Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021/III

Problem set 2

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Fill-in-the-blanks 1. Complete the following arguments, which exemplify certain expressions that *do not* yield norms. It is simplest to use the labels given in Chapter IV of the lecture notes for the conditions required of a norm.

- The formula $\|(x, y)\| = |x + y|$ does not define a norm for points (x, y) in the plane $\mathbb{R}^2$. For example

  $$\text{condition } \big(\_\_\_\_\big)$$

  does not hold, since by choosing

  $$x = \_\_\_\_\_\_\_\_\_\_\_\_, \qquad y = \_\_\_\_\_\_\_\_\_\_\_\_$$

  we find, contrary to the condition, that

  ____________________, although ____________________.

- The formula $\|f\| = |f(0)|$ does not define a norm in the space $C\big([-1, 1]\big)$ of continuous functions on the interval $[-1, 1]$. The reason:

  $$\text{condition } \big(\_\_\_\_\big)$$

  does not hold, because by choosing

  $$f(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad \text{for } t \in [-1, 1],$$

  we find, contrary to the condition, that

  ____.

  The two other conditions are nevertheless satisfied:

  Justification of condition $\big(\_\_\_\_\big)$:

  ____.

  Justification of condition $\big(\_\_\_\_\big)$:

  ____.

Fill-in-the-blanks 2. Complete the following arguments to find out the most general form of an inner product on the vector space $\mathbb{R}^d$. For $x, y \in \mathbb{R}^d$, denote by

$x \cdot y = \sum_{i=1}^{d} x_i y_i$	—	the ordinary Euclidean inner product;
$\langle x, y \rangle$	—	another inner product.

The labels for properties required of an inner product are as in Chapter IV of the lecture notes.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the standard basis vectors in the space $\mathbb{R}^d$, and denote $a_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$, $1 \leq i, j \leq d$. By property (IP1) of inner products, we then have

$$a_{ji} = \underline{\hspace{8cm}}$$

for all $1 \leq i, j \leq d$. This means that the matrix $A = [a_{ij}]$ is a

$$(\star) \qquad \underline{\hspace{6cm}} \; d \times d \text{ matrix.}$$

Writing $x = \sum_i x_i \mathbf{e}_i, \;\; y = \sum_j y_j \mathbf{e}_j$, and using bilinearity of inner products, we get

$$\langle x, y \rangle = \sum_{i,j=1}^{d} \underline{\hspace{3.5cm}} = \sum_{i,j=1}^{d} \underline{\hspace{3cm}}.$$

In matrix notation, this can be written as

$$\langle x, y \rangle = \mathbf{x} \cdot (A\mathbf{y}) = \mathbf{x}^{\top} A \mathbf{y},$$

where $\mathbf{x}$ and $\mathbf{y}$ are x and y interpreted as column vectors.

Property (IP4) implies that for all $\mathbf{x} \in \mathbb{R}^d$, we have

$$\underline{\hspace{4.5cm}} = \underline{\hspace{2.7cm}} \geq 0,$$

so the matrix A is

$$\underline{\hspace{4.5cm}} \; \text{semi-definite.}$$

Finally, property (IP5) implies that for all $\mathbf{x} \neq \vec{0}$ we have

$$\underline{\hspace{9cm}} > 0,$$

so the matrix A is (a more stringent condition than above)

$$(\star\star) \qquad \underline{\hspace{8cm}}.$$

In summary, the most general inner product in $\mathbb{R}^d$ has the form

$$\langle x, y \rangle = \underline{\hspace{4.5cm}},$$

where the $d \times d$ matrix A satisfies conditions $(\star)$ and $(\star\star)$.

In the case $d = 2$, the unit circle $\left\{ x \in \mathbb{R}^2 \;\middle|\; \|x\| = 1 \right\}$ with respect to the norm induced by this inner product is an ellipse, and in the case $d = 3$, the unit sphere $\left\{ x \in \mathbb{R}^3 \;\middle|\; \|x\| = 1 \right\}$ is an ellipsoid.