Aalto University
Department of Mathematics and Systems Analysis
MS-E1200 - Lie groups and Lie algebras

Problem set 3
2024/IV
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Exercise session: Wed 13.3. at 14-16 Hand-in due: Mon 18.3. at 10am

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.

Exercise 1. The set $\mathbb{C}^{n\times n}$ of $n \times n$-matrices is naturally identified with the space of linear maps $\mathbb{C}^n \to \mathbb{C}^n$, and is equipped with the operator norm: the norm of $X \in \mathbb{C}^{n\times n}$ is

$$\|X\|_{\mathrm{op}} = \sup \left\{ \|Xv\| \;\middle|\; v \in \mathbb{C}^n,\ \|v\| = 1 \right\},$$

where $\|\cdot\|$ denotes the usual norm on $\mathbb{C}^n$.

(a) If $X, Y \in \mathbb{C}^{n\times n}$ are two matrices, show that $\|XY\|_{\mathrm{op}} \leq \|X\|_{\mathrm{op}} \, \|Y\|_{\mathrm{op}}$.

(b) For $X \in \mathbb{C}^{n\times n}$, show that the following series is convergent in $\mathbb{C}^{n\times n}$:

$$\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n.$$

(c) For $t, s \in \mathbb{R}$ and $X \in \mathbb{C}^{n\times n}$, show that $\exp(sX)\exp(tX) = \exp\big((s+t)X\big)$.

(d) For any matrix $Z \in \mathbb{C}^{n\times n}$ and an invertible matrix $M \in \mathbb{C}^{n\times n}$, denote by $\mathrm{Ad}_M(Z) = MZM^{-1}$ the conjugation of Z by M. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ad}_{\exp(tX)}(Z) = XZ - ZX.$$

Exercise 2. Let G be a connected Lie group, and $U \subset \mathrm{G}$ an open neighborhood of the neutral element $e \in \mathrm{G}$. Show that the elements in U generate the entire group G (i.e., that the smallest subgroup of G containing U is the entire group).

Hint: Recall that in a connected topological space, the empty set and the whole space are the only subsets which are both open and closed (i.e., complement is open).

✍ **Hand-in 3.** For $X \in \mathbb{C}^{n\times n}$, let $e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$ and when $\|X\|_{\mathrm{op}} < 1$, also $\log(\mathbb{I} + X) = -\sum_{k=1}^{\infty} \frac{1}{k}(-X)^k$.

(a) Let $X \in \mathbb{C}^{n\times n}$, and define
$$f(t) = \det(e^{tX}).$$
for all $t \in \mathbb{R}$. Show that f satisfies the differential equation $f'(t) = c\, f(t)$, where $c = \mathrm{tr}(X)$, and the initial condition $f(0) = 1$. Deduce that
$$\det(e^X) = e^{\mathrm{tr}(X)}.$$

(b) Prove the Baker-Campbell-Hausdorff formula up to third order: for $X, Y \in \mathbb{C}^{n\times n}$ with $\|X\|_{\mathrm{op}}, \|Y\|_{\mathrm{op}}$ sufficiently small, we have
$$\log\left(e^X e^Y\right) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots,$$
where the omitted terms $\cdots$ are $\mathcal{O}\big(\max(\|X\|_{\mathrm{op}}^4, \|Y\|_{\mathrm{op}}^4)\big)$.

✍ **Hand-in 4.** In this exercise we consider the symplectic group and its Lie algebra. The symplectic group is the group of linear transformations which preserves a symplectic form (just like the orthogonal group is the group of linear transformations which preserve an inner product).

Let $J \in \mathbb{R}^{2n\times 2n}$ be an antisymmetric matrix such that $J^2 = -\mathbb{I}$. Define a bilinear form $\boldsymbol{\omega}$ on $\mathbb{R}^{2n}$ by $\boldsymbol{\omega}(v, w) = v^\top J w$. Let $\mathrm{Sp}_{2n}(\mathbb{R})$ be the set of those $M \in \mathbb{R}^{2n\times 2n}$ which preserve the form $\boldsymbol{\omega}$ in the sense that for all $v, w \in \mathbb{R}^{2n}$ we have $\boldsymbol{\omega}(v, w) = \boldsymbol{\omega}(Mv, Mw)$.

(a) Show that we have $\mathrm{Sp}_{2n}(\mathbb{R}) = \left\{M \in \mathbb{R}^{2n\times 2n} \,\middle|\, M^\top J M = J\right\}$, and show that $\mathrm{Sp}_{2n}(\mathbb{R}) \subset \mathrm{GL}_{2n}(\mathbb{R})$ is a closed subgroup[1].

(b) Find a linear condition (on X), which is necessary and sufficient for a matrix $X \in \mathbb{R}^{2n\times 2n}$ to satisfy $\exp(sX) \in \mathrm{Sp}_{2n}(\mathbb{R})$ for all $s \in \mathbb{R}$. Denote the set of such X by $\mathfrak{sp}_{2n}(\mathbb{R})$. Show directly using the linear condition you found that whenever $X, Y \in \mathfrak{sp}_{2n}(\mathbb{R})$, then also $[X,Y] = XY - YX \in \mathfrak{sp}_{2n}(\mathbb{R})$.

✍ **Hand-in 5.** Show that the set $\mathrm{SO}_n = \left\{M \in \mathbb{R}^{n\times n} \,\middle|\, M^\top M = \mathbb{I},\ \det(M) = 1\right\}$ is path-connected[2], i.e., that for any $M \in \mathrm{SO}_n$, there exists a continuous path $\gamma\colon [0,1] \to \mathrm{SO}_n$ such that $\gamma(0) = M$ and $\gamma(1) = \mathbb{I}$.

Hint: There are many possible ways, but one concrete approach is to implement the following idea. Noting that elements of SO_n can be identified with ordered and oriented orthonormal bases $v_1, \ldots, v_n$ in $\mathbb{R}^n$, it is sufficient to continuously deform a given orthonormal basis to the standard basis. To do this, it is convenient to first show that the first basis vector v_1 can be continuously rotated to the first standard basis vector. Then inductively in the dimension n, one can show that in the $n-1$-dimensional orthogonal complement to the first standard basis vector, one can deform the remaining basis vectors.

[1] *Closed subgroup* means a subgroup which is a closed set, topologically. A subgroup is by definition stable under multiplication ("closed under multiplication").

[2] Generally in topology, *path-connected* is a stronger notion than *connected* (path-conn. ⇒ conn.). For manifolds, and Lie groups in particular, the two notions are nevertheless logically equivalent.