Aalto University **Problem set 5B**
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II K Kytölä & A Vavilov

Exercise sessions: 21.-22.11. **Hand-in due: Sun 24.11.2024 at 23:59**

Topic: power series, Taylor series

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ✍): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 5B is Sun 24.11.2024 at 23:59.

Exercise (in class) 1.
The Fibonacci sequence $(a_n)_{n=0,1,2,\ldots}$ is defined recursively by

$$a_0 = 1, \qquad a_1 = 1, \qquad \text{and} \qquad a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2.$$

(a) Prove that $0 \leq a_n \leq 2^n$ for every n, and show that the series

$$F(z) = \sum_{n=0}^{\infty} a_n \, z^n$$

has a radius of convergence $\rho \geq \frac{1}{2}$.

(b) Prove that $F(z)$ from part (a) satisfies

$$F(z) \; = \; 1 + z \, F(z) + z^2 \, F(z).$$

(c) Use the equation in (b) to solve $F(z)$. Use the solution to determine the exact radius of convergence ρ of the series.

Remark: The interested students can think about: What does the value of ρ tell us about the growth rate of the Fibonacci sequence?

Exercise (in class) 2.
Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a non-constant analytic function defined in the entire complex plane, and suppose that there exists a constant $\lambda \neq 1$ such that $f(\lambda z) = f(z)$ for all $z \in \mathbb{C}$.

(a) Prove that there exist a positive integer m such that $\lambda^m = 1$.

(b) Denote by m the smallest positive integer such that $\lambda^m = 1$. Show that there exists an analytic function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = g(z^m)$ for all $z \in \mathbb{C}$.

✍ **Homework exercise 3.**

Consider the function f defined by the formula

$$f(z) = \frac{1}{1+z}.$$

(a) Find a power series representation for f in some nonempty open disk centered at $z_0 = 0$ (series in powers of $z - z_0 = z$). What is the radius of convergence of this series?

(b) Find a power series representation for f in some nonempty open disk centered at $z_0 = -4$ (series in powers of $z - z_0 = z + 4$). What is the radius of convergence of this series?

✍ **Homework exercise 4.**

For any complex number λ and any nonnegative integer n, the *(generalized) binomial coefficient* is defined by

$$\binom{\lambda}{n} = \prod_{j=1}^{n} \frac{\lambda - j + 1}{j} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}.$$

(a) Show that we have

$$(1+z)^\lambda = \sum_{n=0}^{\infty} \binom{\lambda}{n} z^n \qquad \text{when } |z| < 1,$$

where the principal branch of the complex power function is used: $w^\lambda = e^{\lambda \operatorname{Log}(w)}$ with $\operatorname{Log}(w)$ denoting the principal logarithm of $w \in \mathbb{C} \setminus \{0\}$.

(b) Using (a), write down explicitly the first four terms of the power series (developed at $z_0 = 0$, i.e., in powers of $z - z_0 = z$) representing the function

$$z \mapsto \sqrt{1+z} = (1+z)^{1/2}.$$

Remark: The convention is that the empty product equals one, so that $\binom{\lambda}{0} = 1$ for any $\lambda \in \mathbb{C}$.