Aalto University

Problem set 5B

Department of Mathematics and Systems Analysis MS-C1300 — Complex Analysis, 2024-2025/II

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Exercise sessions: 21.-22.11. Hand-in due: Sun 24.11.2024 at 23:59

Topic: (complex) derivatives, analytic functions

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol \triangle): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 5B is Sun 24.11.2024 at 23:59.

Exercise (in class) 1.

The Fibonacci sequence $(a_n)_{n=0,1,2,...}$ is defined recursively by

$$a_0 = 1$$
, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \ge 2$.

(a) Prove that $0 \le a_n \le 2^n$ for every n, and show that the series

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

has a radius of convergence $\rho \geq \frac{1}{2}$.

(b) Prove that F(z) from part (a) satisfies

$$F(z) = 1 + z F(z) + z^2 F(z).$$

(c) Use the equation in (b) to solve F(z). Use the solution to determine the exact radius of convergence ρ of the series.

Remark: The interested students can think about: What does the value of ρ tell us about the growth rate of the Fibonacci sequence?

Exercise (in class) 2.

Suppose that $f: \mathbb{C} \to \mathbb{C}$ is a non-constant analytic function defined in the entire complex plane, and suppose that there exists a constant $\lambda \neq 1$ such that $f(\lambda z) = f(z)$ for all $z \in \mathbb{C}$.

- (a) Prove that there exist a positive integer m such that $\lambda^m = 1$.
- (b) Denote by m the smallest positive integer such that $\lambda^m = 1$. Show that there exists an analytic function $g: \mathbb{C} \to \mathbb{C}$ such that $f(z) = g(z^m)$ for all $z \in \mathbb{C}$.

Consider the function f defined by the formula

$$f(z) = \frac{1}{1+z}.$$

- (a) Find a power series representation for f in some nonempty open disk centered at $z_0 = 0$ (series in powers of $z z_0 = z$). What is the radius of convergence of this series?
- (b) Find a power series representation for f in some nonempty open disk centered at $z_0 = -4$ (series in powers of $z z_0 = z + 4$). What is the radius of convergence of this series?

For any complex number λ and any nonnegative integer n, the (generalized) binomial coefficient is defined by

$$\binom{\lambda}{n} = \prod_{j=1}^{n} \frac{\lambda - j + 1}{j} = \frac{\lambda(\lambda - 1) \cdots (\lambda - n + 1)}{n!}.$$

(a) Show that we have

$$(1+z)^{\lambda} = \sum_{n=0}^{\infty} {\lambda \choose n} z^n$$
 when $|z| < 1$,

where the principal branch of the complex power function is used: $w^{\lambda} = e^{\lambda \operatorname{Log}(w)}$ with $\operatorname{Log}(w)$ denoting the principal logarithm of $w \in \mathbb{C} \setminus \{0\}$.

(b) Using (a), write down explicitly the first four terms of the power series representing

$$\sqrt{1+z} = (1+z)^{1/2}.$$

Remark: The convention is that the empty product equals one, so that $\binom{\lambda}{0} = 1$ for any $\lambda \in \mathbb{C}$.