Aalto University
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2025-2026/II

Problem set 4A

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Exercise sessions: Tue-Wed, Nov 11-12, 2025

Topic: Cauchy's integral theorem, Cauchy's integral formula, contractible paths

The first three exercises are to be discussed and solved in exercise sessions. The last exercise (marked with symbol ✍) is a quiz for which you will write a solution in an exam-like setup (no materials or extra equipment allowed) during the last 20min of the exercise session.

Exercise (in class) 1.
In this problem (and in other problems of this course concerning path homotopy), fully rigorous solutions are not expected; clear intuitive justifications suffice.

(a) Which of the following closed paths are null homotopic (i.e., contractible) in the set

$$A = \Big\{ z \in \mathbb{C} \;\Big|\; |z| < 1 \Big\} \setminus \Big\{ \frac{\mathbb{i}}{n} \;\Big|\; n \in \mathbb{N} \Big\} ?$$

(i) $t \mapsto \frac{1}{\sqrt{2}} e^{\mathbb{i}2\pi t}$ for $t \in [0,1]$;

(ii) $t \mapsto -\frac{1}{2} + \frac{1}{4} e^{\mathbb{i}2\pi t}$ for $t \in [0,1]$;

(iii) $t \mapsto \frac{2\mathbb{i}}{5} + \varepsilon\, e^{\mathbb{i}2\pi t}$ for $t \in [0,1]$, with $\varepsilon > 0$ sufficiently small;

(iv) $\eta \boxplus \overleftarrow{\eta}$ where η is some path in A.

Remark: Here $\boxplus$ denotes path concatenation, and $\overleftarrow{\eta}$ denotes the reverse of the path η.

(b) Which of the following subsets of the complex plane are simply-connected?

(i) $\Big\{ z \in \mathbb{C} \;\Big|\; |z - \mathbb{i}| \leq 1 \Big\} \cup \Big\{ z \in \mathbb{C} \;\Big|\; |z + \mathbb{i}| \leq 1 \Big\}$

(ii) $\Big\{ z \in \mathbb{C} \;\Big|\; |\mathfrak{Re}(z)| + |\mathfrak{Im}(z) - 3| > 2 \Big\}$

(iii) $\mathbb{C} \setminus \bigcup_{k=0}^{5} \Big\{ r e^{\mathbb{i}\pi k/3} \;\Big|\; r > 0 \Big\}.$

Exercise (in class) 2.
Let γ be a positively oriented contour around the circle $|z+\mathbb{i}| = \frac{3}{2}$. Calculate

$$\oint_\gamma \frac{1}{z^4+z^2}\,\mathrm{d}z.$$

Exercise (in class) 3.
Let $a, b > 0$. Show that

$$\int_0^{2\pi} \frac{1}{a^2\,\cos^2(t) + b^2\,\sin^2(t)}\,\mathrm{d}t \;=\; \frac{2\pi}{ab},$$

by considering the contour integral $\oint_\gamma \frac{1}{z}\,\mathrm{d}z$, where the contour γ is a suitable parametrization of the ellipse

$$E := \left\{ x + \mathbb{i}y \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} \subset \mathbb{C}.$$

Quiz 4.
Let γ be a positively oriented contour around the circle $|z| = 1$. Calculate:

(a) $$\oint_\gamma \sqrt{9 - z^2}\,\mathrm{d}z$$

(b) $$\oint_\gamma \frac{1}{z^2+2z}\,\mathrm{d}z$$

Remark: In part (a), $\sqrt{\cdots}$ denotes the principal branch of the square root function, i.e., the choice of complex square roots such that $\mathrm{Arg}(\sqrt{w}) \in (-\pi/2, \pi/2]$ for any $w \in \mathbb{C} \setminus \{0\}$.