(Exercise sessions: 10.-11.2.2022) Hand-in due: Tue 15.2.2022 at 23:59

Denote by ℓ^1 the space of absolutely summable real-number sequences, i.e., the space whose elements are sequences $x = (x_j)_{j\in\mathbb{N}}$ such that $\sum_{j=1}^{\infty} |x_j| < \infty$. We consider it known that ℓ^1 becomes a normed space when equipped with the norm given by the formula $\|x\|_1 := \sum_{j=1}^{\infty} |x_j|$. Equip the space ℓ^1 with the metric induced by $\|\cdot\|_1$.

Note that a sequence $(x^{(n)})_{n\in\mathbb{N}}$ in ℓ^1 is a sequence of sequences: to each $n \in \mathbb{N}$ corresponds an absolutely summable sequence $x^{(n)} = (x^{(n)}_j)_{j\in\mathbb{N}}$.

Fill-in-the-blanks 1. Complete the following proof.

Claim. Let $(x^{(m)})_{m\in\mathbb{N}}$ be a sequence in ℓ^1. Assume:

(A) for every $m \in \mathbb{N}$ we have $\|x^{(m+1)} - x^{(m)}\|_1 \leq 5^{-m}$;

(B) for every $j \in \mathbb{N}$ the sequence $(x^{(m)}_j)_{m\in\mathbb{N}}$ of the j:th coordinates converges to a limit $x_j = \lim_{m\to\infty} x^{(m)}_j$.

Then for every $m \in \mathbb{N}$ we have $\sum_{j=1}^{\infty} \left|x_j - x^{(m)}_j\right| \leq \frac{5}{4}\, 5^{-m}$.

Proof. For any $k > m$, we have "telescopic" cancellations in the sum

$$x^{(k)} - x^{(m)} = \left(x^{(k)} - x^{(k-1)}\right) + \left(x^{(k-1)} - x^{(k-2)}\right) + \cdots + \left(x^{(m+1)} - x^{(m)}\right)$$
$$= \sum_{r=m}^{k-1} \left(x^{(r+1)} - x^{(r)}\right).$$

Using ______________________________ for the above sum we get

$$\|x^{(k)} - x^{(m)}\|_1 \leq \sum_{r=m}^{k-1} \|x^{(r+1)} - x^{(r)}\|_1 \leq \sum_{r=m}^{k-1} 5^{-r} = \frac{5^{-m} - 5^{-k}}{4/5}.$$

For any $J \in \mathbb{N}$ we now get

$$\sum_{j=1}^{J} \left|x^{(k)}_j - x^{(m)}_j\right| \leq \|x^{(k)} - x^{(m)}\|_1 \leq \frac{5^{-m} - 5^{-k}}{4/5} \leq \frac{5}{4}\, 5^{-m}.$$

For every $j \in \mathbb{N}$, we have $\lim_{k\to\infty} |x^{(k)}_j - x^{(m)}_j| = |x_j - x^{(m)}_j|$, because ____________________________________ and because the function $z \mapsto |z - x^{(m)}_j|$ is _________________________. Taking a finite sum, we get $\lim_{k\to\infty} \sum_{j=1}^{J} |x^{(k)}_j - x^{(m)}_j| = \sum_{j=1}^{J} |x_j - x^{(m)}_j|$. The preservation of bounds in the limit $k \to \infty$ then yields $\sum_{j=1}^{J} |x_j - x^{(m)}_j| \leq \frac{5}{4}\, 5^{-m}$. From this, the preservation of bounds in the limit $J \to \infty$ yields the asserted inequality.

Fill-in-the-blanks 2. Complete the following proof.

Claim. The space ℓ^1 is complete.

Proof. To show completeness, we must prove that every Cauchy sequence converges in ℓ^1. So let $(x^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ^1.

Note that it suffices to prove that $(x^{(n)})_{n\in\mathbb{N}}$ has a convergent subsequence, because ____.

Note also that for every $j \in \mathbb{N}$, the real-number sequence $(x_j^{(n)})_{n\in\mathbb{N}}$ is __________________________ , because it is obtained from the Cauchy sequence $(x^{(n)})_{n\in\mathbb{N}}$ in ℓ^1 by applying the 1-Lipschitz function $\mathrm{pr}_j \colon \ell^1 \to \mathbb{R}$ which picks the j:th coordinate of an absolutely summable sequence. By completeness of _______ , we then get that the limit $x_j = \lim_{n\to\infty} x_j^{(n)} \in$ _______ exists for every $j \in \mathbb{N}$. From these limits we form a sequence $x = (x_j)_{j\in\mathbb{N}}$.

Since $(x^{(n)})_{n\in\mathbb{N}}$ is Cauchy, for every $m \in \mathbb{N}$ we can choose a $n_m \in \mathbb{N}$ such that for all $k, l \geq n_m$ we have $\|x^{(k)} - x^{(l)}\|_1 < 5^{-m}$. Moreover, it is possible to choose these indices so that $n_1 < n_2 < n_3 < \ldots$. We will show that the subsequence $(x^{(n_m)})_{m\in\mathbb{N}}$ is convergent.

The subsequence satisfies the assumptions of Fill-in-the-blanks 1, so we get that for every $m \in \mathbb{N}$

$$(\star) \qquad \sum_{j=1}^{\infty} \left| x_j - x_j^{(n_m)} \right| \;\leq\; \frac{5}{4}\, 5^{-m}.$$

This shows in particular that the sequence $x - x^{(n_m)} = (x_j - x_j^{(n_m)})_{j\in\mathbb{N}}$ is absolutely summable. The sequence x can be written as a sum $x = (x - x^{(n_m)}) + x^{(n_m)}$. Since both summands here belong to the vector space _______ , this allows us to conclude the same for x.

Using the estimate $(\star)$ we also get

$$0 \;\leq\; \mathsf{d}(x, x^{(n_m)}) \;=\; \left\| x^{(n_m)} - x \right\|_1 \;=\; \sum_{j=1}^{\infty} |x_j^{(n_m)} - x_j| \;\leq\; \frac{5}{4}\, 5^{-m}.$$

Since ________________________ $= 0$, by the squeeze theorem we get $\lim_{m\to\infty} \mathsf{d}(x, x^{(n_m)}) = 0$. From here we conclude that the subsequence $(x^{(n_m)})_{m\in\mathbb{N}}$ converges to x in ℓ^1. $\square$