Aalto University **Problem set 4**
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MS-C1541 — Metric spaces, 2022-2023/III K Kytölä & T Kelomäki

(Exercise sessions: 2.-3.2.2022) Hand-in due: Tue 7.2.2022 at 23:59

Fill-in-the-blanks 1. Let (X, d) be a metric space. Complete the proof of the following claim.

Claim. For any $z \in X$ and $r \geq 0$, the closed ball $\overline{\mathcal{B}}_r(z) \subset X$ is a closed set.

Proof. Let $z \in X$ and $r \geq 0$. The claim means, by definition of ______________________________, that the set ______________________ is open. We will prove this directly from the definition of open sets. For every $x \in$ ______________________ we must therefore find some $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(x) \subset X \setminus \overline{\mathcal{B}}_r(z)$.

So let $x \in$ ______________________ . Note that by definition of the complement and the closed ball we then have

$$\mathsf{d}(z, x) > r.$$

Let $\varepsilon = \mathsf{d}(z, x) - r$, and notice that from the above we get that $\varepsilon > 0$. Suppose now that $y \in \mathcal{B}_\varepsilon(x)$, i.e., $\mathsf{d}(x, y) < \varepsilon$. Then by the triangle inequality lower bound we get

$$\underline{\hspace{4cm}} \;\geq\; \mathsf{d}(z, x) - \mathsf{d}(x, y) \;>\; \mathsf{d}(z, x) - \varepsilon \;=\; \underline{\hspace{2cm}}.$$

This shows that $\mathcal{B}_\varepsilon(x) \subset X \setminus \overline{\mathcal{B}}_r(z)$. The proof is complete. $\square$

Fill-in-the-blanks 2. Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $M \geq 1$. A function $f\colon X \to Y$ is said to be M-**bilipschitz** if for all $x_1, x_2 \in X$ we have

$$\frac{1}{M}\,\mathsf{d}_X(x_1,x_2) \;\leq\; \mathsf{d}_Y\big(f(x_1),f(x_2)\big) \;\leq\; M\,\mathsf{d}_X(x_1,x_2).$$

Complete the proofs of the following statements.

a) **Claim.** Any M-bilipschitz function is injective.

Proof. Suppose $f\colon X \to Y$ is M-bilipschitz. Let $x_1, x_2 \in X$ be two different points, $x_1 \neq x_2$. Then from ____________________________

____ we get that

$$\mathsf{d}_Y\big(f(x_1),f(x_2)\big) \;\geq\; \frac{1}{M}\,\mathsf{d}_X(x_1,x_2) \;>\; 0.$$

By ____ , this implies $f(x_1) \neq f(x_2)$. Injectivity of f is thus proven. □

b) **Claim.** Any surjective M-bilipschitz function is a homeomorphism.

Proof. Suppose $f\colon X \to Y$ is surjective and M-bilipschitz. Then f is bijective, because it is ________________________ by assumption

and ________________________ by the observation above. It remains

to check that ____ .

The second inequality in the M-bilipschitz property shows that $f\colon X \to Y$ is in particular M-Lipschitz. It follows that f is continuous.

We will show that also $f^{-1}\colon Y \to X$ is M-Lipschitz. For this, let $y_1, y_2 \in Y$. From the first inequality in the bilipschitz property, we get

$$\frac{1}{M}\,\mathsf{d}_X\big(f^{-1}(y_1),f^{-1}(y_2)\big) \;\leq \mathsf{d}_Y\Big(\underline{\hspace{8cm}}\Big)$$
$$= \mathsf{d}_Y(y_1,y_2).$$

Multiplying by $M \geq 1$ we get $\mathsf{d}_X\big(f^{-1}(y_1),f^{-1}(y_2)\big) \leq M\,\mathsf{d}_Y(y_1,y_2)$, showing that f^{-1} is M-Lipschitz. It follows that f^{-1} is continuous. The proof is complete. □