(Exercise sessions: 14.-15.1.2021) Hand-in due: Wed 20.1.2021 at 9:00

Fill-in-the-blanks 1. Complete the following proofs about injectivity, surjectivity, and bijectivity of two functions given by the same formula, but with different domains and codomains.

Claim (a): Consider the function $f\colon \mathbb{R} \to [0,\infty)$ given by $f(x) = x^2$ for $x \in \mathbb{R}$. This f is surjective but not injective. It is not bijective.

Proof of (a). To prove surjectivity of f, we must show that for every $y \in [0,\infty)$ there exists ________________ such that $f(x) = y$.

Let $y \in [0,\infty)$. The non-negative number y has a non-negative square root $x = \sqrt{y} \geq 0$. Then we have

$$f(x) \;=\; x^2 \;=\; (\sqrt{y})^2 \;=\; \underline{\qquad\quad} \,.$$

Since $x \in \mathbb{R}$, this shows that f is surjective.

To prove non-injectivity of f, we must prove that there exists $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$ and ____________________ .

Consider for example $x_1 = -3$ and $x_2 =$ ______. We have

$$f(x_1) \;=\; f(-3) \;=\; (-3)^2 \;=\; 9 \;=\; \big(\underline{\qquad}\big)^2 \;=\; f\big(\underline{\qquad}\big) \;=\; f(x_2).$$

Since $x_1 \neq x_2$, this shows that f is not injective.

Finally, f is not bijective because it __________________________. $\square$

Claim (b): Consider the function $g\colon [0,\infty) \to \mathbb{R}$ given by $g(x) = x^2$ for $x \in \mathbb{R}$. This g is injective but not surjective. It is not bijective.

Proof of (b). To prove non-surjectivity of g, we must show that for some $y \in \mathbb{R}$ there does not exists any ________________ such that $g(x) = y$. For example $y = -1$ works, because for any x we have $g(x) = x^2 \geq$ ______ and thus in particular $g(x) \neq -1 = y$.

To show injectivity of g, we must prove that if for $x_1, x_2 \in$ ________ we have $g(x_1) = g(x_2)$, then necessarily $x_1 = x_2$. So suppose that x_1, x_2 are such a pair. From $x_1^2 = g(x_1) = g(x_2) = x_2^2$, we then get $x_1 = \pm\sqrt{x_2^2} = \pm|x_2|$. However, since __________________________ , this is only possible if $x_1 = x_2$. Injectivity follows.

Now g is not bijective because it ____________________________. $\square$

Fill-in-the-blanks 2. Complete the following proof of the squeeze theorem (sandwich principle, lemma of two policemen).

Claim: If three sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$ of real numbers satisfy $a_n \le b_n \le c_n$ starting from some index, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \beta \in \mathbb{R},$$

then the sequence $(b_n)_{n\in\mathbb{N}}$ also converges, and $\lim_{n\to\infty} b_n = \beta$.

Proof. Since the beginning of a sequence affects neither the convergence nor the limit of the sequence, we may assume that $a_n \le b_n \le c_n$ holds for all $n \in \mathbb{N}$. We will show that $\lim_{n\to\infty} b_n = \beta$.

Let $\varepsilon > 0$. We must show that $|b_n - \beta| < \varepsilon$ from some index on.

Idea: the expression $b_n - \beta$ has to be estimated from both directions; one relying on the sequence $(a_n)_{n\in\mathbb{N}}$, and the other on the sequence $(c_n)_{n\in\mathbb{N}}$. (Draw a figure!)

Since $\lim_{n\to\infty} a_n = \beta$ and $\varepsilon > 0$, there exists an $n'_\varepsilon \in \mathbb{N}$ such that

$$\underline{\hspace{6cm}} \quad \text{for all } n \ge n'_\varepsilon.$$

Since $\lim_{n\to\infty} c_n = \beta$ and $\varepsilon > 0$, there exists an $n''_\varepsilon \in \mathbb{N}$ such that

$$\underline{\hspace{6cm}} \quad \text{for all} \quad \underline{\hspace{4cm}}\,.$$

Now choose

$$n_\varepsilon = \underline{\hspace{6cm}}\,.$$

With this choice, for any $n \ge n_\varepsilon$ we have $n \ge n'_\varepsilon$. Therefore we get

$$\beta - b_n \;\le\; \beta - a_n \;\le\; |\beta - a_n| \;<\; \underline{\hspace{2.5cm}}$$

(the leftmost inequality holds by virtue of the assumption $a_n \le b_n$). Similarly, for $n \ge n_\varepsilon$ we have $n \ge n''_\varepsilon$, so we get

$$b_n - \beta \;\le\; c_n - \beta \;\le\; \underline{\hspace{2.5cm}} \;<\; \underline{\hspace{2.5cm}}$$

(the leftmost inequality holds by virtue of the assumption $b_n \le c_n$). The above inequalities imply that

$$|b_n - \beta| < \underline{\hspace{1.5cm}}$$

for all $n \ge n_\varepsilon$. We have thus proved the claim. $\square$