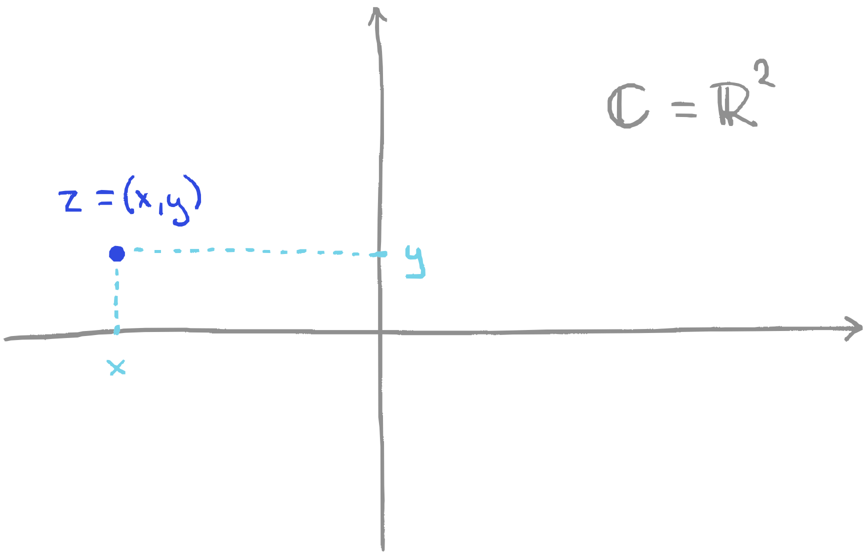
# 1.1 THE FIELD OF COMPLEX NUMBERS




**Def.:**

[complex_numbers]

A **complex number** is an ordered pair $z = (x,y)$ of two real numbers $x, y \in \mathbb{R}$.

The set of complex numbers is denoted by

$$\mathbb{C} = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

$\mathbb{C}$ can be visualized as the 2-dimensional plane, "complex plane". A pair $z = (x,y)$ specifies two coordinates.

For $z = (x, y) \in \mathbb{C}$, we call

$x =: \mathrm{Re}(z)$ the real part of $z$

$y =: \mathrm{Im}(z)$ the imaginary part of $z$.

The following three complex numbers deserve their own names and notation:

$0 = (0,0) \in \mathbb{C}$ "zero"

$1 = (1,0) \in \mathbb{C}$ "one"

$i = (0,1) \in \mathbb{C}$ "imaginary unit".

Generalizing the first two above, any real number $x$ is naturally embedded to the complex plane as $(x, 0) \in \mathbb{C}$. This way, $\mathbb{R}$ is interpreted as a subset of the complex plane, $\mathbb{R} \subset \mathbb{C}$, called the real axis.

Similarly, the subset $\{(0,y) \mid y \in \mathbb{R}\} \subset \mathbb{C}$ is called the imaginary axis.

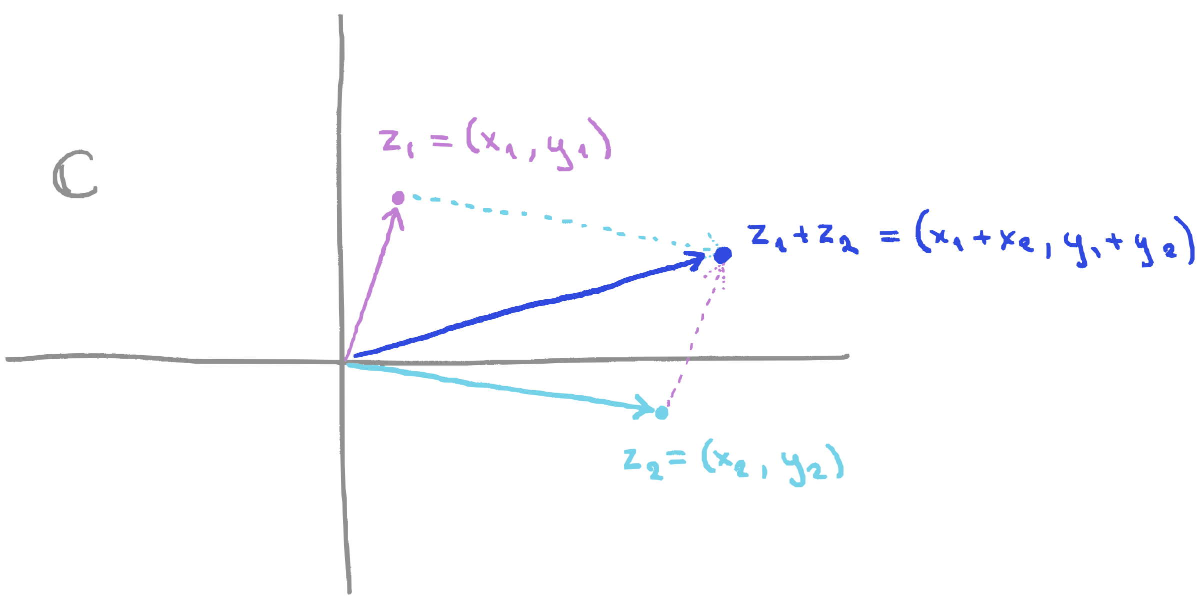
Arithmetic operations on complex numbers:

The sum and product of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2).$$

In particular, the sum of complex numbers corresponds to the usual vector addition in the plane.




The points $0, z_1, z_2, z_1 + z_2$ lie at the four vertices of a parallelogram.

Note that the embedding

$$\mathbb{R} \ni x \longmapsto (x,0) \in \mathbb{C}$$

respects addition and multiplication:

$$\underset{\substack{\uparrow \\ \text{sum in } \mathbb{R}}}{x_1 + x_2} \longmapsto (x_1+x_2, 0) = (x_1,0) \underset{\substack{\uparrow \\ \text{sum in } \mathbb{C}}}{+} (x_2,0)$$

$$\underset{\substack{\uparrow \\ \text{product in } \mathbb{R}}}{x_1 \cdot x_2} \longmapsto (x_1 \cdot x_2, 0) \overset{\circledast}{=} (x_1,0) \underset{\substack{\uparrow \\ \text{product in } \mathbb{C}}}{\cdot} (x_2,0).$$

Therefore there is no risk of a mix-up between arithmetic operations when we interpret

$$\mathbb{R} \subset \mathbb{C}.$$

We will do so from here on without further comments.

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$\circledast$ indeed, by definition of multiplication in $\mathbb{C}$,

$$(x_1,0)\cdot(x_2,0) = (x_1x_2 - 0\cdot 0,\ x_1\cdot 0 + 0\cdot x_1)$$
$$= (x_1x_2,\ 0).$$

Also denote, for $z = (x, y) \in \mathbb{C}$

$$-z = (-x, -y) \in \mathbb{C}$$

and for $z = (x, y) \neq (0, 0)$

$$z^{-1} = \left( \frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

## Theorem

[complex_number_field]

The set $\mathbb{C}$ of complex numbers with its operations of addition (sum) and multiplication (product) forms a field, i.e., for any $z, w, z_1, z_2, z_3 \in \mathbb{C}$, we have

$z + w = w + z$ "commutativity of addition"

$z \cdot w = w \cdot z$ "commutativity of multiplication"

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ "associativity of add."

$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$ "associativity of mult."

$z + 0 = z$ "neutral element for addition"

$z \cdot 1 = z$ "neutral element for multiplication"

$z + (-z) = 0$ "opposite element (additive inverse)"

$z \cdot z^{-1} = 1$ for $z \neq 0$ "(multiplicative) inverse"

$(z_1 + z_2) w = z_1 w + z_2 w$ "distributivity"

Proof: Straightforward calculations, left as exercises to the reader. $\square$

The imaginary unit $i = (0,1) \in \mathbb{C}$ satisfies

$$
\begin{aligned}
i^2 = i \cdot i &= (0,1)\cdot(0,1) \\
&= (0\cdot 0 - 1\cdot 1,\ 0\cdot 1 + 1\cdot 0) \\
&= (-1,\ 0) \\
&= -1,
\end{aligned}
$$

so it is a square root of $-1 \in \mathbb{R}$.

(Note, however, that also $-i = (0,-1)$ satisfies $(-i)^2 = \cdots = -1$, so $i$ is not the only "$\sqrt{-1}$".)

For $x, y \in \mathbb{R}$, a quick calculation gives

$$
\begin{aligned}
x + i\cdot y &= (x,0) + (0,1)\cdot(y,0) \\
&= (x,0) + (0\cdot y - 1\cdot 0,\ 1\cdot y + 0\cdot 0) \\
&= (x,0) + (0,y) \\
&= (x,y).
\end{aligned}
$$

This allows us to write complex numbers $z = (x,y)$ in the form

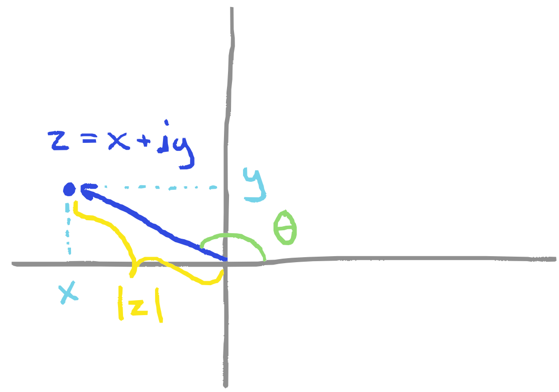
$$z = x + iy \qquad (\text{with } x, y \in \mathbb{R}).$$

Calculations with complex numbers are easy — they satisfy the usual rules of arithmetic, and in addition $i^2 = -1$.

For example, the formula defining products of $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ $\quad (x_1, x_2, y_1, y_2 \in \mathbb{R})$ is easy to recover from

$$
\begin{aligned}
z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\
&= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + \underbrace{i^2}_{=-1} \cdot y_1 y_2 \\
&= x_1 x_2 - y_1 y_2 + i \cdot (x_1 y_2 + y_1 x_2).
\end{aligned}
$$

## 1.2 COMPLEX CONJUGATE, MODULUS, ARGUMENT




A complex number

$$z = x + iy$$

corresponds to a point in the plane with coordinates $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

The modulus or absolute value of $z$ is the distance from the origin,

$$|z| := \sqrt{x^2 + y^2} \geq 0 .$$

If $z \neq 0$, then the angle $\theta$ between the positive real axis and the line from $0$ to $z$ is called an argument of $z$

$$\arg(z) \text{ "="} \, \theta .$$

only well defined modulo integer multiples of $2\pi$

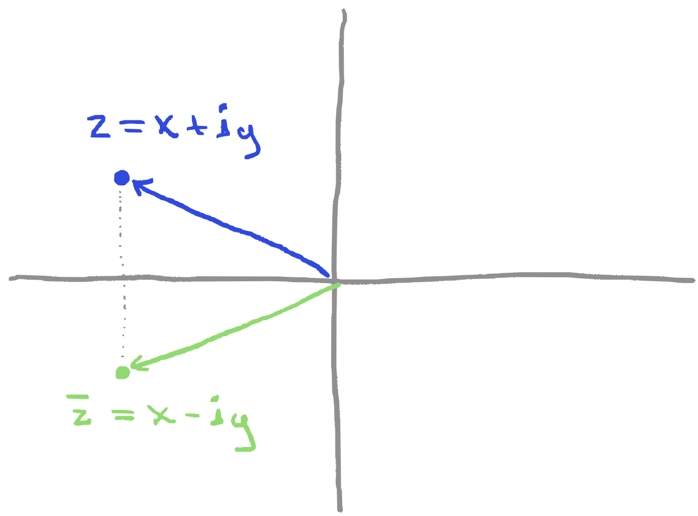
We then have $z = |z| \cdot \cos(\theta) + i \cdot |z| \cdot \sin(\theta)$.

When an unambiguous choice is desirable, we often choose $\theta \in (-\pi, +\pi]$ and call this the principal argument, and denote it by $\mathrm{Arg}(z)$.

The point obtained by a reflection across the real axis,

$$\bar{z} = x - iy$$

is called the complex conjugate of $z = x + iy$




Observations:

[complex_conjugate_properties]

$$z \cdot \bar{z} = (x+iy)(x-iy) = x^2 \cancel{- ixy} \cancel{+ ixy} - i^2 \cdot y^2$$

$$= x^2 + y^2 = |z|^2$$

$$|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$$

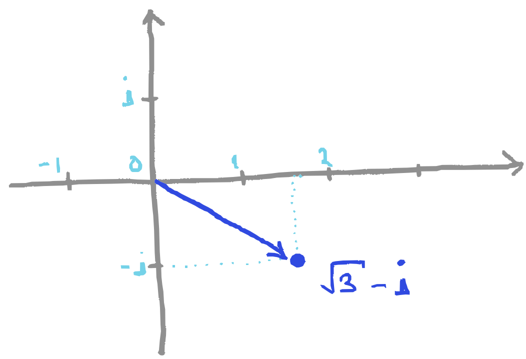
$$z + \bar{z} = (x + \cancel{iy}) + (x - \cancel{iy}) = 2x = 2 \cdot \mathrm{Re}(z)$$

$$z - \bar{z} = (\cancel{x} + iy) - (\cancel{x} - iy) = 2iy = 2i \cdot \mathrm{Im}(z)$$

## Example

Consider the complex number

$$z = \sqrt{3} - i \in \mathbb{C}.$$

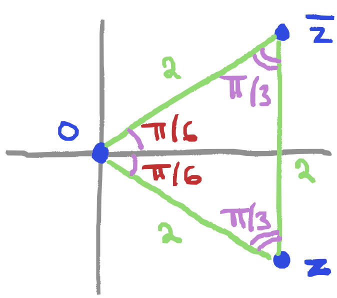



Then $\operatorname{Re}(\sqrt{3} - i) = \sqrt{3}$, $\operatorname{Im}(\sqrt{3} - i) = -1$,

and

$$|\sqrt{3} - i| = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{4} = 2.$$

To determine the argument of $z = \sqrt{3} - i$ observe that also the complex conjugate $\bar{z} = \sqrt{3} + i$ is at distance 2 both from the origin and from $z$, so these three points are the vertices of an equilateral triangle.




So

$$\arg(\sqrt{3} - i) = -\frac{\pi}{6} \pmod{2\pi}.$$

## Example

Consider still $z = \sqrt{3} - i$.

Calculate the square

$$z^2 = (\sqrt{3} - i)(\sqrt{3} - i)$$

$$= \sqrt{3}^2 - 2\sqrt{3}\,i + i^2$$

$$= 3 - 2\sqrt{3}\,i - 1$$

$$= 2 - 2\sqrt{3}\,i$$

and the cube

$$z^3 = z^2 \cdot z = (2 - 2\sqrt{3}\,i)(\sqrt{3} - i)$$

$$= 2\sqrt{3} - 2i - 2\sqrt{3}^2 i + 2\sqrt{3}\,i^2$$

$$= \cancel{2\sqrt{3}} - 2i - 6i - \cancel{2\sqrt{3}}$$

$$= -8i.$$

**Lemma** [complex_conjugate_properties]

Complex conjugation respects the sum and product of complex numbers in the following sense: For any $z, w \in \mathbb{C}$ we have

$$\overline{z+w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

*Proof*: The assertion about the sum follows directly from the definitions (check for yourself).

For the assertion about the product, write $z = x + iy$, $w = x' + iy'$ (with $x, x', y, y' \in \mathbb{R}$) and calculate

$$\begin{aligned}
\overline{z} \cdot \overline{w} &= (x - iy)(x' - iy') \\
&= xx' - ixy' - iyx' + i^2 yy' \\
&= xx' - yy' - i(xy' + yx') \\
&= \overline{(xx' - yy' + i(xy' + yx'))} = \overline{z \cdot w}.
\end{aligned}$$

$\square$

Observation:

Perhaps the easiest way to "remember" the formulas for the real and imaginary parts of the inverse of a nonzero complex number $z = x + iy$ $(x, y \in \mathbb{R})$ is the calculation

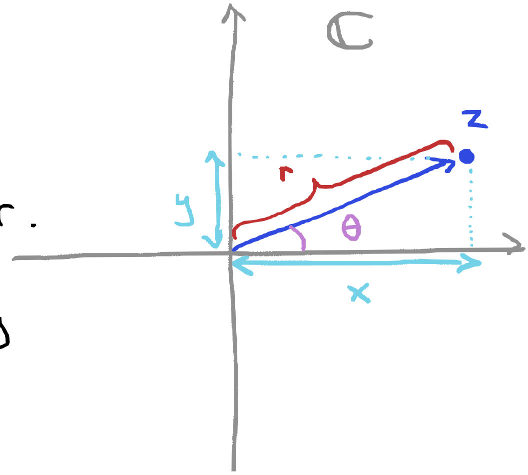
$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{\bar{z}}{|z|^2}$$

$$= \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2+y^2} + i \, \frac{-y}{x^2 + y^2},$$

from which one recovers

$$\mathrm{Re}\left(\frac{1}{x+iy}\right) = \frac{x}{x^2+y^2}, \quad \mathrm{Im}\left(\frac{1}{x+iy}\right) = \frac{-y}{x^2+y^2}.$$

## 1.3 POLAR FORM




Let $z = x + iy$ $(x, y \in \mathbb{R})$ be a complex number.

Denote its modulus by

$$r = |z| = \sqrt{x^2 + y^2}$$

and (any of) its argument(s) by

$$\theta = \arg(z).$$

(If $z = 0$, the choice of $\theta$ will be irrelevant below.)

Then we have $x = r \cdot \cos(\theta)$, $y = r \cdot \sin(\theta)$ and the complex number can be written in the following polar form

$$z = x + iy = r \cdot \left(\cos(\theta) + i \cdot \sin(\theta)\right).$$

Remark: This formula also provides a way to determine $\theta$ by trigonometry, if $z \neq 0$. Since, e.g., $\cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$, the principal argument $\theta = \mathrm{Arg}(z) \in (-\pi, \pi]$ is

$$\mathrm{Arg}(z) = \begin{cases} \mathrm{Arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y \geq 0 \\ -\mathrm{Arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y < 0. \end{cases}$$

**Lemma** (Multiplication in polar form)

[multiplication_polar]

Let $z, w \in \mathbb{C}$ be two complex numbers whose polar forms are

$$z = r \cdot (\cos(\theta) + i \cdot \sin(\theta))$$

$$w = r' \cdot (\cos(\theta') + i \sin(\theta')).$$

Then a polar form of the product is

$$z \cdot w = rr' \cdot (\cos(\theta + \theta') + i \cdot \sin(\theta + \theta')).$$

**Proof**: Calculate, (using the def. of prod. in $\mathbb{C}$)

$$z \cdot w = (r \cdot \cos(\theta) + i r \cdot \sin(\theta))(r' \cdot \cos(\theta') + i r' \cdot \sin(\theta'))$$

$$= r \cdot \cos(\theta) \cdot r' \cdot \cos(\theta') - r \cdot \sin(\theta) \cdot r' \cdot \sin(\theta')$$
$$+ i (r \cdot \cos(\theta) \cdot r' \cdot \sin(\theta') + r \cdot \sin(\theta) \cdot r' \cdot \cos(\theta'))$$

$$= rr' \big(\underbrace{\cos(\theta)\cos(\theta') - \sin(\theta)\sin(\theta')}_{= \cos(\theta + \theta') \text{ by cosine angle sum formula}}\big)$$
$$+ i rr' \big(\underbrace{\cos(\theta)\sin(\theta') + \sin(\theta) \cdot \cos(\theta')}_{= \sin(\theta + \theta') \text{ by sine angle sum formula}}\big)$$

$$= rr' (\cos(\theta + \theta') + i \cdot \sin(\theta + \theta')).$$

□

We now adopt a convenient notation

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta) \qquad \text{for } \theta \in \mathbb{R}$$

(known as Euler's formula)

which anticipates the complex exponential function which we will define later.

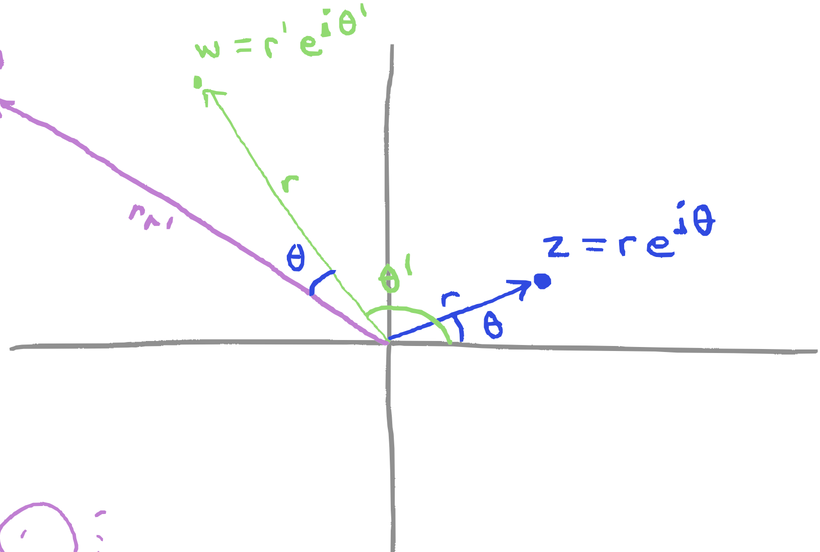
With this notation, the polar form takes the concise form

$$z = r \cdot e^{i\theta}$$

and the polar form product formula of the previous lemma becomes

$$(r \cdot e^{i\theta}) \cdot (r' \cdot e^{i\theta'}) = rr' \cdot e^{i(\theta + \theta')}.$$

Importantly, the lemma above provides a geometric interpretation for the product of complex numbers:




The operation of multiplication by a complex number $z = re^{i\theta}$ consists of a rotation by the angle $\theta$ and a dilatation (= scaling) by the factor $r$.

Put yet another way, the modulus and argument of the product of complex numbers $z, w \in \mathbb{C}$ satisfy

$$|z \cdot w| = |z| \cdot |w|, \quad \arg(z \cdot w) \equiv \arg(z) + \arg(w) \pmod{2\pi}.$$

One particular application if the following, historically noteworthy and occasionally practical formula

**Theorem** (De Moivre's formula)

[de_moivre]

For $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i \cdot \sin(n\theta).$$

**Proof** The case $n=1$ is clear, and induction using the previous lemma then establishes the formula for any positive $n$.

The case $n=0$ is also clear by the definition of the zeroth power,

$$(\cos(\theta) + i \cdot \sin(\theta))^0 = 1.$$

The case $n=-1$ is obtained with the formula for the inverse

$$(\cos(\theta) + i\sin(\theta))^{-1} = \frac{\cos(\theta)}{\cos^2(\theta)+\sin^2(\theta)} + i\,\frac{-\sin(\theta)}{\cos^2(\theta)+\sin^2(\theta)}$$

$$= \cos(\theta) - i \cdot \sin(\theta)$$

$$= \cos(-\theta) + i \cdot \sin(-\theta).$$

Again, induction handles general $n<0$. $\square$

## Example application:

Complex roots of unity

Let $n \in \{3, 4, 5, \dots\}$.

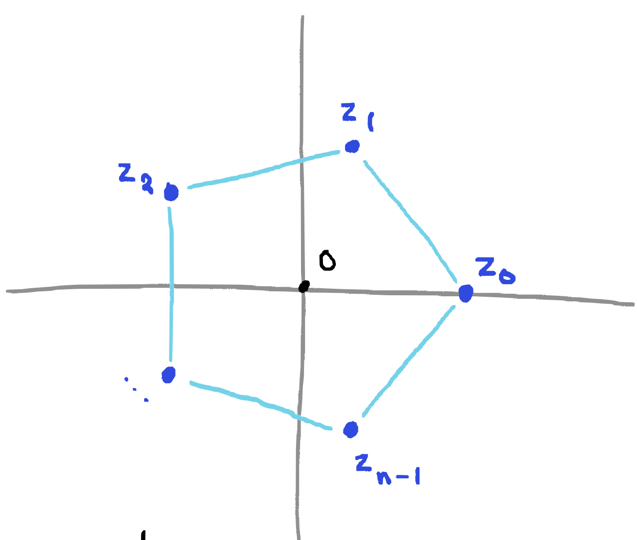
For $j = 0, 1, 2, \dots, n-1$, set

$$z_j = e^{i\frac{2\pi}{n}j} = \cos\left(\frac{2\pi j}{n}\right) + i\sin\left(\frac{2\pi j}{n}\right).$$

De Moivre's formula gives

$$z_j^n = \underbrace{\cos(2\pi j)}_{=1} + i\underbrace{\sin(2\pi j)}_{=0} = 1.$$

The points $z_j$, $j = 0, 1, \dots, n-1$, are the vertices of a regular $n$-gon centered at the origin




The points $z_0, z_1, \dots, z_{n-1}$ are $n$ th **roots of unity**: they satisfy $z^n = 1$.

[roots_of_unity]

## Example calculation

Let us return to the earlier examples with $z = \sqrt{3} - i$. We found out the modulus $|z| = 2$ and (principal) argument $\operatorname{Arg}(z) = -\frac{\pi}{6}$, which give the polar form

$$\sqrt{3} - i = 2 \cdot e^{-i\pi/6}.$$

We also calculated $z^2$ and $z^3$. In polar form, the same calculations are done as follows:

$$(\sqrt{3} - i)^2 = (2 \cdot e^{-i\pi/6})^2 = 2^2 \cdot e^{-2i\pi/6}$$

$$= 4 \cdot e^{-i\pi/3}$$

$$= 4 \cdot \underbrace{\cos\left(-\tfrac{\pi}{3}\right)}_{=1/2} + 4i \cdot \underbrace{\sin\left(-\tfrac{\pi}{3}\right)}_{=-\sqrt{3}/2}$$

$$= 2 - i2\sqrt{3},$$

$$(\sqrt{3} - i)^3 = (2e^{-i\pi/6})^3 = 2^3 \cdot e^{-3i\pi/6}$$

$$= 8e^{-i\pi/2} = 8\underbrace{\cos\left(-\tfrac{\pi}{2}\right)}_{=0} + 8i\underbrace{\sin\left(-\tfrac{\pi}{2}\right)}_{=-1}$$

$$= -8i.$$

If your task was to calculate $(\sqrt{3} - i)^{100}$, would you use Cartesian coordinates (real and imaginary parts) or the polar form ?