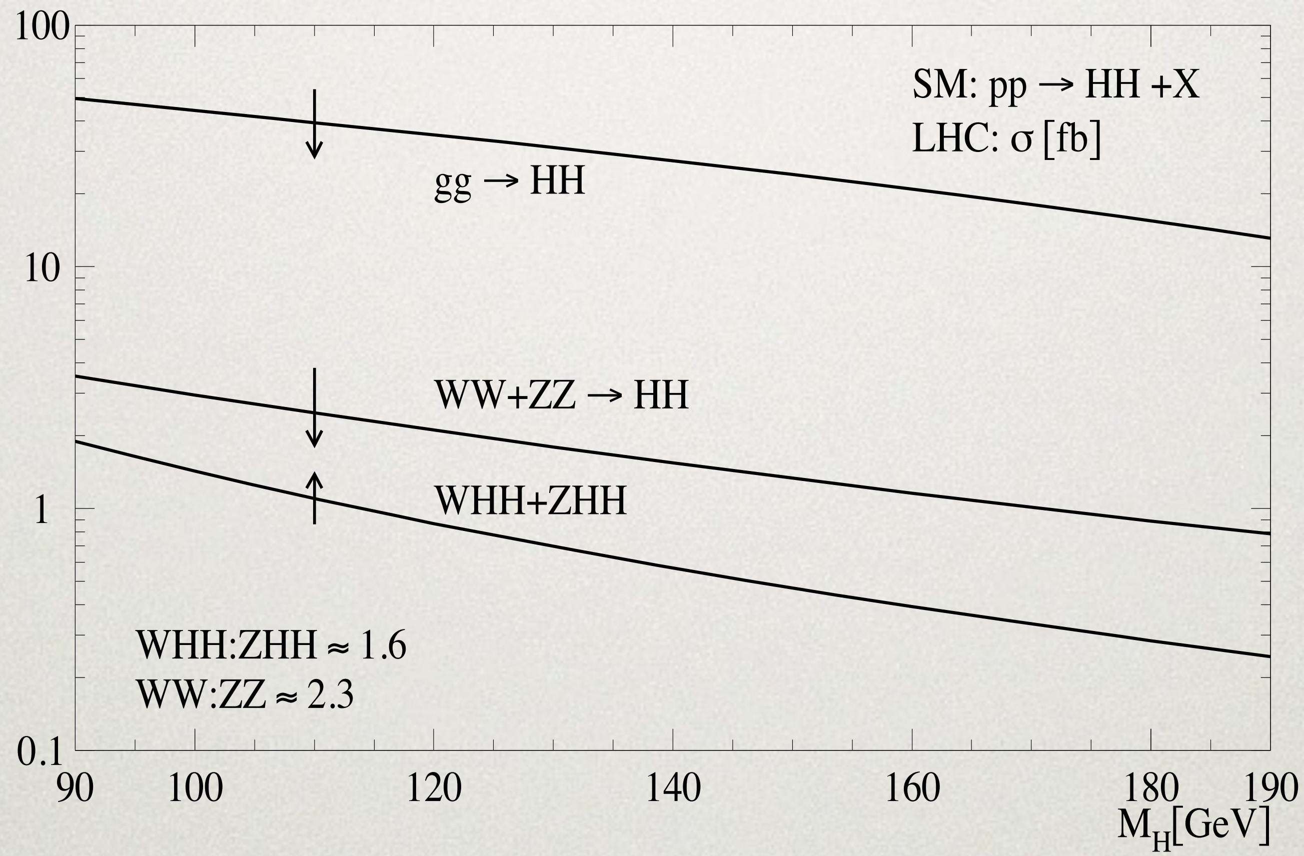


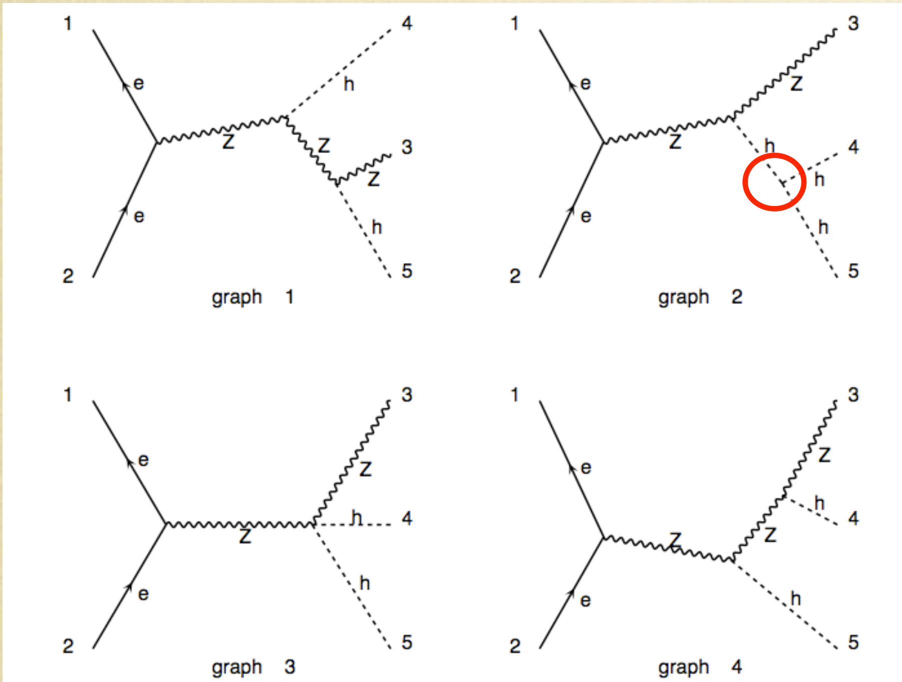
Figure 3.40: The cross sections for gluon fusion, $gg \to HH$, the WW/ZZ fusion $qq \to qqWW/ZZ \to HH$ and the double Higgs–strahlung $q\bar{q} \to WHH + ZHH$ in the SM as a function of M_H. The vertical arrows correspond to a variation of the trilinear Higgs coupling from $\frac{1}{2}$ to $\frac{3}{2}$ of the SM value, $\lambda'_{HHH} = 3M_H^2/M_Z^2$; from Ref. [254].

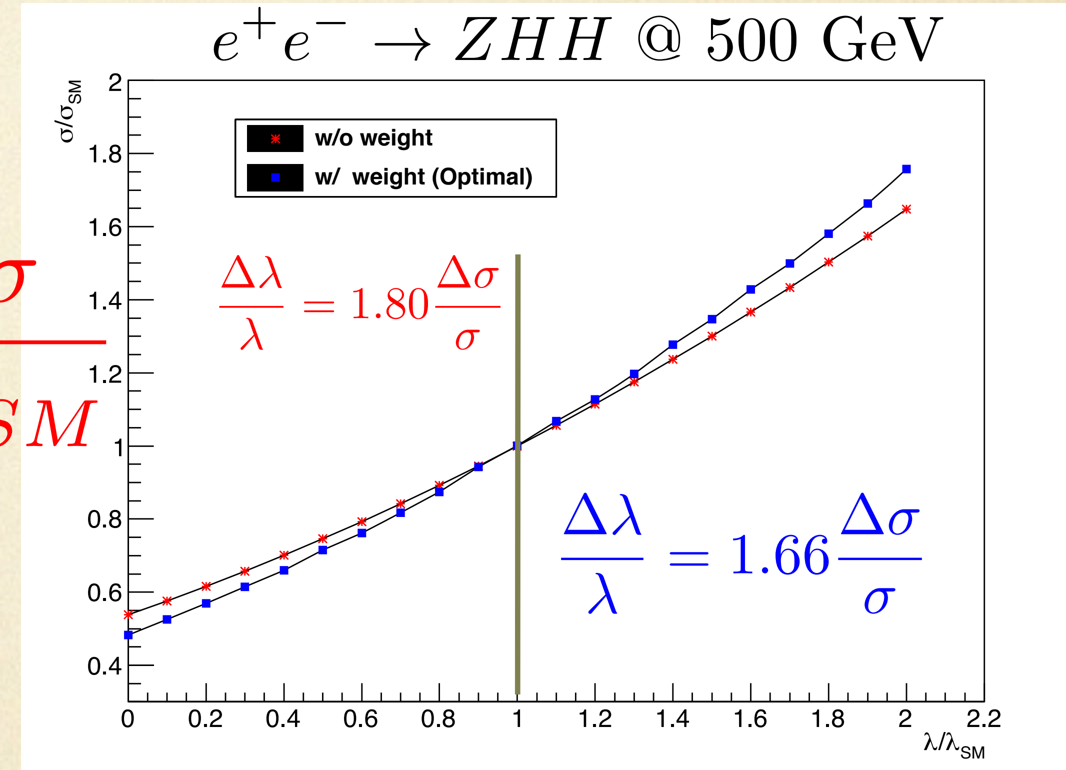
Major Processes

Energy	Reaction	Physics Goal	Polarization
91 GeV	$e^+e^- \to Z$	ultra-precision electroweak	**A**
160 GeV	$e^+e^- \to WW$	ultra-precision W mass	**H**
250 GeV	$e^+e^- \to Zh$	precision Higgs couplings	**H**
350–400 GeV	$e^+e^- \to t\bar{t}$	top quark mass and couplings	**A**
	$e^+e^- \to WW$	precision W couplings	**H**
	$e^+e^- \to \nu\bar{\nu}h$	precision Higgs couplings	**L**
500 GeV	$e^+e^- \to f\bar{f}$	precision search for Z'	**A**
	$e^+e^- \to t\bar{t}h$	Higgs coupling to top	**H**
	$e^+e^- \to Zhh$	Higgs self-coupling	**H**
	$e^+e^- \to \tilde{\chi}\tilde{\chi}$	search for supersymmetry	**B**
	$e^+e^- \to AH, H^+H^-$	search for extended Higgs states	**B**
700–1000 GeV	$e^+e^- \to \nu\bar{\nu}hh$	Higgs self-coupling	**L**
	$e^+e^- \to \nu\nu VV$	composite Higgs sector	**L**
	$e^+e^- \to \nu\bar{\nu}t\bar{t}$	composite Higgs and top	**L**
	$e^+e^- \to \tilde{t}\tilde{t}^*$	search for supersymmetry	**B**

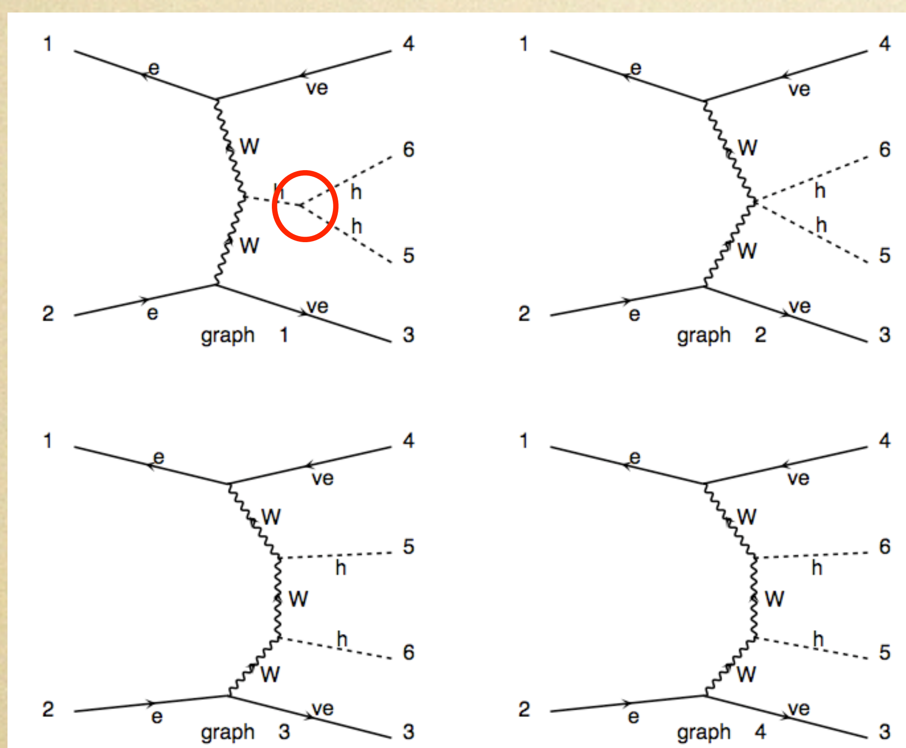
General issue: sensitivity to the cross section

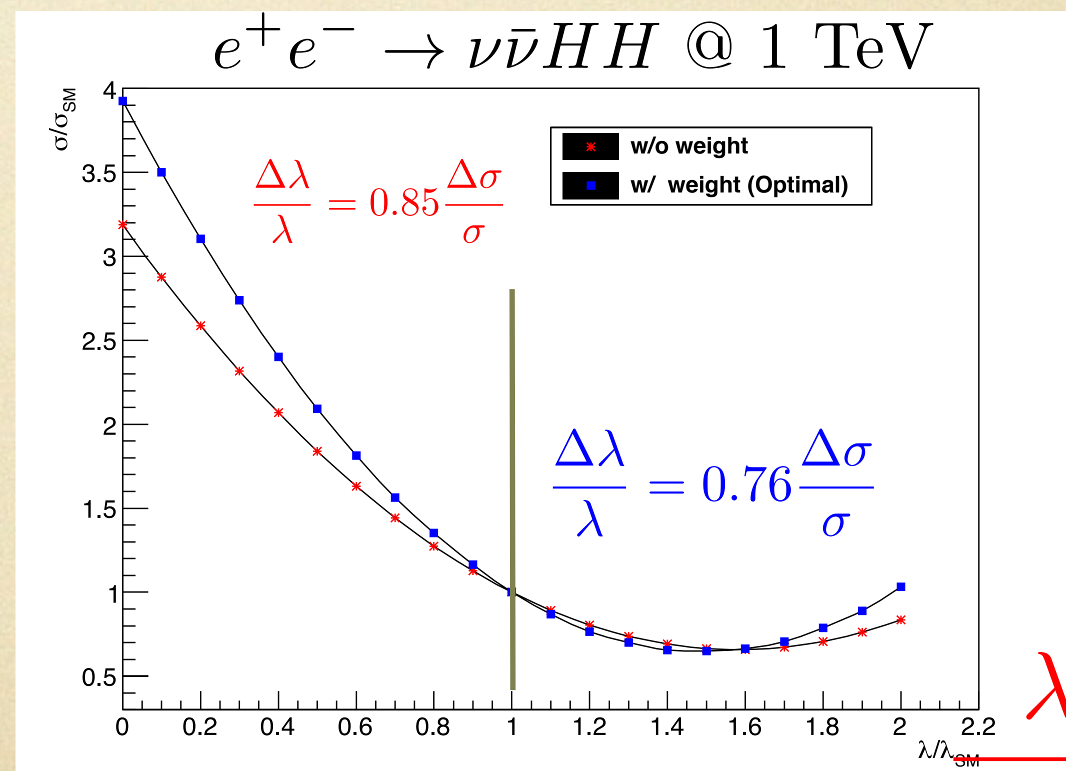
effect of irreducible diagrams





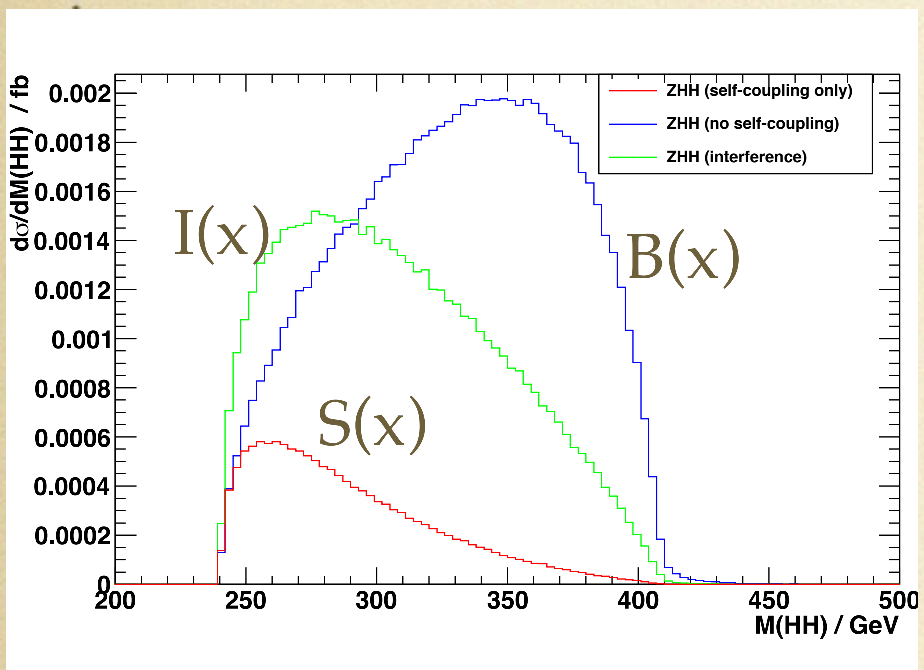
$\frac{\sigma}{\sigma_{SM}}$





$\frac{\lambda}{\lambda_{SM}}$

weighting method to enhance the coupling sensitivity

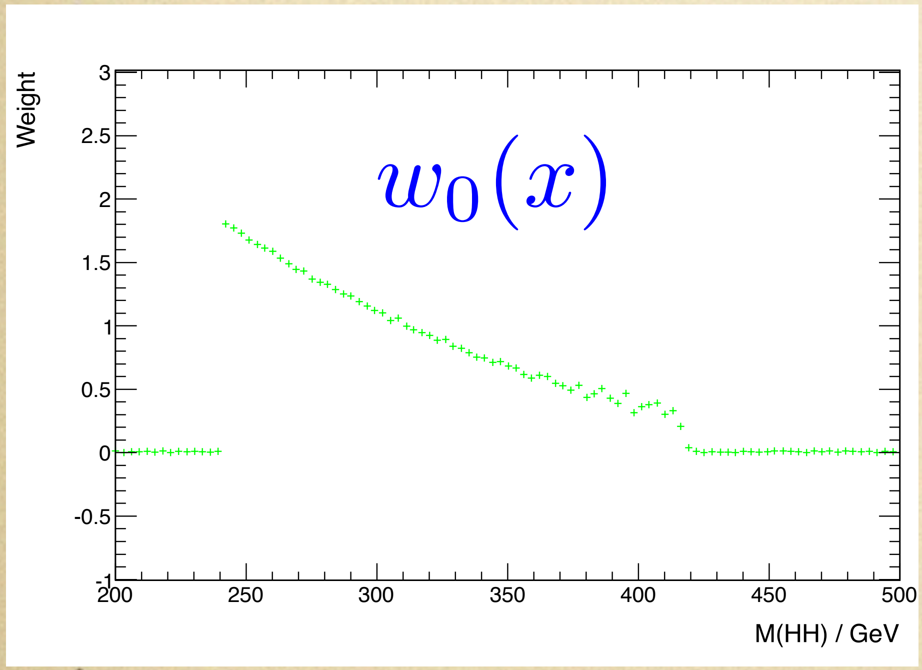


$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} = B(x) + \lambda I(x) + \lambda^2 S(x)$$

irreducible　interference　self-coupling

observable: weighted cross-section

$$\sigma_w = \int \frac{\mathrm{d}\sigma}{\mathrm{d}x} w(x) \mathrm{d}x$$



equation of the optimal w(x):

$$\sigma(x) w_0(x) \int (I(x) + 2S(x)) w_0(x) \mathrm{d}x = (I(x) + 2S(x)) \int \sigma(x) w_0^2(x) \mathrm{d}x$$

general solution:

$$w_0(x) = c \cdot \frac{I(x) + 2S(x)}{\sigma(x)}$$

c: arbitrary normalization factor