MORE ON POWER SERIES

We make a few more general observations about functions defined by power series, before going on to establish that analytic functions must be locally represented by power series.

An important piece of the theory of power series is a general formula for the radius of convergence. This requires the notion of "limit superior" ("limsup") of a sequence of nonnegative real numbers — so we make a small aside.

Aside (limsup)

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

For a given $y\in\mathbb{R}$, we say that

$x_n < y$ eventually if

$\exists N\in\mathbb{N} : \forall n \geq N : x_n < y$

and we say that

$x_n < y$ frequently if

$\forall N\in\mathbb{N} : \exists n \geq N : x_n < y$

(In this case "frequently" is equivalent to "for infinitely many indices", but the former generalizes better to other "filters" so we choose to use that.)

It is natural to generalize "eventually" and "frequently" to any predicates on the members x_n of the sequence — in particular we trust the reader to correctly interpet "$x_n > y$ eventually" and "$x_n > y$ frequently".

Note that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$ is equivalent to the pair of assertions

$$\begin{cases} \forall \varepsilon > 0 : & x_n < x + \varepsilon \text{ eventually} \\ \forall \varepsilon > 0 : & x_n > x - \varepsilon \text{ eventually.} \end{cases}$$

To define limit superior, we relax the latter: we say $\limsup_{n\to\infty} x_n = x \in \mathbb{R}$ if

$$\begin{cases} \forall \varepsilon > 0 : & x_n < x + \varepsilon \text{ eventually} \\ \forall \varepsilon > 0 : & x_n > x - \varepsilon \text{ frequently.} \end{cases}$$

(E.g. for $(x_1, x_2, x_3, x_4, \dots) = (-1, +1, -1, +1, \dots)$ we have $\limsup_{n\to\infty} x_n = +1$ although the limit $\lim_{n\to\infty} x_n$ does not exist.)

We also say $\limsup_{n\to\infty} x_n = +\infty$ if for any y, $x_n > y$ frequently, and $\limsup_{n\to\infty} x_n = -\infty$ if for any y, $x_n < y$ frequently.

It is straightforward to convince oneself that $\limsup_{n\to\infty} x_n$ is always uniquely defined in $[-\infty, \infty] = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

We can then give a general formula for the radius of convergence of a power series in terms of its coefficients.

<u>Theorem</u> (Hadamard's formula: radius of conv.)

The radius of convergence R of a power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

is given by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

where we interpret $\frac{1}{+\infty} = 0$ and $\frac{1}{0} = +\infty$.

Proof

Let us denote

$$s = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

and consider the case $s \in (0, +\infty)$.
The degenerate cases $s = 0$ and $s = +\infty$ are left to the reader.

We will prove separately

$$R \geq \frac{1}{s} \qquad \text{and} \qquad R \leq \frac{1}{s},$$

which together yield the asserted equality $R = \frac{1}{s}$.

proof of $R \geq \frac{1}{s}$:

In order to prove $R \geq \frac{1}{s}$, we must prove that the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges when $|z - z_0| < \frac{1}{s}$.

Fix $z \in \mathbb{C}$ such that $|z - z_0| < \frac{1}{s}$.
Pick $\varepsilon > 0$ small enough so that $|z - z_0| < \frac{1}{s+\varepsilon}$ ⊛ Since $s = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, there exists an $N \in \mathbb{N}$ such that for $n \geq N$ we have $\sqrt[n]{|a_n|} \leq s + \varepsilon$. ⊛

For the tail

$$\sum_{n=N}^{\infty} a_n (z - z_0)^n$$

of the power series at z, we can then estimate the terms by

$$\begin{aligned} |a_n (z-z_0)^n| &= |a_n| \cdot |z - z_0|^n \\ &= \left(\sqrt[n]{|a_n|}\right)^n |z-z_0|^n \\ &\leq (s+\varepsilon)^n |z - z_0|^n \\ &= (\underbrace{(s+\varepsilon)|z-z_0|}_{⊛ < 1})^n . \end{aligned}$$

This gives a convergent geometric series as a majorant, so by Weierstrass M-test the power series indeed converges at z.

proof of $R \leq \frac{1}{s}$:

To prove that $R \leq \frac{1}{s}$, we must show that $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ does not converge when $|z-z_0| > \frac{1}{s}$.

Fix $z \in \mathbb{C}$ such that $|z-z_0| > \frac{1}{s}$.

Choose $\varepsilon > 0$ small enough so that $|z-z_0| > \frac{1}{s-\varepsilon}$ (but still $s-\varepsilon > 0$).

Since $s = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, there are infinitely many indices n such that $\sqrt[n]{|a_n|} > s-\varepsilon$. For such indices n, the term $a_n \cdot (z-z_0)^n$ of the power series satisfies

$$
\begin{aligned}
|a_n (z-z_0)^n| &= |a_n| \cdot |z-z_0|^n \\
&= \left(\sqrt[n]{|a_n|}\right)^n \cdot |z-z_0|^n \\
&> (s-\varepsilon)^n \cdot |z-z_0|^n \geq 1 .
\end{aligned}
$$

It follows that the terms do not tend to zero, so the series does not converge. $\square$

Example

$$\sum_{m=0}^{\infty} z^{2^m} = z + z^2 + z^4 + z^8 + z^{16} + \dots$$

The coefficients of this series are

$$a_n = \begin{cases} 1 & \text{if } n = 2^m \text{ for some } m \\ 0 & \text{otherwise.} \end{cases}$$

Taking n^{th} roots does not affect these, since $\sqrt[n]{1} = 1$ and $\sqrt[n]{0} = 0$:

$$\sqrt[n]{|a_n|} = \begin{cases} 1 & \text{if } n = 2^m \text{ for some } m \\ 0 & \text{otherwise.} \end{cases}$$

Now clearly

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$$

so the radius of convergence is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{1} = 1.$$

The series in this example is rather interesting, in that it defines an analytic function

$$f(z) = \sum_{m=0}^{\infty} z^{2^m}$$

in the unit disk $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ which cannot be extended to any larger open connected set.

What prevents the extension is a dense set of singularities on the boundary $\partial \mathbb{D}$ of the unit disk.

For example at $z = 1$, the series diverges: $f(1) = 1 + 1 + 1 + \dots$.

But also at $z = -1$ we find

$$f(-1) = (-1) + (-1)^2 + (-1)^4 + \dots = -1 + 1 + 1 + 1 + \dots$$

and at $z = \pm i$

$$f(\pm i) = \pm i + (\pm i)^2 + (\pm i)^4 + (\pm i)^8 + \dots$$

$$= \pm i - 1 + 1 + 1 + 1 + 1 + \dots ,$$

The "same" happens at all 2^m th roots of unity.

From the proof of Hadamard's formula, we in fact get:

Lemma

If a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has a positive radius of convergence $R>0$, then for any $r<R$, it converges uniformly in $\overline{B}(z_0;r)$.

In particular, a power series converges uniformly on compacts in its disk of convergence $B(z_0;R)$.

Corollary

A power series determines an analytic function

$$z \mapsto f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

in its disk of convergence $B(z_0;R)$.

It's derivative is given by

$$f'(z) = \sum_{n=1}^{\infty} n \cdot a_n \cdot (z-z_0)^{n-1}$$

$$= \sum_{m=0}^{\infty} (m+1)\, a_{m+1} (z-z_0)^m .$$

Remark:

The radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

and its termwise differentiated series

$$\sum_{n=1}^{\infty} n \cdot a_n (z-z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (z-z_0)^n$$

are equal. Indeed, the former is

$$R = \left(\limsup_n \sqrt[n]{|a_n|}\right)^{-1}$$

and the latter is

$$R' = \left(\limsup_n \sqrt[n]{(n+1)\cdot |a_{n+1}|}\right)^{-1}.$$

But

$$\sqrt[n]{(n+1)|a_{n+1}|} = (n+1)^{1/n} \cdot |a_{n+1}|^{1/n}$$

and since

$$|a_{n+1}|^{1/n} = \Big(\underbrace{|a_{n+1}|^{\frac{1}{n+1}}}_{\text{appears in } R := \cdots}\Big)^{\frac{n+1}{n}} \quad \approx 1$$

and

$$(n+1)^{1/n} = \exp\Big(\underbrace{\tfrac{1}{n}\cdot \log(n+1)}_{\to 0}\Big) \longrightarrow e^0 = 1$$

so it is not difficult to show $R' = R$.

A function f represented by a convergent power series

$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ in a disk around z_0

has, at the center z_0, the value

$f(z_0) = a_0$ and derivative $f'(z_0) = a_1$

and by inductive application of the previous lemma higher derivatives

$$f^{(n)}(z_0) = n! \cdot a_n .$$

In particular the coefficients $a_n = \frac{1}{n!} f^{(n)}(z_0)$ of the power series can be read off of the higher derivatives. Therefore:

Lemma

If two power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z-z_0)^n$ converge and represent the same function in some nonempty open disk $B_r(z_0)$, then their coefficients coincide, $a_n = b_n \; \forall n$.

Absolutely convergent series, and power series in particular, have a natural multiplication:

Lemma

If the two series $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ converge absolutely, then their product is the convergent series

$$\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} z_l w_{k-l}\right).$$

Proof Exercise. $\square$

Lemma

If two power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

developed around $z_0 \in \mathbb{C}$ both have radius of convergence $> r$, then in $B(z_0; r)$ their product is

$$\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)\left(\sum_{n=0}^{\infty} b_n (z-z_0)^n\right)$$
$$= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} a_l b_{k-l}\right)(z-z_0)^k.$$

POWER SERIES FOR ANALYTIC FUNCTIONS

We are now ready to show (using Cauchy's integral formula, of course) that analytic functions can be locally represented by power series.

<u>Theorem</u> (Taylor series for analytic functions)

Let $f : U \to \mathbb{C}$ be analytic on an open set $U \subset \mathbb{C}$. Then in any disk $B(z_0, \rho) \subset U$, f can be represented as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with coefficients $a_n = \frac{1}{n!} f^{(n)}(z_0)$.

Proof

Let $B(z_0;\rho) \subset U$. Fix also $0 < r_0 < r < \rho$. Since $\overline{B}(z_0;r) \subset U$, Cauchy's integral formula gives

$$f(z) = \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f(\xi)}{\xi - z}\, d\xi$$

for all $z \in B(z_0;r)$.

For $z \in \overline{B}(z_0;r_0) \subset B(z_0;r)$

(so that $|\xi - z_0| = r > r_0 \geq |z - z_0|$)

the geometric series in

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}}$$

$$= \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}$$

converges uniformly by the Weierstrass M-test (ratio $\leq \frac{r_0}{r} < 1$).

Therefore the order of integration and limit can be interchanged in

$$f(z) = \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} f(\xi) \cdot \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\xi - z_0)^{n+1}} \, d\xi$$

$$= \sum_{n=0}^{\infty} (z-z_0)^n \cdot \left(\frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \right)$$

$$= \sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n$$

where we denoted

$$a_n = \frac{1}{2\pi i} \oint_{\partial B(z_0;r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi .$$

This is the desired power series representation in $\overline{B}(z_0;r_0)$.

The coefficients must equal

$$a_n = \frac{1}{n!} f^{(n)}(z_0) \qquad \text{for } n \in \{0,1,\dots\}.$$

Since $r_0 < r < \rho$ were arbitrary, the power series converges and represents $f(z)$ for any $z \in B(z_0;\rho)$.

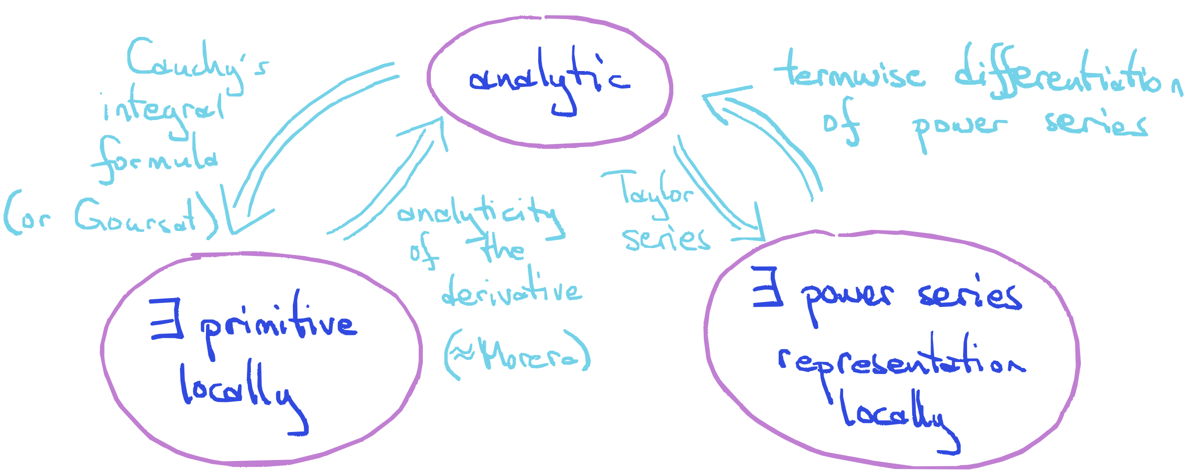
$\square$

It is worth noting that three a priori very different properties of functions considered in this course have now been shown equivalent:

We have defined what it means for a function $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ to

- be analytic
- have a primitive locally in any $B_r(z_0) \subset U$
- be represented by a power series locally in any $B_r(z_0) \subset U$.

But we have also shown their equivalence



In real variable calculus, the analogues of these three properties are plainly inequivalent.

Example

Consider the analytic function

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = e^z.$$

Since $B(0;\varrho) \subset \mathbb{C}$ for any $\varrho > 0$, we have a power series representation

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} f^{(n)}(0) \right) \cdot z^n$$

for any $z \in \mathbb{C}$. And since

$$f'(z) = e^z, \quad f''(z) = e^z, \quad \ldots$$

we have $f^{(n)}(z) = e^z$ and $f^{(n)}(0) = e^0 = 1$.

Therefore

$$e^z = f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

for all $z \in \mathbb{C}$.

(The convergence is uniform on compact subsets of $\mathbb{C}$, but not on all of $\mathbb{C}$.)

Another rigidity property of analytic functions is:

Theorem (Identity principle)

Suppose that $f : D \to \mathbb{C}$ is analytic on a connected open set $D \subset \mathbb{C}$. If there exists a $z_0 \in D$ such that derivatives of f of all orders vanish at z_0, i.e., $f^{(n)}(z_0) = 0 \ \forall n \in \mathbb{N}$, then f is a constant function.

Proof Let us consider the subset

$$D_0 = \{ z \in D \mid f^{(n)}(z) = 0 \text{ for all } n \in \mathbb{N} \}.$$

We will show that both D_0 and its complement $D \setminus D_0$ are open. Then by connectedness, they cannot both be nonempty, and since we assumed $z_0 \in D_0$, we get $D \setminus D_0 = \emptyset$, i.e. $D_0 = D$. In particular $f'(z) = 0$, so f is a constant.

To show that $D \setminus D_0$ is open is easy: if $z \in D \setminus D_0$, then $f^{(n)}(z) \neq 0$ for some $n \in \mathbb{N}$, and by continuity of $f^{(n)}$ the same holds in a small disk $B(z, \varepsilon)$ around z, which is then contained in $D \setminus D_0$.

To show that D_0 is open, suppose $w_0 \in D_0$. Pick $\rho > 0$ s.t. $B(w_0; \rho) \subset D$ and recall that in such a disk, the analytic function f is represented by the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w_0)}{n!} (z - w_0)^n$$

$$= f(w_0) + \sum_{n=1}^{\infty} \underbrace{\frac{f^{(n)}(w_0)}{n!}}_{=0 \text{ since } w_0 \in D_0} (z - w_0)^n$$

$$= f(w_0).$$

This shows f is constant in $B(w_0; \rho)$, and in particular $B(w_0; \rho) \subset D_0$, showing that D_0 is open and completing the proof. $\square$

The power series representations also lead to:

Theorem (The factor theorem)

> Let $f: D \to \mathbb{C}$ be a nonconstant analytic function on a connected open set $D \subset \mathbb{C}$, and let $z_0 \in D$. Then f can be uniquely written in the form
> $$f(z) = f(z_0) + (z-z_0)^m \cdot g(z)$$
> where $m \in \mathbb{N}$ and $g: D \to \mathbb{C}$ is analytic with $g(z_0) \neq 0$.

Remark:

If $f(z_0) = 0$ then this in particular shows $f(z) = (z-z_0)^m g(z)$ with $g(z_0) \neq 0$ (analogous to factorization of polynomials). The number m is then called the order of vanishing of f at z_0 (analoguous to the multiplicity of a root of a polynomial).

Proof

Without loss of generality assume $f(z_0)=0$ (consider $z \mapsto f(z)-f(z_0)$).

Let us first prove the existence of the representation $f(z)=(z-z_0)^m \cdot g(z)$.
Since f is nonconstant, its derivatives of all orders at z_0 cannot vanish.
Let $m \in \mathbb{N}$ be the smallest positive integer such that $f^{(m)}(z_0) \neq 0$.
In a small disk $B(z_0;\rho) \subset D$, f can then be represented as a power series

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

since
$f(z_0)=0$
$f'(z_0)=0$
$\vdots$
$f^{(m-1)}(z_0)=0$

$$= (z-z_0)^m \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

$$= (z-z_0)^m \underbrace{\sum_{k=0}^{\infty} \frac{f^{(m+k)}(z_0)}{(m+k)!}(z-z_0)^k}_{\text{analytic in } B(z_0;\rho),\ \neq 0 \text{ at } z_0}.$$

The function $g : D \to \mathbb{C}$ given by

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^m} & \text{for } z \neq z_0 \\[2ex] f^{(m)}(z_0)/m! & \text{for } z = z_0 \end{cases}$$

is analytic in D (it is clearly analytic in $D \setminus \{z_0\}$ and is given by the power series $\sum_{k=0}^{\infty} \frac{f^{(m+k)}(z_0)}{(m+k)!}(z-z_0)^k$ in $B(z_0;\rho)$). We obtain

$$f(z) = (z-z_0)^m g(z)$$

as desired.

For the uniqueness, suppose that $f(z) = (z-z_0)^\ell \cdot h(z)$ with $\ell \in \mathbb{N}$ and $h : D \to \mathbb{C}$ analytic and $h(z_0) \neq 0$.
("another representation")
We must prove $\ell = m$ and $h = g$.
Let us first prove $\ell \geq m$.

Indeed, if $l > m$ then

$$0 = \lim_{z \to z_0} (z-z_0)^{l-m} \cdot h(z)$$

$$= \lim_{z \to z_0} (z-z_0)^{-m} \cdot f(z)$$

$$= \lim_{z \to z_0} g(z) = g(z_0) \neq 0$$

which is a contradiction.

Similarly one proves $m \geq l$, so combining we get $l = m$.

Then

$$(z-z_0)^m g(z) = f(z) = (z-z_0)^m h(z)$$

for all $z \in D$, which implies

$$g(z) = h(z) \quad \text{for all } z \in D \setminus \{z_0\}.$$

By continuity of both g and h (take limit $z \to z_0$) we must also have $g(z_0) = h(z_0)$, so $g = h$.

This proves the uniqueness. $\square$

In order to formulate the next of our rigidity results of analytic functions, the following notion is needed:

<u>Def</u>: A set $A \subset \mathbb{C}$ has z_0 as its accumulation point if for every $\varepsilon > 0$ there exists a point $z \in A$ with $z \neq z_0$ and $|z - z_0| < \varepsilon$.

(i.e. $\forall \varepsilon > 0 \; \exists z \in (A \setminus \{z_0\}) \cap B_\varepsilon(z_0)$)

With this, we can state:

<u>Theorem</u> (Principle of analytic continuation)

Let $D \subset \mathbb{C}$ be a connected open set and let $f, g : D \to \mathbb{C}$ be two analytic functions on D.

If the set $\{z \in D \mid f(z) = g(z)\}$ has an accumulation point $z_0 \in D$, then we must have $f = g$.

Proof

Assume $z_0 \in D$ is an accumulation point of $\{z \in D \mid f(z) = g(z)\}$.
In particular, for every $n \in \mathbb{N}$ there must exist some $z_n \in D$ such that $z_n \neq z_0$, $|z_n - z_0| < \frac{1}{n}$, and $f(z_n) = g(z_n)$.

Consider the function $h = f - g$ on D. In order to prove that $f = g$, it suffices to prove that $h = 0$. Since f and g are analytic, so is h. So if $r > 0$ is such that $B_r(z_0) \subset D$, then in $B_r(z_0)$, h has a Taylor series representation:

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{when} \quad |z - z_0| < r.$$

We know that $h(z_n) = f(z_n) - g(z_n) = 0$ for each $n \in \mathbb{N}$, and we will use this to show that $a_n = 0$ for all n. This will imply in particular that $h^{(n)}(z_0) = 0$ for all n, so by the identity principle $h = 0$ will follow.

Suppose, by way of contradiction, that $a_m \neq 0$ for some m, and choose m minimal among such indices.

Then for $z \in B_r(z_0)$,

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$= (z - z_0)^m \left(a_m + (z - z_0) \sum_{k=0}^{\infty} a_{m+1+k} (z - z_0)^k \right)$$

$$= (z - z_0)^m \left(a_m + (z - z_0) E(z) \right)$$

where $E : B_r(z_0) \to \mathbb{C}$ is the analytic function defined by the (convergent) power series $E(z) = \sum_{k=0}^{\infty} a_{m+1+k} (z - z_0)^k$.

By continuity of $z \mapsto (z - z_0) E(z)$ and the observation that it vanishes at $z = z_0$, there exists some $\delta > 0$ such that $|(z - z_0) E(z)| < |a_m|$ for all z such that $|z - z_0| < \delta$.

In particular, whenever $0 < |z - z_0| < \delta$, we have

$$|h(z)| = \left|(z-z_0)^m \left(a_m + (z-z_0)E(z)\right)\right|$$

$$\geq \underbrace{|z-z_0|^m}_{>0} \underbrace{\left(|a_m| - |(z-z_0)E(z)|\right)}_{>0} > 0$$

so $h(z) \neq 0$.

But this contradicts $h(z_n) = 0$ for all $n \in \mathbb{N}$ since for n large enough, we have $|z - z_0| < \frac{1}{n} \leq \delta$ and $z_n \neq z_0$.

This shows that $a_m \neq 0$ was impossible, so $a_n = 0$ $\forall n$ and therefore $h \equiv 0$ in $B_r(z_0)$ and from the identity principle indeed $h \equiv 0$ in D. $\square$

As a corollary, we can show an important algebraic well-behavedness of the ring of analytic functions.

Corollary (No zero-divisors of analytic functions)

Let $D \subset \mathbb{C}$ be a connected open set and let $f, g : D \to \mathbb{C}$ be two analytic functions. Then if

$$f \cdot g = 0 \qquad (\text{in } D)$$

we must have either $f = 0$ or $g = 0$.

Proof:

If $D = \emptyset$ there is nothing to prove, so assume $D \neq \emptyset$ and choose $z_0 \in D$. For $n \in \mathbb{N}$, choose $z_n \in D$ s.t. $z_n \neq z_0$ and $|z_n - z_0| < \frac{1}{n}$. (This is possible since $B_r(z_0) \subset D$ for some $r > 0$.)

Since by assumption

$$0 = (f \cdot g)(z_n) = f(z_n) \cdot g(z_n)$$

and $\mathbb{C}$ does not have zero-divisors ($\mathbb{C}$ is a field), we have either $f(z_n) = 0$ or $g(z_n) = 0$.

Therefore we must have $f(z_n) = 0$ for infinitely many $n \in \mathbb{N}$ or $g(z_n) = 0$ for infinitely many $n \in \mathbb{N}$ — without loss of generality assume the former. Since $z_n \longrightarrow z_0$ as $n \to \infty$ and $z_n \neq 0$, this implies that $z_0 \in D$ is an accumulation point of the set $\{ z \in D \mid f(z) = 0 \}$. By the principle of analytic continuation, in this case $f \equiv 0$ on D.
(In the other case we similarly get $g \equiv 0$)

$\square$