Aalto University **Problem set 2B**
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II K Kytölä & A Vavilov

Exercise sessions: 31.10.-1.11. Hand-in due: Sun 3.11.2024 at 23:59

Topic: (complex) derivatives, analytic functions

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ✍): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 2B is Sun 3.11.2024 at 23:59.

Exercise (in class) 1.
Determine the largest open sets in the complex plane on which the functions given by the following formulas are analytic. Calculate the (complex) derivatives of the functions on those sets.

(a): $z \mapsto z^3\,(1+z)^6$

(b): $z \mapsto \dfrac{z-4}{4\mathbb{i}+z}$

(c): $z \mapsto \left(\dfrac{z+2}{125-z^3}\right)^4.$

Exercise (in class) 2.
Of the following functions, which ones are harmonic on some nonempty open subset of the plane $\mathbb{R}^2$? For those that are, find a harmonic conjugate function on some nonempty open set and write down the corresponding analytic function.

(a): $(x,y) \mapsto \dfrac{y}{x^2+y^2}$

(b): $(x,y) \mapsto \dfrac{y^2}{x^2+y^2}.$

Remark: A twice continuously differentiable function $u\colon U \to \mathbb{R}$ is harmonic on an open set $U \subset \mathbb{R}^2$ if it satisfies the Laplace equation $\triangle u = 0$ on U, where $\triangle u := \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u$. A function v is a harmonic conjugate of u if the function $x + \mathbb{i}y \mapsto u(x,y) + \mathbb{i}\,v(x,y)$ is analytic.

✍ Homework exercise 3.

Let $f\colon D \to \mathbb{C}$ be a complex-valued function on an open connected subset $D \subset \mathbb{C}$ of the complex plane. Suppose that both functions

$$z \mapsto f(z) \qquad \text{and} \qquad z \mapsto \overline{f(z)}$$

are analytic in D. Show that f is a constant function.

✍ Homework exercise 4.

Determine the largest open sets in the complex plane on which the functions given by the following formulas are analytic. Calculate the (complex) derivatives of the functions on those sets.

$$\text{(a):} \qquad z \mapsto \frac{e^z - 1}{e^z + 1}$$

$$\text{(b):} \qquad z \mapsto \frac{\sin(\sqrt{\mathrm{i}z + 1})}{z^2 + 1}.$$

Remark: In part (b), $\sqrt{\cdots}$ denotes the principal branch of the square root function, i.e., the choice of complex square roots such that $\mathrm{Arg}(\sqrt{w}) \in (-\pi/2, \pi/2]$ for any $w \in \mathbb{C} \setminus \{0\}$.