Aalto University Problem set 2

Department of Mathematics and Systems Analysis
MS C1541 Matria graces, 2022, 2024 (III

MS-C1541 — Metric spaces, 2023-2024/III — K Kytölä & D Adame-Carrillo

## Exercise sessions: 18.-19.1.2024 Hand-in due: Tue 23.1.2024 at 23:59

Topic: Continuous functions on  $\mathbb{R}$ , inner product spaces, normed spaces.

Written solutions to the exercises marked with symbol △ are to be returned in My-Courses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 2 is Tue 23.1.2024 at 23:59.

## Exercise 1 (Continuous functions on intervals).

Below (as usually), unless explicitly specified, an interval refers to either an open interval, closed interval, or a half-open interval, possibly unbounded. If  $c \in \mathbb{R}$ , we also interpret the singleton  $\{c\} \subset \mathbb{R}$  as a closed interval  $[a,b] \subset \mathbb{R}$  with coinciding endpoints, a = c = b.

- (a) Find an example of a nonempty interval, and a continuous function on this interval which has no maximum.
- (b) Find an example of a continuous function on an *open interval* which has a maximum but has no minimum.
- (c) Using a combination of results from Chapter III, prove that if  $f:[a,b] \to \mathbb{R}$  is a continuous function on a closed interval  $[a,b] \subset \mathbb{R}$ , then the image f[[a,b]] is a closed interval.
- (d) If  $f:(a,b)\to\mathbb{R}$  is a continuous function on an *open interval*  $(a,b)\subset\mathbb{R}$ , is it possible that the image f[(a,b)] is not an open interval?

Exercise 2 (Subpaces of the space of real-valued sequences). We consider it known that the space

$$\mathbb{R}^{\mathbb{N}} = \left\{ (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{R} \text{ for } n \in \mathbb{N} \right\}$$

of sequences of real numbers forms a (real) vector space, with addition and scalar multiplication defined coordinatewise.

- (a) Consider the subset  $\{(x_n)_{n\in\mathbb{N}} \mid \text{the limit } \lim_{n\to\infty} x_n \text{ exists}\} \subset \mathbb{R}^{\mathbb{N}} \text{ consisting of those real-number sequences which are convergent. Show that it forms a vector subspace of the space of all real-number sequences.$
- (b) Consider the subset  $\{(x_n)_{n\in\mathbb{N}} \mid \lim_{n\to\infty} x_n = 0\} \subset \mathbb{R}^{\mathbb{N}}$  consisting of those real-number sequences which converge to zero. Show that it forms a vector subspace of the space of all real-number sequences.
- (c) Consider the subset  $\{(x_n)_{n\in\mathbb{N}} \mid \lim_{n\to\infty} x_n = -\sqrt{\pi}\} \subset \mathbb{R}^{\mathbb{N}}$  consisting of those real-number sequences which converge to  $-\sqrt{\pi}$ . Show that it does *not* form a vector subspace of the space of all real-number sequences.

<sup>&</sup>lt;sup>1</sup>For closed intervals this is a natural convention in the degenerate case of coinciding endpoints, since  $[a,b] \subset \mathbb{R}$  by definition consists of points  $x \in \mathbb{R}$  such that  $a \leq x \leq b$ .

- $\triangle$  Exercise 3 (The  $\ell^1$ -norm in finite dimensional spaces).
  - (a) Prove that the formula

$$||x||_1 = |x_1| + |x_2| + \dots + |x_n|, \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

defines a quantity that satisfies the following conditions:

- (N1)  $||x+y||_1 \le ||x||_1 + ||y||_1$  for all  $x, y \in \mathbb{R}^n$ ;
- (N2)  $||cx||_1 = |c| ||x||_1$  for all  $x \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ ;
- (N3)  $||x||_1 = 0$  if and only if  $x = \vec{0}$ .
- (b) Let n = 2. Sketch the set

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 \mid ||(x_1, x_2)||_1 = |x_1| + |x_2| = 1 \right\},\,$$

i.e., the "unit circle" under the norm of part (a).

- Exercise 4 (Parallelogram law and norms not induced by inner products).
  - (a) Prove that the norm in an inner product space  ${\sf V}$  satisfies the parallelogram law

$$||u+v||^2 + ||u-v||^2 = 2 ||u||^2 + 2 ||v||^2$$
 for all  $u, v \in V$ .

(b) Using part (a), prove that the norm  $\|\cdot\|_1$  of Exercise 3 on  $\mathbb{R}^n$  is not induced by any inner product on  $\mathbb{R}^n$  when  $n \geq 2$ . Why doesn't the same reasoning work in the case n = 1?

## Exercise 5 (Ideas behind Fourier series).

<u>Hint</u>: The solution is shorter than the problem statement!

An indexed collection  $(e_t)_{t\in T}$  of vectors  $e_t \in V$  in an inner product space V is called orthonormal, if for all  $s, t \in T$ , we have

$$\langle e_t, e_s \rangle = \delta_{t,s} = \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{if } t \neq s \end{cases}$$
 (the Kronecker  $\delta$ -symbol).

Remark: Here T can be an arbitrary index set.

(a) Consider a countable orthonormal collection  $(e_n)_{n\in\mathbb{N}}$  in an inner product space V. Assume that a vector  $v\in V$  can be expressed, for some  $m\in\mathbb{N}$ , as a (finite) linear combination

$$v = \alpha_1 e_1 + \cdots + \alpha_m e_m$$
 with some coefficients  $\alpha_n \in \mathbb{R}$ .

Calculate the inner products between the vector v and the vectors  $e_n$  from the orthonormal collection, and deduce a formula for the coefficients  $\alpha_n$ .

*Remark:* The result generalizes to countable linear combinations (i.e. series), which are convergent in a suitable sense.

(b) Define functions  $c_0, s_1, c_1, s_2, c_2, s_3, c_3, \ldots$  of a real variable x by the formulas

$$c_0(x) = \frac{1}{\sqrt{2}},$$

$$s_n(x) = \sin(nx) \qquad \text{(for } n \in \mathbb{N}),$$

$$c_n(x) = \cos(nx) \qquad \text{(for } n \in \mathbb{N}).$$

Prove that the (countably infinite) collection of these functions is orthonormal in the space  $C([-\pi, \pi])$  of continuous functions on  $[-\pi, \pi]$ , with respect to the rescaled inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, \mathrm{d}x.$$

Remark: Since there are 6–9 different cases (depending on the counting convention), it is enough to do the details of just four cases. The complex version would be somewhat easier...

(c) The idea of Fourier series is founded on the following result: Every continuously differentiable  $2\pi$ -periodic function  $f: \mathbb{R} \to \mathbb{R}$  can be represented as

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos(nx) + \beta_n \sin(nx)).$$

Consider for simplicity the case of a function f, for which the above series contains only finitely many terms, i.e., for some  $m \in \mathbb{N}$  we have

$$f(x) = \alpha_0 + \sum_{n=1}^{m} (\alpha_n \cos(nx) + \beta_n \sin(nx)).$$

Using parts (a) and (b), derive the following formula for the coefficients

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \qquad (n \in \mathbb{N}).$$

<u>Hint</u>: Using parts (a) and (b) is indeed the intended approach, instead of calculating everything from scratch again!