Exercise session: Wed 28.2. at 14-16 Hand-in due: Mon 4.3. at 10am

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.

✍ **Hand-in 1.** Make sure that you understand how to interpret the group in part (a) below as the group of rotations of $\mathbb{R}^3$, and the group in part (b) below as the group of (orientation preserving) symmetries of a cube.

(a) Show that the set

$$\mathrm{SO}_3 = \left\{ M \in \mathbb{R}^{3\times 3} \;\middle|\; M^\top M = \mathbb{I}_3,\ \det(M) = 1 \right\}$$

of orthogonal matrices with determinant one is a group, with matrix multiplication as the group operation.

(b) Let e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$. Show the subset $G \subset \mathrm{SO}_3$ consisting of those $M \in \mathrm{SO}_3$ which map the set $F = \{e_1, -e_1, e_2, -e_2, e_3, -e_3\}$ to itself is a finite subgroup of SO_3 of order $|G| = 24$.

✍ **Hand-in 2.** Let G be a group. Two elements $g_1, g_2 \in G$ are said to be conjugates if there exists an element $h \in G$ such that $g_2 = h\, g_1\, h^{-1}$. Being conjugate is an equivalence relation, and the equivalence classes are called conjugacy classes.

(a) Find the conjugacy classes in the symmetric group on three letters, $\mathfrak{S}_3 = \{\sigma\colon\ \{1,2,3\} \to \{1,2,3\} \text{ bijection}\}$.

Hint: Recall that if $g_1, g_2 \in G$ are conjugate elements, then e.g. their orders are equal.

(b) Find the conjugacy classes in the group of orientation preserving symmetries of a cube (see Exercise 1(b)).

Hint: Note that if $M_1, M_2 \in \mathbb{C}^{n\times n}$ are conjugate matrices, then e.g. their eigenvalues coincide.

✍ **Hand-in 3.** As in Exercise 1, let $F = \{e_1, -e_1, e_2, -e_2, e_3, -e_3\}$ (think of the set of faces of a cubic dice), and let G be the group of orientation preserving symmetries of the cube. Let V be the space of complex valued functions on F, i.e.

$$V = \mathbb{C}^F = \{\phi\colon F \to \mathbb{C}\}\,.$$

Equip V with a representation ρ of G as follows: for a function $\phi\colon F \to \mathbb{C}$ and $g \in G$, define $(\rho(g).\phi)(u) = \phi(g^{-1}u)$ for all $u \in F$. Find at least two examples of nontrivial subspaces $V' \subset V$ which are invariant for this action, i.e., $\rho(g).V' \subset V'$ for all $g \in G$ (the trivial cases are the zero subspace and the entire space V).

Hint: To find invariant subspaces, you may want to take some sufficiently symmetric looking functions $\phi\colon F \to \mathbb{C}$, and see what is the subspace spanned by all $\rho(g).\phi$ with $g \in G$.

Exercise 4. The purpose of this exercise is to give one explicit construction of the tensor product $V \otimes W$ of vector spaces V and W, under the simplifying assumption that V and W are function spaces (this can indeed be assumed without loss of generality). Let $\Bbbk$ be the ground field (in this course always $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$).

For any set X, denote by $\Bbbk^X$ the vector space of $\Bbbk$-valued functions on X, with addition and scalar multiplication defined pointwise. Assume that $V \subset \Bbbk^X$ and $W \subset \Bbbk^Y$ for some sets X and Y. For $f \in \Bbbk^X$ and $g \in \Bbbk^Y$, define $f \otimes g \in \Bbbk^{X \times Y}$ by

$$\big(f \otimes g\big)(x, y) \;=\; f(x)\, g(y).$$

Also set

$$V \otimes W \;=\; \mathrm{span}\big\{ f \otimes g \;\big|\; f \in V,\ g \in W \big\}\,,$$

so that the map $(f, g) \mapsto f \otimes g$ is a bilinear map $V \times W \to V \otimes W$.

(a) Show that if $(f_i)_{i \in I}$ is a linearly independent collection in V and $(g_j)_{j \in J}$ is a linearly independent collection in W, then the collection $(f_i \otimes g_j)_{(i,j) \in I \times J}$ is linearly independent in $V \otimes W$.

(b) Show that if $(f_i)_{i \in I}$ is a collection that spans V and $(g_j)_{j \in J}$ is collection that spans W, then the collection $(f_i \otimes g_j)_{(i,j) \in I \times J}$ spans $V \otimes W$.

(c) Conclude that if $(f_i)_{i \in I}$ is a basis of V and $(g_j)_{j \in J}$ is a basis of W, then $(f_i \otimes g_j)_{(i,j) \in I \times J}$ is a basis of $V \otimes W$, and for finite dimensional V, W one in particular has $\dim(V \otimes W) = \dim(V)\dim(W)$.

(d) Define a bilinear map $\phi\colon V \times W \to V \otimes W$ by $\phi(f, g) = f \otimes g$. Show that if $\beta\colon V \times W \to U$ is a bilinear map to any vector space U, then there exists a unique linear map $\bar{\beta}\colon V \otimes W \to U$ such that the following diagram commutes

$$\begin{tikzcd} V \times W \arrow[rr, "\beta"] \arrow[dr, "\phi"'] & & U \,, \\ & V \otimes W \arrow[ur, dotted, "\bar{\beta}"'] & \end{tikzcd}$$

i.e., $\beta = \bar{\beta} \circ \phi$. (This is called the universal property of the tensor product.)

Exercise 5. Let V and W be two vector spaces, and denote the dual of V by

$$V^* = \mathrm{Hom}(V, \Bbbk) = \{\varphi\colon V \to \Bbbk \text{ linear}\}\,.$$

The purpose of this exercise is to work out the relation between the tensor product space $W \otimes V^*$, and the space $\mathrm{Hom}(V, W)$ of linear maps from V to W.

(a) For $w \in W$ and $\varphi \in V^*$, we associate to $w \otimes \varphi$ the following map $V \to W$

$$v \;\mapsto\; \langle \varphi, v \rangle\, w.$$

Show that the linear extension of this defines an injective linear map

$$W \otimes V^* \;\longrightarrow\; \mathrm{Hom}(V, W).$$

Show that if both V and W are finite dimensional, then this in fact gives an isomorphism $W \otimes V^* \;\cong\; \mathrm{Hom}(V, W)$.

In general, the rank $r = \mathrm{rank}(t)$ of a tensor $t \in V \otimes W$ is the smallest number $r \in \mathbb{Z}_{\geq 0}$ such that for some $v_1, \ldots, v_r \in V$ and $w_1, \ldots, w_r \in W$ we can write $t = \sum_{i=1}^r v_i \otimes w_i$.

(b) Show that, in general, $\mathrm{rank}(t) \leq \min\big(\dim(V), \dim(W)\big)$. Then return to the case in part (a): assume that V and W are finite-dimensional, and identify $W \otimes V^* \cong \mathrm{Hom}(V, W)$. Show that under this identification, the rank of a tensor $t \in W \otimes V^*$ is the same as the rank of a matrix of the corresponding linear map $T \in \mathrm{Hom}(V, W)$.