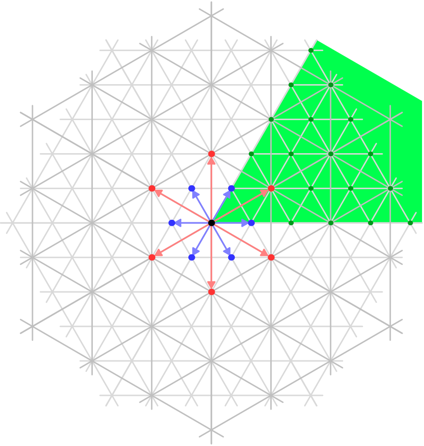
Aalto University
School of Science
and Technology

Lie groups and Lie algebras (MS-E1200)

Kalle Kytölä

Department of Mathematics and Systems Analysis
Aalto University
email: kalle.kytola@aalto.fi

February 23, 2026



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February 23 – April 15, 2026
(period III)

Zulip chat:

`https://lie2026.zulip.aalto.fi/`

Course practicalities

MyCourses web page:

- `https://mycourses.aalto.fi/course/view.php?id=47241`

Prerequisites:

- linear algebra
- abstract algebra
- calculus (+ε of topology)

Lectures: Kalle Kytölä

- Mon 12-14 (hall R2), Wed 12-14 (hall R2)

Exercises: Osama Abuzaid

- Thu 14-16 (Y228b)

(i) exercises (present solutions in exercise class)
(ii) hand-ins (written solutions due by Monday at 12)

Credits: 5 cr

Material:

- lecture notes
- textbooks

Grading:

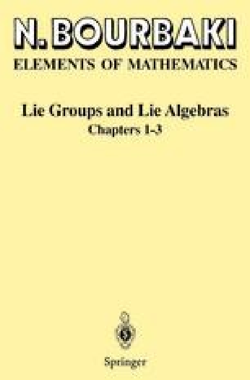
- exam 5 problems à 6 points (course total 30pts)
- up to 6 bonus points for exercise solutions (bonus bonus for corrections)

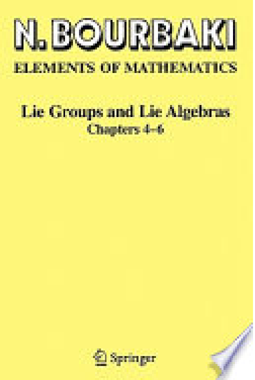
Exam dates:

- April 15 (or June 4)

Your feedback:

- any time!
- feedback questionnaire


N. BOURBAKI
ELEMENTS OF MATHEMATICS
Lie Groups and Lie Algebras
Chapters 1-3
Springer


N. BOURBAKI
ELEMENTS OF MATHEMATICS
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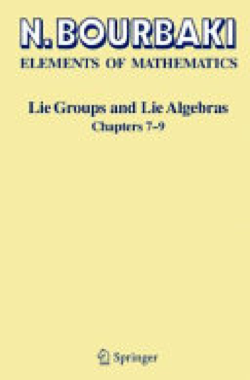
Chapters 4-6
Springer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Chapters 7-9
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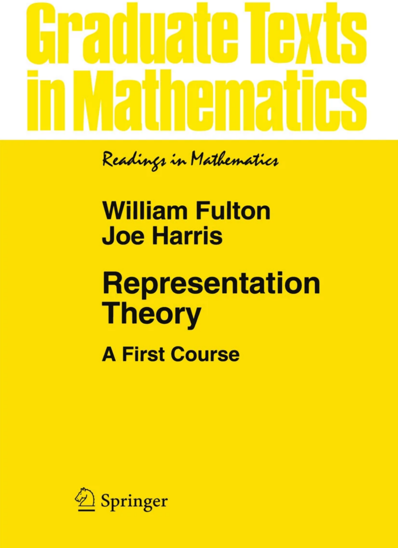
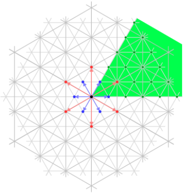

Graduate Texts in Mathematics
Readings in Mathematics
William Fulton
Joe Harris
Representation Theory
A First Course
Springer

An introduction to

Lie groups and Lie algebras

and representation theory



Kalle Kytölä

February 22, 2026

Aalto University
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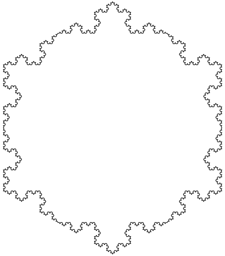
Lie groups & Lie algebras (MS-E1200)
period III: February 23 – April 15, 2026
Aalto University, Dept. Mathematics and Systems Analysis

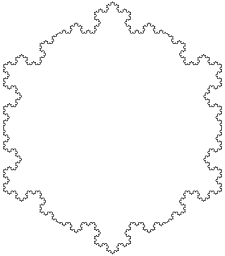
Course topic in a nutshell: symmetry

Symmetries



Course topic in a nutshell: symmetry

Symmetries (of an object)



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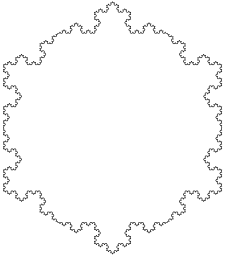
Symmetries (of an object)

$\approx$ collection of transformations (usually a group) leaving some property of interest unchanged (invariant)



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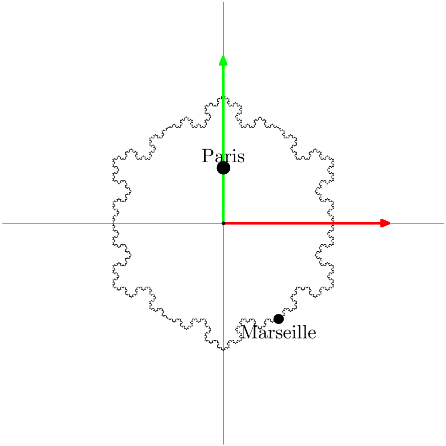

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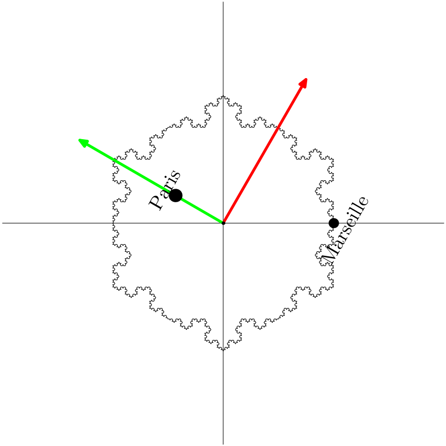

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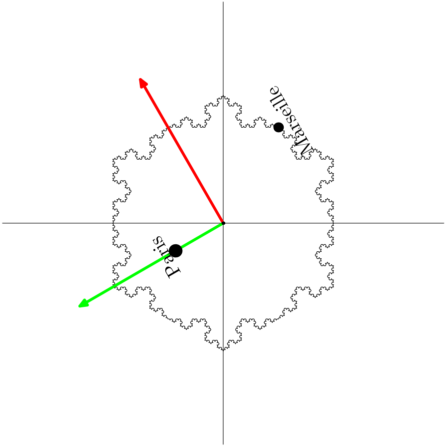

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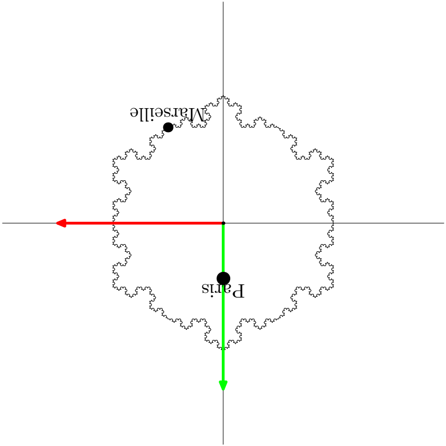

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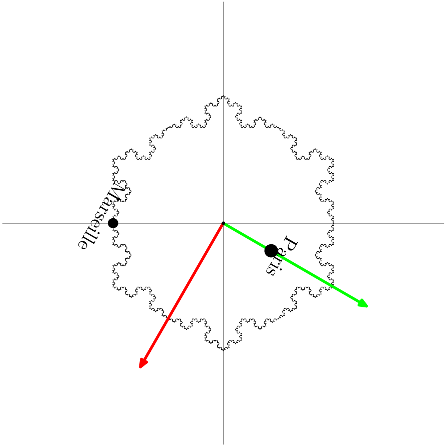

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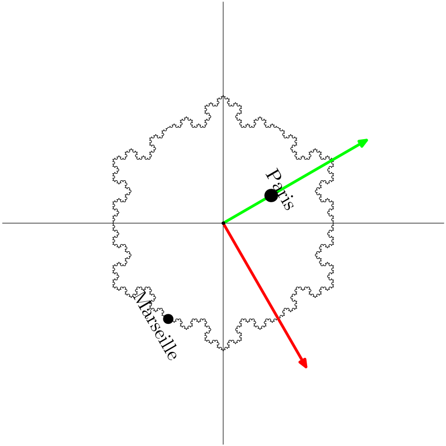

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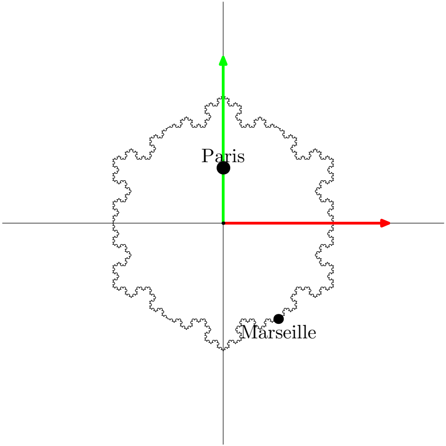

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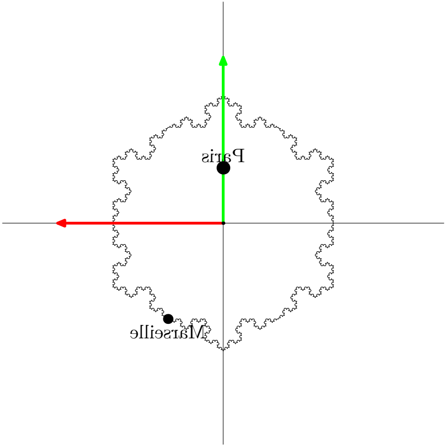

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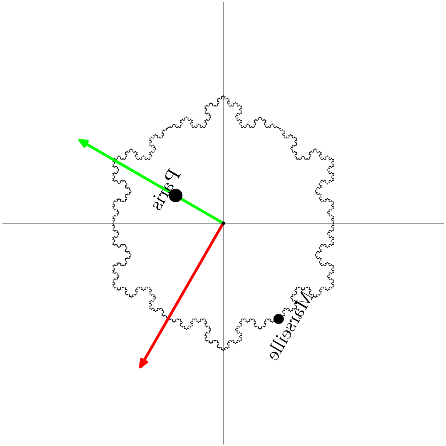

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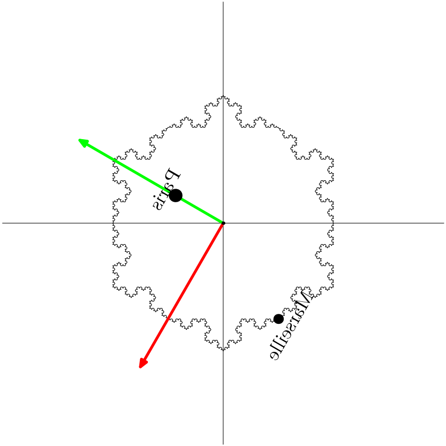
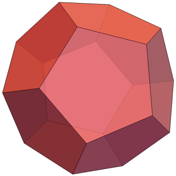


Types of symmetries

- discrete symmetries
  $\rightsquigarrow$ (typically) finite groups
- continuous symmetries
  $\rightsquigarrow$ (typically) Lie groups

Example 1: Dodecahedron



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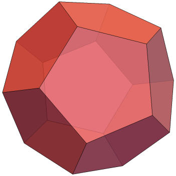
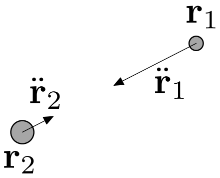


symmetry group $\cong \mathfrak{A}_5 \rtimes (\mathbb{Z}/2\mathbb{Z})$

Example 2: Newton's law of gravitation

$$\begin{cases} \ddot{\mathbf{r}}_1 = m_2 G \frac{\mathbf{r}_2 - \mathbf{r}_1}{\|\mathbf{r}_2 - \mathbf{r}_1\|^3} \\ \ddot{\mathbf{r}}_2 = m_1 G \frac{\mathbf{r}_1 - \mathbf{r}_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|^3}, \end{cases}$$



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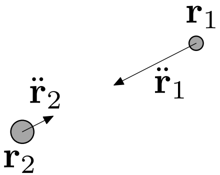
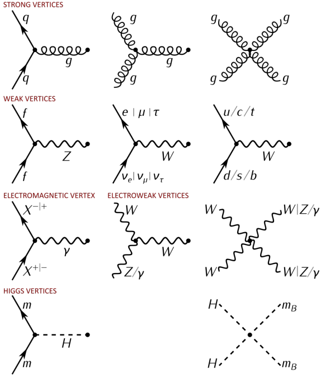


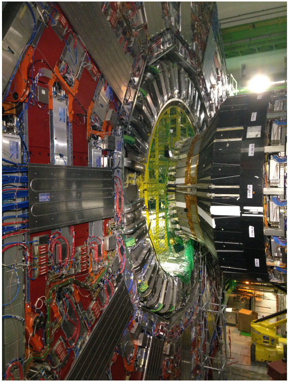
$$\text{symmetry group} \;\cong\; \mathbb{R} \times \left(\mathbb{R}^3 \rtimes \mathrm{SO}(3)\right)$$

Example 3: Standard model of particle physics

[Ponor, Wikipedia, CC BY-SA 4.0]



[Tighef, Wikipedia, CC BY-SA 3.0]



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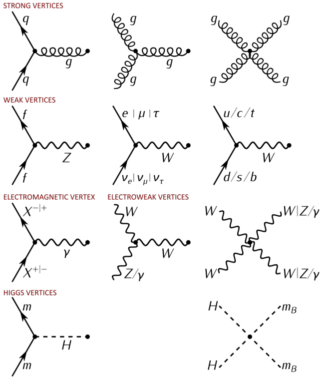

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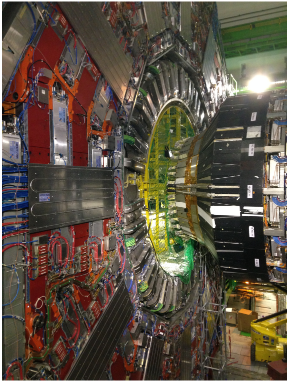


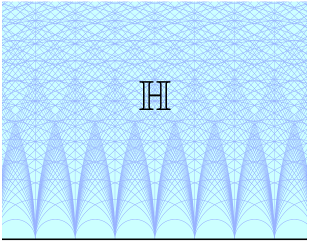
$$\text{internal gauge symmetry group} \;\cong\; \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

$$\text{global relativistic symmetry group} \;\cong\; \mathbb{R}^4 \rtimes \mathrm{O}(1,3)$$

Example 4: Hyperbolic upper half plane

$$\mathbb{H} := \left\{ z \in \mathbb{C} \;\middle|\; \mathfrak{Im}(z) > 0 \right\}$$

length of $\gamma : [0,1] \to \mathbb{H}$ $\qquad \ell(\gamma) := \displaystyle\int_0^1 \frac{|\dot{\gamma}(t)|}{\mathfrak{Im}(\gamma(t))}\,\mathrm{d}t.$



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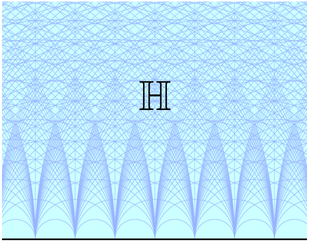
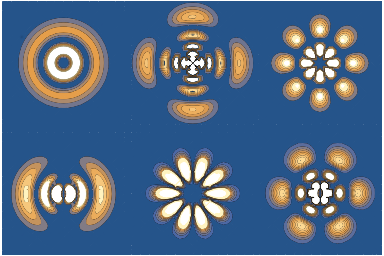


symmetry group $\cong$ $\mathrm{PSL}(2,\mathbb{R})$

A key theme: representation theory

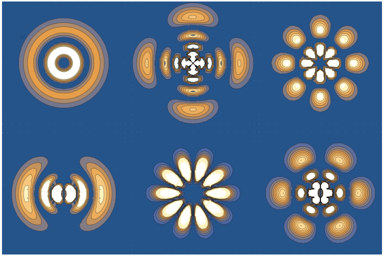
Representation theory



A key theme: representation theory

Representation theory

= the case when the symmetries act linearly on a vector space



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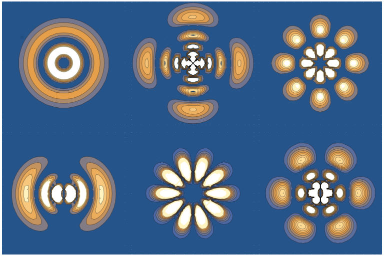
Why?!?



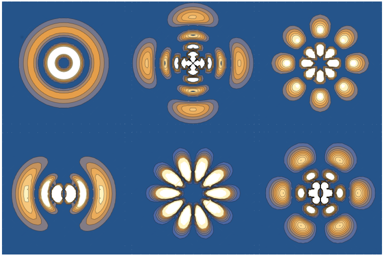
A key theme: representation theory

Representation theory

= the case when the symmetries act linearly on a vector space

Why?!?

- concrete realization of the symmetry



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Representation theory

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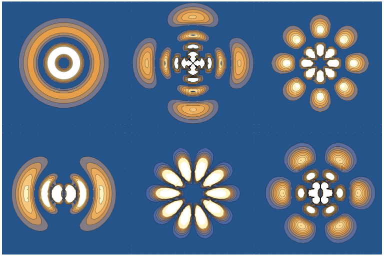
Why?!?

- concrete realization of the symmetry
- often natural
  (quantum mechanics, Markov processes, function spaces, . . .)



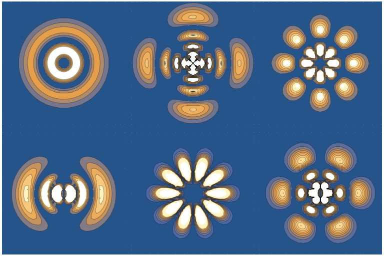
A key theme: representation theory

Representation theory

= the case when the symmetries act linearly on a vector space

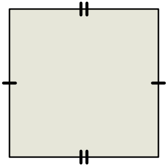
Why?!?

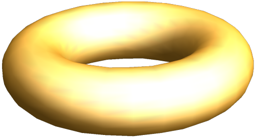
- concrete realization of the symmetry
- often natural
  (quantum mechanics, Markov processes, function spaces, . . .)
- powerful theory with many applications!



Idea: Torus & Fourier series

$$\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$$





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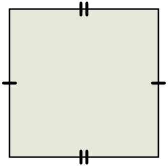

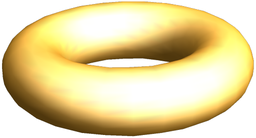

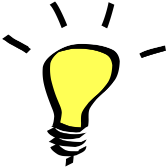

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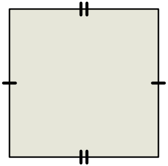

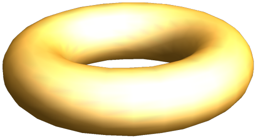

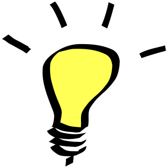

Fourier series: Decomposition of the representation of translational symmetries on a space of functions $f\colon \mathbb{T}^2 \to \mathbb{C}$.

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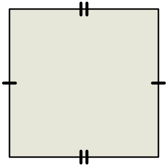
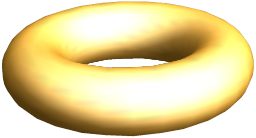
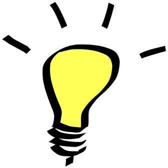






Fourier series: Decomposition of the representation of translational symmetries on a space of functions $f \colon \mathbb{T}^2 \to \mathbb{C}$.

Representation theory: Generalization of Fourier analysis to non-commutative symmetries!

Lie groups and representation theory used in…

- Group theory
  - Burnside's theorem
  - Monster simple group
- Number theory
  - Dirichlet's theorem
  - Langlands program
- Geometry and topology
  - Symmetric spaces and homogeneous spaces
  - Singularities of Kleinian surfaces
  - principal fiber bundles
  - knot theory
- Analysis
  - differential equations
- Physics
  - Gauge theories, e.g. the standard model of particle physics
  - Quantum mechanics
  - Classical physics: Noether's theorem about symmetries and conservation laws
- Chemistry
  - Molecule spectra
  - Crystallographic groups
- …

Structure of the course ($\approx$ learning outcomes)

- **Representations of finite groups** ($\approx$ 1.5 weeks)

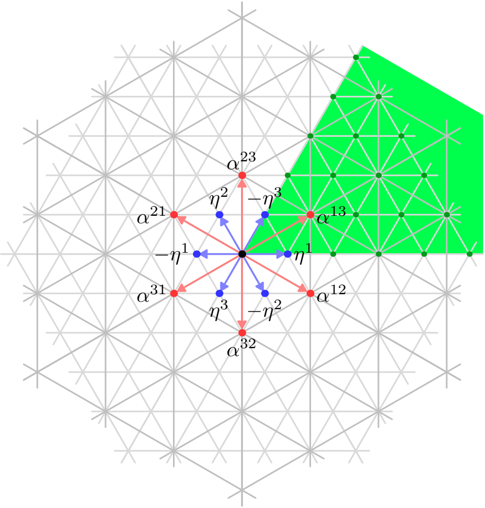
  $\mathfrak{S}_n, \ldots$
  - decomposition to irreducible subrepresentations
  - character theory

- **Lie groups and their Lie algebras**

  $\mathrm{GL}_n(\mathbb{R}), \quad \mathrm{SL}_n(\mathbb{R}), \quad \mathrm{SO}_n, \quad \mathrm{SU}_n, \ldots$
  - infinitesimal calculus on a Lie group ($\rightsquigarrow$ Lie algebra)
  - global topological properties of a Lie group

- **Representations of Lie algebras (and Lie groups)**

  $\mathfrak{sl}_n(\mathbb{C}), \quad \mathfrak{so}_n(\mathbb{C}), \quad \mathfrak{sp}_n(\mathbb{C}), \quad \mathfrak{g}_2, \quad \mathfrak{f}_4, \quad \mathfrak{e}_6, \quad \mathfrak{e}_7, \quad \mathfrak{e}_8, \ldots$
  - complex semisimple Lie algebras
  - …and their (irreducible) highest weight representations


α^23
η^2
−η^3
α^21
α^13
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η^1
α^31
α^12
η^3
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α^32