Aalto University Problem set 5

Department of Mathematics and Systems Analysis

MS-C1541 — Metric spaces, 2023-2024/III

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Exercise sessions: 8.-9.2.2024 Hand-in due: Tue 13.2.2024 at 23:59

Topic: pointwise and uniform convergence, completeness and Cauchy sequences

Written solutions to the exercises marked with symbol ≤ are to be returned in My-Courses. Each exercise is graded on a scale 0-3. The deadline for returning solutions to problem set 5 is Tue 13.2.2024 at 23:59.

Exercise 1 (Limits of function sequences — exam problem in 2021).

(a) For  $n \in \mathbb{N}$ , let  $f_n : [0, \infty) \to \mathbb{R}$  be the function given by

$$f_n(x) = \frac{n}{1 + nx + n^2 (x^2 - x)^2}$$
 for  $x \in [0, \infty)$ .

Does the function sequence  $(f_n)_{n\in\mathbb{N}}$  converge pointwise? Does the function sequence  $(f_n)_{n\in\mathbb{N}}$  converge uniformly?

(b) For  $n \in \mathbb{N}$ , let  $g_n : [1, \infty) \to \mathbb{R}$  be the function given by

$$g_n(x) = \frac{n}{1 + nx + n^2 (x^2 - x)^2}$$
 for  $x \in [1, \infty)$ .

Does the function sequence  $(g_n)_{n\in\mathbb{N}}$  converge pointwise? Does the function sequence  $(g_n)_{n\in\mathbb{N}}$  converge uniformly?

(c) For  $n \in \mathbb{N}$ , let  $h_n : [2, \infty) \to \mathbb{R}$  be the function given by

$$h_n(x) = \frac{n}{1 + nx + n^2 (x^2 - x)^2}$$
 for  $x \in [2, \infty)$ .

Does the function sequence  $(h_n)_{n\in\mathbb{N}}$  converge pointwise? Does the function sequence  $(h_n)_{n\in\mathbb{N}}$  converge uniformly?

Exercise 2 (Solving an equation by fixed point iteration).

The equation  $\tan x = 1/x$  has a unique solution on the interval  $x \in (0, \pi/2)$  (sketch the graphs!). This equation can be rewritten as a fixed point equation f(x) = x in (at least) two ways.

- (a) Which way leads to a convergent fixed point iteration, when the initial value is  $x_0 = 1$ ? (You do not need to be precise about the domain and codomain in this problem).
- (b) By fixed point iteration, determine an approximate solution to the equation, accurate to two decimal places.

Remark: If a function f has an inverse function, then at a fixed point f(a) = a we also have  $f^{-1}(a) = a$ . However, according to the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

only one of the quantities |f'(a)| and  $|(f^{-1})'(a)| = 1/|f'(a)|$  can be < 1, and therefore lead to a contraction in the vicinity of the fixed point a.

Exercise 3 (Another function sequence).

Define, for  $n \in \mathbb{N}$ , a function  $f_n : [0, 100] \to \mathbb{R}$  by setting

$$f_n(x) = n^2 - n^2 \cos(x/n)$$
 for  $x \in [0, 100]$ .

(a) Determine the limit function  $f: [0, 100] \to \mathbb{R}$  of the sequence  $(f_n)_{n \in \mathbb{N}}$ ,

$$f(x) = \lim_{n \to \infty} f_n(x)$$
 for  $x \in [0, 100]$ .

(b) Does the sequence  $(f_n)$  converge uniformly on the interval [0, 100] to the function f?

<u>Hint</u>: The 2nd order Taylor polynomial of the function cos should be helpful. Finding the maximum/supremum is probably most straightforwardly done by calculating derivatives.

Exercise 4 (A sequence in a function space).

Let

$$f_n(x) = x^n$$
, for  $0 \le x \le 1$  and  $n \in \mathbb{N}$ .

- (a) Prove (directly from the definition), that  $(f_n)_{n\in\mathbb{N}}$  is not a Cauchy sequence in the space  $\mathcal{C}([0,1])$ , when the norm  $||f||_{\infty} = \sup_{x\in[0,1]} |f(x)|$  is used.

  Hint: It suffices to consider the case  $\ell = 2k$  in the definition of Cauchy sequences, and to maximize the difference  $x^k x^{2k}$ .
- (b) Show that  $(f_n)_{n\in\mathbb{N}}$  is a Cauchy sequence in the space  $\mathcal{C}([0,1])$ , when the norm  $||f||_1 = \int_0^1 |f(x)| dx$  is used.

Remark: The space  $\mathcal{C}([0,1])$  is complete with respect to the norm in part (a), but not with respect to the norm in part (b). Moreover, convergence in the norm of part (a) is equivalent to uniform convergence, so  $(f_n)_{n\in\mathbb{N}}$  cannot be Cauchy, since its pointwise limit is discontinuous. In the norm of part (b), however,  $(f_n)_{n\in\mathbb{N}}$  converges to the zero function, so to prove the failure of completeness in this case requires a more elaborate example.

## **Exercise 5** (Weierstrass' M test).

(a) Let  $I \subset \mathbb{R}$  be an interval and  $g_k \colon I \to \mathbb{R}$   $(k \in \mathbb{N})$  continuous functions. Assume that  $|g_k(x)| \leq M_k$  for all  $x \in I$  and that the series  $\sum_{k=1}^{\infty} M_k$  is convergent. Prove that the infinite series

$$f(x) = \sum_{k=1}^{\infty} g_k(x),$$

defines a continuous function  $f: I \to \mathbb{R}$ .

<u>Hint</u>: Uniform convergence of the partial sums and a result concerning uniform convergence, or a direct argument of the same type. The continuity of the partial sums (finitely many terms) is considered known. The remainder of a convergent series tends to zero as the starting index increases.

(b) Using part (a), prove that the Fourier series

$$\sum_{k=1}^{\infty} \frac{4(-1)^k \cos(kx)}{k^2}$$

defines a continuous function on  $\mathbb{R}$ .

Remark: Apart from an additive constant, this is the Fourier series of the  $2\pi$ -periodic extension of  $f(x) = x^2$ ,  $-\pi \le x \le \pi$ , which can be used to calculate the sum of the series  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$  (by first determining the additive constant and then substituting x = 0).