Aalto University
Department of Mathematics and Systems Analysis
Problem set 5

MS-C1541 — Metric spaces, 2021/III

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Fill-in-the-blanks 1. In this exercise we work out an example, which shows:

In the definition of completeness it is not sufficient to consider only the distances $\mathsf{d}(x_{k+1}, x_k)$ between consecutive members of a sequence, but one must consider all distances $\mathsf{d}(x_k, x_\ell)$, for $k, \ell \geq n_0$.

Justification. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ on the real line, which consists of the partial sums

$$x_n = \sum_{j=1}^{n} \frac{1}{j}$$

of the harmonic series. From Calculus-1, we know that the sequence $(x_n)_{n\in\mathbb{N}}$

$$\underline{\hspace{6cm}} \; .$$

Since $\mathbb{R}$ is complete, the sequence $(x_n)_{n\in\mathbb{N}}$ therefore can not be Cauchy. Let us show that it nevertheless satisfies the following property: For all $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$|x_{k+1} - x_k| < \varepsilon \quad \text{whenever} \quad k \geq n_0.$$

Let $\varepsilon > 0$. Since we have

$$|x_{k+1} - x_k| = \underline{\hspace{5cm}} \; ,$$

the condition $|x_{k+1} - x_k| < \varepsilon$ holds for all indices

$$k > \underline{\hspace{4cm}} \; .$$

As n_0, we can therefore choose any natural number greater than the real number on the right hand side above, and the asserted property then holds for $k \geq n_0$.

Fill-in-the-blanks 2. Complete the proof of the following claim.

Claim. Let (X, d) be a complete metric space, and $A \subset X$ a subset of it. Then $A \subset X$ is closed if and only if the subspace $A \subset X$ (with the inherited metric) is a complete metric space.

Remark: In determining the completeness of a subspace $A \subset X$, the limits of Cauchy sequences must lie in the subspace A itself (rather than the whole space X).

Proof.

"only if": Assume that $A \subset X$ is closed. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the subspace A. Then $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence also in the space ________ , so it ____________________ to a limit $a = \lim_{n\to\infty} a_n \in$ ______ . Since A is closed, according to a characterization of closedness, the limit satisfies ____________ . This shows that the subspace A is complete.

"if": Assume that the subspace A is complete. Then assume, by contrapositive, that $A \subset X$ is not closed, i.e., that $X \setminus A \subset X$ is not open. Then there exists a point $b \in X \setminus A$ such that for all $n \in \mathbb{N}$ we have

$$\mathcal{B}_{1/n}(b) \cap A \neq \text{\_\_\_\_\_\_\_\_} .$$

Thus for each $n \in \mathbb{N}$ we can choose $a_n \in \mathcal{B}_{1/n}(b) \cap A$. This gives rise to a sequence $(a_n)_{n\in\mathbb{N}}$ in A. According to the choice, we have

$$0 \leq \mathsf{d}(a_n, b) \text{ \_\_\_\_ \_\_\_\_\_\_\_\_\_\_ } \underset{n\to\infty}{\longrightarrow} \text{ \_\_\_\_\_\_\_\_ },$$

so the sequence $(a_n)_{n\in\mathbb{N}}$ converges to the limit $\lim_{n\to\infty} a_n = b$. Since a convergent sequence $(a_n)_{n\in\mathbb{N}}$ is ____________________ and the subspace A was assumed complete, we get that the limit satisfies

$$\text{\_\_\_\_\_\_\_\_} \in \text{\_\_\_\_\_\_\_\_} .$$

This is a contradiction, since we originally chose a point $b \in$ ________ .

□