(Exercise sessions: 3.-4.2.2022) **Hand-in due: Tue 8.2.2022 at 23:59**

Fill-in-the-blanks 1. Let (X, d) be a metric space. Complete the proof of the following claim.

Claim. For any $z \in X$ and $r \geq 0$, the closed ball $\overline{\mathcal{B}}_r(z) \subset X$ is a closed set.

Proof. Let $z \in X$ and $r \geq 0$. The claim means, by definition of ________________________, that the set __________________ is open. We will prove this directly from the definition of open sets. For every $x \in$ __________________ we must therefore find some $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(x) \subset X \setminus \overline{\mathcal{B}}_r(z)$.

So let $x \in$ __________________ . Note that by definition of the complement and the closed ball we then have

$$\mathsf{d}(z, x) > r.$$

Let $\varepsilon = \mathsf{d}(z, x) - r$, and notice that from the above we get that $\varepsilon > 0$. Suppose now that $y \in \mathcal{B}_\varepsilon(x)$, i.e., $\mathsf{d}(x, y) < \varepsilon$. Then by the triangle inequality lower bound we get

$$\underline{\qquad\qquad\qquad} \;\geq\; \mathsf{d}(z, x) - \mathsf{d}(x, y) \;>\; \mathsf{d}(z, x) - \varepsilon \;=\; \underline{\qquad\quad}.$$

This shows that $\mathcal{B}_\varepsilon(x) \subset X \setminus \overline{\mathcal{B}}_r(z)$. The proof is complete. $\square$

Fill-in-the-blanks 2. Complete the proofs of the following statements.

a) **Claim.** If (X, d) is a metric space and $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in the space X, then the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X and denote its limit by

$$a = \lim_{n\to\infty} x_n.$$

Apply the definition of limit by choosing $\varepsilon = 1$. Then there exists an $n_1 \in \mathbb{N}$, such that

$$\underline{\qquad\qquad\qquad\qquad} < 1 \quad \text{whenever} \quad \underline{\qquad\qquad\qquad\qquad}.$$

Since $x_1, x_2, \ldots, x_{n_1-1}$ is a finite list of points, we can define

$$R = \max\{1,\ \mathsf{d}(x_1, a),\ \underline{\qquad\qquad\qquad}, \ldots, \underline{\qquad\qquad\qquad}\} < \infty.$$

Then for all members x_k of the sequence we have

$$\mathsf{d}(x_k, a) \leq \underline{\qquad\qquad\qquad},$$

so the members of the sequence are contained in the closed ball

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}.$$

This proves that the sequence is bounded. □

b) **Claim.** Consider the metric space $(\mathbb{R}, \mathsf{d}_{0/1})$, where on the real line $\mathbb{R}$ we use the discrete 0/1-metric $\mathsf{d}_{0/1}$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ in $(\mathbb{R}, \mathsf{d}_{0/1})$ defined by the formula $x_n = \frac{1}{n}$ does not have 0 as its limit.

Proof. For all $n \in \mathbb{N}$ we have

$$\mathsf{d}_{0/1}(x_n, 0) = \underline{\qquad\qquad\qquad},$$

so there does not exists $\underline{\qquad\qquad\qquad}$ such that

$$\underline{\qquad\qquad\qquad\qquad\qquad} < 1/2 \quad \text{for all} \quad \underline{\qquad\qquad\qquad\qquad\qquad}.$$

The defining condition of limit therefore is not fulfilled for $\varepsilon = 1/2$, so the sequence $(x_n)_{n\in\mathbb{N}}$ does not converge to 0. □