

(2024 - 2025)

MS-C1300

Complex Analysis

Lecturer: **Kalle Kytölä**

Home page:

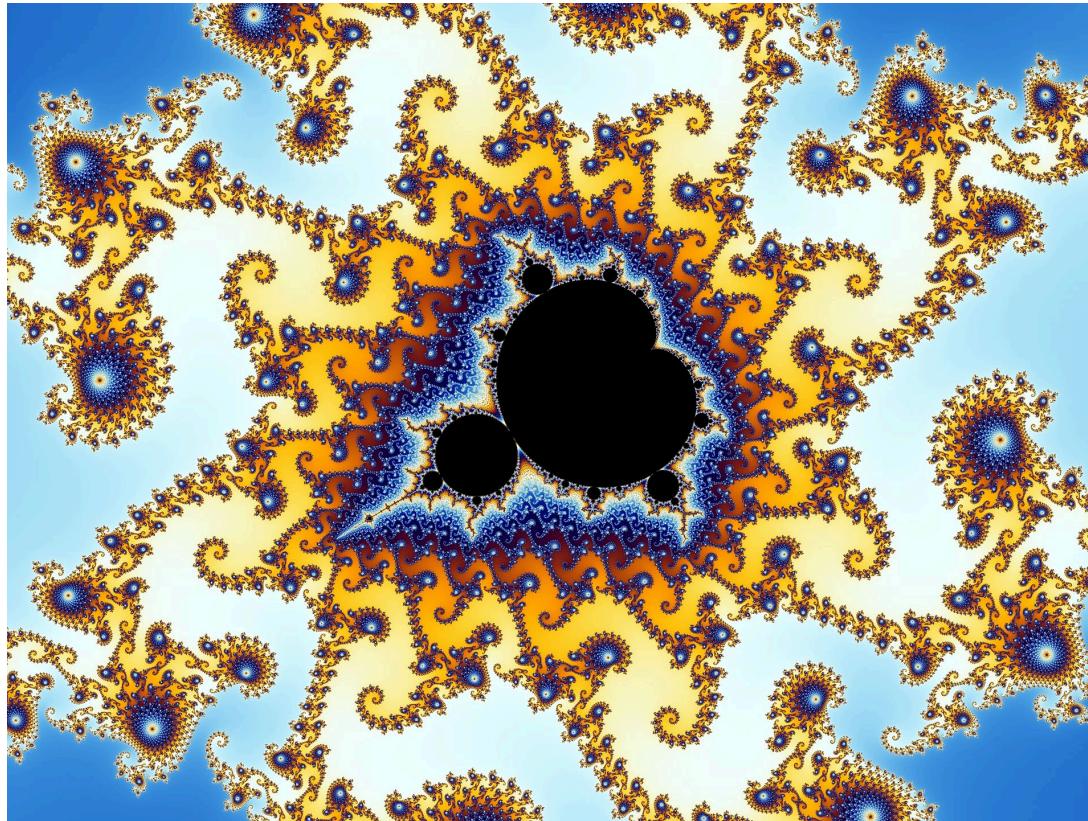
<https://mycourses.aalto.fi/courseview.php?id=44640>

Zulip chat:

<https://ms-c1300-2024.zulip.aalto.fi>

The beauty of complex analysis, some examples

Mandelbrot set



Circle Limit III (M. C. Escher, 1959)

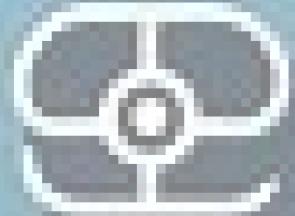
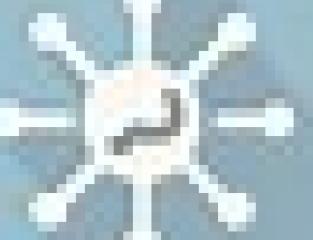


https://en.wikipedia.org/wiki/Mandelbrot_set

The true beauty of complex analysis

$$\frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz$$

Course arrangements



Course arrangements

Lecturer: Kalle Kytölä

Teaching assistants:

Anton Vavilov (Head TA)

Mikail Müftüoglu (TA)

Matvei Zhukov (TA)

Lectures: Mon 12-14, Thu 10-12

Exercises: twice a week, 3 groups

Home page:

<https://mycourses.aalto.fi/course/view.php?id=44640>

Exam

- 5 problems à 6 pts
⇒ course max 30 pts
exercises ↵ 0-6 pts added
- w/ hand-written memory aid sheet
- 2024-2025 exam dates
 - T0: Mon 2.12.2024 09:00-12:00
 - T01: Mon 17.2.2025 13:00-16:00
 - T02: Wed 4.6.2025 16:30-19:30

Exercises

Exercise sessions in 3 groups

Per week: two rounds (two problem sets, two exercise sessions)

Per round

- 2 in-class exercises (in the exercise session)
 - Mark your participation in a MyCourses Quiz with a password
- 2 homeworks to submit in written (folders on MyCourses, pdf format)

Submission folders close on:

- Wednesdays at 23:59 (first weekly round)
- Sundays at 23:59 (second weekly round)
- Each student's two lowest rounds (2 out of 12) in both of the above categories are omitted from grading. No other individual exceptions will be made.

Materials

During the course:

- hand-written lecture sketches?
- the theory in an interactive chart

Textbook:

An Introduction to Complex Function Theory by Bruce Palka (1991)

Other resources:

see course home page

Prerequisites

calculus in one and two real variables

- **MS-A01XX**
Differential and integral calculus 1
- **MS-A02XX**
Differential and integral calculus 2

(*derivatives, partial derivatives, differentials, integrals, ...*)

Review by yourself as necessary!



Course contents, learning outcomes

Course core contents (\rightsquigarrow learning outcomes)

Complex numbers \mathbb{C}

algebra, geometry, topology

Analytic functions $f: U \rightarrow \mathbb{C}$



- \mathbb{C} -differentiability
- existence of primitives (locally)
- power series representations

...all turn out logically equivalent!

Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz$$



- Taylor series (and Laurent series)
- fundamental theorem of algebra
- residue calculus
- ...

Planned lecture schedule

1A: Overview, complex numbers

2A: Analytic functions (1/2)

3A: Contour integration

4A: Cauchy's thm, consequences (2/3)

5A: Series, power series (1/2)

6A: Isolated singularities, residues (1/2)

1B: Topology of \mathbb{C} , mappings

2B: Analytic functions (2/2)

3B: Cauchy's thm, consequences (1/3)

4B: Cauchy's thm, consequences (3/3)

5B: Series, power series (1/2)

6B: Isolated singularities, residues (2/2)

Some history of complex numbers

(prehistory to complex analysis)

A complex issue?

$$x \in \mathbb{R} \implies x^2 \geq 0$$

$$\sqrt{-1} ?!?$$

"imaginary unit" i such that

$$i^2 = -1$$

**Heron of
Alexandria**

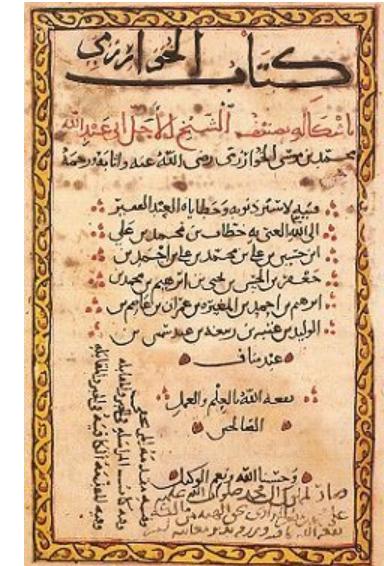
(~100 AD)

...is lead to a
square root of a
negative number.



Al-Khwarizmi
(ca. 780 - ca. 850)

solution of
quadratic
equations





Gerolamo Cardano (1501-1576)

Imaginary numbers are “as subtle as they are useless”. (1545)

...in **Tartaglia**'s and **del Ferro**'s solutions to cubic equations.

Equation:

$$x^3 = 15x + 4$$

Solution?

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

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DI RAFAEL BOMBELLI
DA BOLOGNA.

Nouamente posta in luce.



IN BOLOGNA

Nella stamperia di Giovani Rossi
M D L X X I I.

Con Licentia delli RR. VV. del Vesc. & Inquisit.

Rafael Bombelli (1526 - 1572)
calculations with complex numbers
(1572)

*...as long as imaginary numbers
cancel out by the end
of the calculation!*



Thomas Harriot (ca. 1560 - 1621)

complex numbers appeared in unpublished manuscripts, but were removed by the editors for posthumous publication (ca. 1600)

René Descartes

(1596 - 1650)

term: "*imaginary*"

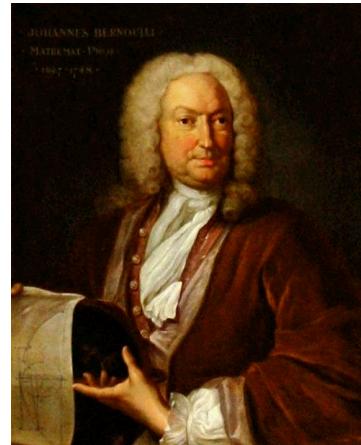
(1637)



Johann Bernoulli

(1667 - 1748)

complex
substitution in an
integral (1702)



John Wallis

(1616 – 1703)

insights from
geometry (1685)



Gottfried Wilhelm Leibniz

(1646 - 1716)



"The imaginary number is a fine and wonderful resource of the human spirit, almost an amphibian between being and not being." (Leibniz, 1702)



Leonhard Euler (1707 - 1783)

rather free use of complex numbers

- the symbol i for the imaginary unit
- geometry for complex roots of unity
- complex substitutions in calculus can also be justified in other ways
- statement (but no proof) of the fundamental theorem of algebra in a letter to Nicolaus Bernoulli (1759)



Carl Friedrich Gauss (1777 - 1855)

- fundamental theorem of algebra (1799)
(earlier attempts by d'Alembert, Euler, de Foncenex, Lagrange, Laplace...)
- complex plane
- arithmetic of complex numbers
- the term "complex number"

Caspar Wessel (1745 - 1818)

geometric interpretation of complex numbers presented to the Royal Danish Academy of Sciences and Letters (1797)

Jean-Robert Argand (1768 - 1822)

geometric interpretation (1806), proof of the fundamental theorem of algebra



CONSTRUCTION GÉOMÉTRIQUE, etc. 133

PHILOSOPHIE MATHÉMATIQUE.
Essai sur une manière de représenter les quantités imaginaires, dans les constructions géométriques;
Par M. ARGAND.

AU RÉDACTEUR DES ANNALES,

MONSIEUR,

Le mémoire de M. J. F. Français qui a paru à la page 61 du 4^e volume des *Annales*, a pour objet d'exposer quelques nouveaux principes de géométrie de position, dont les conséquences tendent particulièrement à modifier les notions admises jusqu'ici sur la nature des quantités imaginaires.

Ensuite que je l'apprends, M. Français assume qu'il a trouvé le fond de ces nouvelles idées dans une lettre de M. Legendre qui en parlait comme d'une chose qui lui avait été communiquée, et il témoigne le désir que le premier auteur de ces idées mette au jour son travail sur ce sujet. Il y a tout lieu de croire que le vœu de M. Français est depuis long-temps rempli. J'ai publié en 1806, un opuscule sous le titre d'*Essai sur une manière de représenter les quantités imaginaires, dans les constructions géométriques*, dont les principes sont entièrement analogues à ceux de M. Français, ainsi que l'ensemble des applications pratiques. J'ai l'honneur de vous adresser (1). M. Legendre a eu, dans le temps, la bonté d'examiner mon manuscrit et de me donner ses avis, et ce doit être là, si je ne m'abuse, la source de la communication dont parle M. Français.

(1) L'ouvrage se trouve à Paris, chez l'auteur, Imbouog. St-Marcos, rue de Clémens de Gentilly, n° 12.
Tom. IV, n^o V, 1^{re} novembre 1812.

J. D. G.
13



William Rowan Hamilton (1805 - 1865)

algebraic construction of complex numbers as pairs of real numbers (1837)



Augustus De Morgan (1806 - 1871)

"We have shown the symbol $\sqrt{-a}$ to be void of meaning, or rather self-contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility." (1831)

...still an issue?

Construction of complex numbers



Constructing \mathbb{C}

One possible construction, as a subset of real 2×2 matrices

$$\mathbb{C} := \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \subset \mathbb{R}^{2 \times 2}$$

usual matrix operations (addition, multiplication), e.g.

$$\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 - y_1y_2 & x_1y_2 + y_1x_2 \\ -x_1y_2 - y_1x_2 & x_1x_2 - y_1y_2 \end{bmatrix}$$

Constructing \mathbb{C}

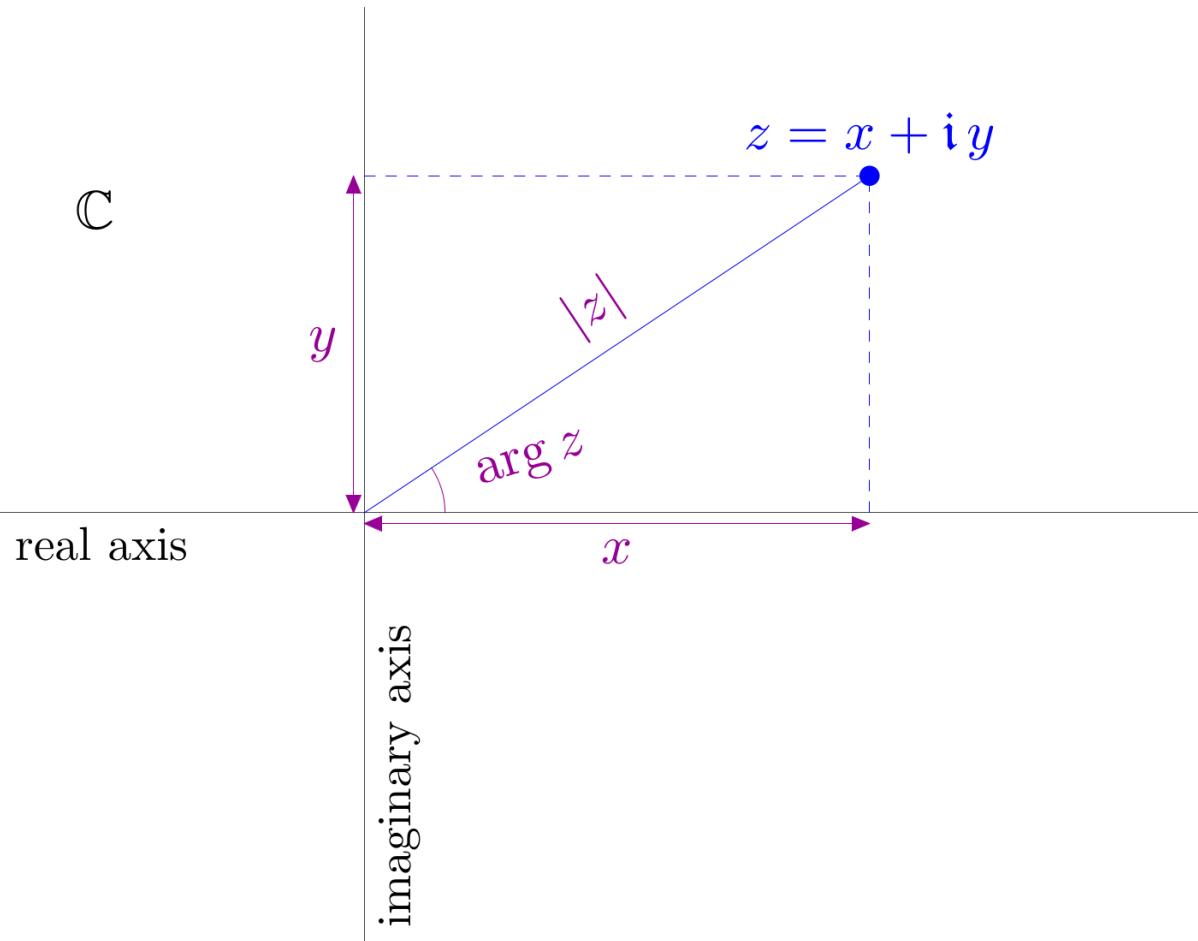
A more common construction

$$\mathbb{C} := \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

a complex number $z \in \mathbb{C}$ is an ordered pair of real numbers $z = (x, y)$

- addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
- multiplication: $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$
- $\mathbb{R} \subset \mathbb{C}$ via the embedding $\mathbb{R} \hookrightarrow \mathbb{C}$ given by $x \mapsto (x, 0)$
 - respects addition and multiplication!
- imaginary unit $i = (0, 1)$
 - $x + i \cdot y = (x, 0) + (0, 1) \cdot (y, 0) = (x, 0) + (0, y) = (x, y)$

Complex numbers



Complex plane

$$\mathbb{C} \cong \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

$$z = x + iy \in \mathbb{C}$$

- $x = \Re(z) \in \mathbb{R}$
- $y = \Im(z) \in \mathbb{R}$
- $|z| = \sqrt{x^2 + y^2}$
- $\arg(z) = \theta$:
 $\cos \theta = \frac{x}{|z|}, \sin \theta = \frac{y}{|z|}$



Some history of complex analysis

Augustin-Louis Cauchy (1789 - 1857)

- complex differentiability
- integrals and residues



Bernhard Riemann (1826 - 1866)

- a geometric approach to complex analysis



Karl Weierstrass (1815–1897)

- use of power series



Weierstrass



Why complex analysis?

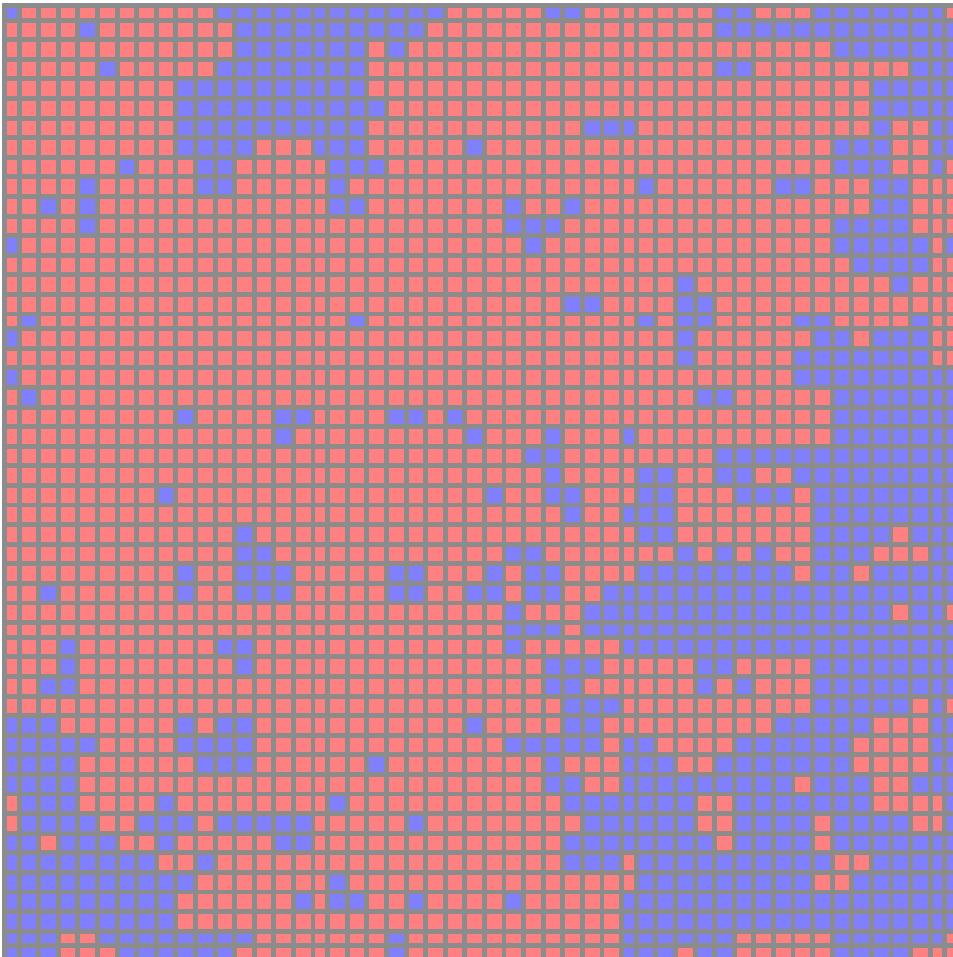
What is complex analysis good for?

- real and harmonic analysis
 - explicit calculations, Fourier theory, ...
- functional analysis
 - spectral theory of operators, ...
- (analytic) number theory, (analytic) combinatorics
- geometry and topology
 - conformal maps, Riemann surfaces, algebraic geometry, ...
- physics
 - in statistical physics, hydrodynamics, quantum mechanics, quantum field theory, string theory and conformal field theory, ...

"In order to simplify, one should complexify!"

Complex analysis in my own research topics

Ising model on \mathbb{Z}^2 at criticality



Uniform random labyrinth

