Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2022-2023/III

Problem set 2

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Exercise sessions: 19.-20.1.2023 Hand-in due: Tue 24.1.2023 at 23:59

Topic: Continuous functions on $\mathbb{R}$, inner product spaces, normed spaces.

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 2 is Tue 24.1.2023 at 23:59.

Exercise 1 (Continuous functions on intervals)**.**
Below (as usually), unless explicitly specified, an interval refers to either an open interval, closed interval, or a half-open interval, possibly unbounded. If $c \in \mathbb{R}$, we also interpret the singleton $\{c\} \subset \mathbb{R}$ as a closed interval $[a, b] \subset \mathbb{R}$ with coinciding endpoints, $a = c = b$.[1]

(a) Find an example of a continuous function on an interval which has no maximum.
(b) Find an example of a continuous function on an *open interval* which has a maximum but has no minimum.
(c) Using a combination of results from Chapter III, prove that if $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed interval $[a, b] \subset \mathbb{R}$, then the image $f\big[[a, b]\big]$ is a closed interval.
(d) If $f : (a, b) \to \mathbb{R}$ is a continuous function on an *open interval* $(a, b) \subset \mathbb{R}$, is it possible that the image $f\big[(a, b)\big]$ is not an open interval?

Exercise 2 (Subpaces of of the space of real-valued sequences)**.**
We consider it known that the space

$$\mathbb{R}^{\mathbb{N}} = \Big\{ (x_n)_{n\in\mathbb{N}} \;\Big|\; x_n \in \mathbb{R} \text{ for } n \in \mathbb{N} \Big\}$$

of sequences of real numbers forms a (real) vector space, with addition and scalar multiplication defined coordinatewise.

(a) Show that the subset $\Big\{ (x_n)_{n\in\mathbb{N}} \;\Big|\; \text{the limit } \lim_{n\to\infty} x_n \text{ exists} \Big\} \subset \mathbb{R}^{\mathbb{N}}$ consisting of those real-number sequences which are convergent forms a vector subspace of the space of all real-number sequences.
(b) Show that the subset $\Big\{ (x_n)_{n\in\mathbb{N}} \;\Big|\; \lim_{n\to\infty} x_n = 0 \Big\} \subset \mathbb{R}^{\mathbb{N}}$ consisting of those real-number sequences which converge to zero forms a vector subspace of the space of all real-number sequences.
(c) Show that the subset $\Big\{ (x_n)_{n\in\mathbb{N}} \;\Big|\; \lim_{n\to\infty} x_n = -\sqrt{\pi} \Big\} \subset \mathbb{R}^{\mathbb{N}}$ consisting of those real-number sequences which converge to $-\sqrt{\pi}$ does *not* form a vector subspace of the space of all real-number sequences.

[1] For *closed intervals* this is a natural convention in the degenerate case of coinciding endpoints, since $[a, b] \subset \mathbb{R}$ by definition consists of points $x \in \mathbb{R}$ such that $a \leq x \leq b$.

✍ **Exercise 3** (The ℓ^1-norm in finite dimensional spaces)**.**

(a) Prove that the formula

$$\|x\|_1 \;=\; |x_1| + |x_2| + \cdots + |x_n|, \qquad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

defines a quantity that satisfies the following conditions:

(N1) $\|x + y\|_1 \le \|x\|_1 + \|y\|_1$ for all $x, y \in \mathbb{R}^n$;

(N2) $\|cx\|_1 = |c|\,\|x\|_1$ for all $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$;

(N3) $\|x\|_1 = 0$ if and only if $x = \vec{0}$.

(b) Let $n = 2$. Sketch the set

$$\Big\{(x_1, x_2) \in \mathbb{R}^2 \;\Big|\; \|(x_1, x_2)\|_1 = |x_1| + |x_2| = 1\Big\},$$

i.e., the "unit circle" under the norm of part (a).

✍ **Exercise 4** (Parallelogram law and norms not induced by inner products)**.**

(a) Prove that the norm in an inner product space V satisfies the *parallelogram law*

$$\|u + v\|^2 + \|u - v\|^2 = 2\,\|u\|^2 + 2\,\|v\|^2 \qquad \text{for all } u, v \in \mathsf{V}.$$

(b) Using part (a), prove that the norm $\|\cdot\|_1$ of Exercise 3 on $\mathbb{R}^n$ is not induced by any inner product on $\mathbb{R}^n$ when $n \ge 2$. Why doesn't the same reasoning work in the case $n = 1$?

✍ **Exercise 5** (Ideas behind Fourier series)**.**

Hint: *The solution is shorter than the problem statement!*

An indexed collection $(e_t)_{t \in T}$ of vectors $e_t \in \mathsf{V}$ in an inner product space V is called *orthonormal*, if for all $s, t \in T$, we have

$$\langle e_t, e_s \rangle = \delta_{t,s} = \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{if } t \neq s \end{cases} \qquad \text{(the Kronecker } \delta\text{-symbol).}$$

Remark: Here T can be an arbitrary index set.

(a) Consider a countable orthonormal collection $(e_n)_{n \in \mathbb{N}}$ in an inner product space V. Assume that a vector $v \in \mathsf{V}$ can be expressed, for some $m \in \mathbb{N}$, as a (finite) linear combination

$$v = \alpha_1 e_1 + \cdots + \alpha_m e_m \qquad \text{with some coefficients } \alpha_n \in \mathbb{R}.$$

Calculate the inner products between the vector v and the vectors e_n from the orthonormal collection, and deduce a formula for the coefficients α_n.

Remark: The result generalizes to countable linear combinations (i.e. series), which are convergent in a suitable sense.

(b) Define functions $c_0, s_1, c_1, s_2, c_2, s_3, c_3, \ldots$ of a real variable x by the formulas

$$\begin{aligned} c_0(x) &= \frac{1}{\sqrt{2}}, \\ s_n(x) &= \sin(nx) \qquad \text{(for } n \in \mathbb{N}), \\ c_n(x) &= \cos(nx) \qquad \text{(for } n \in \mathbb{N}). \end{aligned}$$

Prove that the (countably infinite) collection of these functions is orthonormal in the space $C([-\pi, \pi])$ of continuous functions on $[-\pi, \pi]$, with respect to the rescaled inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x.$$

Remark: Since there are 6–9 different cases (depending on the counting convention), **it is enough to do the details of just four cases**. The complex version would be somewhat easier…

(c) The idea of Fourier series is founded on the following result: Every continuously differentiable 2π-periodic function $f \colon \mathbb{R} \to \mathbb{R}$ can be represented as

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \big(\alpha_n \cos(nx) + \beta_n \sin(nx) \big).$$

Consider for simplicity the case of a function f, for which the above series contains only finitely many terms, i.e., for some $m \in \mathbb{N}$ we have

$$f(x) = \alpha_0 + \sum_{n=1}^{m} \big(\alpha_n \cos(nx) + \beta_n \sin(nx) \big).$$

Using parts (a) and (b), derive the following formula for the coefficients

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x \qquad (n \in \mathbb{N}).$$

Hint: *Using parts (a) and (b) is indeed the intended approach, instead of calculating everything from scratch again!*