

Exercise session: Wed 23.3. at 14-16 Hand-in due: Mon 28.3. at 10am

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.

Exercise 1. Let \mathfrak{g} be a Lie algebra over a field \mathbb{k} . A *Lie subalgebra* of \mathfrak{g} is a vector subspace $\mathfrak{s} \subset \mathfrak{g}$ such that $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}$.¹ A (Lie algebra) *ideal* of \mathfrak{g} is a vector subspace $\mathfrak{j} \subset \mathfrak{g}$ such that $[\mathfrak{j}, \mathfrak{g}] \subset \mathfrak{j}$.

- (a) Show that if $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then the quotient vector space $\mathfrak{g}/\mathfrak{i}$ becomes a Lie algebra by defining a bracket $[X + \mathfrak{i}, Y + \mathfrak{i}]_{\mathfrak{g}/\mathfrak{i}} = [X, Y] + \mathfrak{i}$.
- (b) Show that if $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$ is a homomorphism of Lie algebras, then $\text{Im}(\phi) \subset \mathfrak{h}$ is a Lie subalgebra, and $\text{Ker}(\phi) \subset \mathfrak{g}$ is an ideal. Show that $\mathfrak{g}/\text{Ker}(\phi) \cong \text{Im}(\phi)$ as Lie algebras.

Assume now that G is a matrix Lie group and $G' \subset G$ is a closed subgroup (i.e., $G' \subset G \subset \text{GL}_n(\mathbb{R})$ where each inclusion is as a closed subgroup). Denote by $\mathfrak{g} = \mathcal{L}(G)$ and $\mathfrak{g}' = \mathcal{L}(G')$ the Lie algebras of G and G' (here $\mathbb{k} = \mathbb{R}$).

- (c) Show that $\mathfrak{g}' \subset \mathfrak{g}$ is a Lie subalgebra.
- (d) Show that if $G' \trianglelefteq G$ is a normal subgroup, then $\mathfrak{g}' \subset \mathfrak{g}$ is a Lie algebra ideal.

Hint: Recall Exercise 1 of Problem set 3.

Exercise 2. Consider the four dimensional space \mathbb{R}^4 equipped with the Minkowski metric: the bilinear form $B: \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}$ of signature $(3, 1)$ given explicitly by

$$B\left((t, x, y, z), (t', x', y', z')\right) = tt' - xx' - yy' - zz'.$$

Let $O_{3,1}$ be the Lorentz group, consisting of those linear maps $M: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ which preserve the Minkowski metric in the sense that $B(v, v') = B(Mv, Mv')$ for all $v, v' \in \mathbb{R}^4$. Let $\text{SL}_2(\mathbb{C})$ be the set of 2×2 complex matrices with unit determinant.


- (a) Encode the point $v = (t, x, y, z) \in \mathbb{R}^4$ in the 2×2 Hermitian matrix


$$X = \begin{bmatrix} t + z & x - iy \\ x + iy & t - z \end{bmatrix}.$$

Show the points $v \in \mathbb{R}^4$ thus correspond bijectively to Hermitian matrices X , and show that under this correspondence we have $B(v, v) = \det(X)$.

- (b) Define an action of $\text{SL}_2(\mathbb{C})$ on \mathbb{R}^4 as follows. A matrix $A \in \text{SL}_2(\mathbb{C})$ acts on a Hermitian 2×2 -matrix X by conjugation $X \mapsto AXA^\dagger$, and the action on \mathbb{R}^4 is obtained using the identification given in part (a). Check that this indeed defines an action $\alpha: \text{SL}_2(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{R}^4)$, and show that the image consists of Lorentz transformations, $\mathfrak{Im}(\alpha) \subset O_{3,1} \subset \text{Aut}(\mathbb{R}^4)$.
- (c) Describe the (real) Lie algebra \mathfrak{g} of the (real) Lie group $\text{SL}_2(\mathbb{C})$, and the Lie algebra $\mathfrak{o}_{3,1}$ of $O_{3,1}$. Show in particular that their dimensions are equal.
- (d) Show that for $\alpha: \text{SL}_2(\mathbb{C}) \rightarrow O_{3,1}$ as in part (b), the derivative at the neutral element $d\alpha|_e: \mathfrak{g} \rightarrow \mathfrak{o}_{3,1}$ is an isomorphism.

¹For $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$, we denote by $[\mathfrak{h}_1, \mathfrak{h}_2]$ the linear span of $[X_1, X_2]$ with $X_1 \in \mathfrak{h}_1$ and $X_2 \in \mathfrak{h}_2$.

 **Hand-in 3.** Let $J \in \mathbb{R}^{2n \times 2n}$ be an antisymmetric matrix such that $J^2 = -\mathbb{I}$. The real symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{R})$ consists of those matrices $X \in \mathbb{R}^{2n \times 2n}$ which satisfy $X^\top J + JX = 0$. Compute the dimension $\dim(\mathfrak{sp}_{2n}(\mathbb{R}))$ of this Lie algebra.

 **Hand-in 4.** Let $C^\infty(\mathbb{R}^3)$ denote the space of smooth complex valued functions on \mathbb{R}^3 , and on this space, consider the differential operators $\mathcal{J}_x, \mathcal{J}_y, \mathcal{J}_z$ given by $\mathcal{J}_x = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}$, $\mathcal{J}_y = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}$, and $\mathcal{J}_z = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$.


- (a) By direct calculation, show that the commutators of the above differential operators are $[\mathcal{J}_x, \mathcal{J}_y] = \mathcal{J}_z$, $[\mathcal{J}_y, \mathcal{J}_z] = \mathcal{J}_x$, $[\mathcal{J}_z, \mathcal{J}_x] = \mathcal{J}_y$.
- (b) For $M \in \text{SO}_3$ and $f \in C^\infty(\mathbb{R}^3)$, define $M.f \in C^\infty(\mathbb{R}^3)$ by $(M.f)(\vec{x}) = f(M^{-1}\vec{x})$. Show that $C^\infty(\mathbb{R}^3)$ thus becomes a representation of the group SO_3 .
- (c) Let

$$M_x^{(\theta)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix},$$

be the rotation by angle θ around the x axis in the positive direction, and let $M_y^{(\theta)}$ and $M_z^{(\theta)}$ be the rotations by θ around y and z -axes, respectively. Show that for any $f \in C^\infty(\mathbb{R}^3)$ we have

$$\left. \frac{d}{d\theta} \right|_{\theta=0} (M_x^{(\theta)}.f) = \mathcal{J}_x f,$$

and perform similar calculations for the actions of $M_y^{(\theta)}$ and $M_z^{(\theta)}$.

 **Hand-in 5.** Suppose that V is a complex vector space, and $\mathcal{E}, \mathcal{F}, \mathcal{H}$ are linear operators $V \rightarrow V$ which satisfy the commutation relations

$$\mathcal{H}\mathcal{E} - \mathcal{E}\mathcal{H} = 2\mathcal{E}, \quad \mathcal{H}\mathcal{F} - \mathcal{F}\mathcal{H} = -2\mathcal{F}, \quad \mathcal{E}\mathcal{F} - \mathcal{F}\mathcal{E} = \mathcal{H}.$$

- (a) Show that for any $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{E}\mathcal{F}^k = \mathcal{F}^k\mathcal{E} + k\mathcal{F}^{k-1}(\mathcal{H} - k + 1).$$

- (b) Show that for any $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{E}^k\mathcal{F}^k = k! \mathcal{H}(\mathcal{H} - 1) \cdots (\mathcal{H} - (k - 1)) + \mathcal{P}\mathcal{E},$$

where \mathcal{P} is some operator $V \rightarrow V$ (depending on k) that can be written as a polynomial in the operators $\mathcal{E}, \mathcal{F}, \mathcal{H}$.