Aalto University
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II

Problem set 3B

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Exercise sessions: 7.-8.11. **Hand-in due: Sun 10.11.2024 at 23:59**

Topic: (complex) derivatives, analytic functions

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ✍): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 3B is Sun 10.11.2024 at 23:59.

Exercise (in class) 1.
Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be the path given by the formula $\gamma(t) = -2e^{\mathbb{i}t}$. Evaluate the integral

$$\int_\gamma \frac{1}{z^2 - 1}\,\mathrm{d}z.$$

Exercise (in class) 2.
Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be the path given by the formula $\gamma(t) = e^{\mathbb{i}t}$, and let $a \in \mathbb{R}$ be such that $0 < a < 1$.

(a) Show that we have

$$\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos(t)}\,\mathrm{d}t = \oint_\gamma \frac{\mathbb{i}}{(z-a)(az-1)}\,\mathrm{d}z.$$

(b) Evaluate the real integral

$$\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos(t)}\,\mathrm{d}t.$$

✍ **Homework exercise 3.**
Let $a \in \mathbb{R}$ be a real number such that $|a| > 1$. With the help of the complex integral

$$\int_\gamma \frac{1}{z+a}\,\mathrm{d}z$$

along the path $\gamma\colon [0, 2\pi] \to \mathbb{C}$ given by the formula $\gamma(t) = e^{\mathbb{i}t}$, show that

$$\int_0^{2\pi} \frac{1 + a\,\cos(t)}{1 + 2a\,\cos(t) + a^2}\,\mathrm{d}t \;=\; 0.$$

✍ **Homework exercise 4.**
Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be the path given by the formula $\gamma(t) = e^{\mathbb{i}t}$. Evaluate the following integrals along γ:

(a) $$\int_\gamma \frac{1}{(z-2)^2}\,\mathrm{d}z$$

(b) $$\int_\gamma \frac{1}{z^2-4}\,\mathrm{d}z$$

(c) $$\int_\gamma \left(z + \frac{1}{z}\right)^n \mathrm{d}z \qquad \text{where } n \in \mathbb{N}.$$