3. COMPLEX INTEGRATION

Integration is the core of complex analysis — especially via one key result: Cauchy's integral formula.

Our first goal here is to set the stage for needed theory of integration, and in particular define

- integrals $\int_a^b f(t)\,dt$ of complex-valued functions on closed intervals $[a,b] \subset \mathbb{R}$
  (This is auxiliary for the following.)
- contour integrals $\int_\gamma f(z)\,dz$ and arc length integrals $\int_\gamma f(z)\,|dz|$ along curves γ in $A \subset \mathbb{C}$ for functions $f : A \to \mathbb{C}$
- primitives, i.e., "antiderivatives", i.e., "integral functions" for functions $f : U \to \mathbb{C}$ on open sets $U \subset \mathbb{C}$.

Riemann integrals of complex-valued functions

Recall from calculus that we can define Riemann integrals

$$\int_a^b f(t)\, dt$$

over closed intervals $[a,b] \subset \mathbb{R}$ on the real line of real-valued continuous functions $f : [a,b] \longrightarrow \mathbb{R}$.

(One could relax the continuity assumption, especially if Riemann integrals were replaced by Lebesgue integrals, but for our present purposes, continuous integrands will be entirely sufficient.)

For a continuous complex-valued function $f : [a,b] \to \mathbb{C}$, it is then natural to define the integral over $[a,b] \subset \mathbb{R}$ by splitting to real and imaginary parts.

[complex_valued_integral]

$$\int_a^b f(t)\,dt := \int_a^b \mathrm{Re}(f(t))\,dt + i\int_a^b \mathrm{Im}(f(t))\,dt.$$

<u>Rmk</u> Recall that $t \mapsto \mathrm{Re}(f(t))$ and $t \mapsto \mathrm{Im}(f(t))$ are continuous if (and only if) $f: [a,b] \to \mathbb{C}$ is. So the real integrals on the right hand side are defined.

This natural definition makes integration $\mathbb{C}$-linear:

[Linearity-of-complex-integral]

<u>Lemma</u> (Linearity of complex-valued integration)

(a): If $f_1, f_2 : [a,b] \to \mathbb{C}$ are continuous, then
$$\int_a^b (f_1(t) + f_2(t))\,dt = \int_a^b f_1(t)\,dt + \int_a^b f_2(t)\,dt.$$

(b): If $f: [a,b] \to \mathbb{C}$ is continuous and $\lambda \in \mathbb{C}$ is a scalar, then
$$\int_a^b \lambda \cdot f(t)\,dt = \lambda \cdot \int_a^b f(t)\,dt.$$

Proof: Part (a) is a direct consequence of the $\mathbb{R}$-linearity of integration and of the real and imaginary part operations.

For part (b), write $f(t) = u(t) + i\, v(t)$ (where $u(t) = \mathrm{Re}(f(t))$ and $v(t) = \mathrm{Im}(f(t))$) and $\lambda = \xi + i\eta$ ($\xi = \mathrm{Re}(\lambda)$, $\eta = \mathrm{Im}(\lambda)$).

Now for example

$$\mathrm{Re}(\lambda \cdot f(t)) = \mathrm{Re}\big((\xi + i\eta)(u(t) + i\, v(t))\big)$$
$$= \mathrm{Re}\big(\xi \cdot u(t) + i\,\xi \cdot v(t) + i\,\eta \cdot u(t) - \eta \cdot v(t)\big)$$
$$= \xi \cdot u(t) - \eta \cdot v(t),$$

which by definition of complex integrals gives

$$\mathrm{Re}\Big(\int_a^b \lambda \cdot f(t)\, dt\Big) = \int_a^b \big(\xi \cdot u(t) - \eta \cdot v(t)\big)\, dt$$
$$= \xi \cdot \int_a^b u(t)\, dt - \eta \cdot \int_a^b v(t)\, dt$$
$$= \xi \cdot \mathrm{Re}\Big(\int_a^b f(t)\, dt\Big) - \eta \cdot \mathrm{Im}\Big(\int_a^b f(t)\, dt\Big)$$
$$= \mathrm{Re}\Big((\xi + i\eta)\Big(\mathrm{Re}\Big(\int_a^b f(t)\, dt\Big) + i\,\mathrm{Im}\Big(\int_a^b f(t)\, dt\Big)\Big)\Big)$$
$$= \mathrm{Re}\Big(\lambda \cdot \int_a^b f(t)\, dt\Big).$$

Imaginary parts are handled similarly, with the result

$$\operatorname{Im}\left(\int_a^b \lambda \cdot f(t)\, dt\right) = \operatorname{Im}\left(\lambda \cdot \int_a^b f(t)\, dt\right).$$

Together these show the desired

$$\int_a^b \lambda \cdot f(t)\, dt = \lambda \cdot \int_a^b f(t)\, dt.$$

$\square$

The derivative of $f : [a,b] \to \mathbb{C}$ with respect to its real parameter $t \in [a,b]$ is denoted by

$$\dot{f}(t) = \frac{d}{dt} f(t) = \frac{d}{dt} \operatorname{Re}(f(t)) + j \frac{d}{dt} \operatorname{Im}(f(t)).$$

From the fundamental theorem of calculus in real variables, we easily get

<u>Theorem</u> (Fundamental theorem of calculus)

[ftc_for_complex_valued_integrals]

If $f : [a,b] \to \mathbb{C}$ is continuously differentiable, then

$$\int_a^b \dot{f}(t)\, dt = f(b) - f(a).$$

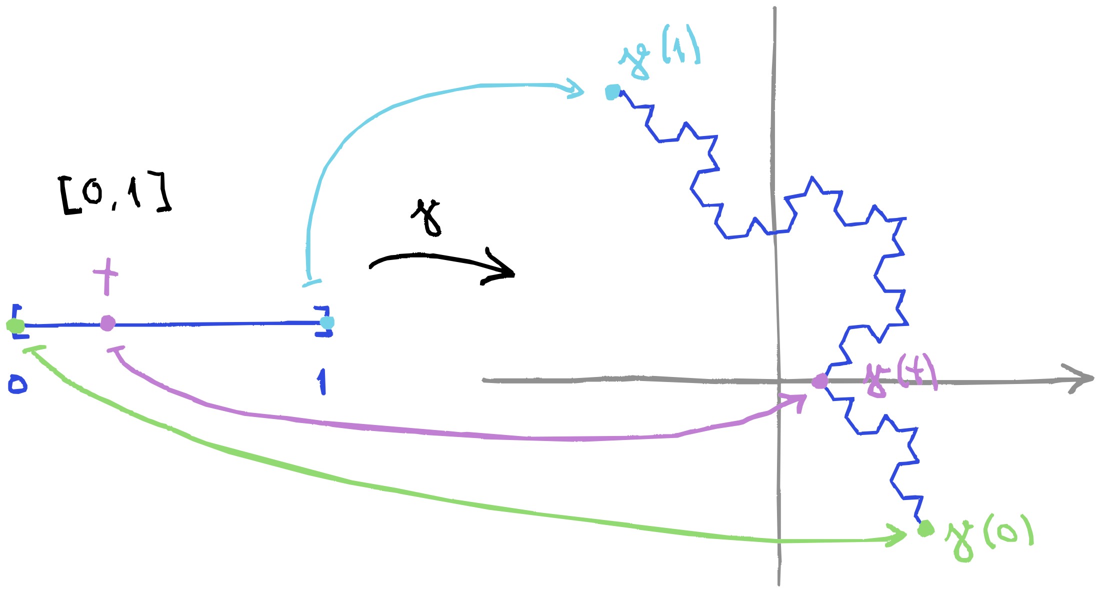
Smooth paths and contours

The integration theory needed for complex analysis is integration along (suitable) paths in the complex plane, so let us recall what paths are and address some of their regularity properties.

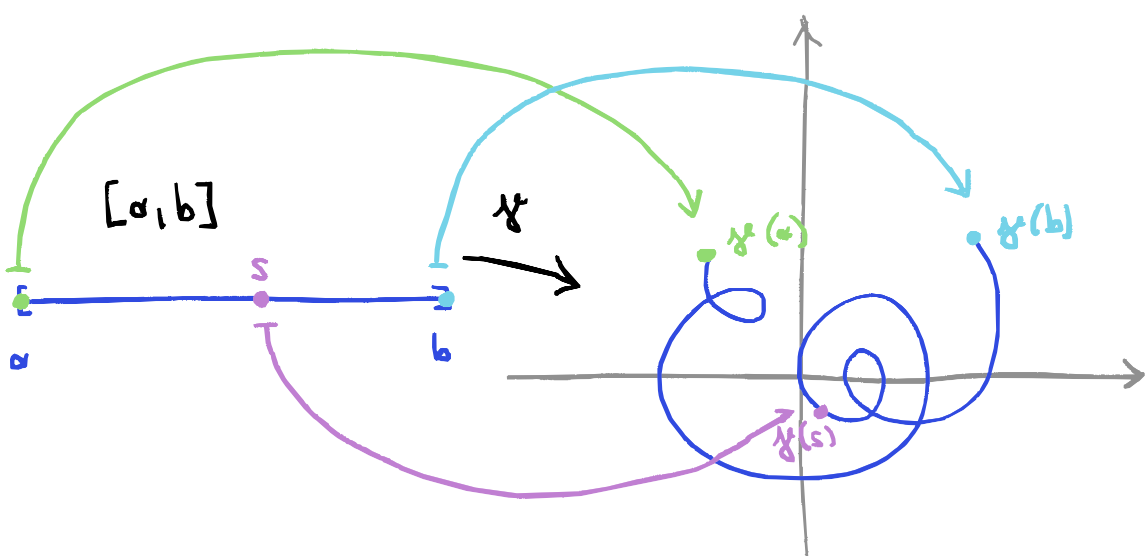
Def:

[path] A (parametrized) path in $\mathbb{C}$ is a continuous function $\gamma : [a,b] \to \mathbb{C}$ defined on some closed interval $[a,b] \subset \mathbb{R}$.

(By a path in a subset $A \subset \mathbb{C}$ we mean a path $\gamma : [a,b] \to \mathbb{C}$ such that $\gamma(t) \in A$ for every $t \in [a,b]$.)



a (non-smooth) path in $\mathbb{C}$

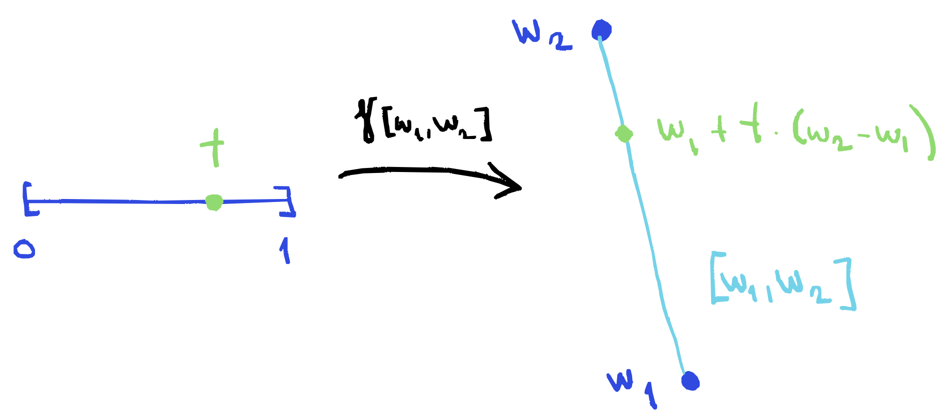


a smooth non-simple path in $\mathbb{C}$

Example (Line segment path)

Let $w_1, w_2 \in \mathbb{C}$ be two points in the complex plane. The line segment path from w_1 to w_2 is the path

$$\gamma_{[w_1,w_2]} : [0,1] \longrightarrow \mathbb{C}$$
$$\gamma_{[w_1,w_2]}(t) = w_1 + t \cdot (w_2 - w_1).$$



The image of this path is the subset of the complex plane denoted by

$$[w_1, w_2] = \{ w_1 + t(w_2 - w_1) \mid t \in [0,1] \} \subset \mathbb{C}$$

and called the line segment from w_1 to w_2.

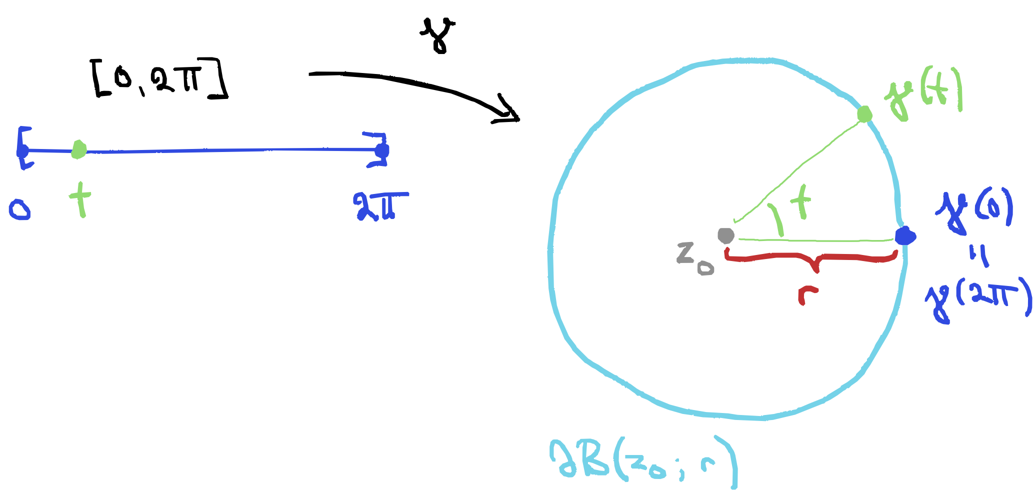
Example (Parametrized circle)

Let $z_0 \in \mathbb{C}$ and $r > 0$.

The path

$$\gamma : [0, 2\pi] \to \mathbb{C}$$
$$\gamma(t) = z_0 + r \cdot e^{it}$$

parametrizes the circle $\partial B(z_0; r)$ counterclockwise (i.e., in the positive orientation).



Def

[smooth - path]

A path $\gamma : [a,b] \to \mathbb{C}$ is $(C^1\text{-})$ smooth if the derivative

$$\dot{\gamma}(t) = \frac{d}{dt}\gamma(t) = \frac{d}{dt}\mathrm{Re}(\gamma(t)) + i\frac{d}{dt}\mathrm{Im}(\gamma(t))$$

exists at every $t \in [a,b]$ (one-sided derivatives at the end points a and b) and $t \mapsto \dot{\gamma}(t)$ is continuous.

Def

[contour]

A contour (or a piecewise smooth path) is a path $\gamma : [a,b] \to \mathbb{C}$ such that for some subdivision

$$a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$$

of the parameter interval, the restrictions $\gamma|[t_{j-1}, t_j] : [t_{j-1}, t_j] \to \mathbb{C}$ to the subintervals $[t_{j-1}, t_j] \subset [a,b]$ for $j = 1, 2, \dots, n$ are smooth.

Contour integrals

Let $A \subset \mathbb{C}$ be a subset of the complex plane. For suitably regular functions $f : A \to \mathbb{C}$ on A and paths γ in A, we want to define the integral of f along γ.

We start from the case of smooth γ.

Def (Contour integral along a smooth path)

[smooth_contour_integral]

If $\gamma : [a,b] \to A \subset \mathbb{C}$ is a smooth path and $f : A \to \mathbb{C}$ is a continuous function, then we define the integral of f along γ to be

$$\int_\gamma f(z)\, dz := \underbrace{\int_a^b f(\gamma(t))\, \dot{\gamma}(t)\, dt}.$$

(The right hand side is the integral of the complex-valued continuous function $t \mapsto f(\gamma(t))\, \dot{\gamma}(t)$ on the parameter interval $[a,b]$.)

The idea of the definition and the notation dz used in it is the following:

The points z on the path γ are of the form $z = \gamma(t) \in A$ and the function f is defined at such points, $f(z) = f(\gamma(t))$.

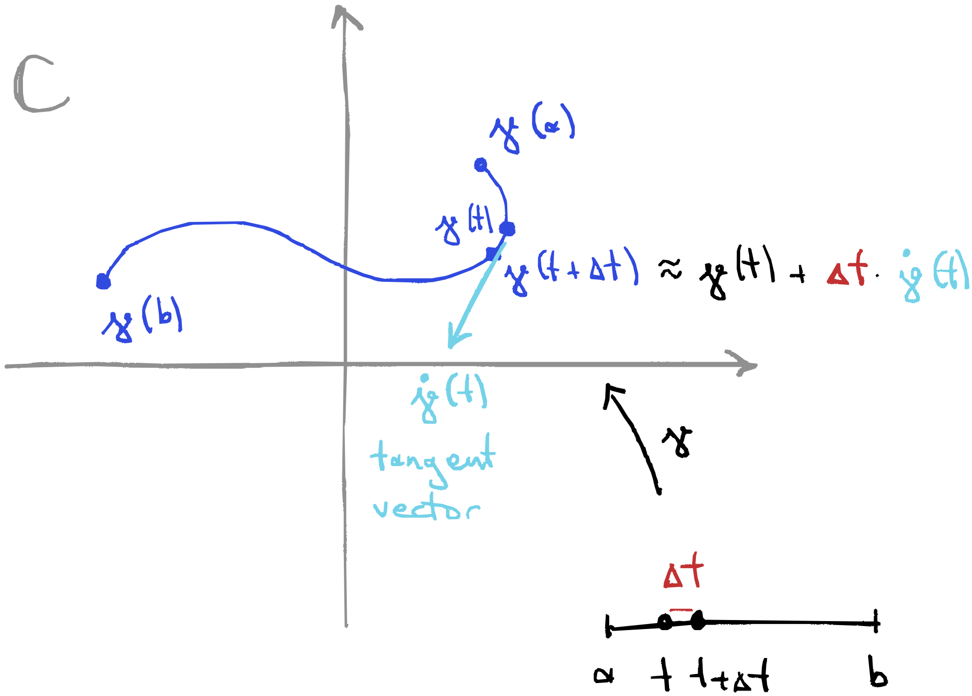
The "differential" dz describes a small change in the position of the point z along the smooth path γ, in the direction of its tangent vector,

$$\text{"} \quad dz = \dot{\gamma}(t) \cdot dt \quad \text{"}$$

tangent vector to the path γ at parameter value t

"a small increment of the parameter"

There is a precise differential-geometric meaning to these as differentials and one-forms, but it is beyond the scope of the present course.



Lemma

For a smooth path $\gamma : [a,b] \to A$ and a continuous function $f : A \to \mathbb{C}$, if we write $x(t) = \mathrm{Re}(\gamma(t))$, $y(t) = \mathrm{Im}(\gamma(t))$, $u(z) = \mathrm{Re}(f(z))$, $v(z) = \mathrm{Im}(f(z))$, then

$$\int_\gamma f(z)\,dz = \int_a^b \big(u(\gamma(t))\,\dot{x}(t) - v(\gamma(t))\,\dot{y}(t)\big)\,dt + i\int_a^b \big(u(\gamma(t))\,\dot{y}(t) + v(\gamma(t))\,\dot{x}(t)\big)\,dt$$

" $dz = dx + i\,dy$ " with $\begin{cases} \text{"} dx = \dot{x}(t)\,dt \text{"} \\ \text{"} dy = \dot{y}(t)\,dt \text{"} \end{cases}$

Proof Direct calculation (left as an exercise). $\square$

Example

The path

$\gamma : [0, 2\pi] \to \mathbb{C}$

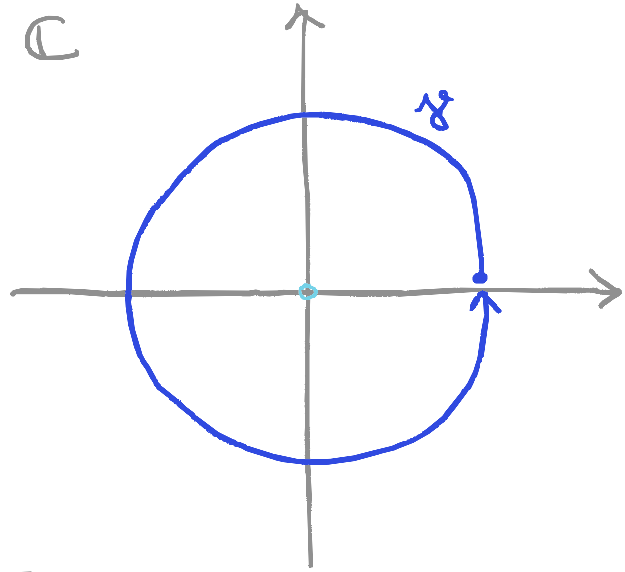
given by

$\gamma(t) = e^{it}$

parametrizes the

unit circle $\{|z| = 1\} \subset \mathbb{C}$

in the positive direction (counterclockwise).



The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$

given by $f(z) = \frac{1}{z}$ is continuous

in the punctured plane $\mathbb{C} \setminus \{0\}$.

The integral of f along γ is

$$\int_\gamma f(z)\, dz = \int_0^{2\pi} \underbrace{f(\gamma(t))}_{= \frac{1}{e^{it}}} \underbrace{\dot{\gamma}(t)}_{= i e^{it}}\, dt$$

$$= \int_0^{2\pi} \frac{i e^{it}}{e^{it}}\, dt = \int_0^{2\pi} i\, dt = 2\pi \cdot i .$$

Integral with respect to arclength

Although of less fundamental importance, we also use integrals with respect to the arclength of a path.
Again, starting from the case of smooth paths, we define:

Def:

[smooth_arc_length_integral]

If $\gamma : [a,b] \to A \subset \mathbb{C}$ is a smooth path, and $f : A \to \mathbb{C}$ is a continuous function, we define the integral of f with respect to the arclength of γ as

$$\int_\gamma f(z)\, |dz| := \int_a^b f(\gamma(t)) \cdot |\dot{\gamma}(t)|\, dt .$$

$$``\ |dz| = \underbrace{|\dot{\gamma}(t)|} \cdot dt = \underbrace{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}\, dt$$

the length of the tangent vector of γ, i.e., the "velocity" of the parametrized path

If we take the integrand f to be 1, the integral with respect to arclenght gives the length of the path:

$$\ell(\gamma) := \int_\gamma 1 \cdot |dz| = \int_a^b |\dot{\gamma}(t)|\, dt .$$

<u>Example</u>

Consider again the path $\gamma : [0, 2\pi] \to \mathbb{C}$ parametrizing the circle of radius $r > 0$ centered at the origin, $\gamma(t) = re^{it}$. Then $\dot{\gamma}(t) = i r e^{it}$ and $|\dot{\gamma}(t)| = r$, so

$$\ell(\gamma) = \int_\gamma 1 \cdot |dz| = \int_0^{2\pi} r\, dt = 2\pi r .$$

This matches the circumference of the circle of radius r, as expected.

The main use of arclength integration is for estimating integrals along paths by the following "triangle inequality for integrals".

<u>Lemma</u> (Triangle inequality for integrals)

[integral_triangle_inequality]

For a smooth path $\gamma : [a,b] \to A$ and a continuous function $f : A \to \mathbb{C}$ we have

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\, |dz| .$$

<u>Proof</u>: If $\int_\gamma f(z)\, dz = 0$ then the assertion is clear (the RHS is nonnegative) so assume $\int_\gamma f(z)\, dz \neq 0$. Note that for any complex number $w \in \mathbb{C} \setminus \{0\}$, we have $e^{-i\operatorname{Arg}(w)} \cdot w = |w|$ (since $w = |w| \cdot e^{i\operatorname{Arg}(w)}$). In particular denoting $\phi = \operatorname{Arg}\left(\int_\gamma f(z)\, dz \right)$, we have

$$\left|\int_\gamma f(z)\,dz\right| = e^{-i\phi}\int_\gamma f(z)\,dz$$
$$= \mathrm{Re}\left(e^{-i\phi}\int_\gamma f(z)\,dz\right)$$
$$= \mathrm{Re}\left(e^{-i\phi}\int_a^b f(\gamma(t))\,\dot\gamma(t)\,dt\right)$$
$$= \int_a^b \underbrace{\mathrm{Re}\left(e^{-i\phi} f(\gamma(t))\,\dot\gamma(t)\right)}_{\le \left|e^{-i\phi}\cdot f(\gamma(t))\cdot\dot\gamma(t)\right| = |f(\gamma(t))|\cdot|\dot\gamma(t)|}\,dt$$
$$\le \int_a^b |f(\gamma(t))|\cdot|\dot\gamma(t)|\cdot dt$$
$$= \int_\gamma |f(z)|\,|dz|.$$

$\square$

A very useful simple special case is:

<u>Corollary</u>

[contour_integral_bound]

If $|f(z)| \le M$ for all z on a smooth path γ, then
$$\left|\int_\gamma f(z)\,dz\right| \le M\cdot \ell(\gamma).$$

Linearity of integrals along paths

A simple consequence of the definitions of contour integrals and arclength integrals is their $\mathbb{C}$-linearity:

For a smooth path $\gamma : [a,b] \to A$ and for continuous functions $f, f_1, f_2 : A \to \mathbb{C}$ and scalars $\lambda \in \mathbb{C}$ we have

$$\int_\gamma \left(f_1(z) + f_2(z)\right) dz = \int_\gamma f_1(z)\, dz + \int_\gamma f_2(z)$$

$$\int_\gamma \left(f_1(z) + f_2(z)\right) |dz| = \int_\gamma f_1(z)\, |dz| + \int_\gamma f_2(z)\, |dz|$$

$$\int_\gamma \left(\lambda \cdot f(z)\right) dz = \lambda \cdot \int_\gamma f(z)\, dz$$

$$\int_\gamma \left(\lambda \cdot f(z)\right) |dz| = \lambda \cdot \int_\gamma f(z)\, |dz| .$$

(We leave it to the reader to perform the simple derivation of these, starting from the $\mathbb{C}$-linearity of integrals of $\mathbb{C}$-valued functions on $[a,b]$.)

Path reparametrization

If $\gamma : [a,b] \to A$ is a path and $\varphi : [c,d] \to [a,b]$ is a continuous bijection, then $\gamma \circ \varphi : [c,d] \to A$ is a reparametrization of the path γ.

If φ is moreover increasing, then the reparametrization preserves the orientation of the original path.

Also if γ is smooth and φ is continuously differentiable, then also $\gamma \circ \varphi$ is smooth.

Contour integrals are invariant under orientation-preserving reparametrizations in the following sense.

[reparametrization-invariance-of-integral]

Lemma (Reparametrization invariance)

Let $f : A \to \mathbb{C}$ be continuous on $A \subset \mathbb{C}$ and let $\gamma : [a,b] \to A$ be a smooth path in A. Then for any continuously differentiable increasing bijection $\varphi : [c,d] \to [a,b]$, setting $\tilde\gamma = \gamma \circ \varphi : [c,d] \to A$, we have

$$\int_{\tilde\gamma} f(z)\, dz = \int_\gamma f(z)\, dz \quad \text{and}$$

$$\int_{\tilde\gamma} f(z)\, |dz| = \int_\gamma f(z)\, |dz| .$$

Sketch: Observe

$$\dot{\tilde\gamma}(s) = \frac{d}{ds}\tilde\gamma(s) = \frac{d}{ds}\gamma(\varphi(s)) = \dot\varphi(s)\cdot\dot\gamma(\varphi(s)) .$$

Use the change of variables $t = \varphi(s)$ in

$$\int_{\tilde\gamma} f(z)\,dz = \int_c^d f(\tilde\gamma(s))\,\dot{\tilde\gamma}(s)\,ds$$

$$= \int_c^d f(\gamma(\varphi(s)))\,\dot\gamma(\varphi(s))\,\dot\varphi(s)\,ds = \int_a^b f(\gamma(t))\,\dot\gamma(t)\,dt$$

$$= \int_\gamma f(z)\,dz .$$

The case of arclength int. is similar. $\square$

Observe $\dot\varphi(s) \geq 0$ for increasing φ.

Path-reversal and path concatenation

Given a path $\gamma : [a,b] \longrightarrow A$, it is also possible to reparametrize it so that the orientation is reversed, e.g. by setting $\overleftarrow{\gamma} : [a,b] \longrightarrow A$

$$\overleftarrow{\gamma}(t) = \gamma(b+a-t).$$

Contour integrals and arclength integrals behave differently under such a path-reversal:

For $f : A \to \mathbb{C}$ continuous on $A \subset \mathbb{C}$ and $\gamma : [a,b] \longrightarrow A$ smooth path, it is easy to verify from the definitions (by change of variables $s = b+a-t$) that

$$\int_{\overleftarrow{\gamma}} f(z)\,dz = -\int_{\gamma} f(z)\,dz \qquad \text{and}$$

$$\int_{\overleftarrow{\gamma}} f(z)\,|dz| = \int_{\gamma} f(z)\,|dz|.$$

[integral_reversal]

As usual, integration can also be decomposed to pieces, so that if $\gamma : [a,b] \to A$ is a smooth path and

$$a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$$

is a subdivision of the parameter interval, then we have

[integral_concatenation]

$$\int_\gamma f(z)\, dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz \qquad \text{and}$$

$$\int_\gamma f(z)\, |dz| = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, |dz| \qquad \text{where}$$

$\gamma_j : [t_{j-1}, t_j] \to A$ are the restrictions $\gamma_j = \gamma|_{[t_{j-1}, t_j]}$ of γ to the subintervals.

This additivity with respect to path allows to define contour integrals and arclength integrals for piecewise smooth paths.

Primitives

In calculus in one real variable, definite integrals can be calculated in terms of "integral functions", or primitives. The questions about the existence of primitives are more subtle in complex analysis, but the definition itself is formally similar.

Def:

[primitive]

Let $f : U \to \mathbb{C}$ be a complex-valued function on an open set $U \subset \mathbb{C}$. A primitive of f is an analytic function $F : U \to \mathbb{C}$ such that

$$F'(z) = f(z) \qquad \forall z \in U.$$

Remark:

If $U \subset \mathbb{C}$ is open and connected, then a primitive F of $f: U \to \mathbb{C}$ is unique up to an additive constant. Indeed, suppose that both F_1 and F_2 are primitives of f. Then their difference $F_2 - F_1$ has derivative

$$(F_2 - F_1)'(z) = F_2'(z) - F_1'(z)$$
$$= f(z) - f(z) = 0$$

at any point $z \in U$. This implies that $F_2 - F_1$ is constant $c \in \mathbb{C}$, i.e., for any $z \in U$

$$F_2(z) = F_1(z) + c,$$

i.e., the two primitives only differ by an additive constant.

Examples

1.) Let $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$.

The power function $f(z) = z^n$ has a primitive in the whole complex plane $\mathbb{C}$ given by

$$F(z) = \frac{1}{n+1} z^{n+1} + c$$

since $F'(z) = z^n$ for any $z \in \mathbb{C}$.

2.) Let $n \in \{-2, -3, -4, \dots\}$.

The power function $f(z) = z^n$ has a primitive in the punctured complex plane $\mathbb{C} \setminus \{0\}$ given by

$$F(z) = \frac{1}{n+1} z^{n+1} + c,$$

since $F'(z) = z^n$ for any $z \in \mathbb{C} \setminus \{0\}$.

⚠ The power function $f(z) = z^{-1} = \frac{1}{z}$ does <u>not</u> have a primitive in the punctured plane $\mathbb{C} \setminus \{0\}$ — as we will soon see.

3.) The exponential function $f(z) = e^z$ has a primitive in the whole complex plane $\mathbb{C}$ given by

$$F(z) = e^z + c$$

since $F'(z) = e^z$ for any $z \in \mathbb{C}$.

Remark

A primitive of a function is necessarily continuous (because it is analytic!).

As one might expect, there is a version of the fundamental theorem of calculus for contour integrals. (... but not for arc length integrals!)

The key observation for it is the chain rule: if F has (complex) derivative $F'(z_0)$ at a point $z_0 = \gamma(t_0)$ and γ has (time) derivative $\dot{\gamma}(t_0)$ at time t_0, then $\frac{d}{dt} F(\gamma(t))\big|_{t=t_0}$

Indeed, one straightforward way to verify the chain rule in this context (combining a 1-dimensional time derivative with a complex derivative of a function) is to write local linear approximations

$$\begin{aligned} \gamma(t) &= \gamma(t_0) + (t-t_0)\cdot\dot{\gamma}(t_0) + (t-t_0)\,\varepsilon(t) \\ &= z_0 + (t-t_0)\,\dot{\gamma}(t_0) + (t-t_0)\,\varepsilon(t) \end{aligned}$$

with $\varepsilon(t) \longrightarrow 0$ as $t \to t_0$ and

$$F(z) = F(z_0) + (z-z_0)F'(z_0) + |z-z_0|\cdot E(z)$$

with $E(z) \longrightarrow 0$ as $z \to z_0$, which can be combined to

$$\begin{aligned} F(\gamma(t)) &= F(z_0 + (t-t_0)\,\dot{\gamma}(t_0) + (t-t_0)\,\varepsilon(t)) \\ &= F(z_0) + \left[(t-t_0)\,\dot{\gamma}(t_0) + (t-t_0)\,\varepsilon(t)\right]F'(z_0) \\ &\quad + \left|(t-t_0)\,\dot{\gamma}(t_0) + (t-t_0)\,\varepsilon(t)\right|\cdot E(\gamma(t)) \\ &= F(z_0) + (t-t_0)\,\dot{\gamma}(t_0)\,F'(z_0) + |t-t_0|\cdot(\cdots) \end{aligned}$$

where $\cdots$ contains many terms, but each of them $\longrightarrow 0$ as $t \longrightarrow t_0$.

Theorem (FTC for contour integrals)

[ftc_for_contour_integrals]

Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a continuous function which has a primitive $F : U \to \mathbb{C}$ on U. Then for any contour $\gamma : [a,b] \to U$ in U we have

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof: By additivity of contour integrals under concatenation, it suffices to prove this for smooth paths γ. Observe also, by chain rule, that $\frac{d}{dt}F(\gamma(t)) = \dot{\gamma}(t) \cdot F'(\gamma(t))$. Then calculate

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\dot{\gamma}(t)\,dt$$

$$= \int_a^b \left(\frac{d}{dt}F(\gamma(t))\right)dt = F(\gamma(b)) - F(\gamma(a))$$

by the "usual" fundamental thm of calculus. $\square$

Corollary

If a continuous function $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ has a primitive, then for all closed contours γ in U we have

$$\oint_\gamma f(z)\,dz = 0.$$

Proof If F is a primitive of f and $\gamma : [a,b] \to U$ is a closed contour (i.e. $\gamma(a) = \gamma(b)$) then

$$\oint_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0. \qquad \square$$

equal, cancel

Observation By an earlier calculation,

$\oint_\gamma \frac{1}{z}\,dz = 2\pi i \neq 0$ when γ is the closed contour parametrizing the unit circle ($\subset \mathbb{C} \setminus \{0\}$). Therefore the function $z \mapsto \frac{1}{z}$ cannot have a primitive in $\mathbb{C} \setminus \{0\}$.