Aalto University
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2025-2026/II

Problem set 4B

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Exercise sessions: Thu-Fri, Nov 13-14, 2025

Topic: Cauchy's integral formula and its consequences

The first two exercises (marked with symbol ♮) are to be solved before the exercise sessions so that you are ready to present your solution on the blackboard. The last two exercises are to be discussed and solved in exercise sessions.

♮ Blackboard exercise 1.

Use Cauchy's integral formulas to evaluate the following contour integrals when the circles are positively (counterclockwise) oriented:

$$\text{(a)} \qquad \oint_{|z|=1} \frac{\cos(z)}{z} \, \mathrm{d}z$$

$$\text{(b)} \qquad \int_{|z|=2} \frac{e^{z+1}}{(z+1)^2} \, \mathrm{d}z \, .$$

♮ Blackboard exercise 2.

Let $f \colon \mathbb{C} \to \mathbb{C}$ be an analytic function defined in the entire complex plane, with the property that for some nonnegative constants $c, d \geq 0$ we have

$$|f(z)| \; \leq \; c\sqrt{|z|} + d \qquad \text{for all } z \in \mathbb{C}.$$

Prove that f must be a constant function.

Hint: *Try to get bounds on the derivative of f.*

Exercise (in class) 3.
Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a non-constant analytic function defined in the entire complex plane. Prove that the range $f[\mathbb{C}] \subset \mathbb{C}$ must be dense in the following sense: for every $w_0 \in \mathbb{C}$ and any $\varepsilon > 0$ there exists a $z \in \mathbb{C}$ such that $\left|f(z) - w_0\right| < \varepsilon$.

Hint: *Do a proof by contradiction. If for some w_0 and ε no such points z exist, then what can be said about the function $z \mapsto \frac{1}{f(z)-w_0}$?*

Exercise (in class) 4.
Let $B = \mathcal{B}(r; 0)$ be the disk of radius $r > 0$ centered at the origin. Let $u\colon B \to \mathbb{R}$ be a harmonic function. Prove that for $0 < \rho < r$, we have

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{\mathrm{i}t})\, \mathrm{d}t.$$

Hint: *We consider it known that a harmonic function in a disk has a harmonic conjugate.*