Aalto University
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2025-2026/II

Problem set 2B

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Exercise sessions: Thu-Fri, Oct 30-31, 2025

Topic: (complex) derivatives, analytic functions

The first two exercises (marked with symbol ♮) are to be solved before the exercise sessions so that you are ready to present your solution on the blackboard. The last two exercises are to be discussed and solved in exercise sessions.

♮ Blackboard exercise 1.

Determine the largest open sets in the complex plane on which the functions given by the following formulas are analytic. Calculate the (complex) derivatives of the functions on those sets.

(a): $z \mapsto z^3 (1+z)^6$

(b): $z \mapsto \dfrac{z-4}{4\mathbb{i}+z}$

(c): $z \mapsto \left(\dfrac{z+2}{125-z^3}\right)^4.$

♮ Blackboard exercise 2.

Of the following functions, which ones are harmonic on some nonempty open subset of the plane $\mathbb{R}^2$? For those that are, find a harmonic conjugate function on some nonempty open set and write down the corresponding analytic function.

(a): $(x,y) \mapsto \dfrac{y}{x^2+y^2}$

(b): $(x,y) \mapsto \dfrac{y^2}{x^2+y^2}.$

Remark: A twice continuously differentiable function $u \colon U \to \mathbb{R}$ is harmonic on an open set $U \subset \mathbb{R}^2$ if it satisfies the Laplace equation $\triangle u = 0$ on U, where $\triangle u := \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u$. A function v is a harmonic conjugate of u if the function $x + \mathbb{i} y \mapsto u(x,y) + \mathbb{i}\, v(x,y)$ is analytic.

Exercise (in class) 3.
Let $f \colon D \to \mathbb{C}$ be a complex-valued function on an open connected subset $D \subset \mathbb{C}$ of the complex plane. Suppose that both functions

$$z \mapsto f(z) \qquad \text{and} \qquad z \mapsto \overline{f(z)}$$

are analytic in D. Show that f is a constant function.

Exercise (in class) 4.
Determine the largest open sets in the complex plane on which the functions given by the following formulas are analytic. Calculate the (complex) derivatives of the functions on those sets.

$$\text{(a):} \qquad z \mapsto \frac{e^z - 1}{e^z + 1}$$

$$\text{(b):} \qquad z \mapsto \frac{\sin(\sqrt{\mathbb{i}z + 1})}{z^2 + 1}.$$

Remark: In part (b), $\sqrt{\cdots}$ denotes the principal branch of the square root function, i.e., the choice of complex square roots such that $\mathrm{Arg}(\sqrt{w}) \in (-\pi/2, \pi/2]$ for any $w \in \mathbb{C} \setminus \{0\}$.