(Exercise sessions: 4.-5.2.2021) **Hand-in due: Tue 9.2.2021 at 23:59**

Fill-in-the-blanks 1. Let (X, d) be a metric space and $(\mathsf{V}, \|\cdot\|)$ a normed space. On V, we use the metric induced by the norm, $\mathsf{d}_{\mathsf{V}}(u, v) = \|v - u\|$ for $u, v \in \mathsf{V}$. Complete the proof of the following statement.

Claim. If $f, g \colon X \to \mathsf{V}$ are two continuous functions, then the function $f + g \colon X \to \mathsf{V}$ is also continuous.

Remark: The function $f + g \colon X \to \mathsf{V}$ is the pointwise sum of the vector valued functions f and g, defined by the formula $(f + g)(x) = f(x) + g(x)$ (the right hand side is vector addition in vector space V).

Proof. First fix $x \in X$. Let us show that $f + g$ is continuous at the point x. Let $\varepsilon > 0$. Then also $\frac{\varepsilon}{2} > 0$, and since by assumption f is continuous at x, there exists a

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such that we have $\|f(x') - f(x)\| < \frac{\varepsilon}{2}$ whenever $x' \in X$ and

____________________ .

Similarly, since g is also continuous at x by assumption, there exists a

______________ such that __________________________________ .

Now let $\delta =$ ________________________ > 0.

Then whenever $x' \in X$ and $\mathsf{d}(x, x') < \delta$, we have both

$\mathsf{d}(x, x') <$ ____________ and $\mathsf{d}(x, x') <$ ____________ , so

$\|(f + g)(x') - (f + g)(x)\| =$ ________________________________

$\leq$ ________________________________

$<$ __________________________ $= \varepsilon.$

This implies that $f + g$ is continuous at x. Since $x \in X$ was arbitrary, $f + g$ is therefore a continuous function. $\square$

Fill-in-the-blanks 2. Complete the proofs of the following statements.

a) **Claim.** If (X, d) is a metric space and $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in the space X, then the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X and denote its limit by

$$a = \lim_{n\to\infty} x_n.$$

Apply the definition of limit by chooseing $\varepsilon = 1$. Then there exists an $n_1 \in \mathbb{N}$, such that

$$\underline{\hspace{4cm}} < 1 \quad \text{whenever} \quad \underline{\hspace{4cm}}\ .$$

Since $x_1, x_2, \ldots, x_{n_1-1}$ is a finite list of points, we can define

$$R = \max\{1,\ \mathsf{d}(x_1, a),\ \underline{\hspace{3cm}}\ , \ldots,\ \underline{\hspace{3cm}}\ \} < \infty.$$

Then for all members x_k of the sequence we have

$$\mathsf{d}(x_k, a) \leq \ \underline{\hspace{2.5cm}}\ ,$$

so the members of the sequence are contained in the closed ball

$$\underline{\hspace{6cm}}\ .$$

This proves that the sequence is bounded. $\square$

b) **Claim.** Consider the metric space $(\mathbb{R}, \mathsf{d}_{0/1})$, where on the real line $\mathbb{R}$ we use the discrete 0/1-metric $\mathsf{d}_{0/1}$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ in $(\mathbb{R}, \mathsf{d}_{0/1})$ defined by the formula $x_n = \frac{1}{n}$ does not have 0 as its limit.

Proof. For all $n \in \mathbb{N}$ we have

$$\mathsf{d}_{0/1}(x_n, 0) = \ \underline{\hspace{3cm}}\ ,$$

so there does not exists $\underline{\hspace{3cm}}$ such that

$$\underline{\hspace{5cm}} < 1/2 \quad \text{for all} \quad \underline{\hspace{5cm}}\ .$$

The defining condition of limit therefore is not fulfilled for $\varepsilon = 1/2$, so the sequence $(x_n)_{n\in\mathbb{N}}$ does not converge to 0. $\square$