Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021/III

Problem set 1

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Exercise sessions: 14.-15.1.2021 **Hand-in due: ??.1.2021 at ??:??**

Topic: Sets, functions, real numbers

Written solutions to the exercises marked with symbol ✍ are to be returned in My-Courses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 1 is ??.1.2021 at ??:??.

Exercise 1 (Examine properties of some sets)**.**
Consider the following subsets of the real line $\mathbb{R}$:

$$A_1 = [0, \sqrt{2}] \cap \mathbb{Q}\,, \qquad A_2 = \left\{ e^{-x} \;\middle|\; x \in \mathbb{R} \right\}, \qquad A_3 = \left\{ \frac{7n(-1)^n + 4}{6n} \;\middle|\; n \in \mathbb{N} \right\}.$$

Pick two of these three sets A_j, $j \in \{1, 2, 3\}$, and answer the following questions about each of them.

- Is the set A_j bounded from above?
- Is it bounded from below?
- Does it have a maximum, and if it does, what is $\max A_j$?
- Does it have a supremum, and if it does, what is $\sup A_j$?
- Does it have a minimum, and if it does, what is $\min A_j$?
- Does it have an infimum, and if it does, what is $\inf A_j$?

Exercise 2 (Images and preimages of unions and intersections)**.**
Let X and Y be sets and $f : X \to Y$ a function.

(a) Show that for any $C, D \subset Y$, the preimages satisfy
$$f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D].$$

(b) Show that for any $C, D \subset Y$, the preimages satisfy
$$f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D].$$

(c) Show that for any $A, B \subset X$, the images satisfy
$$f[A \cup B] = f[A] \cup f[B].$$

(d) Give an example in which for the images of subsets $A, B \subset X$ we have
$$f[A \cap B] \neq f[A] \cap f[B].$$

Hint: *In parts (a)–(c) it is possible to argue by a chain of equivalent conditions*

$$x \in \textit{left hand side set} \iff \cdots \iff x \in \textit{right hand side set.}$$

✍ **Exercise 3** (Non-zero limit implies members are eventually non-zero)**.**

(a) Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers which tends to a limit

$$\lim_{n\to\infty} a_n \;=\; 1.$$

Show that there exists some $N \in \mathbb{N}$ such that $b_n > \frac{1}{2}$ for all $n \geq N$.

(b) Suppose that $(b_n)_{n\in\mathbb{N}}$ is a sequence of real numbers which tends to a non-zero limit

$$\beta \;=\; \lim_{n\to\infty} b_n \;\neq\; 0.$$

Show that there exists some $M \in \mathbb{N}$ such that $b_n \neq 0$ for all $n \geq M$.

✍ **Exercise 4** (Examine properties of some sequences)**.**
Let

$$a_n = \frac{n+1}{2n+1} \qquad \text{and} \qquad b_n = \sqrt[n]{3^n + 2^n} \qquad \text{for } n \in \mathbb{N}.$$

Pick one of the two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ and answer the following questions about it. Is the sequence increasing, decreasing, bounded from above, bounded from below? Does it converge, and if it does, then what is the limit?

You may use known properties of limits.

✍ **Exercise 5** (A variant of a formulation of the completeness axiom)**.**
Recall that one of the formulations of the completeness axiom of real numbers is:

(C2) Every increasing real number sequence $(a_n)_{n\in\mathbb{N}}$ which is bounded from above has a limit $\lim_{n\to\infty} a_n \in \mathbb{R}$.

Consider the statement

(C2') Every decreasing real number sequence $(b_n)_{n\in\mathbb{N}}$ which is bounded from below has a limit $\lim_{n\to\infty} b_n \in \mathbb{R}$.

Prove that (C2) implies (C2').

Hint: *Given a sequence $(b_n)_{n\in\mathbb{N}}$ as above, what can be said about the numbers $-b_n$ for $n \in \mathbb{N}$?*