Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021-2022/III

Problem set 5

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Exercise sessions: 10.-11.2.2022 **Hand-in due: Tue 15.2.2022 at 23:59**

Topic: pointwise and uniform convergence, completeness and Cauchy sequences

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 5 is Tue 15.2.2022 at 23:59.

Exercise 1 (A function sequence)**.**
Consider the sequence (f_n) of functions $f_n\colon [0,\infty) \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} x-n+1, & \text{for } n-1 \le x \le n, \\ n+1-x, & \text{for } n \le x \le n+1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the graphs of a few of the functions. Determine the pointwise limit function f of the function sequence $(f_n)_{n\in\mathbb{N}}$. (After guessing the limit with the help of the sketches, prove pointwise convergence from the definition).
(b) Does the function sequence $(f_n)_{n\in\mathbb{N}}$ converge uniformly on $[0,\infty)$ to the limit function f?
(c) Do we have

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,\mathrm{d}x \;=\; \int_0^\infty f(x)\,\mathrm{d}x\ ?$$

(d) By a minor modification one obtains a similar sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$ of functions which has the same limit function f, but the the integrals of $\tilde{f}_n$ tend to $+\infty$ as $n\to\infty$. Find such a modification (an illustration suffices).

Exercise 2 (Solving an equation by fixed point iteration)**.**
The equation $\tan x = 1/x$ has a unique solution on the interval $x \in (0,\pi/2)$ (sketch the graphs!). This equation can be rewritten as a fixed point equation $f(x) = x$ in (at least) two ways.

(a) Which way leads to a convergent fixed point iteration, when the initial value is $x_0 = 1$? (You do not need to be precise about the domain and codomain in this problem).
(b) By fixed point iteration, determine an approximate solution to the equation, accurate to two decimal places.

Remark: If a function f has an inverse function, then at a fixed point $f(a) = a$ we also have $f^{-1}(a) = a$. However, according to the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

only one of the quantities $|f'(a)|$ and $|(f^{-1})'(a)| = 1/|f'(a)|$ can be < 1, and therefore lead to a contraction in the vicinity of the fixed point a.

✍ **Exercise 3** (Another function sequence).
Define $f_n \colon [0,1] \to \mathbb{R}$ by setting

$$f_n(x) = n \sin(x/n), \qquad \text{for } x \in [0,1] \text{ and } n \in \mathbb{N}.$$

(a) Determine the limit function $f \colon [0,1] \to \mathbb{R}$ of the sequence (f_n),

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x \in [0,1].$$

(b) Does the sequence (f_n) converge uniformly on the interval $[0,1]$ to the function f?

Hint: *Either the 1:st order Taylor polynomial of the fucntion* sin *or L'Hospital's rule should be helpful. Finding the maximum/supremum is probably most straightforwardly done by calculating derivatives.*

✍ **Exercise 4** (A sequence in a function space).
Let

$$f_n(x) = x^n, \qquad \text{for } 0 \le x \le 1 \text{ and } n \in \mathbb{N}.$$

(a) Prove (directly from the definition), that $(f_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence in the space $\mathcal{C}([0,1])$, when the norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ is used.
Hint: *It suffices to consider the case $\ell = 2k$ in the definition of Cauchy sequences, and to maximize the difference $x^k - x^{2k}$.*

(b) Show that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $\mathcal{C}([0,1])$, when the norm $\|f\|_1 = \int_0^1 |f(x)|\,\mathrm{d}x$ is used.

Remark: The space $\mathcal{C}([0,1])$ is complete with respect to the norm in part (a), but not with respect to the norm in part (b). Moreover, convergence in the norm of part (a) is equivalent to uniform convergence, so $(f_n)_{n\in\mathbb{N}}$ cannot be Cauchy, since its pointwise limit is discontinuous. In the norm of part (b), however, $(f_n)_{n\in\mathbb{N}}$ converges to the zero function, so to prove the failure of completeness in this case requires a more elaborate example.

✍ **Exercise 5** (Weierstrass' M test).

(a) Let $I \subset \mathbb{R}$ be an interval and $g_k \colon I \to \mathbb{R}$ ($k \in \mathbb{N}$) continuous functions. Assume that $|g_k(x)| \le M_k$ for all $x \in I$ and that the series $\sum_{k=1}^\infty M_k$ is convergent. Prove that the infinite series

$$f(x) = \sum_{k=1}^{\infty} g_k(x),$$

defines a continuous function $f \colon I \to \mathbb{R}$.
Hint: *Uniform convergence of the partial sums and a result concerning uniform convergence, or a direct argument of the same type. The continuity of the partial sums (finitely many terms) is considered known. The remainder of a convergent series tends to zero as the starting index increases.*

(b) Using part (a), prove that the Fourier series

$$\sum_{k=1}^{\infty} \frac{4\,(-1)^k \cos(kx)}{k^2}$$

defines a continuous function on $\mathbb{R}$.
Remark: Apart from an additive constant, this is the Fourier series of the 2π-periodic extension of $f(x) = x^2$, $-\pi \le x \le \pi$, which can be used to calculate the sum of the series $\sum_{k=1}^\infty \frac{(-1)^k}{k^2}$ (by first determining the additive constant and then substituting $x = 0$).