Aalto University **Problem set 1**
Department of Mathematics and Systems Analysis 2025-2026/IV
MS-E1200 - Lie groups and Lie algebras K Kytölä & O Abuzaid

Exercise session: Thu 26.2. at 14-16 Hand-in due: Mon 2.3. at 12:00

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.

Exercise 1. Make sure that you understand how to interpret the group in part (a) below as the group of rotations of $\mathbb{R}^3$, and the group in part (b) below as the group of (orientation preserving) symmetries of a cube.

(a) Show that the set
$$\mathrm{SO}_3 = \left\{ M \in \mathbb{R}^{3\times 3} \;\middle|\; M^\top M = \mathbb{I}_3,\ \det(M) = 1 \right\}$$
of orthogonal matrices with determinant one is a group, with matrix multiplication as the group operation.

(b) Let e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$. Show the subset $G \subset \mathrm{SO}_3$ consisting of those $M \in \mathrm{SO}_3$ which map the set $F = \{e_1, -e_1, e_2, -e_2, e_3, -e_3\}$ to itself is a finite subgroup of SO_3 of order $|G| = 24$.

✍ **Hand-in 2.** Let G be a group. Two elements $g_1, g_2 \in G$ are said to be conjugates if there exists an element $h \in G$ such that $g_2 = h\, g_1\, h^{-1}$. Being conjugate is an equivalence relation, and the equivalence classes are called conjugacy classes.

(a) Find the conjugacy classes in the symmetric group on three letters, $\mathfrak{S}_3 = \{\sigma \colon \{1,2,3\} \to \{1,2,3\} \text{ bijection}\}$.

Hint: Recall that if $g_1, g_2 \in G$ are conjugate elements, then e.g. their orders are equal.

(b) Find the conjugacy classes in the group of orientation preserving symmetries of a cube (see Exercise 1(b)).

Hint: Note that if $M_1, M_2 \in \mathbb{C}^{n\times n}$ are conjugate matrices, then e.g. their eigenvalues coincide.

✍ **Hand-in 3.** As in Exercise 1, let $F = \{e_1, -e_1, e_2, -e_2, e_3, -e_3\}$ (think of the set of faces of a cubic dice), and let G be the group of orientation preserving symmetries of the cube. Let V be the space of complex valued functions on F, i.e.
$$V = \mathbb{C}^F = \{\phi \colon F \to \mathbb{C}\}.$$
Equip V with a representation ρ of G as follows: for a function $\phi \colon F \to \mathbb{C}$ and $g \in G$, define $(\rho(g).\phi)(u) = \phi(g^{-1}u)$ for all $u \in F$. Find at least two examples of nontrivial subspaces $V' \subset V$ which are invariant for this action, i.e., $\rho(g).V' \subset V'$ for all $g \in G$ (the trivial cases are the zero subspace and the entire space V).

Hint: To find invariant subspaces, you may want to take some sufficiently symmetric looking functions $\phi \colon F \to \mathbb{C}$, and see what is the subspace spanned by all $\rho(g).\phi$ with $g \in G$.

Exercise 4. The purpose of this exercise is to give one explicit construction of the tensor product $V \otimes W$ of vector spaces V and W, under the simplifying assumption that V and W are function spaces (this can indeed be assumed without loss of generality). Let $\Bbbk$ be the ground field (in this course always $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$).

For any set X, denote by $\Bbbk^X$ the vector space of $\Bbbk$-valued functions on X, with addition and scalar multiplication defined pointwise. Assume that $V \subset \Bbbk^X$ and $W \subset \Bbbk^Y$ for some sets X and Y. For $f \in \Bbbk^X$ and $g \in \Bbbk^Y$, define $f \otimes g \in \Bbbk^{X \times Y}$ by

$$\big(f \otimes g\big)(x, y) \;=\; f(x)\, g(y).$$

Also set

$$V \otimes W \;=\; \mathrm{span}\big\{ f \otimes g \;\big|\; f \in V,\; g \in W \big\},$$

so that the map $(f, g) \mapsto f \otimes g$ is a bilinear map $V \times W \to V \otimes W$.

(a) Show that if $(f_i)_{i \in I}$ is a linearly independent collection in V and $(g_j)_{j \in J}$ is a linearly independent collection in W, then the collection $(f_i \otimes g_j)_{(i,j) \in I \times J}$ is linearly independent in $V \otimes W$.

(b) Show that if $(f_i)_{i \in I}$ is a collection that spans V and $(g_j)_{j \in J}$ is collection that spans W, then the collection $(f_i \otimes g_j)_{(i,j) \in I \times J}$ spans $V \otimes W$.

(c) Conclude that if $(f_i)_{i \in I}$ is a basis of V and $(g_j)_{j \in J}$ is a basis of W, then $(f_i \otimes g_j)_{(i,j) \in I \times J}$ is a basis of $V \otimes W$, and for finite dimensional V, W one in particular has $\dim(V \otimes W) = \dim(V)\dim(W)$.

(d) Define a bilinear map $\phi \colon V \times W \to V \otimes W$ by $\phi(f, g) = f \otimes g$. Show that if $\beta \colon V \times W \to U$ is a bilinear map to any vector space U, then there exists a unique linear map $\bar{\beta} \colon V \otimes W \to U$ such that the following diagram commutes

$$\begin{tikzcd} V \times W \arrow[rr, "\beta"] \arrow[dr, "\phi"'] & & U \\ & V \otimes W \arrow[ur, dotted, "\bar{\beta}"'] & \end{tikzcd},$$

i.e., $\beta = \bar{\beta} \circ \phi$. (This is called the universal property of the tensor product.)

Exercise 5. Let V and W be two vector spaces, and denote the dual of V by

$$V^* = \mathrm{Hom}(V, \Bbbk) = \{\varphi \colon V \to \Bbbk \text{ linear}\}.$$

The purpose of this exercise is to work out the relation between the tensor product space $W \otimes V^*$, and the space $\mathrm{Hom}(V, W)$ of linear maps from V to W.

(a) For $w \in W$ and $\varphi \in V^*$, we associate to $w \otimes \varphi$ the following map $V \to W$

$$v \;\mapsto\; \langle \varphi, v \rangle\, w.$$

Show that the linear extension of this defines an injective linear map

$$W \otimes V^* \;\longrightarrow\; \mathrm{Hom}(V, W).$$

Show that if both V and W are finite dimensional, then this in fact gives an isomorphism $W \otimes V^* \;\cong\; \mathrm{Hom}(V, W)$.

In general, the rank $r = \mathrm{rank}(t)$ of a tensor $t \in V \otimes W$ is the smallest number $r \in \mathbb{Z}_{\geq 0}$ such that for some $v_1, \ldots, v_r \in V$ and $w_1, \ldots, w_r \in W$ we can write $t = \sum_{i=1}^{r} v_i \otimes w_i$.

(b) Show that, in general, $\mathrm{rank}(t) \leq \min\big(\dim(V), \dim(W)\big)$. Then return to the case in part (a): assume that V and W are finite-dimensional, and identify $W \otimes V^* \cong \mathrm{Hom}(V, W)$. Show that under this identification, the rank of a tensor $t \in W \otimes V^*$ is the same as the rank of a matrix of the corresponding linear map $T \in \mathrm{Hom}(V, W)$.