Aalto University
Department of Mathematics and Systems Analysis

MS-C1541 — Metric spaces, 2022-2023/III

K Kytölä & T Kelomäki

Problem set 4

Exercise sessions: 2.-3.2.2023 Hand-in due: Tue 7.2.2023 at 23:59

Topic: Continuous functions, homeomorphisms, sequences in metric spaces

Written solutions to the exercises marked with symbol △ are to be returned in My-Courses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 4 is Tue 7.2.2023 at 23:59.

Exercise 1 (Verifying openness and closedness of subsets).

(a) Show that the set $U = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 < y^2 + z^2 - xyz + 3\} \subset \mathbb{R}^3$ is open and that the set $F = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 2 \text{ and } x \le \frac{1}{3}\sin(\pi y)\} \subset \mathbb{R}^2$ is closed.

<u>Hint</u>: Express the sets U and F appropriately using preimages; for the latter it is best to use two different functions and an intersection of preimages.

The continuity of the functions involved can be considered known.

(b) Consider the space C([-1,1]) of all continuous functions $f: [-1,1] \to \mathbb{R}$, equipped with the metric induced by the sup-norm $\|\cdot\|_{\infty}$. Consider the subset¹

$$D = \left\{ p \in \mathcal{C}([-1,1]) \mid p(x) \ge 0 \ \forall x \in [-1,1], \ \int_{-1}^{1} p(x) \, \mathrm{d}x = 1 \right\}.$$

Show that $D \subset \mathcal{C}([-1,1])$ is a closed set.

<u>Hint</u>: You may use the facts that evaluation functions $f \mapsto f(x)$ (for an arbitrary $x \in [-1, 1]$), and the integration function $f \mapsto \int_{-1}^{1} f(x) dx$ are continuous functions $C([-1, 1]) \to \mathbb{R}$ with the chosen metric. Otherwise the ideas are similar to part (a).

Exercise 2 (The closure of a set).

Let (X, d) be a metric space and $A \subset X$ a subset. The closure of A is by definition the set

$$\overline{A} = X \setminus \text{ext}(A), \quad \text{where}$$

$$\text{ext}(A) = \left\{ x \in X \mid \exists r > 0 : \ \mathcal{B}_r(x) \subset X \setminus A \right\}.$$

- (a) Prove that the set \overline{A} is closed.
- (b) Prove that for $x \in X$, the following two conditions are equivalent:
 - (i) $x \in A$;
 - (ii) there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n\in A$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}a_n=x$.

¹This subset D could be interpreted as the set of all continuous probability density functions supported on the interval [-1, +1].

Exercise 3.

(a) Let X be a set, $J \neq \emptyset$ a nonempty index set, and $A_j \subset X$ subsets for each $j \in J$. Prove De Morgan's laws for arbitrary unions and intersections:

$$X \setminus \bigcup_{j \in J} A_j = \bigcap_{j \in J} (X \setminus A_j)$$
 and $X \setminus \bigcap_{j \in J} A_j = \bigcup_{j \in J} (X \setminus A_j)$.

<u>Hint</u>: Often the equality A=B between two sets is most straightforward to prove in two stages: $A\subset B$ and $B\subset A$. In this problem the two stages can be combined, by directly establishing a chain of equivalent conditions $x\in A\Leftrightarrow \cdots\Leftrightarrow x\in B$.

- (b) Using part (a) and known properties of open sets, prove the following (stated as Theorem VII.16 in the lecture notes):
 - Arbitrary intersections of closed sets are closed.
 - Finite unions of closed sets are closed.

<u>Hint</u>: The first proof could start with: "Let J be an index set, and $F_j \subset X$ closed sets, for $j \in J$."

Livercise 4 (Some homeomorphisms).

- (a) Prove that [0,1) and $[0,\infty)$ are homeomorphic, $[0,1)\approx [0,\infty)$. Remark: The subsets of the real axis above are equipped with the ordinary metric. The continuity of the functions involved can be considered known.
- (b) Prove that the open unit disk $B(\vec{0},1) \subset \mathbb{R}^2$ in the Euclidean plane \mathbb{R}^2 and the whole plane \mathbb{R}^2 are homeomorphic, $B(\vec{0},1) \approx \mathbb{R}^2$.

 Hint: One option is a radial mapping in polar coordinates, using part (a).

 \triangle Exercise 5 (Coordinatewise convergence is not sufficient for convergence in ℓ^1). Consider the space

$$\ell^1 = \left\{ x = (x_j)_{j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid \sum_{j=1}^{\infty} |x_j| < \infty \right\}$$

of absolutely summable real sequences. We consider it known that the formula $||x||_1 = \sum_{j=1}^{\infty} |x_j|$ for $x = (x_j)_{j \in \mathbb{N}} \in \ell^1$ defines a norm on ℓ^1 . We equip ℓ^1 with the metric induced by the norm $||\cdot||_1$.

(a) Show that if a sequence $(x^{(n)})_{n\in\mathbb{N}}$ of elements $x^{(n)}=(x_j^{(n)})_{j\in\mathbb{N}}\in\ell^1$ converges in ℓ^1 to $x=(x_j)_{j\in\mathbb{N}}$, then for every $k\in\mathbb{N}$, the sequence $(x_k^{(n)})_{n\in\mathbb{N}}$ of the k:th coordinates of $x^{(n)}$'s converges to $\lim_{n\to\infty}x_k^{(n)}=x_k$ (limit in \mathbb{R}).

<u>Hint</u>: You can start by showing that the k:th coordinate projection function $(x_j)_{j\in\mathbb{N}} \mapsto x_k$ is a 1-Lipschitz function $\ell^1 \to \mathbb{R}$.

(b) For $n \in \mathbb{N}$ let $x^{(n)} = (x_j^{(n)})_{j \in \mathbb{N}} \in \ell^1$ be the element given by

$$x_j^{(n)} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } j \neq n. \end{cases}$$

Show that for any $k \in \mathbb{N}$ we have $\lim_{n\to\infty} x_k^{(n)} = 0$ but in the space $(\ell^1, \|\cdot\|_1)$ the sequence $(x^{(n)})_{n\in\mathbb{N}}$ does not converge.

<u>Hint</u>: If the sequence would converge in ℓ^1 , then part (a) together with the first calculation of (b) identifies the only possibility for a limit $x \in \ell^1$. Now show directly from the definition of limits that we do not have convergence to that candidate limit.