


Exercise sessions: 18.-19.2.2021 Hand-in due: Tue 23.2.2021 at 23:59

Topic: compactness, connectedness

Written solutions to the exercises marked with symbol  are to be returned in My-Courses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 6 is Tue 23.2.2021 at 23:59.

Exercise 1 (Compactness and continuous functions).

Prove the following assertions related to compactness by forming a continuous function, which settles the matter.

- (a) If $A \subset \mathbb{R}^3$ is a compact subset, then it contains a highest point $a \in A$, i.e., a point $a = (a_1, a_2, a_3)$ such that we have $a_3 \geq x_3$ for all $x = (x_1, x_2, x_3) \in A$.
- (b) If $B \subset \mathbb{R}^2$ is a compact subset, then it contains points $a, b, c \in B$, for which the perimeter

$$\|a - b\| + \|b - c\| + \|c - a\|$$

of the triangle is maximal among triangles whose vertices lie in B .

Hint: You can use the fact that Cartesian products of compact sets are compact.

- (c) If X is a compact metric space and $g: X \rightarrow X$ is a continuous function that has no fixed points, then there exists a $c > 0$ such that $d(g(x), x) \geq c$ for all $x \in X$.

Hint: You can consider the following continuity statements known:

- the function $X \rightarrow X \times X$ given by $x \mapsto (x, g(x))$ is continuous (since its component functions id_X and g are continuous);
- the metric $d: X \times X \rightarrow [0, \infty)$ is a continuous function.


Remark: An alternative strategy in each part would be to first pick a sequence which approaches the relevant sup/inf-value, and then use compactness to extract a convergent subsequence.

Exercise 2 (Connectedness and a midpoint between two subsets).

Let X be a connected metric space and $A, B \subset X$ two non-empty subsets. Show that there exists (at least one) point $x \in X$ such that

$$\text{dist}(\{x\}, A) = \text{dist}(\{x\}, B).$$

Hint: Consider the continuous function defined by $f(x) = \text{dist}(\{x\}, A) - \text{dist}(\{x\}, B)$.

 **Exercise 3** (A closed and bounded non-compact set).

Consider the space $\mathcal{C}([0, 1])$ of continuous real valued functions on $[0, 1]$, with the metric d induced by the sup-norm $\|\cdot\|_\infty$

$$d(f, g) = \|f - g\|_\infty = \sup \left\{ |f(x) - g(x)| \mid x \in [0, 1] \right\}.$$


Prove that the closed unit ball $\overline{B}(\vec{0}, 1)$ is not compact.

Hint: Construct a sequence (f_n) for which $\|f_n\|_\infty \leq 1$ for all $n \in \mathbb{N}$ and

$$\|f_k - f_n\|_\infty \geq 1$$

whenever $k \neq n$. Such a sequence cannot have a convergent subsequence (why?).

Remark: While in Euclidean spaces \mathbb{R}^d , compactness of a subset is equivalent to its closedness and boundedness (Bolzano-Weierstrass theorem / Heine-Borel theorem), this exercise shows that the same is not true in general metric spaces! Infinite-dimensional normed spaces (like $\mathcal{C}([0, 1])$ in this exercise) are typical counterexamples.

 **Exercise 4** (The union of two compact sets is compact).

Prove that for any two compact subsets $A, B \subset X$ of a metric space X , the union $A \cup B$ is compact, ...

- (a) ... directly from the definition of (sequential) compactness;
- (b) ... by using the characterization of compactness by open covers.

Hint: The proofs should begin as follows:

a) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $A \cup B$

b) Let $(U_j)_{j \in J}$ be an open cover of $A \cup B$

 **Exercise 5** (Decreasing sequences of compact sets).

- (a) Let X be a metric space and $(A_n)_{n \in \mathbb{N}}$ a decreasing sequence (i.e., $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$) of non-empty compact subsets of the space X . Prove that the intersection of these subsets

$$A = \bigcap_{n=1}^{\infty} A_n$$

- is (i) non-empty; and (ii) compact.
- (b) Give an example of a decreasing sequence of non-empty closed subsets, whose intersection is empty.