Aalto University **Problem set 4A**
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II K Kytölä & A Vavilov

Exercise sessions: 12.-13.11. Hand-in due: Wed 13.11.2024 at 23:59

Topic: complex contour integration, Cauchy's integral theorem, Cauchy's integral formula, contractible paths

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ✍): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 4A is Wed 13.11.2024 at 23:59.

Exercise (in class) 1.
In this problem (and in other problems of this course concerning path homotopy), fully rigorous solutions are not expected; clear intuitive justifications suffice.

(a) Which of the following closed paths are null homotopic (i.e., contractible) in the set

$$A = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 \right\} \setminus \left\{ \frac{\mathbb{i}}{n} \;\middle|\; n \in \mathbb{N} \right\}?$$

(i) $t \mapsto \frac{1}{\sqrt{2}} e^{\mathbb{i}2\pi t}$ for $t \in [0,1]$;

(ii) $t \mapsto -\frac{1}{2} + \frac{1}{4} e^{\mathbb{i}2\pi t}$ for $t \in [0,1]$;

(iii) $t \mapsto \frac{2\mathbb{i}}{5} + \varepsilon\, e^{\mathbb{i}2\pi t}$ for $t \in [0,1]$, with $\varepsilon > 0$ sufficiently small;

(iv) $\eta \boxplus \overleftarrow{\eta}$ where η is some path in A.

Remark: Here $\boxplus$ denotes path concatenation, and $\overleftarrow{\eta}$ denotes the reverse of the path η.

(b) Which of the following subsets of the complex plane are simply-connected?

(i) $$\left\{ z \in \mathbb{C} \;\middle|\; |z - \mathbb{i}| \leq 1 \right\} \cup \left\{ z \in \mathbb{C} \;\middle|\; |z + \mathbb{i}| \leq 1 \right\}$$

(ii) $$\left\{ z \in \mathbb{C} \;\middle|\; |\,\mathfrak{Re}(z)| + |\,\mathfrak{Im}(z) - 3| > 2 \right\}$$

(iii) $$\mathbb{C} \setminus \bigcup_{k=0}^{5} \left\{ r e^{\mathbb{i}\pi k/3} \;\middle|\; r > 0 \right\}.$$

Exercise (in class) 2.
Let γ be a positively oriented contour around the circle $|z + \mathbb{i}| = \frac{3}{2}$. Calculate

$$\oint_\gamma \frac{1}{z^4 + z^2} \, \mathrm{d}z.$$

✍ **Homework exercise 3.**
Let $a, b > 0$. Show that

$$\int_0^{2\pi} \frac{1}{a^2\,\cos^2(t) + b^2\,\sin^2(t)}\,\mathrm{d}t \;=\; \frac{2\pi}{ab},$$

by considering the contour integral $\oint_\gamma \frac{1}{z}\,\mathrm{d}z$, where the contour γ is a suitable parametrization of the ellipse

$$E := \left\{ x + \mathbb{i}y \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} \subset \mathbb{C}.$$

✍ **Homework exercise 4.**
Let γ be a positively oriented contour around the circle $|z| = 1$. Calculate:

(a) $$\oint_\gamma \sqrt{9 - z^2}\,\mathrm{d}z$$

(b) $$\oint_\gamma \frac{1}{z^2 + 2z}\,\mathrm{d}z$$

Remark: In part (a), $\sqrt{\cdots}$ denotes the principal branch of the square root function, i.e., the choice of complex square roots such that $\mathrm{Arg}(\sqrt{w}) \in (-\pi/2, \pi/2]$ for any $w \in \mathbb{C} \setminus \{0\}$.