Aalto University **Problem set 5B**
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2025-2026/II K Kytölä

**Exercise sessions: Thu-Fri, Nov 20-21, 2025**

Topic: power series, Taylor series

*The first two exercises (marked with symbol ♮) are to be solved before the exercise sessions so that you are ready to present your solution on the blackboard. The last two exercises are to be discussed and solved in exercise sessions.*

## ♮ Blackboard exercise 1.

Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a non-constant analytic function defined in the entire complex plane, and suppose that there exists a constant $\lambda \neq 1$ such that $f(\lambda z) = f(z)$ for all $z \in \mathbb{C}$.

(a) Prove that there exist a positive integer $m$ such that $\lambda^m = 1$.
(b) Denote by $m$ the smallest positive integer such that $\lambda^m = 1$. Show that there exists an analytic function $g\colon \mathbb{C} \to \mathbb{C}$ such that $f(z) = g(z^m)$ for all $z \in \mathbb{C}$.

## ♮ Blackboard exercise 2.

The Fibonacci sequence $(a_n)_{n=0,1,2,\ldots}$ is defined recursively by

$$a_0 = 1, \qquad a_1 = 1, \qquad \text{and} \qquad a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2.$$

(a) Prove that $0 \leq a_n \leq 2^n$ for every $n$, and show that the series

$$F(z) = \sum_{n=0}^{\infty} a_n \, z^n$$

has a radius of convergence $\rho \geq \frac{1}{2}$.
(b) Prove that $F(z)$ from part (a) satisfies

$$F(z) \;=\; 1 + z\,F(z) + z^2\,F(z).$$

(c) Use the equation in (b) to solve $F(z)$. Use the solution to determine the exact radius of convergence $\rho$ of the series.
*Remark:* The interested students can think about: What does the value of $\rho$ tell us about the growth rate of the Fibonacci sequence?

**Exercise (in class) 3.**
Consider the function $f$ defined by the formula

$$f(z) \;=\; \frac{1}{1+z}.$$

(a) Find a power series representation for $f$ in some nonempty open disk centered at $z_0 = 0$ (series in powers of $z - z_0 = z$). What is the radius of convergence of this series?

(b) Find a power series representation for $f$ in some nonempty open disk centered at $z_0 = -4$ (series in powers of $z - z_0 = z + 4$). What is the radius of convergence of this series?

**Exercise (in class) 4.**
For any complex number $\lambda$ and any nonnegative integer $n$, the *(generalized) binomial coefficient* is defined by

$$\binom{\lambda}{n} \;=\; \prod_{j=1}^{n} \frac{\lambda - j + 1}{j} \;=\; \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}.$$

(a) Show that we have

$$(1+z)^{\lambda} = \sum_{n=0}^{\infty} \binom{\lambda}{n} z^n \qquad \text{when } |z| < 1,$$

where the principal branch of the complex power function is used: $w^{\lambda} = e^{\lambda \operatorname{Log}(w)}$ with $\operatorname{Log}(w)$ denoting the principal logarithm of $w \in \mathbb{C} \setminus \{0\}$.

(b) Using (a), write down explicitly the first four terms of the power series (developed at $z_0 = 0$, i.e., in powers of $z - z_0 = z$) representing the function

$$z \;\mapsto\; \sqrt{1+z} \;=\; (1+z)^{1/2}.$$

*Remark:* The convention is that the empty product equals one, so that $\binom{\lambda}{0} = 1$ for any $\lambda \in \mathbb{C}$.