Aalto University **Problem set 4**
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2023-2024/III K Kytölä & D Adame-Carrillo

Exercise sessions: 1.-2.2.2024 **Hand-in due: Tue 6.2.2024 at 23:59**

Topic: Continuous functions, homeomorphisms, sequences in metric spaces

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 4 is Tue 6.2.2024 at 23:59.

Exercise 1 (Verifying openness and closedness of subsets)**.**

(a) Show that the set $U = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 < y^2 + z^2 - xyz + 3\right\} \subset \mathbb{R}^3$ is open and that the set $F = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 \leq 2 \text{ and } x \leq \frac{1}{3}\sin(\pi y)\right\} \subset \mathbb{R}^2$ is closed.

Hint: *Express the sets U and F appropriately using preimages; for the latter it is best to use two different functions and an intersection of preimages.*
The continuity of the functions involved can be considered known.

(b) Consider the space $\mathcal{C}([-1,1])$ of all continuous functions $f \colon [-1,1] \to \mathbb{R}$, equipped with the metric induced by the sup-norm $\|\cdot\|_\infty$. Consider the subset[1]

$$D = \left\{p \in \mathcal{C}([-1,1]) \;\middle|\; p(x) \geq 0 \;\forall x \in [-1,1],\; \int_{-1}^{1} p(x)\,\mathrm{d}x = 1\right\}.$$

Show that $D \subset \mathcal{C}([-1,1])$ is a closed set.

Hint: *You may use the facts that evaluation functions $f \mapsto f(x)$ (for an arbitrary $x \in [-1,1]$), and the integration function $f \mapsto \int_{-1}^{1} f(x)\,\mathrm{d}x$ are continuous functions $\mathcal{C}([-1,1]) \to \mathbb{R}$ with the chosen metric. Otherwise the ideas are similar to part (a).*

Exercise 2 (The closure of a set)**.**

Let (X,d) be a metric space and $A \subset X$ a subset. The closure of A is by definition the set

$$\overline{A} = X \setminus \mathrm{ext}(A), \qquad \text{where}$$
$$\mathrm{ext}(A) = \left\{x \in X \;\middle|\; \exists r > 0 :\; \mathcal{B}_r(x) \subset X \setminus A\right\}.$$

(a) Prove that the set $\overline{A}$ is closed.

(b) Prove that for $x \in X$, the following two conditions are equivalent:

(i) $x \in \overline{A}$;

(ii) there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = x$.

[1] This subset D could be interpreted as the set of all continuous probability density functions supported on the interval $[-1,+1]$.

✍ **Exercise 3.**

(a) Let X be a set, $J \neq \emptyset$ a nonempty index set, and $A_j \subset X$ subsets for each $j \in J$. Prove De Morgan's laws for arbitrary unions and intersections:

$$X \setminus \bigcup_{j \in J} A_j = \bigcap_{j \in J} (X \setminus A_j) \qquad \text{and} \qquad X \setminus \bigcap_{j \in J} A_j = \bigcup_{j \in J} (X \setminus A_j).$$

(b) Using part (a) and known properties of open sets, prove the following (stated as Theorem VII.16 in the lecture notes):

- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

✍ **Exercise 4** (Some homeomorphisms)**.**

(a) Let $a, b \in \mathbb{R}$ with $a < b$. Prove that the open interval $(a, b) \subset \mathbb{R}$ and the real line $\mathbb{R}$ are homeomorphic, $(a, b) \approx \mathbb{R}$.

Remark: The continuity of the functions involved can be considered known (you can for example use judiciously chosen rational functions).

(b) Consider the cylinder surface $C \subset \mathbb{R}^3$ and the annulus $A \subset \mathbb{R}^2$ given by

$$C = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 = 1 \right\}$$
$$A = \left\{ (\xi, \eta) \in \mathbb{R}^2 \;\middle|\; 1 < \xi^2 + \eta^2 < 2 \right\}$$

(we named the coordinates in $\mathbb{R}^3$ and $\mathbb{R}^2$ differently here to avoid confusion). Prove that the cylinder C and the annulus A are homeomorphic, $C \approx A$.

Hint: *To construct a homomorphism between C and A, you can use a homeomorphism $(1, 2) \approx \mathbb{R}$ from part (a) to make the radial direction of the annulus correspond to the z-coordinate of the cylider, while doing something simpler with the angular parts.*

✍ **Exercise 5** (Coordinatewise convergence is not sufficient for convergence in ℓ^1)**.**
Consider the space

$$\ell^1 = \left\{ x = (x_j)_{j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \sum_{j=1}^{\infty} |x_j| < \infty \right\}$$

of absolutely summable real sequences. We consider it known that the formula $\left\|x\right\|_1 = \sum_{j=1}^{\infty} |x_j|$ for $x = (x_j)_{j \in \mathbb{N}} \in \ell^1$ defines a norm on ℓ^1. We equip ℓ^1 with the metric induced by the norm $\| \cdot \|_1$.

(a) Show that if a sequence $(x^{(n)})_{n \in \mathbb{N}}$ of elements $x^{(n)} = (x_j^{(n)})_{j \in \mathbb{N}} \in \ell^1$ converges in ℓ^1 to $x = (x_j)_{j \in \mathbb{N}}$, then for every $k \in \mathbb{N}$, the sequence $(x_k^{(n)})_{n \in \mathbb{N}}$ of the k:th coordinates of $x^{(n)}$'s converges to $\lim_{n \to \infty} x_k^{(n)} = x_k$ (limit in $\mathbb{R}$).

Hint: *You can start by showing that the k:th coordinate projection function $(x_j)_{j \in \mathbb{N}} \mapsto x_k$ is a 1-Lipschitz function $\ell^1 \to \mathbb{R}$.*

(b) For $n \in \mathbb{N}$ let $x^{(n)} = (x_j^{(n)})_{j \in \mathbb{N}} \in \ell^1$ be the element given by

$$x_j^{(n)} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } j \neq n. \end{cases}$$

Show that for any $k \in \mathbb{N}$ we have $\lim_{n \to \infty} x_k^{(n)} = 0$ but in the space $(\ell^1, \| \cdot \|_1)$ the sequence $(x^{(n)})_{n \in \mathbb{N}}$ does not converge.

Hint: *If the sequence would converge in ℓ^1, then part (a) together with the first calculation of (b) identifies the only possibility for a limit $x \in \ell^1$. Now show directly from the definition of limits that we do not have convergence to that candidate limit.*