Aalto University **Problem set 6A**
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2025-2026/II K Kytölä

Exercise sessions: Tue-Wed, Nov 25-26, 2025

Topic: Taylor series, Laurent series, isolated singularities

The first three exercises are to be discussed and solved in exercise sessions. The last exercise (marked with symbol ✍) is a quiz for which you will write a solution in an exam-like setup (no materials or extra equipment allowed) during the last 20min of the exercise session.

Hint: *(…to all exercises.) Two power series can be multiplied "termwise" inside the disks of convergence of both factors. This fact can be considered known; it essentially follows from Exercise 5A.3(ii).*

Exercise (in class) 1.

Find the Laurent series expansions of the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in the following regions:

(a) $\{z \in \mathbb{C} \mid 0 < |z-2| < 1\}$;
(b) $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.

Exercise (in class) 2.

The *Bernoulli numbers* B_n are defined as the coefficients in the following series expansion, valid when $|z| \neq 0$ is small enough:

$$\frac{z}{e^z - 1} = B_0 + \frac{B_1}{1!}\, z + \frac{B_2}{2!}\, z^2 + \frac{B_3}{3!}\, z^3 + \frac{B_4}{4!}\, z^4 + \cdots .$$

Show that the Bernoulli numbers satisfy the recurrence

$$\binom{k}{0} B_0 + \binom{k}{1} B_1 + \cdots + \binom{k}{k-1} B_{k-1} = 0$$

for any $k \geq 2$. Calculate the first 7 Bernoulli numbers $B_0, B_1, \ldots, B_6$.

Remark: A natural follow-up consideration would be to examine the growth rate of the Bernoulli numbers. A starting point for it could be the radius of convergence of the power series defining them (or computer assisted calculations).

Exercise (in class) 3.

Determine the types of singularities of the following functions at the given points:

(a): $z \mapsto z \cos(1/z)$ at $z = 0$

(b): $z \mapsto \dfrac{z^2 + 1 - e^{z^2}}{z^4}$ at $z = 0$

(c): $z \mapsto \dfrac{37}{\cos(\frac{\pi z^3}{2})}$ at $z = 1$.

✍ **Quiz 4.**

Find the Laurent series expansions of the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in the following regions:

(a) $\{z \in \mathbb{C} \mid 0 < |z-1| < 1\}$;

(b) $\{z \in \mathbb{C} \mid |z| > 2\}$.