

(Exercise sessions: 20.-21.1.2022) Hand-in due: Tue 25.1.2022 at 23:59

Fill-in-the-blanks 1. Complete the following arguments, which exemplify certain expressions that *do not* yield norms. It is simplest to use the labels given in Chapter IV of the lecture notes for the conditions required of a norm.

- The formula $\|(x, y)\| = |x + y|$ does not define a norm for points (x, y) in the plane \mathbb{R}^2 . For example

condition $\left(\rule{1cm}{0.4pt}\right)$

does not hold, since by choosing

$$x = \rule{2cm}{0.4pt}, \quad y = \rule{2cm}{0.4pt}$$

we find, contrary to the condition, that

$\rule{2cm}{0.4pt}$, although $\rule{2cm}{0.4pt}$.

- The formula $\|f\| = |f(0)|$ does not define a norm in the space $C([-1, 1])$ of continuous functions on the interval $[-1, 1]$. The reason:

condition $\left(\rule{1cm}{0.4pt}\right)$

does not hold, because by choosing

$$f(t) = \rule{2cm}{0.4pt} \quad \text{for } t \in [-1, 1],$$

we find, contrary to the condition, that

$\rule{2cm}{0.4pt}$.

The two other conditions are nevertheless satisfied:

Justification of condition $\left(\rule{1cm}{0.4pt}\right)$:

$\rule{2cm}{0.4pt}$.

Justification of condition $\left(\rule{1cm}{0.4pt}\right)$:

$\rule{2cm}{0.4pt}$.

Fill-in-the-blanks 2. Complete the following arguments to show that subsequences of a convergent real-number sequence are convergent, and they have the same limit as the original sequence. Recall that a subsequence of a sequence $(x_n)_{n \in \mathbb{N}}$ is a sequence of the form $(x_{\varphi(n)})_{n \in \mathbb{N}}$, where $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing function.

Lemma: If $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing function, then for all $n \in \mathbb{N}$ we have $\varphi(n) \geq n$.

Proof of lemma: Let $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function. We prove the claim by induction on n .

For the base case, $n = 1$, it suffices to note that

$$\underline{\hspace{2cm}} \geq \underline{\hspace{2cm}} \quad \text{since } \varphi(1) \in \mathbb{N}.$$

For the induction step, assume that for a natural number $n \in \mathbb{N}$ we have $\varphi(n) \geq n$. We must prove the corresponding property for $n + 1$. Since φ is strictly increasing, we have

$$\underline{\hspace{2cm}} > \underline{\hspace{2cm}},$$

which implies $\varphi(n + 1) \geq \varphi(n) + 1$, since the strict inequality above concerns natural numbers. To conclude the proof, we estimate

$$\varphi(n + 1) \geq \varphi(n) + 1 \geq \underline{\hspace{2cm}},$$

using $\underline{\hspace{4cm}}$ in the last step. \square

Theorem: If $(x_n)_{n \in \mathbb{N}}$ is a sequence of real numbers which converges to $\alpha \in \mathbb{R}$, then any subsequence of it also converges to α .

Proof of theorem: Let $(x_n)_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \rightarrow \infty} x_n = \alpha$, and let $(x_{\varphi(n)})_{n \in \mathbb{N}}$ be a subsequence of it. We must prove

$$\lim_{n \rightarrow \infty} \underline{\hspace{2cm}} = \alpha.$$

To prove this from the definition of limit, let $\varepsilon > 0$. From the assumption $\lim_{n \rightarrow \infty} x_n = \alpha$ we get that there exists an $N \in \mathbb{N}$

such that for all $\underline{\hspace{2cm}}$ we have $\underline{\hspace{4cm}}$.

The same N will work for our present goal. So let $n \geq N$. Note that by the lemma we then get $\varphi(n) \geq n \geq N$. In particular from the above, we obtain $|x_{\varphi(n)} - \alpha| < \varepsilon$. This finishes the proof, directly by the definition of a limit. \square