Problem set 1

Department of Mathematics and Systems Analysis

MS-C1541 — Metric spaces, 2023-2024/III

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## Exercise sessions: 11.-12.1.2024 Hand-in due: Tue 16.1.2024 at 23:59

Topic: Sets, functions, real numbers

Written solutions to the exercises marked with symbol  $\triangle$  are to be returned in My-Courses. Each exercise is graded on a scale 0-3. The deadline for returning solutions to problem set 1 is Tue 16.1.2024 at 23:59.

**Exercise 1** (Images and preimages of unions and intersections). Let X and Y be sets and  $f: X \to Y$  a function.

(a) Show that for any  $C, D \subset Y$ , the preimages satisfy

$$f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D].$$

(b) Show that for any  $C, D \subset Y$ , the preimages satisfy

$$f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D].$$

(c) Show that for any  $A, B \subset X$ , the images satisfy

$$f[A \cup B] = f[A] \cup f[B].$$

(d) Give an example in which for the images of subsets  $A, B \subset X$  we have

$$f[A \cap B] \neq f[A] \cap f[B].$$

<u>Hint</u>: In parts (a)-(c) it is possible to argue by a chain of equivalent conditions

 $x \in left \ hand \ side \ set \iff \cdots \iff x \in right \ hand \ side \ set.$ 

## Exercise 2 (Some geometric sums).

(a) Consider the real-number sequences  $(a_n)_{n\in\mathbb{N}}$  and  $(b_n)_{n\in\mathbb{N}}$  defined by

$$a_n = \sum_{j=0}^n \left(\frac{4}{5}\right)^j = 1 + \frac{4}{5} + \frac{16}{25} + \dots + \frac{4^n}{5^n}$$
 and  $b_n = \sum_{j=0}^n \left(\frac{-1}{2}\right)^j$ .

Find simplified formulas for  $a_n$  and  $b_n$ . Find the limits  $\lim_{n\to\infty} a_n$  and  $\lim_{n\to\infty} b_n$ , and prove directly using the definition of limits of real-number sequences that your answers indeed are those limits.

Hint: To simplify, recall finite geometric sums.

(b) Let  $(a_n)_{n\in\mathbb{N}}$  be as in part (a). Does the set

$$A = \{a_n \mid n \in \mathbb{N}\} \subset \mathbb{R}$$

have a supremum, and if it does, what is  $\sup(A)$ ? Does the set A have a maximum, and if it does, what is  $\max(A)$ ?

(c) Let  $(b_n)_{n\in\mathbb{N}}$  be as in part (a). Does the set

$$B = \{b_n \mid n \in \mathbb{N}\} \subset \mathbb{R}$$

have an infimum, and if it does, what is  $\inf(A)$ ? Does the set B have a minimum, and if it does, what is  $\min(A)$ ?

Exercise 3 (Calculating limits of sequences).

Calculate the limits of the real-number sequences  $(a_n)_{n\in\mathbb{N}}$  and  $(b_n)_{n\in\mathbb{N}}$ , where

$$a_n = \frac{3n+4}{5n+6}$$
 and  $b_n = \frac{(1+2n)(1+3n)e^{-n}}{n^3e^{-2n}+n^2e^{-n}}$  for  $n \in \mathbb{N}$ .

You may use known properties of limits and limits that are well-known from earlier courses.

- Exercise 4 (A variant of a formulation of the completeness axiom). Recall that one of the formulations of the completeness axiom of real numbers is:
  - (C2) Every increasing real number sequence  $(a_n)_{n\in\mathbb{N}}$  which is bounded from above has a limit  $\lim_{n\to\infty} a_n \in \mathbb{R}$ .

Consider the statement

(C2') Every decreasing real number sequence  $(b_n)_{n\in\mathbb{N}}$  which is bounded from below has a limit  $\lim_{n\to\infty} b_n \in \mathbb{R}$ .

Prove that (C2) implies (C2).

<u>Hint</u>: Given a sequence  $(b_n)_{n\in\mathbb{N}}$  as above, what can be said about the numbers  $-b_n$  for  $n\in\mathbb{N}$ ?

Exercise 5 (Some points in the Cantor set).

For a fixed  $n \in \mathbb{N}$  and any  $b_1, \ldots, b_n \in \{0, 1\}$ , consider the closed interval of length  $\frac{1}{3^n}$  whose left endpoint is  $\sum_{j=1}^n \frac{2b_j}{3^j} = \frac{2b_1}{3} + \frac{2b_2}{9} + \frac{2b_3}{27} + \cdots + \frac{2b_n}{3^n}$ . Let  $C_n \subset \mathbb{R}$  be the union of these  $2^n$  intervals (see figure below)

$$C_n = \bigcup_{b_1, \dots, b_n \in \{0,1\}} \left[ \sum_{j=1}^n \frac{2 \, b_j}{3^j} \,, \, \sum_{j=1}^n \frac{2 \, b_j}{3^j} + \frac{1}{3^n} \right].$$

The Cantor set  $C \subset \mathbb{R}$  is the intersection of these sets over all  $n \in \mathbb{N}$ ,

$$C = \bigcap_{n \in \mathbb{N}} C_n$$

 $C = \bigcap_{n \in \mathbb{N}} C_n.$  Show that  $\frac{3}{4} \in C$  and  $\frac{4}{5} \notin C$ . (Or if  $\frac{3}{4}$  and  $\frac{4}{5}$  are too boring, use  $\frac{11}{12}$  and  $\frac{10}{11}$  instead.)

<u>Hint</u>: To get started, it helps to note that  $\frac{3}{4} = \sum_{k=0}^{\infty} \frac{2}{3^{1+2k}}$  (use geometric series), which can be written as an infinite series  $\sum_{j=1}^{\infty} \frac{2b_j}{3^j}$  for a suitably chosen sequence  $b_1, b_2, b_3, \ldots$  of zeros and ones.

