Aalto University **Problem set 2**
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MS-C1541 — Metric spaces, 2021/III K Kytölä & D Adame-Carrillo

Exercise sessions: 21.-22.1.2021 **Hand-in due: Tue 26.1.2021 at 23:59**

Topic: Sets, functions, real numbers

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 2 is Tue 26.1.2021 at 23:59.

Exercise 1 (Continuous functions on intervals)**.**
Below (as usually), unless explicitly specified, an interval refers to either an open interval, closed interval, or a half-open interval, possibly unbounded. If $c \in \mathbb{R}$, we also interpret the singleton $\{c\} \subset \mathbb{R}$ as a closed interval $[a, b] \subset \mathbb{R}$ with coinciding endpoints, $a = c = b$.[1]

(a) Find an example of a continuous function on an interval which has no maximum.
(b) Find an example of a continuous function on an *open interval* which has a maximum but has no minimum.
(c) Using a combination of results from Chapter III, prove that if $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed interval $[a, b] \subset \mathbb{R}$, then the image $f\big[[a, b]\big]$ is a closed interval.
(d) If $f : (a, b) \to \mathbb{R}$ is a continuous function on an *open interval* $(a, b) \subset \mathbb{R}$, is it possible that the image $f\big[(a, b)\big]$ is not an open interval?

Exercise 2 (The dihedral angle of a regular tetrahedron)**.**
Calculate the dihedral angle (the angle between two adjacent faces) of a regular tetrahedron using the following idea: Place the vertices of the tetrahedron in the 4-dimensional space $\mathbb{R}^4$ at points $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ ja $\mathbf{e}_4$ (standard basis vectors).

(i) Verify that the lengths of the edges $\|\mathbf{e}_i - \mathbf{e}_j\|$, for $i \neq j$, are all equal.
(ii) By symmetry, the angle in question is the same as (for example) the angle between the vectors
$$\mathbf{u} = \mathbf{e}_4 - \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) \qquad \text{and} \qquad \mathbf{v} = \mathbf{e}_3 - \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2),$$
so it is easily calculated.

Hint: *Draw a figure! Answer:* arccos(1/3).

[1] For *closed intervals* this is a natural convention in the degenerate case of coinciding endpoints, since $[a, b] \subset \mathbb{R}$ by definition consists of points $x \in \mathbb{R}$ such that $a \leq x \leq b$.

✍ **Exercise 3** (Some Legendre polynomials)**.**
Consider the three first *Legendre polynomials*

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= (3x^2-1)/2. \end{aligned}$$

(a) Show that these polynomials are pairwise *orthogonal* with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x,$$

i.e., that we have

$$\langle P_i, P_j\rangle = 0, \qquad \text{for } i \neq j.$$

(b) Determine coefficients $c_0, c_1, c_2 \in \mathbb{R}$ such that $\|c_k P_k\| = 1$ for all $k \in \{0,1,2\}$, where as norm we use the one arising from the inner product in part (a).

Remark: Together with orthogonality from part (a), in part (b) we obtain *orthonormal vectors*.

✍ **Exercise 4** (The ℓ^1-norm in finite dimensional spaces)**.**

(a) Prove that the formula

$$\|x\|_1 \;=\; |x_1| + |x_2| + \cdots + |x_n|, \qquad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

defines a quantity that satisfies the following conditions:

(N1) $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$ for all $x, y \in \mathbb{R}^n$;
(N2) $\|cx\|_1 = |c|\,\|x\|_1$ for all $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$;
(N3) $\|x\|_1 = 0$ if and only if $x = \vec{0}$.

Remark: This is an example of a norm (which does not arise from an inner product).

(b) Let $n = 2$. Sketch the set

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \|(x_1,x_2)\|_1 = |x_1| + |x_2| = 1 \right\},$$

i.e., the “unit circle” under the norm of part (a).

✍ **Exercise 5** (Ideas behind Fourier series)**.**

Hint: *The solution is shorter than the problem statement! Compare also with Problem 3.*

An indexed collection $(e_t)_{t \in T}$ of vectors $e_t \in \mathsf{V}$ in an inner product space V is called *orthonormal*, if for all $s, t \in T$, we have

$$\langle e_t, e_s \rangle = \delta_{t,s} = \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{if } t \neq s \end{cases} \qquad \text{(the Kronecker } \delta\text{-symbol).}$$

Remark: Here T can be an arbitrary index set.

(a) Consider a countable orthonormal collection $(e_n)_{n \in \mathbb{N}}$ in an inner product space V. Assume that a vector $v \in \mathsf{V}$ can be expressed, for some $m \in \mathbb{N}$, as a (finite) linear combination

$$v = \alpha_1 \, e_1 + \cdots + \alpha_m \, e_m \qquad \text{with some coefficients } \alpha_n \in \mathbb{R}.$$

Calculate the inner products between the vector v and the vectors e_n from the orthonormal collection, and deduce a formula for the coefficients α_n.

Remark: The result generalizes to countable linear combinations (i.e. series), which are convergent in a suitable sense.

(b) Define functions $c_0, s_1, c_1, s_2, c_2, s_3, c_3, \ldots$ of a real variable x by the formulas

$$\begin{aligned} c_0(x) &= \frac{1}{\sqrt{2}}, \\ s_n(x) &= \sin(nx) \qquad (\text{for } n \in \mathbb{N}), \\ c_n(x) &= \cos(nx) \qquad (\text{for } n \in \mathbb{N}). \end{aligned}$$

Prove that the (countably infinite) collection of these functions is orthonormal in the space $C([-\pi, \pi])$ of continuous functions on $[-\pi, \pi]$, with respect to the rescaled inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x.$$

Remark: Since there are 6–9 different cases (depending on the counting convention), **it is enough to do the details of just four cases**. The complex version would be somewhat easier...

(c) The idea of Fourier series is founded on the following result: Every continuously differentiable 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ can be represented as

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \big(\alpha_n \cos(nx) + \beta_n \sin(nx) \big).$$

Consider for simplicity the case of a function f, for which the above series contains only finitely many terms, i.e., for some $m \in \mathbb{N}$ we have

$$f(x) = \alpha_0 + \sum_{n=1}^{m} \big(\alpha_n \cos(nx) + \beta_n \sin(nx) \big).$$

Using parts (a) and (b), derive the following formula for the coefficients

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x \qquad (n \in \mathbb{N}).$$

Hint: *Using parts (a) and (b) is indeed the intended approach, instead of calculating everything from scratch again!*