Aalto University
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II

Problem set 1B

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Exercise sessions: 24.-25.10. Hand-in due: Sun 27.10.2024 at 23:59

Topic: topological notions (limits, continuity), and calculations in the complex plane

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol \triangle): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0-3. The deadline for returning solutions to problem set 1B is Sun 27.10.2024 at 23:59.

Exercise (in class) 1.

Determine whether the following limits exist:

(a)
$$\lim_{n \to \infty} \left(i^{n!} + 2^{-n} \right) \qquad \text{(limit along } n \in \mathbb{N} \text{)};$$

(b)
$$\lim_{z \to 0} \left(\frac{1}{|z|^2} \sin \left(\frac{z^2 - \overline{z}^2}{4i} \right) \right) \quad \text{(limit along } z \in \mathbb{C} \setminus \{0\} \text{)}.$$

<u>Hint</u>: It may be helpful to start by simplifying $\frac{z^2 - \overline{z}^2}{4i}$ when z = x + iy. Observe in particular that this expression takes real values, so only the familiar real sin function appears above.

Exercise (in class) 2.

Let $U \subset \mathbb{C}$ be a nonempty open subset of the complex plane. Consider possible functions $f: U \to \mathbb{C}$ which satisfy

$$(\star): \qquad e^{f(z)} = 1 \qquad \text{ for all } z \in U.$$

- (a) Show that there exist nonconstant functions $f: U \to \mathbb{C}$ satisfying the condition (\star) .
- (b) Show that if U is not connected, then there exist nonconstant continuous functions $f: U \to \mathbb{C}$ satisfying the condition (\star) .
- (c) Show that if U is connected, then any continuous function $f: U \to \mathbb{C}$ satisfying the condition (\star) is a constant function.

For $n \in \mathbb{N}$, show that

$$1 + \cos(\theta) + \cos(2\theta) + \dots + \cos(n\theta) = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)}.$$

<u>Hint</u>: Recall Exercise 1A.1 and the formulas $\sin(\theta) = \frac{1}{2i} \left(e^{i\theta} - e^{-i\theta} \right)$ and $\cos(\theta) = \frac{1}{2} \left(e^{i\theta} + e^{-i\theta} \right)$ for trigonometric functions in terms of complex exponentials.

Find all points of discontinuity of the following functions:

(a)
$$f: \mathbb{C} \setminus \{0\} \to \mathbb{R}$$
 given by

$$f(z) = \operatorname{Arg}(z^2)$$

where Arg is the principal branch (with values in $(-\pi, \pi]$) of the argument;

(b) $g: \mathbb{C} \to \mathbb{C}$ given by

$$g(z) = \sqrt{1 - z^2} \; ;$$

where $\sqrt{\cdot}$ denotes the principal branch (with arguments in $(-\pi/2, \pi/2]$) of the complex square root.