(Exercise sessions: 28.-29.1.2021) Hand-in due: Tue 2.2.2021 at 23:59

Fill-in-the-blanks 1. Let (X, d) be a metric space. Complete the proof of the following claim.

Claim. Every finite subset

$$\{x_1, \ldots, x_n\} \subset X$$

in X is closed.

Proof. We seek to prove that $F = \{x_1, \ldots, x_n\} \subset X$ is closed. By definition this means that its complement $X \setminus F$ is

$$\underline{\hspace{6cm}}.$$

Therefore, let $y \in X \setminus F$. Then for all $k = 1, \ldots, n$ we have $y \neq x_k$, and thus

$$r_k = \mathsf{d}(y, x_k) > \underline{\hspace{2.5cm}} .$$

With the choice

$$r = \underline{\hspace{7cm}} ,$$

we have $r > 0$ and $r \leq r_k$ for all $k = 1, 2, \ldots, n$. We then get

$$\mathcal{B}_r(y) \underline{\hspace{1cm}} \mathcal{B}_{r_k}(y) \subset \underline{\hspace{5cm}} ,$$

where the latter inclusion holds by the choice of r_k. Then we have also $\mathcal{B}_r(y) \subset X \setminus \{x_1, \ldots, x_n\}$.

The above reasoning shows that the set $X \setminus F$ is open. This proves the claim. $\square$

Fill-in-the-blanks 2. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, denote

$$\|x\|_p = \Big(\sum_{k=1}^{d} |x_k|^p\Big)^{1/p} \qquad \text{and} \qquad \|x\|_\infty = \max\{|x_1|, \ldots, |x_d|\}.$$

Complete the proof of the following result pertaining to the above norms on the n-dimensional space $\mathbb{R}^d$.

Claim. For all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have

$$\lim_{p\to\infty} \|x\|_p \;=\; \|x\|_\infty.$$

Proof. Consider a fixed $x \in \mathbb{R}^d$. (The parameter p, on the other hand, is thought of as a variable here.) Our goal is to prove that for all $p \geq 1$ we have

$$(\star) \qquad \|x\|_\infty \;\leq\; \|x\|_p \;\leq\; d^{1/p}\,\|x\|_\infty.$$

In view of

$$\lim_{p\to\infty} d^{1/p} \;=\; \underline{\hspace{4cm}}\,,$$

the claim then follows from ____.

(i) Let us first prove the left inequality in $(\star)$. From the definition of the norm $\|x\|_\infty$ it follows that there exists an index $m \in \{1, \ldots, d\}$ such that $|x_m| = \|x\|_\infty$. Since the function $u \mapsto u^{1/p}$, $u \geq 0$, is increasing, we have

$$\begin{aligned} \|x\|_\infty = |x_m| = \big(|x_m|^p\big)^{1/p} &\leq \Big(\underline{\hspace{6cm}}\Big)^{1/p} \\ &= \underline{\hspace{6cm}}. \end{aligned}$$

This proves the left inequality in $(\star)$.

(ii) Let us then prove the right inequality in $(\star)$. Choose the index m as above. Since the function $u \mapsto u^p$, $u \geq 0$, is increasing, for all indices $k = 1, \ldots, d$ we have $|x_k|^p \leq |x_m|^p$, so we get

$$\begin{aligned} \|x\|_p = (|x_1|^p + \cdots + |x_d|^p)^{1/p} &\leq \Big(\underline{\hspace{6cm}}\Big)^{1/p} \\ &= \underline{\hspace{6cm}} \\ &= \underline{\hspace{6cm}}. \end{aligned}$$

The right inequality in $(\star)$ is thus also proven. $\square$