Aalto University   **Problem set 3A**
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II   K Kytölä & A Vavilov

Exercise sessions: 5.-6.11.   **Hand-in due: Wed 6.11.2024 at 23:59**

Topic: complex contour integration, paths, primitives

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ✐): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 3A is Wed 6.11.2024 at 23:59.

Exercise (in class) 1.
Let $\gamma\colon [0,\pi] \to \mathbb{C}$ be given by $\gamma(t) = t\,e^{\mathrm{i}t}$. Calculate the following integrals:

$$\text{(a)} \qquad \int_\gamma \overline{z}\,\mathrm{d}z$$

$$\text{(b)} \qquad \int_\gamma |z|\,|\mathrm{d}z|$$

$$\text{(c)} \qquad \int_\gamma z\,\mathrm{d}z.$$

Exercise (in class) 2.
Find functions $F\colon \mathbb{C} \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \mathbb{C}$ when

$$\text{(a)} \qquad\qquad f(z) = z\,e^{z^2}$$
$$\text{(b)} \qquad\qquad f(z) = z\,e^{z}$$
$$\text{(c)} \qquad\qquad f(z) = z^2\,\sin(4z).$$

✍ **Homework exercise 3.**

Consider the path $\gamma\colon [-1, 1] \to \mathbb{C}$ given by

$$\gamma(t) = \frac{1 - t^2}{1 + t^2} + \mathrm{i}\,\frac{2t}{1 + t^2}.$$

Show that the formula

$$\phi(s) = \tan(s/2)$$

defines a smooth increasing function $\phi\colon [-\pi/2,\,\pi/2] \to [-1, 1]$. Consider the reparametrization

$$\eta = \gamma \circ \phi\colon [-\pi/2,\,\pi/2] \to \mathbb{C},$$

of γ by ϕ, and simplify the formula for $\eta(s) = \gamma\big(\phi(s)\big)$.

Hint: *The simplified answer should make it apparent that η (and thus also γ) is a path that parametrizes a certain semicircle.*

✍ **Homework exercise 4.**

Prove the following estimates for the contour integrals:

(a)
$$\left| \int_\gamma \frac{\mathrm{d}z}{z^2 - \mathrm{i}} \right| \leq \frac{3\pi}{4}$$

when γ parametrizes the circle $|z| = 3$ once in the positive direction;

(b)
$$\left| \int_\gamma \frac{e^{3z}}{1 + e^z}\,\mathrm{d}z \right| \leq \frac{2\pi e^{3R}}{e^R - 1}$$

when γ parametrizes a line segment from $z = R > 0$ to $z = R + 2\pi\mathrm{i}$;

(a)
$$\left| \int_\gamma e^{\sin(z)}\,\mathrm{d}z \right| \leq 1$$

when γ parametrizes a line segment from $z = 0$ to $z = \mathrm{i}$.

Hint: *"Triangle inequality for integrals".*