# Policy Gradient

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# Parameterizing Policy

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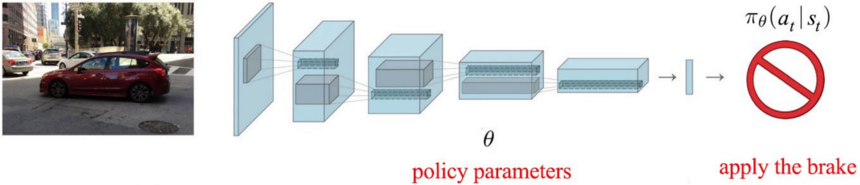

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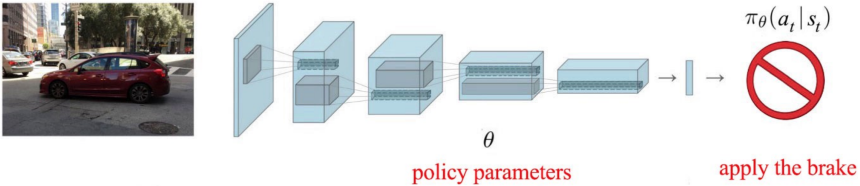


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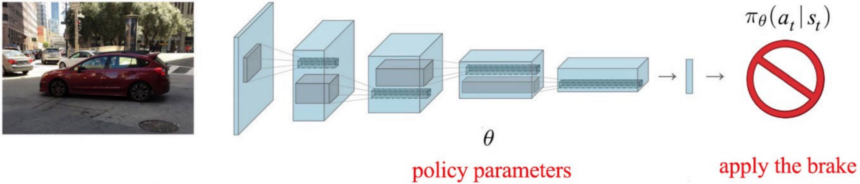
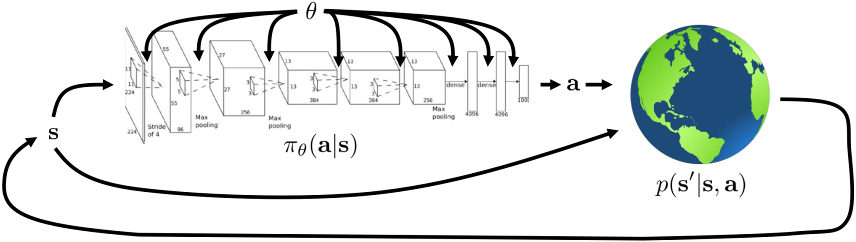



Q) How can we find a good $\pi(a|s)$, which is a **function**?

Idea:

- Parameterize policy by a parameter vector $\theta \in \mathbb{R}^\ell$: $\pi_\theta(a|s)$
- Find an optimal $\theta$

# How to find optimal parameters $\theta$?




[S. Levine, CS285]

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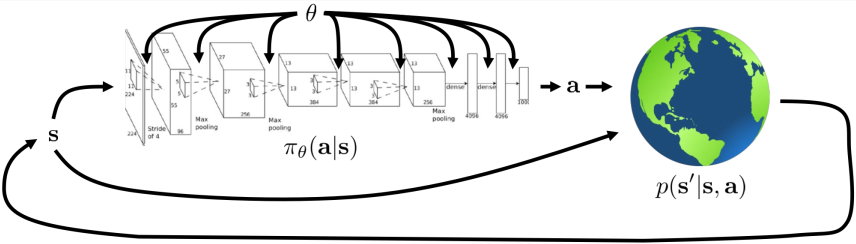


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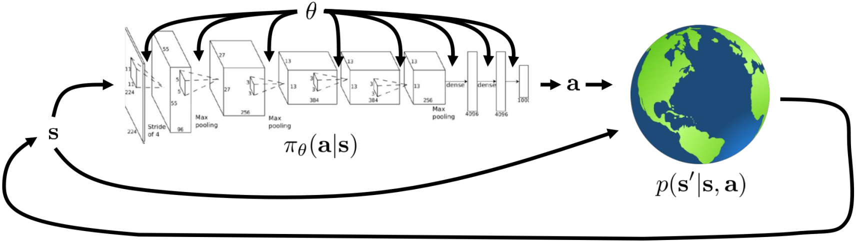


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- Let $\tau := (s_0, a_0, \ldots, s_T, a_T)$ denote the state-action trajectory
- By Markov property,

$$p_\theta(\tau) = p(s_0) \prod_{t=0}^{T} \pi_\theta(a_t|s_t) p(s_{t+1}|s_t, a_t)$$

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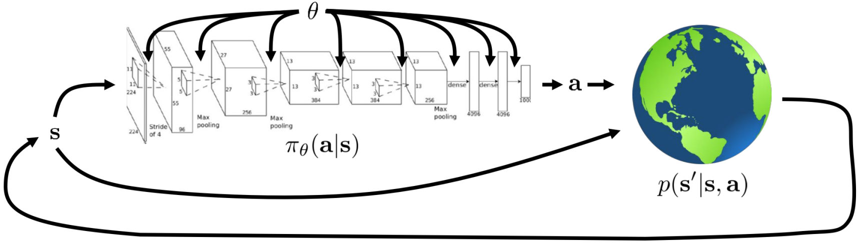


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- Approximate MDP problem:

$$\max_\theta \mathbb{E}_{\tau \sim p_\theta(\tau)} \left[ \sum_t r(s_t, a_t) \right] =: J(\theta)$$

# Basics of Optimization

$$\max_{\theta} \; J(\theta)$$

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   - Set $k \leftarrow k + 1$;

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- Differentiate $J$ w.r.t $\theta$:

$$\nabla_\theta J(\theta) = \int \nabla_\theta p_\theta(\tau) r(\tau) d\tau$$

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Therefore, the gradient can be written as

$$\begin{aligned}
\nabla_\theta J(\theta) &= \int \nabla_\theta p_\theta(\tau) r(\tau) d\tau \\
&= \int p_\theta(\tau) \nabla_\theta \log p_\theta(\tau) r(\tau) d\tau \\
&= \mathbb{E}_{\tau \sim p_\theta(\tau)} \left[ \nabla_\theta \log p_\theta(\tau) r(\tau) \right]
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- So far, we have the **policy gradient theorem**:

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- REINFORCE algorithm: using empirical estimate of $\mathbb{E}_{\tau \sim p_\theta(\tau)}$

Machine Learning, 8, 229–256 (1992)


Simple Statistical Gradient-Following Algorithms for Connectionist Reinforcement Learning

RONALD J. WILLIAMS rjw@corwin.ccs.northeastern.edu
*College of Computer Science, 161 CN, Northeastern University, 360 Huntington Ave., Boston, MA 02115*

# REINFORCE algorithm

Initialize $\theta$;

1. Sample $\{\tau^i\}_{i=1}^N := \{(s_0^i, a_0^i, \ldots, s_T^i, a_T^i)\}_{i=1}^N$ using the current policy $\pi_\theta(a_t|s_t)$

2. Estimate the gradient

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^N \left( \sum_{t=0}^T \nabla_\theta \log \pi_\theta(a_t^i|s_t^i) \right) \left( \sum_{t=0}^T r(s_t^i, a_t^i) \right)$$

3. Perform gradient ascent:

$$\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta);$$

# Example: Gaussian Policy

Set

$$\pi_\theta(\cdot|s_t) \sim \mathcal{N}(f_{NN}(s_t); \Sigma)$$

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- $\nabla_\theta \log \pi_\theta(a_t|s_t) = \frac{1}{2}(a_t - f_{NN}(s_t))\nabla_\theta f_{NN}(s_t)$

# Advantages and Disadvantages

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Disadvantages:

- High variance of the gradient
- On policy: Must use the most recent policy
  (Huge # of samples required)

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2. Use a baseline, $b$, not related to $\theta$:

$$
\begin{aligned}
&\mathbb{E}_{\tau\sim p_\theta}[\nabla_\theta \log p_\theta(\tau)(r(\tau)-b)] = \nabla_\theta J(\theta) - \mathbb{E}_{\tau\sim p_\theta}[\nabla_\theta \log p_\theta(\tau) b] \\
&= \nabla_\theta J(\theta) - \int p_\theta(\tau)\nabla_\theta \log p_\theta(\tau) b d\tau \\
&= \nabla_\theta J(\theta) - \int \nabla_\theta p_\theta(\tau) b d\tau \\
&= \nabla_\theta J(\theta) - b\nabla_\theta \int p_\theta(\tau) d\tau \\
&= \nabla_\theta J(\theta) - b\nabla_\theta 1 \\
&= \nabla_\theta J(\theta)
\end{aligned}
$$

$\Longrightarrow$ Subtracting a baseline $b$ is unbiased in expectation!

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No baseline:

$$\mathrm{Var}[\nabla_\theta J^{NB}(\theta)] = \mathbb{E}[(\nabla_\theta \log p_\theta(\tau) r(\tau))^2] - \mathbb{E}[\nabla_\theta \log p_\theta(\tau) r(\tau)]^2$$

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With baseline:

$$\begin{aligned}\mathrm{Var}[\nabla_\theta J^{B}(\theta)] &= \mathbb{E}[(\nabla_\theta \log p_\theta(\tau)(r(\tau) - b))^2] - \mathbb{E}[\nabla_\theta \log p_\theta(\tau)(r(\tau) - b)]^2 \\ &= \mathbb{E}[(\nabla_\theta \log p_\theta(\tau)(r(\tau) - b))^2] - \mathbb{E}[\nabla_\theta \log p_\theta(\tau) r(\tau)]^2\end{aligned}$$

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Therefore,

$$\mathrm{Var}[\nabla_\theta J^{B}(\theta)] \leq \mathrm{Var}[\nabla_\theta J^{NB}(\theta)]$$

if $b \in [0, 2r(\tau)]$.

# Which baseline to choose?

- Recall

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_i \left( \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \right) \left( \sum_{t=0}^{T} r(s_t^i, a_t^i) \right)$$

- Further approximate it by

$$\begin{aligned} \nabla_\theta J(\theta) &\approx \frac{1}{N} \sum_i \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \sum_{t'=t}^{T} r(s_{t'}^i, a_{t'}^i) \\ &= \frac{1}{N} \sum_i \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) Q(s_t^i, a_t^i) \end{aligned}$$

- Choose baseline $b := v(s_t^i)$:

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_i \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \underbrace{\left[ Q(s_t^i, a_t^i) - v(s_t^i) \right]}_{=: A(s_t^i, a_t^i)}$$

# Intuition behind policy gradient with baseline

1. Case I: Trajectory A receives +10 rewards and Trajectory B receives -10 rewards
2. Case II: Trajectory A receives +10 rewards and Trajectory B receives +1 rewards

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Now, Consider $\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_i \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) [Q(s_t^i, a_t^i) - b]$ with baseline $b = 5$

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1. Case I: Trajectory A receives $+10$ rewards and Trajectory B receives -10 rewards
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1. Case I: Trajectory A receives $+5$ rewards and Trajectory B receives -15 rewards
2. Case II: Trajectory A receives $+5$ rewards and Trajectory B receives -4 rewards

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1. Case I: Trajectory A receives +5 rewards and Trajectory B receives -15 rewards
2. Case II: Trajectory A receives +5 rewards and Trajectory B receives -4 rewards

$\Longrightarrow$ PG will increase the probability of Trajectory A but decrease the probability of Trajectory B

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- Use much larger batches ($100\times$ larger than DQN)
- Tuning learning rates is very hard
  Adaptive size rule like Adam can be fine (but not the best)
- Use Actor-Critic with advanced PG methods
  Will learn DDPG, TRPO, SAC