CONSEQUENCES OF CAUCHY'S FORMULA

Cauchy's integral formula is the heart of complex analysis. We now start deriving some of its remarkable and elegant consequences, and we will continue with more throughout the rest of the course.

To appreciate the scope of what Cauchy's formula enables, as an overview we already mention some of the results that we will obtain with it:

- Analytic functions are infinitely differentiable.
- Analytic functions are locally represented by their Taylor series.
- Analytic functions satisfy the mean value property and a maximum principle.
- Bounded analytic functions in the whole complex plane must be constant "Liouville's theorem"
- Every nonconstant complex polynomial has a root. "Fundamental theorem of algebra"
- ...

(We will have much more even in this course especially once we combine Cauchy's integral formula with the theory of power series.)

Let us recall a simple version of Cauchy's integral formula

(This simple special case is how the theorem is most often applied, although we had much more general versions, too.)

Theorem (Cauchy's integral formula with circles)

[cauchy-formula-circle]

Let $f: U \to \mathbb{C}$ be an analytic function and $\overline{B}(z_0; r) \subset U$ a closed disk contained in its domain.

Then for any $z \in B(z_0; r)$ we have

$$(\star): \quad f(z) = \frac{1}{2\pi i} \oint_{\partial B(z_0; r)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

where the contour integral is taken over the circle $\partial B(z_0; r)$ with positive (counterclockwise) orientation.

An almost immediate consequence of this special case is :

Theorem (Mean value property)

[mean_value_property]

Let $f: U \to \mathbb{C}$ be an analytic function in an open set $U \subset \mathbb{C}$, and let $z \in U$ and $r > 0$ be such that $B_r(z) \subset U$. Then for any $\rho \in (0, r)$ we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + \rho e^{i\theta}) \, d\theta .$$

Proof: Exercise (using the previous special case of Cauchy's integral formula and the definition of contour integrals).

"$\square$"

A differentiation lemma for contour integrals

We next record a lemma which may not appear particularly interesting in its own right, but which is the key to a few important subsequent developments

Lemma

[derivative_auxiliary_integral]

Let γ be a contour in $\mathbb{C}$, and let h be a continuous function on the range of γ, and let $k \in \mathbb{N}$ be a positive integer.
Then the function H defined by

$$H(z) = \int_\gamma \frac{h(\xi)}{(\xi - z)^k} d\xi$$

is analytic in $\mathbb{C} \setminus \gamma$ and its derivative is given by

$$H'(z) = k \cdot \int_\gamma \frac{h(\xi)}{(\xi - z)^{k+1}} d\xi .$$

We will only need the cases $k=1$ and $k=2$, so we do not give the details of the general case, but instead focus on the concrete case $k=2$.

Proof for the case $k=2$:

In this case we have

$$H(z) = \int_\gamma \frac{h(\xi)}{(\xi - z)^2} d\xi$$

and we seek to prove that

$$H'(z) = 2 \cdot \int_\gamma \frac{h(\xi)}{(\xi - z)^3} d\xi ,$$

or equivalently (since $H'(z) = \lim\limits_{w \to z} \frac{H(w) - H(z)}{w - z}$)

$$\lim_{w \to z} \left| \frac{H(w) - H(z)}{w - z} - 2 \int_\gamma \frac{h(\xi)}{(\xi - z)^3} d\xi \right| = 0.$$

We calculate:

$$\frac{H(w)-H(z)}{w-z} = \frac{1}{w-z}\int_\gamma h(\xi)\left(\frac{1}{(\xi-w)^2}-\frac{1}{(\xi-z)^2}\right)d\xi$$

$$\overset{\circledast}{=} \int_\gamma \frac{(2\xi-w-z)\cdot h(\xi)}{(\xi-w)^2(\xi-z)^2}\,d\xi\,.$$

$\circledast$ Here we used the calculation

$$\frac{1}{(\xi-w)^2}-\frac{1}{(\xi-z)^2} = \frac{(\xi-z)^2-(\xi-w)^2}{(\xi-w)^2(\xi-z)^2}$$

$$= \frac{\xi^2-2\xi z+z^2-(\xi^2-2\xi w+w^2)}{(\xi-w)^2(\xi-z)^2}$$

$$= \frac{z^2-w^2+2\xi(w-z)}{(\xi-w)^2(\xi-z)^2} = \frac{(w-z)(2\xi-w-z)}{(\xi-w)^2(\xi-z)^2}\,.$$

Further, we get:

$$\frac{H(w)-H(z)}{w-z} - 2\int_\gamma \frac{h(\xi)}{(\xi-z)^3}\,d\xi$$

$$= \int_\gamma h(\xi)\left(\frac{2\xi-w-z}{(\xi-w)^2(\xi-z)^2} - 2\frac{1}{(\xi-z)^3}\right)d\xi$$

$$\overset{\circledast}{=} (w-z)\cdot\int_\gamma h(\xi)\,\frac{3\xi-2w-z}{(\xi-w)^2(\xi-z)^3}\,d\xi$$

and the prefactor $(w-z)$ above is ultimately the reason why this $\longrightarrow 0$ as $w \longrightarrow z$.

$\circledast$ Here we used the calculation

$$\frac{2\xi-w-z}{(\xi-w)^2(\xi-z)^2} - 2\frac{1}{(\xi-z)^3}$$

$$= \frac{(\xi-z)(2\xi-w-z)-2(\xi-w)^2}{(\xi-w)^2(\xi-z)^3}$$

$$= \frac{2\xi^2-\xi w-3\xi z+zw+z^2-2(\xi^2-2\xi w+w^2)}{(\xi-w)^2(\xi-z)^3}$$

$$= (w-z)\,\frac{3\xi-2w-z}{(\xi-w)^2(\xi-z)^3}$$

To make the rest of the argument precise, for the fixed $z \in \mathbb{C} \setminus \gamma$, choose an $r > 0$ such that $B(z;r) \subset \mathbb{C} \setminus \gamma$

(possible: γ is compact, so $z \notin \gamma$ implies $\operatorname{dist}(z,\gamma) > 0$)

and choose an $R > 0$ such that $\gamma \subset B(z;R)$.

(also possible: γ is compact, so bounded, and contained in $B(z;R)$ for R large enough)

Then first of all $|\zeta - z| \geq r$ for any ζ on the contour γ. Moreover, for w such that $0 < |w - z| < \frac{2r}{3}$ we also have $|\zeta - w| \geq \frac{r}{3}$ for any ζ on γ.

For such w, we estimate

$$\left| \frac{H(w) - H(z)}{w - z} - 2\int_\gamma \frac{h(\zeta)}{(\zeta - z)^3}\, d\zeta \right|$$

$$\leq \left| (w-z) \cdot \int_\gamma h(\zeta) \frac{3\zeta - 2w - z}{(\zeta - w)^2 (\zeta - z)^2}\, d\zeta \right|$$

$$\leq |w - z| \cdot \underbrace{\ell(\gamma) \cdot \left(\sup_\zeta |h(\zeta)|\right) \cdot \frac{R + 2\cdot\left(R + \frac{2r}{3}\right)}{(r/3)^2 \; r^3}}_{\text{const.}}$$

$\longrightarrow 0$ as $w \to z$. $\square$

Here is an immediate interesting consequence:

Theorem (Cauchy's integral formula for derivative)

[cauchy-formula-derivative]

Let $f : U \to \mathbb{C}$ be an analytic function on an open set $U \subset \mathbb{C}$, and let $\overline{B}(z_0; r) \subset U$ be a closed disk contained in the domain of f. Then for any $z \in B(z_0; r)$ we have

$$f'(z) = \frac{1}{2\pi i} \oint_{\partial B(z;r)} \frac{f(\xi)}{(\xi - z)^2} \, d\xi$$

where the integration is along the circle $\partial B(z;r)$ in positive orientation (counterclockwise).

Proof Cauchy's integral formula gives

$$f(z) = \frac{1}{2\pi i} \oint_{\partial B(z;r)} \frac{f(\xi)}{\xi - z} \, d\xi ,$$

so the asserted formula for $f'(z)$ follows from the previous lemma (case $k=1$). $\square$

And here is another:

Lemma (Analyticity of the derivative)

[derivative analytic]

Let $f : U \to \mathbb{C}$ be an analytic function on an open set $U \subset \mathbb{C}$. Then also the derivative $f' : U \to \mathbb{C}$ is analytic.

Proof For $z_0 \in U$ choose $r > 0$ such that $\overline{B}(z_0; r) \subset U$. By Cauchy's integral formula for the derivative, we have

$$f'(z) = \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f(\xi)}{(\xi - z)^2} d\xi$$

for $z \in B(z_0; r)$. Applying the $k=2$ case of the previous lemma yields

$$f''(z_0) = \frac{1}{2\pi i} \cdot 2 \cdot \oint_{\partial B(z_0;r)} \frac{f(\xi)}{(\xi - z_0)^3} d\xi,$$

and in particular the derivative f' has a derivative $f''(z_0)$ at an arbitrary point $z_0 \in U$. $\square$

We thus get a striking result about analytic functions

(... and it is remarkable how easily we arrived to it from Cauchy's integral formula !)

Corollary

[higher_derivatives_analytic]

An analytic function $f : U \to \mathbb{C}$ is infinitely differentiable, i.e., it has derivatives of all orders.

Proof This at once follows by induction from the analyticity of f'. $\square$

Even more concretely, the inductive application of the earlier lemma yields a formula for the m^{th} derivative:

$$f^{(m)}(z) = \frac{m!}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f(\xi)}{(\xi - z)^{m+1}} \, d\xi .$$

From Cauchy's formula for the m^{th} derivative, we get a very useful estimate for the higher order derivatives of an analytic function in terms of the function itself.

[cauchy-derivative-estimate]

Lemma (Cauchy's derivative estimate)

Let $f: U \to \mathbb{C}$ be analytic in an open set $U \subset \mathbb{C}$, and let $z_0 \in U$ and $r > 0$ be such that $B(z_0; r) \subset U$. If $|f(z)| \leq M$ for all $z \in B(z_0; r)$, then

$$|f^{(m)}(z)| \leq \frac{m! \cdot M \cdot r}{(r - |z - z_0|)^{m+1}}$$

for every $z \in B(z_0; r)$, and in particular

$$|f^{(m)}(z_0)| \leq \frac{m!\, M}{r^m}.$$

Proof: Fix $z \in B(z_0; r)$ and ρ such that $|z - z_0| < \rho < r$. The integral formula for the m^{th} derivative gives

$$f^{(m)}(z) = \frac{m!}{2\pi i} \oint_{\partial B(z_0;\rho)} \frac{f(\zeta)}{(\zeta - z)^{m+1}}\, d\zeta .$$

For $\zeta \in \partial B(z_0; \rho)$ we have $|\zeta - z_0| = \rho$ and so $|\zeta - z| \geq |\zeta - z_0| - |z - z_0| = \rho - |z - z_0|$.

The triangle inequality for integrals thus gives

$$\left| f^{(m)}(z) \right| = \frac{m!}{2\pi} \left| \oint_{\partial B(z_0;\rho)} \frac{f(\zeta)}{(\zeta - z)^{m+1}}\, d\zeta \right|$$

$$\leq \frac{m!}{2\pi} \oint_{\partial B(z_0;\rho)} \underbrace{\frac{|f(\zeta)|}{|\zeta - z|^{m+1}}}_{\leq \frac{M}{(\rho - |z - z_0|)^{m+1}}} |d\zeta|$$

$$\leq \frac{m!}{2\pi} \cdot \underbrace{\ell(\partial B(z_0;\rho))}_{= 2\pi\rho} \cdot \frac{M}{(\rho - |z - z_0|)^{m+1}}$$

$$= \frac{m! \cdot M \cdot \rho}{(\rho - |z - z_0|)^{m+1}} .$$

Since $\rho < r$ was arbitrary, we may let $\rho \nearrow r$ to get the asserted bound. $\square$

The previous estimate easily gives:

Theorem (Liouville's theorem)

[Liouville]

Suppose that $f: \mathbb{C} \to \mathbb{C}$ is an analytic function in the whole plane $\mathbb{C}$ such that for some $M \geq 0$ we have $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f is a constant.

Proof: Note that for any $z_0 \in \mathbb{C}$ and any $r > 0$ we have $B_r(z_0) \subset \mathbb{C}$. Cauchy's estimate for the first derivative thus gives (for any z_0 and r)

$$|f'(z_0)| \leq \frac{1! \cdot M}{r} \xrightarrow[r \to \infty]{} 0,$$

which implies $f'(z_0) = 0$ for any $z_0 \in \mathbb{C}$. Therefore the derivative f' vanishes everywhere in $\mathbb{C}$, and by connectedness of $\mathbb{C}$, this implies that f is a constant function. $\square$

Interestingly, the previous (surprising but) easy result easily in turn gives:

[fundamental_theorem_of_algebra]

Theorem (The fundamental theorem of algebra) Suppose that $P : \mathbb{C} \to \mathbb{C}$ is a polynomial function which is not constant. Then there exists a $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

Proof Assume, by way of contradiction, that P is a nonconstant polynomial which does not vanish anywhere, $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $f(z) = \frac{1}{P(z)}$ defines an analytic function $f : \mathbb{C} \to \mathbb{C}$. We claim that f is bounded. Once that is shown, Liouville's theorem implies that f is a constant, and we get that

$P(z) = \frac{1}{f(z)}$ is also a constant, contradicting our assumption.

To show that f is bounded, first write P explicitly as

$$P(z) = c_d z^d + c_{d-1} z^{d-1} + \cdots + c_2 z^2 + c_1 z + c_0$$

$$= \sum_{n=0}^{d} c_n z^n$$

with $c_0, c_1, \cdots, c_{d-1}, c_d \in \mathbb{C}$ and $c_d \neq 0$.

Write $Q(z) = P(z) - c_d z^d = \sum_{n=0}^{d-1} c_n z^n$,

and note that

$$\frac{|Q(z)|}{|z|^d} \leq \frac{|c_{d-1}|\,|z|^{d-1} + \cdots + |c_1|\,|z| + |c_0|}{|z|^d}$$

$$\xrightarrow[|z|\to\infty]{} 0$$

(the numerator is a lower order polynomial).

In particular, there exists an $R > 0$ such that whenever $|z| > R$ we have $\frac{|Q(z)|}{|z|^d} \leq \frac{|c_d|}{3}$.

Then for $|z| > R$ we also have

$$|P(z)| = |c_d z^d + Q(z)|$$
$$\geq |c_d| |z|^d - |Q(z)|$$
$$\geq |c_d| |z|^d - \frac{|c_d|}{3} |z|^d = \frac{2|c_d|}{3} |z|^d$$
$$\geq \frac{2|c_d|}{3} R^d$$

and in turn

$$|f(z)| = \frac{1}{|P(z)|} \leq \frac{3R^d}{2|c_d|}.$$

On $\overline{B_R(0)} = \{ z \in \mathbb{C} \mid |z| \leq R \}$ on the other hand, the continuous function f must be bounded by compactness, so there exists some $A > 0$ such that $|f(z)| \leq A$ for $|z| \leq R$.

Setting $M = \max\left\{ A, \frac{3R^d}{2|c_d|} \right\}$, we have $|f(z)| \leq M$ for all $z \in \mathbb{C}$, proving boundedness of f and thus finishing the proof. $\square$

By general theory of polynomials (the division algorithm), we get a very concrete version of the Fundamental theorem of algebra:

<u>Corollary</u> (Factorization of complex polynomials)

[complex_polynomial_factorization]

Let $P(z) = \sum_{n=0}^{d} c_n z^n$ be a polynomial of degree d (so $c_d \neq 0$) with complex coefficients $c_0, c_1, \dots, c_d \in \mathbb{C}$.
Then there exist $\underbrace{z_1, z_2, \dots, z_d}_{\text{the "roots" of } P \text{ (with multiplicity)}} \in \mathbb{C}$ such that

$$P(z) = c_d \cdot \prod_{j=1}^{d} (z - z_j)$$
$$= c_d \cdot (z - z_1) \cdot \dots \cdot (z - z_d).$$

<u>Proof</u> Proceed by induction on the degree d.

<u>Base case</u>:

For $d = 1$, the assertion is clear:

$P(z) = c_1 z + c_0 = c_1 \left(z - \frac{c_0}{c_1}\right)$.

(Also $d = 0$ is clear: the empty product is 1.)

<u>Induction step</u>:

Assume the claim holds for all polynomials of degree d and consider a polynomial P of degree $d+1$.
Such a polynomial is not a constant (the degree $d+1$ is not 0), so by the Fundamental theorem of algebra, there exists a $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

Then by polynomial division, we get

$$P(z) = (z-z_0)\,\tilde{P}(z),$$

where $\tilde{P}$ is a polynomial of degree d. By induction hypothesis

$$\tilde{P}(z) = \tilde{c}_d \cdot \prod_{j=1}^{d} (z-\tilde{z}_j)$$

for some $\tilde{c}_d \neq 0$ and $\tilde{z}_1, \ldots, \tilde{z}_d \in \mathbb{C}$.

For P we thus get the expression

$$P(z) = (z-z_0)\cdot \tilde{c}_d \cdot (z-\tilde{z}_1)(z-\tilde{z}_2)\cdots(z-\tilde{z}_d)$$

which (up to renaming the roots) is the asserted factorization.

$\square$

Among the consequences of Cauchy's integral formula, we now turn to a different kind of application: a sufficient condition for a continuous function to be analytic:

Theorem (Morera's theorem)

[morera]

Suppose that a continuous function $f: U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ satisfies $\oint_{\partial\Delta} f(z)\,dz = 0$ for all closed triangles $\Delta \subset U$ contained in its domain. Then f is analytic.

Proof If $\oint_{\partial\Delta} f(z)\,dz = 0$ for all Δ, then in any disk $B \subset U$ (or more generally in any star-shaped subdomain) we are able to construct a (local) primitive $F: B \to \mathbb{C}$ for $f|_B$ by the same method as in the proof of Cauchy's integral theorem (for star-shaped domains). Now $F: B \to \mathbb{C}$ is analytic, with $F' = f$ in B. Since the derivative of an analytic function is analytic, we conclude that f is analytic in B. Since $B \subset U$ was arbitrary, analyticity in U follows. $\square$

Winding numbers

For the general form of Cauchy's integral formula, it is convenient to have a notion which counts the number of revolutions made by a closed contour γ around a point $z \in \mathbb{C}$ which does not lie on γ.

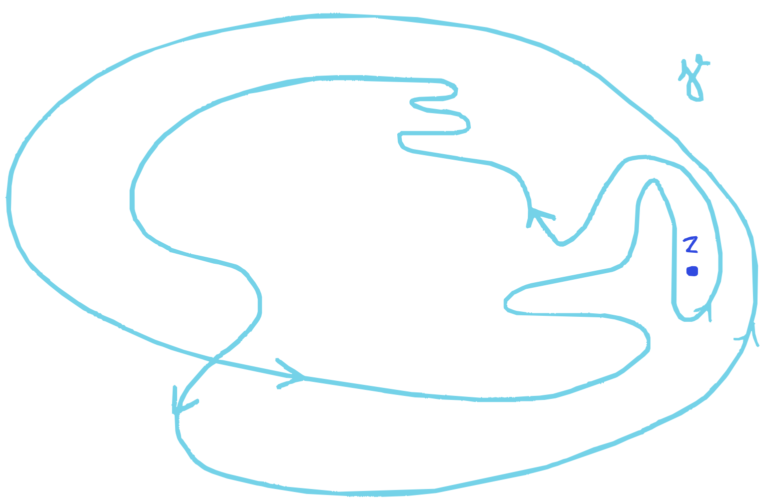

Figure: $n_\gamma(z) = 2$

Def

[winding-number]

The winding number of a closed contour γ around a point $z \in \mathbb{C} \setminus \gamma$ is $n_\gamma(z) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - z} \, d\zeta$.

Lemma

The winding number of γ around z is an integer, $n_\gamma(z) \in \mathbb{Z}$.

Proof Let us for simplicity assume that $\gamma : [a,b] \to \mathbb{C}$ is a smooth closed path. (The reader can modify the details of the proof to accommodate also piecewise smooth γ.)

Define $h(t) = \int_a^t \frac{1}{\gamma(t) - z} \dot{\gamma}(t)\, dt$, so that $h(a) = 0$ and $h(b) = 2\pi i \cdot n_\gamma(z)$

Then $\dot{h}(t) = \frac{\dot{\gamma}(t)}{\gamma(t) - z}$ for any t.

Consider also $k(t) = e^{-h(t)} \cdot (\gamma(t) - z)$ and observe that

$$\dot{k}(t) = -\dot{h}(t)\, e^{-h(t)} (\gamma(t) - z) + e^{-h(t)} \dot{\gamma}(t)$$

$$= -\dot{\gamma}(t)\, e^{-h(t)} + e^{-h(t)} \dot{\gamma}(t) = 0,$$

so k is in fact a constant.

In particular $k(a) = k(b)$, which gives

$$\begin{array}{ccc} k(a) & = & k(b) \\ \| & & \| \\ e^{-h(a)} \cdot (\gamma(a) - z) & & e^{-h(b)} \cdot (\gamma(b) - z) \\ (h(a)=0) \longrightarrow \| & & (h(b) = 2\pi i\, n_\gamma(z)) \rightarrow \| \\ 1 \cdot \underbrace{(\gamma(a) - z)}_{\neq 0} & & e^{-2\pi i\, n_\gamma(z)} \cdot \underbrace{(\gamma(b) - z)}_{\neq 0} . \end{array}$$

(since z is not on the path)

$\gamma(a) = \gamma(b)$ (closed path)

This shows $1 = e^{-2\pi i\, n_\gamma(z)}$, which by properties of complex exp is only possible if $n_\gamma(z) \in \mathbb{Z}$.

$\square$

Lemma

[winding-number-properties]

Let γ be a closed contour. Then

(i): $n_\gamma(z)$ is constant for z in each connected component of $\mathbb{C} \setminus \gamma$.

(ii): $n_\gamma(z) = 0$ for z in the unbounded component of $\mathbb{C} \setminus \gamma$.

(iii): If γ is a simple closed contour, then either $n_\gamma(z) = 1$ or $n_\gamma(z) = -1$ for all z in the bounded component of $\mathbb{C} \setminus \gamma$.

Proof of (i):

In order to prove that $z \mapsto n_\gamma(z)$ is constant in each connected component, it suffices to show that it has vanishing complex derivatives, $n_\gamma'(z) = 0$ for all $z \in \mathbb{C} \setminus \gamma$. By an earlier lemma,

$$n_\gamma'(z) = \frac{d}{dz}\left(\frac{1}{2\pi i}\oint_\gamma \frac{1}{\xi - z}\, d\xi\right)$$

$$= \frac{1}{2\pi i}\oint_\gamma \frac{1}{(\xi - z)^2}\, d\xi \overset{*}{=} 0$$

* since the function $\xi \mapsto \frac{1}{(\xi - z)^2}$ has a primitive $\xi \mapsto \frac{-1}{\xi - z}$ in $\mathbb{C} \setminus \{z\}$, and γ is a closed contour in $\mathbb{C} \setminus \{z\}$ (since $z \in \mathbb{C} \setminus \gamma$).

$\square$