Aalto University

Problem set 5A

Department of Mathematics and Systems Analysis MS-C1300 — Complex Analysis, 2024-2025/II

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Exercise sessions: 26.-27.11. Hand-in due: Wed 27.11.2024 at 23:59

Topic: Taylor series, Laurent series, isolated singularities

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol \triangle): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0-3. The deadline for returning solutions to problem set 5A is Wed 27.11.2024 at 23:59.

Hint: (... to all exercises.) Two power series can be multiplied "termwise" inside the disks of convergence of both factors. This fact can be considered known; it essentially follows from Exercise 5A.4(ii).

Exercise (in class) 1.

Find the Laurent series expansions of the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in the following regions:

- (a) $\{z \in \mathbb{C} \mid 0 < |z 1| < 1\};$ (b) $\{z \in \mathbb{C} \mid 0 < |z 2| < 1\}.$

What do these expansions say about the types of singularities of f?

Exercise (in class) 2.

The Bernoulli numbers B_n are defined as the coefficients in the following series expansion, valid when $|z| \neq 0$ is small enough:

$$\frac{z}{e^z - 1} = B_0 + \frac{B_1}{1!} z + \frac{B_2}{2!} z^2 + \frac{B_3}{3!} z^3 + \frac{B_4}{4!} z^4 + \cdots$$

Show that the Bernoulli numbers satisfy the recurrence

$$\binom{k}{0}B_0 + \binom{k}{1}B_1 + \dots + \binom{k}{k-1}B_{k-1} = 0$$

for any $k \geq 2$. Calculate the first 7 Bernoulli numbers B_0, B_1, \ldots, B_6 .

Remark: The students who want to think about this further may examine the growth rate of the Bernoulli numbers. A starting point can be the radius of convergence of the power series defining them or computer assisted calculations.

Find the Laurent series expansions of the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in the following regions:

(a)
$$\{z \in \mathbb{C} \mid 1 < |z| < 2\};$$

(b) $\{z \in \mathbb{C} \mid |z| > 2\}.$

(b)
$$\{z \in \mathbb{C} \mid |z| > 2\}.$$

△ Homework exercise 4.

Find the Laurent series expansions of the following functions in the given regions, and determine the types of singularities of the functions.

(a):
$$z \mapsto z \cos(1/z)$$

in
$$\{z \in \mathbb{C} \mid |z| > 0\}$$

(b):
$$z \mapsto \frac{z^2 + 1 - e^{z^2}}{z^4}$$

(c): $z \mapsto \frac{37}{\cos(\pi z^3)}$

in
$$\{z \in \mathbb{C} \mid |z| > 0\}$$

(c):
$$z \mapsto \frac{37}{\cos(\pi z^3)}$$

in
$$\{z \in \mathbb{C} \mid |z - 1/2| > 0\}$$
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