Aalto University
Department of Mathematics and Systems Analysis
MS-E1200 - Lie groups and Lie algebras

Problem set 4
2022/IV
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Exercise session: Wed 23.3. at 14-16 Hand-in due: Mon 28.3. at 10am

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.

Exercise 1. Let $\mathfrak{g}$ be a Lie algebra over a field $\Bbbk$. A *Lie subalgebra* of $\mathfrak{g}$ is a vector subspace $\mathfrak{s} \subset \mathfrak{g}$ such that $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}$.[1] A (Lie algebra) *ideal* of $\mathfrak{g}$ is a vector subspace $\mathfrak{j} \subset \mathfrak{g}$ such that $[\mathfrak{j}, \mathfrak{g}] \subset \mathfrak{j}$.

(a) Show that if $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then the quotient vector space $\mathfrak{g}/\mathfrak{i}$ becomes a Lie algebra by defining a bracket $[X + \mathfrak{i}, Y + \mathfrak{i}]_{\mathfrak{g}/\mathfrak{i}} = [X, Y] + \mathfrak{i}$.
(b) Show that if $\lambda \colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, then $\mathrm{Im}(\lambda) \subset \mathfrak{h}$ is a Lie subalgebra, and $\mathrm{Ker}(\lambda) \subset \mathfrak{g}$ is an ideal. Show that $\mathfrak{g}/\mathrm{Ker}(\lambda) \cong \mathrm{Im}(\lambda)$ as Lie algebras.

Assume now that G is a matrix Lie group ($G \subset \mathrm{GL}_n\mathbb{R}$ a closed subgroup) and $G' \subset G$ is a closed subgroup. Denote by $\mathfrak{g} = \mathcal{L}(G)$ and $\mathfrak{g}' = \mathcal{L}(G')$ the Lie algebras of G and G' (these are real Lie algebras, i.e., $\Bbbk = \mathbb{R}$ here).

(c) Show that $\mathfrak{g}' \subset \mathfrak{g}$ is a Lie subalgebra.
(d) Show that if $G' \trianglelefteq G$ is a normal subgroup, then $\mathfrak{g}' \subset \mathfrak{g}$ is a Lie algebra ideal.
Hint: Recall Exercise 1 of Problem set 3.

Exercise 2. Consider the four dimensional space $\mathbb{R}^4$ equipped with the Minkowski metric: the bilinear form $B \colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ of signature $(3, 1)$ given explicitly by

$$B\Big((t, x, y, z),\, (t', x', y', z')\Big) = tt' - xx' - yy' - zz'.$$

Let $\mathrm{O}_{3,1}$ be the Lorentz group, consisting of those linear maps $M \colon \mathbb{R}^4 \to \mathbb{R}^4$ which preserve the Minkowski metric in the sense that $B(v, v') = B(Mv, Mv')$ for all $v, v' \in \mathbb{R}^4$. Let $\mathrm{SL}_2(\mathbb{C})$ be the set of 2×2 complex matrices with unit determinant.

(a) Encode the point $v = (t, x, y, z) \in \mathbb{R}^4$ in the 2×2 Hermitian matrix
$$X = \begin{bmatrix} t + z & x - \mathbb{i}y \\ x + \mathbb{i}y & t - z \end{bmatrix}.$$
Show the points $v \in \mathbb{R}^4$ thus correspond bijectively to Hermitian matrices X, and show that under this correspondence we have $B(v, v) = \det(X)$.
(b) Define an action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{R}^4$ as follows. A matrix $A \in \mathrm{SL}_2(\mathbb{C})$ acts on a Hermitian 2×2-matrix X by conjugation $X \mapsto AXA^\dagger$, and the action on $\mathbb{R}^4$ is obtained using the identification given in part (a). Check that this indeed defines an action $\alpha \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathbb{R}^4)$, and show that the image consists of Lorentz transformations, $\mathfrak{Im}(\alpha) \subset \mathrm{O}_{3,1} \subset \mathrm{Aut}(\mathbb{R}^4)$.
(c) Describe the (real) Lie algebra $\mathfrak{g}$ of the (real) Lie group $\mathrm{SL}_2(\mathbb{C})$, and the Lie algebra $\mathfrak{o}_{3,1}$ of $\mathrm{O}_{3,1}$. Show in particular that their dimensions are equal.
(d) Show that for $\alpha \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}_{3,1}$ as in part (b), the derivative at the neutral element $\mathrm{d}\alpha|_e \colon \mathfrak{g} \to \mathfrak{o}_{3,1}$ is an isomorphism.

[1] Here and in what follows, $[\mathfrak{h}, \mathfrak{h}']$ means the linear span of $[X, X']$ with $X \in \mathfrak{h}$ and $X' \in \mathfrak{h}'$.

✍ **Hand-in 3.** Let $J \in \mathbb{R}^{2n\times 2n}$ be an antisymmetric matrix such that $J^2 = -\mathbb{I}$. The real symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{R})$ consists of those matrices $X \in \mathbb{R}^{2n\times 2n}$ which satisfy $X^\top J + JX = 0$. Compute the dimension $\dim(\mathfrak{sp}_{2n}(\mathbb{R}))$ of this Lie algebra.

✍ **Hand-in 4.** Let $C^\infty(\mathbb{R}^3)$ denote the space of smooth complex valued functions on $\mathbb{R}^3$, and on this space, consider the differential operators $\mathcal{J}_x, \mathcal{J}_y, \mathcal{J}_z$ given by $\mathcal{J}_x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$, $\mathcal{J}_y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$, and $\mathcal{J}_z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.

(a) By direct calculation, show that the commutators of the above differential operators are $[\mathcal{J}_x, \mathcal{J}_y] = \mathcal{J}_z$, $[\mathcal{J}_y, \mathcal{J}_z] = \mathcal{J}_x$, $[\mathcal{J}_z, \mathcal{J}_x] = \mathcal{J}_y$.

(b) For $M \in \mathrm{SO}_3$ and $f \in C^\infty(\mathbb{R}^3)$, define $M.f \in C^\infty(\mathbb{R}^3)$ by $(M.f)(\vec{x}) = f(M^{-1}\vec{x})$. Show that $C^\infty(\mathbb{R}^3)$ thus becomes a representation of the group SO_3.

(c) Let

$$M_x^{(\theta)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix},$$

be the rotation by angle θ around the x axis in the positive direction, and let $M_y^{(\theta)}$ and $M_z^{(\theta)}$ be the rotations by θ around y and z-axes, respectively. Show that for any $f \in C^\infty(\mathbb{R}^3)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Big|_{\theta=0} \big(M_x^{(\theta)}.f\big) = \mathcal{J}_x f,$$

and perform similar calculations for the actions of $M_y^{(\theta)}$ and $M_z^{(\theta)}$.

✍ **Hand-in 5.** Suppose that V is a complex vector space, and $\mathcal{E}, \mathcal{F}, \mathcal{H}$ are linear operators $V \to V$ which satisfy the commutation relations

$$\mathcal{H}\mathcal{E} - \mathcal{E}\mathcal{H} = 2\,\mathcal{E}, \qquad \mathcal{H}\mathcal{F} - \mathcal{F}\mathcal{H} = -2\,\mathcal{F}, \qquad \mathcal{E}\mathcal{F} - \mathcal{F}\mathcal{E} = \mathcal{H}.$$

(a) Show that for any $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{E}\mathcal{F}^k = \mathcal{F}^k\mathcal{E} + k\mathcal{F}^{k-1}(\mathcal{H} - k + 1).$$

(b) Show that for any $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{E}^k\mathcal{F}^k = k!\,\mathcal{H}\big(\mathcal{H} - 1\big)\cdots\big(\mathcal{H} - (k-1)\big) + \mathcal{P}\,\mathcal{E},$$

where $\mathcal{P}$ is some operator $V \to V$ (depending on k) that can be written as a polynomial in the operators $\mathcal{E}, \mathcal{F}, \mathcal{H}$.