Aalto University
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II

Problem set 2A

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Exercise sessions: 29.-30.10. **Hand-in due: Wed 30.10.2024 at 23:59**

Topic: complex differentiability, functions as mappings

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ✍): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 2A is Wed 30.10.2024 at 23:59.

Exercise (in class) 1.
Consider the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(x+\mathbb{i}y) = \begin{cases} \frac{xy(x+\mathbb{i}y)}{x^2+y^2} & \text{for } x,y \in \mathbb{R} \text{ such that } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

(a) Is f continuous at the origin?
(b) Do the Cauchy-Riemann equations hold for f at the origin?
(c) Is f complex-differentiable at the origin?
(d) Is f analytic at the origin?

Exercise (in class) 2.
Let $A \subset \mathbb{C}$ be the part of the plane that lies to the right of the hyperbola $x^2 - y^2 = 1$. Show that the function f defined by the formula

$$f(z) = z^2$$

maps the region A bijectively onto the half-plane $R = \{ w \in \mathbb{C} \mid \mathfrak{Re}(w) > 1 \}$. Sketch the curves in A which map onto the vertical lines $\mathfrak{Re}(w) = \text{const.}$, and the curves which map onto horizontal half-lines $\mathfrak{Im}(w) = \text{const.}$, $\mathfrak{Re}(w) > 1$.

✍ **Homework exercise 3.**
Define $f\colon \mathbb{C} \to \mathbb{C}$ by

$$f(x+\mathbb{i}y) = 3 - 2y + x^2 + \mathbb{i}\big(2x + 4xy + 2y^2 + x^2\big) \qquad \text{for } x, y \in \mathbb{R}.$$

Locate all points $z = x + \mathbb{i}y$ at which f is complex-differentiable, and determine the complex derivative $f'(z)$ for each such point.

✍ **Homework exercise 4.**
Consider the disk

$$D = \Big\{ z \in \mathbb{C} \;\Big|\; |z - \mathbb{i}| < 1 \Big\}.$$

Show that the function f defined by the formula

$$f(z) = \frac{-2}{z}$$

maps the disk D bijectively onto the half-plane $U = \big\{ w \in \mathbb{C} \,\big|\, \Im\mathfrak{m}(w) > 1 \big\}$.