Aalto University **Problem set 5A**
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II K Kytölä & A Vavilov

Exercise sessions: 19.-20.11. **Hand-in due: Wed 20.11.2024 at 23:59**

Topic: series of analytic functions, power series, uniform convergence

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ✍): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 5A is Wed 20.11.2024 at 23:59.

Exercise (in class) 1.
Determine the disks of convergence of the following power series in a complex variable z:

$$\text{(i)} \qquad \sum_{n=1}^{\infty} \sqrt[n]{n}\,(z-1)^n$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} n!\,z^n$$

$$\text{(iii)} \qquad \sum_{n=1}^{\infty} n^4\,(z+3\mathbb{i})^{2^n} .$$

Exercise (in class) 2.
Determine the radius of convergence of the following power series in a complex variable z, and find simplified formulas for the analytic functions determined by the series:

$$\text{(a):} \quad \sum_{n=1}^{\infty} z^n \qquad \text{(b):} \quad \sum_{n=1}^{\infty} n\,z^n \qquad \text{(c):} \quad \sum_{n=1}^{\infty} n^2\,z^n \qquad \text{(d):} \quad \sum_{n=1}^{\infty} n^3\,z^n .$$

✍ Homework exercise 3.

Determine the disks of convergence of the following power series in a complex variable z:

$$\text{(i)} \qquad \sum_{n=1}^{\infty} \frac{\mathbb{i}^n}{n} (z-1)^n$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} \frac{(2n)!}{n!\,(n+1)!} z^n$$

$$\text{(iii)} \qquad \sum_{n=1}^{\infty} n!\, z^{n!} .$$

✍ Homework exercise 4.

Let $A \subset \mathbb{C}$ be a subset of the complex plane, and let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequences[1] of complex-valued functions on A which converge uniformly to limit functions $f \colon A \to \mathbb{C}$ and $g \colon A \to \mathbb{C}$, respectively.

(i) Show that the sequence $(f_n + g_n)_{n\in\mathbb{N}}$ of the pointwise sum functions $f_n + g_n \colon A \to \mathbb{C}$ converges uniformly on A to the limit function which is the pointwise sum $f + g$ of the limits (i.e., the function $z \mapsto f(z) + g(z)$).

(ii) Assuming furthermore that both sequences of functions are uniformly bounded, show that the sequence $(f_n\, g_n)_{n\in\mathbb{N}}$ of their pointwise product functions $f_n\, g_n \colon A \to \mathbb{C}$ converges uniformly on A to the limit function which is the pointwise product fg of the limits (i.e., the function $z \mapsto f(z)\, g(z)$).

Remark: Here, e.g., a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f \colon A \to \mathbb{C}$ is said to be uniformly bounded if there exists a constant $c < \infty$ such that $|f_n(z)| \leq c$ for all $n \in \mathbb{N}$ and all $z \in A$.

[1] Yes — sequences, not series in exercise 4!