(Exercise sessions: 16.-17.2.2023) Hand-in due: Tue 21.2.2023 at 23:59

Fill-in-the-blanks 1. A *line segment* in $\mathbb{R}^n$ between points $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$ is the set

$$J = [\vec{x}_1, \vec{x}_2] = \left\{ \vec{x}_1 + t(\vec{x}_2 - \vec{x}_1) \;\middle|\; t \in [0,1] \right\} \subset \mathbb{R}^n.$$

A *broken line* in $\mathbb{R}^n$ through points $\vec{x}_1, \ldots, \vec{x}_\ell$ is the union of line segments

$$M = [\vec{x}_1, \vec{x}_2] \cup [\vec{x}_2, \vec{x}_3] \cup \cdots \cup [\vec{x}_{\ell-1}, \vec{x}_\ell] \subset \mathbb{R}^n.$$

A broken line M of this form is said to connect the points $\vec{x}_1$ and $\vec{x}_\ell$.

Claim. If $D \subset \mathbb{R}^n$ is open and connected, then for all points $\vec{x}, \vec{y} \in D$ there exists a broken line $M \subset D$ connecting the points $\vec{x}$ and $\vec{y}$.

Proof. Assume $D \subset \mathbb{R}^n$ is open and connected, and $\vec{x}, \vec{y} \in D$.

Let $U \subset D$ be the set of all points $\vec{z} \in D$ to which the point $\vec{x}$ can be connected by some broken line. Below we will show that both $U \subset D$ and its complement $D \setminus U \subset D$ are open. Moreover, we clearly have $U \neq \emptyset$, since at least ________ $\in U$. With the connectedness of D these imply that $D \setminus U =$ ________ . In particular $\vec{y} \in U$, i.e., the point $\vec{x}$ can be connected to $\vec{y}$ by a broken line, completing the proof.

Let us first show that $U \subset D$ is open. Let $\vec{z} \in U$. Then there exists a broken line $M \subset D$ connecting points ________ and $\vec{z}$. Since $D \subset \mathbb{R}^n$ is open and $\vec{z} \in D$, for some $r > 0$ we have $\mathcal{B}_r(\vec{z})$ __________ . For any $\vec{w} \in \mathcal{B}_r(\vec{z})$, the line segment $[\vec{z}, \vec{w}]$ is contained in the ball $\mathcal{B}_r(\vec{z})$. Therefore, by setting $M' = M \cup [\vec{z}, \vec{w}]$, we obtain a broken line $M' \subset$ ________ connecting points $\vec{x}$ and $\vec{w}$. This implies ________ $\in U$. We conclude that $\mathcal{B}_r(\vec{z}) \subset U$, showing that U is open.

Let us then show that $D \setminus U \subset D$ is open. Let $\vec{z} \in D \setminus U$. Again by openness of D, there exists an $r > 0$ such that ________________ .

If $\mathcal{B}_r(\vec{z}) \subset D \setminus U$, then the openness of $D \setminus U$ follows. Assume the converse; that there exists $\vec{w} \in \mathcal{B}_r(\vec{z}) \cap U$. Then there exists a broken line $M \subset D$ connecting the points ________ and ________ . But since $[\vec{w}, \vec{z}] \subset \mathcal{B}_r(\vec{z})$, by setting $M' = M \cup \Big[$ ______ , ______ $\Big]$ we would obtain a broken line $M' \subset D$ connecting the points $\vec{x}$ and $\vec{z}$, which is a contradiction since ____________ . □

Fill-in-the-blanks 2. Consider the d-dimensional vector space $\mathbb{R}^d$ with different norms. Denote the Euclidean norm on $\mathbb{R}^d$ by $\|\cdot\|$, and denote another arbitrary norm on $\mathbb{R}^d$ by $|||\cdot|||$. Unless explicitly otherwise specified, we use the metric induced by the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^d$. Denote by $\vec{e}_1, \ldots, \vec{e}_d$ the standard basis of $\mathbb{R}^d$.

Claim (i): $f : \mathbb{R}^d \to [0, \infty)$ defined by $f(\vec{v}) = |||\vec{v}|||$ is Lipschitz.

Proof. Write $\vec{w} \in \mathbb{R}^d$ in the standard basis as $\vec{w} = w_1\vec{e}_1 + \cdots + w_d\vec{e}_d$. Then estimate, using properties of the norm $|||\cdot|||$,

$$\begin{aligned} f(\vec{w}) &= |||w_1\vec{e}_1 + \cdots + w_d\vec{e}_d||| \le |||w_1\vec{e}_1||| + \cdots + |||w_d\vec{e}_d||| \\ &= |w_1|\,|||\vec{e}_1||| + \cdots + |w_d|\,|||\vec{e}_d|||. \end{aligned}$$

Note that for each $j = 1, \ldots, d$ we have $|w_j| \le \|\vec{w}\|$, because the j:th coordinate projection $\vec{w} \mapsto w_j$ is 1-Lipschitz $\mathbb{R}^d \to \mathbb{R}$. Let $M := \max\{|||e_1|||, \ldots, |||e_d|||\}$, so that $|||e_j||| \le M$ for all $j = 1, \ldots, d$.

Combined with the earlier estimate, we get $f(\vec{w}) \le$ __________ $\|\vec{w}\|$.

Now let $\vec{u}, \vec{v} \in \mathbb{R}^d$. Since $|||\cdot|||$ is 1-Lipschitz $\mathbb{R}^d \to [0, \infty)$ when the domain is equipped with the metric induced by the norm $|||\cdot|||$, we get

$$\big|f(\vec{v}) - f(\vec{u})\big| = \Big||||\vec{v}||| - |||\vec{u}|||\Big| \le |||\vec{v} - \vec{u}||| \le dM\,\|\vec{v} - \vec{u}\|.$$

This shows that $f\colon \mathbb{R}^d \to [0, \infty)$ is dM-Lipschitz. $\square$

Claim (ii): There exists constants $A, B > 0$ such that

$$A\,\|\vec{v}\| \le |||\vec{v}||| \le B\,\|\vec{v}\| \qquad \text{for all } \vec{v} \in \mathbb{R}^d.$$

Proof. Take B to be ______________________________.

Then it follows from part (i) that the second inequality then holds for all $\vec{v}$. It remains to prove the first one inequality.

The unit sphere $S = \{\vec{u} \in \mathbb{R}^d \mid \|\vec{u}\| = 1\}$ in the Euclidean space $\mathbb{R}^d$ is ________________ and ________________ . Thus S is compact, by ____________________________________ .

The restriction of f to the compact set S remains Lipschitz and in particular ________________ , so it has a minimum at some $\vec{u}_0 \in S$.

We claim that with $A = \min_{\vec{u} \in S} f(\vec{u}) = f(\vec{u}_0) = |||\vec{u}_0||| > 0$, the first inequality holds for all $\vec{v} \in \mathbb{R}^d$. If $\vec{v} = \vec{0}$, the inequality is trivial, so we may assume $\vec{v} \ne \vec{0}$. Then define $\vec{w} = \frac{1}{\|\vec{v}\|}\vec{v} \in S$. Observe

$$A = \min_{\vec{u} \in S} f(\vec{u}) \le f(\vec{w}) = |||\vec{w}||| = |||\frac{1}{\|\vec{v}\|}\vec{v}||| = \frac{1}{\|\vec{v}\|}\,|||\vec{v}|||$$

Multiplying by $\|\vec{v}\|$ yields the desired inequality $A\,\|\vec{v}\| \le |||\vec{v}|||$. $\square$