Aalto University **Problem set 4**
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021/III K Kytölä & D Adame-Carrillo

Exercise sessions: 4.-5.2.2021 **Hand-in due: Tue 9.2.2021 at 23:59**

Topic: Continuous functions, homeomorphisms, sequences in metric spaces

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 4 is Tue 9.2.2021 at 23:59.

Exercise 1 (Verifying openness and closednes of subsets).

(a) Show that the set $U = \big\{(x,y,z) \in \mathbb{R}^3 \;\big|\; x^2 < y^2 + z^2 - xyz + 3\big\} \subset \mathbb{R}^3$ is open and that the set $F = \big\{(x,y) \in \mathbb{R}^2 \;\big|\; x^2 + y^2 \leq 2 \text{ and } x \leq \frac{1}{3}\sin(\pi y)\big\} \subset \mathbb{R}^2$ is closed.

<u>Hint</u>: *Express the sets U and F appropriately using preimages; for the latter it is best to use two different functions and an intersection of preimages.*

The continuity of the functions involved can be considered known.

(b) Consider the space $\mathcal{C}([-1,1])$ of all continuous functions $f\colon [-1,1] \to \mathbb{R}$, equipped with the metric induced by the sup-norm $\|\cdot\|_\infty$. Consider the subset[1]

$$D \;=\; \left\{ p \in \mathcal{C}([-1,1]) \;\middle|\; p(x) \geq 0 \;\forall x \in [-1,1],\;\; \int_{-1}^{1} p(x)\,\mathrm{d}x = 1 \right\}.$$

Show that $D \subset \mathcal{C}([-1,1])$ is a closed set.

<u>Hint</u>: *You may use the facts that evaluation functions $f \mapsto f(x)$ (for an arbitrary $x \in [-1,1]$), and the integration function $f \mapsto \int_{-1}^{1} f(x)\,\mathrm{d}x$ are continuous functions $\mathcal{C}([-1,1]) \to \mathbb{R}$ with the chosen metric. Otherwise the ideas are similar to part (a).*

Exercise 2 (Coordinatewise convergence is not sufficient for convergence in ℓ^1).
Consider the space

$$\ell^1 \;=\; \left\{ x = (x_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \sum_{j=1}^{\infty} |x_j| < \infty \right\}$$

of absolutely summable real sequences. We consider it known that the formula $\|x\|_1 = \sum_{j=1}^{\infty} |x_j|$ for $x = (x_j)_{j\in\mathbb{N}} \in \ell^1$ defines a norm on ℓ^1. We equip ℓ^1 with the metric induced by the norm $\|\cdot\|_1$.

(a) Show that if a sequence $(x^{(n)})_{n\in\mathbb{N}}$ of elements $x^{(n)} = (x^{(n)}_j)_{j\in\mathbb{N}} \in \ell^1$ converges in ℓ^1 to $x = (x_j)_{j\in\mathbb{N}}$, then for every $k \in \mathbb{N}$, the sequence $(x^{(n)}_k)_{n\in\mathbb{N}}$ of the k:th coordinates of $x^{(n)}$'s converges to $\lim_{n\to\infty} x^{(n)}_k = x_k$ (limit in $\mathbb{R}$).

<u>Hint</u>: *You can start by showing that the k:th coordinate projection function $(x_j)_{j\in\mathbb{N}} \mapsto x_k$ is a 1-Lipschitz function $\ell^1 \to \mathbb{R}$.*

[1] This subset D could be interpreted as the set of all continuous probability density functions supported on the interval $[-1,+1]$.

(b) For $n \in \mathbb{N}$ let $x^{(n)} = (x^{(n)}_j)_{j \in \mathbb{N}} \in \ell^1$ be the element given by

$$x^{(n)}_j = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } j \neq n. \end{cases}$$

Show that for any $k \in \mathbb{N}$ we have $\lim_{n \to \infty} x^{(n)}_k = 0$ but in the space $(\ell^1, \|\cdot\|_1)$ the sequence $(x^{(n)})_{n \in \mathbb{N}}$ does not converge.

Hint: *If the sequence would converge in ℓ^1, then part (a) together with the first calculation of (b) identifies the only possibility for a limit $x \in \ell^1$. Now show directly from the definition of limits that we do not have convergence to that candidate limit.*

✍ **Exercise 3** (Hölder functions are continuous)**.**
Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is said to be (M, α)-Hölder with parameters $M > 0$ and $\alpha > 0$ if

$$\mathsf{d}_Y\big(f(x_1), f(x_2)\big) \;\leq\; M\, \mathsf{d}_X(x_1, x_2)^\alpha \qquad \text{for all } x_1, x_2 \in X.$$

Prove that if $f\colon X \to Y$ is (M, α)-Hölder with some $M > 0$ and $\alpha > 0$, then f is continuous.

✍ **Exercise 4** (Some homeomorphisms)**.**

(a) Prove that $[0, 1)$ and $[0, \infty)$ are homeomorphic, $[0, 1) \;\approx\; [0, \infty)$.

Remark: The subsets of the real axis above are equipped with the ordinary metric. The continuity of the functions involved can be considered known.

(b) Prove that the open unit disk $B(\vec{0}, 1) \subset \mathbb{R}^2$ in the Euclidean plane $\mathbb{R}^2$ and the whole plane $\mathbb{R}^2$ are homeomorphic, $B(\vec{0}, 1) \approx \mathbb{R}^2$.

Hint: *One option is a radial mapping in polar coordinates, using part (a).*

✍ **Exercise 5** (Some functions of functions)**.**

(a) Define a function $f\colon \mathcal{C}([0, 10]) \to \mathcal{C}([0, 10])$ by setting

$$[f(x)](t) = \int_0^t s\, x(s)\, \mathrm{d}s, \qquad \text{for } x \in \mathcal{C}([0, 10]) \text{ and } t \in [0, 10].$$

(Here $f(x)$ is a function $[0, 10] \to \mathbb{R}$, whose value at $t \in [0, 10]$ is obtained by the above formula.)
Prove that f is K-Lipschitz for a suitable $K \geq 0$, when the space $\mathcal{C}([0, 10])$ (both as the domain and codomain of f) is equipped with the metric induced by the norm $\|\cdot\|_\infty$ — i.e., show that for all $x, y \in C([0, 10])$ we have

$$\|f(x) - f(y)\|_\infty \leq\; K\, \|x - y\|_\infty.$$

Hint: *The "triangle equality for integrals" $\big|\int_a^b h(s)\, \mathrm{d}s\big| \;\leq\; \int_a^b |h(s)|\, \mathrm{d}s$ can be used here.*

(b) Let

$$\mathcal{C}^1([-1, 1]) = \Big\{ x\colon [-1, 1] \to \mathbb{R} \;\Big|\; x' \text{ is continuous} \Big\}$$

be the set of all continuously differentiable functions on the interval $[-1, 1]$. Interpret it as a subset $\mathcal{C}^1([-1, 1]) \subset \mathcal{C}([-1, 1])$, where the space $\mathcal{C}([-1, 1])$ of continuous functions is equipped again with the metric induced by the norm $\|\cdot\|_\infty$. Consider the function $g\colon \mathcal{C}^1([-1, 1]) \to \mathcal{C}([-1, 1])$ given by

$$[g(x)](t) = x'(t), \qquad \text{for } x \in \mathcal{C}^1([-1, 1]) \text{ and } t \in [-1, 1].$$

Show that g is not K-Lipschitz for any $K \geq 0$.

Hint: *Consider e.g. the zero function and $x_n(t) = \sin(\pi n t)$, for $n \in \mathbb{N}$, in the Lipschitz condition.*