

(Exercise sessions: 8.-9.2.2024) Hand-in due: Tue 13.2.2024 at 23:59

Denote by ℓ^1 the space of absolutely summable real-number sequences, i.e., the space whose elements are sequences $x = (x_j)_{j \in \mathbb{N}}$ such that $\sum_{j=1}^{\infty} |x_j| < \infty$. We consider it known that ℓ^1 becomes a normed space when equipped with the norm given by the formula $\|x\|_1 := \sum_{j=1}^{\infty} |x_j|$. Equip the space ℓ^1 with the metric induced by $\|\cdot\|_1$.

Note that a sequence $(x^{(n)})_{n \in \mathbb{N}}$ in ℓ^1 is a sequence of sequences: to each $n \in \mathbb{N}$ corresponds an absolutely summable sequence $x^{(n)} = (x_j^{(n)})_{j \in \mathbb{N}}$.

Fill-in-the-blanks 1. Complete the following proof.

Claim. Let $(x^{(m)})_{m \in \mathbb{N}}$ be a sequence in ℓ^1 . Assume:

- (A) for every $m \in \mathbb{N}$ we have $\|x^{(m+1)} - x^{(m)}\|_1 \leq 5^{-m}$;
- (B) for every $j \in \mathbb{N}$ the sequence $(x_j^{(m)})_{m \in \mathbb{N}}$ of the j :th coordinates converges to a limit $x_j = \lim_{m \rightarrow \infty} x_j^{(m)}$.

Then for every $m \in \mathbb{N}$ we have $\sum_{j=1}^{\infty} |x_j - x_j^{(m)}| \leq \frac{5}{4} 5^{-m}$.

Proof. For any $k > m$, we have “telescopic” cancellations in the sum

$$\begin{aligned} x^{(k)} - x^{(m)} &= (x^{(k)} - x^{(k-1)}) + (x^{(k-1)} - x^{(k-2)}) + \dots + (x^{(m+1)} - x^{(m)}) \\ &= \sum_{r=m}^{k-1} (x^{(r+1)} - x^{(r)}). \end{aligned}$$

Using _____ for the above sum we get

$$\|x^{(k)} - x^{(m)}\|_1 \leq \sum_{r=m}^{k-1} \|x^{(r+1)} - x^{(r)}\|_1 \leq \sum_{r=m}^{k-1} 5^{-r} = \frac{5^{-m} - 5^{-k}}{4/5}.$$

For any $J \in \mathbb{N}$ we now get

$$\sum_{j=1}^J |x_j^{(k)} - x_j^{(m)}| \leq \|x^{(k)} - x^{(m)}\|_1 \leq \frac{5^{-m} - 5^{-k}}{4/5} \leq \frac{5}{4} 5^{-m}.$$

For every $j \in \mathbb{N}$, we have $\lim_{k \rightarrow \infty} |x_j^{(k)} - x_j^{(m)}| = |x_j - x_j^{(m)}|$, because _____ and because the function

$z \mapsto |z - x_j^{(m)}|$ is _____. Taking a finite sum, we get $\lim_{k \rightarrow \infty} \sum_{j=1}^J |x_j^{(k)} - x_j^{(m)}| = \sum_{j=1}^J |x_j - x_j^{(m)}|$. The preservation of bounds in the limit $k \rightarrow \infty$ then yields $\sum_{j=1}^J |x_j - x_j^{(m)}| \leq \frac{5}{4} 5^{-m}$. From this, the preservation of bounds in the limit $J \rightarrow \infty$ yields the asserted inequality.

Fill-in-the-blanks 2. Complete the following proof.

Claim. The space ℓ^1 is complete.

Proof. To show completeness, we must prove that every Cauchy sequence converges in ℓ^1 . So let $(x^{(n)})_{n \in \mathbb{N}}$ be a Cauchy sequence in ℓ^1 .

Note that it suffices to prove that $(x^{(n)})_{n \in \mathbb{N}}$ has a convergent subsequence, because _____.

Note also that for every $j \in \mathbb{N}$, the real-number sequence $(x_j^{(n)})_{n \in \mathbb{N}}$ is _____, because it is obtained from the

Cauchy sequence $(x^{(n)})_{n \in \mathbb{N}}$ in ℓ^1 by applying the 1-Lipschitz function $\text{pr}_j: \ell^1 \rightarrow \mathbb{R}$ which picks the j :th coordinate of an absolutely summable sequence. By completeness of _____, we then get that the limit

$x_j = \lim_{n \rightarrow \infty} x_j^{(n)} \in \text{_____}$ exists for every $j \in \mathbb{N}$. From these limits we form a sequence $x = (x_j)_{j \in \mathbb{N}}$.

Since $(x^{(n)})_{n \in \mathbb{N}}$ is Cauchy, for every $m \in \mathbb{N}$ we can choose a $n_m \in \mathbb{N}$ such that for all $k, l \geq n_m$ we have $\|x^{(k)} - x^{(l)}\|_1 < 5^{-m}$. Moreover, it is possible to choose these indices so that $n_1 < n_2 < n_3 < \dots$. We will show that the subsequence $(x^{(n_m)})_{m \in \mathbb{N}}$ is convergent.

The subsequence satisfies the assumptions of Fill-in-the-blanks 1, so we get that for every $m \in \mathbb{N}$

$$(\star) \quad \sum_{j=1}^{\infty} |x_j - x_j^{(n_m)}| \leq \frac{5}{4} 5^{-m}.$$

This shows in particular that the sequence $x - x^{(n_m)} = (x_j - x_j^{(n_m)})_{j \in \mathbb{N}}$ is absolutely summable. The sequence x can be written as a sum $x = (x - x^{(n_m)}) + x^{(n_m)}$. Since both summands here belong to the vector space _____, this allows us to conclude the same for x .

Using the estimate (\star) we also get

$$0 \leq d(x, x^{(n_m)}) = \|x^{(n_m)} - x\|_1 = \sum_{j=1}^{\infty} |x_j^{(n_m)} - x_j| \leq \frac{5}{4} 5^{-m}.$$

Since _____ = 0, by the squeeze theorem we get

$\lim_{m \rightarrow \infty} d(x, x^{(n_m)}) = 0$. From here we conclude that the subsequence $(x^{(n_m)})_{m \in \mathbb{N}}$ converges to x in ℓ^1 . \square