Aalto University
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II

Problem set 1B

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Exercise sessions: 24.-25.10. **Hand-in due: Sun 27.10.2024 at 23:59**

Topic: calculation with and geometric interpretation of complex numbers

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ✍): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 1B is Sun 27.10.2024 at 23:59.

Exercise (in class) 1.
Determine whether the following limits exist:

(a)
$$\lim_{n\to\infty} \left(\mathbb{i}^{n!} + 2^{-n} \right) \qquad \text{(limit along } n \in \mathbb{N}\text{)} \; ;$$

(b)
$$\lim_{z\to 0} \left(\frac{1}{|z|^2} \sin\left(\frac{z^2 - \overline{z}^2}{4\mathbb{i}} \right) \right) \qquad \text{(limit along } z \in \mathbb{C} \setminus \{0\}\text{)} \; .$$

<u>Hint</u>: *It may be helpful to start by simplifying* $\frac{z^2 - \overline{z}^2}{4\mathbb{i}}$ *when* $z = x + \mathbb{i}y$. *Observe in particular that this expression takes real values, so only the familiar real* sin *function appears above.*

Exercise (in class) 2.
Let $U \subset \mathbb{C}$ be a nonempty open subset of the complex plane. Consider possible functions $f : U \to \mathbb{C}$ which satisfy

$$(\star) : \qquad e^{f(z)} = 1 \qquad \text{for all } z \in U.$$

(a) Show that there exist nonconstant functions $f : U \to \mathbb{C}$ satisfying the condition $(\star)$.
(b) Show that if U is not connected, then there exist nonconstant continuous functions $f : U \to \mathbb{C}$ satisfying the condition $(\star)$.
(c) Show that if U is connected, then any continuous function $f : U \to \mathbb{C}$ satisfying the condition $(\star)$ is a constant function.

✍ **Homework exercise 3.**
For $n \in \mathbb{N}$, show that

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) \;=\; \frac{1}{2} + \frac{\sin\big((2n+1)\theta/2\big)}{2\sin(\theta/2)}\,.$$

Hint: *Recall Exercise 1A.1 and the formulas* $\sin(\theta) = \frac{1}{2\,\mathrm{i}}\big(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}\big)$ *and* $\cos(\theta) = \frac{1}{2}\big(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}\big)$ *for trigonometric functions in terms of complex exponentials.*

✍ **Homework exercise 4.**
Find all points of discontinuity of the following functions:

(a) $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{R}$ given by

$$f(z) \;=\; \mathrm{Arg}(z^2)$$

where Arg is the principal branch (with values in $(-\pi,\pi]$) of the argument;

(b) $g\colon \mathbb{C} \to \mathbb{C}$ given by

$$g(z) \;=\; \sqrt{1-z^2}\;;$$

where $\sqrt{\cdot}$ denotes the principal branch (with arguments in $(-\pi/2,\pi/2]$) of the complex square root.