EXAMPLE CALCULATION OF REAL INTEGRALS

Complex analysis techniques can be used (among other things) for the calculation of (many) real integrals. The most systematic ways of doing such calculations are based on residue calculus, which we will introduce soon.

First, however, let us give two illustrative examples which only require Cauchy's integral formula.

(The hope is that the general method of residues is both easier to appreciate and easier to understand with such pedestrian examples worked out "by hand".)

Example:

Calculate $\boxed{\displaystyle\int_0^{2\pi} \frac{1}{3 + 2 \cdot \sin(t)} \, dt = ?}$

Parametrize the unit circle $\partial B(0;1)$
by $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.

Note that for $z = \gamma(t) = e^{it}$, then,

$$\sin(t) = \frac{1}{2i}(e^{it} - e^{-it}) = \frac{z - z^{-1}}{2i}.$$

Also $\dot{\gamma}(t) = i \cdot e^{it} = i \cdot \gamma(t)$. So

$$\oint_\gamma \frac{1}{3 + 2 \cdot \frac{1}{2i}(z - z^{-1})} \quad \frac{dz}{iz}$$

$$= \int_0^{2\pi} \frac{1}{3 + 2 \sin(t)} \underbrace{\frac{1}{i\,\gamma(t)} \, \dot{\gamma}(t)}_{=1} \, dt$$

$$= \int_0^{2\pi} \frac{1}{3 + 2 \sin(t)} \, dt, \qquad \text{which is}$$

the integral we seek to calculate.

It therefore suffices to calculate

$$\oint_{\partial B(0,1)} \frac{1}{3 + 2 \cdot \frac{1}{2i}(z - \bar{z}^{-1})} \frac{dz}{iz}$$

$$= \oint_{\partial B(0;1)} \frac{1}{z^2 + 3iz - 1} \, dz$$

Denote $\quad f(z) = \dfrac{1}{z^2 + 3iz - 1}$.

The two roots of the denominator are

$$\omega_{\pm} = \frac{1}{2}\left(-3i \pm \sqrt{(3i)^2 + 4}\right)$$

$$= \frac{-3i \pm i\sqrt{5}}{2}.$$

$\underbrace{} = \sqrt{-9+4} = \sqrt{-5}$

(the choice $\sqrt{-5} = \pm i\sqrt{5}$
only affects the $\pm$ label)

and we can factorize the denominator

$$(z - \omega_+)(z - \omega_-) = \left(z + \frac{3}{2}i - \frac{\sqrt{5}}{2}i\right)\left(z + \frac{3}{2}i + \frac{\sqrt{5}}{2}i\right)$$

$$= z^2 + 3iz - 1.$$

So
$$f(z) = \frac{1}{(z - \omega_+)(z - \omega_-)}.$$

Note that ω_+ is inside the unit disk,
$|\omega_+| = \dfrac{3 - \sqrt{5}}{2} < 1$, whereas ω_- is outside it.

Applying Cauchy's integral formula
to the function $g(z) = \dfrac{1}{z - \omega_-}$

gives

$$\oint_{\partial B(0;1)} f(z)\, dz \;=\; \oint_{\partial B(0;1)} \frac{g(z)}{z - \omega_+}\, dz$$

$$= 2\pi i \cdot g(\omega_+)$$

$$= 2\pi i \underbrace{\frac{1}{\omega_+ - \omega_-}}_{= \frac{1}{i\sqrt{5}}} \;=\; \frac{2\pi}{\sqrt{5}} \;.$$

We conclude

$$\int_0^{2\pi} \frac{1}{3 + 2\sin(t)}\, dt \;=\; \frac{2\pi}{\sqrt{5}} \;.$$

Example

Calculate
$$\int_0^\infty \frac{1}{x^2+1} dx = ?$$

Observe by symmetry

$\left(R(x) = \frac{1}{x^2+1}\right.$ is even: $\left. R(-x) = R(x)\right)$
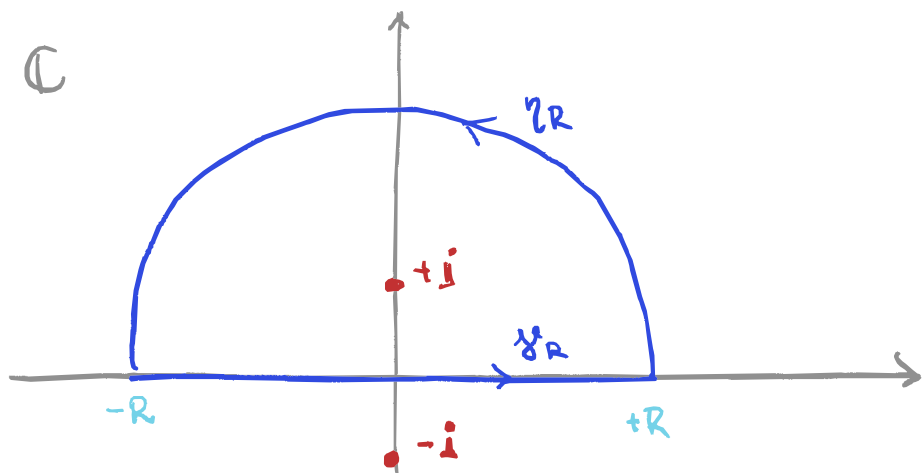
$$\int_0^\infty \frac{1}{x^2+1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{x^2+1} dx$$

$$= \frac{1}{2} \lim_{R \to \infty} \int_{-R}^R \frac{1}{x^2+1} dx.$$

Integrate $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$

$f: \mathbb{C} \setminus \{-i, +i\} \longrightarrow \mathbb{C}$ analytic

over suitably chosen contours.

If γ_R is the (left-to-right oriented) line segment $[-R, R]$ contour, then

$$\int_{\gamma_R} f(z)\, dz = \int_{-R}^{R} \frac{1}{x^2 + 1}\, dx .$$

But γ_R alone is not a closed contour, so concatenate to it a semicircle contour β_R along $\partial B(0; R)$

$$\beta_R(t) = R \cdot e^{it} \qquad t \in [0, \pi] .$$

Since $\gamma_R \boxplus \beta_R$ is counterclockwise and surrounds$^{\oplus}$ i (but not $-i$), Cauchy's integral theorem gives (the half-disk surrounded is star-shaped)

$^{\oplus}$when $R > 1$

$$\oint_{\gamma_R \boxplus \gamma_R} f(z)\, dz = \oint_{\gamma_R \boxplus \gamma_R} \frac{1}{(z-i)(z+i)}\, dz$$

$g(z)$ — analytic in the half-disk

$$= 2\pi i \cdot g(i) = 2\pi i \frac{1}{i + i} = \pi .$$

The added contribution from η_R can be estimated:

$$\int_{\eta_R} f(z)\, dz = \int_{\eta_R} \frac{1}{z^2 + 1}\, dz$$

$$\left| \frac{1}{z^2+1} \right| \leq \frac{1}{R^2-1} \quad \text{on } \eta_R$$

$$\leq \underbrace{\ell(\eta_R)}_{= \pi R} \cdot \frac{1}{R^2-1} = \frac{\pi R}{R^2 - 1} \xrightarrow[R \to \infty]{} 0.$$

So we conclude

$$\int_0^\infty \frac{1}{x^2+1}\, dx = \frac{1}{2} \lim_{R \to \infty} \int_{\gamma_R} \frac{1}{z^2+1}\, dz + 0$$

$$= \frac{1}{2} \lim_{R \to \infty} \left(\int_{\gamma_R} \frac{1}{z^2+1}\, dz + \underbrace{\int_{\eta_R} \frac{1}{z^2+1}\, dz}_{\text{vanishes in the } R \to \infty \text{ limit}} \right)$$

$$= \frac{1}{2} \lim_{R \to \infty} \underbrace{\oint_{\gamma_R \boxplus \eta_R} \frac{1}{z^2+1}\, dz}_{= \pi \text{ when } R > 1}$$

$$= \frac{\pi}{2}.$$

RESIDUE THEOREM

The above kinds of calculations are both generalized and systematized with the method of residues.

There are two main aspects of this generalization. In the examples above, the key was to calculate $\oint_\gamma f(z)\, dz$ where $f(z) = \frac{g(z)}{z-w}$ and g was analytic in the (star-shaped) domain surrounded by γ. In particular, f had one isolated singular point in that domain: a pole of order one at w (a "simple pole"). We could envision:

- multiple isolated singular points for f
- poles of higher order or even essential singularities for f.

- also z could wind around the singular points in more complicated ways than just once counterclockwise.

The residue theorem allows to handle all of the above.

First let us make:

Definition:

Let $f: U \to \mathbb{C}$ have an isolated singular point z_0, and let
$$f(z) = \sum_{n=-\infty}^{+\infty} a_n \cdot (z - z_0)^n$$
be its Laurent expansion in some punctured disk $B^*(z_0; \delta)$. The coefficient $a_{-1} \in \mathbb{C}$ of this expansion is called the residue of f at z_0 and denoted
$$a_{-1} = \operatorname{Res}_{z_0}(f) = \operatorname{Res}_{z=z_0}(f(z)).$$

Example

Consider $f(z) = \dfrac{1}{z^2 + 3iz - 1}$ and

the isolated singular point

$$\omega_+ = \frac{-3i + i\sqrt{5}}{2} \qquad \text{Denote also}$$

$$\omega_- = \frac{-3i - i\sqrt{5}}{2} \qquad \text{so that}$$

$$f(z) = \frac{1}{(z - \omega_+)(z - \omega_-)}$$

$$= \frac{1}{z - \omega_+} \cdot \frac{1}{\omega_+ - \omega_-} \cdot \frac{1}{1 + \dfrac{z - \omega_+}{\omega_+ - \omega_-}}$$

$$= \frac{1}{\omega_+ - \omega_-} (z - \omega_+)^{-1} \sum_{k=0}^{\infty} \left(-\frac{z - \omega_+}{\omega_+ - \omega_-} \right)^k$$

$$= - \sum_{k=0}^{\infty} \left(\frac{i}{\sqrt{5}} \right)^{k+1} (z - \omega_+)^{k-1}$$

for $0 < |z - \omega_+| < |\omega_+ - \omega_-|$.

The coefficient of $(z - \omega_+)^{-1}$ $(k=0)$

gives

$$\operatorname{Res}_{\omega_+} f = -\frac{i}{\sqrt{5}} = \frac{1}{i\sqrt{5}} .$$

(This number appeared in the first

example above.)

Example

Let $m \in \mathbb{N}$ and $z_0 \in \mathbb{C}$.

Consider functions of the form

$$f(z) = \frac{g(z)}{(z-z_0)^m} \quad \text{where } g \text{ is analytic}$$

in some disk $B(z_0; \delta)$ around z_0.

Since g can be written as a Taylor series

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(z_0) \cdot (z-z_0)^k,$$

f can be written as

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(z_0) \cdot (z-z_0)^{k-m}.$$

The coefficient of $(z-z_0)^{-1}$ is $(k=m-1)$

$$\text{Res}_{z_0}(f) = \frac{1}{(m-1)!} g^{(m-1)}(z_0).$$

The reason why the single number $\text{Res}_{z_0}(f)$ at an isolated singular point z_0 of f is a fruitful quantity is :

Theorem (Residue theorem)

Let U be an open set and let $f: U \setminus S \to \mathbb{C}$ be an analytic function with S a set of isolated singular points of S. Let γ be a closed contour in $U \setminus S$ which is <u>contractible in U</u>

(not necessarily contractible in $U \setminus S$, though!)

Then we have

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{w \in S} n_\gamma(w) \operatorname{Res}_w(f),$$

where $n_\gamma(w)$ are the winding numbers of γ around the points $w \in S$.

<u>Remark</u>: Easy compactness arguments (and the assumption of <u>isolated</u> singular points) show that only finitely many of the $w \in S$ can have $n_\gamma(w) \neq 0$. Therefore the sum only has finitely many nonzero terms (so it is well-defined).

(The reader familiar with compactness can take justifying this as an additional exercise.)

Recall: $n_\gamma(\omega) = \dfrac{1}{2\pi i} \oint_\gamma \dfrac{1}{z-\omega} dz$ for $\omega \notin \gamma$.

Sketch of proof:

By the remark above, there are only finitely many $\omega \in S$ such that ω is not in the unbounded component of $\mathbb{C} \sim \gamma$, in which case $n_\gamma(\omega) = 0$. Let us denote them by $\omega_1, \ldots, \omega_m \in S$.

For each $j = 1, \ldots, m$, we consider the Laurent expansions

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n^{(j)} (z-\omega_j)^n$$

of f in punctured disks $B^*(\omega_j, \delta_j)$.

Denote the series of only the negative powers of these by

$$g_j(z) = \sum_{n=-\infty}^{-1} a_n^{(j)} (z - w_j)^n$$

$$= \sum_{k=0}^{\infty} a_{-1-k}^{(j)} \left(\frac{1}{z - w_j}\right)^{k+1}$$

and note that the radius of convergence $\left(1/\limsup_{k \to \infty} \sqrt[k]{|a_{-1-k}^{(j)}|}\right)$ of this power series in $\frac{1}{z - w_j}$ is $+\infty$ (since the inner radius of annuli around the isolated singular point w_j can be taken as 0). So the series defines analytic functions

$$g_j : \mathbb{C} \setminus \{w_j\} \longrightarrow \mathbb{C}.$$

Moreover, $f - \sum_{j=1}^{m} g_j$ has removable singular points at $w_1, \ldots, w_m$, so we may view it (by extension to these points) as an analytic

function on $V = (U \setminus S) \cup \{w_1, \ldots, w_m\}$.

In V, γ is again contractible (the rest of the points of S were in the unbounded component of $\mathbb{C} \setminus \gamma$ and γ was assumed contractible in U) so Cauchy's integral formula (in the general form) gives

$$\oint_\gamma \left(f(z) - \sum_{j=1}^m g_j(z) \right) dz = 0 .$$

We rearrange this to

$$\oint_\gamma f(z) \, dz = \sum_{j=1}^m \oint_\gamma g_j(z) \, dz ,$$

and then it only remains to prove that the terms on the RHS are $2\pi i \cdot n_\gamma(w_j) \cdot \text{Res}_{w_j}(f)$.

This is indeed straightforward: the series defining g_j converges uniformly on γ, so we get

$$\oint_\gamma g_j(z)\, dz \;=\; \oint_\gamma \left(\sum_{n=-\infty}^{-1} a_n^{(j)} (z-w_j)^n \right) dz$$

$$= \sum_{n=-\infty}^{-1} a_n^{(j)} \cdot \oint_\gamma (z-w_j)^n\, dz \;.$$

Now for $n \neq -1$ we have

$$\oint_\gamma (z-w_j)^n\, dz \;=\; 0$$

$\left(z \mapsto \dfrac{1}{n+1}(z-w_j)^{n+1} \right.$ is a primitive and of the integrand in $\left. \mathbb{C} \setminus \{w_j\} \right)$

and for $n = -1$ we have

$$\oint_\gamma (z-w_j)^{-1}\, dz \;=\; 2\pi i \cdot n_\gamma(w_j)$$

by the definition of winding numbers.

We get the desired

$$\oint_\gamma f(z)\, dz \;=\; \sum_{j=1}^{m} \oint_\gamma g_j(z)\, dz$$

$$= \sum_{j=1}^{m} a_{-1}^{(j)} \cdot 2\pi i \cdot n_\gamma(w_j)$$

$$= 2\pi i \sum_{j=1}^{m} n_\gamma(w_j) \cdot \mathrm{Res}_{w_j}(f) \;.$$

$\square$

Let us give some example applications
which would have been difficult to
obtain "by hand" using just Cauchy's
integral formula.

Example

To generalize our earlier calculation
of $\int_0^{2\pi} \frac{1}{3-2\sin(t)} dt$, consider

any rational function $R(x,y)$
of two variables which is
well-defined (nonvanishing denominator)
on $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and
the integral

$$\int_0^{2\pi} R(\cos(t), \sin(t)) \, dt .$$

Using $\gamma(t) = e^{it}$ $(t \in [0, 2\pi])$,

$\cos(t) = \frac{1}{2}\left(\gamma(t) + \gamma(t)^{-1}\right)$,

$\sin(t) = \frac{1}{2i}\left(\gamma(t) - \gamma(t)^{-1}\right)$,

and $\dfrac{\dot{g}(t)}{j \cdot g(t)} = 1,$ we write just as before

$$\int\limits_{\partial B(0,1)} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2j}\right) \frac{dz}{jz}$$

$$= \int_0^{2\pi} R(\cos(t), \sin(t)) \, dt \quad,$$

so the calculation of our integral involving a rational function of the trigonometric functions reduces to the calculation of a contour integral of

$$f(z) = \frac{1}{jz} \cdot R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2j}\right).$$

This is a rational function of z with finitely many isolated singular points w_j inside the unit disk, so

$$\int_0^{2\pi} R(\cos(t), \sin(t)) \, dt = 2\pi i \sum_j \text{Res}_{w_j}(f)$$

by the residue theorem (since

$n_\gamma(w_j) = 1$ for w_j in the unit disk).

Note that calculating the residues $\operatorname{Res}_{w_j}(f)$ of the explicit rational function f is routine (at least provided we can factorize the denominator explicitly) so we have in effect made the calculation of integrals of rational functions of trigonometric functions routine!

Let us then give another example, this time concretely generalizing our second example.

Example

Calculate :

$$\int_0^\infty \frac{1}{(x^2+1)^2}\, dx \ = \ ?$$

Observe , by symmetry,

$$\int_0^\infty \frac{1}{(x^2+1)^2}\, dx \ = \ \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^2}\, dx .$$

💡 Integrate $\quad f(z) = \dfrac{1}{(z^2+1)^2}$

$$= \frac{1}{(z-i)^2 (z+i)^2}$$

along the closed contour $\gamma_R \boxplus \eta_R$
where γ_R is the line segment
$[-R,R]$ and η_R is the semicircle
along $\partial B(0;R)$ from R to $-R$.

Note: $f : \mathbb{C} \smallsetminus \{-i, +i\} \longrightarrow \mathbb{C}$ is
analytic and only the isolated
singular point $+i$ is inside the
contour $\gamma_R \boxplus \eta_R$ (when $R > 1$).

So

$$\oint_{\gamma_R \boxplus \gamma_R} f(z)\, dz \;=\; 2\pi i \cdot \mathrm{Res}_{z=+i}\, f(z).$$

To calculate the residue, observe

$$f(z) = \frac{1}{(z-i)^2 (z+i)^2} = \frac{1}{(z-i)^2} g(z)$$

where $g(z) = \frac{1}{(z+i)^2}$ is analytic

near $+i$ (indeed analytic in $\mathbb{C} \setminus \{-i\}$).

So developing g as a Taylor series

near $+i$ we find

$$f(z) = \frac{1}{(z-i)^2}\left(\sum_{n=0}^{\infty} \frac{g^{(n)}(i)}{n!} (z-i)^n \right)$$

$$= \frac{1}{(z-i)^2}\left(g(i) + g'(i)\cdot(z-i) \right.$$
$$\left. + \frac{g''(i)}{2}(z-i)^2 + \cdots \right)$$

$$= \frac{g(i)}{(z-i)^2} + \underbrace{\frac{g'(i)}{z-i}}_{} + \tfrac{1}{2}g''(i) + \cdots$$

← this term gives
the residue

Since $g'(z) = \dfrac{-2}{(z+j)^3}$, we find

$$\text{Res}_{z=i} f(z) = g'(i) = \dfrac{-2}{(2j)^3} = \dfrac{1}{4j}$$

Therefore $\displaystyle\oint_{\gamma_R \uplus \gamma_R} f(z)\, dz = 2\pi i \cdot \dfrac{1}{4j} = \dfrac{\pi}{2}$.

The contribution from the semicircle η_R can be estimated by $(R>1)$

$$\left| \int_{\eta_R} \dfrac{1}{(z^2+1)^2}\, dz \right|$$

$$\underbrace{\left| \dfrac{1}{(z^2+1)^2} \right| \leq \dfrac{1}{(R^2-1)^2} \text{ for } z \in \partial B(0;R)}$$

$$\leq \underbrace{\ell(\eta_R)}_{=\pi \cdot R} \cdot \dfrac{1}{(R^2-1)^2} = \dfrac{\pi R}{(R^2-1)^2} \xrightarrow[R \to \infty]{} 0.$$

Also,

$$\int_{\gamma_R} \dfrac{1}{(z^2+1)^2}\, dz = \int_{-R}^{R} \dfrac{1}{(x^2+1)^2}\, dx.$$

With these, we calculate the original integral :

$$\int_0^\infty \frac{1}{(x^2+1)^2} dx = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^R \frac{1}{(x^2+1)^2} dx + \textcolor{red}{\bigcirc}$$

$$= \frac{1}{2} \lim_{R\to\infty} \left(\int_{\gamma_R} \frac{1}{(z^2+1)^2} dz + \int_{\gamma_R} \frac{1}{(z^2+1)^2} dz \right)$$

$$\underbrace{\phantom{\int_{\gamma_R} \frac{1}{(z^2+1)^2} dz}}_{\to 0 \text{ as } R\to\infty}$$

$$= \frac{1}{2} \lim_{R\to\infty} \oint_{\gamma_R \boxplus \gamma_R} \frac{1}{(z^2+1)^2} dz$$

$$\underbrace{\phantom{\oint \frac{1}{(z^2+1)^2}}}_{\displaystyle = 2\pi i \cdot \operatorname{Res}_{z=+i} \frac{1}{(z^2+1)^2} = \frac{\pi}{2}}$$

$$\text{(when } R>1)$$

$$= \frac{\pi}{4}.$$

Example

Even more generally, we could calculate

$$\int_0^\infty \frac{1}{(x^2+1)^m}\, dx \qquad \text{for} \quad m \in \mathbb{N}.$$

(The previous examples gave $m=1$, $m=2$.)

With the residue theorem, the argument for any m is almost entirely similar to the previous case, and we get

$$\int_0^\infty \frac{1}{(x^2+1)^m}\, dx = \frac{1}{2} \cdot 2\pi i \cdot \operatorname{Res}_{z=i} \frac{1}{(z^2+1)^m}.$$

The main difference is the calculation of the residue: we now observe

$$f(z) = \frac{1}{(z^2+1)^m} = \frac{1}{(z-i)^m}\, g_m(z)$$

where $g_m(z) = \dfrac{1}{(z+i)^m}$ is

analytic in $\mathbb{C} \setminus \{-i\}$. So the calculation of the residue of f at i reduces to the calculation of the $(m-1)^{th}$ derivative of g at i (see example before!)

$$\text{Res}_i(f) = \frac{1}{(m-1)!} g_m^{(m-1)}(i).$$

Now
$$g_m^{(k)}(z) = \frac{d^k}{dz^k} (z+i)^{-m}$$

$$= (-m)(-m-1)\cdots(-m-k+1) \cdot (z+i)^{-m-k}$$

$$= (-1)^k \frac{(m+k-1)!}{(m-1)!} (z+i)^{-m-k}$$

so
$$g_m^{(m-1)}(i) = (-1)^{m-1} \frac{(2m-2)!}{(m-1)!} (2i)^{-2m+1}$$

$$= -i \frac{(2m-2)! \cdot 2}{(m-1)! \cdot 2^{2m}}.$$

Finally, the value of the integral is

$$\int_0^\infty \frac{1}{(x^2+1)^m} dx = i\pi \cdot \text{Res}_i(f)$$

$$= i\pi \cdot \frac{1}{(m-1)!} g_m^{(m-1)}(i) = \frac{2\pi}{4^m} \cdot \frac{(2m-2)!}{((m-1)!)^2}.$$

Example

Calculate $\displaystyle\int_0^\infty \frac{x}{1+x^m}\, dx$

where $m \in \{3,4,5,\dots\}$.

⚠ Now just $x \longleftrightarrow -x$ symmetry does not help : for odd m there simply is no symmetry, and for even m the integrand is odd (changes sign under $x \longleftrightarrow -x$) so the integral $\int_{-\infty}^{+\infty}$ just vanishes and teaches us nothing.

The key is to integrate

$$f(z) = \frac{z}{1+z^m}$$

along some more cleverly chosen contour.

Note first that the roots of the denominator $1+z^M$ are

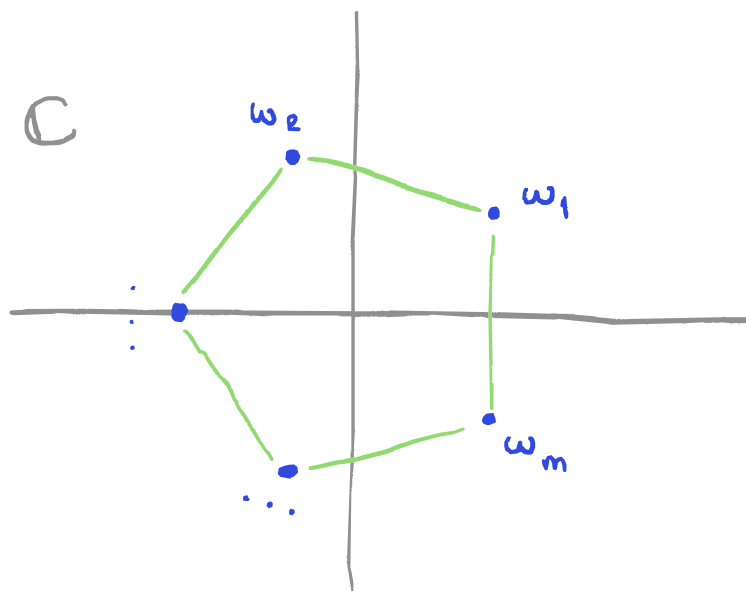$$\omega_j = e^{j\pi \frac{2j-1}{m}} \qquad (j=1,2,\ldots,m)$$

Indeed $\omega_j^{\,m} = e^{j\pi(2j-1)} = e^{-j\pi} = -1$

for each $j = 0,1,2,\ldots,m$. We may thus factorize the denominator

$$z^m + 1 = (z-\omega_1)(z-\omega_2)\cdots(z-\omega_m)$$

$$= \prod_{j=1}^{m} (z-\omega_j).$$

The roots of the denominator are the isolated singular points of
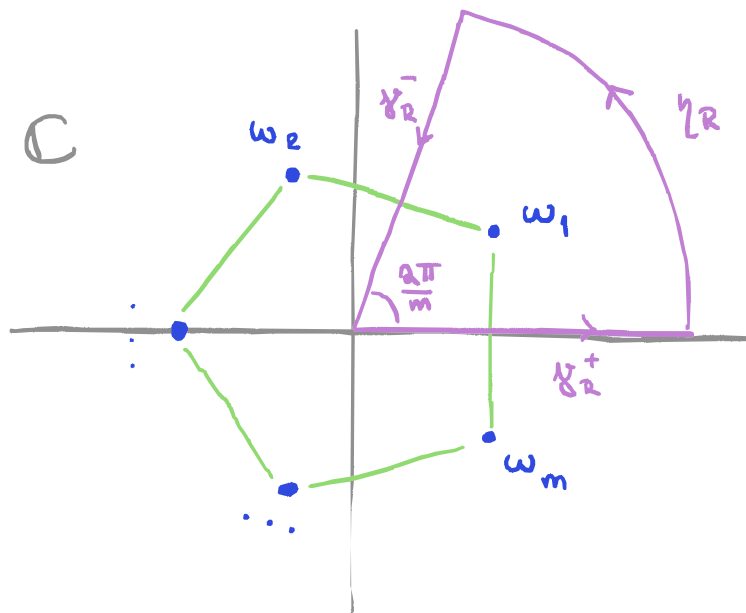
$$f: \mathbb{C} \smallsetminus \{\omega_1, \omega_2, \ldots, \omega_m\} \longrightarrow \mathbb{C}.$$

Observe the rotational symmetry by angle $\frac{2\pi}{m}$. The function f is also almost unchanged under such a rotation:

$$f\left(z \cdot e^{i 2\pi/m}\right) = \frac{z \cdot e^{i 2\pi/m}}{1 + \left(z \cdot e^{i 2\pi/m}\right)^m}$$

$$= \frac{z \cdot e^{i 2\pi/m}}{1 + z^m} = e^{i 2\pi/m} \cdot f(z).$$

The clever choice of contour is a pizza slice with angle $\frac{2\pi}{m}$, surrounding the isolated singular point ω_1 and no others.

The pizza slice contour is the
concatenation of

γ_R^+ : line segment $[0, R]$

η_R : arc of circle $\partial B(0; R)$

γ_R^- : line segment $[Re^{i2\pi/m}, 0]$.

The contribution of the arc of circle
can be estimated

$$\left| \int_{\eta_R} f(z) \, dz \right| = \left| \int_{\eta_R} \frac{z}{1 + z^m} \, dz \right|$$

$$\leq \underbrace{\ell(\eta_R)}_{= \frac{2\pi}{m} R} \cdot \frac{R}{R^m - 1} = \frac{2\pi}{m} \frac{R^2}{R^m - 1} \longrightarrow 0$$

as $R \rightarrow \infty$.

The integrals along the line
segments γ_R^+, γ_R^- are related
to the original integral:

$$\int_{\gamma_R^+} f(z) \, dz = \int_0^R \frac{x}{1 + x^m} \, dx$$

$$\int_{\gamma_{\bar{R}}} f(z)\, dz \;=\; -\, e^{i 2\pi/m} \cdot \int_0^R \frac{x}{1+x^m}\, dx \; .$$

With these in place, we calculate

$$2\pi i \cdot \operatorname{Res}_{z=\omega_1} f(z)$$

$$= \lim_{R \to \infty} \oint_{\gamma_R^+ \boxplus \gamma_R \boxplus \gamma_{\bar{R}}} f(z)\, dz$$

$$= \lim_{R \to \infty} \left(\int_{\gamma_R^+} f(z)\, dz \;+\; \int_{\gamma_R} f(z)\, dz \;+\; \int_{\gamma_{\bar{R}}} f(z)\, dz \right)$$

$$= \int_0^\infty \frac{x}{1+x^m}\, dx \;+\; 0 \;-\; e^{i 2\pi/m} \int_0^\infty \frac{x}{1+x^m}\, dx$$

$$= \left(1 - e^{i 2\pi/m} \right) \cdot \int_0^\infty \frac{x}{1+x^m}\, dx \; .$$

It just remains to calculate the residue. But this is straightforward: f has a first order pole at $z = \omega_1$

So $g(z) = (z - w_1) f(z)$ has a removable singularity and extends analytically to $z = w_1$, and then

$$f(z) = \frac{1}{z - w_1} g(z)$$

$$= \frac{1}{z - w_1} \left(g(w_1) + g'(w_1) \cdot (z - w_1) + \cdots \right)$$

$$= \frac{g(w_1)}{z - w_1} + \sum_{k=0}^{\infty} (\text{some coeffs}) (z - w_1)^k$$

$\hookleftarrow$ this gives the residue

To explicitly extract the residue, simplify

$$g(z) = \frac{(z - w_1) \cdot z}{1 + z^m}$$

using the observation ("geometric series") that if $\xi = e^{-i\pi/m} z$ then

$$1 + z^m = 1 - \xi^m = (1 + \xi + \cdots + \xi^{m-1})(1 - \xi)$$

so

$$g(z) = \frac{(z - w_1) z}{1 + z^m} = \frac{e^{i\pi/m} (\xi - 1) e^{i\pi/m} \xi}{(1 + \xi + \cdots + \xi^{m-1})(1 - \xi)}$$

$$= - e^{i 2\pi/m} \frac{\xi}{1 + \xi + \cdots + \xi^{m-1}}$$

$$= -\frac{e^{\pi i/m} \cdot z}{\sum_{k=0}^{m-1} \left(e^{-i\pi/m} z\right)^k} .$$

At $z = \omega_1 = e^{i\pi/m}$ $\quad (\xi = 1)$ we find

$$\mathrm{Res}_{z=\omega_1} f(z) = g(\omega_1) = -\frac{e^{\pi i/m}}{1+1+\cdots+1}$$

$$= -\frac{e^{2\pi i/m}}{m} .$$

There would be many more clever applications of the residue theorem to calculations as well as to the theory of complex analysis.

But this course ends here, and the deeper subsequent developments have to wait...

Fin