On the existence of primitives

Recall that a primitive of a continuous function $f: U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ is an analytic function $F: U \to \mathbb{C}$ such that $F'(z) = f(z)$ for every $z \in U$.
Our goal is to understand when do such primitives exist.
The first key observation was already made: if F is a primitive of f, then for any contour γ from z_1 to z_2 we have

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1),$$

i.e. the contour integrals of f only depend on the two endpoints of the contour, and in particular for closed contours γ, the contour integral must vanish,

$$\oint_\gamma f(z)\,dz = 0.$$

This observation shows that two necessary conditions for the existence of a primitive of $f : U \to \mathbb{C}$ are

- $\int_\gamma f(z)\,dz$ only depends on the starting and ending points of γ
- $\oint_\gamma f(z)\,dz = 0$ for all closed contours γ.

The first example application was that $z \mapsto \frac{1}{z}$ does not have a primitive in $\mathbb{C} \setminus \{0\}$, because by a direct calculation

$$\oint_\gamma \frac{1}{z}\,dz = 2\pi i \neq 0$$

when γ is the positively oriented unit circle (closed contour), $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$.

Let us give another, different example.

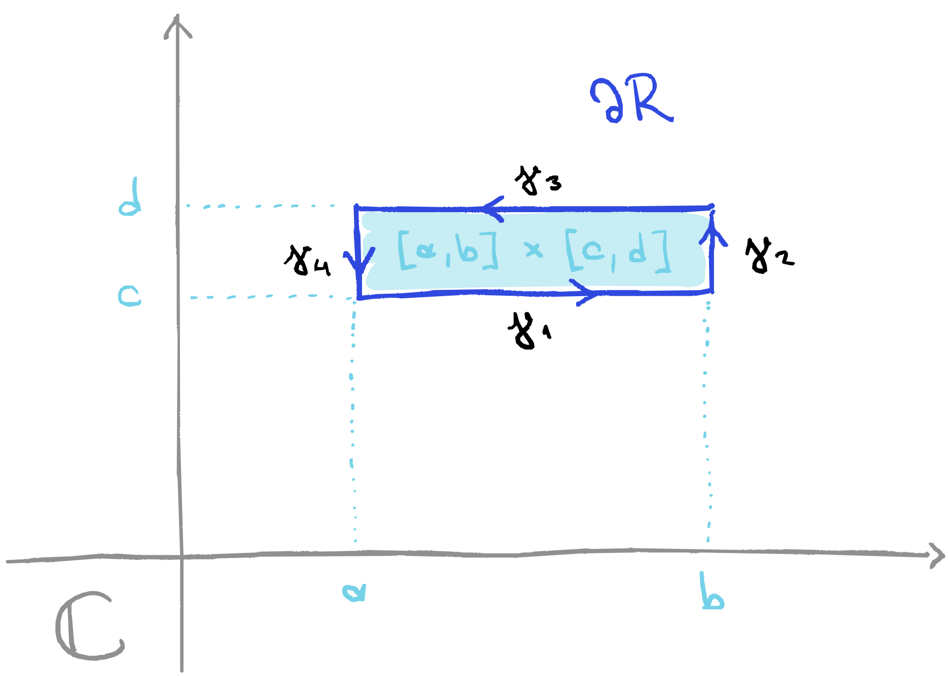
Example

Consider the function $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$.

Let $a<b$, $c<d$, and consider the boundary ∂R of the rectangle

$$R = [a,b] \times [c,d]$$

as a closed (piecewise smooth) path.



We can parametrize ∂R as the concatenation of four line segments,

$$\gamma_1(t) = a + ic + t(b-a) \qquad t \in [0,1]$$
$$\gamma_2(t) = b + ic + it(d-c) \qquad t \in [0,1]$$
$$\gamma_3(t) = b + id - t(b-a) \qquad t \in [0,1]$$
$$\gamma_4(t) = a + id - it(d-c) \qquad t \in [0,1].$$

The contour integral of f along the closed contour $\gamma_1 \boxplus \gamma_2 \boxplus \gamma_3 \boxplus \gamma_4$ is the sum

$$\oint_{\gamma_1 \boxplus \cdots \boxplus \gamma_4} \bar{z}\, dz = \int_{\gamma_1} \bar{z}\, dz + \cdots + \int_{\gamma_4} \bar{z}\, dz.$$

Let us calculate the integral of $z \mapsto \bar{z}$ along a general line segment, from w_1 to w_2, $\gamma_{[w_1,w_2]}(t) = w_1 + t\cdot(w_2 - w_1)$ for $t \in [0,1]$. We get

$$\int_{\gamma_{[w_1,w_2]}} \bar{z}\, dz = \int_0^1 \overline{\gamma_{[w_1,w_2]}(t)} \cdot \dot{\gamma}_{[w_1,w_2]}(t)\, dt$$

$$= \int_0^1 \overline{\left(w_1 + t\cdot(w_2-w_1)\right)} \cdot (w_2-w_1)\, dt$$

$$= \bar{w}_1 \cdot (w_2-w_1) \underbrace{\int_0^1 dt}_{=1} + (\bar{w}_2-\bar{w}_1)(w_2-w_1) \underbrace{\int_0^1 t\, dt}_{=1/2}$$

$$= \bar{w}_1 \cdot (w_2-w_1) + \frac{1}{2}(\bar{w}_2-\bar{w}_1)(w_2-w_1)$$

$$= (w_2-w_1)\,\frac{\bar{w}_1+\bar{w}_2}{2}.$$

The integral along ∂R is then

$$\int_{\gamma_1 \boxplus \cdots \boxplus \gamma_4} \bar{z}\, dz = \int_{\gamma_1} \bar{z}\, dz + \int_{\gamma_2} \bar{z}\, dz + \int_{\gamma_3} \bar{z}\, dz + \int_{\gamma_4} \bar{z}\, dz$$

$$= (b-a)\,\frac{a+b-2ic}{2} + (id-ic)\,\frac{-ic-id+2b}{2} + (a-b)\,\frac{a+b-2id}{2} + (ic-id)\,\frac{-ic-id+2a}{2}$$

$$= 2i\cdot(b-a)(d-c)$$

$$= 2i\cdot \operatorname{Area}\left([a,b]\times[c,d]\right) \neq 0.$$

The example shows that $z \mapsto \bar{z}$ cannot have a primitive in $\mathbb{C}$. (In fact that $z \mapsto \bar{z}$ cannot have a primitive in any nonempty open set, because the above can be done with arbitrarily tiny rectangles.)

It is useful to observe that the two necessary conditions are in fact equivalent.

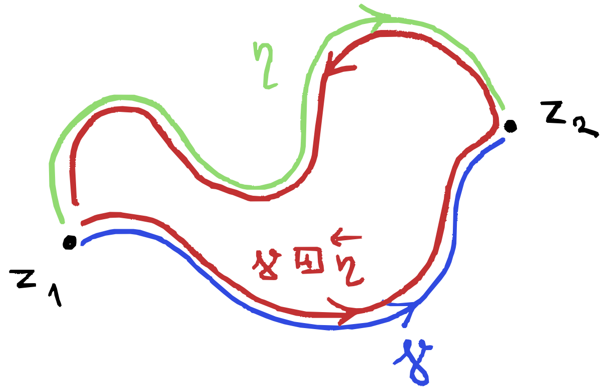
Lemma

For a continuous function $f: U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$, the following are equivalent:

(i): The contour integrals $\int_\gamma f(z)\,dz$ only depend on the starting and end points of γ.

(ii): For all closed contours γ, $\oint_\gamma f(z)\,dz = 0$.

Proof



(ii) $\Rightarrow$ (i) : Assume (ii).

To prove (i), let γ and η be two contours in U with the same starting and end points, say z_1 (start) and z_2 (end).

Observe that the concatenation $\gamma \boxplus \overleftarrow{\eta}$ of one path γ with the reversal $\overleftarrow{\eta}$ of the other path is a closed contour (starting and ending at z_1). Assumption (ii) gives

$$0 = \oint_{\gamma \boxplus \overleftarrow{\eta}} f(z)\,dz = \int_{\gamma} f(z)\,dz + \int_{\overleftarrow{\eta}} f(z)\,dz$$

$$= \int_{\gamma} f(z)\,dz - \int_{\eta} f(z)\,dz,$$

and therefore $\displaystyle\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz$,

showing (i): the integrals along the paths are equal whenever the starting and end points coincide.

(i) $\Rightarrow$ (ii): Assume (i).

To prove (ii), let γ be a closed contour starting and ending at z_1. The constant path $\sigma(t) = z_1$ for $t \in [0,1]$ also has z_1 as its starting and end point, so by assumption (i)

$$\int_\gamma f(z)\,dz = \int_\sigma f(z)\,dz = \int_0^1 f(\sigma(t))\,\underbrace{\dot\sigma(t)}_{=0}\,dt$$

$$= \int_0^1 0\,dt = 0,$$

showing (ii).

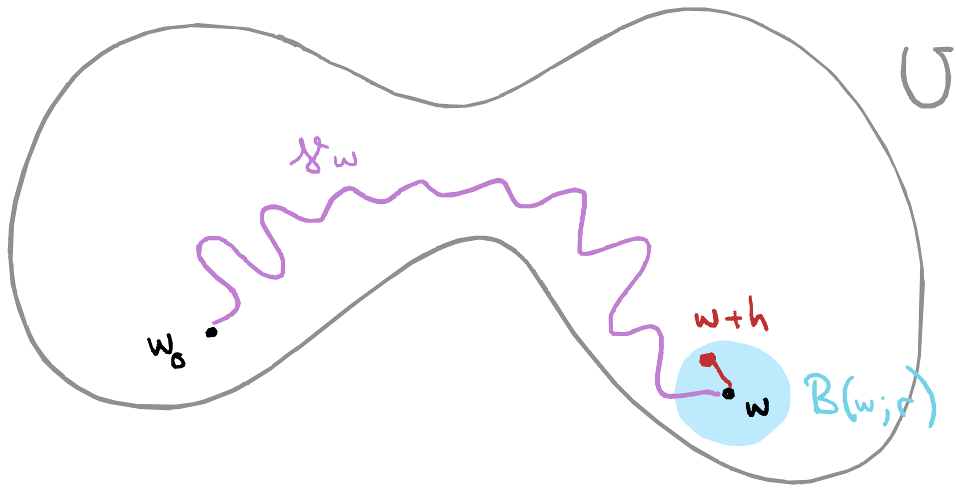
$\square$

Importantly, the conditions are not just necessary but also sufficient for the existence of a primitive:

Lemma

Suppose that $f : U \to \mathbb{C}$ is a continuous function on a connected open set $U \subset \mathbb{C}$ such that the contour integrals $\int_\gamma f(z)\,dz$ only depend on the starting and end points of γ. Then f has a primitive on U.

Proof



Pick a reference point $w_0 \in U$ arbitrarily.

(We are right away assuming $U \neq \emptyset$ because a primitive in an empty set exists for trivial and uninteresting reasons.)

By (path-) connectedness, choose also paths γ_w from this w_0 to w for every $w \in U$.

We claim that the formula

$$F(w) = \int_{\gamma_w} f(z)\,dz \qquad (\text{for } w \in U)$$

defines a primitive of f in U.

For this, we must show that

$$F'(w) = f(w) \quad \text{for every } w \in U.$$

Fix $w \in U$. Since $U \subset \mathbb{C}$ is open, we have $B(w;r) \subset U$ for some $r > 0$. We will calculate the difference quotients

$$\frac{F(w+h) - F(w)}{h}$$

when $0 < |h| < r$, so that

$$w + h \in B(w;r) \subset U.$$

For this purpose, let η_h be a parametrization of the line segment from w to $w+h$,

$\eta_h(t) = w + t \cdot h$ for $t \in [0,1]$. Note that η_h is contained in $B(w;r) \subset U$. Note also that

$$F(w+h) = \int_{\gamma_{w+h}} f(z)\,dz = \int_{\gamma_w \boxplus \eta_h} f(z)\,dz$$

because also the concatenation $\gamma_w \boxplus \eta_h$ is a path from w_0 to $w+h$, and by assumption the contour integral of f is the same for two paths whose starting and end points are the same.

We can therefore simplify

$$F(w+h) - F(w)$$

$$= \int_{\gamma_w \boxplus \eta_h} f(z)\,dz - \int_{\gamma_w} f(z)\,dz = \int_{\eta_h} f(z)\,dz .$$

Let $\varepsilon > 0$. By continuity of f at w, there exists a $\delta > 0$ such that when $|z-w| < \delta$, we have $|f(z) - f(w)| < \varepsilon$. For $|h| < \delta$, then, we have $|f(z) - f(w)| < \varepsilon$ for all z on the line segment η_h from w to $w+h$. We can then estimate

$$\left| \frac{F(w+h)-F(w)}{h} - f(w) \right|$$

$$= \left| \frac{1}{h} \int_{\gamma_h} f(z)\,dz - f(w) \right|$$

$$= \left| \frac{1}{h} \int_{\gamma_h} \left(f(z) - f(w) \right) dz \right|$$

$$\leq \frac{1}{|h|} \int_{\gamma_h} \underbrace{|f(z)-f(w)|}_{<\varepsilon \text{ on } \gamma_h} |dz|$$

$$\leq \frac{1}{|h|} \underbrace{\ell(\gamma_h)}_{=|h|} \cdot \varepsilon = \varepsilon .$$

Since for an arbitrary $\varepsilon > 0$ we achieved this for h s.t. $0 < |h| < \delta$ (with some $\delta > 0$), we have proved

$$F'(w) = \lim_{h \to 0} \frac{F(w+h)-F(w)}{h} = f(w) .$$

Since also $w \in U$ was arbitrary, we have shown that F is a primitive of f in U. $\square$

Combining the observations so far about the necessary and sufficient conditions for the existence of a primitive, we have proved:

Theorem (Characterizing existence of primitives)

[primitive_characterization]

Let $f : U \to \mathbb{C}$ be a continuous function on an open set $U \subset \mathbb{C}$. Then the following are equivalent:

(i) the contour integrals $\int_\gamma f(z)\,dz$ only depend on the starting and end points of γ;

(ii) for every closed contour γ in U we have $\oint_\gamma f(z)\,dz = 0$;

(iii) f has a primitive in U.

Proof This is just a combination of earlier results, once one also observes that the connectedness assumption can be dropped: primitives can be constructed separately in each connected component. $\square$

This characterization is a good starting for analyzing the question of the existence of primitives. But it is in addition important to appreciate the two different sources of failure for the existence of primitives in the two counterexamples we had:

- $z \mapsto \bar{z}$ does not have a primitive in $\mathbb{C}$ — or in any nonempty open set $U \subset \mathbb{C}$.
  - This is a "local" failure of the existence of primitives — we cannot even find primitives in small regions.

    (Infinitesimally, the contributions to $\oint f(z)\,dz$ are proportional to $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}$ which is nonzero unless the Cauchy-Riemann equations hold. For $z \mapsto \bar{z}$, they do not hold.)

(In differential geometry terms, the failure is due to the fact that the 1-form $\bar{z}\cdot dz$ is not "closed", which is a local property of forms.)

- $z \mapsto \frac{1}{z}$ does not have a primitive in the punctured plane $\mathbb{C} \setminus \{0\}$

The function $z \mapsto \frac{1}{z}$ however does turn out to have a primitive in, e.g., any disk $B \subset \mathbb{C} \setminus \{0\}$, i.e., locally — the failure here is due to the "global" topology of $\mathbb{C} \setminus \{0\}$, namely because $\mathbb{C} \setminus \{0\}$ is not simply connected.

(In differential geometry terms, the failure is that the 1-form $\frac{1}{z} dz$ is not "exact" although it is "closed" — the nontrivial cohomology of $\mathbb{C} \setminus \{0\}$ allows for such differential forms to exist.)

CAUCHY'S INTEGRAL THEOREM

Cauchy's integral theorem asserts the existence of primitives when there is no local obstruction (the function satisfies C-R equations, i.e., is analytic) and no global obstruction (the domain is simply connected).

In order to simplify the topological considerations(*), we work under the geometric hypotheses of convexity or star-shapedness, which are more restrictive than simple connectedness, but sufficient for most applications.

(*) (We do this to avoid invoking homotopy theory for piecewise smooth paths.)

Convex and star-shaped sets

Def

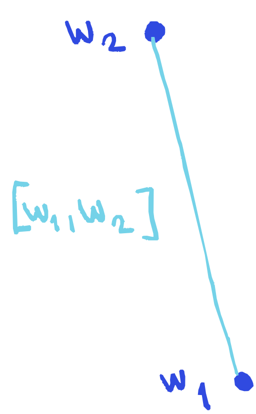
Let $w_1, w_2 \in \mathbb{C}$ be two points in the complex plane. The line segment path from w_1 to w_2 is the path

$$\begin{aligned} [0,1] &\longrightarrow \mathbb{C} \\ t &\longmapsto w_1 + t\cdot(w_2 - w_1). \end{aligned}$$

The image of this path is the subset of the complex plane denoted by

$$[w_1, w_2] = \{ w_1 + t(w_2 - w_1) \mid t \in [0,1] \} \subset \mathbb{C}$$

and called the line segment from w_1 to w_2.



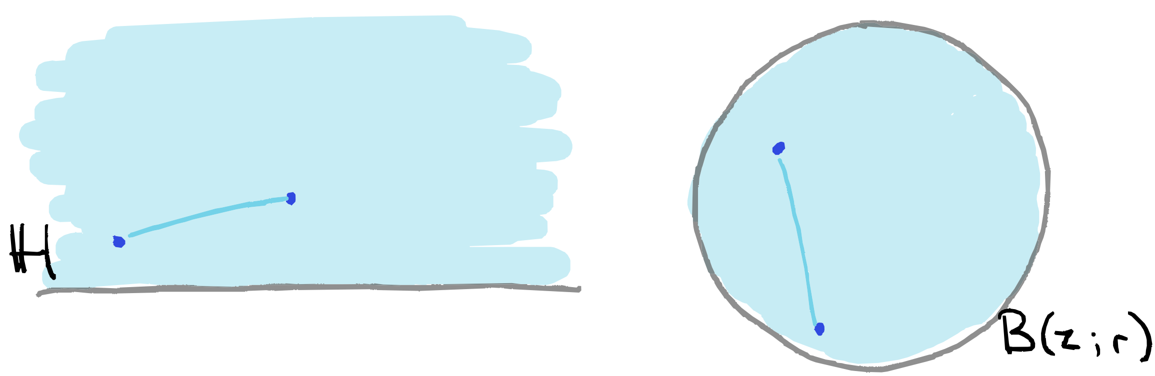
Def

A set $A \subset \mathbb{C}$ is convex if for any two points $z_1, z_2 \in A$ the line segment between them is contained in the set, $[z_1, z_2] \subset A$.

Examples

The following sets are convex:

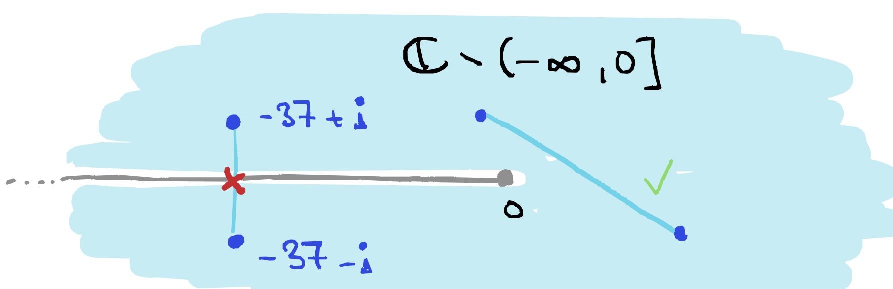
- the complex plane $\mathbb{C}$
- the upper half-plane
$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \} \subset \mathbb{C}$$
- any disk $B(z;r) \subset \mathbb{C}$



(some line segments in convex sets)

Example

The set $\mathbb{C} \setminus (-\infty, 0]$ in not convex: for example the points $z_1 = -37 + i$ and $z_2 = -37 - i$ belong to $\mathbb{C} \setminus (-\infty, 0]$ but the line segment between them also contains the (mid)point $-37 \in (-\infty, 0]$ in the complement.



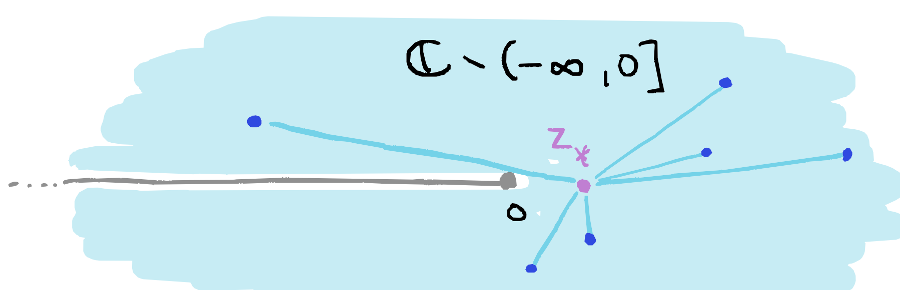
In addition to convexity, there is also another, less restrictive, geometric condition that is still convenient and easy to work with. (i.e., we can still avoid homotopy theory with it.)

Def:

A set $A \subset \mathbb{C}$ is star-shaped if there exists a point $z_* \in A$ such that for all $z \in A$ the line segment from z_* to z is contained in the set, $[z_*, z] \subset A$.

Example

The set $\mathbb{C} \setminus (-\infty, 0]$ is star-shaped: indeed, if we take $z_* = 1$, then it is easy to see that for any $z \in \mathbb{C} \setminus (-\infty, 0]$ we have $[z_*, z] \subset \mathbb{C} \setminus (-\infty, 0]$.



The following observations are immediate from the definitions:

- every nonempty convex set is star-shaped (we can choose any z_* !)
- every star-shaped set (and in particular every convex set) is path-connected (and in particular connected).

Although we have not yet precisely defined simple connectedness, to place the notion in context for our purposes we note that:

- every star-shaped set is simply connected.

Example

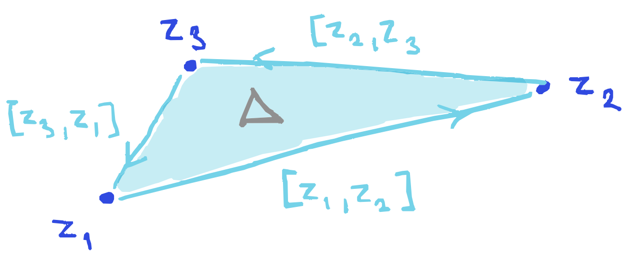
The punctured plane $\mathbb{C} \setminus \{0\}$ is not star-shaped. (... it is not even simply connected.)

To prove Cauchy's integral theorem in star-shaped domains, a special case of triangle contours (Goursat's lemma) suffices!

Given any three points $z_1, z_2, z_3 \in \mathbb{C}$, consider the (closed) triangle

$$\Delta = \left\{ r z_1 + s z_2 + t z_3 \;\middle|\; \begin{array}{l} 0 \le r, s, t \le 1, \\ r + s + t = 1 \end{array} \right\}$$

with these points as its vertices.



Its boundary $\partial\Delta$ is the union of three line segments

$$\partial\Delta = [z_1, z_2] \cup [z_2, z_3] \cup [z_3, z_1].$$

With slight abuse of notation, we view this as a parametrized path (in the counterclockwise direction) obtained as the concatenation of the three parametrized line segment paths.

__Lemma__ (Goursat's lemma)

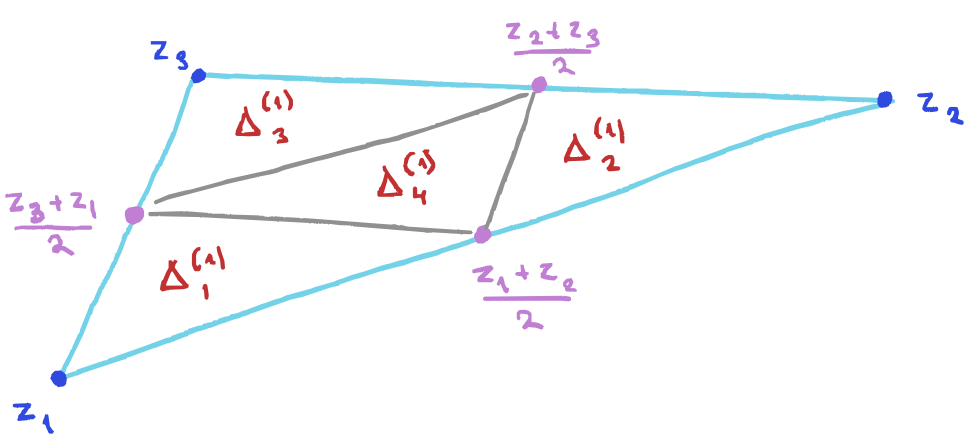
[goursat]

Let $f: U \to \mathbb{C}$ be an analytic function on an open set $U \subset \mathbb{C}$ and $\Delta \subset U$ a closed triangle contained in U. Then we have

$$\oint_{\partial\Delta} f(z)\,dz = 0.$$

__Proof__ Denote $I(\Delta) = \oint_{\partial\Delta} f(z)\,dz$.

Divide Δ into four similar triangles $\Delta_1^{(1)}, \Delta_2^{(1)}, \Delta_3^{(1)}, \Delta_4^{(1)}$ by lines connecting the midpoints of the three sides:



Observe that (with all integrals along counterclockwise paths)

$$I(\Delta) = \oint_{\partial\Delta} f(z)\,dz$$

$$\overset{(*)}{=} \underbrace{\oint_{\partial\Delta_1^{(1)}} f(z)\,dz}_{I(\Delta_1^{(1)})} + \oint_{\partial\Delta_2^{(1)}} f(z)\,dz + \oint_{\partial\Delta_3^{(1)}} f(z)\,dz + \underbrace{\int_{\partial\Delta_4^{(1)}} f(z)\,dz}_{I(\Delta_4^{(1)})}$$

$$= \sum_{j=1}^{4} I(\Delta_j^{(1)})$$

(*) Since the line segments in the interior of Δ appear twice with the opposite orientations in the sum of contour integrals along the boundaries of the smaller triangles, so the apparent excess cancels.

By the triangle inequality, we estimate

$$|I(\Delta)| \leq \sum_{j=1}^{4} |I(\Delta_j^{(1)})| \leq 4 \cdot \max_{j=1,\dots,4} |I(\Delta_j^{(1)})|.$$

Now choose the index $j_1 \in \{1, \dots, 4\}$ for which the sub-triangle $\Delta^{(1)}_{j_1} \subset \Delta$ has the maximal absolute value of contour integral, so that

$$|I(\Delta)| \leq 4 \cdot |I(\Delta^{(1)}_{j_1})| .$$

Also note that side-lengths were halved, so $\quad \ell(\partial \Delta^{(1)}_{j_1}) = \frac{1}{2} \ell(\partial \Delta)$.

Then repeat a similar argument for the triangle $\Delta^{(1)}_{j_1}$: divide to four similar smaller triangles $\Delta^{(2)}_1, \Delta^{(2)}_2, \Delta^{(2)}_3, \Delta^{(2)}_4 \subset \Delta^{(1)}_{j_1}$ by connecting the midpoints of the sides, and estimate similarly, choosing $j_2 \in \{1, \dots, 4\}$ to be the index of the maximal absolute contribution

$$|I(\Delta^{(1)}_{j_1})| \leq 4 \cdot |I(\Delta^{(2)}_{j_2})| \qquad \text{where}$$

$$I(\Delta^{(2)}_{j_2}) = \oint_{\partial \Delta^{(2)}_{j_2}} f(z)\, dz .$$

Note again

$$\ell(\partial\Delta^{(2)}_{j_2}) = \frac{1}{2}\ell(\partial\Delta^{(1)}_{j_1}) = \frac{1}{2^2}\ell(\partial\Delta)$$

and $|I(\partial\Delta)| \leq 4\,|I(\partial\Delta^{(1)}_{j_1})| \leq 4^2\cdot|I(\partial\Delta^{(2)}_{j_2})|$.

Continue to recursively find subtriangles $\Delta \supset \Delta^{(1)}_{j_1} \supset \Delta^{(2)}_{j_2} \supset \Delta^{(3)}_{j_3} \supset \dots$ such that

$$\ell(\partial\Delta^{(m)}_{j_m}) = 2^{-m}\cdot\ell(\partial\Delta)$$

and

$$|I(\partial\Delta)| \leq 4^m\cdot|I(\partial\Delta^{(m)}_{j_m})|.$$

By Cantor's intersection theorem, the nested sequence of nonempty compact sets $(\Delta^{(m)}_{j_m})_{m\in\mathbb{N}}$ has a nonempty intersection, and we may choose a point

$$z_0 \in \bigcap_{m\in\mathbb{N}} \Delta^{(m)}_{j_m}.$$

(In fact, since $\operatorname{diam}(\Delta^{(m)}_{j_m}) \longrightarrow 0$ as $m\to\infty$ we have $\bigcap_{m\in\mathbb{N}} \Delta^{(m)}_{j_m} = \{z_0\}$, only one point.)

Now we use the assumption of analyticity of f : the derivative $f'(z_0)$ exists so we may write the local linear approximation

$$f(z) = f(z_0) + f'(z_0)\cdot(z-z_0) + |z-z_0|\,\epsilon(z)$$

where $\lim_{z\to z_0} \epsilon(z) = 0$.

(💡: This is how analyticity guarantees the absence of local obstructions!)

We use this approximation to estimate the contour integral along boundaries of triangles $\tilde{\Delta} \subset U$. We have

$$I(\tilde{\Delta}) = \oint_{\partial\tilde{\Delta}} f(z)\,dz$$

$$= f(z_0)\cdot \underbrace{\oint_{\partial\tilde{\Delta}} 1\cdot dz}_{=0} {}^{*} + f'(z_0)\cdot \underbrace{\oint_{\partial\tilde{\Delta}} (z-z_0)\,dz}_{=0} {}^{*}$$

$$+ \oint_{\partial\tilde{\Delta}} |z-z_0|\cdot\epsilon(z)\,dz$$

$$= \oint_{\partial\tilde{\Delta}} |z-z_0|\,\epsilon(z)\,dz .$$

* The polynomial functions $z\mapsto 1$ and $z\mapsto z-z_0$ have primitives so these closed contour integrals vanish.

This allows us to estimate by the triangle inequality for integrals

$$|I(\tilde{\Delta})| = \left| \oint_{\partial\tilde{\Delta}} |z-z_0|\, \epsilon(z)\, dz \right|$$

$$\leq \ell(\partial\tilde{\Delta}) \cdot \sup_{z\in\partial\tilde{\Delta}} \left(|z-z_0| \cdot |\epsilon(z)|\right).$$

We now apply this to the triangles $\tilde{\Delta} = \Delta_{j_m}^{(m)}$ in the nested sequence.

Denote $S_m := \sup_{z\in\partial\Delta_{j_m}^{(m)}} |\epsilon(z)|$.

Note that since $z_0 \in \Delta_{j_m}^{(m)}$ and

$\mathrm{diam}(\Delta_{j_m}^{(m)}) = 2^{-m}\,\mathrm{diam}(\Delta) \longrightarrow 0$ as $m\to\infty$

we have $S_m \longrightarrow 0$ as $m \longrightarrow \infty$ by the error term property $\lim_{z\to z_0} \epsilon(z) = 0$.

The estimate thus gives

$$|I(\Delta_{j_m}^{(m)})| \leq \ell(\partial\Delta_{j_m}^{(m)}) \cdot \sup_{z\in\partial\Delta_{j_m}^{(m)}} \left(|z-z_0| \cdot |\epsilon(z)|\right)$$

$$\leq \ell(\partial\Delta_{j_m}^{(m)}) \cdot \mathrm{diam}(\Delta_{j_m}^{(m)}) \cdot S_m$$

$$= 2^{-m}\,\ell(\partial\Delta) \cdot 2^{-m}\,\mathrm{diam}(\Delta) \cdot S_m.$$

Combining with the estimate which relates $I(\Delta_{j_m}^{(m)})$ to the original integral $I(\Delta)$, we get the desired

$$|I(\Delta)| \leq 4^m \cdot |I(\Delta_{j_m}^{(m)})|$$

$$\leq \underbrace{4^m \cdot 2^{-2m}}_{=1} \cdot \underbrace{\ell(\partial\Delta) \cdot \mathrm{diam}(\Delta)}_{=\text{const.}} \cdot \underbrace{\delta'_m}_{\to 0}$$

$$\longrightarrow 0 \quad \text{as} \quad m \to \infty.$$

Therefore $I(\Delta) = 0$ and the proof is complete.

$\square$

Next we will use this towards ....

Theorem (Cauchy's integral theorem)

[cauchy-theorem]

If $U \subset \mathbb{C}$ is open and simply connected and $f : U \to \mathbb{C}$ is analytic, then for all closed contours γ in U we have

$$\oint_\gamma f(z)\, dz = 0 .$$

... but only in the slightly more restrictive case where the domain U is assumed star-shaped.