CAUCHY'S INTEGRAL THEOREM

The following theorem is our first substantial result of complex analysis. Note that it relates complex differentiability (more precisely analyticity) to complex contour integration. In real-variable calculus there is no meaningful analogue of such a result. (Perhaps some variants of Stokes' theorem are the closest equivalents.)

<u>Theorem</u> (Cauchy's integral theorem for star-shaped domains)

[cauchy_theorem_star_shaped]

Let $f: U \to \mathbb{C}$ be an analytic function in a star-shaped open set $U \subset \mathbb{C}$. Then for any closed contour γ in U we have

$$\boxed{\oint_\gamma f(z)\, dz = 0.}$$

Proof:

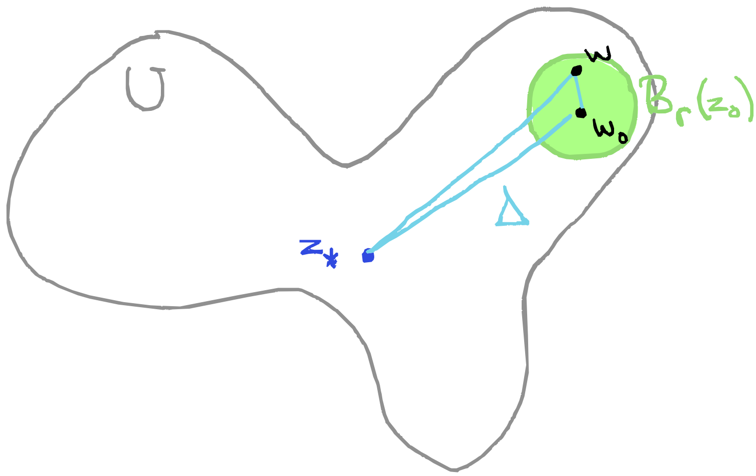
Goursat's lemma is a special case of this, which is in fact sufficient to construct a primitive for f, and then we may just appeal to a characterization of the existence of primitives.

Recall that $\oint_\gamma f(z)\,dz = 0$ for all closed contours γ in U is equivalent to the existence of a primitive $F : U \to \mathbb{C}$ of f in U. It therefore suffices to show that f has a primitive in U.

Since U is assumed star-shaped, there exists a point $z_* \in U$ such that for all $z \in U$ the line segment $[z_*, z]$ is contained in U.
To simplify notation, below we interpret $[w_0, w_1]$ as the parametrized path $t \mapsto w_0 + t(w_1 - w_0)$ $(t \in [0,1])$, where convenient.

We define $F: U \to \mathbb{C}$ by setting

$$F(w) = \int_{[z_*, w]} f(z)\, dz \qquad \text{for } w \in U.$$



To prove that this F is a primitive of f, we must show $F'(w_0) = f(w_0)$ for every $w_0 \in U$. So fix $w_0 \in U$.

By openness of U, there exists an $r > 0$ such that $B_r(w_0) \subset U$. Note also that by star-shapedness of U we have that for any $w \in B_r(w_0)$ the closed triangle Δ with vertices z_*, w_0, w is contained in U.

The sides of Δ are the line segments $[z_*, w_0]$, $[w_0, w]$, $[w, z_*]$ which we view below as oriented paths which parametrize the boundary $\partial\Delta$ of Δ seen as a closed path. By Goursat's lemma we have

$$\oint_{\partial\Delta} f(z)\, dz = 0.$$

We now rewrite this equality as

$$\begin{aligned} 0 &= \int_{\partial\Delta} f(z)\, dz \\ &= \underbrace{\int_{[z_*, w_0]} f(z)\, dz}_{= F(w_0)} + \int_{[w_0, w]} f(z)\, dz + \underbrace{\int_{[w, z_*]} f(z)\, dz}_{= -F(w)} \\ &= F(w_0) - F(w) + \int_{[w_0, w]} f(z)\, dz \end{aligned}$$

which gives (for $w \in B_r(w_0)$, $w \neq w_0$)

$$\frac{F(w) - F(w_0)}{w - w_0} = \frac{1}{w - w_0} \int_{[w_0, w]} f(z)\, dz.$$

From here we conclude just like in the characterization of existence of primitives (using continuity of f at w_0) that

$$F'(w_0) = \lim_{w \to w_0} \frac{F(w) - F(w_0)}{w - w_0} = f(w_0).$$

Since $w_0 \in U$ was arbitrary, this shows that F is a primitive of f and the proof is complete.

$\square$

A (stronger) version of Cauchy's integral theorem drops the star-shapedness hypothesis and even simple connectedness hypothesis of U but requires the closed contours γ to be contractible in U. The proper treatment of this version requires some homotopy theory, so we only briefly describe the relevant notions here.

<u>Def</u> (Contractible paths)

[contractible]

A closed path $\gamma : [a,b] \longrightarrow U$ in U is said to be contractible (or null-homotopic) in U if there exists a continuous function

$$\Gamma : [a,b] \times [0,1] \longrightarrow U$$

such that (denoting $z_0 = \gamma(a) = \gamma(b)$)

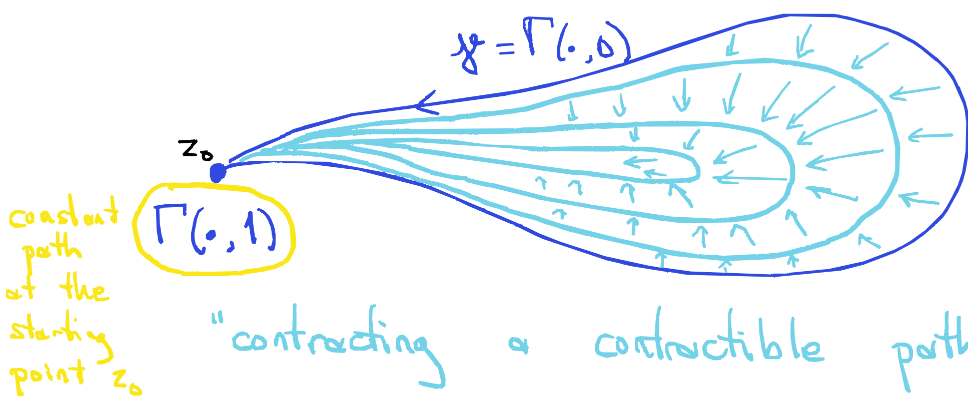
$$\Gamma(t,0) = \gamma(t) \qquad \forall t \in [a,b]$$

$$\Gamma(a,s) = \Gamma(b,s) = z_0 \qquad \forall s \in [0,1]$$

$$\Gamma(t,1) = z_0 \qquad \forall t \in [a,b].$$

Γ provides a continuous deformation of the original path $t \mapsto \gamma(t) = \Gamma(t,0)$ to the constant path $t \mapsto \Gamma(t,1)$ keeping the endpoints fixed.

Such a deformation exists is γ "does not surround any holes of U", in particular if "U does not have any holes" (i.e. U is simply connected).



"contracting a contractible path"

Thm (Stronger version of Cauchy's integral theorem)

[Cauchy-Theorem]

If $f : U \to \mathbb{C}$ is analytic on an open set $U \subset \mathbb{C}$ and γ is a contractible contour in U, then $\oint_\gamma f(z)\,dz = 0$.

The idea of the proof is to show that the value of the integral does not change under the continuous deformation, i.e., that

$$s \longmapsto \oint_{\Gamma(\cdot,s)} f(z)\, dz$$

is a constant function.

At $s=0$ the value is $\oint_\gamma f(z)\, dz$ and at $s=1$ the value is 0, (a contour integral along a constant path).

(The technical details of proving this are slightly annoying unless one assumes that $\Gamma : [a,b] \times [0,1] \longrightarrow U$ is continuously differentiable in both its variables. The main ideas are already present in the slightly weaker version which we proved using Goursat's lemma.)

Let us give a nontrivial application of Cauchy's integral theorem now.

<u>Example</u> (Fresnel's integrals)

We will use Cauchy's integral theorem to calculate the real improper Riemann integrals

$$\int_0^\infty \cos(t^2)\,dt = \lim_{T\to+\infty} \int_0^T \cos(t^2)\,dt \quad \text{and}$$

$$\int_0^\infty \sin(t^2)\,dt = \lim_{T\to+\infty} \int_0^T \sin(t^2)\,dt .$$

Observing that for $t \in \mathbb{R}$ we have

$$e^{it^2} = \cos(t^2) + i \cdot \sin(t^2)$$

it suffices to calculate

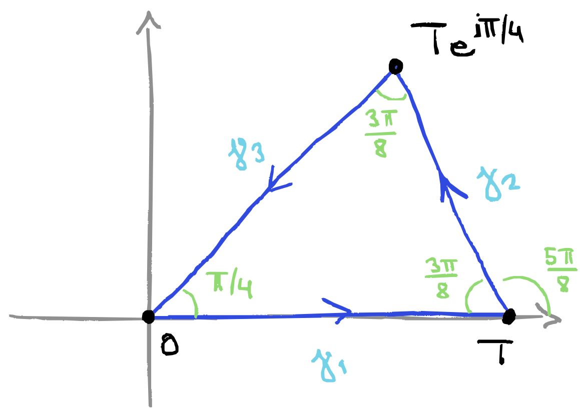
$$\lim_{T\to+\infty} \int_0^T e^{it^2}\,dt$$

and take real and imaginary parts.

It is therefore natural to define

$$f : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad f(z) = e^{iz^2},$$

which is an analytic function on the domain $\mathbb{C}$ (which is star-shaped).
The key idea is to consider a cleverly chosen closed contour:



$\gamma = \gamma_1 \boxplus \gamma_2 \boxplus \gamma_3$ with three pieces

$$\gamma_1 : [0,T] \to \mathbb{C} \qquad \gamma_1(t) = t$$

$$\gamma_2 : [0,1] \to \mathbb{C} \qquad \gamma_2(t) = T + t \cdot (Te^{i\pi/4} - T)$$

$$\overleftarrow{\gamma_3} : [0,T] \to \mathbb{C} \qquad \overleftarrow{\gamma_3}(t) = e^{i\pi/4} \cdot t$$

Cauchy's integral theorem gives

$$0 = \oint_\gamma f(z)\,dz = \int_{\gamma_1} e^{iz^2} dz + \int_{\gamma_2} e^{iz^2} dz - \int_{\overleftarrow{\gamma_3}} e^{iz^2} dz .$$

The integrals over γ_1 and $\overleftarrow{\gamma_3}$ are related to the one we want to calculate. The integral over γ_2, in turn, is negligible in the $T \to +\infty$ limit: for $t \in [0,1]$ we have

$$|f(\gamma_2(t))| = |e^{i\gamma_2(t)^2}|$$

$$= |e^{-2\cdot \mathrm{Im}(\gamma_2(t))\mathrm{Re}(\gamma_2(t)) + i(\text{something real})}|$$

$$= e^{-2\,\mathrm{Im}(\gamma_2(t))\mathrm{Re}(\gamma_2(t))} \leq e^{-T^2 t}.$$

$\gamma_2(t)$
$= T(1+t(e^{i\pi/4}-1))$
$= T(1+t(\frac{1+i}{\sqrt{2}}-1))$
$= T((1-t(1-\frac{1}{\sqrt{2}})) + i\frac{t}{\sqrt{2}})$ $\implies$ $\mathrm{Im}(\gamma_2(t))\mathrm{Re}(\gamma_2(t)) = T^2(1-t(1-\frac{1}{\sqrt{2}}))\frac{1}{\sqrt{2}}t \geq \frac{1}{2}T^2 t$

(for $t \in [0,1]$)

Breaking γ_2 further into two pieces, $\gamma_{2;A}$ and $\gamma_{2;B}$,
$\gamma_{2;A} : [0, T^{-3/2}] \to \mathbb{C}$ and
$\gamma_{2;B} : [T^{-3/2}, 1] \to \mathbb{C}$ given by the same formula as γ_2 (assuming $T \geq 1$ to have $0 \leq T^{-3/2} \leq 1$) we can bound the contribution

from γ_2 by triangle inequalities :

$$\left| \int_{\gamma_2} e^{iz^2} dz \right| \leq \left| \int_{\gamma_{2:A}} e^{iz^2} dz \right| + \left| \int_{\gamma_{2:B}} e^{iz^2} dz \right|$$

$$\leq \underbrace{\ell(\gamma_{2;A})}_{\substack{= T^{-3/2} \cdot |Te^{i\pi/4} - T| \\ = T^{-1/2} \cdot |e^{i\pi/4} - 1|}} \cdot e^{-0} + \underbrace{\ell(\gamma_{2;B})}_{\substack{= (1 - T^{-3/2}) |Te^{i\pi/4} - T| \\ \leq T |e^{i\pi/4} - 1|}} \cdot e^{-T^2 T^{-3/2}}$$

$$\leq |e^{i\pi/4} - 1| \Big(\underbrace{T^{-1/2}}_{\xrightarrow[T \to +\infty]{} 0} + \underbrace{T \cdot e^{-T^{1/2}}}_{\xrightarrow[T \to +\infty]{} 0} \Big)$$

$$\longrightarrow 0 \quad \text{as} \quad T \longrightarrow +\infty.$$

This allows us to disregard the contribution from γ_2 in the limit $T \to +\infty$.

The other two contributions are

$$\int_{\gamma_1} f(z)\,dz = \int_0^T e^{i\gamma_1(t)^2}\,\dot{\gamma}_1(t)\,dt$$

$$= \int_0^T e^{it^2} \cdot 1 \cdot dt$$

This is the integral we wanted!

and

$$\int_{\overleftarrow{\gamma_3}} f(z)\,dz = \int_0^T e^{i\gamma_3(t)^2}\,\dot{\gamma}_3(t)\,dt$$

$$= \int_0^T e^{i(e^{i\pi/4}t)^2}\, e^{i\pi/4}\,dt$$

$$= e^{i\pi/4}\int_0^T e^{-t^2}\,dt$$

$$\xrightarrow[T\to\infty]{} \frac{\sqrt{\pi}}{2}$$ by a known* Gaussian integral.

* For completeness, let us calculate $I = \int_0^\infty e^{-t^2}\,dt$. Note that

$$I^2 = \left(\int_0^\infty e^{-t^2}\,dt\right)^2 = \int_0^\infty\!\!\int_0^\infty dx\,dy\, e^{-x^2-y^2} \overset{*}{=} \int_0^\infty dr \cdot \frac{\pi}{2} r \cdot e^{-r^2}\,dz$$

* change to radial coordinates

$$= \frac{\pi}{4}\int_0^\infty \frac{d}{dr}\left(e^{-r^2}\right) dr = \frac{\pi}{4}.$$ So $I = \sqrt{\frac{\pi}{4}}$.

Therefore we can take a limit $T \to \infty$ in

$$\int_{\gamma_1} e^{iz^2}\,dz = -\int_{\gamma_2} e^{iz^2}\,dz + \int_{\overleftarrow{\gamma_3}} e^{iz^2}\,dz$$

(where the T-dependence is implicitly in the contours $\gamma_1 = \gamma_1^T$, $\gamma_2 = \gamma_2^T$, $\gamma_3 = \gamma_3^T$)

to get

$$\lim_{T\to\infty} \int_0^T e^{it^2}\,dt = \lim_{T\to\infty}\left(-\int_{\gamma_2^T} e^{iz^2}\,dz + e^{i\pi/4}\int_0^T e^{-t^2}\,dt\right)$$

$$= -0 + e^{i\pi/4}\,\frac{\sqrt{\pi}}{2}$$

$$= \frac{1+i}{\sqrt{2}}\,\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}\,(1+i)}{2\sqrt{2}}$$

Taking real and imaginary parts gives

$$\int_0^\infty \cos(t^2)\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}$$ and

$$\int_0^\infty \sin(t^2)\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

CAUCHY'S INTEGRAL FORMULA

The next result, Cauchy's integral formula, is really the key which unlocks the magical power of complex analysis.

(We will see powerful consequences starting from the next lecture.)

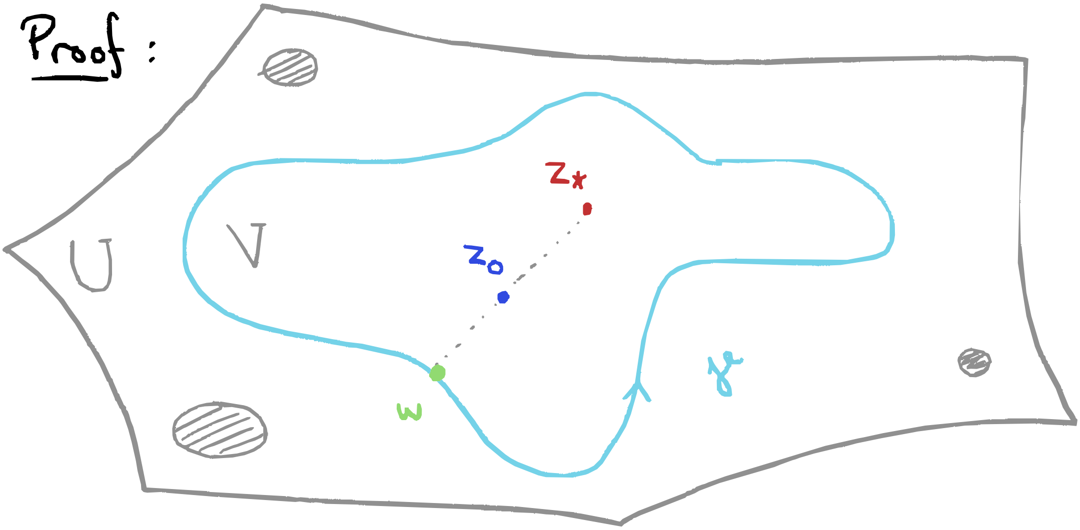
Theorem (Cauchy's integral formula in a star-shaped setup)

[cauchy-formula-star-shaped]

Let $f : U \to \mathbb{C}$ be an analytic function and γ a closed contour in U surrounding a star-shaped region counterclockwise. Then for any z_0 in the region surrounded by γ we have

$$\boxed{\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i \cdot f(z_0)}\,.$$

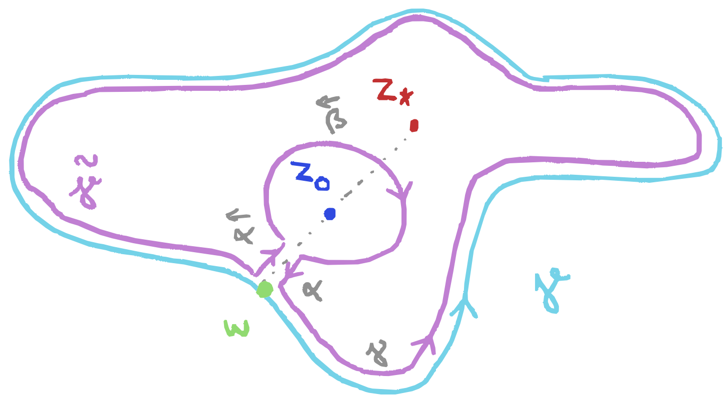
Proof:



Let $V \subset U$ denote the star-shaped region surrounded (counterclockwise) by γ, and let $z_* \in V$ denote a point such that $[z_*, z] \subset V$ for all $z \in V$. If the line segment from z_* to z_0 is extended, it meets γ at some (nearest) point $w \in \partial V$. (If $z_0 = z_*$ we may use any line from z_* here.)

Let $\varepsilon > 0$. By continuity of f at z_0 and since z_0 is an interior point of V, we can choose a $\delta > 0$

such that $\overline{B}(z_0;\delta) \subset V$ and $|f(z)-f(z_0)| < \varepsilon$ when $z \in \overline{B}(z_0;\delta)$.



We now form a contour $\tilde{\gamma}$ by concatenating the following paths:

- the clockwise (negatively oriented) path $\overleftarrow{\beta}$ traversing the circle $\partial B(z_0;\delta)$, starting and ending at the point which lies on the line segment $[z_0,w]$
- the line segment α from the circle $\partial B(z_0;\delta)$ to ∂V along $[z_0,w]$
- the contour γ parametrizing ∂V counterclockwise from w to w (w.l.o.g. start γ from w)
- the line segment $\overleftarrow{\alpha}$ (reverse of α).

The contour $\tilde{\gamma} = \overleftarrow{\beta} \boxplus \alpha \boxplus \gamma \boxplus \overleftarrow{\alpha}$ is closed, and it stays in the closure of the star-shaped domain $V \setminus [z_0, w]$ so that the conclusion of Cauchy's integral theorem remains valid for the analytic function $z \longmapsto \frac{f(z)}{z - z_0}$ (the precise justification is left to the reader) so we have

$$0 = \oint_{\tilde{\gamma}} \frac{f(z)}{z - z_0}\, dz$$

$$= \oint_{\overleftarrow{\beta}} \frac{f(z)}{z-z_0} dz + \cancel{\int_{\alpha} \frac{f(z)}{z-z_0} dz} + \oint_{\gamma} \frac{f(z)}{z-z_0} dz + \cancel{\int_{\overleftarrow{\alpha}} \frac{f(z)}{z-z_0} dz}$$

$$= - \oint_{\beta} \frac{f(z)}{z-z_0} dz + \oint_{\gamma} \frac{f(z)}{z-z_0} dz .$$

positively oriented circle $\partial B(z_0; \delta)$

From this, we get

$$\oint_{\gamma} \frac{f(z)}{z - z_0} dz = \oint_{\beta} \frac{f(z)}{z - z_0} dz .$$

pos. oriented.

A straightforward calculation shows

$$\oint_\beta \frac{1}{z-z_0}\,dz = 2\pi i,$$

(cf. the calculation $\oint_{\partial B(0;1)} \frac{1}{z}\,dz = 2\pi i$)

and multiplying this by $f(z_0)$ allows us to write $f(z_0) = \frac{f(z_0)}{2\pi i}\oint_\beta \frac{1}{z-z_0}\,dz$.

Then we estimate

$$\left| \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}\,dz - f(z_0) \right|$$

$$= \left| \frac{1}{2\pi i}\oint_\beta \frac{f(z)}{z-z_0}\,dz - \frac{f(z_0)}{2\pi i}\oint_\beta \frac{1}{z-z_0}\,dz \right|$$

$$= \left| \frac{1}{2\pi i}\oint_\beta \frac{f(z)-f(z_0)}{z-z_0}\,dz \right|$$

$$\leq \frac{1}{2\pi}\oint_\beta \underbrace{\frac{|f(z)-f(z_0)|}{|z-z_0|}}_{<\frac{\varepsilon}{\delta}}\,|dz|$$

since $|f(z)-f(z_0)| < \varepsilon$ and $|z-z_0| = \delta$

$$\leq \frac{1}{2\pi}\,\frac{\varepsilon}{\delta}\cdot \underbrace{\ell(\beta)}_{=2\pi\delta} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz = f(z_0)$$

and the proof is complete. $\square$

Example application:

Let us show how Cauchy's integral formula can be used to calculate

$$\int_0^\infty \frac{\cos(t)}{t^2+1}\,dt = \lim_{R\to\infty} \int_0^R \frac{\cos(t)}{t^2+1}\,dt .$$

Observe first that $t \mapsto \frac{\cos(t)}{t^2+1}$ is an even function, so

$$\int_0^R \frac{\cos(t)}{t^2+1}\,dt = \frac{1}{2}\int_{-R}^R \frac{\cos(t)}{t^2+1}\,dt ,$$

Observe also that for $t \in \mathbb{R}$,

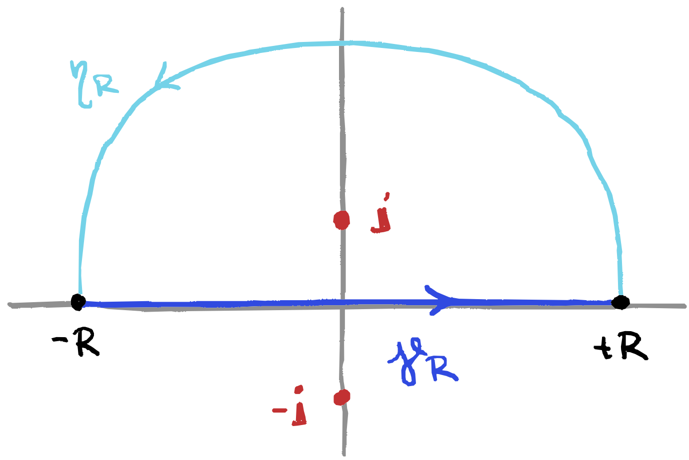
$$\frac{\cos(t)}{t^2+1} = \mathrm{Re}\left(\frac{e^{it}}{t^2+1}\right) \quad \text{so}$$

$$\int_{-R}^R \frac{\cos(t)}{t^2+1}\,dt = \mathrm{Re}\left(\int_{\gamma_R} \frac{e^{iz}}{z^2+1}\,dz\right)$$

where γ_R is the line segment $[-R,R]$ from $-R$ to R. This makes it natural to consider the function given by

$$g(z) = \frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z-i)(z+i)} .$$

The contour γ_R is not closed, so we concatenate to it a semicircle contour $\eta_R : [0, \pi] \to \mathbb{C}$ given by $\eta_R(t) = R \cdot e^{it}$.



The contour $\gamma_R \boxplus \eta_R$ is closed, and it surrounds a half-disk of radius R counterclockwise (and the half-disk is convex, so in particular star-shaped).

Since $g(z) = \dfrac{e^{iz}}{(z-i)(z+i)} = \dfrac{f(z)}{z-i}$

where $f(z) = \dfrac{e^{iz}}{z+i}$ is analytic in the surrounded half-disk, Cauchy's integral formula (with $z_0 = i$) gives (when $R > 1$ so that i is surrounded)

$$\underbrace{2\pi i\, f(z_0)}_{= \frac{e^{i\cdot i}}{i+i} = \frac{e^{-1}}{2i}} = \oint_{\gamma_R \boxplus \eta_R} \underbrace{\frac{f(z)}{z-i}}_{= g(z) = \frac{e^{iz}}{z^2+1}} \, dz$$

$$= \int_{\gamma_R} \frac{e^{iz}}{z^2+1} \, dz + \int_{\eta_R} \frac{e^{iz}}{z^2+1} \, dz .$$

Rearranging, we find

$$\int_{\gamma_R} \frac{e^{iz}}{z^2+1} \, dz = \frac{\pi}{e} - \int_{\eta_R} \frac{e^{iz}}{z^2+1} \, dz .$$

The original integral requires taking the limit $R \to \infty$.

The last term can be estimated by the triangle inequality for integrals:

for z on η_R we have $\operatorname{Im}(z) \geq 0$

so $\left|e^{iz}\right| = e^{\operatorname{Re}(iz)} = e^{-\operatorname{Im}(z)} \leq 1$

and when $R > 1$, $\left|\frac{1}{z^2-1}\right| \leq \frac{1}{R^2-1}$, so

$$\left|\int_{\eta_R} \frac{e^{iz}}{z^2+1}\,dz\right| \leq \ell(\eta_R) \cdot \frac{1}{R^2-1}$$

$$= \frac{\pi R}{R^2-1} \xrightarrow[R\to\infty]{} 0.$$

We conclude that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z^2+1}\,dz = \lim_{R\to\infty} \left(\frac{\pi}{e} + \int_{\eta_R} \frac{e^{iz}}{z^2+1}\,dz\right)$$

$$= \frac{\pi}{e} + 0 = \frac{\pi}{e}.$$

The original integral is therefore

$$\int_0^\infty \frac{\cos(t)}{t^2+1}\,dt = \lim_{R\to\infty} \int_0^R \frac{\cos(t)}{t^2+1}\,dt$$

$$= \lim_{R\to\infty} \frac{1}{2} \operatorname{Re}\left(\int_{\gamma_R} \frac{e^{iz}}{z^2+1}\,dz\right)$$

$$= \frac{1}{2}\operatorname{Re}\left(\frac{\pi}{e}\right) = \frac{\pi}{2e}.$$