


**Exercise sessions: 18.-19.2.2021**

**Hand-in due: ???**

*Topic: compactness, connectedness*

*Written solutions to the exercises marked with symbol  are to be returned in My-Courses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 6 is ???.*

**Exercise 1** (Compactness and continuous functions).

Prove the following assertions related to compactness by forming a continuous function, which settles the matter.

- (a) If  $A \subset \mathbb{R}^3$  is a compact subset, then it contains a highest point  $a \in A$ , i.e., a point  $a = (a_1, a_2, a_3)$  such that we have  $a_3 \geq x_3$  for all  $x = (x_1, x_2, x_3) \in A$ .
- (b) If  $B \subset \mathbb{R}^2$  is a compact subset, then it contains points  $a, b, c \in B$ , for which the perimeter

$$\|a - b\| + \|b - c\| + \|c - a\|$$

of the triangle is maximal among triangles whose vertices lie in  $B$ .

Hint: You can use the fact that Cartesian products of compact sets are compact.

- (c) If  $X$  is a compact metric space and  $g: X \rightarrow X$  is a continuous function that has no fixed points, then there exists a  $c > 0$  such that  $\mathbf{d}(g(x), x) \geq c$  for all  $x \in X$ .

Hint: You can consider the following continuity statements known:

- the function  $X \rightarrow X \times X$  given by  $x \mapsto (x, g(x))$  is continuous (since its component functions  $\text{id}_X$  and  $g$  are continuous);
- the metric  $\mathbf{d}: X \times X \rightarrow [0, \infty)$  is a continuous function.


*Remark:* An alternative strategy in each part would be to first pick a sequence which approaches the relevant sup/inf-value, and then use compactness to extract a convergent subsequence.

**Exercise 2** (Connectedness and a midpoint between two subsets).

Let  $X$  be a connected metric space and  $A, B \subset X$  two non-empty subsets. Show that there exists (at least one) point  $x \in X$  such that

$$\text{dist}(\{x\}, A) = \text{dist}(\{x\}, B).$$

Hint: Consider the continuous function defined by  $f(x) = \text{dist}(\{x\}, A) - \text{dist}(\{x\}, B)$ .

 **Exercise 3** (A closed and bounded non-compact set).

Consider the space  $\mathcal{C}([0, 1])$  of continuous real valued functions on  $[0, 1]$ , with the metric  $d$  induced by the sup-norm  $\|\cdot\|_\infty$

$$d(f, g) = \|f - g\|_\infty = \sup \left\{ |f(x) - g(x)| \mid x \in [0, 1] \right\}.$$


Prove that the closed unit ball  $\overline{B}(\vec{0}, 1)$  is not compact.

Hint: Construct a sequence  $(f_n)$  for which  $\|f_n\|_\infty \leq 1$  for all  $n \in \mathbb{N}$  and

$$\|f_k - f_n\|_\infty \geq 1$$

whenever  $k \neq n$ . Such a sequence cannot have a convergent subsequence (why?).

*Remark:* While in Euclidean spaces  $\mathbb{R}^d$ , compactness of a subset is equivalent to its closedness and boundedness (Bolzano-Weierstrass theorem / Heine-Borel theorem), this exercise shows that the same is not true in general metric spaces! Infinite-dimensional normed spaces (like  $\mathcal{C}([0, 1])$  in this exercise) are typical counterexamples.

 **Exercise 4** (The union of two compact sets is compact).

Prove that for any two compact subsets  $A, B \subset X$  of a metric space  $X$ , the union  $A \cup B$  is compact, ...

- (a) ... directly from the definition of (sequential) compactness;
- (b) ... by using the characterization of compactness by open covers.

Hint: The proofs should begin as follows:

a) Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $A \cup B$ . ...

b) Let  $(U_j)_{j \in J}$  be an open cover of  $A \cup B$ . ...

 **Exercise 5** (Decreasing sequences of compact sets).

- (a) Let  $X$  be a metric space and  $(A_n)_{n \in \mathbb{N}}$  a decreasing sequence (i.e.,  $A_{n+1} \subset A_n$  for all  $n \in \mathbb{N}$ ) of non-empty compact subsets of the space  $X$ . Prove that the intersection of these subsets

$$A = \bigcap_{n=1}^{\infty} A_n$$

- is (i) non-empty; and (ii) compact.
- (b) Give an example of a decreasing sequence of non-empty closed subsets, whose intersection is empty.