

Exercise session: Wed 27.3. at 14-16 Hand-in due: Mon 8.4. at 10am

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.

Recall: The basis E, F, H of the complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is as in the lectures, and $L(\lambda)$ denotes the finite-dimensional irreducible representation of highest weight $\lambda \in \mathbb{Z}_{\geq 0}$.

Exercise 1.

- Suppose that V is a complex vector space, and $\mathcal{E}, \mathcal{F}, \mathcal{H} \in \text{End}(V)$ satisfy $\mathcal{H}\mathcal{E} - \mathcal{E}\mathcal{H} = 2\mathcal{E}$, $\mathcal{H}\mathcal{F} - \mathcal{F}\mathcal{H} = -2\mathcal{F}$, $\mathcal{E}\mathcal{F} - \mathcal{F}\mathcal{E} = \mathcal{H}$. Show that the operator $\mathcal{Q} = \mathcal{E}\mathcal{F} + \mathcal{F}\mathcal{E} + \frac{1}{2}\mathcal{H}^2$ commutes with \mathcal{E} , \mathcal{F} , and \mathcal{H} .
- Let $\vartheta: \mathfrak{sl}_2(\mathbb{C}) \rightarrow \text{End}(V)$ be a finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Show that $\mathcal{Q} = \vartheta(E)\vartheta(F) + \vartheta(F)\vartheta(E) + \frac{1}{2}\vartheta(H)^2$ is a scalar multiple of the identity operator on V , that is $\mathcal{Q} = q \times \text{id}_V$ for some $q \in \mathbb{C}$.
- For each finite-dimensional irreducible representation $L(\lambda)$ of $\mathfrak{sl}_2(\mathbb{C})$, find the value of the scalar q in part (b).

Exercise 2. Let G be a matrix Lie group with Lie algebra \mathfrak{g} (over \mathbb{R}), and let $\rho_V: G \rightarrow \text{Aut}(V)$ and $\rho_W: G \rightarrow \text{Aut}(W)$ be representations of G (on \mathbb{R} -vector spaces V, W). Denote $\vartheta_V = (d\rho_V)|_e: \mathfrak{g} \rightarrow \text{End}(V)$, i.e.,

$$(\vartheta_V(X))v = \frac{d}{dt}\rho(\exp(tX))v|_{t=0} \quad \text{for } X \in \mathfrak{g}, v \in V,$$

and similarly for $\vartheta_W: \mathfrak{g} \rightarrow \text{End}(W)$. Consider the associated group representations ρ_{\oplus} , ρ_{\otimes} , and ρ_{Hom} of G on $V \oplus W$, $V \otimes W$ and $\text{Hom}(V, W)$, respectively.

- Calculate $(d\rho_{\oplus})|_e: \mathfrak{g} \rightarrow \text{End}(V \oplus W)$, $(d\rho_{\otimes})|_e: \mathfrak{g} \rightarrow \text{End}(V \otimes W)$, and $(d\rho_{\text{Hom}})|_e: \mathfrak{g} \rightarrow \text{End}(\text{Hom}(V, W))$, in terms of ϑ_V and ϑ_W .

Let \mathfrak{g} be a Lie algebra over \mathbb{k} , and let $\vartheta_1: \mathfrak{g} \rightarrow \text{End}(V_1)$ and $\vartheta_2: \mathfrak{g} \rightarrow \text{End}(V_2)$ be two representations of \mathfrak{g} on \mathbb{k} -vector spaces V_1 and V_2 , respectively.

- Show that the direct sum $V_1 \oplus V_2$ becomes a representation of \mathfrak{g} by setting $X.(v_1 + v_2) = \vartheta_1(X)v_1 + \vartheta_2(X)v_2$ for all $X \in \mathfrak{g}$, $v_1 \in V_1$ and $v_2 \in V_2$.
- Show that the tensor product $V_1 \otimes V_2$ becomes a representation of \mathfrak{g} by setting $X.(v_1 \otimes v_2) = \vartheta_1(X)v_1 \otimes v_2 + v_1 \otimes \vartheta_2(X)v_2$ for all $X \in \mathfrak{g}$, $v_1 \in V_1$ and $v_2 \in V_2$, and extending linearly.
- Show that the space of linear maps $\text{Hom}(V_1, V_2)$ becomes a representation of \mathfrak{g} by defining, for any $X \in \mathfrak{g}$ and linear map $T: V_1 \rightarrow V_2$,

$$X.T = \vartheta_2(X) \circ T - T \circ \vartheta_1(X).$$


 **Hand-in 3.** Let $V = \mathbb{C}[x, y]$ be the polynomial algebra in two indeterminates, x and y .

- (a) Define a linear map $\mathfrak{sl}_2(\mathbb{C}) \rightarrow \text{End}(V)$ by setting

$$E \mapsto x \frac{\partial}{\partial y}, \quad F \mapsto y \frac{\partial}{\partial x}, \quad H \mapsto x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$


Show that V thus becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$.

- (b) Let $V_m \subset V$ be the subspace of homogeneous polynomials of degree $m \in \mathbb{Z}_{\geq 0}$, i.e., the linear span of monomials $x^i y^j$ with $i + j = m$. Show that $V_m \subset V$ is a finite-dimensional subrepresentation of the representation in part (a).
(c) Show that the finite dimensional subrepresentation $V_m \subset V$ in part (b) is irreducible. Conclude that V_m must be isomorphic to an irreducible highest weight representation $L(\lambda)$ with some highest weight λ , and find the actual value of λ .

 **Hand-in 4.** Consider the tensor product $L(\lambda_1) \otimes L(\lambda_2)$ of two irreducible highest weight representations of $\mathfrak{sl}_2(\mathbb{C})$.

Recall: The tensor product of representations of a Lie algebra is defined as in Exercise 2(c).

- (a) Find the multiplicities of all H -eigenvalues in $L(\lambda_1) \otimes L(\lambda_2)$.
(b) Admitting complete reducibility of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ and the classification of irreducible finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$, deduce from part (a) the multiplicities of all irreducible representations in the decomposition of $L(\lambda_1) \otimes L(\lambda_2)$ into a direct sum of irreducible subrepresentations.
(c) Find explicit formulas (in terms of the bases of $L(\lambda_1)$ and $L(\lambda_2)$) for all vectors $v \in L(\lambda_1) \otimes L(\lambda_2)$ which satisfy $E.v = 0$ and $H.v = \lambda v$ for some λ .

 **Hand-in 5.** The purpose of this exercise is to study the roots of $\mathfrak{sl}_n(\mathbb{C})$.

We denote by $\mathfrak{h} \subset \mathfrak{sl}_n(\mathbb{C})$ the Lie subalgebra consisting of diagonal matrices. By definition, a *root* of $\mathfrak{sl}_n(\mathbb{C})$ is a nonzero element $\alpha \in \mathfrak{h}^* \setminus \{0\}$ in the dual of \mathfrak{h} such that the corresponding weight space in the adjoint representation of $\mathfrak{sl}_n(\mathbb{C})$

$$(\mathfrak{sl}_n(\mathbb{C}))_\alpha = \left\{ X \in \mathfrak{sl}_n(\mathbb{C}) \mid \forall H \in \mathfrak{h} : [H, X] = \alpha(H)X \right\}$$

is non-zero, i.e., $(\mathfrak{sl}_n(\mathbb{C}))_\alpha \neq \{0\}$. The set of all roots is denoted by $\Phi \subset \mathfrak{h}^*$. The *root lattice* $\Lambda_{\mathbb{R}} \subset \mathfrak{h}^*$ is defined as the set of integer linear combinations of roots $\alpha \in \Phi$.

- (a) Find all the roots of $\mathfrak{sl}_n(\mathbb{C})$.
(b) For $i \in \{1, \dots, n\}$, denote by η^i the functional $\eta^i \in \mathfrak{h}^*$ picking the i th diagonal entry: $\eta^i(H) = H_{ii}$. Let $r_1 > r_2 > \dots > r_n$ be chosen such that $\sum_{i=1}^n r_i = 0$ and that the functional ℓ on \mathfrak{h}^* (i.e., $\ell \in (\mathfrak{h}^*)^*$) determined by the condition $\ell(\eta^i) = r_i$ does not vanish on any nonzero element of the root lattice: $\ell(\mu) \neq 0$ for all $\mu \in \Lambda_{\mathbb{R}} \setminus \{0\}$. Find the set $\Phi^+ = \{\alpha \in \Phi \mid \ell(\alpha) > 0\}$ of positive roots. Find a subset $\Delta \subset \Phi^+$ of $n - 1$ positive roots such that any $\alpha \in \Phi^+$ is a nonnegative integer linear combination of elements of Δ .
(c) Suppose that $\vartheta: \mathfrak{sl}_n(\mathbb{C}) \rightarrow \text{End}(V)$ is a representation of $\mathfrak{sl}_n(\mathbb{C})$. The *weights* of V are those $\mu \in \mathfrak{h}^*$ such that the weight space

$$V_\mu = \{v \in V \mid \forall H \in \mathfrak{h} : \vartheta(H)v = \mu(H)v\}$$

is non-zero, i.e., $V_\mu \neq \{0\}$. Show that if V is irreducible and finite-dimensional, then for any two weights μ and μ' of V we have $\mu' - \mu \in \Lambda_{\mathbb{R}}$.