Problem set 6
2022/IV
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Exercise session: Wed 6.4. at 14-16 Hand-in due: Mon 11.4. at 10am

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.

A *fundamental system* of rank r is a basis $\alpha_1, \ldots, \alpha_r$ of the r-dimensional Euclidean space (i.e., of $\mathbb{R}^r$ equipped with the usual inner product $\langle \cdot, \cdot \rangle$) such that for all $i \neq j$ we have $n_{ij} := 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \in \{0, -1, -2, -3\}$, and $n_{ji} = -1$ if $n_{ij} \in \{-2, -3\}$.

Exercise 1. Consider fundamental systems of rank $r = 2$, i.e., fundamental systems in the Euclidean plane $\mathbb{R}^2$.

(a) With rotations, scalings and reflections, we may assume that $\alpha_1 = (1, 0)$ and $\alpha_2 = (x, y)$ with $y > 0$. Find all possible α_2 in this case such that α_1, α_2 is a fundamental system.

(b) Define the reflections σ_j, $j \in \{1, 2\}$, of the plane $\mathbb{R}^2$ by $\sigma_j(\beta) := \beta - 2\frac{\langle \alpha_j, \beta \rangle}{\langle \alpha_j, \alpha_j \rangle}\alpha_j$. For each of the fundamental systems in part (a), find the smallest set of points $\Phi \subset \mathbb{R}^2$ which contains the fundamental system, $\alpha_1, \alpha_2 \in \Phi$, and has the property that $\sigma_j(\beta) \in \Phi$ for all $\beta \in \Phi$ and $j \in \{1, 2\}$.

Recall: The irreducible representations of $\mathfrak{sl}_3(\mathbb{C})$ are the highest weight representations $L(\lambda_{a,b})$ with highest weights $\lambda_{a,b} = a\,\eta^1 - b\,\eta^3$, where $a, b \in \mathbb{Z}_{\geq 0}$.

Exercise 2. Consider $\mathfrak{sl}_3(\mathbb{C})$ and its standard representation $V = \mathbb{C}^3$

(a) Show that the standard representation V and its dual V^* are both irreducible, and isomorphic to $V \cong L(\lambda_{1,0})$ and $V^* \cong L(\lambda_{0,1})$, respectively.

(b) Write down the weights and their multiplicities in $V \otimes V$. Which ones of these weights take non-negative integer values on H^{12} and H^{23}? Show that $V \otimes V$ contains a highest weight vector of each of the weights satisfying that condition. Deduce that $V \otimes V \cong L(\lambda_{2,0}) \oplus L(\lambda_{0,1})$, and that $\dim\big(L(\lambda_{2,0})\big) = 6$.

(c) Write down the weights and their multiplicities in $V \otimes V^*$. Which ones of these weights take non-negative integer values on H^{12} and H^{23}? Show that $V \otimes V^*$ contains a highest weight vector of each such weight. Deduce that $V \otimes V^* \cong L(\lambda_{1,1}) \oplus L(\lambda_{0,0})$, and that $\dim\big(L(\lambda_{1,1})\big) = 8$.

✍ **Hand-in 3.** Let $\mathfrak{sl}_2(\mathbb{R}) := \{X \in \mathbb{R}^{2\times 2} \mid \mathrm{tr}(X) = 0\}$. Show that the real Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is not of compact type. Show that its complexification $(\mathfrak{sl}_2(\mathbb{R}))_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{i}\,\mathfrak{sl}_2(\mathbb{R})$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Remark: Note that the Lie algebra $\mathfrak{so}_3$ is of compact type and its complexification is also isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. This shows that the same complex Lie algebra can be the complexification of both real Lie algebras of compact type and real Lie algebras that are not of compact type.

✍ **Hand-in 4.** Let $C^\infty(\mathbb{R}^3 \setminus \{0\})$ denote the space of smooth differentiable complex valued functions on $\mathbb{R}^3 \setminus \{0\}$, and on this space, consider the differential operators $\mathcal{J}_x, \mathcal{J}_y, \mathcal{J}_z$ as in Exercise 4 of Problem set 4.

(a) Show that one can define a representation $\rho \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}\big(C^\infty(\mathbb{R}^3 \setminus \{0\})\big)$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by setting

$$\rho(E) = \mathbb{i}\,\mathcal{J}_x - \mathcal{J}_y, \qquad \rho(F) = \mathbb{i}\,\mathcal{J}_x + \mathcal{J}_y, \qquad H \mapsto 2\mathbb{i}\,\mathcal{J}_z,$$

and in this representation the operator $\mathcal{Q} = \rho(E)\rho(F) + \rho(F)\rho(E) + \frac{1}{2}\rho(H)^2$ studied in Exercise **??** reads $\mathcal{Q} = -2 \times (\mathcal{J}_x^2 + \mathcal{J}_y^2 + \mathcal{J}_z^2)$.

(b) Define also the radial derivative operator $\mathcal{R} = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}$, where $r(x,y,z) = \sqrt{x^2+y^2+z^2}$. Show that the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ admits the expression

$$\Delta = \frac{-1}{2r^2}\,\mathcal{Q} + \mathcal{R}^2 + \frac{2}{r}\,\mathcal{R}.$$

Recall: The irreducible representations of $\mathfrak{sl}_3(\mathbb{C})$ are the highest weight representations $L(\lambda_{a,b})$ with highest weights $\lambda_{a,b} = a\,\eta^1 - b\,\eta^3$, where $a, b \in \mathbb{Z}_{\geq 0}$. We consider complete reducibility for $\mathfrak{sl}_3(\mathbb{C})$ known.

✍ **Hand-in 5.** Consider $\mathfrak{sl}_3(\mathbb{C})$ and its irreducible representations $L(\lambda_{a,b})$, $a, b \in \mathbb{Z}_{\geq 0}$.

(a) Use Schur's lemma to show that the multiplicity of the irreducible representation $L(\lambda')$ in $L(\lambda) \otimes L(\eta^1)$ equals the multiplicity of the irreducible representation $L(\lambda)$ in $L(\lambda') \otimes L(-\eta^3)$.

(b) Show that the multiplicity in part (a) is zero unless for some $m, n \in \mathbb{Z}_{\geq 0}$ we have $\lambda' - \lambda = \eta^3 + n\alpha^{12} + m\alpha^{23}$, and for some $k, l \in \mathbb{Z}_{\geq 0}$ we have $\lambda' - \lambda = \eta^1 - k\alpha^{12} - l\alpha^{23}$. Find all m, n, k, l satisfying these conditions.

(c) Let $a, b > 0$, and $\lambda = \lambda_{a,b}$. Assume that W is a representation with the properties: *(i)* all weights of W lie in the cone $\lambda - \mathbb{Z}_{\geq 0}\,\alpha^{12} - \mathbb{Z}_{\geq 0}\,\alpha^{23}$, *(ii)* the multiplicities of the weights λ, $\lambda - \alpha^{12}$, and $\lambda - \alpha^{23}$ are one, and *(iii)* the multiplicity of the weight $\lambda - \alpha^{12} - \alpha^{23}$ is two. Show that in the tensor product $L(\eta^1) \otimes W$, the weights $\lambda + \eta^1$ and $\lambda + \eta^1 - \alpha^{23}$ have multiplicity one, the weight $\lambda + \eta^1 - \alpha^{12}$ has multiplicity two, and the weight $\lambda + \eta^1 - \alpha^{12} - \alpha^{23}$ has multiplicity four.

(d) Use parts (b) and (c) to prove that for all $a, b > 0$, we have

$$L(\lambda_{a,b}) \otimes L(\eta^1) \cong L(\lambda_{a+1,b}) \oplus L(\lambda_{a-1,b+1}) \oplus L(\lambda_{a,b-1}).$$