Exercise sessions: 14.-15.1.2021 **Hand-in due: Wed 20.1.2021 at 9:00**

Topic: Compactness, connectedness (see, e.g., [Carothers: Chapters 8 and 6])
Exercises 1 and 2 are to be solved prior to the exercise session, and you are asked to mark the exercises you have solved in a list. The teaching assistant chooses one of the students to present his/her solution to one exercise on the blackboard, and the TA presents the solution to the other problem himself/herself. The rest of the exercise session can be used for working on the remaining exercises, which are to be turned in for grading.

Written solutions to the exercises marked with symbol ✍ are to be returned to the mailbox opposite to Laskutupa (Y190c). Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 1P is Wed 20.1.2021 at 9:00.

Exercise 1. Prove the following assertions related to compactness by forming a continuous function, which settles the matter.

(a) If $A \subset \mathbb{R}^3$ is a compact subset, then it contains a highest point $a \in A$, i.e., a point $a = (a_1, a_2, a_3)$ such that we have $a_3 \geq x_3$ for all $x = (x_1, x_2, x_3) \in A$.
(b) If $B \subset \mathbb{R}^2$ is a compact subset, then it contains points $a, b, c \in B$, for which the perimeter
$$\|a - b\| + \|b - c\| + \|c - a\|$$
of the triangle is maximal among triangles whose vertices lie in B.
(c) If $\mathfrak{X}$ is a compact metric space and $g\colon \mathfrak{X} \to \mathfrak{X}$ is a continuous function that has no fixed points, then there exists a $c > 0$ such that $d(g(x), x) \geq c$ for all $x \in \mathfrak{X}$.

Remark: Each part can alternatively be solved by first picking a sequence which approaches the sup/inf-value.

Exercise 2. Let X be a connected metric space and $A, B \subset X$ two non-empty subsets. Show that there exists (at least one) point $x \in X$ such that $d(x, A) = d(x, B)$.
Hint: *Consider the continuous function $f(x) = d(x, A) - d(x, B)$.*

✍ **Exercise 3.** Prove that for any two compact subsets $A, B \subset \mathfrak{X}$ of a metric space $\mathfrak{X}$, the union $A \cup B$ is compact, ...

(a) ...directly from the definition of (sequential) compactness [Carothers, Thm 8.2].
(b) ...by using the characterization of compactness by open covers [Carothers, Thm 8.9].

Hint: *The proofs should begin as follows:*
a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $A \cup B$.
b) Let $\mathcal{D}$ be an open cover of $A \cup B$.

✍ **Exercise 4.** Let X and Y be two non-empty metric spaces. Show that if either X or Y is disconnected, then the Cartesian product $X \times Y$ is also disconnected.

✍ **Exercise 5.**

(a) Let X be a metric space and $(A_n)_{n\in\mathbb{N}}$ a decreasing sequence (i.e., $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$) of non-empty compact subsets of the space X. Prove that the intersection of these subsets

$$A = \bigcap_{n=1}^{\infty} A_n$$

is (i) non-empty; and (ii) compact.

(b) Give an example of a decreasing sequence of non-empty closed subsets, whose intersection is empty.