FUNCTIONS OF A COMPLEX VARIABLE

Complex analysis is the study of (analytic) complex-valued functions of a complex variable. An important point of view to such functions is to consider them as mappings — they transform regions of the plane to other regions.

Let us next just introduce a number of examples (which we will return to later).

The exponential function

Def:

The complex exponential function

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}$$

is defined by the formula

$$\exp(x+iy) = e^x \cdot \cos(y) + i \cdot e^x \cdot \sin(y)$$

for $x, y \in \mathbb{R}$.

Remark: The e^x in the defining formula is the usual real exponential of $x \in \mathbb{R}$ and the $\cos(y)$ and $\sin(y)$ are the usual real trigonometric functions of $y \in \mathbb{R}$.

Remark: When $y=0$, we have $\cos(y)=1$ and $\sin(y)=0$, so $\exp(x+i0) = e^x$.
In other words, on the real line $\mathbb{R} \subset \mathbb{C}$, the complex exponential function coincides with the real exponential. Without risk of confusion we thus also denote

$$e^z = \exp(z) \quad \text{for } z \in \mathbb{C}.$$

Remark: If $x=0$, then $e^{iy} = \cos(y) + i \cdot \sin(y)$ $\boxed{\text{Euler's formula}}$

Let us record a few easy properties of the complex exponential right away.

<u>Lemma</u> (Some properties of the complex exp.)

For any $z \in \mathbb{C}$ we have

$$|e^z| = e^{\mathrm{Re}(z)}$$

$$\arg(e^z) = \mathrm{Im}(z) \pmod{2\pi}$$

$$e^{\bar{z}} = \overline{e^z}.$$

<u>Proof</u>: Write $z = x + iy$ with $x, y \in \mathbb{R}$.

Calculate

$$|e^z| = |e^x \cos(y) + i e^x \sin(y)|$$

$$= \sqrt{(e^x \cos(y))^2 + (e^x \sin(y))^2}$$

$$= \sqrt{e^{2x} \underbrace{(\cos^2(y) + \sin^2(y))}_{=1}}$$

$$= \sqrt{e^{2x}} = e^x,$$

$$e^{\bar{z}} = e^{x - iy} = e^x \cdot \underbrace{\cos(-y)}_{= \cos(y)} + i \cdot e^x \cdot \underbrace{\sin(-y)}_{= -\sin(y)}$$

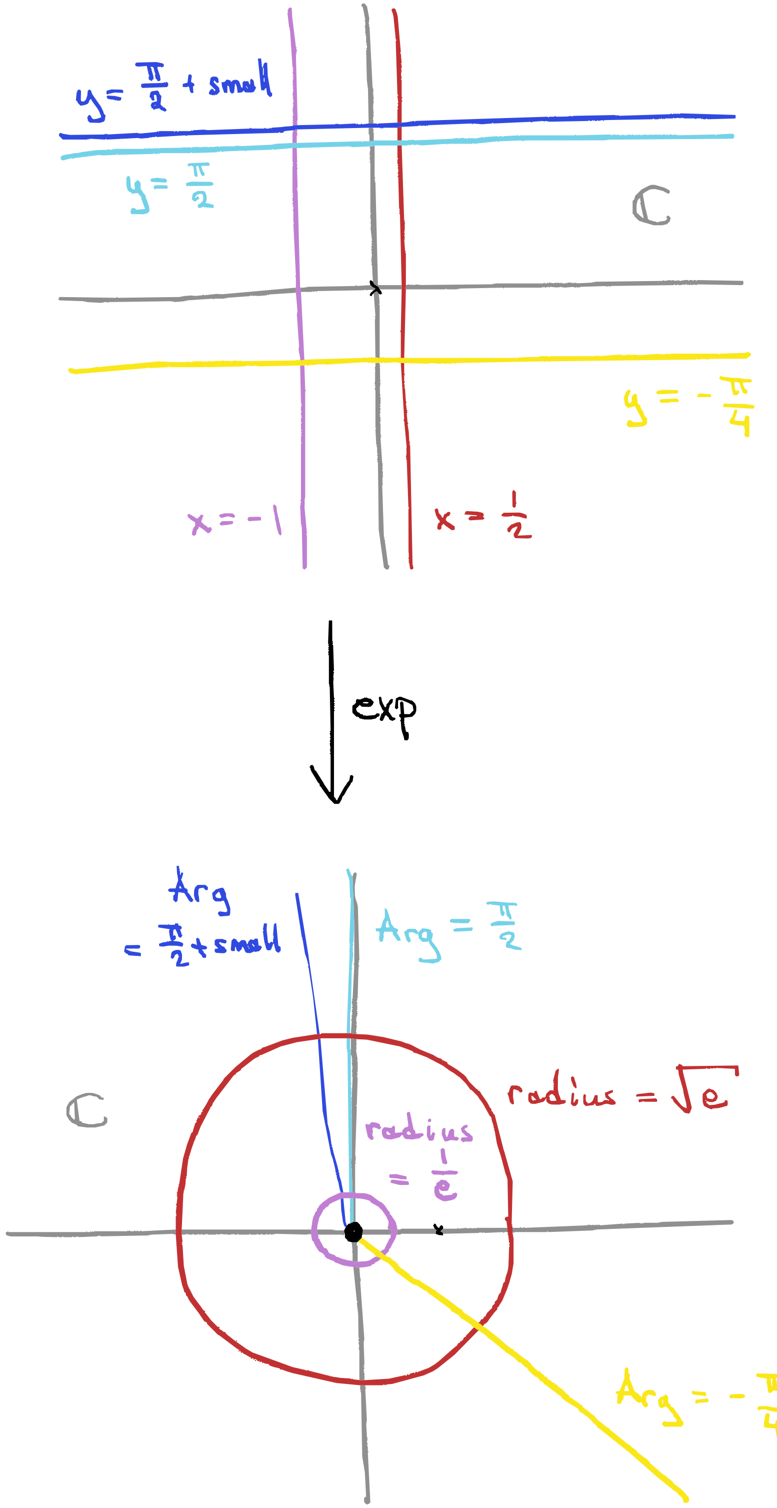
$$= e^x \cdot \cos(y) - i e^x \cdot \sin(y) = \overline{\exp(x+iy)}.$$

The claim about $\arg(e^{x+iy})$ is clear from $e^{x+iy} = e^x(\cos(y) + i\sin(y))$. $\square$

To understand the mapping properties of $\exp : \mathbb{C} \to \mathbb{C}$, note that:

- Vertical lines ($x = \text{const.}$, $y \in \mathbb{R}$) get mapped to circes of radius e^x centered at 0. Indeed $|e^{x+iy}| = e^x$ by the above, and as y increases by 2π, the point e^{x+iy} travels once around the circle in the positive (counterclockwise) direction.

- Horizontal lines ($y = \text{const.}$, $x \in \mathbb{R}$) get mapped to rays from the origin in a fixed direction. Indeed e^{x+iy} is a positive multiple of the unit vector $(\cos(y) + i \sin(y))$.
  As x increases "from $-\infty$ to $+\infty$", the distance $|e^{x+iy}| = e^x$ from the origin grows "from 0 to ∞".



y = π/2 + small
y = π/2
ℂ
y = -π/4
x = -1
x = 1/2
exp
Arg = π/2 + small
Arg = π/2
ℂ
radius = √e
radius = 1/e
Arg = -π/4

Crucially, the complex exponential still satisfies the familiar:

Lemma (Functional equation for exponentials)

For any $z, w \in \mathbb{C}$ we have

$$\exp(z+w) = \exp(z) \cdot \exp(w).$$

(The proof is basically the same as the calculation for multiplication in polar form.)

Proof: Write $z = x_1 + i y_1$, $w = x_2 + i y_2$ (with $x_1, x_2, y_1, y_2 \in \mathbb{R}$). By definition,

$$\begin{aligned}
&\exp(z) \cdot \exp(w) \\
&= e^{x_1}(\cos(y_1) + i \cdot \sin(y_1)) \cdot e^{x_2}(\cos(y_2) + i \cdot \sin(y_2)) \\
&= e^{x_1} e^{x_2} \big((\cos(y_1)\cos(y_2) - \sin(y_1)\sin(y_2)) \\
&\qquad\qquad + i(\cos(y_1)\sin(y_2) + \sin(y_1)\cos(y_2)) \big) \\
&= e^{x_1 + x_2} \left(\cos(y_1 + y_2) + i \cdot \sin(y_1 + y_2) \right) \\
&= \exp\left(x_1 + x_2 + i(y_1 + y_2) \right) = \exp(z + w).
\end{aligned}$$

$\square$

Logarithms

Note first that $|e^z| = e^{\mathrm{Re}(z)} > 0$ is always positive, so the complex exponential never attains the value 0.

Note then that for a fixed $w \in \mathbb{C} \setminus \{0\}$, written in polar form as $w = r(\cos(\theta) + i\sin(\theta))$, the equation

$$\boxed{e^z = w}$$

is solved by $z = x + iy$ $(x, y \in \mathbb{R})$ if and only if

$$\underbrace{|e^z|}_{= e^x} = \underbrace{|w|}_{= r} \quad \text{and} \quad \underbrace{\arg(e^z)}_{= y \ (\mathrm{mod}\ 2\pi)} = \underbrace{\arg(w)}_{= \theta \ (\mathrm{mod}\ 2\pi)} \ (\mathrm{mod}\ 2\pi),$$

i.e. iff $x = \log(r)$ (real logarithm of $r > 0$!)

and $y = \theta + 2\pi n$ for some $n \in \mathbb{Z}$.

So there are infinitely many solutions.

There is no one canonical way of choosing the logarithm z of $w \in \mathbb{C} \setminus \{0\}$.

But one possible choice is to select $y = \mathrm{Arg}(w) \in (-\pi, \pi]$, the principal argument.

This defines the principal branch of the complex logarithm

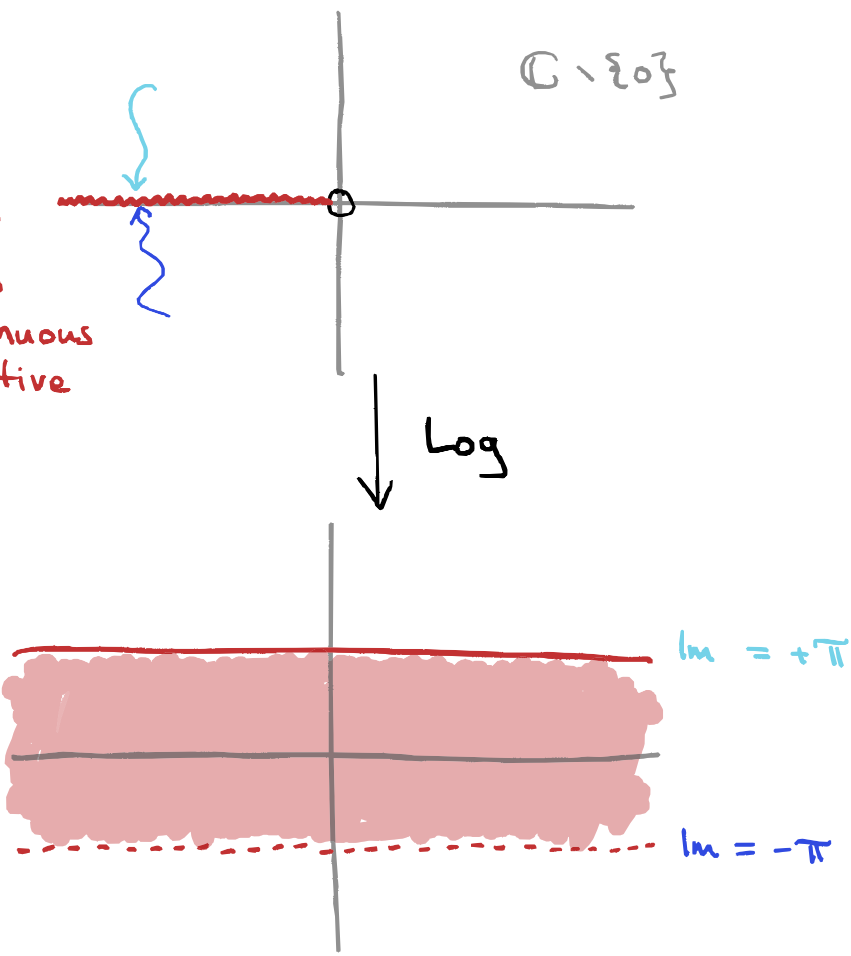
$$\mathrm{Log} : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C},$$

$$\mathrm{Log}(w) := \log(|w|) + i \cdot \mathrm{Arg}(w).$$

Note that since $\mathrm{Arg}(w) \in (-\pi, \pi]$, the values of the principal logarithm lie in the horizontal strip

$$\{ z \in \mathbb{C} \mid -\pi < \mathrm{Im}(z) \leq \pi \}.$$

WARNING: The principal Arg and Log are discontinuous on the negative real axis!



By construction, for any $w \in \mathbb{C} \setminus \{0\}$ we have

$$\exp(\mathrm{Log}(w)) = \exp(\log|w| + i\,\mathrm{Arg}(w))$$
$$= e^{\log|w|} \cdot (\cos(\mathrm{Arg}(w)) + i \sin(\mathrm{Arg}(w)))$$
$$= w.$$

However, it is not always true that $\mathrm{Log}(\exp(z))$ equals z. For example for $z = \frac{123}{2}\pi i$ we have

$$\exp\left(\tfrac{123}{2}\pi i\right) = \overbrace{\cos\Big(\underbrace{\tfrac{123}{2}\pi}_{\equiv \frac{3\pi}{2} \ (\mathrm{mod}\ 2\pi)}\Big)}^{=0} + i \cdot \overbrace{\sin\Big(\underbrace{\tfrac{123}{2}\pi}_{-\text{ll}-}\Big)}^{=-1} = -i.$$

Now $\mathrm{Arg}(-i) = -\frac{\pi}{2}$ and $|-i| = 1$, so

$$\mathrm{Log}\left(\exp\left(\tfrac{123}{2}\pi i\right)\right) = \mathrm{Log}(-i) = \log(1) + i\,\frac{-\pi}{2}$$
$$= -i\frac{\pi}{2} \neq \frac{123}{2}\pi i.$$

(these are equal modulo integer multiples of $2\pi i$, however.)

$$\boxed{\exp(\mathrm{Log}(w)) = w \ \checkmark}$$

$$\boxed{\mathrm{Log}(\exp(z)) \neq z \ \text{(IN GENERAL)}}$$

Power functions and n^{th} roots

Logarithms and exponentials allow one to define general power functions — and n^{th} root functions in particular.
These typically exhibit similar difficulties as complex logarithms, related to (continuous) branch choices.

For $\alpha \in \mathbb{C}$, we define the principal branch of the α - power function $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$ by

$$z \longmapsto z^{\alpha} := e^{\alpha \cdot \mathrm{Log}(z)}$$

Note that the discontinuity of the principal logarithm Log typically causes this to be discontinuous on the negative real axis, so sometimes other "branch choices" are preferable.
Note that for $\alpha, \beta \in \mathbb{C}$, we have

$$z^{\alpha} \cdot z^{\beta} = e^{\alpha \cdot \mathrm{Log}(z)} \cdot e^{\beta \cdot \mathrm{Log}(z)}$$

$$= e^{(\alpha+\beta)\mathrm{Log}(z)} = z^{\alpha+\beta}.$$

(But beware: $(zw)^{\alpha}$ may not be $z^{\alpha} \cdot w^{\alpha}$!)

For $\alpha = \frac{1}{n}$ with $n \in \mathbb{N}$, we get the principal n^{th} root function

$$z \mapsto \sqrt[n]{z} := z^{1/n} = e^{\frac{1}{n}\operatorname{Log}(z)},$$

which at least satisfies

$$\left(\sqrt[n]{z}\right)^n = \left(e^{\frac{1}{n}\operatorname{Log}(z)}\right)^n = e^{\operatorname{Log}(z)} = z.$$

Note however that the choice of the n^{th} root is not unique — if $\xi = e^{i2\pi j/n}$, $j \in \{0, 1, \ldots, n-1\}$, is any of the n n^{th} roots of unity, then also

$$\left(\xi \cdot \sqrt[n]{z}\right)^n = \xi^n \cdot \left(\sqrt[n]{z}\right)^n = 1 \cdot z = z.$$

(Remark: For $n=2$, this is just the familiar observation that $(\sqrt{z})^2 = z = (-\sqrt{z})^2$.)

WARNING:
Careless manipulation (without appropriately accounting for branch choices) leads to "paradoxes" (mistakes, really!) such as

$$-1 = i^2 = \sqrt{-1} \cdot \sqrt{-1} \overset{?}{=} \sqrt{(-1)\cdot(-1)} = \sqrt{1} = 1.$$

FALSE!

($\sqrt{(-1)\cdot(-1)} \neq \sqrt{-1} \cdot \sqrt{-1}$ (with principal $\sqrt{\cdot}$) is one case of $(z \cdot w)^\alpha \neq z^\alpha \cdot w^\alpha$)

Trigonometric functions

Note that Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ for $\theta \in \mathbb{R}$ allows us to write

$$\cos(\theta) = \operatorname{Re}(e^{i\theta}) = \tfrac{1}{2}(e^{i\theta} + \overline{e^{i\theta}})$$
$$= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta})$$

and

$$\sin(\theta) = \operatorname{Im}(e^{i\theta}) = \tfrac{1}{2i}(e^{i\theta} - \overline{e^{i\theta}})$$
$$= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

This suggests that the following definition of complex trigonometric functions is reasonable (agrees with the real trigonom. functions on the real line $\mathbb{R} \subset \mathbb{C}$)

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}.$$

We have thus generalized sine and cosine to functions

$$\sin : \mathbb{C} \to \mathbb{C}, \qquad \cos : \mathbb{C} \to \mathbb{C}.$$

Many formulas and identities for these functions can now be proven simply with the arithmetic of complex numbers and the property $e^{z+w} = e^z e^w$ of the complex exponential. In particular, identities for real trigonometric functions are obtained — as special cases, restricting to the real line $\mathbb{R} \subset \mathbb{C}$.

As an example, consider, for $z \in \mathbb{C}$

$$
\begin{aligned}
&\cos(z)^2 + \sin(z)^2 \\
&= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 \\
&= \frac{\cancel{e^{2iz}} + 2e^{i0} + \cancel{e^{-2iz}}}{4} + \frac{\cancel{e^{2iz}} - 2e^{i0} + \cancel{e^{-2iz}}}{-4} \\
&= \frac{4 \cdot e^{i0}}{4} = 1.
\end{aligned}
$$

A.1 DISTANCES (METRIC) IN $\mathbb{C}$

In order to make rigorous sense of, e.g., limits and continuity in $\mathbb{C}$, we need to endow the complex plane with a topology. For this, it suffices to have a reasonable notion of distances between points in the plane, i.e., a metric,

$$d : \mathbb{C} \times \mathbb{C} \longrightarrow [0, \infty)$$

$$d(z, w) := |w - z| = \text{"the distance from } z \text{ to } w\text{"}$$

The reasonability amounts to the following properties:

<u>Theorem</u> (The metric axioms for $\mathbb{C}$)

For any $z_1, z_2, z_3 \in \mathbb{C}$ we have:

$$|z_3 - z_1| \leq |z_3 - z_2| + |z_2 - z_1| \quad \text{"triangle inequality"}$$

$$|z_2 - z_1| = |z_1 - z_2| \quad \text{"symmetricity"}$$

$$|z_2 - z_1| = 0 \iff z_1 = z_2. \quad \text{"separation of points"}$$

The metric in $\mathbb{C}$ is constructed using the absolute value (modulus), so to prove the above and to work with the metric, we need some auxiliary observations about absolute values.

Lemma

For any $z \in \mathbb{C}$ we have:

$$|\mathrm{Re}(z)| \leq |z|$$

$$|\mathrm{Im}(z)| \leq |z|$$

Proof: Write $z = x + iy$ with $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

Then

$$|\mathrm{Re}(z)| = |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |z|.$$

(since $y^2 \geq 0$ and $\sqrt{\cdot} : [0,\infty) \to [0,\infty)$ is increasing)

Similarly

$$|\mathrm{Im}(z)| = |y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} = |z|.$$

$\square$

Lemma (Triangle inequality for the modulus)

For any $z, w \in \mathbb{C}$, we have
$$|z+w| \leq |z| + |w|.$$

Proof: Recall the formula $|z|^2 = z \cdot \bar{z}$ for the modulus squared. Apply this to calculate the square of $|z+w|$

$$
\begin{aligned}
|z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\
&= \underbrace{z\bar{z}}_{=|z|^2} + \underbrace{z\bar{w} + w\bar{z}}_{=2\mathrm{Re}(z\bar{w})} + \underbrace{w\bar{w}}_{=|w|^2} \\
&= |z|^2 + 2 \cdot \underbrace{\mathrm{Re}(z\bar{w})}_{\leq |z\bar{w}| \text{ (by lemma above)}} + |w|^2 \\
&\leq |z|^2 + 2|z\bar{w}| + |w|^2 \\
&= |z|^2 + 2|z| \cdot |w| + |w|^2 \\
&= (|z| + |w|)^2.
\end{aligned}
$$

Taking square roots(*), we obtain the desired
$$|z+w| \leq |z| + |w|. \qquad \square$$

((*) The increasing function $\sqrt{\cdot} : [0,\infty) \to [0,\infty)$ preserves the inequality.)

Proof of thm (metric axioms)

The triangle inequality for the metric,

$$|z_3 - z_1| \le |z_3 - z_2| + |z_2 - z_1|$$

follows from the previous lemma by taking $z = z_3 - z_2$, $w = z_2 - z_1$.

The other two properties are very easy. $\square$

Another useful consequence of the previous lemma is:

Lemma

For any $z \in \mathbb{C}$,

$$|z| \le |\mathrm{Re}(z)| + |\mathrm{Im}(z)|.$$

Proof Write $z = x + iy$ with $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. Then apply the previous lemma to x and iy:

$$|z| = |x + iy| \le |x| + |iy|$$

$$= |x| + \underbrace{|i|}_{=1} \cdot |y| = |x| + |y|$$

$$= |\mathrm{Re}(z)| + |\mathrm{Im}(z)|. \qquad \square$$

DISKS AND CIRCLES

In metric topology, the notion of a "ball" is central. In the complex plane $\mathbb{C}$, balls with respect to the metric $d(z,w) = |z-w|$ are disks, so in complex analysis, the latter term is commonly used.

<u>Def</u>: Let $z_0 \in \mathbb{C}$ and $r > 0$.

The subset ("the set of points whose distance to z_0 is less than r")

$$B(z_0, r) := \{ z \in \mathbb{C} \mid |z - z_0| < r \}$$

is called an open disk (open ball) with center z_0 and radius r.

The subset

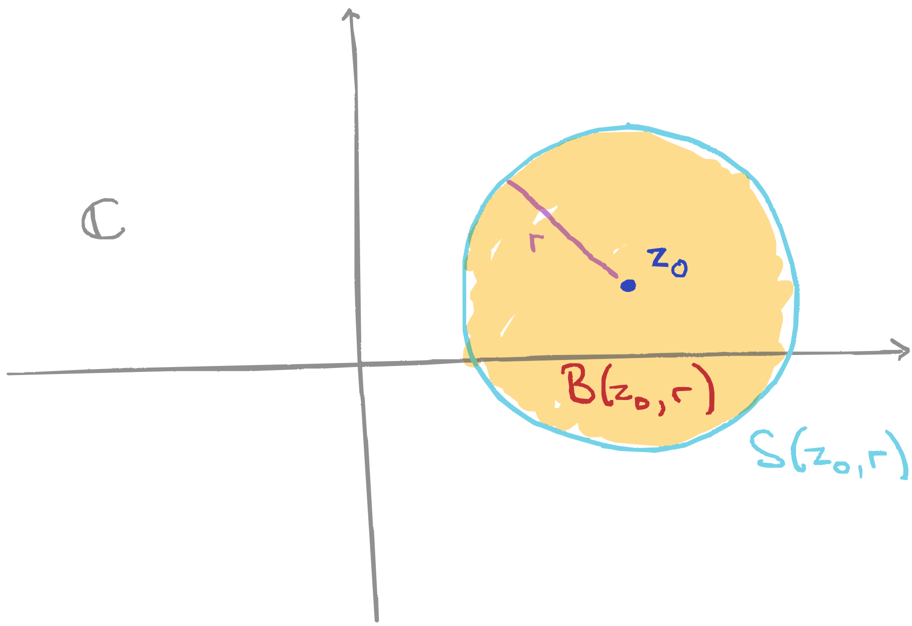
$$\overline{B}(z_0, r) := \{ z \in \mathbb{C} \mid |z - z_0| \leq r \}$$

is called a closed disk (closed ball).

The subset

$$S(z_0, r) := \{ z \in \mathbb{C} \mid |z - z_0| = r \}$$

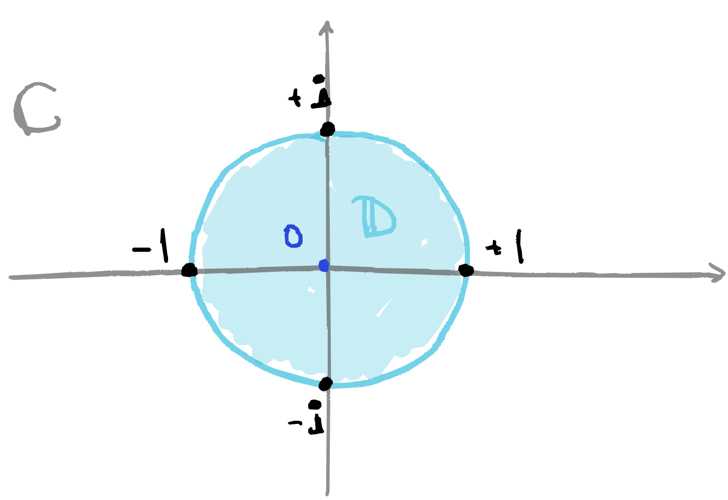
is called a circle (a sphere).



It is often particularly convenient to use the unit disk

$$\mathbb{D} = B(0,1) = \{z \in \mathbb{C} \mid |z| < 1\}$$

in the complex plane, $\mathbb{D} \subset \mathbb{C}$.

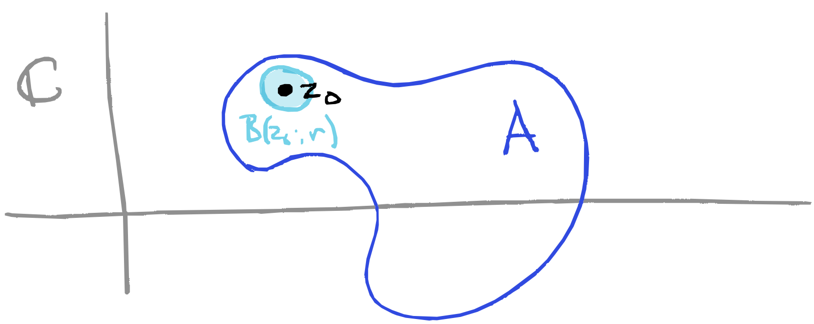


TOPOLOGY

With balls (disks), we define the most fundamental concepts of topology.

Def:

Let $A \subset \mathbb{C}$ be a subset of the complex plane. A point $z_0 \in A$ is said to be an interior point of A if for some $r > 0$ we have $B(z_0; r) \subset A$.



There is a little bit of room in A around an interior point z_0 of A. Such a point can therefore be approached from all directions within the set A. This makes it so that for example derivatives at z_0 carry meaningful information about the change of the function in all directions.

The best kinds of sets for analysis are:

Def

A set $U \subset \mathbb{C}$ is called an open set if every point $z_0 \in U$ is an interior point of U.

A related useful notion is:

Def

A set $F \subset \mathbb{C}$ is called a closed set if its complement $\mathbb{C} \setminus F$ is an open set.

Examples

The following are typical examples (see Metric Spaces course for details)

- the empty set $\emptyset$ is both open and closed
- the whole space $\mathbb{C}$ is both open and closed
- an open disk $B(z_0;r)$ is open
- a closed disk $\overline{B}(z_0;r)$ is closed
- any finite set is closed.

Lemma

(i): The union of an arbitrary collection of open sets is open.

(ii): The intersection of an arbitrary finite collection of open sets is open.

(iii): The intersection of an arbitrary collection of closed sets is closed.

(iv): The union of an arbitrary finite collection of closed sets is closed.

(The proof is straightforward, see Metric Spaces)

Limits

Analysis relies heavily on the notion of limits, so let us spell out this notion clearly in a complex-variable context

Def.

A sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers $z_n \in \mathbb{C}$ converges to the limit $w \in \mathbb{C}$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$|z_n - w| < \varepsilon \quad \text{whenever} \quad n \geq N.$$

We then denote $\lim_{n\to\infty} z_n = w$.

Def:

If $f : A \to \mathbb{C}$ is a complex-valued function on a subset $A \subset \mathbb{C}$ of the complex plane, then we say that f has limit $w \in \mathbb{C}$ at a point z_0 if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - w| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

We then denote $\lim_{z\to z_0} f(z) = w$.

For calculations with limits, the following properties are essential in practice:

Lemma

Suppose $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ are two sequences of complex numbers with

$$\lim_{n\to\infty} z_n = a, \quad \lim_{n\to\infty} w_n = b.$$

Then

$$\lim_{n\to\infty} (z_n + w_n) = a+b, \quad \lim_{n\to\infty} (z_n \cdot w_n) = a\cdot b,$$

and if $b \neq 0$ then $\displaystyle\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{a}{b}$.

Lemma

Suppose $f, g : A \to \mathbb{C}$ are two complex-valued functions on $A \subset \mathbb{C}$ such that

$$\lim_{z\to z_0} f(z) = \alpha, \quad \lim_{z\to z_0} g(z) = \beta.$$

Then

$$\lim_{z\to z_0} (f(z) + g(z)) = \alpha + \beta \quad \text{and}$$

$$\lim_{z\to z_0} (f(z)\cdot g(z)) = \alpha\beta.$$

If $\beta \neq 0$, then also $\displaystyle\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\alpha}{\beta}$.

Continuity

Def

A function $f : A \to \mathbb{C}$ defined on a subset $A \subset \mathbb{C}$ of the complex plane is continuous at a point $z_0 \in A$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

We say that f is a continuous function if it is continuous at every $z_0 \in A$.

From properties of limits we easily get:

Lemma

Suppose that $f, g : A \to \mathbb{C}$ are two functions which are continuous at $z_0 \in A$. Then:

$z \mapsto f(z) + g(z)$ is continuous at z_0

$z \mapsto f(z) \cdot g(z)$ is continuous at z_0

and if moreover $g(z_0) \neq 0$ then

$z \mapsto \dfrac{f(z)}{g(z)}$ is continuous at z_0

Also the composition of functions preserves continuity:

Lemma

Let $A, B \subset \mathbb{C}$ be subsets of the complex plane and let

$$f: A \longrightarrow B \quad \text{and} \quad g: B \longrightarrow \mathbb{C}$$

be two functions.

Consider the composed function

$$g \circ f : A \longrightarrow \mathbb{C} \quad \text{defined by}$$

$$(g \circ f)(z) = g(f(z)).$$

If f is continuous at $z_0 \in A$ and g is continuous at $f(z_0) \in B$, then $g \circ f$ is continuous at z_0.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & \mathbb{C} \end{tikzcd}$$

$$z \underset{f}{\longmapsto} f(z) \underset{g}{\longmapsto} g(f(z))$$

Later on in the course, we will also make use of the following topological notions — for now we just quickly mention:

(We will be more precise about these later. A systematic study, however, is done in other courses — e.g. Metric Spaces, General Topology)

- compactness

  (A set $A \subset \mathbb{C}$ is compact if and only if it is closed and bounded.)

  - continuous real-valued functions on compact sets have maxima/minima (in particular such functions are bounded)
  - continuous functions on compact sets are uniformly continuous ($\rightsquigarrow$ integrals can be defined)
  - intersections of nested non-empty compact sets are non-empty (a key fact used in the proof of Goursat's lemma, which in turn is key to the proof of Cauchy's integral formula)

- **connectedness** and **path-connectedness**

(a set $A \subset \mathbb{C}$ is connected if every continuous $f : A \to \{0,1\}$ is constant)

(A set $A \subset \mathbb{C}$ is path connected if for any $z, w \in A$ there exists a cont. $\gamma : [0,1] \to A$ such that $\gamma(0) = z$, $\gamma(1) = w$)

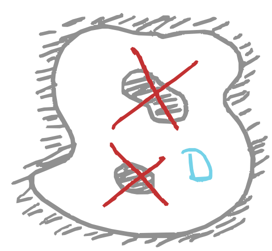
(Fact: path-connected $\Longrightarrow$ connected)

(Fact: open & connected $\Longrightarrow$ path-connected)

- Important e.g. for integrating along paths.
- A function $f : D \to \mathbb{R}$ on a connected open set $D \subset \mathbb{C}$ is constant if and only if its differential is zero.

- **simple connectedness**

$\Longleftrightarrow$ "no holes"



- Holes (failure of simply connectedness) are often the (only) global obstruction to "integral functions existence"

(See e.g. harmonic conjugates and existence of primitives later.)