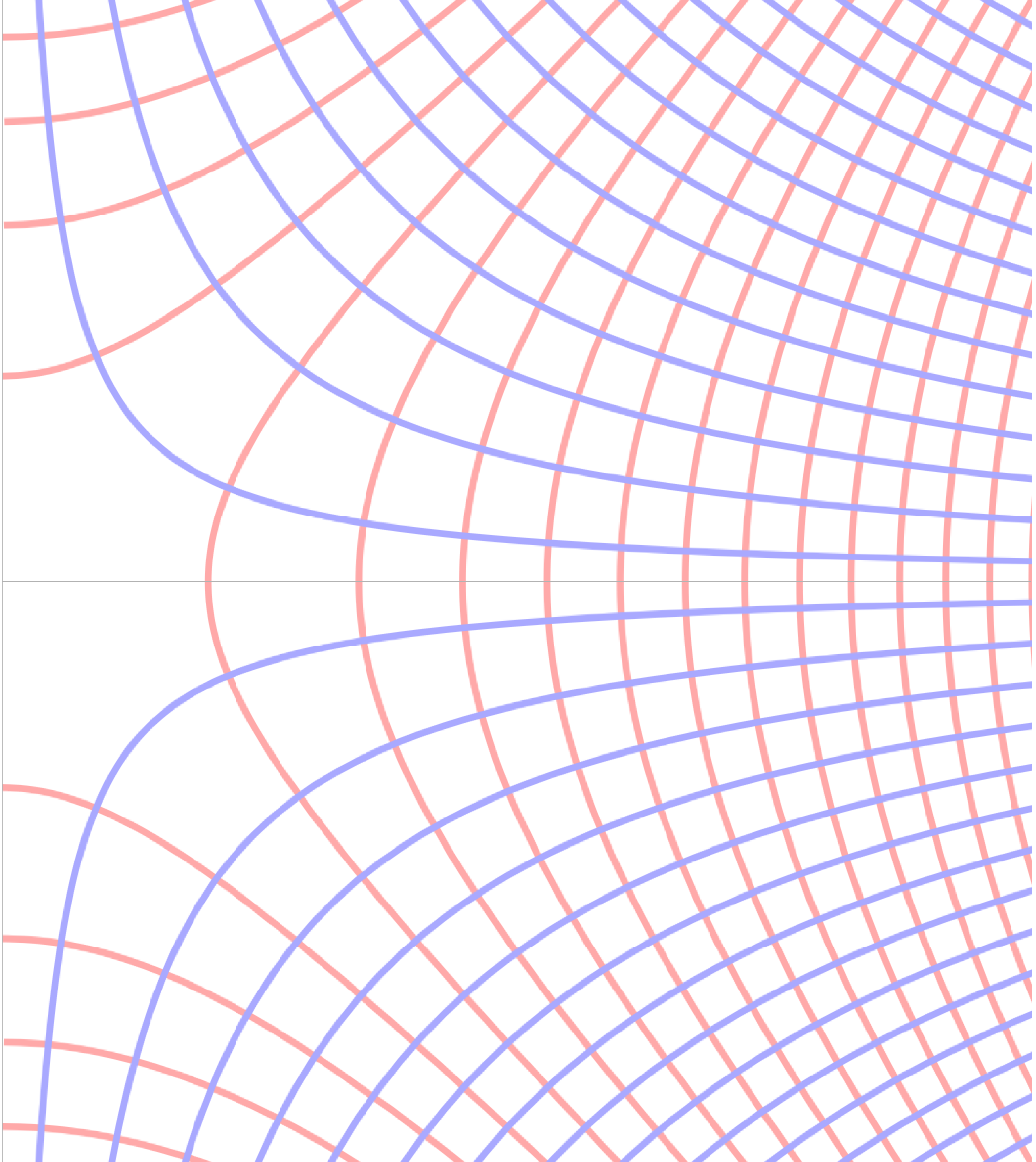


(2025 - 2026)

MS-C1300

Complex Analysis

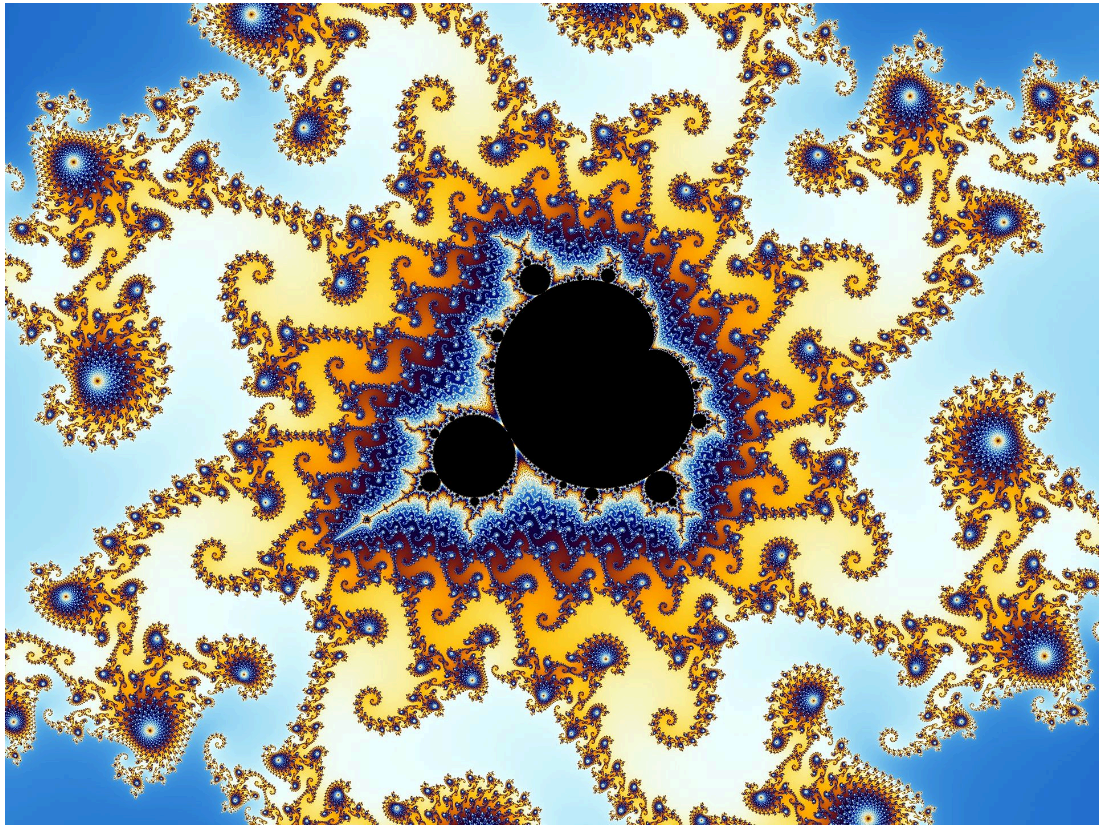
Lecturer: **Kalle Kytölä**

Home page:
https://mycourses.aalto.fi/course/view.php?id=47329

Zulip chat:
https://ms-c1300-2025.zulip.aalto.fi

The beauty of complex analysis, some examples

Mandelbrot set



Circle Limit III (M. C. Escher, 1959)

The true beauty of complex analysis

$$\frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} \, dz$$

Course arrangements

Course arrangements

Lecturer: Kalle Kytölä

Teaching assistants:
Kalle Heinonen (Head TA)
Eero Härmä (TA)
Reetta Leinonen (TA)
Mikail Müftüoglu (TA)

Lectures: Mon 12-14, Thu 10-12

Exercises: twice a week, 3 groups

Home page:
https://mycourses.aalto.fi/course/view.php?id=47329

Exam

- 2025-2026 exam dates
  - T0: Fri 5.12.2025 09:00-12:00
  - T01: Mon 16.2.2026 13:00-16:00
  - T02: Wed 3.6.2026 17:00-20:00
- w/ hand-written memory aid sheet

Grading

- more favorable of the options
  - 100% exam
  - 60% exam + 40% exercises

Exercises

Per week: two problem sets

A sets (exercise sessions Tue-Wed)

- 3 *regular problems* (solve in the exercise session)
- 1 *written quiz* (pen-and-paper solution in 20min individually without material)

B sets (exercise sessions Thu-Fri)

- 2 *blackboard exercises* (prepare in advance and be ready to present)
- 2 *regular problems* (solve in the exercise session)

Each student's lowest score of both A and B rounds are omitted from grading.
No other individual exceptions will be made.

Materials

During the course:

- the theory in an interactive chart
- hand-written lecture sketches?

Textbook:

An Introduction to Complex Function Theory by Bruce Palka (1991)

Other resources:

see course home page

Prerequisites

calculus in one and two real variables *(derivatives, partial derivatives, differentials, integrals, ...)*

- **MS-A01XX**
  Differential and integral calculus 1
- **MS-A02XX**
  Differential and integral calculus 2

also (very) helpful but not required

- MS-C1541 Metric spaces

Review by yourself as necessary!

Course contents, learning outcomes

Course core contents ($\rightsquigarrow$ learning outcomes)

Complex numbers $\mathbb{C}$

algebra, geometry, topology

Analytic functions $f\colon U \to \mathbb{C}$

- $\mathbb{C}$-differentiability
- existence of primitives (locally)
- power series representations

Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \oint \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

- Taylor series (and Laurent series)
- fundamental theorem of algebra
- residue calculus
- $\ldots$

Planned lecture schedule

1A: Overview, complex numbers
2A: Analytic functions (1/2)
3A: Contour integration
4A: Cauchy's thm, consequences (2/3)
5A: Series, power series (1/2)
6A: Isolated singularities, residues (1/2)

1B: Topology of $\mathbb{C}$, mappings
2B: Analytic functions (2/2)
3B: Cauchy's thm, consequences (1/3)
4B: Cauchy's thm, consequences (3/3)
5B: Series, power series (2/2)
6B: Isolated singularities, residues (2/2)

Some history of complex numbers

(prehistory to complex analysis)

A complex issue?

$$x \in \mathbb{R} \implies x^2 \geq 0$$

$$\sqrt{-1}\ ?!?$$

"imaginary unit" i such that

$$i^2 = -1$$

Heron of Alexandria

(~100 AD)

...is lead to a square root of a negative number.

Al-Khwarizmi

(ca. 780 - ca. 850)

solution of quadratic equations

Gerolamo Cardano (1501-1576)

Imaginary numbers are “as subtle as they are useless”. (1545)

...in **Tartaglia**'s and **del Ferro**'s solutions to cubic equations.

Equation:

$$x^3 = 15x + 4$$

Solution?

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$


L'ALGEBRA
PARTE MAGGIORE
DELL'ARIMETICA
DIVISA IN TRE LIBRI
DI RAFAEL BOMBELLI
DA BOLOGNA.

Nouamente posta in luce.

IN BOLOGNA
Nella stamperia di Giouanni Rossi
M D L X X I I.
Con Licentia delli RR. VV. del Vesc. & Inquisit.

Rafael Bombelli (1526 - 1572)

calculations with complex numbers (1572)

...as long as imaginary numbers
cancel out by the end
of the calculation!

Thomas Harriot (ca. 1560 - 1621)

complex numbers appeared in unpublished manuscripts, but were removed by the editors for posthumous publication (ca. 1600)

René Descartes
(1596 - 1650)

term: "*imaginary*"
(1637)

John Wallis
(1616 – 1703)

insights from
geometry (1685)

Johann Bernoulli
(1667 - 1748)

complex
substitution in an
integral (1702)

**Gottfried
Wilhelm Leibniz**
(1646 - 1716)

"The imaginary number is a fine and wonderful resource of the human spirit, almost an amphibian between being and not being." (Leibniz, 1702)

Leonhard Euler (1707 - 1783)

rather free use of complex numbers

- the symbol i for the imaginary unit
- geometry for complex roots of unity
- complex substitutions in calculus can also be justified in other ways
- statement (but no proof) of the fundamental theorem of algebra in a letter to Nicolaus Bernoulli (1759)

Carl Friedrich Gauss (1777 - 1855)

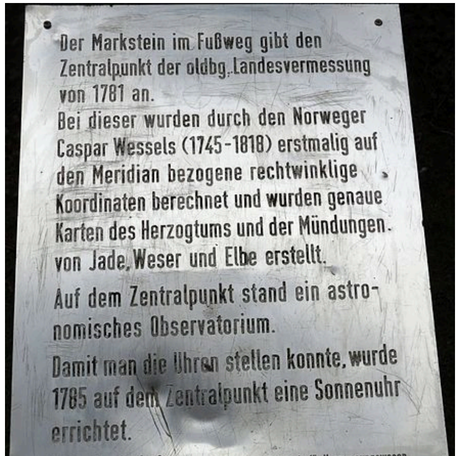
- fundamental theorem of algebra (1799)
  (earlier attempts by d'Alembert, Euler, de Foncenex, Lagrange, Laplace...)
- complex plane
- arithmetic of complex numbers
- the term "complex number"

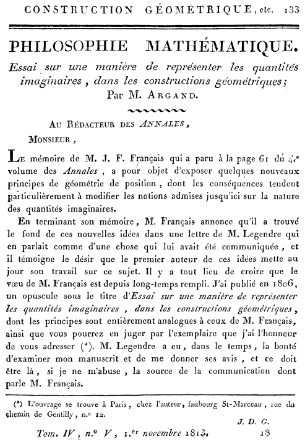
Caspar Wessel (1745 - 1818)

geometric interpretation of complex numbers presented to the Royal Danish Academy of Sciences and Letters (1797)

Jean-Robert Argand (1768 - 1822)

geometric interpretation (1806), proof of the fundamental theorem of algebra





CONSTRUCTION GÉOMÉTRIQUE, etc. 133

PHILOSOPHIE MATHÉMATIQUE.

Essai sur une manière de représenter les quantités imaginaires, dans les constructions géométriques;
Par M. ARGAND.

AU RÉDACTEUR DES *ANNALES*,

MONSIEUR,

LE mémoire de M. J. F. Français qui a paru à la page 61 du 4.e volume des *Annales*, a pour objet d'exposer quelques nouveaux principes de géométrie de position, dont les conséquences tendent particulièrement à modifier les notions admises jusqu'ici sur la nature des quantités imaginaires.

En terminant son mémoire, M. Français annonce qu'il a trouvé le fond de ces nouvelles idées dans une lettre de M. Legendre qui en parlait comme d'une chose qui lui avait été communiquée, et il témoigne le désir que le premier auteur de ces idées mette au jour son travail sur ce sujet. Il y a tout lieu de croire que le vœu de M. Français est depuis long-temps rempli. J'ai publié en 1806, un opuscule sous le titre d'*Essai sur une manière de représenter les quantités imaginaires, dans les constructions géométriques*, dont les principes sont entièrement analogues à ceux de M. Français, ainsi que vous pourrez en juger par l'exemplaire que j'ai l'honneur de vous adresser (*). M. Legendre a eu, dans le temps, la bonté d'examiner mon manuscrit et de me donner ses avis, et ce doit être là, si je ne m'abuse, la source de la communication dont parle M. Français.

(*) L'ouvrage se trouve à Paris, chez l'auteur, faubourg St-Marceau, rue du chemin de Gentilly, n.o 12.

J. D. G.

Tom. IV, n.o V, 1.er novembre 1813. 18

William Rowan Hamilton (1805 - 1865)

algebraic construction of complex numbers as pairs of real numbers (1837)

Augustus De Morgan (1806 - 1871)

"We have shown the symbol $\sqrt{-a}$ *to be void of meaning, or rather self-contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility." (1831)*

...still an issue?

Construction of complex numbers

Constructing $\mathbb{C}$

One possible construction, as a subset of real 2×2 matrices

$$\mathbb{C} := \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} \subset \mathbb{R}^{2\times 2}$$

usual matrix operations (addition, multiplication), e.g.

$$\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 - y_1y_2 & x_1y_2 + y_1x_2 \\ -x_1y_2 - y_1x_2 & x_1x_2 - y_1y_2 \end{bmatrix}$$

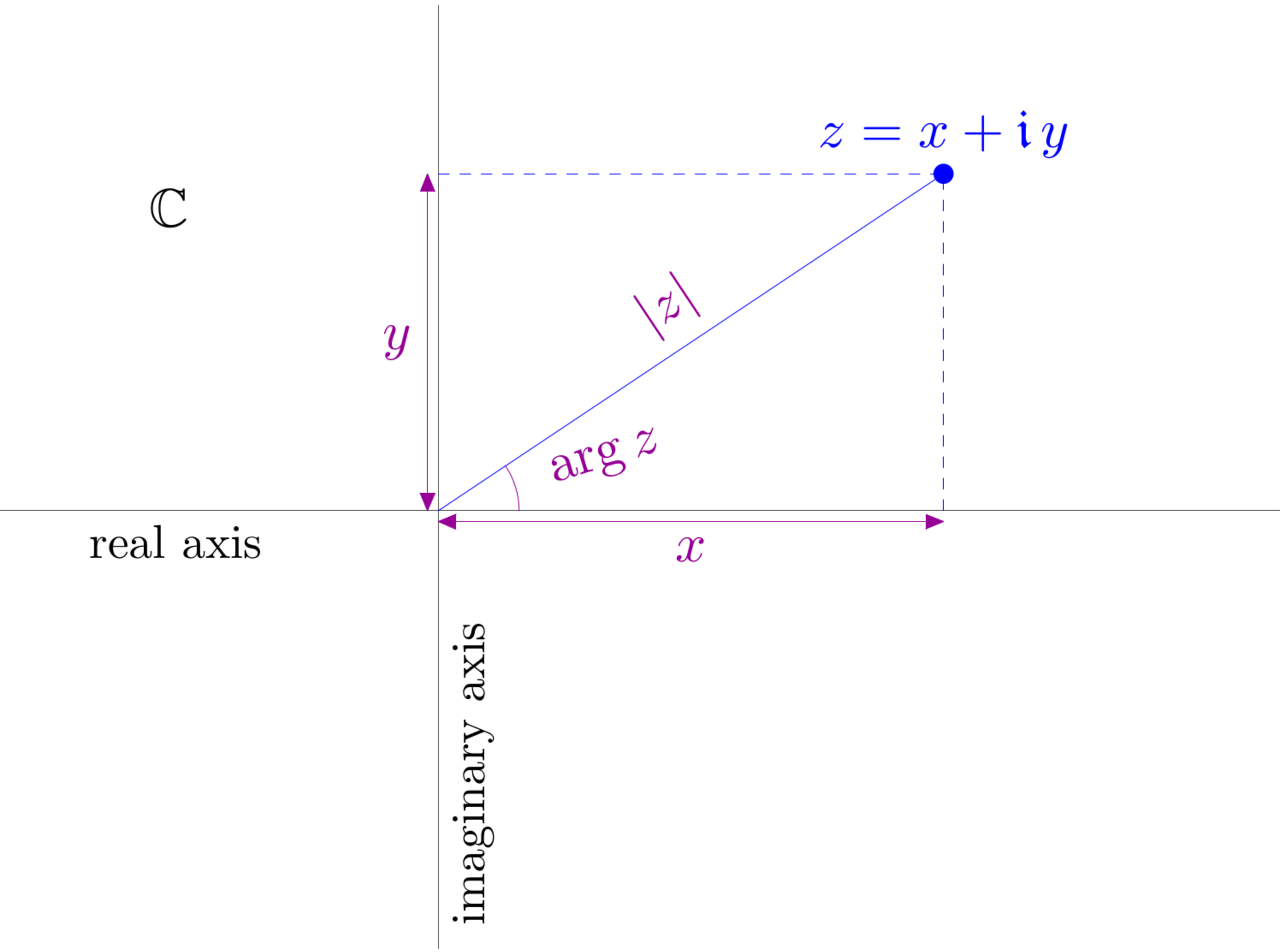
Constructing $\mathbb{C}$

A more common construction

$$\mathbb{C} := \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

a complex number $z \in \mathbb{C}$ is an ordered pair of real numbers $z = (x, y)$

- addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$
- multiplication: $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + y_1 x_2)$
- $\mathbb{R} \subset \mathbb{C}$ via the embedding $\mathbb{R} \hookrightarrow \mathbb{C}$ given by $x \mapsto (x, 0)$
  - respects addition and multiplication!
- imaginary unit $\mathfrak{i} = (0, 1)$
  - $x + \mathfrak{i} \cdot y = (x, 0) + (0, 1) \cdot (y, 0) = (x, 0) + (0, y) = (x, y)$



Complex numbers

Complex plane

$\mathbb{C} \cong \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$$z = x + \mathrm{i}y \in \mathbb{C}$$

- $x = \Re\mathfrak{e}(z) \in \mathbb{R}$
- $y = \Im\mathfrak{m}(z) \in \mathbb{R}$
- $|z| = \sqrt{x^2 + y^2}$
- $\arg(z) = \theta$:
  $\cos\theta = \frac{x}{|z|}$, $\sin\theta = \frac{y}{|z|}$

Some history of complex analysis

Augustin-Louis Cauchy (1789 - 1857)

- complex differentiability
- integrals and residues

Bernhard Riemann (1826 - 1866)

- a geometric approach to complex analysis

Karl Weierstrass (1815–1897)

- use of power series

Lars Ahlfors (1936)

Kodaira Kunihiko (1954)

Charles Fefferman (1978)

Wendelin Werner (2006)

conformal invariance in probability

Stanislav Smirnov (2010)

discrete complex analysis

Maryam Mirzakhani (2014)

Riemann surfaces

James Maynard (2022)

analytic number theory

Why complex analysis?

What is complex analysis good for?

- real and harmonic analysis
    - explicit calculations, Fourier theory, ...
- functional analysis
    - spectral theory of operators, ...
- (analytic) number theory, (analytic) combinatorics
- geometry and topology
    - conformal maps, Riemann surfaces, algebraic geometry, ...
- physics
    - in statistical physics, hydrodynamics, quantum mechanics, quantum field theory, string theory and conformal field theory, ...

"In order to simplify, one should complexify!"