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Master Thesis

APPLICATIONS OF WAVELETS IN DATA MINING

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keywords:
wavelets
statistics
learning and adaptive systems

Short summary:
*** To be changed ***
The aim of this thesis is an analysis and comparison of statistical methods employing wavelets. Special attention will be paid to data mining and machine learning methods.

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Contents

Introduction

*** Few words about my thesis... Generally what is a data mining, what are the wavelets, what are the applications in data mining and finally, what is the main purpose - edge detection. ***

Data mining is connected with computing huge amount of data, thus to analyse those data it is require to use algorithms with low computational complexity. Wavelet transform is very efficient.

In today world more and more times we are operating on data that tend to be big and high-dimensional. In addiction most of them can be really messy. Missing values, noises or inconsistent data are only few of the problems we have to face. That is why working with Data Mining algorithms are often difficult or even impossible. At this point wavelets are usually the solution. Thanks to their properties we are able to underlying function from the data. Then Data Mining tasks are way easier to carry out. But how it is done? Every wavelet consists with many different wavelet coefficients. We can specify, thanks to vanishing moment property, which of wavelet coefficients are significant in our case. Knowing this we can perform wavelet transform which makes our data more less noisy and dimensional reduced. It is possible because wavelet coefficients are generally decorrelated.

Chapter 1

Wavelets theory

A "wavelet" literally means a small wave. This term says a lot about its nature. Wavelets are a family of functions which oscillates like wave and should be compactly supported.

Definition 1.1. A mother wavelet is a function $\psi(t)$, which satisfies the condition that the set of functions

$$\left\{ \psi(2^j t - m), \ j, m \in \mathbb{Z} \right\} \tag{1.1}$$

is an orthonormal basis of $\mathbf{L}^2(\mathbb{R})$.

Definition 1.2. Wavelet functions are created by scaling and shifting of the mother wavelet $\psi(t)$. They are defined as

$$\psi^{(a,b)}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \ a > 0, \tag{1.2}$$

where a is a scale parameter and b translation parameter.

There are plenty of different mother wavelets, for example
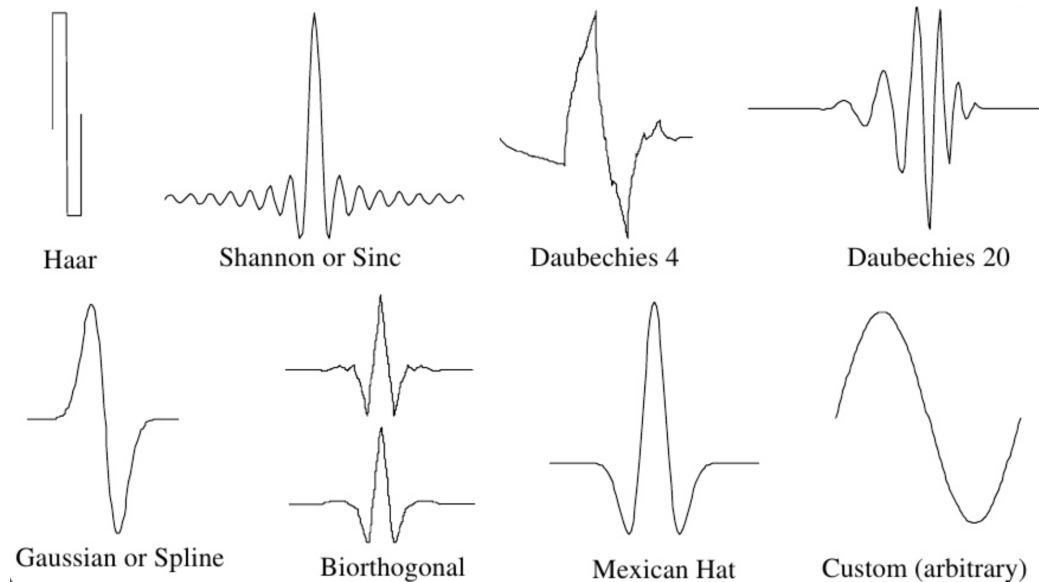

Figure 1.1: Different types of wavelets.

The wavelet functions have specific properties which are useful in data analysis. According to the [1] article, the most important are:

- Vanishing moments. Wavelet has n vanishing moments, when $\int t^j \psi(t)dt = 0$ for $j = 0, 1, \ldots, n-1$. For all wavelets at least the first moment, the mean value, is equal zero. Particular wavelets have more vanishing moments, e.g. Daubechies family, the *Daub4* and *Daub20* presented on the figure 1.1 has respectively, 2 and 10 vanishing moments. This property is also known as approximation order, because for the part of data which are represented as n-degree polynomials, the wavelet coefficients are also equal zero (according to the definition 1.4). So the bigger n, the more data is neglected. ** *Add that the bigger coeffs are more significant?* **
- Compact support. Wavelets domain is a compact set, most of the time it is a finite interval. This property protects from leaving a data region during wavelet processing.
- Decorrelated coefficients. Wavelet transformations diminish the time correlation, thus the wavelet coefficients are less depended than the analysed data.

1.1. Haar and Daubechies wavelets

Each type of wavelet function is more suitable for different applications. As it is shown in [6], the Daubechies wavelets are very useful in compression, denoising and enhancement audio signals or images.

Definition 1.3. Daubechies wavelets are collection of orthogonal and compactly supported functions. A denotations for those wavelets are *dbn* or *DaubN*, where n is the number of vanishing moments and $N = 2n$. Moreover, the support in on the interval $[0, 2n-1]$.

The Haar wavelet is the simplest of Daubechies ($db1$), and also of all, wavelets. The mother function is illustrated on the graph 1.2 and has the following formula

$$\psi(t) = \begin{cases} 1, & \text{for } 0 \le t < \frac{1}{2}, \\ -1, & \text{for } \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$
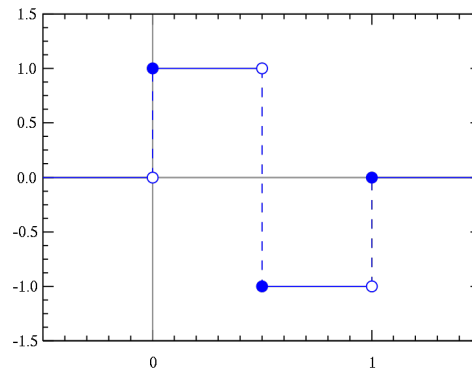

Figure 1.2: The Haar wavelet.

Thanks to this simple form the Haar wavelet has wide application in signal and image analysis. However, the simplest wavelet is not always enough. There are cases which require more complex wavelets. Examples of those Daubechies wavelets are shown on the figure 1.3. We can see that the bigger n, the function is smoother and more regular.
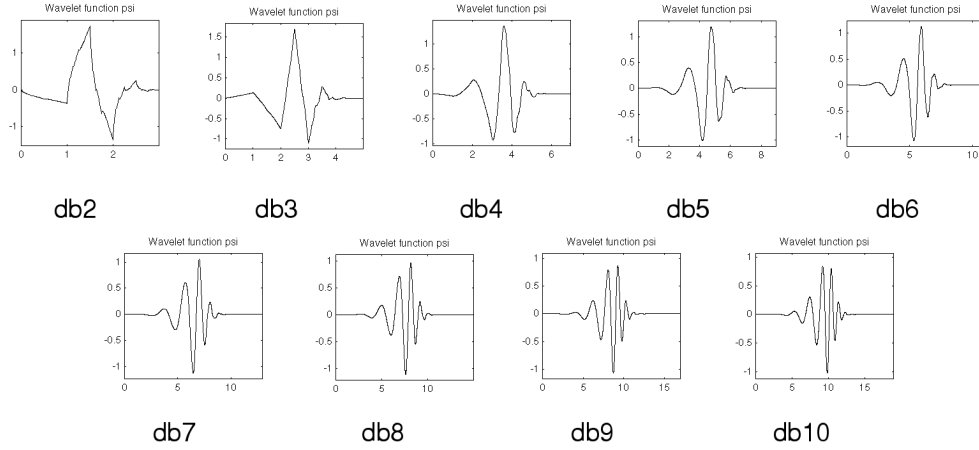

Figure 1.3: Daubechies wavelets.

1.2. Wavelet Transform

Generally, integral transforms are useful in data processing. The idea is to convert data into another domain, where it is easier to manipulate them, finding desired informations, etc. Finally, the results are transformed back to the original domain by inverse integral transform.

We can define integral transform based on the wavelet functions - Wavelet Transform. However, the most popular kind of integral transform is Fourier Transform. Here the question arises, what are the differences between both operators. Lets start from the definitions.

Definition 1.4. Continuous Wavelet Transform is expressed by the formula

$$W(a,b) = \int_{-\infty}^{\infty} y(t)a^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right)dt, \qquad (1.4)$$

where a is scale parameter, b translation parameter and $y(t)$ original signal.

Definition 1.5. Fourier transform is defined as

$$Y(f) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}dt, \qquad (1.5)$$

where $y(t)$ is time domain signal and $Y(f)$ is frequency domain signal.

The most important differences are presented in the table below.

The obvious distinction between both transforms is the type of function. In Fourier case there are sine and cosine functions, wherein wavelet transform uses wavelets. Sine function oscillates on the whole real axis, thus it cannot

Wavelet transform	Fourier transform
Suitable for stationary and non--stationary signals	Suitable for stationary signals
High time and frequency resolution	Zero time resolution and very high frequency resolution
Very suitable for studying the local behaviours of the signal	No suitable
Scaled and translated mother wavelets	Sine and cosine waves

Table 1.1: The comparison of Wavelet and Fourier Transform.

represent abrupt changes. However, the Wavelet transform is localized in space and time, so it can be used to detect trends or sudden changes in signals and images. Moreover, wide range of wavelet functions is a main advantage of wavelet analysis, because we can adjust the type of wavelet to the specific case and thank to that obtain more accurate results.

1.3. Discrete Wavelet transform

There are two types of Wavelet Transform:
- Continuous Wavelet Transform (CWT),
- Discrete Wavelet Transform (DWT).

The Discrete Wavelet Transform has a wide range of applications in image processing, for example denoising and compressing, also in detecting shapes on images.
*** I'll add a little more about applications in pre-processing and machine learning ***
Scale and translation parameters are defined as

$$a = 2^j \text{ and } b = 2^j m, \ j, m = 1, 2, \dots. \tag{1.6}$$

Therefore, assuming that $0 \leq t < 2^K$, $K \in \mathbb{Z}$, the coefficients of Discrete Wavelet Transform are defined as follow

$$d^{(j,m)} = \sum_{t=0}^{2^K-1} y(t) 2^{-\frac{j}{2}} \psi \left(2^{-j} t - m \right). \tag{1.7}$$

The figure 1.4 on a page 11 shows how DWT works. Discrete Wavelet Transform splits signal with two filters: $g(n)$ - low pass filter (LPF) and $h(n)$ - high pass filter (HPF). The LPF captures a part with lower frequencies which refers to the main signal. Whereas, the HPF captures higher frequencies - a noise of the signal. Subsequently, both parts are downsampled by a factor of 2. This decomposition can be repeated on the LPF part of the signal. Hence, the next levels of DWT coefficients.
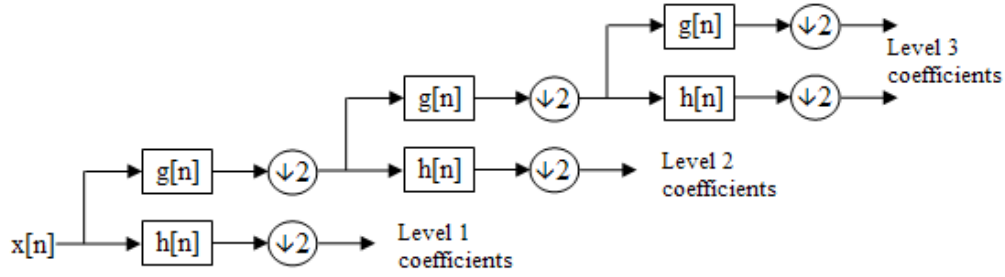
Figure 1.4: Discrete Wavelet transform on a signal $x(n)$.

1.4. 2-D Discrete Wavelet transform

2-Dimensional Discrete Wavelet Transform works similar way as 1-D with High Pass Filter, Low Pass Filter and downsampling, except that one level of the decomposition includes double filtering, on columns and rows. The figure 1.5 shows an image decomposition. Firstly, the DWT is applied on columns of the input image and then on the rows of the both outputs. Ultimately, there are four results:

- LL - result of LPF applied on both, columns and rows,
- LH - result of LPF applied on columns and HPF on rows,
- HL - result of HPF applied on columns and LPF on rows,
- HH - result of HPF applied on both, columns and rows.
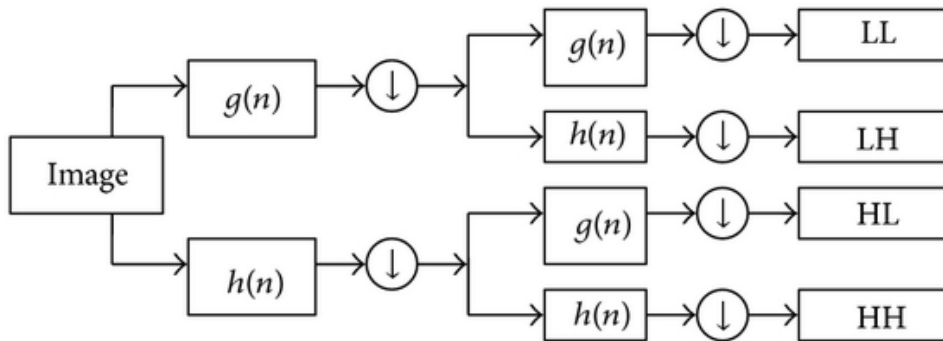
Figure 1.5: 2-D Discrete Wavelet transform on an image.

Recall that the outcome of Low Pass Filter in the previous case was the main signal (without a noise). Thus, a 2-Dimensional equivalent is an approximation of an analysed image. The High Pass Filter captures high frequencies, then for an image the outcome are sudden changes in the image contrast. Now, lets focus on what exactly each result represent. First one, the LL is just an approximation of the initial image. Next, the LH shows abrupt changes in a horizontal direction, whereas the HL part presents similar issues but in a vertical direction. The HH shows sudden changes in a diagonal

direction. In conclusion, the output of 2-D DWT gives us an approximation of the image and three parts with abrupt changes in different directions.

1.5. Inverse Wavelet Transform

Majority applications of Wavelet Transform assume that data is converted to wavelet domain and then we can receive the desired information. Although, we must return to the original domain for the information to be useful. The Inverse Wavelet Transform allows for such operation.

Definition 1.6. Inverse Discrete Wavelet Transform is defined as

$$y(t) = \sum_{j=1}^{K} \sum_{m=0}^{2^{K-j}-1} d^{(j,m)} 2^{-\frac{j}{2}} \psi \left(2^{-j} t - m \right) + \psi_0, \qquad (1.8)$$

where $y(t)$ is reconstruction of the input data and $d^{(j,m)}$ coefficients of wavelet transform. The ψ_0 is an average value of $y(t)$ over $t \in [0, 2^K - 1]$. This parameter can be approximated by zero, without loss of generality.

Summarizing, now we are able to transform data to a wavelet domain. Process them simply and efficiently, as well as, we can back to the original domain to read the obtained results.

Chapter 2

Edge detection

In this chapter let us focus on the main goal of the thesis, which is edge detection. An edge is a place where image brightness changes rapidly. There are various methods to identify such discontinuities. The most popular are gradient based (e.g. Canny, Prewitt, Sobel) and Laplacian based. However, there is also another method, which provides similar results and can be more efficient in terms of computation. This method is based on the 2-Dimensional Discrete Wavelet Transform described in section 1.4.

The mentioned method contains the following steps. *** Link to article Edge detection ***
1. Convert image to grey scale.
2. Apply 2-D DWT on an image.
3. Remove the LL part.
4. Denoise the LH, HL and HH coefficients.
5. Reconstruct the initial image.
6. Post-processing - modify contrast to emphasize obtained edges.

2.1. Implementation of an algorithm

The algorithm is implemented in `Python` using `PyWavelets` package. There are used also auxiliary packages like: `numpy`, `matplotlib`, `PIL` and `scipy`. The `PyWavelets` package contains all features required to edge detection algorithm, i.e. 2D Forward and Inverse Discrete Wavelet Transform, build-in many wavelet functions and thresholding functionality (used to denoise coefficients).
Lets go deeper into algorithm, using a simple example - white square on a black background.
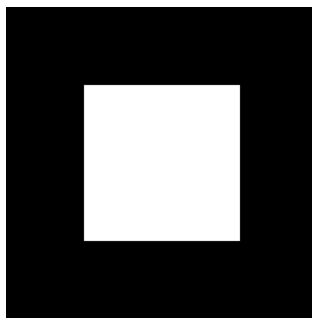

Figure 2.1: An initial image - a white square.

Initially, a colour image must be simplified by conversion to grey scale. Edges are recognised as changes in brightness, so a single pixel should contains only information about a colour (black) intensity. Our initial image is already black and white so we do not need to convert it. Now, we can apply the 2D DWT function. As a result, according to the description in section 1.4, we obtain four components of wavelet coefficients: LL, LH, HL and HH shown on a graph 2.2. It is clearly visible that the LL part reflects the initial image and the rest components contains information about rapid brightness changes in particular directions.
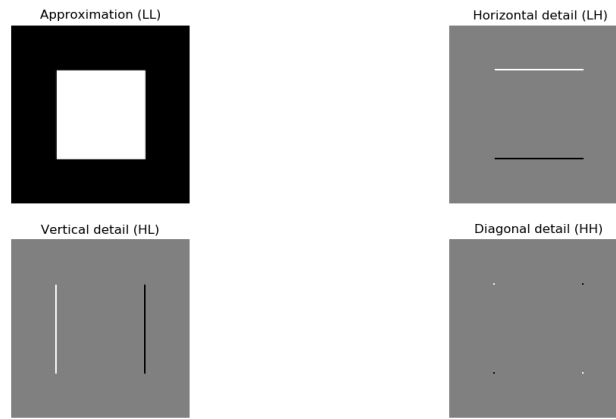
Figure 2.2: 2-D DWT coefficients.

Thus, we are interested only in components which gives information about edges, so we remove the approximation part - figure 2.3.
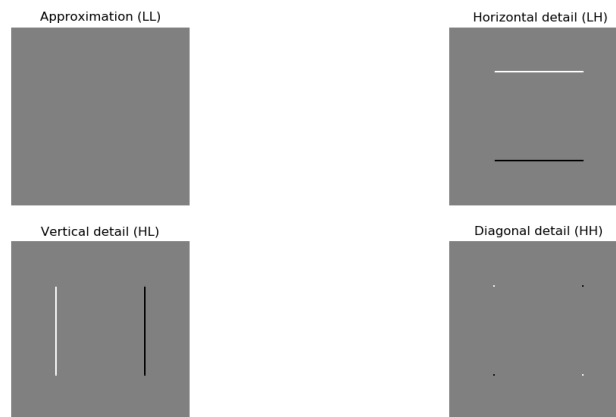
Figure 2.3: 2-D DWT coefficients with removed the LL part.

Subsequently, remaining components can be denoised. It means, we can get rid off small, insignificant coefficients by thresholding. More about setting

the threshold is described in section 2.2. This simple example do not require any denoising, so lets go further.

The last main step is reconstruction of the initial image, i.e. application an inverse DWT on the denoised coefficients. In the result, we obtain edges of the initial image presented on a graph 2.4.
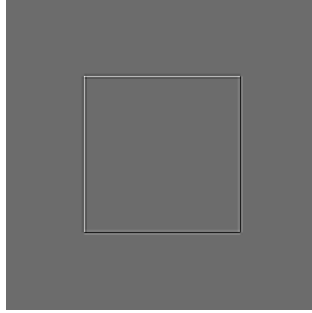

Figure 2.4: A reconstructed image showing the edges.

At the end, we can manipulate contrast to emphasize obtained lines. Currently, a background has grey colour and the edges are white or black. Therefore, using simple mathematical calculations we can modify image to have black background and white edges. It is enough to get an absolute value, subtract 128 and then scale by multiplying 2 times.

*** Calculations taken from article Edge detection. Maybe add some more about values in an image array ***

Finally, we obtain the edges of the square shown on the figure 2.5.
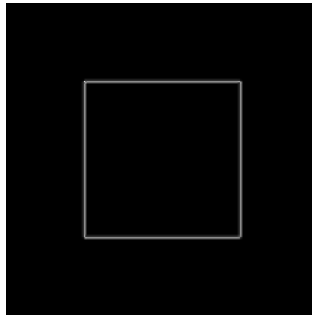
Figure 2.5: The edges of the square image after post-processing.

2.2. Thresholding

There are two types of thresholding: hard and soft. Lets denote λ as threshold value and d as wavelet coefficient. The hard thresholding is defined as follow

$$D^H(d|\lambda) = \begin{cases} 0, & \text{for } |d| \leq \lambda, \\ d, & \text{for } |d| > \lambda. \end{cases} \qquad (2.1)$$

It assigns zero value for coefficients below the set threshold. The soft thresholding works in the same way on the coefficients smaller than λ, but additionally the coefficient bigger than the set threshold are "shrinked" towards zero, as it is defined below.

$$D^S(d|\lambda) = \begin{cases} 0, & \text{for } |d| \leq \lambda, \\ d - \lambda, & \text{for } d > \lambda, \\ d + \lambda, & \text{for } d < -\lambda. \end{cases} \tag{2.2}$$

2.2.1. Hard or soft thresholding

To see the differences between both types of thresholding lets consider a noisy image (fig. 2.6).
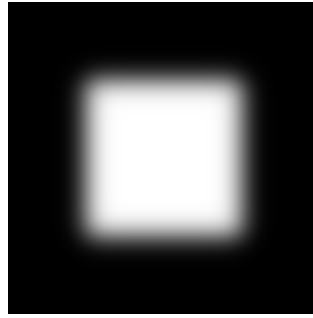
Figure 2.6: A noisy square image.

The coefficients of 2-D DWT are shown on the graph 2.7.
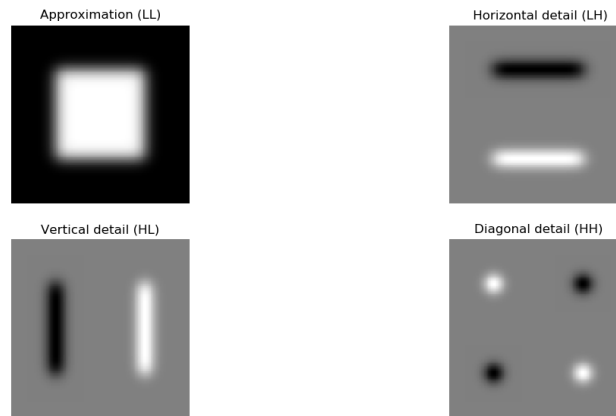
Figure 2.7: 2-D DWT coefficients.

Now, according to the edge detection algorithm, we remove the LL component and we can denoise the other ones. Results of hard and soft thresholding are presented respectively on a figures 2.8 and 2.9.

We can see the distinction between both thresholding, especially on the LH and HL components. After soft one the lines are smoother and slightly
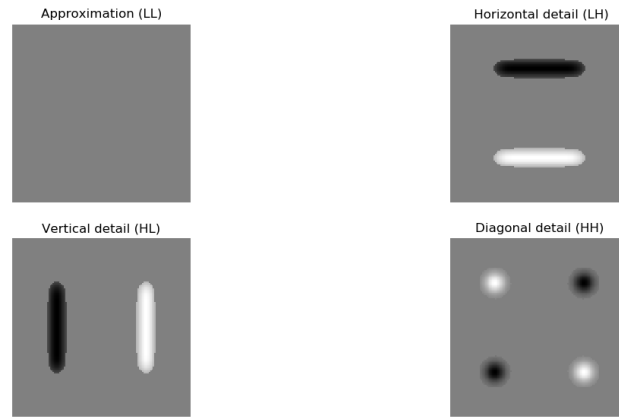
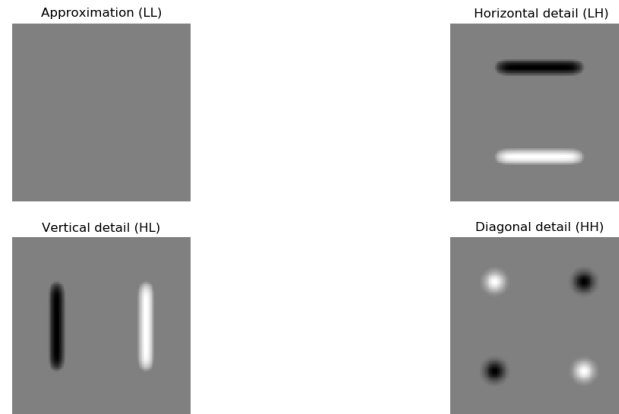Figure 2.8: 2-D DWT coefficients after hard thresholding.

Figure 2.9: 2-D DWT coefficients after soft thresholding.

narrower, then in the other case. Therefore, the soft thresholding seems to be better for edge detection. Because of the smoother results, it is especially useful for the real life images, where edges are often blurred. To confirm this conclusion lets compare the final results, after inverse DWT (fig. 2.10)

2.2.2. Setting the threshold

The corollary from the previous subsection is that the soft thresholding provides better results for edge detection. However, there is also a question how to choose the value of a threshold λ to denoise coefficients and do not loose any significant information. A simple idea is to get a quantile of the coefficients in the specific components.
** Add histograms of the coefficients **
We decided to start with a quantile on level 0.95. It occurs to be accurate for the majority of standard images. All the earlier results are computed

(a) hard   (b) soft

Figure 2.10: Results of edge detection with hard and soft thresholding.

for threshold set as 0.95 quantile. Although, images which are noisier could require changing the quantile level.

Lets consider three images of basic diamond shape with different noisiness, shown on the figure 2.11.
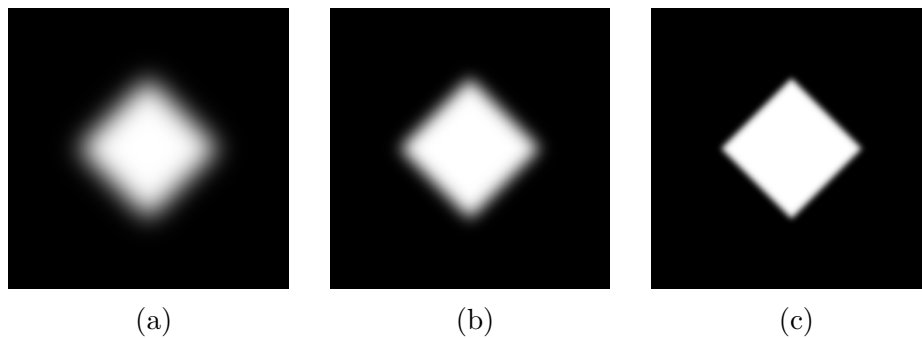

(a)   (b)   (c)

Figure 2.11: The initial images of diamond with different noise level.

At first, see how looks the result with mentioned earlier threshold set as 0.95 quantile (fig. 2.12). Algorithm found the edges properly but they are quite wide, especially for the noisier image. Lets try to increase the threshold to quantile on the 0.99 level. The results are presented on the graph 2.13. It can be seen that for (b) and (c) images the edges are narrower and visible. Nevertheless, result for the (a) image is incorrect, threshold is too high and too much information is removed. Hence, lets consider one more threshold on a 0.98 quantile level. The recognised edges for (a) and (b) images are presented on the figure 2.14. For this threshold edges are correctly detected. In conclusion, for noisier images the increased threshold provides more accurate result, but increasing the threshold must be done carefully to not ignore crucial informations. Additionally, when image have one sharp object and noisy background, increasing the threshold we can control the amount of background details in the result.

2.3. Selection of a wavelet type

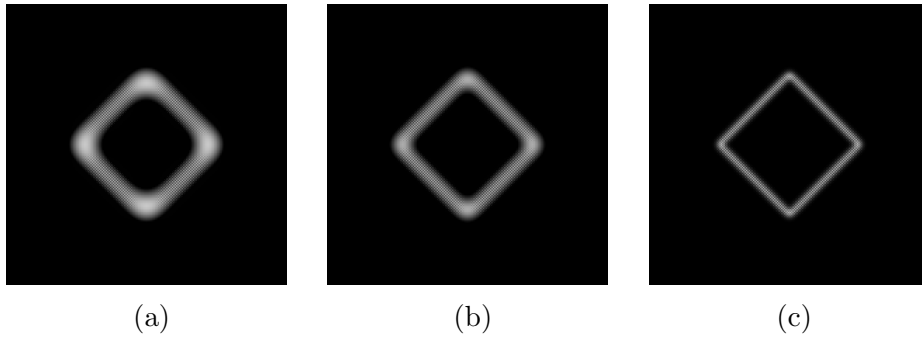*** Show results for different wavelet types: haar and db2? ***

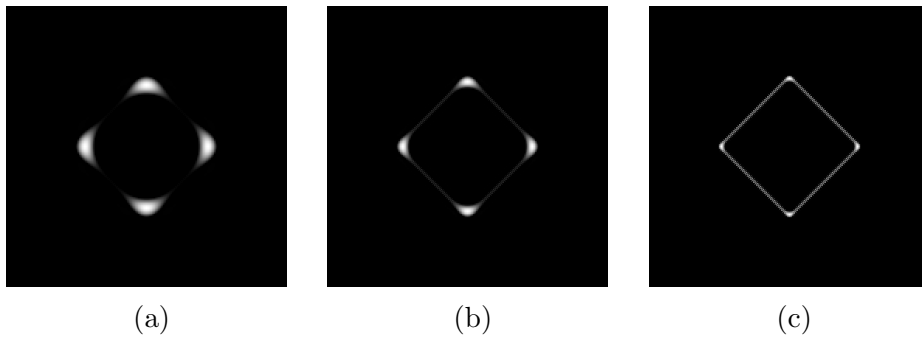Figure 2.12: The recognized edges for the diamonds with λ equals 0.95 quantile.

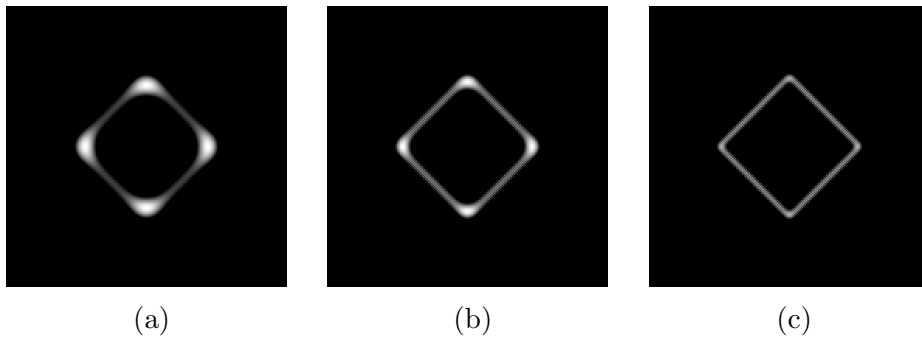Figure 2.13: The recognized edges for the diamonds with λ equals 0.99 quantile.

Figure 2.14: The recognized edges for the diamonds with λ equals 0.98 quantile.

Chapter 3

*** Results. Maybe something about medical applications ***

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