


Exercise session: Wed 2.3 at 14-16

Hand-in due: Mon 7.3 at 10am

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.


 **Hand-in 1.** Make sure that you understand how to interpret the group in part (a) below as the group of rotations of \mathbb{R}^3 , and the group in part (b) below as the group of (orientation preserving) symmetries of a cube.

(a) Show that the set

$$\mathrm{SO}_3 = \{M \in \mathbb{R}^{3 \times 3} \mid M^\top M = \mathbb{I}_3, \det(M) = 1\}$$

of orthogonal matrices with determinant one is a group, with matrix multiplication as the group operation.

(b) Let e_1, e_2, e_3 denote the standard basis of \mathbb{R}^3 . Show the subset $G \subset \mathrm{SO}_3$ consisting of those $M \in \mathrm{SO}_3$ which map the set $F = \{e_1, -e_1, e_2, -e_2, e_3, -e_3\}$ to itself is a finite subgroup of SO_3 of order $|G| = 24$.


 **Hand-in 2.** Let G be a group. Two elements $g_1, g_2 \in G$ are said to be conjugates if there exists an element $h \in G$ such that $g_2 = h g_1 h^{-1}$. Being conjugate is an equivalence relation, and the equivalence classes are called conjugacy classes.

(a) Find the conjugacy classes in the symmetric group on three letters, $\mathfrak{S}_3 = \{\sigma: \{1, 2, 3\} \rightarrow \{1, 2, 3\} \text{ bijection}\}$.

Hint: Recall that if $g_1, g_2 \in G$ are conjugate elements, then e.g. their orders are equal.

(b) Find the conjugacy classes in the group of orientation preserving symmetries of a cube (see Exercise 1(b)).

Hint: Note that if $M_1, M_2 \in \mathbb{C}^{n \times n}$ are conjugate matrices, then e.g. their eigenvalues coincide.

 **Hand-in 3.** As in Exercise 1, let $F = \{e_1, -e_1, e_2, -e_2, e_3, -e_3\}$ (think of the set of faces of a cubic dice), and let G be the group of orientation preserving symmetries of the cube. Let V be the space of complex valued functions on F , i.e.

$$V = \mathbb{C}^F = \{\phi: F \rightarrow \mathbb{C}\}.$$

Equip V with a representation ρ of G as follows: for a function $\phi: F \rightarrow \mathbb{C}$ and $g \in G$, define $(\rho(g).\phi)(u) = \phi(g^{-1}u)$ for all $u \in F$. Find at least two examples of nontrivial subspaces $V' \subset V$ which are invariant for this action, i.e., $\rho(g).V' \subset V'$ for all $g \in G$ (the trivial cases are the zero subspace and the entire space V).

Hint: To find invariant subspaces, you may want to take some sufficiently symmetric looking functions $\phi: F \rightarrow \mathbb{C}$, and see what is the subspace spanned by all $\rho(g).\phi$ with $g \in G$.

Exercise 4. The purpose of this exercise is to give one explicit construction of the tensor product $V \otimes W$ of vector spaces V and W , under the simplifying assumption

that V and W are function spaces (this can indeed be assumed without loss of generality). Let \mathbb{k} be the ground field (in this course always $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$).

For any set X , denote by \mathbb{k}^X the vector space of \mathbb{k} -valued functions on X , with addition and scalar multiplication defined pointwise. Assume that $V \subset \mathbb{k}^X$ and $W \subset \mathbb{k}^Y$ for some sets X and Y . For $f \in \mathbb{k}^X$ and $g \in \mathbb{k}^Y$, define $f \otimes g \in \mathbb{k}^{X \times Y}$ by

$$(f \otimes g)(x, y) = f(x)g(y).$$

Also set

$$V \otimes W = \text{span} \{f \otimes g \mid f \in V, g \in W\},$$

so that the map $(f, g) \mapsto f \otimes g$ is a bilinear map $V \times W \rightarrow V \otimes W$.

- Show that if $(f_i)_{i \in I}$ is a linearly independent collection in V and $(g_j)_{j \in J}$ is a linearly independent collection in W , then the collection $(f_i \otimes g_j)_{(i,j) \in I \times J}$ is linearly independent in $V \otimes W$.
- Show that if $(f_i)_{i \in I}$ is a collection that spans V and $(g_j)_{j \in J}$ is a collection that spans W , then the collection $(f_i \otimes g_j)_{(i,j) \in I \times J}$ spans $V \otimes W$.
- Conclude that if $(f_i)_{i \in I}$ is a basis of V and $(g_j)_{j \in J}$ is a basis of W , then $(f_i \otimes g_j)_{(i,j) \in I \times J}$ is a basis of $V \otimes W$, and for finite dimensional V, W one in particular has $\dim(V \otimes W) = \dim(V) \dim(W)$.
- Define a bilinear map $\phi: V \times W \rightarrow V \otimes W$ by $\phi(f, g) = f \otimes g$. Show that if $\beta: V \times W \rightarrow U$ is a bilinear map to any vector space U , then there exists a unique linear map $\bar{\beta}: V \otimes W \rightarrow U$ such that the following diagram commutes

$$\begin{array}{ccc} V \times W & \xrightarrow{\beta} & U \\ & \searrow \phi & \nearrow \bar{\beta} \\ & V \otimes W & \end{array}$$

i.e., $\beta = \bar{\beta} \circ \phi$. (This is called the universal property of the tensor product.)

Exercise 5. Let V and W be two vector spaces, and denote the dual of V by

$$V^* = \text{Hom}(V, \mathbb{k}) = \{\varphi: V \rightarrow \mathbb{k} \text{ linear}\}.$$

The purpose of this exercise is to work out the relation between the tensor product space $W \otimes V^*$, and the space $\text{Hom}(V, W)$ of linear maps from V to W .

- For $w \in W$ and $\varphi \in V^*$, we associate to $w \otimes \varphi$ the following map $V \rightarrow W$

$$v \mapsto \langle \varphi, v \rangle w.$$

Show that the linear extension of this defines an injective linear map

$$W \otimes V^* \longrightarrow \text{Hom}(V, W).$$

Show that if both V and W are finite dimensional, then this in fact gives an isomorphism $W \otimes V^* \cong \text{Hom}(V, W)$.

In general, the rank $r = \text{rank}(t)$ of a tensor $t \in V \otimes W$ is the smallest number $r \in \mathbb{Z}_{\geq 0}$ such that for some $v_1, \dots, v_r \in V$ and $w_1, \dots, w_r \in W$ we can write $t = \sum_{i=1}^r v_i \otimes w_i$.

- Show that, in general, $\text{rank}(t) \leq \min(\dim(V), \dim(W))$. Then return to the case in part (a): assume that V and W are finite-dimensional, and identify $W \otimes V^* \cong \text{Hom}(V, W)$. Show that under this identification, the rank of a tensor $t \in W \otimes V^*$ is the same as the rank of a matrix of the corresponding linear map $T \in \text{Hom}(V, W)$.