Aalto University | Problem set 5
Department of Mathematics and Systems Analysis | 2022/IV
MS-E1200 - Lie groups and Lie algebras | K Kytölä & D Adame-Carrillo

Exercise session: Wed 30.3. at 14-16 Hand-in due: Mon 4.4. at 10am

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.

Recall: The basis E, F, H of the complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is as in the lectures, and $L(\lambda)$ denotes the finite-dimensional irreducible representation of highest weight $\lambda \in \mathbb{Z}_{\geq 0}$.

Exercise 1.

(a) Suppose that V is a complex vector space, and $\mathcal{E}, \mathcal{F}, \mathcal{H} \in \mathrm{End}(V)$ satisfy $\mathcal{H}\mathcal{E} - \mathcal{E}\mathcal{H} = 2\,\mathcal{E}$, $\mathcal{H}\mathcal{F} - \mathcal{F}\mathcal{H} = -2\,\mathcal{F}$, $\mathcal{E}\mathcal{F} - \mathcal{F}\mathcal{E} = \mathcal{H}$. Show that the operator $\mathcal{Q} = \mathcal{E}\mathcal{F} + \mathcal{F}\mathcal{E} + \frac{1}{2}\mathcal{H}^2$ commutes with $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{H}$.

(b) Let $\vartheta\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}(V)$ be a finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Show that $\mathcal{Q} = \vartheta(E)\vartheta(F) + \vartheta(F)\vartheta(E) + \frac{1}{2}\vartheta(H)^2$ is a scalar multiple of the identity operator on V, that is $\mathcal{Q} = q \times \mathrm{id}_V$ for some $q \in \mathbb{C}$.

(c) For each finite-dimensional irreducible representation $L(\lambda)$ of $\mathfrak{sl}_2(\mathbb{C})$, find the value of the scalar q in part (b).

Exercise 2. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ (over $\mathbb{R}$), and let $\rho_V\colon \mathrm{G} \to \mathrm{Aut}(V)$ and $\rho_W\colon \mathrm{G} \to \mathrm{Aut}(W)$ be representations of G (on $\mathbb{R}$-vector spaces V, W). Denote $\vartheta_V = (\mathrm{d}\rho_V)|_e\colon \mathfrak{g} \to \mathrm{End}(V)$, i.e.,

$$\big(\vartheta_V(X)\big)v = \frac{\mathrm{d}}{\mathrm{d}t}\rho\big(\exp(tX)\big)\,v\Big|_{t=0} \qquad \text{for } X \in \mathfrak{g},\ v \in V,$$

and similarly for $\vartheta_W\colon \mathfrak{g} \to \mathrm{End}(W)$. Consider the associated group representations $\rho_\oplus$, $\rho_\otimes$, and ρ_{Hom} of G on $V \oplus W$, $V \otimes W$ and $\mathrm{Hom}(V, W)$, respectively.

(a) Calculate $(\mathrm{d}\rho_\oplus)|_e\colon \mathfrak{g} \to \mathrm{End}(V \oplus W)$, $(\mathrm{d}\rho_\otimes)|_e\colon \mathfrak{g} \to \mathrm{End}(V \otimes W)$, and $(\mathrm{d}\rho_{\mathrm{Hom}})|_e\colon \mathfrak{g} \to \mathrm{End}(\mathrm{Hom}(V, W))$, in terms of ϑ_V and ϑ_W.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{k}$, and let $\vartheta_1\colon \mathfrak{g} \to \mathrm{End}(V_1)$ and $\vartheta_2\colon \mathfrak{g} \to \mathrm{End}(V_2)$ be two representations of $\mathfrak{g}$ on $\mathbb{k}$-vector spaces V_1 and V_2, respectively.

(b) Show that the direct sum $V_1 \oplus V_2$ becomes a representation of $\mathfrak{g}$ by setting $X.(v_1 + v_2) = \vartheta_1(X)v_1 + \vartheta_2(X)v_2$ for all $X \in \mathfrak{g}$, $v_1 \in V_1$ and $v_2 \in V_2$.

(c) Show that the tensor product $V_1 \otimes V_2$ becomes a representation of $\mathfrak{g}$ by setting $X.(v_1 \otimes v_2) = \vartheta_1(X)v_1 \otimes v_2 + v_1 \otimes \vartheta_2(X)v_2$ for all $X \in \mathfrak{g}$, $v_1 \in V_1$ and $v_2 \in V_2$, and extending linearly.

(d) Show that the space of linear maps $\mathrm{Hom}(V_1, V_2)$ becomes a representation of $\mathfrak{g}$ by defining, for any $X \in \mathfrak{g}$ and linear map $T\colon V_1 \to V_2$,

$$X.T = \vartheta_2(X) \circ T - T \circ \vartheta_1(X).$$

✍ **Hand-in 3.** Let $V = \mathbb{C}[x, y]$ be the polynomial algebra in two indeterminates, x and y.

(a) Define a linear map $\mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}(V)$ by setting

$$E \;\mapsto\; x\,\frac{\partial}{\partial y}, \qquad F \;\mapsto\; y\,\frac{\partial}{\partial x}, \qquad H \;\mapsto\; x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y}.$$

Show that V thus becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$.

(b) Let $V_m \subset V$ be the subspace of homogeneous polynomials of degree $m \in \mathbb{Z}_{\geq 0}$, i.e., the linear span of monomials $x^i y^j$ with $i + j = m$. Show that $V_m \subset V$ is a finite-dimensional subrepresentation of the representation in part (a).

(c) Show that the finite dimensional subrepresentation $V_m \subset V$ in part (b) is irreducible. Conclude that V_m must be isomorphic to an irreducible an irreducible highest weight representation $L(\lambda)$ with some highest weight λ, and find the actual value of λ.

✍ **Hand-in 4.** Consider the tensor product $L(\lambda_1) \otimes L(\lambda_2)$ of two irreducible highest weight representations of $\mathfrak{sl}_2(\mathbb{C})$.

Recall: The tensor product of representations of a Lie algebra is defined as in Exercise 2(c).

(a) Find the multiplicities of all H-eigenvalues in $L(\lambda_1) \otimes L(\lambda_2)$.

(b) Admitting complete reducibility of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ and the classification of irreducible finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$, deduce from part (a) the multiplicities of all irreducible representations in the decomposition of $L(\lambda_1) \otimes L(\lambda_2)$ into a direct sum of irreducible subrepresentations.

(c) Find explicit formulas (in terms of the bases of $L(\lambda_1)$ and $L(\lambda_2)$) for all vectors $v \in L(\lambda_1) \otimes L(\lambda_2)$ which satisfy $E.v = 0$ and $H.v = \lambda\, v$ for some λ.

✍ **Hand-in 5.** The purpose of this exercise is to study the roots of $\mathfrak{sl}_n(\mathbb{C})$.

We denote by $\mathfrak{h} \subset \mathfrak{sl}_n(\mathbb{C})$ the Lie subalgebra consisting of diagonal matrices. By definition, a *root* of $\mathfrak{sl}_n(\mathbb{C})$ is a nonzero element $\alpha \in \mathfrak{h}^* \setminus \{0\}$ in the dual of $\mathfrak{h}$ such that the corresponding weight space in the adjoint representation of $\mathfrak{sl}_n(\mathbb{C})$

$$\big(\mathfrak{sl}_n(\mathbb{C})\big)_\alpha = \Big\{ X \in \mathfrak{sl}_n(\mathbb{C}) \;\Big|\; \forall H \in \mathfrak{h} : [H, X] = \alpha(H) X \Big\}$$

is non-zero, i.e., $(\mathfrak{sl}_n(\mathbb{C}))_\alpha \neq \{0\}$. The set of all roots is denoted by $\Phi \subset \mathfrak{h}^*$. The *root lattice* $\Lambda_{\mathrm{R}} \subset \mathfrak{h}^*$ is defined as the set of integer linear combinations of roots $\alpha \in \Phi$.

(a) Find all the roots of $\mathfrak{sl}_n(\mathbb{C})$.

(b) For $i \in \{1, \ldots, n\}$, denote by η^i the functional $\eta^i \in \mathfrak{h}^*$ picking the ith diagonal entry: $\eta^i(H) = H_{ii}$. Let $r_1 > r_2 > \cdots > r_n$ be chosen such that $\sum_{i=1}^n r_i = 0$ and that the functional ℓ on $\mathfrak{h}^*$ (i.e., $\ell \in (\mathfrak{h}^*)^*$) determined by the condition $\ell(\eta^i) = r_i$ does not vanish on any nonzero element of the root lattice: $\ell(\mu) \neq 0$ for all $\mu \in \Lambda_{\mathrm{R}} \setminus \{0\}$. Find the set $\Phi^+ = \big\{ \alpha \in \Phi \;\big|\; \ell(\alpha) > 0 \big\}$ of positive roots. Find a subset $\Delta \subset \Phi^+$ of $n-1$ positive roots such that any $\alpha \in \Phi^+$ is a nonnegative integer linear combination of elements of Δ.

(c) Suppose that $\vartheta \colon \mathfrak{sl}_n(\mathbb{C}) \to \mathrm{End}(V)$ is a representation of $\mathfrak{sl}_n(\mathbb{C})$. The *weights* of V are those $\mu \in \mathfrak{h}^*$ such that the weight space

$$V_\mu = \big\{ v \in V \;\big|\; \forall H \in \mathfrak{h} : \vartheta(H) v = \mu(H) v \big\}$$

is non-zero, i.e., $V_\mu \neq \{0\}$. Show that if V is irreducible and finite-dimensional, then for any two weights μ and μ' of V we have $\mu' - \mu \in \Lambda_{\mathrm{R}}$.