Aalto University **Problem set 4**
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MS-C1541 — Metric spaces, 2021-2022/III K Kytölä & M Orlich

Exercise sessions: 3.-4.2.2022 **Hand-in due: Tue 8.2.2022 at 23:59**

Topic: Continuous functions, homeomorphisms, sequences in metric spaces

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 4 is Tue 8.2.2022 at 23:59.

Exercise 1 (Verifying openness and closedness of subsets)**.**

(a) Show that the set $U = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2 < y^2 + z^2 - xyz + 3\right\} \subset \mathbb{R}^3$ is open and that the set $F = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \leq 2 \text{ and } x \leq \frac{1}{3}\sin(\pi y)\right\} \subset \mathbb{R}^2$ is closed.

Hint: *Express the sets U and F appropriately using preimages; for the latter it is best to use two different functions and an intersection of preimages.*

The continuity of the functions involved can be considered known.

(b) Consider the space $\mathcal{C}([-1,1])$ of all continuous functions $f\colon [-1,1] \to \mathbb{R}$, equipped with the metric induced by the sup-norm $\|\cdot\|_\infty$. Consider the subset[1]

$$D = \left\{p \in \mathcal{C}([-1,1]) \,\middle|\, p(x) \geq 0 \;\forall x \in [-1,1],\; \int_{-1}^{1} p(x)\,\mathrm{d}x = 1\right\}.$$

Show that $D \subset \mathcal{C}([-1,1])$ is a closed set.

Hint: *You may use the facts that evaluation functions $f \mapsto f(x)$ (for an arbitrary $x \in [-1,1]$), and the integration function $f \mapsto \int_{-1}^{1} f(x)\,\mathrm{d}x$ are continuous functions $\mathcal{C}([-1,1]) \to \mathbb{R}$ with the chosen metric. Otherwise the ideas are similar to part (a).*

Exercise 2 (The closure of a set)**.**

Let (X, d) be a metric space and $A \subset X$ a subset. The closure of A is by definition the set

$$\overline{A} = X \setminus \mathrm{ext}(A), \qquad \text{where}$$
$$\mathrm{ext}(A) = \left\{x \in X \,\middle|\, \exists r > 0 :\; \mathcal{B}_r(x) \subset X \setminus A\right\}.$$

(a) Prove that the set $\overline{A}$ is closed.

(b) Prove that for $x \in X$, the following two conditions are equivalent:

(i) $x \in \overline{A}$;

(ii) there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = x$.

[1] This subset D could be interpreted as the set of all continuous probability density functions supported on the interval $[-1,+1]$.

✍ **Exercise 3.**

(a) Let X be a set, $J \neq \emptyset$ a nonempty index set, and $A_j \subset X$ subsets for each $j \in J$. Prove De Morgan's laws for arbitrary unions and intersections:

$$X \setminus \bigcup_{j \in J} A_j \;=\; \bigcap_{j \in J} (X \setminus A_j) \qquad \text{and} \qquad X \setminus \bigcap_{j \in J} A_j \;=\; \bigcup_{j \in J} (X \setminus A_j).$$

Hint: *Often the equality $A = B$ between two sets is most straightforward to prove in two stages: $A \subset B$ and $B \subset A$. In this problem the two stages can be combined, by directly establishing a chain of equivalent conditions $x \in A \Leftrightarrow \cdots \Leftrightarrow x \in B$.*

(b) Using part (a) and known properties of open sets, prove the following (stated as Theorem VII.14 in the lecture notes):

- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Hint: *The first proof could start with: "Let J be an index set, and $F_j \subset X$ closed sets, for $j \in J$."*

✍ **Exercise 4** (Some homeomorphisms)**.**

(a) Prove that $[0, 1)$ and $[0, \infty)$ are homeomorphic, $[0, 1) \approx [0, \infty)$.

Remark: The subsets of the real axis above are equipped with the ordinary metric. The continuity of the functions involved can be considered known.

(b) Prove that the open unit disk $B(\vec{0}, 1) \subset \mathbb{R}^2$ in the Euclidean plane $\mathbb{R}^2$ and the whole plane $\mathbb{R}^2$ are homeomorphic, $B(\vec{0}, 1) \approx \mathbb{R}^2$.

Hint: *One option is a radial mapping in polar coordinates, using part (a).*

✍ **Exercise 5** (Coordinatewise convergence is not sufficient for convergence in ℓ^1)**.** Consider the space

$$\ell^1 \;=\; \left\{ x = (x_j)_{j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \sum_{j=1}^{\infty} |x_j| < \infty \right\}$$

of absolutely summable real sequences. We consider it known that the formula $\|x\|_1 = \sum_{j=1}^{\infty} |x_j|$ for $x = (x_j)_{j \in \mathbb{N}} \in \ell^1$ defines a norm on ℓ^1. We equip ℓ^1 with the metric induced by the norm $\|\cdot\|_1$.

(a) Show that if a sequence $(x^{(n)})_{n \in \mathbb{N}}$ of elements $x^{(n)} = (x_j^{(n)})_{j \in \mathbb{N}} \in \ell^1$ converges in ℓ^1 to $x = (x_j)_{j \in \mathbb{N}}$, then for every $k \in \mathbb{N}$, the sequence $(x_k^{(n)})_{n \in \mathbb{N}}$ of the k:th coordinates of $x^{(n)}$'s converges to $\lim_{n \to \infty} x_k^{(n)} = x_k$ (limit in $\mathbb{R}$).

Hint: *You can start by showing that the k:th coordinate projection function $(x_j)_{j \in \mathbb{N}} \mapsto x_k$ is a 1-Lipschitz function $\ell^1 \to \mathbb{R}$.*

(b) For $n \in \mathbb{N}$ let $x^{(n)} = (x_j^{(n)})_{j \in \mathbb{N}} \in \ell^1$ be the element given by

$$x_j^{(n)} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } j \neq n. \end{cases}$$

Show that for any $k \in \mathbb{N}$ we have $\lim_{n \to \infty} x_k^{(n)} = 0$ but in the space $(\ell^1, \|\cdot\|_1)$ the sequence $(x^{(n)})_{n \in \mathbb{N}}$ does not converge.

Hint: *If the sequence would converge in ℓ^1, then part (a) together with the first calculation of (b) identifies the only possibility for a limit $x \in \ell^1$. Now show directly from the definition of limits that we do not have convergence to that candidate limit.*