Aalto University
Department of Mathematics and Systems Analysis
MS-E1200 - Lie groups and Lie algebras

Problem set 2
2024/IV
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Exercise session: Wed 6.3. at 14-16 Hand-in due: Mon 11.3. at 10am

In the weekly exercise sheets there are two types of problems: problems whose solutions are to be presented in the exercise session, and hand-in problems for which written solutions are to be returned.

Exercise 1.
In this exercise the ground field is complex numbers, $\mathbb{K} = \mathbb{C}$. We denote by $\mathbb{C}^\times$ the multiplicative group of non-zero complex numbers.

(a) Let G be group. Construct a "bijective correspondence between equivalence classes of one-dimensional representations of G and homomomorphisms $G \to \mathbb{C}^\times$"; or more precisely an injective mapping from the set of all group homomorphisms $G \to \mathbb{C}^\times$ to the set of representations of G on $\mathbb{C}$, such that the equivalence class of any one-dimensional representation of G is in the range of the mapping.

(b) Let G be a finite abelian (=commutative) group. Show that any irreducible representation of G is one dimensional. Conclude that the equivalence classes of irreducible representations of G can be identified with group homomorphisms $G \to \mathbb{C}^\times$.

(c) Let $C_n \cong \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n, i.e., the group with one generator c and relation $c^n = e$. Find all irreducible representations of C_n up to equivalence.

Exercise 2.
In this exercise the ground field is complex numbers, $\mathbb{K} = \mathbb{C}$.

(a) Let G be a group acting on a set X via a group homomorphism
$$\alpha \colon G \to \mathfrak{S}(X) = \{\sigma \colon X \to X \text{ bijection}\},$$
denoted briefly by $\big(\alpha(g)\big)(x) = g.x$. Form the complex vector space V with basis $(u_x)_{x\in X}$ indexed by the set X. For each $g \in G$, define $\rho(g) \colon V \to V$ by linear extension of
$$\rho(g)\, u_x = u_{g.x}.$$
Show that ρ is a representation of G.

(b) Let $G = \mathfrak{S}_n$ be the symmetric group on n letters, acting naturally on the set $X = \{1, 2, \ldots, n\}$. Let V be the n-dimensional vector space equipped with the representation ρ constructed as in part (a). Find a one-dimensional trivial representation as a subrepresentation of V.

(c) Let V be the representation of $\mathfrak{S}_n$ in part (b). For any $g \in \mathfrak{S}_n$, compute the trace of $\rho(g)$ on V. Explicitly for all $n \le 5$, compute also
$$\frac{1}{n!} \sum_{g \in \mathfrak{S}_n} \Big(\mathrm{tr}\big(\rho(g)\big)\Big)^2.$$

✍ Hand-in 3.

Fix a ground field $\mathbb{K}$ for all representations considered in this exercise (if you wish, you are allowed to assume $\mathbb{K} = \mathbb{C}$).

State and prove the isomorphism theorem for representations of a group G (i.e., the analogue of the isomorphism theorem for groups, from lecture 1, where one replaces group by a representation, homomorphism by intertwining map, etc.).

✍ Hand-in 4.

Let D_4 be the dihedral group of order 8, generated by r, m subject to relations $r^4 = e$, $m^2 = e$ and $mrm = r^{-1}$.

(a) Show that the conjugacy classes of D_4 are

$$\{e\}, \quad \{r, r^3\}, \quad \{r^2\}, \quad \{m, mr^2\}, \quad \{mr, mr^3\}.$$

(b) Calculate the character of the 2-dimensional representation of D_4 defined by

$$r \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad m \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and use the result to conclude that this representation is irreducible.

(c) Find four non-isomorphic one-dimensional complex representations of D_4.

(d) Check that the characters of the five irreducible representations given in parts (b) and (c) form an orthonormal basis of the set of class functions on D_4. Write down the character table of D_4.

(e) Let V be the 2-dimensional representation as in part (b). Calculate the character of the tensor product representation $V \otimes V$, and use the result to infer what irreducible subrepresentations $V \otimes V$ has.

✍ Hand-in 5.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) > 0\}$ be the upper half-plane, interpreted as a subset of the complex plane $\mathbb{C}$. Let $\mathrm{SL}_2(\mathbb{R})$ be the group of 2×2 matrices with real entries and determinant one.

(a) For any

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}),$$

define a function α_M on the complex plane by $\alpha_M(z) = \frac{az+b}{cz+d}$. Show that $M \mapsto \alpha_M$ defines an action of the group $\mathrm{SL}_2(\mathbb{R})$ on the set $\mathbb{H}$. What is the kernel of the homomorphism $\alpha \colon \mathrm{SL}_2(\mathbb{R}) \to \mathfrak{S}(\mathbb{H})$?

(b) Show that the action α in part (a) is transitive, i.e., for all $z, w \in \mathbb{H}$ there exists an $M \in \mathrm{SL}_2(\mathbb{R})$ such that $\alpha_M(z) = w$.

(c) The stabilizer of a given point $z_0 \in \mathbb{H}$ is the subgroup consisting of those M for which $\alpha_M(z_0) = z_0$. Show that the stabilizer of $z_0 = \mathbb{i}$ is the special orthogonal group $\mathrm{SO}_2 \subset \mathrm{SL}_2(\mathbb{R})$ (i.e. the group of 2×2 orthogonal matrices with determinant one). Discuss identifying $\mathbb{H}$ with the quotient $\mathrm{SL}_2(\mathbb{R}) \,/\, \mathrm{SO}_2$ (the set of left cosets) via a suitable bijection.