Recall that the complex derivative of a function $f: A \to \mathbb{C}$ at an interior point $z_0 \in A \subset \mathbb{C}$ is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

if this limit exists.

Recall also that $f: U \to \mathbb{C}$ is said to be analytic on an open set $U \subset \mathbb{C}$ if it has a complex derivative at every point z_0 in U. We also showed that the analyticity of f on U is equivalent to differentiability of f in the real sense and the Cauchy-Riemann equations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

holding everywhere on U for the component functions

$$u(x,y) = \mathrm{Re}(f(x+iy)), \quad v(x,y) = \mathrm{Im}(f(x+iy)).$$

Constantness criteria for analytic functions

As the first consequences of Cauchy-Riemann equations, let us note some rigidity results for analytic functions.

Recall first from calculus:

Lemma [vanishing_partial_derivatives]

> If a function $u : D \to \mathbb{R}$ on a connected open subset $D \subset \mathbb{R}^2$ of the plane is such that
> $$\frac{\partial}{\partial x} u \equiv 0 \quad \text{and} \quad \frac{\partial}{\partial y} u \equiv 0 \quad \text{on } D$$
> then u is a constant on D.

A straightforward consequence is:

Lemma [zero_derivative_implies_constant]

> If a function $f : D \to \mathbb{C}$ on a connected open subset $D \subset \mathbb{C}$ has vanishing complex derivative $f'(z) = 0$ at every $z \in D$, then f is a constant.

Using this and Cauchy-Riemann equations, we get somewhat surprinsing sufficient criteria for an analytic function to be constant.

<u>Lemma</u>

[constantness_criteria]

Let $D \subset \mathbb{C}$ be a connected open subset of the complex plane, and let $f : D \to \mathbb{C}$ be an analytic function. If any one of the functions $z \mapsto \mathrm{Re}(f(z))$, $z \mapsto \mathrm{Im}(f(z))$, or $z \mapsto |f(z)|$ is constant, then f itself is constant.

Proof: Write $u(x,y) = \mathrm{Re}(f(x+iy))$ and $v(x,y) = \mathrm{Im}(f(x+iy))$ for the component functions.

Assume first that u is constant on D. Then $\frac{\partial u}{\partial x} \equiv 0$ and $\frac{\partial u}{\partial y} \equiv 0$. By C-R equations we get

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \equiv 0, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \equiv 0,$$

which implies that v is also a constant on D, and thus $f = u + iv$ is indeed a constant.

Assuming v constant similarly leads to u constant (by C-R) and thus f constant.

For the final case, assume

$$|f(x+iy)|^2 = u(x,y)^2 + v(x,y)^2 = c.$$

If $c = 0$ then immediately $f \equiv 0$, so we may assume $c > 0$.

Then differentiating this equation w.r.t. x and y we get

$$2 \cdot u \cdot \frac{\partial u}{\partial x} + 2 \cdot v \cdot \frac{\partial v}{\partial x} = 0$$

$$2 \cdot u \cdot \frac{\partial u}{\partial y} + 2 \cdot v \cdot \frac{\partial v}{\partial y} = 0 .$$

Using C-R equations, these imply

$$u \cdot \frac{\partial u}{\partial x} - v \cdot \frac{\partial u}{\partial y} = 0 ,$$

$$u \cdot \frac{\partial u}{\partial y} + v \cdot \frac{\partial u}{\partial x} = 0 .$$

Multiply the first of these by u and the second by v and add to get

$$0 = u^2 \frac{\partial u}{\partial x} - \cancel{uv} \frac{\partial u}{\partial y} + \cancel{uv} \frac{\partial u}{\partial y} + v^2 \frac{\partial u}{\partial x}$$

$$= (u^2 + v^2) \frac{\partial u}{\partial x} = c \cdot \frac{\partial u}{\partial x} .$$

We conclude $\frac{\partial u}{\partial x} \equiv 0$. Similarly, multiplying the first by $-v$ and the second by u and adding, we get $\frac{\partial u}{\partial y} \equiv 0$. Thus u is a constant on D. By the first part of the proof, then, f is a constant again. $\square$

Harmonicity and harmonic conjugates

Def:

[harmonic] A C^2-function* $u : U \to \mathbb{R}$ on an open set $U \subset \mathbb{R}^2$ is harmonic if
$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0 \quad \text{on } U.$$

* The notation C^k ($k \in \mathbb{N}$) is used for k times continuously differentiable functions, i.e., functions whose all partial derivatives of order k are continuous functions. So C^2 means "twice continuously differentiable". With this assumption, in particular the Laplacian
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
is well-defined (and continuous).

Lemma

[real-and-imaginary-parts-harmonic]

Suppose that $f : U \to \mathbb{C}$ is analytic and its component functions $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ are twice continuously differentiable. Then u and v are harmonic.

Remark We will prove later that analytic functions have derivatives of all orders, so the assumption that u and v are C^2 turns out to be unnecessary (automatically guaranteed already by the analyticity of f).

Proof: Consider, e.g., u. Exchanging the order of partial derivatives (see calculus courses for justification) and using C-R equations, we find

$$\frac{\partial^2}{\partial y^2} u = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$
$$= -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 u}{\partial x^2}. \quad \square$$

As another rigidity property, one may try to reconstruct an analytic function from its real part u (or imaginary part v) only. By the previous lemma, the real part must at least be harmonic.

Def

[harmonic_conjugate]

Let $u : U \to \mathbb{R}$ be a harmonic function on an open set $U \subset \mathbb{R}^2$. Then a function $v : U \to \mathbb{R}$ is called a harmonic conjugate of u if the function

$$x + iy \longmapsto u(x,y) + i\,v(x,y)$$

is analytic in U.

Let us consider an example of how to find harmonic conjugates.

Example

Let $u: \mathbb{R}^2 \to \mathbb{R}$ be the function

$$u(x,y) = x^2 y - \frac{1}{3} y^3.$$

Let us try to (systematically) find a harmonic conjugate to u, i.e., a function $v: \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$f(x+iy) = u(x,y) + i\, v(x,y)$$

defines an analytic function $\mathbb{C} \to \mathbb{C}$.

For this to be possible, u has to at least be harmonic, so let us verify that first. Calculate:

$$\frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y)$$

$$= \frac{\partial^2}{\partial x^2}\left(x^2 y - \frac{1}{3} y^3\right) + \frac{\partial^2}{\partial y^2}\left(x^2 y - \frac{1}{3} y^3\right)$$

$$= \frac{\partial}{\partial x}\left(2xy - 0\right) + \frac{\partial}{\partial y}\left(x^2 - y^2\right)$$

$$= 2y + (-2y) = 0.$$

So indeed u is harmonic.

If v is a harmonic conjugate, then together with u it has to satisfy C-R equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\left(x^2 y - \frac{1}{3}y^3\right) = -x^2 + y^2$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2 y - \frac{1}{3}y^3\right) = 2xy.$$

Considering a fixed $x \in \mathbb{R}$, the function $y \mapsto v(x,y)$ has to be (by the second equation above) of the form

$$v(x,y) = \int 2xy\, dy = C(x) + xy^2$$

where $C(x) \in \mathbb{R}$ is an integration constant that may depend on x.

The first equation above then requires that

$$-x^2 + y^2 = \frac{\partial v}{\partial x} = C'(x) + y^2,$$

from which we get $C'(x) = -x^2$.

This implies $C(x) = -\frac{1}{3}x^2 + c$

with $c \in \mathbb{R}$. Now simplifying, we have found that v must be of the form

$$v(x,y) = -\frac{1}{3}x^3 + xy^2 + c.$$

It remains to check that v of the above form indeed makes $f = u + iv$ analytic, but this indeed follows, since C-R equations hold for these continuously differentiable functions u, v.

The analytic functions $f = u + iv$ take the form

$$f(x+iy) = u(x,y) + i\,v(x,y)$$

$$= x^2 y - \frac{1}{3}y^3 + i x y^2 - \frac{i}{3}x^3 + ic$$

$$= -\frac{i}{3}(x+iy)^3 + ic$$

i.e. $f(z) = -\frac{i}{3}z^3 + ic.$

A natural question: Is it always possible to find a harmonic conjugate to a given harmonic function $u: U \to \mathbb{R}$?

Not quite. For example the function

$$u : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$$
$$u(x,y) = \log(x^2+y^2)$$

is harmonic, but does not have a (single-valued) harmonic conjugate on $U = \mathbb{R}^2 \setminus \{(0,0)\}$. The obstruction is related to topology: U is not simply-connected (there are non-contractible loops in U).

(we leave it to the reader to think why)

In simply connected domains, it turns out that harmonic conjugates always exist. For concreteness, we state this only for disks.

Lemma (Local existence of harmonic conj.)

[harmonic_conjugate_existence]

Let $B = B(z_0, R) \subset \mathbb{C}$ be a disk, and let $u : B \to \mathbb{R}$ be a harmonic function. Then there exists another harmonic function $v : B \to \mathbb{R}$ such that $x + iy \mapsto u(x,y) + i\,v(x,y)$ is analytic in B.

The proof is not very difficult (uses Stokes' formula from calculus) but we omit it here.

Rules of differentiation

The complex derivatives obey many familiar differentiation rules.

Linearity

Suppose that $f, g : A \to \mathbb{C}$ have complex derivatives $f'(z_0)$ and $g'(z_0)$ at $z_0 \in A$. Then

[derivative_linearity]

$z \mapsto f(z) + g(z)$ has derivative

$$(f+g)'(z_0) = f'(z_0) + g'(z_0) \quad \text{at } z_0.$$

Also for any $c \in \mathbb{C}$,

$z \mapsto c \cdot f(z)$ has derivative

$$(c \cdot f)'(z_0) = c \cdot f'(z_0) \quad \text{at } z_0.$$

(These properties are direct consequences of the linearity of limits and the definition of complex derivatives.)

Chain rule

[chain_rule]

If $A, B \subset \mathbb{C}$ are subsets of $\mathbb{C}$ and $f: A \to B$, $g: B \to \mathbb{C}$ are functions which have derivatives $f'(z_0)$ and $g'(w_0)$ at interior points $z_0 \in A$ and $w_0 = f(z_0) \in B$, then

$z \longmapsto g(f(z)) = (g \circ f)(z)$ has deriv.

$$(g \circ f)'(z_0) = f'(z_0) \cdot g'(f(z_0)) \quad \text{at } z_0.$$

Leibniz product rule

[leibniz_rule]

If $f, g : A \to \mathbb{C}$ have derivatives $f'(z_0)$ and $g'(z_0)$ at $z_0 \in A$, then

$z \longmapsto f(z) \cdot g(z)$ has derivative

$$(f \cdot g)'(z_0) = f'(z_0) \cdot g(z_0) + f(z_0) \cdot g'(z_0).$$

Quotient rule

[quotient_derivative]

If $f, g : A \to \mathbb{C}$ have derivatives $f'(z_0)$ and $g'(z_0)$ at $z_0 \in A$ and $g(z_0) \neq 0$, then $z \longmapsto f(z)/g(z)$ has derivative

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)\,g(z_0) - f(z_0)\,g'(z_0)}{g(z_0)^2}.$$

Finally, for derivatives of local inverse functions we have:

Lemma

[inverse_derivative]

Suppose that an analytic function f has in some neighborhood of a point z_0 a continuous inverse function f^{-1}, and suppose that the derivative $f'(z_0)$ is nonzero. Then the inverse function f^{-1} has a derivative

$$(f^{-1})'(f(z_0)) = \frac{1}{f'(z_0)}.$$

Remark We will later see that analytic functions f have continuous local inverse functions near any point z_0 where the derivative $f'(z_0)$ is nonzero.

Proof: Let $U \subset \mathbb{C}$ be a neighborhood of z_0 such that $f: U \to V$ has an inverse function $f^{-1}: V \to U$ which is continuous. Denote $w_0 = f(z_0)$.

Now for $k \in \mathbb{C}$ with $|k|$ small enough, we have $w_0 + k \in V$, and we may write

$$f^{-1}(w_0+k) = z_0 + h(k) \in U$$

where $h(k) := f^{-1}(w_0+k) - z_0$. By continuity of f^{-1} we have

$$\lim_{k \to 0} (h(k)) = \lim_{k \to 0} (f^{-1}(w_0+k)) - z_0$$
$$= f^{-1}(w_0) - z_0 = 0.$$

Let us then consider the limit of difference quotients that defines the derivative of f^{-1} at w_0:

$$\lim_{k\to 0} \frac{f^{-1}(w_0+k) - f^{-1}(w_0)}{k}$$

$$= \lim_{k\to 0} \frac{z_0 + h(k) - z_0}{w_0 + k - w_0}$$

$$= \lim_{k\to 0} \frac{h(k)}{f(z_0+h(k)) - f(z_0)} = \frac{1}{f'(z_0)}$$

by definition of the derivative $f'(z_0)$ and by properties of limits.

This shows $(f^{-1})'(w_0)$ exists and equals $1/f'(z_0)$.

$\square$

Examples of analytic functions

Polynomials

Let us progressively make some easy observations:

- constant functions are analytic on $\mathbb{C}$ (zero derivative)
- the identity function $z \mapsto z$ is analytic on $\mathbb{C}$ (derivative const. 1)
- inductively with Leibniz' rule we get that $z \mapsto z^n$ $(= z \cdot z^{n-1})$ is analytic on $\mathbb{C}$ with derivative $n \cdot z^{n-1}$
- using linearity of derivatives, we find that any polynomial function $p : \mathbb{C} \to \mathbb{C}$

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

(with coefficients $a_0, a_1, \dots, a_n \in \mathbb{C}$) is analytic on $\mathbb{C}$.

[polynomial_analytic]

Rational functions

[rational-function-analytic]

Let $p, q : \mathbb{C} \to \mathbb{C}$ be polynomial functions, and let $U = \{z \in \mathbb{C} \mid q(z) \neq 0\}$ (open set in $\mathbb{C}$). The rational function $r : U \to \mathbb{C}$, $r(z) = \frac{p(z)}{q(z)}$ is analytic on U by the derivative rule for quotients.

Exponential function

[exp-analytic]

We saw that $\exp : \mathbb{C} \to \mathbb{C}$, $z \mapsto e^z$, is analytic on $\mathbb{C}$ (by verifying Cauchy-Riemann equations).

Trigonometric functions

The trigonometric functions given by $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$ and $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$ are analytic on $\mathbb{C}$ by analyticity of exp and linearity of derivatives.

Logarithms (NOTE: branch choice difficulties!)

If $\ell : U \to \mathbb{C}$ is a continuous function on an open set $U \subset \mathbb{C}$ such that $e^{\ell(w)} = w$ for any $w \in U$, then it provides a local inverse to the analytic function exp in a neighborhood of any point $\ell(w)$ in its image. Since

$$\exp'(\ell(w)) = \exp(\ell(w)) = w,$$

by the inverse function derivative rule we find

$$\ell'(w) = \frac{1}{\exp'(\ell(w))} = \frac{1}{w}.$$

Such continuous local inverses to exp are called branches of the complex logarithm.

The principal branch

$$\mathrm{Log} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$$

is one, but there are others.

n^{th} roots (NOTE: branch choice difficulties!)

Fix $n \in \mathbb{N}$. If $r : U \to \mathbb{C}$ is a continuous function on an open set $U \subset \mathbb{C}$ such that $r(w)^n = w$ for any $w \in U$, then it provides a local inverse to the analytic function $z \mapsto z^n$ in a neighborhood of any point $r(w)$ in its image. Since

$$\frac{d}{dz}(z^n) = n \cdot z^{n-1}$$

by the inverse function derivative rule we find

$$r'(w) = \frac{1}{n \cdot r(w)^{n-1}}.$$

(Informally, $\frac{d}{dw} \sqrt[n]{w} = \frac{1}{n \cdot (\sqrt[n]{w})^{n-1}}$.)

Such continuous inverses to $z \mapsto z^n$ are called branches of the complex n^{th} root function.