UNIFORM CONVERGENCE

Analysis often features situations where convergence of function sequences (and series) is needed.
The most straightforward notion is that of pointwise convergence.

Def

A sequence $(f_n)_{n\in\mathbb{N}}$ of complex-valued functions $f_n : A \longrightarrow \mathbb{C}$ defined on a common domain A converges pointwise to a function $f : A \to \mathbb{C}$ if for every $z \in A$ we have

$$\lim_{n\to\infty} f_n(z) = f(z).$$

Pointwise convergence does not, however, behave well in relation to notions needed in analysis, such as continuity or integration. A much better behaved notion is uniform convergence.

Def [uniform_limit]

A sequence $(f_n)_{n\in\mathbb{N}}$ of complex-valued functions $f_n : A \to \mathbb{C}$ on a common domain A converges uniformly on A to a function $f : A \to \mathbb{C}$ if for any $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$ and all $z \in A$ we have $|f_n(z) - f(z)| < \varepsilon$.

As the first easy observation, we record:

Lemma (Uniform convergence implies pointwise conv.)

If $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, then it converges also pointwise to f.

Proof Easy exercise. $\square$

Let us then state some crucial well-behavedness properties of uniform convergence. (... which do not hold for pointwise convergence in general.)

Lemma (Preservation of continuity)

[uniform_limit_continuous]

Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence of continuous functions $f_n : A \to \mathbb{C}$ which converges uniformly to a function $f : A \to \mathbb{C}$. Then also the limit function f is continuous.

Proof See MS-C1541 Metric Spaces. $\square$

(The proof is a typical $\frac{\varepsilon}{3}$ argument.)

Lemma (Integration and uniform limits)

[uniform_limit_contour_integral]

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n : A \to \mathbb{C}$ defined on $A \subset \mathbb{C}$, and let γ be a contour in A.
If the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on A, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma \left(\lim_{n\to\infty} f_n(z)\right) dz .$$

Proof Note first of all that from the uniform convergence assumption and continuity of each f_n it follows that $f = \lim_{n\to\infty} f_n$ is a continuous function, so that the right hand side of the asserted equation is at least well-defined.

Let us then prove the limit assertion.

Let $\varepsilon > 0$. Denote by $\ell(\gamma)$ the length of the contour γ. Since $\frac{\varepsilon}{\ell(\gamma)+1} > 0$, by uniform convergence there exists an N such that $|f_n(z) - f(z)| < \frac{\varepsilon}{\ell(\gamma)+1}$ for all $n \geq N$ and $z \in A$. Then by the triangle inequality for contour integrals, for $n \geq N$

$$\left| \int_\gamma f_n(z)\, dz - \int_\gamma f(z)\, dz \right|$$

$$= \left| \int_\gamma \left(f_n(z) - f(z)\right) dz \right|$$

$$\leq \int_\gamma \underbrace{|f_n(z) - f(z)|}_{< \frac{\varepsilon}{\ell(\gamma)+1}}\, |dz| \leq \frac{\varepsilon}{\ell(\gamma)+1} \cdot \ell(\gamma) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves

$$\lim_{n \to \infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz.$$

$\square$

Combining this with Cauchy's integral formula will easily give the preservation of analyticity in uniform limits!

We can in fact do even slightly better — it suffices to assume less than uniform convergence on the whole domain of definition of the analytic functions.

[uoc_limit]

<u>Def</u> (Convergence uniformly on compacts)

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : A \to \mathbb{C}$ defined on $A \subset \mathbb{C}$ and let also $f : A \to \mathbb{C}$ be a function. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compacts (UOC) to f if for every compact subset $K \subset A$ the restrictions $f_n|_K : K \to \mathbb{C}$ converge uniformly to $f|_K : K \to \mathbb{C}$. We then denote $f_n \xrightarrow{\text{UOC}} f$.

Theorem (Preservation of analyticity)

[uoc_limit_analytic]

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of analytic functions $f_n : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$. If the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compacts to $f : U \to \mathbb{C}$, then also the limit function f is analytic.

Proof Suppose $f_n \xrightarrow{\text{uoc}} f$.

Let $z_0 \in U$. Choose $r > 0$ small enough so that $\overline{B}(z_0; r) \subset U$.

Then for any $z \in B(z_0; r)$, we find

$$f(z) \overset{\circledast}{=} \lim_{n\to\infty} f_n(z)$$

$\circledast$ since UOC convergence implies pointwise conv.

$$\overset{\circledast}{=} \lim_{n\to\infty} \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f_n(\xi)}{\xi - z} \, d\xi$$

$\circledast$ Cauchy's integral formula

$$\overset{\circledast}{=} \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f(\xi)}{\xi - z} \, d\xi$$

$\circledast$ since $\partial B(z_0; r) \subset U$ is compact and the integrand converges uniformly on it.

This representation of f in $B(z_0;r)$ is of the form that an earier lemma shows f is analytic inside $B(z_0;r)$ — in particular the derivative $f'(z_0) = \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f(\xi)}{(\xi - z_0)^2} d\xi$ exists. Since $z_0 \in U$ was arbitrary, we have proved analyticity of f. $\square$

From the proof, we actually easily obtain even a stronger conclusion.

<u>Corollary</u>

[uoc_limit_analytic]

If $(f_n)_{n\in\mathbb{N}}$ is a sequence of analytic functions $f_n : U \to \mathbb{C}$ which converges UOC to $f : U \to \mathbb{C}$ then also the derivatives f_n' converge UOC to f'.

<u>Proof</u>: To prove UOC convergence of the derivatives $f_n' : U \to \mathbb{C}$, it sufices to prove that on any closed disk $\overline{B}(z_0;\rho) \subset U$ the derivatives $(f_n')_{n\in\mathbb{N}}$ converge uniformly to f' — because any compact $K \subset U$ is contained in a finite union of such closed disks.

So fix such a disk $\overline{B}(z_0;\rho) \subset U$ and observe that we may in fact choose a slightly larger radius $r > \rho$ such that still $\overline{B}(z_0;r) \subset U$.

Then Cauchy's integral formula for derivatives yields

$$f_n'(z) = \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f_n(\xi)}{(\xi - z)^2}\, d\xi$$

and

$$f'(z) = \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f(\xi)}{(\xi - z)^2}\, d\xi$$

for any $z \in \overline{B}(z_0;\rho) \subset B(z_0;r)$.

Let $\varepsilon > 0$, and note that by UOC convergence $f_n \xrightarrow{\text{UOC}} f$ and the compactness of $\partial B(z_0; r)$, we can choose $N \in \mathbb{N}$ such that for $n \geq N$ and $\xi \in \partial B(z_0; r)$ we have $|f_n(\xi) - f(\xi)| < \frac{\varepsilon \cdot (r-\rho)^2}{2r}$ Then for $z \in \overline{B}(z_0; \rho)$ we have $|\xi - z| \geq r - \rho$ in

$$
\begin{aligned}
&|f_n'(z) - f'(z)| \\
&= \left| \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f_n(\xi)}{(\xi - z)^2} d\xi - \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f(\xi)}{(\xi - z)^2} d\xi \right| \\
&= \left| \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^2} d\xi \right| \\
&\leq \frac{1}{2\pi} \int_{\partial B(z_0;r)} \underbrace{\frac{|f_n(\xi) - f(\xi)|}{|\xi - z|^2}}_{\leq \frac{\varepsilon \cdot (r-\rho)^2}{2r} \cdot \frac{1}{(r-\rho)^2}} |d\xi| \\
&\leq \frac{1}{2\pi} \cdot \frac{\varepsilon}{2r} \cdot \underbrace{\ell(\partial B(z_0;r))}_{= 2\pi r} = \frac{\varepsilon}{2} < \varepsilon .
\end{aligned}
$$

Uniform convergence $f_n' \to f$ on $\overline{B}(z_0;\rho)$ follows. □

SERIES

As usual, limits of sequences allow us to define sums of (infinite) series, by considering the sequences of partial sums.

Def:

[complex_series]

Let $z_1, z_2, z_3, \ldots \in \mathbb{C}$ be complex numbers. For $N \in \mathbb{N}$, define the N^{th} partial sum of these to be

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N .$$

If the sequence $(S_N)_{N\in\mathbb{N}}$ of partial sums has a limit, then we use this limit as the definition of the sum of the infinite series

$$\sum_{n=1}^{\infty} z_n = \lim_{N\to\infty} \sum_{n=1}^{N} z_n$$

(If the indexing starts, e.g., from $n=0$, then $\sum_{n=0}^{\infty}$ is defined with obvious modifications.)

With series, there is a sufficient condition for convergence which only looks at the magnitudes of the terms involved.

[absolute_convergence]

Def: (Absolute convergence of a complex series)

A complex series $\sum_{n=1}^{\infty} z_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |z_n|$ of the absolute values of its terms converges.

[absolute_convergence_implies_convergence]

Lemma (Absolute convergence implies convergence)

If a series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then it is convergent.

Sketch of proof:

The key here is the completeness of $\mathbb{C}$ as a metric space, which says that every Cauchy sequence in $\mathbb{C}$ converges.

(The metric structure of $\mathbb{C}$ is the same as of $\mathbb{R}^2$, so this is a special case of the completeness of the Euclidean spaces $\mathbb{R}^d$. See MS-C1541.)

It suffices to verify that the sequence $(S_N)_{N \in \mathbb{N}}$ of partial sums $S_N = z_1 + \cdots + z_N$ is Cauchy, which means that for every $\varepsilon > 0$ there exists an N_ε such that for $k, l \geq N_\varepsilon$ we have $|S_k - S_l| < \varepsilon$.

Verifying this is quite straightforward — a similar analysis appears in the proof of Weierstrass' M-test below, so we omit the details here.

$\square$

There is also a very straightforward necessary condition for convergence of series: at least the terms must be tending to zero.

__Lemma__

[terms_tend_to_zero]

If a complex series $\sum_{n=1}^{\infty} z_n$ converges, then
$$\lim_{n \to \infty} z_n = 0.$$

__Proof__ Suppose $\sum_{n=1}^{\infty} z_n$ converges, i.e., the limit $s = \lim_{N \to \infty} S_N$ of the partial sums $S_N = z_1 + \dots + z_N$ exists.

Let $\varepsilon > 0$. Since $\lim_{N \to \infty} S_N = s$, there exists a $N' \in \mathbb{N}$ such that for $N \geq N'$ we have $|S_N - s| < \frac{\varepsilon}{2}$. In particular since $z_n = S_N - S_{N-1}$, for $n > N'$ we have
$$|z_n| = |S_n - S_{n-1}| = |S_n - s - (S_{n-1} - s)|$$
$$\leq |S_n - s| + |S_{n-1} - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
This shows $\lim_{n \to \infty} z_n = 0$ as claimed.

□

Function series

The various notions of convergence of function series are also straightforwardly defined in terms of the sequence of partial sum functions.

Def

[function_series]

Let $(f_n)_{n\in\mathbb{N}}$ be functions $f_n : A \to \mathbb{C}$ defined on $A \subset \mathbb{C}$. For $N \in \mathbb{N}$, define $F_N : A \to \mathbb{C}$ by

$$F_N(z) = \sum_{n=1}^{N} f_n(z) = f_1(z) + f_2(z) + \dots + f_N(z).$$

We say that the function series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, uniformly on compacts of A, or pointwise, if the sequence $(F_N)_{N\in\mathbb{N}}$ of partial sum functions converges in the corresponding sense.

An often convenient way to verify uniform convergence of a function series is

[weierstrass_test]

Lemma (Weierstrass' M-test)

Suppose that $(f_n)_{n\in\mathbb{N}}$ are functions $f_n : A \longrightarrow \mathbb{C}$ and $M_1, M_2, M_3, \ldots \geq 0$ are constants. If $\sum_{n=1}^{\infty} M_n < \infty$ and $|f_n(z)| \leq M_n$ for all $n \in \mathbb{N}$ and all $z \in A$, then the series

$$\sum_{n=1}^{\infty} f_n(z)$$ converges uniformly for $z \in A$.

Proof: By Cauchy's criterion, it suffices to prove that the sequence $(F_N)_{N\in\mathbb{N}}$ of partial sum functions $F_N(z) = \sum_{n=1}^{N} f_n(z)$ is uniformly Cauchy on A.

So let $\varepsilon > 0$.

Since $\sum_{n=1}^{\infty} M_n$ is a convergent series, its remainders tend to zero:

$$\sum_{n=N+1}^{\infty} M_n \xrightarrow[N\to\infty]{} 0 .$$

We may therefore choose an N such that $\sum_{n=N+1}^{\infty} M_n < \varepsilon$.

Then for any $k, l \geq N$ $(k \geq l)$ and any $z \in A$ we have

$$|F_k(z) - F_l(z)| = \left| \sum_{n=1}^{k} f_n(z) - \sum_{n=1}^{l} f_n(z) \right|$$

$$= \left| \sum_{n=l+1}^{k} f_n(z) \right| \leq \sum_{n=l+1}^{k} |f_n(z)|$$

$$\leq \sum_{n=l+1}^{k} M_n \leq \sum_{n=N+1}^{\infty} M_n < \varepsilon .$$

This shows that $(F_N)_{N\in\mathbb{N}}$ is a uniform Cauchy sequence, and uniform convergence of the series $\sum_{n=1}^{\infty} f_n$ on A follows. $\square$

Example (The Riemann zeta function)

Recall from calculus that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, whereas for any $\varepsilon > 0$ the series $\sum_{n=1}^{\infty} n^{-1-\varepsilon}$ converges.

Consider then the complex series

$$\sum_{n=1}^{\infty} n^{-z}$$

where $z \in \mathbb{C}$ and we use the principal complex powers (writing $\begin{matrix} x = \mathrm{Re}(z) \\ y = \mathrm{Im}(z) \end{matrix}$)

$$n^{-z} = \exp(-z \cdot \underbrace{\mathrm{Log}(n)}_{= \log(n)}) \quad \text{"real log"}$$

$$= \exp(-x \cdot \log(n) - iy \cdot \log(n))$$

$$= n^{-x} \cdot e^{-iy \cdot \log(n)}.$$

Observe in particular that

$$|n^{-z}| = n^{-\mathrm{Re}(z)}.$$

For any $\varepsilon > 0$, Weierstrass' M-test can therefore be applied in

$$\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 1+\varepsilon\}$$

since if $\mathrm{Re}(z) \geq 1+\varepsilon$, we have

$$|n^{-z}| = n^{-\mathrm{Re}(z)} \leq n^{-1-\varepsilon}$$

and $\sum_{n=1}^{\infty} n^{-1-\varepsilon}$ converges.

We find that $\boxed{\zeta(z) = \sum_{n=1}^{\infty} n^{-z}}$

converges uniformly on $\{\mathrm{Re}(z) \geq 1+\varepsilon\}$, and it easily follows that it converges uniformly on compacts of

$$\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 1\}.$$

Since all terms $z \mapsto n^{-z} = e^{-z \cdot \mathrm{Log}(z)}$ are analytic on this domain, the function ζ is analytic there as well (preservation of analyticity).

It turns out that ζ can be uniquely extended to an analytic function $\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C}$.

"the Riemann zeta function"

POWER SERIES

The most important class of function series in complex analysis is power series.

Def

[power_series]

Let $z_0 \in \mathbb{C}$ be a point in the complex plane, and let $a_0, a_1, a_2, \dots \in \mathbb{C}$ be coefficients. A function series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

(seen as a function of z) is called a power series developed around z_0.

Example

Polynomial functions $z \longmapsto \sum_{n=0}^{d} a_n z^n$ can be seen as power series where only finitely many of the coefficients are $\neq 0$.

<u>Example</u> (Geometric series)

The geometric series with ratio $z \in \mathbb{C}$ is the power series

$$\sum_{n=0}^{\infty} z^n .$$

(developed around $z_0 = 0$ and all coefficients $= 1$)

The N^{th} partial sum is (exercise)

$$S_N = \sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z} \quad \text{if } z \neq 1.$$

If $|z| < 1$, then $|z^{N+1}| = |z|^{N+1} \xrightarrow[N \to \infty]{} 0$ and we find that the series converges to

$$\sum_{n=0}^{\infty} z^n = \lim_{N \to \infty} \frac{1 - z^{N+1}}{1 - z} = \frac{1}{1 - z} .$$

If $|z| \geq 1$ then $|z^n| = |z|^n \geq 1$, so the general term of the series does not tend to zero, and the series cannot converge.

We also note that if $0 < r < 1$, then for $z \in \overline{B}(0;r)$ we have $|z^n| = |z|^n \leq r^n$, and since $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} < \infty$, Weierstrass' M-test gives that the geometric series $\sum_{n=1}^{\infty} z^n$ converges uniformly in $\overline{B}(0;r)$. It easily follows that $\sum_{n=0}^{\infty} z^n$ convergen uniformly on compact subsets of the unit disk $\mathbb{D} = B(0;1)$. Preservation of analyticity would therefore imply that the geometric series defines an analytic function in $\mathbb{D}$, but we of course saw this already, since

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad (\text{for } |z|<1)$$

is clearly analytic as a rational function.

A key observation about power series is:

Lemma (Abel's theorem)

[abel]

Suppose that a power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

converges at $z = w \in \mathbb{C}$. Then it also converges absolutely at any $z \in \mathbb{C}$ such that $|z-z_0| < |w-z_0|$.

Before the proof, let us note the contrapositive.

Corollary (Contrapositive of Abel's theorem)

[abel_contrapositive]

Suppose that a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ does not converge at $z = w$. Then it also does not converge at any $z \in \mathbb{C}$ such that $|z-z_0| > |w-z_0|$.

Proof: Convergence at such a z would imply convergence at w by the lemma above. $\square$

__Proof of Abel's theorem:__

Suppose $\sum_{n=0}^{\infty} a_n (w-z_0)^n$ converges.

In particular, the terms tend to zero,

$$\lim_{n\to\infty} |a_n \cdot (w-z_0)^n| = 0.$$

It follows in particular that there exists an $N \in \mathbb{N}$ such that for $n \geq N$ we have $|a_n (w-z_0)^n| < 1$, i.e. $|a_n| < |w-z_0|^{-n}$.

Let z be such that $|z-z_0| < |w-z_0|$.

For the tail $\sum_{n=N}^{\infty} a_n \cdot (z-z_0)^n$ of the power series evaluated at z, we can bound the terms by

$$|a_n \cdot (z-z_0)^n| = \underbrace{|a_n|}_{\leq |w-z_0|^{-n}} \cdot |z-z_0|^n \leq \Bigg(\underbrace{\frac{|z-z_0|}{|w-w_0|}}_{<1}\Bigg)^n.$$

This gives a convergent geometric series as a majorant in Weierstrass' M-test, proving absolute convergence of the tail and therefore of the whole series. $\square$

The observations based on Abel's theorem motivate the following definition.

<u>Def</u>

[radius-of-convergence]

The **radius of convergence** of a power series $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ is

$$R := \sup \left\{ |z-z_0| \;\middle|\; \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ converges} \right\}.$$

Namely, from Abel's theorem and its contrapositive we get that the power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

- converges for any $z \in \mathbb{C}$ such that $|z-z_0| < R$;
- does not converge for any $z \in \mathbb{C}$ such that $|z-z_0| > R$.

(- no conclusions are directly obtained for z such that $|z-z_0| = R$.)

We call $B(z_0; R)$ the disk of convergence of the power series (with the interpretation $B(z_0;R) = \mathbb{C}$ if $R = +\infty$): the convergence is guaranteed in this disk of convergence.

In fact a closer look at the proof of Abel's theorem gives a strengthened conclusion:

<u>Lemma</u>

Let R be the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$. Then for any $r<R$, this power series converges uniformly in $B(z_0;r) \subset B(z_0;R)$.

(We invite the reader to adapt the previous proof to get this stronger version.)

It easily follows that a power series converges uniformly on compacts in its disk of convergence. We get

[power_series_analytic]

<u>Corollary</u>

A power series defines an analytic function in its disk of convergence.

<u>Proof</u> Analyticity is preserved in UOC limits, and the partial sums of a power series are polynomials, and therefore indeed analytic. $\square$

We also get that a power series can be differentiated term by term.

Lemma

[power-series-analytic]

Consider a function f defined by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

in its disk of convergence $B(z_0;R)$.
Then its derivative is given by the termwise differentiated power series

$$f'(z) = \sum_{n=1}^{\infty} a_n \cdot n \cdot (z-z_0)^{n-1}$$

in the disk $B(z_0;R)$.

Proof: We saw that UOC convergence of analytic functions implies UOC convergence of the derivatives, and for the partial sums (polynomials), differentiation obviously gives
$$\frac{d}{dz}\left(\sum_{n=0}^{N} a_n (z-z_0)^n\right) = \sum_{n=1}^{N} a_n \cdot n \cdot (z-z_0)^{n-1}.$$
$\square$

Finally, we record a general formula for determining the radius of convergence of a power series.

Theorem (Hadamard's formula)

[hadamard_formula]

The radius of convergence of a power series

$$\sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n$$

is given by the formula

$$R = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{|a_n|}}$$

where we interpret $\frac{1}{+\infty} = 0$ and $\frac{1}{0} = +\infty$.