2. COMPLEX DERIVATIVES AND ANALYTIC FUNCTIONS

We now turn to the fundamental notion of differentiability of complex-valued functions of a complex variable.

It is surprising that while the definition

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

of complex derivatives as limits of difference quotients appears identical to the definition in real-variable calculus, the consequences of complex differentiability are far stronger.

To clearly appreciate the differences, we start by recalling the definition and properties of differentiability in the real sense.

Component functions

We identify the set $\mathbb{C}$ of complex numbers with the 2-dimensional plane $\mathbb{R}^2$ as usual

$$\mathbb{R}^2 \ni (x,y) \longleftrightarrow x+iy \in \mathbb{C}.$$

A complex-valued function

$$f : A \longrightarrow \mathbb{C}$$

defined on a subset $A \subset \mathbb{C}$ of the complex plane then corresponds to a two-component (so vector-valued) function of two real variables:

$$f(x+iy) = u(x,y) + i \cdot v(x,y)$$

where $u,v : A \to \mathbb{R}$ are the real-valued component functions,

$$u(x,y) := \mathrm{Re}\big(f(x+iy)\big)$$
$$v(x,y) := \mathrm{Im}\big(f(x+iy)\big).$$

Let us therefore recall notions of calculus in $\underbrace{\text{several}}_{\text{(two, in the case at hand)}}$ real variables.

Differentiability in the real sense

Recall :

Def :

[differential]

Let $u : A \to \mathbb{R}$ be a function defined on a subset $A \subset \mathbb{R}^2$ of the plane, and let $(x_0, y_0) \in A$ be a point in that subset.

A linear map $L : \mathbb{R}^2 \to \mathbb{R}$ is called a differential of u at (x_0, y_0) if we can write

linear approximation of u locally

$$u(x_0 + \xi, y_0 + \eta) = u(x_0, y_0) + L(\xi, \eta) + E(\xi, \eta)$$

where the error term E is small in the sense that

$$\lim_{(\xi,\eta) \to (0,0)} \frac{|E(\xi, \eta)|}{\|(\xi, \eta)\|} = 0 .$$

(Euclidean norm of a vector: $\|(\xi, \eta)\| := \sqrt{\xi^2 + \eta^2}$)

It is not difficult to check that if (x_0, y_0) is an interior point of A, then L is uniquely determined in the above. We then call it the differential of u at (x_0, y_0) and denote it by

$$du(x_0, y_0) = L : \mathbb{R}^2 \to \mathbb{R} .$$

Recall: Linearity of $L: \mathbb{R}^2 \to \mathbb{R}^m$ means that

$$L(\xi_1+\xi_2,\ \eta_1+\eta_2) = L(\xi_1,\eta_1) + L(\xi_2,\eta_2)$$
$$\forall\ (\xi_1,\eta_1),\ (\xi_2,\eta_2) \in \mathbb{R}^2$$

and

$$L(\lambda\xi,\ \lambda\eta) = \lambda \cdot L(\xi,\eta)$$
$$\forall (\xi,\eta)\in\mathbb{R}^2,\ \lambda\in\mathbb{R}.$$

Fixing choices of bases, linear maps ⊛ can be represented by matrices. (between finite-dim. vector spaces)

Choosing the standard basis of $\mathbb{R}^2$

$\left(e_1 = \begin{bmatrix}1\\0\end{bmatrix},\ e_2 = \begin{bmatrix}0\\1\end{bmatrix}\right.$ so that $\left.\begin{bmatrix}\xi\\\eta\end{bmatrix} = \xi e_1 + \eta e_2\right)$

and considering limits $(\xi,\eta)\to(0,0)$ along coordinate axes $\left(\begin{bmatrix}\xi\\0\end{bmatrix} \xrightarrow[\xi\to 0]{} \begin{bmatrix}0\\0\end{bmatrix},\ \begin{bmatrix}0\\\eta\end{bmatrix} \xrightarrow[\eta\to 0]{} \begin{bmatrix}0\\0\end{bmatrix}\right)$

one finds the matrix of the differential expressed in terms of partial derivatives:

Lemma [jacobian-matrix]

If $u: A\to\mathbb{R}$ has differential L at an interior point (x_0,y_0) of $A\subset\mathbb{R}^2$, then u has both partial derivatives $\frac{\partial u}{\partial x}(x_0,y_0)$ and $\frac{\partial u}{\partial y}(x_0,y_0)$ at that point and

$$L = du(x_0,y_0) \longleftrightarrow \left[\frac{\partial u}{\partial x}(x_0,y_0)\quad \frac{\partial u}{\partial y}(x_0,y_0)\right] \in \mathbb{R}^{1\times 2}.$$

(matrix rep. in standard basis)

Example

The function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$u(x,y) = x^2 + y^2$$

has, at a point $(x_0, y_0) \in \mathbb{R}^2$, the differential

$$L(\xi, \eta) = 2x_0 \xi + 2y_0 \eta$$

since

$$E(\xi,\eta) = u(x_0+\xi, y_0+\eta) - u(x_0,y_0) - L(\xi,\eta)$$

$$= \underbrace{(x_0+\xi)^2}_{= x_0^2 + 2x_0\xi + \xi^2} + \underbrace{(y_0+\eta)^2}_{= y_0^2 + 2y_0\eta + \eta^2} - x_0^2 - y_0^2 - 2x_0\xi - 2y_0\eta$$

$$= \xi^2 + \eta^2$$

satisfies

$$\lim_{(\xi,\eta)\to(0,0)} \frac{E(\xi,\eta)}{\|(\xi,\eta)\|} = \lim_{(\xi,\eta)\to(0,0)} \frac{\xi^2+\eta^2}{\sqrt{\xi^2+\eta^2}} = 0 .$$

And indeed the matrix of L (in std basis)

$$\begin{bmatrix} 2x_0 & 2y_0 \end{bmatrix}$$

has components given by the partial derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2) = 2x , \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2) = 2y$$

at (x_0, y_0).

Let us record an easy but important observation:

Lemma

differentiable - implies - continuous

If $u: A \to \mathbb{R}$ defined on $A \subset \mathbb{R}^2$ is differentiable at $(x_0, y_0) \in A$, then it is also continuous at (x_0, y_0)

Proof: For any linear map $L: \mathbb{R}^2 \to \mathbb{R}$ we have

$$\lim_{(\xi,\eta)\to(0,0)} L(\xi,\eta) = 0$$

and for the error term E we also have

$$\lim_{(\xi,\eta)\to(0,0)} E(\xi,\eta) = 0$$

(Indeed, from $\lim_{(\xi,\eta)\to(0,0)} \frac{|E(\xi,\eta)|}{\|(\xi,\eta)\|} = 0$ it follows that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|E(\xi,\eta)| < \varepsilon \cdot \|(\xi,\eta)\|$ when $\|(\xi,\eta)\| < \delta$.)

Using the linearity of limits, we therefore get

$$\lim_{(\xi,\eta)\to(0,0)} u(x_0+\xi, y_0+\eta)$$

$$= \lim_{(\xi,\eta)\to(0,0)} \Big(u(x_0,y_0) + L(\xi,\eta) + E(\xi,\eta)\Big)$$

$$= u(x_0,y_0) + 0 + 0 = u(x_0,y_0),$$

showing that u is continuous at (x_0,y_0). $\square$

Differentiability of vector-valued functions

$$f : A \to \mathbb{R}^2$$

simply amounts to the differentiability of each of the component functions. Specifically, if

$$f(x,y) = (u(x,y),\ v(x,y))$$

then f has a differential at $(x_0,y_0) \in A$ if

$$u(x_0+\xi, y_0+\eta) = u(x_0,y_0) + L^{(u)}(\xi,\eta) + E^{(u)}(\xi,\eta),$$
$$v(x_0+\xi, y_0+\eta) = v(x_0,y_0) + L^{(v)}(\xi,\eta) + E^{(v)}(\xi,\eta),$$

where

$$\lim_{(\xi,\eta)\to(0,0)} \frac{|E^{(u)}(\xi,\eta)|}{\|(\xi,\eta)\|} = 0, \quad \lim_{(\xi,\eta)\to(0,0)} \frac{|E^{(v)}(\xi,\eta)|}{\|(\xi,\eta)\|} = 0.$$

The clearest form is obtained by collecting the differentials and error terms in vectors as

$$L(\xi,\eta) = (L^{(u)}(\xi,\eta),\ L^{(v)}(\xi,\eta)) \in \mathbb{R}^2$$
$$E(\xi,\eta) = (E^{(u)}(\xi,\eta),\ L^{(v)}(\xi,\eta)) \in \mathbb{R}^2,$$

so that $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map, and the local linear approximation reads

$$f(x_0+\xi, y_0+\eta) = f(x_0,y_0) + L(\xi,\eta) + E(\xi,\eta)$$

with the error term satisfying

$$\lim_{(\xi,\eta)\to(0,0)} \frac{\|E(\xi,\eta)\|}{\|(\xi,\eta)\|} = 0 .$$

The matrix representation of the differential

$$df(x_0,y_0) = L$$

is again expressible in terms of the partial derivatives (of the component functions):

Lemma

[jacobian - matrix]

If $f: A \to \mathbb{R}^2$ is written in terms of its component functions $u, v : A \to \mathbb{R}$, and f has a differential $L: \mathbb{R}^2 \to \mathbb{R}^2$ at an interior point (x_0,y_0) of A, then both partial derivatives of both component functions exist at (x_0,y_0) and the matrix of the differential reads

$$L = df(x_0,y_0) \longleftrightarrow \begin{bmatrix} \frac{\partial u}{\partial x}(x_0,y_0) & \frac{\partial u}{\partial y}(x_0,y_0) \\ \frac{\partial v}{\partial x}(x_0,y_0) & \frac{\partial v}{\partial y}(x_0,y_0) \end{bmatrix} \in \mathbb{R}^{2\times 2} .$$

(matrix in standard basis of $\mathbb{R}^2$)

Example

The function $z \mapsto \bar{z}$ $(\mathbb{C} \to \mathbb{C})$ corresponds to the vector-valued function

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$f(x,y) = (x, -y)$$

of two real variables, whose component functions are given by

$$u(x,y) = x \qquad v(x,y) = -y.$$

Clearly this f is differentiable in the real sense at any $(x_0, y_0) \in \mathbb{R}^2$, and its differential (at any point) is the linear map determined by the matrix

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(Just note that the existence of partial derivatives is not sufficient for differentiability — but once one writes things out, the error term in this example is identically zero, so differentiability has been justified, "clearly".)

i.e. $L(\xi, \eta) = (\xi, -\eta) \quad \forall (\xi, \eta) \in \mathbb{R}^2.$

Complex derivatives

Let us then turn our attention to derivatives in the complex sense.

Def

[complex_derivative]

A complex-valued function $f : A \to \mathbb{C}$ defined on a subset $A \subset \mathbb{C}$ of the complex plane has (complex) derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at an interior point z_0 of A if the limit above exists in $\mathbb{C}$.

Remark:

Another equivalent way to write the derivative is obtained by setting $h = z - z_0$ above:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

The limit that defines the complex derivative must exist when h tends to 0 from any direction in the complex plane.

Example

Consider the function $f : \mathbb{C} \to \mathbb{C}$
$f(z) = z^2$.

It has a derivative at any $z_0 \in \mathbb{C}$:

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{(z_0+h)^2 - z_0^2}{h}$$

$$= \frac{\cancel{z_0^2} + 2h \cdot z_0 + h^2 - \cancel{z_0^2}}{h}$$

$$= 2z_0 + h \xrightarrow[h \to 0]{} 2z_0 .$$

So $f'(z_0) = 2z_0$.

Example

Consider the function $f: \mathbb{C} \longrightarrow \mathbb{C}$
$f(z) = \bar{z}$.

This function does not have a (complex) derivative at any point $z_0 \in \mathbb{C}$:

Approaching $z_0 = x_0 + iy_0$ $(x_0, y_0 \in \mathbb{R})$ from a "real" direction $(h = \xi \in \mathbb{R})$ we find a limit of difference quotients

$$\frac{f(z_0+\xi) - f(z_0)}{\xi} = \frac{(x_0 + \xi - iy_0) - (x_0 - iy_0)}{\xi}$$

$$= \frac{\xi}{\xi} = 1 \xrightarrow[\xi \to 0]{} 1$$

whereas from an "imaginary" direction $(h = i\eta,\ \eta \in \mathbb{R})$ we find

$$\frac{f(z_0+i\eta) - f(z_0)}{i\eta} = \frac{(x_0 - i(y_0+\eta)) - (x_0 - iy_0)}{i\eta}$$

$$= \frac{-i\eta}{i\eta} = -1 \xrightarrow[\eta \to 0]{} -1.$$

The different directional limits show that $\lim\limits_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ does not exist.

The previous example shows that differentiability in the real sense is not sufficient to guarantee the existence of a complex derivative.
($z \mapsto \bar{z}$ is differentiable in the real sense but does not have a complex derivative anywhere)

The following shows that differentiability in the real sense is, however, a necessary condition for the existence of a complex derivative.

Cauchy-Riemann equations

Theorem

[complex_derivative_implies_differentiable]

If a function $f : A \to \mathbb{C}$ defined on $A \subset \mathbb{C}$ has a complex derivative $f'(z_0)$ at an interior point $z_0 = x_0 + i y_0$ then f viewed as a vector-valued function of two real variables is differentiable at (x_0, y_0) and the partial derivatives of its component functions $u(x,y) = \mathrm{Re}(f(x+iy))$ and $v(x,y) = \mathrm{Im}(f(x+iy))$ satisfy the Cauchy-Riemann equations

$$\boxed{\begin{cases} \dfrac{\partial u}{\partial x}(x_0,y_0) = \dfrac{\partial v}{\partial y}(x_0,y_0) \\[2ex] \dfrac{\partial v}{\partial x}(x_0,y_0) = -\dfrac{\partial u}{\partial y}(x_0,y_0) \end{cases}}$$

Proof: Suppose that

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists. Write $f'(z_0) = a + ib$ with $a, b \in \mathbb{R}$. Also write $h = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}$ and consider the error term

$$E(\xi, \eta) = f(z_0 + \xi + i\eta) - f(z_0) - (\xi + i\eta) f'(z_0).$$

We then find

$$\lim_{(\xi,\eta) \to (0,0)} \frac{\| E(\xi,\eta) \|}{\| (\xi,\eta) \|}$$

$$= \lim_{(\xi,\eta) \to (0,0)} \frac{| f(z_0 + \xi + i\eta) - f(z_0) - (\xi + i\eta) f'(z_0) |}{| \xi + i\eta |}$$

$$= \lim_{(\xi,\eta) \to (0,0)} \left| \frac{f(z_0 + \xi + i\eta) - f(z_0)}{\xi + i\eta} - f'(z_0) \right| = 0$$

using the existence of the complex derivative.

this pair corresponds to the complex number $(\xi + i\eta) \cdot f'(z_0) = (\xi + i\eta)(a + ib)$

The mapping

$$(\xi, \eta) \longmapsto L(\xi, \eta) = (a\xi - b\eta, \; a\eta + b\xi)$$

is real-linear and from the above we see that it is the differential $df(x_0, y_0)$.

In matrix form, this differential L reads

$$L \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

So comparing with the matrix of partial derivatives

$$\begin{bmatrix} \frac{\partial u}{\partial x}(x_0,y_0) & \frac{\partial u}{\partial y}(x_0,y_0) \\ \frac{\partial v}{\partial x}(x_0,y_0) & \frac{\partial v}{\partial y}(x_0,y_0) \end{bmatrix}$$

(which by a previous lemma also represents $L = df(x_0,y_0)$) we find

$$\frac{\partial u}{\partial x}(x_0,y_0) = a = \frac{\partial v}{\partial y}(x_0,y_0)$$

$$\frac{\partial v}{\partial x}(x_0,y_0) = b = -\frac{\partial u}{\partial y}(x_0,y_0),$$

which are the asserted Cauchy-Riemann equations.

$\square$

From the proof we observe:

If the complex derivative $f'(z_0)$ exists, then the differential

$$df(x_0,y_0): \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is not merely $\mathbb{R}$-linear but also a $\mathbb{C}$-linear map

$$\xi + i\eta \longmapsto (a+ib)(\xi+i\eta)$$

once we identify the plane $\mathbb{R}^2$ with $\mathbb{C}$ as usual:

$$\mathbb{R}^2 \ni (\xi,\eta) \longleftrightarrow \xi + i\eta \in \mathbb{C}.$$

The Cauchy-Riemann equations are exactly the additional requirement to promote the $\mathbb{R}$-linearity of the differential to $\mathbb{C}$-linearity.

Geometric interpretation:

Write the complex derivative in polar form, $f'(z_0) = \rho \cdot e^{i\phi}$, where $\rho = |f'(z_0)| \geq 0$, $\phi = \arg(f'(z_0))$ (mod 2π).

Then the differential $df(z_0)$ is the $\mathbb{C}$-linear map $\xi + i\eta \longmapsto \rho \cdot e^{i\phi} \cdot (\xi + i\eta)$ which performs a rotation by angle ϕ and dilatation by factor ρ.

Example

We saw that $f(z) = z^2$ defines a function $f : \mathbb{C} \to \mathbb{C}$ which has a complex derivative $f'(z_0) = 2z_0$ at $z_0 \in \mathbb{C}$.

Writing

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy$$

we find that the component functions of f are given by

$$u(x,y) = x^2 - y^2 \,, \qquad v(x,y) = 2xy.$$

Now also a direct calculation of the partial derivatives

$$\frac{\partial u}{\partial x} = 2x \qquad\qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial v}{\partial x} = 2y \qquad\qquad -\frac{\partial u}{\partial y} = 2y$$

shows that the Cauchy-Riemann equations hold for f everywhere.

The previous result showed that the existence of a complex derivative implied differentiability and Cauchy-Riemann equations. Also the converse holds:

Theorem

If $f : A \to \mathbb{C}$ defined on $A \subset \mathbb{C}$ is differentiable at an interior point (x_0, y_0) and if the Cauchy-Riemann equations hold at (x_0, y_0) for its component functions

$$u(x,y) = \mathrm{Re}(f(x+iy)), \quad v(x,y) = \mathrm{Im}(f(x+iy))$$

then f has a complex derivative at $z_0 = x_0 + i y_0$ given by any of the following expressions:

$$\begin{aligned} f'(z_0) &= \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0) \\ &= \frac{\partial v}{\partial y}(x_0,y_0) - i \frac{\partial u}{\partial y}(x_0,y_0) \\ &= \frac{\partial u}{\partial x}(x_0,y_0) - i \frac{\partial u}{\partial y}(x_0,y_0) \\ &= \frac{\partial v}{\partial y}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0). \end{aligned}$$

Sketch of proof:

The equality of the 4 different expressions follows directly from the Cauchy-Riemann equations.
The proof is otherwise very similar to the converse proven before. □

To conclude the existence of the complex derivative at a point, the hypothesis (in the theorem above) about differentiability at that point cannot be relaxed to the mere existence of partial derivatives (even if they would satisfy the Cauchy-Riemann equations).
For a counterexample, see the exercises.

Example

Recall that the complex exponential function was defined by

$$\exp : \mathbb{C} \to \mathbb{C}$$

$$\exp(x+iy) = \underbrace{e^x \cdot \cos(y)}_{=u(x,y)} + i \cdot \underbrace{e^x \cdot \sin(y)}_{=v(x,y)}.$$

Claim The function $z \mapsto \exp(z) = e^z$ is complex differentiable at every $z_0 \in \mathbb{C}$ and its derivative is $\exp'(z_0) = \exp(z_0)$.

To check this, observe first that the component functions u, v have continuous partial derivatives everywhere, so exp is differentiable everywhere (see calculus courses).

By the previous theorem, then, it suffices to verify the Cauchy-Riemann equations. For this, calculate

$$\frac{\partial u}{\partial x} = e^x \cdot \cos(y), \quad \frac{\partial v}{\partial y} = e^x \cdot \cos(y)$$

$$\frac{\partial v}{\partial x} = e^x \cdot \sin(y), \quad \frac{\partial u}{\partial y} = -e^x \cdot \sin(y).$$

$\square$

Analyticity (= holomorphicity)

The existence of the (complex) derivative at a single point does not yet have drastic implications.
A more fruitful starting point is :

Def

A function $f : U \to \mathbb{C}$ defined on an open set $U \subset \mathbb{C}$ is analytic if it has a complex derivative at every point $z \in U$.
(We say that f is analytic at z_0 if for some $\varepsilon > 0$ it has a complex derivative at every point $z \in B(z_0; \varepsilon)$ of the ε-radius disk centered at z_0.)

[analytic - function]

Example

The complex exponential function

$$\exp : \mathbb{C} \to \mathbb{C}$$

is analytic in the whole complex plane $\mathbb{C}$ (by the previous example).

Example:

Consider the function defined by the formula

$$f(z) = |z|^2.$$

Its component functions are

$$u(x,y) = \mathrm{Re}(|x+iy|^2) = x^2+y^2 \quad \text{and}$$

$$v(x,y) = \mathrm{Im}(|x+iy|^2) = 0.$$

The partial derivatives are

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 2y.$$

The continuity of the partial derivatives implies differentiability.

Cauchy-Riemann equations hold at the point $(x,y) = (0,0)$ but nowhere else. The function $f(z) = |z|^2$ is not analytic anywhere (although at $z=0$ it has complex derivative $f'(0) = 0$).

We can combine some earlier observations to obtain the first important properties of analytic functions:

Lemma

[analytic-implies-continuous]

Every analytic function $f : U \to \mathbb{C}$ is continuous.

Proof: Analyticity implies, by definition, the existence of a complex derivative at any point $z \in U$, which by an earlier lemma implies differentiability at z, which by another earlier lemma implies continuity at z. $\square$

Theorem

[cauchy-riemann]

A function $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ is analytic if and only if its component functions $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$ are differentiable at every $z \in U$ and their partial derivatives satisfy the Cauchy-Riemann equations.

Proof This follows from the definition of analyticity and an earlier lemma. $\square$