Problem set 1A

Department of Mathematics and Systems Analysis MS-C1300 — Complex Analysis, 2024-2025/II

K Kytölä & A Vavilov

Exercise sessions: 22.-23.10. Hand-in due: Wed 23.10.2024 at 23:59

Topic: calculation with and geometric interpretation of complex numbers

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ♠): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 1A is Wed 23.10.2024 at 23:59.

Exercise (in class) 1.

(a) Let $z \in \mathbb{C} \setminus \{1\}$ and let $n \in \mathbb{N}$. Prove that

$$\sum_{j=0}^{n} z^{j} = \frac{1 - z^{n+1}}{1 - z} .$$

(b) Let $n \in \mathbb{N}$, $n \geq 2$. Find all solutions $z \in \mathbb{C}$ to the equation

$$z^{n-1} + z^{n-2} + \dots + z + 1 = 0.$$

Exercise (in class) 2.

Visualize and describe in geometric terms the following subsets of the complex plane:

$$A = \left\{ z \in \mathbb{C} \mid |z - 3i| < 2 \right\};$$

$$B \ = \ \Big\{z \in \mathbb{C} \ \Big| \ |z+1| \ = \ |z+\mathrm{i}| \Big\} \ ;$$

$$C \ = \ \Big\{z \in \mathbb{C} \ \Big| \ |z| \ = \ 2 \, |z-1| \Big\} \ ;$$

$$D = \left\{ z \in \mathbb{C} \mid z^2 = \overline{z}^2 \right\} .$$

Calculate:

- (a) all complex square roots of $-1 + i \sqrt{3}$; (b) all complex cube roots of -8.

Express the answers in Cartesian coordinates (i.e., as z = x + iy with $x, y \in \mathbb{R}$) and in polar coordinates (i.e., as $z = r e^{i\theta} = r(\cos(\theta) + i\sin(\theta))$ with $r \ge 0$ and $\theta \in \mathbb{R}$).

Show geometrically (recommended) or by a calculation (also possible) that if $z \in \mathbb{C}$ is such that |z| = 1 and $z \neq 1$, then

$$\Im \mathfrak{m} \left(\frac{z}{(z+1)^2} \right) = 0.$$

Which other points $z \in \mathbb{C}$ satisfy the above equation?