Problem set 3B

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Exercise sessions: Thu-Fri, Nov 6-7, 2025

Topic: (complex) derivatives, analytic functions

The first two exercises (marked with symbol ♮) are to be solved before the exercise sessions so that you are ready to present your solution on the blackboard. The last two exercises are to be discussed and solved in exercise sessions.

♮ Blackboard exercise 1.

Let $a \in \mathbb{R}$ be a real number such that $|a| > 1$. With the help of the complex integral

$$\int_\gamma \frac{1}{z+a}\,\mathrm{d}z$$

along the path $\gamma\colon [0, 2\pi] \to \mathbb{C}$ given by the formula $\gamma(t) = e^{\mathbb{i}t}$, show that

$$\int_0^{2\pi} \frac{1 + a\,\cos(t)}{1 + 2a\,\cos(t) + a^2}\,\mathrm{d}t \;=\; 0.$$

♮ Blackboard exercise 2.

Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be the path given by the formula $\gamma(t) = -2e^{\mathbb{i}t}$. Evaluate the integral

$$\int_\gamma \frac{1}{z^2 - 1}\,\mathrm{d}z.$$

Exercise (in class) 3.
Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be the path given by the formula $\gamma(t) = e^{\mathrm{i}t}$, and let $a \in \mathbb{R}$ be such that $0 < a < 1$.

(a) Show that we have
$$\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos(t)}\,\mathrm{d}t = \oint_\gamma \frac{\mathrm{i}}{(z-a)(az-1)}\,\mathrm{d}z.$$

(b) Evaluate the real integral
$$\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos(t)}\,\mathrm{d}t.$$

Exercise (in class) 4.
Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be the path given by the formula $\gamma(t) = e^{\mathrm{i}t}$. Evaluate the following integrals along γ:

(a) $$\int_\gamma \frac{1}{(z-2)^2}\,\mathrm{d}z$$

(b) $$\int_\gamma \frac{1}{z^2-4}\,\mathrm{d}z$$

(c) $$\int_\gamma \left(z + \frac{1}{z}\right)^n \mathrm{d}z \qquad \text{where } n \in \mathbb{N}.$$