Aalto University

Problem set 3A

Department of Mathematics and Systems Analysis MS-C1300 — Complex Analysis, 2024-2025/II

K Kytölä & A Vavilov

Exercise sessions: 5.-6.11. Hand-in due: Wed 6.11.2024 at 23:59

Topic: complex contour integration, paths, primitives

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol \(\sigma \): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 3A is Wed 6.11.2024 at 23:59.

Exercise (in class) 1.

Let $\gamma \colon [0,\pi] \to \mathbb{C}$ be given by $\gamma(t) = t e^{it}$. Calculate the following integrals:

(a)
$$\int_{\gamma} \overline{z} \, dz$$

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(b)
$$\int_{\gamma} |z| \, |dz|$$
(c)
$$\int_{\gamma} z \, dz.$$

(c)
$$\int_{\gamma} z \, \mathrm{d}z.$$

Exercise (in class) 2.

Find functions $F \colon \mathbb{C} \to \mathbb{C}$ such that F'(z) = f(z) for all $z \in \mathbb{C}$ when

(a)
$$f(z) = z e^{z^2}$$

(b)
$$f(z) = z e^z$$

(c)
$$f(z) = z^2 \sin(4z).$$

Consider the path $\gamma \colon [-1,1] \to \mathbb{C}$ given by

$$\gamma(t) = \frac{1 - t^2}{1 + t^2} + i \frac{2t}{1 + t^2}.$$

Show that the formula

$$\phi(s) = \tan(s/2)$$

defines a smooth increasing function $\phi \colon [-\pi/2, \, \pi/2] \to [-1, 1]$. Consider the reparametrization

$$\eta = \gamma \circ \phi \colon [-\pi/2, \, \pi/2] \to \mathbb{C},$$

of γ by ϕ , and simplify the formula for $\eta(s) = \gamma(\phi(s))$.

<u>Hint</u>: The simplified answer should make it apparent that η (and thus also γ) is a path that parametrizes a certain semicircle.

Prove the following estimates for the contour integrals:

(a)

$$\left| \int_{\gamma} \frac{\mathrm{d}z}{z^2 - i} \right| \le \frac{3\pi}{4}$$

when γ parametrizes the circle |z|=3 once in the positive direction;

(b)

$$\left| \int_{\gamma} \frac{e^{3z}}{1 + e^z} \, \mathrm{d}z \right| \le \frac{2\pi e^{3R}}{e^R - 1}$$

when γ parametrizes a line segment from z = R > 0 to $z = R + 2\pi i$;

(c)

$$\left| \int_{\gamma} e^{\sin(z)} \, \mathrm{d}z \right| \leq 1$$

when γ parametrizes a line segment from z=0 to $z=\mathrm{i}.$

 $\underline{\mathrm{Hint}} \hbox{: ``Triangle inequality for integrals''}.$