PRELIMINARIES ON ISOLATED SINGULARITIES

We will work towards analyzing and classifying "isolated singular points" of analytic functions. Informally, these are points z_0 such that our function f of interest is defined and analytic near z_0 but not (necessarily) at z_0. We are then interested in the behavior of f near z_0.

For the definition of isolated singular points as well as for their analysis, it is convenient to introduce:

Def:

For $z_0 \in \mathbb{C}$ and $r > 0$, denote by

$$B^*(z_0;r) = B(z_0;r) \setminus \{z_0\}$$
$$= \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$$

the punctured disk of radius r centered at z_0.

The precise meaning of the isolated singularities that we seek to analyze is:

Def:

A point $z_0 \in \mathbb{C}$ is an isolated singular point of a function $f: U \to \mathbb{C}$ if there exists an $\varepsilon > 0$ such that $B^*(z_0;\varepsilon) \subset U$ and f is analytic in $B^*(z_0;\varepsilon)$.

To start classifying the behavior of complex-valued functions near their isolated singular points, we extend the notion of limits of complex-valued functions slightly — we give a notion of "tending to ∞" of complex values.

Specifically, we write

$$\lim_{z \to z_0} f(z) = \infty$$

if for any $M > 0$ (large) there exists a $\delta > 0$ (small) such that

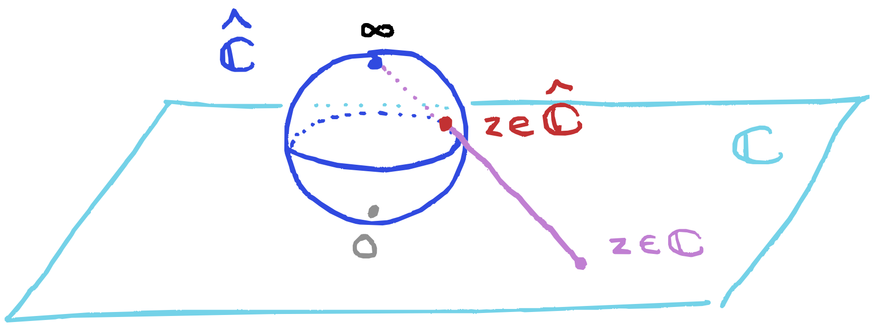
$$|f(z)| > M \quad \text{whenever} \quad 0 < |z - z_0| < \delta .$$

Although we do not make this formal at the moment, the idea is to "one-point compactify" the complex plane to

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} .$$

"the Riemann sphere"

It is natural to give this extended complex plane the topology of a sphere.



The stereographic projection can be used to make points of the complex plane $\mathbb{C}$ correspond bijectively (and homeomorphically) to points on a sphere $\hat{\mathbb{C}}$ "sitting on top of the complex plane at the origin", except for one point, the "north pole" $\infty \in \hat{\mathbb{C}}$.

The stereographic projection maps points in $\mathbb{C}$ far from the origin to points on the sphere $\hat{\mathbb{C}}$ near the "north pole" ∞.

One can observe that our definition of $\lim_{z \to z_0} f(z) = \infty$ is equivalent to $\lim_{z \to z_0} \frac{1}{f(z)} = 0$. The differential geometric idea underlying this observation is that the values $w = f(z)$ of f are on the Riemann sphere $\hat{\mathbb{C}}$, and $\frac{1}{w}$ is a good local coordinate at the point $\infty \in \hat{\mathbb{C}}$ where the standard coordinate w cannot be used.

Note further that yet another equivalent phrasing of $\lim_{z \to z_0} f(z) = \infty$ is $\lim_{z \to z_0} |f(z)| = +\infty$.

(This equivalence is rather direct from the definition, when limit $+\infty$ is defined as usually in real-variable calculus.)

Definition (Types of isolated singularities)

Suppose that $z_0 \in \mathbb{C}$ is an isolated singular point of a function $f: U \to \mathbb{C}$. Then, depending on $\lim_{z \to z_0} f(z)$, we introduce the following

- if $\lim_{z \to z_0} f(z) \in \mathbb{C}$, then we say that f has a removable singularity at z_0;
- if $\lim_{z \to z_0} f(z) = \infty$, then we say that f has a pole at z_0;
- if $\lim_{z \to z_0} f(z)$ does not exist in the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ then we say that f has an essential singularity at z_0.

(Every isolated singular point is of one and only one of the above three types.)

Let us give three prototypical examples of functions with an isolated singular point at $z_0 = 0$.

Example R:

Consider $f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$ given by

$$f(z) = \frac{\sin(z)}{z}.$$

Observe that from $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ and $\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ we get the Taylor series

$$\sin(z) = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \frac{1}{7!} z^7 + \dots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} z^{2m+1}.$$

Consequently

$$f(z) = \frac{\sin(z)}{z} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} z^{2m}$$

and then $\lim_{z \to 0} f(z) = \frac{(-1)^0}{(2\cdot 0+1)!} = \frac{1}{1} = 1$.

The singularity of f at 0 is removable.

Example P:

Fix $m \in \mathbb{N} = \{1, 2, 3, \dots\}$. Consider $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z^m}.$$

It is easy to see that

$$\lim_{z \to 0} f(z) = \infty.$$

(For example

$$\lim_{z \to 0} \frac{1}{f(z)} = \lim_{z \to 0} \frac{1}{1/z^m} = \lim_{z \to 0} z^m = 0.$$

Or directly from the definition, for $M > 0$ let $\delta = \frac{1}{2 \sqrt[m]{M}}$ and observe that when $0 < |z| < \delta$ we have

$$|f(z)| = \left| \frac{1}{z^m} \right| = \frac{1}{|z|^m} > \frac{1}{\delta^m} = 2^m M > M.)$$

Therefore f has a pole at 0.

Example E

Consider $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = e^{1/z}$.

We claim that f has an essential singularity at 0.

To justify this claim, we must show that $\lim_{z \to 0} f(z)$ does not exist. A natural way is to argue by contradiction based on obtaining different limits along particular sequences tending to 0.

For example letting $z_n = \frac{-1}{n}$ for $n \in \mathbb{N}$ we have $z_n \to 0$ as $n \to \infty$ and

$$f(z_n) = e^{1/z_n} = e^{1/(-1/n)} = e^{-n} \xrightarrow[n\to\infty]{} 0.$$

On the other hand letting $w_n = \frac{i}{2\pi n}$ for $n \in \mathbb{N}$ we also have $w_n \xrightarrow[n\to\infty]{} 0$ but

$$f(w_n) = e^{1/w_n} = e^{1/(i/2\pi n)} = e^{-2\pi n i} = 1 \xrightarrow[n\to\infty]{} 1.$$

Indeed the limit $\lim_{z\to 0} f(z)$ cannot exist.

Let us still give two more examples.

<u>Example:</u>

Consider the function

$$f : \mathbb{C} \setminus \{\tfrac{\pi}{2}\} \longrightarrow \mathbb{C}$$

$$f(z) = \frac{1 - \sin(z)}{(z - \pi/2)^2}.$$

We claim that the point $\frac{\pi}{2}$ is a removable singularity of f.

For that, note first that

$$\sin(z) = \cos\left(z - \tfrac{\pi}{2}\right)$$

$$= 1 - \frac{1}{2!}\left(z - \tfrac{\pi}{2}\right)^2 + \frac{1}{4!}\left(z - \tfrac{\pi}{2}\right)^4 - \frac{1}{6!}\left(z - \tfrac{\pi}{2}\right)^6 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(z - \tfrac{\pi}{2}\right)^{2k}.$$

Therefore

$$1 - \sin(z) = -\frac{1}{2!}\left(z - \tfrac{\pi}{2}\right)^2 + \frac{1}{4!}\left(z - \tfrac{\pi}{2}\right)^4 - \frac{1}{6!}\left(z - \tfrac{\pi}{2}\right)^6 + \dots$$

$$= \left(z - \tfrac{\pi}{2}\right)^2 \cdot \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{(2(l+1))!}\left(z - \tfrac{\pi}{2}\right)^{2l}$$

$$= \left(z - \tfrac{\pi}{2}\right)^2 \cdot g(z)$$

where g is analytic in a neighborhood of $z = \frac{\pi}{2}$ (it is given by a convergent power series) and $g(\frac{\pi}{2}) = -\frac{1}{2}$. In particular g is continuous at $z = \frac{\pi}{2}$ and we can use this to calculate

$$\lim_{z \to \pi/2} f(z) = \lim_{z \to \pi/2} \frac{1 - \sin(z)}{(z - \pi/2)^2}$$

$$= \lim_{z \to \pi/2} \frac{\cancel{(z - \pi/2)^2} \cdot g(z)}{\cancel{(z - \pi/2)^2}}$$

$$= \lim_{z \to \pi/2} g(z) = g(\tfrac{\pi}{2}) = -\tfrac{1}{2} \in \mathbb{C}.$$

The existence of this limit ($\in \mathbb{C}$) shows that the singularity of f at $\frac{\pi}{2}$ is removable.

Example

Consider the function

$$f : \mathbb{C} \setminus \{2\pi i m \mid m \in \mathbb{Z}\} \longrightarrow \mathbb{C}$$

defined by

$$f(z) = \frac{z}{e^z - 1}.$$

(This function is related to the so called Bernoulli numbers.)

We claim that the point $-2\pi i$ (and in fact any point of the form $2\pi i m$ with $m \in \mathbb{Z} \setminus \{0\}$) is a pole of f.

To see that, observe

$$\begin{aligned} e^z - 1 &= e^{z+2\pi i} - 1 \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(z+2\pi i)^n - 1 \\ &= \sum_{n=1}^{\infty} \frac{1}{n!}(z+2\pi i)^n \\ &= (z+2\pi i) \cdot \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(z+2\pi i)^k \end{aligned}$$

This expresses $e^z - 1$ in the form

$$e^z - 1 = (z + 2\pi i) \cdot g(z)$$

where g is analytic in a neighborhood of $-2\pi i$ (convergent power series) and $g(-2\pi i) = \frac{1}{2!} = \frac{1}{2}$.

We note that also $h(z) = \frac{g(z)}{z}$ defines an analytic function in a neighborhood of $-2\pi i$ and

$$h(-2\pi i) = \frac{1/2}{-2\pi i} = \frac{i}{4\pi}.$$

In particular h is continuous and so we may find a $\delta > 0$ such that $0 < |h(z)| < 1$ for $|z - (-2\pi i)| < \delta$. (Note: $0 < |h(-2\pi i)| < 1$.)

Now for z such that $0 < |z - (-2\pi i)| < \delta$ we have

$$|f(z)| = \left| \frac{z}{e^z - 1} \right| = \left| \frac{z}{(z + 2\pi i)\, g(z)} \right|$$

$$= \underbrace{\frac{1}{|z + 2\pi i|}}_{\to +\infty} \cdot \underbrace{\frac{1}{|h(z)|}}_{> 1} \xrightarrow[z \to -2\pi i]{} +\infty .$$

So $\lim_{z \to -2\pi i} f(z) = \infty$ and $-2\pi i$ is a pole.

LAURENT SERIES

There is a generalization of power series in complex analysis which is particularly important for the analysis of isolated singularities — the generalization allows both positive and negative powers, and the series are then called Laurent series.

To make the notion precise, we should first give a meaning to "doubly-infinite series" of the form

$$\sum_{n=-\infty}^{+\infty} z_n$$ where the terms are

$$\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots \in \mathbb{C}.$$

(So far we treated $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=m}^{\infty} z_n$ for starting at index $m \in \mathbb{Z}$.)

Def

Let $(z_n)_{n\in\mathbb{Z}}$ be complex numbers, $z_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. We say that the doubly infinite series $\sum_{n=-\infty}^{+\infty} z_n$ converges to $s \in \mathbb{C}$ if for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\left| \sum_{n=-m_-}^{m_+} z_n - s \right| < \varepsilon$$

whenever $m_+ \geq N$ and $m_- \leq -N$.

In practice it is often convenient to split a doubly-infinite series into two infinite series — one with nonnegative indices and the other with negative ones. The following lemma justifies this practice.

Lemma

The doubly-infinite series

$$\sum_{n=-\infty}^{+\infty} z_n$$

of complex numbers $z_n \in \mathbb{C}$, $n \in \mathbb{Z}$, converges if and only if both infinite series

$$\sum_{n=0}^{\infty} z_n \quad \text{and} \quad \sum_{n=1}^{\infty} z_{-n}$$

converge. In that case we have

$$\sum_{n=-\infty}^{+\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n .$$

Proof

Assume first that both $\sum_{n=0}^{\infty} z_n = s_+$ and $\sum_{n=1}^{\infty} z_{-n} = s_-$ are convergent.

Let $\varepsilon > 0$. By convergence of the two series, we can choose $N_+, N_- \in \mathbb{N}$ such that for all $k_+ \geq N_+$ and $k_- \geq N_-$ we have

$$\left| \sum_{n=0}^{k_+} z_n - s_+ \right| < \frac{\varepsilon}{2} \quad \text{and}$$

$$\left| \sum_{n=1}^{k_-} z_{-n} - s_- \right| < \frac{\varepsilon}{2}.$$

Now with $N = \max\{N_+, N_-\}$ we have for all $m_+ \geq N$, $m_- \leq -N$

$$\left| \sum_{n=m_-}^{m_+} z_n - (s_- + s_+) \right|$$

$$= \left| \left(\sum_{n=1}^{-m_-} z_{-n} - s_- \right) + \left(\sum_{n=0}^{m_+} z_n - s_+ \right) \right|$$

$$\leq \left| \sum_{n=1}^{-m_-} z_{-n} - s_- \right| + \left| \sum_{n=0}^{m_+} z_n - s_+ \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\sum_{n=-\infty}^{+\infty} z_n$ converges to $S_- + S_+$.

Conversely, assume that $\sum_{n=-\infty}^{\infty} z_n$ converges to $s \in \mathbb{C}$. To show the convergence of $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$, we prove that the sequences of partial sums are Cauchy — completeness of $\mathbb{C}$ then gives convergence. The argument is similar for both, so focus on $\sum_{n=0} z_n$.

Let $\varepsilon > 0$ and choose N (by assumed convergence of $\sum_{n=-\infty}^{+\infty} z_n$) such that

$$\left| \sum_{n=m_-}^{m_+} z_n - s \right| < \frac{\varepsilon}{2} \quad \text{whenever} \quad \begin{cases} m_+ \geq N \\ m_- \leq -N \end{cases}.$$

Now for $k, l \geq N$ we have

$$\left| \sum_{n=0}^{k} z_n - \sum_{n=0}^{l} z_n \right| = \left| \sum_{n=-N}^{k} z_n - \sum_{n=-N}^{l} z_n \right|$$

$$= \left| \left(\sum_{n=-N}^{k} z_n - s \right) - \left(\sum_{n=-N}^{l} z_n - s \right) \right|$$

$$\leq \left| \sum_{n=-N}^{k} z_n - s \right| + \left| \sum_{n=-N}^{l} z_n - s \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows the seq. of partial sums is Cauchy.

$\square$

Laurent series are essentially "doubly infinite power series", i.e., function series of the form

$$z \longmapsto \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where $z_0 \in \mathbb{C}$ is the center of the expansion and $\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots \in \mathbb{C}$ are coefficients.

Note that by our results on power series, the nonnegative index part

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

converges for z s.t. $|z-z_0| < R_+$ where

$$R_+ = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} .$$

Likewise, the negative index part

$$\sum_{m=1}^{\infty} a_{-m} (z-z_0)^{-m}$$

converges for z s.t.

$$\left| (z-z_0)^{-1} \right| < \frac{1}{\limsup\limits_{m \to \infty} \sqrt[m]{|a_{-m}|}}$$

i.e. $$|z-z_0| > \limsup_{m \to \infty} \sqrt[m]{|a_{-m}|} =: R_- .$$

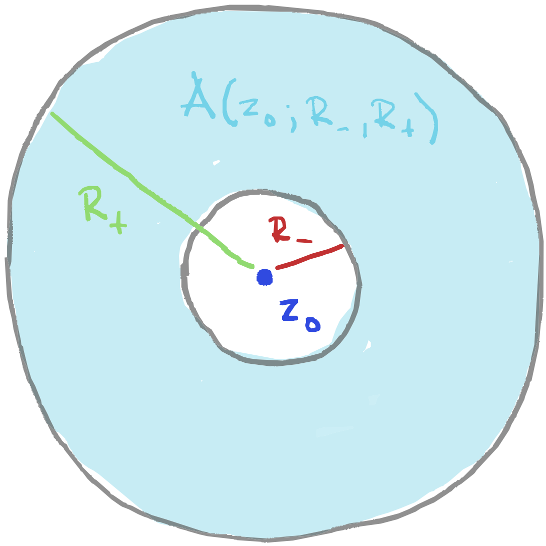
In other words, the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

converges in the annulus

$$A(z_0; R_-, R_+)$$
$$= \{ z \in \mathbb{C} \mid R_- < |z-z_0| < R_+ \} .$$

A particularly important case for analyzing isolated singularities is when $R_- = 0$ — then $A(z_0; 0, R_+)$ is a punctured disk $B^*(z_0; R_+)$.



Arguments similar to the power series case also show that a Laurent series converges uniformly on compact subsets of its annulus of convergence $A(z_0;R_-,R_+)$, and therefore defines an analytic function $A(z_0;R_-,R_+) \longrightarrow \mathbb{C}$, which can be differentiated term by term etc.

Let us now show that analytic functions have Laurent series representations in any annulus contained in their domain of def.

Theorem (Laurent series for analytic functions)

Let $f: U \to \mathbb{C}$ be an analytic function on a subset $U \subset \mathbb{C}$ of the complex plane which contains an annulus

$$A(z_0; r_-, r_+) = \{ z \in \mathbb{C} \mid r_- < |z - z_0| < r_+ \} \subset U.$$

Then f can be represented in the annulus uniquely as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \cdot (z - z_0)^n$$

for $z \in A(z_0; r_-, r_+)$.

The coefficients a_n, for $n \in \mathbb{Z}$, are given by

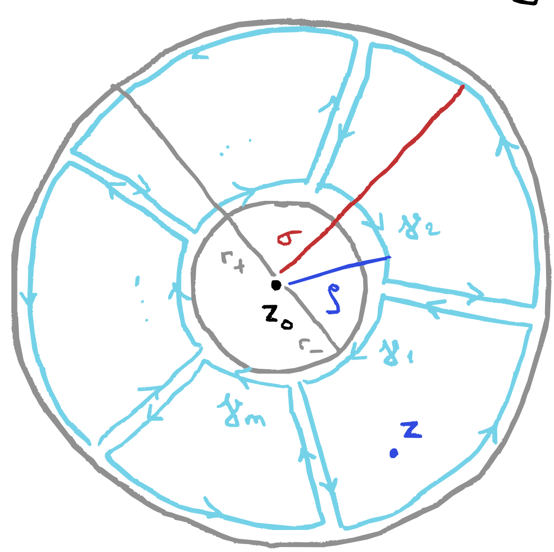
$$a_n = \frac{1}{2\pi i} \oint_{\partial B(z_0; r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

where $r \in (r_-, r_+)$ is arbitrary.

Sketch of proof:

Let $z \in A(z_0; r_-, r_+)$. Choose σ and ϱ so that $r_- < \sigma < |z - z_0| < \varrho < r_+$.

Now the closure of the thinner annulus $A(z_0; \sigma, \varrho)$, in particular its boundary circles $\partial B(z_0; \sigma)$ and $\partial B(z_0; \varrho)$, is contained in the original annulus.



Now divide $A(z_0; \sigma, \varrho)$ into finitely many star-shaped sectors as illustrated, and let $\gamma_1, \gamma_2, \dots, \gamma_m$ be the contours parametrizing their boundaries counterclockwise. We may assume that z is in the sector surrounded by γ_1.

The function $\xi \mapsto \dfrac{f(\xi)}{\xi - z}$ is analytic in the other sectors (even slightly extended) so Cauchy's integral theorem gives

$$\oint_{\gamma_j} \frac{f(\xi)}{\xi - z}\, d\xi = 0 \qquad \text{for } j = 2, 3, \dots, m$$

In the sector containing z, we instead use Cauchy's integral formula to get

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\xi)}{\xi - z}\, d\xi .$$

Summing the integrals around all sectors and noting cancellations on line segments between sectors traversed in opposite directions, we therefore get

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} \frac{f(\xi)}{\xi - z} \, d\xi$$

$$= \frac{1}{2\pi i} \oint_{\partial B(z_0; \sigma)} \frac{f(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \oint_{\partial B(z_0; \rho)} \frac{f(\xi)}{\xi - z} \, d\xi .$$

For $\xi \in \partial B(z_0; \rho)$ we have

$$|\xi - z_0| = \rho > |z - z_0|$$

so the following (geometric) series converges:

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 - (z - z_0)} = \frac{1}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}}$$

$$= \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} .$$

For $\xi \in \partial B(z_0; \sigma)$ by contrast

$|\xi - z_0| = \sigma < |z - z_0|$, so then

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 - (z - z_0)} = \frac{-1}{z - z_0} \cdot \frac{1}{1 - \frac{\xi - z_0}{z - z_0}}$$

$$= \frac{-1}{z - z_0} \sum_{m=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0} \right)^m = - \sum_{m=0}^{\infty} \frac{(\xi - z_0)^m}{(z - z_0)^{m+1}} .$$

Therefore (since uniform convergence of the series allows interchanging the order of summation and integration)

$$f(z) = \frac{1}{2\pi i} \oint_{\partial B(z_0; \rho)} f(\xi) \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\xi - z_0)^{n+1}} \, d\xi$$

$$- \frac{1}{2\pi i} \oint_{\partial B(z_0; \sigma)} f(\xi) \left(- \sum_{m=0}^{\infty} \frac{(\xi - z_0)^m}{(z - z_0)^{1+m}} \right) d\xi$$

$$= \sum_{n=0}^{\infty} (z - z_0)^n \cdot \left(\frac{1}{2\pi i} \oint_{\partial B(z_0; \rho)} \frac{f(\xi)}{(\xi - z_0)^{1+n}} \, d\xi \right)$$

$$+ \sum_{m=0}^{\infty} (z - z_0)^{-1-m} \left(\frac{1}{2\pi i} \oint_{\partial B(z_0; \sigma)} \frac{f(\xi)}{(\xi - z_0)^{1+(-1-m)}} \, d\xi \right)$$

$$= \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

with $a_n = \frac{1}{2\pi i} \oint_{\partial B(z_0; r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$. $\square$

the choice of $r \in (r_-, r_+)$ does not affect the value of this integral.

CLASSIFICATION OF ISOLATED SINGULARITIES

We saw that the behavior of an analytic function near an isolated singular point can be of three different types, depending on the limit of the function values in the extended complex plane. We now present characterizations of each of the three cases, giving deeper insights into the behavior in each case.

We first state the characterizations for all three cases, and then provide the proofs.

__Theorem__ (Characterization of removable singularities)

Let $f: U \to \mathbb{C}$ be an analytic function for which $z_0 \in \mathbb{C}$ is an isolated singular point. Then the following are equivalent

(R-1): The singularity of f at z_0 is removable, i.e., the limit $\lim_{z \to z_0} f(z)$ exists in $\mathbb{C}$

(R-2): The function f is bounded in some punctured disk $B(z_0; \delta) \setminus \{z_0\}$ with $\delta > 0$.

(R-3): The Laurent series development $f(z) = \sum_{n=-\infty}^{\infty} a_n \cdot (z-z_0)^n$ of f in a punctured disk around z_0 is such that $a_n = 0$ for all $n < 0$.

(R-4): There exists an analytic function $\tilde{f}: U \cup \{z_0\} \to \mathbb{C}$ such that $f(z) = \tilde{f}(z)$ for all $z \in U$.

Theorem (Characterization of poles)

Let $f: U \to \mathbb{C}$ be an analytic function for which $z_0 \in \mathbb{C}$ is an isolated singular point. Then the following are equivalent

(P-1): The singularity of f at z_0 is a pole, i.e., f has the limit $\lim_{z \to z_0} f(z) = \infty$ at z_0.

(P-2): The Laurent series development $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ of f in a punctured disk around z_0 has finitely many (but not zero) nonzero coeficients of the negative powers.

(P-3): There exists an $m \in \mathbb{N} = \{1, 2, \ldots\}$ such that the function $z \mapsto (z-z_0)^m \cdot f(z)$ has a removable singularity and a nonzero limit $\lim_{z \to z_0} (z-z_0)^m f(z) \in \mathbb{C} \setminus \{0\}$ at z_0.

Theorem (Characterization of essential singularities)

Let $f: U \to \mathbb{C}$ be an analytic function for which $z_0 \in \mathbb{C}$ is an isolated singular point. Then the following are equivalent

(E-1) The singularity of f at z_0 is essential, i.e., the limit $\lim_{z \to z_0} f(z)$ does not exist in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

(E-2): The Laurent series development $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ of f in a punctured disk around z_0 has infinitely many nonzero coefficients of negative powers, $\limsup_{m \to \infty} |a_{-m}| \neq 0$.

(E-3): For any $\delta > 0$, the image $f[B(z_0;\delta) \setminus \{z_0\}]$ of the punctured disk $B(z_0;\delta) \setminus \{z_0\}$ is dense in $\mathbb{C}$.

Proof of the removable singularity characterizations:

We show

$$\begin{tikzcd}
& (R\text{-}1) \arrow[dr, Rightarrow] & \\
(R\text{-}4) \arrow[ur, Rightarrow] & & (R\text{-}2) \arrow[dl, Rightarrow] \\
& (R\text{-}3) \arrow[ul, Rightarrow] &
\end{tikzcd}$$

$(R\text{-}1) \Longrightarrow (R\text{-}2)$:

Assuming the limit $\lim_{z \to z_0} f(z) = \alpha \in \mathbb{C}$ exists, we in particular get that for some $\delta > 0$, whenever $0 < |z - z_0| < \delta$, we have $|f(z) - \alpha| < 1$. By triangle inequality we get boundedness

$$|f(z)| = |f(z) - \alpha + \alpha| \leq |f(z) - \alpha| + |\alpha| < 1 + |\alpha|$$

in the punctured disk $B^*(z_0; \delta)$.

$(R\text{-}2) \Rightarrow (R\text{-}3)$:

Assume $|f(z)| \le M$ for all $z \in B^*(z_0, \delta)$. The coefficients a_n of the Laurent series $f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$ can be represented as integrals

$$a_n = \frac{1}{2\pi i} \oint_{\partial B(z_0; r)} \frac{f(\xi)}{(\xi - z_0)^{1+n}} \, d\xi$$

for any $0 < r < \delta$. Since

$$\left| \frac{f(\xi)}{(\xi - z_0)^{1+n}} \right| \le \frac{M}{r^{1+n}} \quad \text{on} \quad \partial B(z_0; r),$$

we get

$$|a_n| \le \frac{1}{2\pi} \cdot \underbrace{\ell(\partial B(z_0; r))}_{= 2\pi r} \cdot \frac{M}{r^{1+n}} = \frac{M}{r^n}.$$

For $n < 0$ we have $\frac{M}{r^n} \longrightarrow 0$ as $r \to 0^+$, so we must have $|a_n| = 0$. This shows that only nonnegative powers in the Laurent series may have nonzero coefficients.

$(R-3) \Rightarrow (R-4)$:

If $a_n = 0$ for all $n<0$ in the Laurent series representation

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n, \quad \text{then}$$

f coincides in the punctured disk with the function $z \mapsto \sum_{n=0}^{\infty} a_n (z-z_0)^n$ given by a convergent power series in the disk. The power series defines an analytic function in the disk around z_0, with value a_0 at z_0. So

$$\tilde{f}(z) = \begin{cases} f(z) & \text{for } z \in U \\ a_0 & \text{for } z = z_0 \end{cases}$$

defines an analytic function in $U \cup \{z_0\}$.

(R-4) $\Longrightarrow$ (R-1)

Suppose there exists an analytic function $\tilde{f} : U \cup \{z_0\} \to \mathbb{C}$ such that $\tilde{f}(z) = f(z)$ for all $z \in U$.
In particular $\tilde{f}$ is continuous at z_0, so the limit

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \tilde{f}(z) = \tilde{f}(z_0)$$

exists. This shows that the singularity of f at z_0 is removable.

$\square$

Proof of the pole characterization:

We prove

$$\begin{array}{ccc} & (P\text{-}1) & \\ \nearrow & & \searrow \\ (P\text{-}3) & \Longleftarrow & (P\text{-}2) \end{array}.$$

(P-1) $\Rightarrow$ (P-2):

If $\lim_{z \to z_0} f(z) = \infty$, then the function $z \mapsto \frac{1}{f(z)}$ is bounded and analytic in some punctured disk $B^*(z_0, \delta)$ and $\lim_{z \to z_0} \frac{1}{f(z)} = 0$. By the characterization of removable singularities, $z \mapsto \frac{1}{f(z)}$ can be expanded in a Laurent series with no negative powers,

$$\frac{1}{f(z)} = \sum_{n=0}^{\infty} b_n (z - z_0)^n \qquad \text{for } z \in B^*(z_0, \delta)$$

and in fact $b_0 = \lim_{z \to z_0} \frac{1}{f(z)} = 0$.

Let $m \in \mathbb{N}$ be the smallest index for which $b_m \neq 0$. Then

$$\frac{1}{f(z)} = \sum_{n=m}^{\infty} b_n (z-z_0)^n$$

$$= (z-z_0)^m \underbrace{\sum_{k=0}^{\infty} b_{m+k} \cdot (z-z_0)^k}_{=: g(z)}$$

$$= (z-z_0)^m \cdot g(z)$$

where $g(z_0) = b_m \neq 0$ and g is analytic in $B(z_0; \delta)$

Therefore also $z \mapsto \frac{1}{g(z)}$ is analytic in some neighborhood of z_0 and we can write

$$f(z) = \frac{1}{1/f(z)} = \frac{1}{(z-z_0)^m g(z)}$$

$$= (z-z_0)^{-m} \cdot \frac{1}{g(z)} = (z-z_0)^{-m} \underbrace{\sum_{n=0}^{\infty} c_n (z-z_0)^n}_{\text{Taylor series of } z \mapsto \frac{1}{g(z)}}$$

$$= \sum_{k=-m}^{\infty} c_{k+m} (z-z_0)^k .$$

There are finitely many negative powers and the coeff. $c_0 = \frac{1}{g(z_0)}$ of $(z-z_0)^{-m}$ is $\neq 0$.

(P-2) $\Longrightarrow$ (P-3)

Suppose $f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n$

with $m \in \mathbb{N}$ and $a_{-m} \neq 0$.

(the Laurent series contains finitely many negative powers with $-m$ being the "largest negative index" with nonzero coefficient)

Then

$$(z-z_0)^m f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^{m+n}$$

$$= \sum_{k=0}^{\infty} a_{-m+k} (z-z_0)^k$$

and

$$\lim_{z \to z_0} (z-z_0)^m f(z) = a_{-m} \neq 0 .$$

By the removable singularity characterization this also implies that the singularity of $z \longmapsto (z-z_0)^m f(z)$ is removable.

(P-3) $\Rightarrow$ (P-1):

Suppose there exists an $m \in \mathbb{N}$ such that $\lim_{z \to z_0} (z-z_0)^m f(z) = \alpha \neq 0$ exists. Then there exists a $\delta > 0$ such that when $0 < |z-z_0| < \delta$ we have

$$\left| (z-z_0)^m f(z) - \alpha \right| < \frac{|\alpha|}{2}.$$

Then we can estimate

$$|f(z)| = \left| (z-z_0)^{-m} \left(\alpha + (z-z_0)^m f(z) - \alpha \right) \right|$$

$$= \frac{1}{|z-z_0|^m} \left| \alpha + \left((z-z_0)^m f(z) - \alpha \right) \right|$$

$$\geq \frac{1}{|z-z_0|^m} \left(|\alpha| - \underbrace{|(z-z_0)^m f(z) - \alpha|}_{< |\alpha|/2} \right)$$

$$\geq \frac{1}{|z-z_0|^m} \frac{|\alpha|}{2} \xrightarrow[|z-z_0| \to 0]{} \infty .$$

This shows the singularity of f at z_0 is a pole.

$\square$

Proof of essential singularity characterization:

We prove

$$\begin{array}{ccc} & (E\text{-}1) & \\ \nearrow\!\!\!\nearrow & & \searrow\!\!\!\searrow \\ (E\text{-}3) & \Longleftarrow & (E\text{-}2) \end{array}$$

(E-1) $\Rightarrow$ (E-2):

If the limit $\lim_{z \to z_0} f(z)$ does not exist in $\hat{\mathbb{C}}$, then z_0 cannot be a removable singularity or a pole of f. There must then be infinitely many negative indices $n < 0$ such that $a_n \neq 0$. (Otherwise the earlier characterizations would show z_0 to be a removable singularity or a pole.)

$(E\text{-}2) \Rightarrow (E\text{-}3)$:

Suppose there are infinitely many negative indices n s.t. $a_n \neq 0$.

Fix $\delta > 0$. (s.t. $B^*(z_0;\delta) \subset U$)

By way of contradiction, suppose that some $c \in \mathbb{C}$ is not in the closure of $f[B^*(z_0;\delta)]$, i.e., there exists an $\varepsilon > 0$ such that $|f(z) - c| > \varepsilon$ for all $z \in B^*(z_0;\delta)$.

Then the function

$$z \longmapsto \frac{1}{f(z) - c}$$

is analytic and bounded (by $\frac{1}{\varepsilon} < \infty$) in $B^*(z_0;\delta)$. It follows from the earlier characterization that the singularity of $z \longmapsto \frac{1}{f(z)-c}$ at z_0 is removable and there exists an analytic function

$g : B(z_0;\delta) \to \mathbb{C}$ (defined also at z_0!)
such that $g(z) = \dfrac{1}{f(z)-c}$ for $z \in B^*(z_0;\delta)$.

If $g(z_0) \neq 0$ then also

$z \mapsto \dfrac{1}{g(z)}$ is well-defined and analytic

in some neighborhood of z_0. But
then $f(z) = c + \dfrac{1}{g(z)}$ also has a
removable singularity at z_0
which contradicts having $a_{-n} \neq 0$
for infinitely many $n < 0$.

If, on the other hand, $g(z_0) = 0$,
then $\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to z_0} \left(c + \dfrac{1}{g(z)}\right) = \infty$
which implies z_0 is a pole of f,
also contradicting having $a_{-n} \neq 0$
for infinitely many $n < 0$.

In either case we obtained a
contradiction. So every $c \in \mathbb{C}$
must in fact be in the closure
of $f[B^*(z_0;\delta)]$ for any $\delta > 0$.

$\underline{(E\text{-}3) \Longrightarrow (E\text{-}1)}$:

If for any $\delta > 0$ there are points in $B^*(z_0; \delta)$ arbitrarily close to any $c \in \mathbb{C}$ (density) then clearly the limit $\lim_{z \to z_0} f(z)$ cannot exist.

$\square$