Aalto University
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2025-2026/II

Problem set 1B

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Exercise sessions: Thu-Fri, Oct 23-24, 2025

Topic: topological notions (limits, continuity), and calculations in the complex plane

The first two exercises (marked with symbol ♮) are to be solved before the exercise sessions so that you are ready to present your solution on the blackboard. The last two exercises are to be discussed and solved in exercise sessions.

♮ Blackboard exercise 1.

Determine whether the following limits exist:

(a)
$$\lim_{n\to\infty} \big(\mathbb{i}^{n!} + 2^{-n}\big) \qquad \text{(limit along } n \in \mathbb{N}\text{)} \;;$$

(b)
$$\lim_{z\to 0} \left(\frac{1}{|z|^2} \sin\left(\frac{z^2 - \overline{z}^2}{4\mathbb{i}} \right) \right) \qquad \text{(limit along } z \in \mathbb{C} \setminus \{0\}\text{)} \;.$$

Hint: *It may be helpful to start by simplifying* $\frac{z^2-\overline{z}^2}{4\mathbb{i}}$ *when* $z = x + \mathbb{i}y$. *Observe in particular that this expression takes real values, so only the familiar real* sin *function appears above.*

♮ Blackboard exercise 2.

For $n \in \mathbb{N}$, show that

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) \;=\; \frac{1}{2} + \frac{\sin\big((2n+1)\theta/2\big)}{2\sin(\theta/2)} \;.$$

Hint: *Recall Exercise 1A.2 and the formulas* $\sin(\theta) = \frac{1}{2\mathbb{i}}\big(e^{\mathbb{i}\theta} - e^{-\mathbb{i}\theta}\big)$ *and* $\cos(\theta) = \frac{1}{2}\big(e^{\mathbb{i}\theta} + e^{-\mathbb{i}\theta}\big)$ *for trigonometric functions in terms of complex exponentials.*

Exercise (in class) 3.
Let $U \subset \mathbb{C}$ be a nonempty open subset of the complex plane. Consider possible functions $f : U \to \mathbb{C}$ which satisfy

$$(\star) : \qquad e^{f(z)} = 1 \qquad \text{for all } z \in U.$$

(a) Show that there exist nonconstant functions $f : U \to \mathbb{C}$ satisfying the condition $(\star)$.
(b) Show that if U is not connected, then there exist nonconstant continuous functions $f : U \to \mathbb{C}$ satisfying the condition $(\star)$.
(c) Show that if U is connected, then any continuous function $f : U \to \mathbb{C}$ satisfying the condition $(\star)$ is a constant function.

Exercise (in class) 4.
Find all points of discontinuity of the following functions:

(a) $f : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ given by

$$f(z) = \operatorname{Arg}(z^2)$$

where Arg is the principal branch (with values in $(-\pi, \pi]$) of the argument;
(b) $g : \mathbb{C} \to \mathbb{C}$ given by

$$g(z) = \sqrt{1 - z^2}\ ;$$

where $\sqrt{\cdot}$ denotes the principal branch (with arguments of values in $(-\pi/2, \pi/2]$) of the complex square root.