(Exercise sessions: 27.-28.1.2022) Hand-in due: Tue 1.2.2022 at 23:59

Fill-in-the-blanks 1. Let (X, d) be a metric space and $(\mathsf{V}, \|\cdot\|)$ a normed space. On V, we use the metric induced by the norm, $\mathsf{d}_{\mathsf{V}}(u, v) = \|v - u\|$ for $u, v \in \mathsf{V}$. Complete the proof of the following statement.

Claim. If $f, g \colon X \to \mathsf{V}$ are two continuous functions, then the function $f + g \colon X \to \mathsf{V}$ is also continuous.

Remark: The function $f + g \colon X \to \mathsf{V}$ is the pointwise sum of the vector valued functions f and g, defined by the formula $(f + g)(x) = f(x) + g(x)$ (the right hand side is vector addition in vector space V).

Proof. First fix $x \in X$. Let us show that $f + g$ is continuous at the point x. Let $\varepsilon > 0$. Then also $\frac{\varepsilon}{2} > 0$, and since by assumption f is continuous at x, there exists a

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such that we have $\|f(x') - f(x)\| < \frac{\varepsilon}{2}$ whenever $x' \in X$ and

____________________ .

Similarly, since g is also continuous at x by assumption, there exists a

______________ such that __________________________________ .

Now let $\delta =$ ________________________ > 0.

Then whenever $x' \in X$ and $\mathsf{d}(x, x') < \delta$, we have both

$\mathsf{d}(x, x') <$ ____________ and $\mathsf{d}(x, x') <$ ____________ , so

$\|(f + g)(x') - (f + g)(x)\| =$ ________________________________

$\leq$ ________________________________

$<$ __________________________ $= \varepsilon.$

This implies that $f + g$ is continuous at x. Since $x \in X$ was arbitrary, $f + g$ is therefore a continuous function. $\square$

Fill-in-the-blanks 2. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, denote

$$\|x\|_p = \Big(\sum_{k=1}^{d} |x_k|^p\Big)^{1/p} \qquad \text{and} \qquad \|x\|_\infty = \max\{|x_1|, \ldots, |x_d|\}\,.$$

Complete the proof of the following result pertaining to the above norms on the n-dimensional space $\mathbb{R}^d$.

Claim. For all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have

$$\lim_{p\to\infty} \|x\|_p \;=\; \|x\|_\infty.$$

Proof. Consider a fixed $x \in \mathbb{R}^d$. (The parameter p, on the other hand, is thought of as a variable here.) Our goal is to prove that for all $p \geq 1$ we have

$$(\star) \qquad \|x\|_\infty \;\leq\; \|x\|_p \;\leq\; d^{1/p}\,\|x\|_\infty.$$

In view of

$$\lim_{p\to\infty} d^{1/p} \;=\; \underline{\hspace{4cm}}\,,$$

the claim then follows from $\underline{\hspace{8cm}}$.

(i) Let us first prove the left inequality in $(\star)$. From the definition of the norm $\|x\|_\infty$ it follows that there exists an index $m \in \{1, \ldots, d\}$ such that $|x_m| = \|x\|_\infty$. Since the function $u \mapsto u^{1/p}$, $u \geq 0$, is increasing, we have

$$\|x\|_\infty = |x_m| = \big(|x_m|^p\big)^{1/p} \leq \Big(\underline{\hspace{6cm}}\Big)^{1/p}$$

$$= \underline{\hspace{6cm}}.$$

This proves the left inequality in $(\star)$.

(ii) Let us then prove the right inequality in $(\star)$. Choose the index m as above. Since the function $u \mapsto u^p$, $u \geq 0$, is increasing, for all indices $k = 1, \ldots, d$ we have $|x_k|^p \leq |x_m|^p$, so we get

$$\|x\|_p = (|x_1|^p + \cdots + |x_d|^p)^{1/p} \leq \Big(\underline{\hspace{6cm}}\Big)^{1/p}$$

$$= \underline{\hspace{6cm}}$$

$$= \underline{\hspace{6cm}}.$$

The right inequality in $(\star)$ is thus also proven. $\square$