(Exercise sessions: 12.-13.1.2023)   Hand-in due: Tue 17.1.2023 at 23:59

Fill-in-the-blanks 1. Complete the following proofs about injectivity, surjectivity, and bijectivity of two functions given by the same formula, but with different domains and codomains.

Claim (a): Consider the function $f \colon \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$ for $x \in \mathbb{R}$. This f is surjective but not injective. It is not bijective.

Proof of (a). To prove surjectivity of f, we must show that for every $y \in [0, \infty)$ there exists _____ such that $f(x) = y$.

Let $y \in [0, \infty)$. The non-negative number y has a non-negative square root $x = \sqrt{y} \geq 0$. Then we have

$$f(x) \;=\; x^2 \;=\; (\sqrt{y})^2 \;=\; \underline{} \,.$$

Since $x \in \mathbb{R}$, this shows that f is surjective.

To prove non-injectivity of f, we must prove that there exists $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$ and _____ .

Consider for example $x_1 = -3$ and $x_2 = \underline{}$. We have

$$f(x_1) \;=\; f(-3) \;=\; (-3)^2 \;=\; 9 \;=\; (\underline{})^2 \;=\; f(\underline{}) \;=\; f(x_2).$$

Since $x_1 \neq x_2$, this shows that f is not injective.

Finally, f is not bijective because it _____. □

Claim (b): Consider the function $g \colon [0, \infty) \to \mathbb{R}$ given by $g(x) = x^2$ for $x \in [0, \infty)$. This g is injective but not surjective. It is not bijective.

Proof of (b). To prove non-surjectivity of g, we must show that for some $y \in \mathbb{R}$ there does not exists any _____ such that $g(x) = y$. For example $y = -1$ works, because for any x we have $g(x) = x^2 \geq \underline{}$ and thus in particular $g(x) \neq -1 = y$.

To show injectivity of g, we must prove that if for $x_1, x_2 \in \underline{}$ we have $g(x_1) = g(x_2)$, then necessarily $x_1 = x_2$. So suppose that x_1, x_2 are such a pair. From $x_1^2 = g(x_1) = g(x_2) = x_2^2$, we then get $x_1 = \pm\sqrt{x_2^2} = \pm|x_2|$. However, since _____ , this is only possible if $x_1 = x_2$. Injectivity follows.

Now g is not bijective because it _____. □

Fill-in-the-blanks 2. Complete the following proof of the squeeze theorem (sandwich principle, lemma of two policemen).

Claim: If three sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$ of real numbers satisfy $a_n \leq b_n \leq c_n$ starting from some index, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \beta \in \mathbb{R},$$

then the sequence $(b_n)_{n\in\mathbb{N}}$ also converges, and $\lim_{n\to\infty} b_n = \beta$.

Proof. Since the beginning of a sequence affects neither the convergence nor the limit of the sequence, we may assume that $a_n \leq b_n \leq c_n$ holds for all $n \in \mathbb{N}$. We will show that $\lim_{n\to\infty} b_n = \beta$.

Let $\varepsilon > 0$. We must show that $|b_n - \beta| < \varepsilon$ from some index on.

Idea: the expression $b_n - \beta$ has to be estimated from both directions; one relying on the sequence $(a_n)_{n\in\mathbb{N}}$, and the other on the sequence $(c_n)_{n\in\mathbb{N}}$. (Draw a figure!)

Since $\lim_{n\to\infty} a_n = \beta$ and $\varepsilon > 0$, there exists an $n'_\varepsilon \in \mathbb{N}$ such that

_____ for all $n \geq n'_\varepsilon$.

Since $\lim_{n\to\infty} c_n = \beta$ and $\varepsilon > 0$, there exists an $n''_\varepsilon \in \mathbb{N}$ such that

_____ for all _____ .

Now choose

$$n_\varepsilon = \text{\underline{\hspace{5cm}}} .$$

With this choice, for any $n \geq n_\varepsilon$ we have $n \geq n'_\varepsilon$. Therefore we get

$$\beta - b_n \ \leq \ \beta - a_n \ \leq \ |\beta - a_n| \ < \ \text{\underline{\hspace{2cm}}}$$

(the leftmost inequality holds by virtue of the assumption $a_n \leq b_n$). Similarly, for $n \geq n_\varepsilon$ we have $n \geq n''_\varepsilon$, so we get

$$b_n - \beta \ \leq \ c_n - \beta \ \leq \ \text{\underline{\hspace{2cm}}} \ < \ \text{\underline{\hspace{2cm}}}$$

(the leftmost inequality holds by virtue of the assumption $b_n \leq c_n$). The above inequalities imply that

$$|b_n - \beta| < \text{\underline{\hspace{2cm}}}$$

for all $n \geq n_\varepsilon$. We have thus proved the claim. $\square$