

**Exercise session: Wed 10.4. at 14-16 Hand-in due: Mon 15.4. at 10am**

*In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.*

A *fundamental system* of rank  $r$  is a basis  $\alpha_1, \dots, \alpha_r$  of the  $r$ -dimensional Euclidean space (i.e., of  $\mathbb{R}^r$  equipped with the usual inner product  $\langle \cdot, \cdot \rangle$ ) such that for all  $i \neq j$  we have  $n_{ij} := 2 \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \in \{0, -1, -2, -3\}$ , and  $n_{ji} = -1$  if  $n_{ij} \in \{-2, -3\}$ .


**Exercise 1.** Consider fundamental systems of rank  $r = 2$ , i.e., fundamental systems in the Euclidean plane  $\mathbb{R}^2$ .

- (a) With rotations, scalings and reflections, we may assume that  $\alpha_1 = (1, 0)$  and  $\alpha_2 = (x, y)$  with  $y > 0$ . Find all possible  $\alpha_2$  in this case such that  $\alpha_1, \alpha_2$  is a fundamental system.
- (b) Define the reflections  $\sigma_j$ ,  $j \in \{1, 2\}$ , of the plane  $\mathbb{R}^2$  by  $\sigma_j(\beta) := \beta - 2 \frac{\langle \alpha_j, \beta \rangle}{\langle \alpha_j, \alpha_j \rangle} \alpha_j$ . For each of the fundamental systems in part (a), find the smallest set of points  $\Phi \subset \mathbb{R}^2$  which contains the fundamental system,  $\alpha_1, \alpha_2 \in \Phi$ , and has the property that  $\sigma_j(\beta) \in \Phi$  for all  $\beta \in \Phi$  and  $j \in \{1, 2\}$ .


*Recall:* Chapter III.3 in the lecture notes contains more details about the representations of  $\mathfrak{sl}_3(\mathbb{C})$ . In the following exercise, we use in particular that the irreducible representations of  $\mathfrak{sl}_3(\mathbb{C})$  are the highest weight representations  $L(\lambda_{a,b})$  with highest weights  $\lambda_{a,b} = a\eta^1 - b\eta^3$ , where  $a, b \in \mathbb{Z}_{\geq 0}$ .

**Exercise 2.** Consider  $\mathfrak{sl}_3(\mathbb{C})$  and its standard representation  $V = \mathbb{C}^3$

- (a) Show that the standard representation  $V$  and its dual  $V^*$  are both irreducible, and isomorphic to  $V \cong L(\lambda_{1,0})$  and  $V^* \cong L(\lambda_{0,1})$ , respectively.
- (b) Write down the weights and their multiplicities in  $V \otimes V$ . Which ones of these weights take non-negative integer values on  $H^{12}$  and  $H^{23}$ ? Show that  $V \otimes V$  contains a highest weight vector of each of the weights satisfying that condition. Deduce that  $V \otimes V \cong L(\lambda_{2,0}) \oplus L(\lambda_{0,1})$ , and that  $\dim(L(\lambda_{2,0})) = 6$ .
- (c) Write down the weights and their multiplicities in  $V \otimes V^*$ . Which ones of these weights take non-negative integer values on  $H^{12}$  and  $H^{23}$ ? Show that  $V \otimes V^*$  contains a highest weight vector of each such weight. Deduce that  $V \otimes V^* \cong L(\lambda_{1,1}) \oplus L(\lambda_{0,0})$ , and that  $\dim(L(\lambda_{1,1})) = 8$ .

 **Hand-in 3.** Let  $\mathfrak{sl}_2(\mathbb{R}) := \{X \in \mathbb{R}^{2 \times 2} \mid \text{tr}(X) = 0\}$ . Show that the real Lie algebra  $\mathfrak{sl}_2(\mathbb{R})$  is not of compact type. Show that its complexification  $(\mathfrak{sl}_2(\mathbb{R}))_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{R}) \oplus i\mathfrak{sl}_2(\mathbb{R})$  is isomorphic to  $\mathfrak{sl}_2(\mathbb{C})$ .

*Remark:* Note that the Lie algebra  $\mathfrak{so}_3$  is of compact type and its complexification is also isomorphic to  $\mathfrak{sl}_2(\mathbb{C})$ . This shows that the same complex Lie algebra can be the complexification of both real Lie algebras of compact type and real Lie algebras that are not of compact type.

 **Hand-in 4.** Let  $C^\infty(\mathbb{R}^3 \setminus \{0\})$  denote the space of smooth differentiable complex valued functions on  $\mathbb{R}^3 \setminus \{0\}$ , and on this space, consider the differential operators  $\mathcal{J}_x, \mathcal{J}_y, \mathcal{J}_z$  as in Exercise 4 of Problem set 4.

- (a) Show that one can define a representation  $\rho: \mathfrak{sl}_2(\mathbb{C}) \rightarrow \text{End}(C^\infty(\mathbb{R}^3 \setminus \{0\}))$  of the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$  by setting


$$\rho(E) = i\mathcal{J}_x - \mathcal{J}_y, \quad \rho(F) = i\mathcal{J}_x + \mathcal{J}_y, \quad H \mapsto 2i\mathcal{J}_z,$$

and in this representation the operator  $\mathcal{Q} = \rho(E)\rho(F) + \rho(F)\rho(E) + \frac{1}{2}\rho(H)^2$  studied in Exercise 1 of Problem set 5 reads  $\mathcal{Q} = -2 \times (\mathcal{J}_x^2 + \mathcal{J}_y^2 + \mathcal{J}_z^2)$ .

- (b) Define also the radial derivative operator  $\mathcal{R} = \frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y} + \frac{z}{r} \frac{\partial}{\partial z}$ , where  $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ . Show that the Laplacian  $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$  admits the expression

$$\Delta = \frac{-1}{2r^2} \mathcal{Q} + \mathcal{R}^2 + \frac{2}{r} \mathcal{R}.$$

*Recall:* Chapter III.3 in the lecture notes contains more details about the representations of  $\mathfrak{sl}_3(\mathbb{C})$ . In the following exercise, we use in particular that the irreducible representations of  $\mathfrak{sl}_3(\mathbb{C})$  are the highest weight representations  $L(\lambda_{a,b})$  with highest weights  $\lambda_{a,b} = a\eta^1 - b\eta^3$ , where  $a, b \in \mathbb{Z}_{\geq 0}$ . We also consider complete reducibility for finite-dimensional representations of  $\mathfrak{sl}_3(\mathbb{C})$  known.

 **Hand-in 5.** Consider  $\mathfrak{sl}_3(\mathbb{C})$  and its irreducible representations  $L(\lambda_{a,b})$ ,  $a, b \in \mathbb{Z}_{\geq 0}$ .

- (a) Use Schur's lemma to show that the multiplicity of the irreducible representation  $L(\lambda')$  in  $L(\lambda) \otimes L(\eta^1)$  equals the multiplicity of the irreducible representation  $L(\lambda)$  in  $L(\lambda') \otimes L(-\eta^3)$ .
- (b) Show that the multiplicity in part (a) is zero unless for some  $m, n \in \mathbb{Z}_{\geq 0}$  we have  $\lambda' - \lambda = \eta^3 + n\alpha^{12} + m\alpha^{23}$ , and for some  $k, l \in \mathbb{Z}_{\geq 0}$  we have  $\lambda' - \lambda = \eta^1 - k\alpha^{12} - l\alpha^{23}$ . Find all  $m, n, k, l$  satisfying these conditions.
- (c) Let  $a, b > 0$ , and  $\lambda = \lambda_{a,b}$ . Assume that  $W$  is a representation with the properties: (i) all weights of  $W$  lie in the cone  $\lambda - \mathbb{Z}_{\geq 0}\alpha^{12} - \mathbb{Z}_{\geq 0}\alpha^{23}$ , (ii) the multiplicities of the weights  $\lambda$ ,  $\lambda - \alpha^{12}$ , and  $\lambda - \alpha^{23}$  are one, and (iii) the multiplicity of the weight  $\lambda - \alpha^{12} - \alpha^{23}$  is two. Show that in the tensor product  $L(\eta^1) \otimes W$ , the weights  $\lambda + \eta^1$  and  $\lambda + \eta^1 - \alpha^{23}$  have multiplicity one, the weight  $\lambda + \eta^1 - \alpha^{12}$  has multiplicity two, and the weight  $\lambda + \eta^1 - \alpha^{12} - \alpha^{23}$  has multiplicity four.
- (d) Use parts (b) and (c) to prove that for all  $a, b > 0$ , we have

$$L(\lambda_{a,b}) \otimes L(\eta^1) \cong L(\lambda_{a+1,b}) \oplus L(\lambda_{a-1,b+1}) \oplus L(\lambda_{a,b-1}).$$