Aalto University
Department of Mathematics and Systems Analysis
MS-C1300 — Complex Analysis, 2024-2025/II

Problem set 4B

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Exercise sessions: 14.-15.11. **Hand-in due: Sun 17.11.2024 at 23:59**

Topic: (complex) derivatives, analytic functions

The first two exercises are to be discussed and solved in exercise sessions. The last two exercises are homework (marked with symbol ✍): written solutions to them are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 4B is Sun 17.11.2024 at 23:59.

Exercise (in class) 1.
Use Cauchy's integral formulas to evaluate the following contour integrals when the circles are positively (counterclockwise) oriented:

$$\text{(a)} \qquad \oint_{|z|=1} \frac{\cos(z)}{z} \, \mathrm{d}z$$

$$\text{(b)} \qquad \int_{|z|=2} \frac{e^{z+1}}{(z+1)^2} \, \mathrm{d}z \,.$$

Exercise (in class) 2.
Suppose that $f \colon \mathbb{C} \to \mathbb{C}$ is a non-constant analytic function defined in the entire complex plane. Prove that the range $f[\mathbb{C}] \subset \mathbb{C}$ must be dense in the following sense: for every $w_0 \in \mathbb{C}$ and any $\varepsilon > 0$ there exists a $z \in \mathbb{C}$ such that $\big|f(z) - w_0\big| < \varepsilon$.

Hint: *Do a proof by contradiction. If for some w_0 and ε no such points z exist, then what can be said about the function $z \mapsto \frac{1}{f(z)-w_0}$?*

✍ **Homework exercise 3.**
Let $B = \mathcal{B}(r;0)$ be the disk of radius $r > 0$ centered at the origin. Let $u\colon B \to \mathbb{R}$ be a harmonic function. Prove that for $0 < \rho < r$, we have

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{\mathrm{i}t})\,\mathrm{d}t.$$

Hint: *We consider it known that a harmonic function in a disk has a harmonic conjugate.*

✍ **Homework exercise 4.**
Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function defined in the entire complex plane, with the property that for some nonnegative constants $c, d \geq 0$ we have

$$|f(z)| \;\leq\; c\sqrt{|z|} + d \qquad \text{for all } z \in \mathbb{C}.$$

Prove that f must be a constant function.

Hint: *Try to get bounds on the derivative of f.*