Aalto University **Problem set 6**
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2023-2024/III K Kytölä & D Adame-Carrillo

Exercise sessions: 15.-16.2.2024 Hand-in due: Tue 20.2.2024 at 23:59

Topic: compactness, connectedness

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 6 is Tue 20.2.2024 at 23:59.

Exercise 1 (Compactness and continuous functions)**.**
Prove the following assertions related to compactness by forming a continuous function, which settles the matter.

(a) If $A \subset \mathbb{R}^3$ is a compact subset, then it contains a highest point $a \in A$, i.e., a point $a = (a_1, a_2, a_3)$ such that we have $a_3 \geq x_3$ for all $x = (x_1, x_2, x_3) \in A$.

(b) If $B \subset \mathbb{R}^2$ is a compact subset, then it contains points $a, b, c \in B$, for which the perimeter

$$\|a - b\| + \|b - c\| + \|c - a\|$$

of the triangle is maximal among triangles whose vertices lie in B.

Hint: *You can use the fact that Cartesian products of compact sets are compact.*

(c) If X is a compact metric space and $g \colon X \to X$ is a continuous function that has no fixed points, then there exists a $c > 0$ such that $\mathsf{d}\big(g(x), x\big) \geq c$ for all $x \in X$.

Hint: *You can consider the following continuity statements known:*
- *the function $X \to X \times X$ given by $x \mapsto (x, g(x))$ is continuous (since its component functions id_X and g are continuous);*
- *the metric $\mathsf{d} \colon X \times X \to [0, \infty)$ is a continuous function.*

Remark: An alternative strategy in each part would be to first pick a sequence which approaches the relevant sup/inf-value, and then use compactness to extract a convergent subsequence.

Exercise 2 (Connectedness and a midpoint between two subsets)**.**
Let X be a connected metric space and $A, B \subset X$ two non-empty subsets. Show that there exists (at least one) point $x \in X$ such that

$$\mathrm{dist}\big(\{x\}, A\big) = \mathrm{dist}\big(\{x\}, B\big).$$

Hint: *Consider the continuous function defined by $f(x) = \mathrm{dist}\big(\{x\}, A\big) - \mathrm{dist}\big(\{x\}, B\big)$.*

✍ **Exercise 3** (A closed and bounded non-compact set).
Consider the space $\mathcal{C}([0,1])$ of continuous real valued functions on $[0,1]$, with the metric d induced by the sup-norm $\|\cdot\|_\infty$

$$\mathsf{d}(f,g) = \|f-g\|_\infty = \sup\left\{|f(x)-g(x)| \;\middle|\; x \in [0,1]\right\}.$$

Prove that the closed unit ball $\overline{B}(\vec{0},1)$ is not compact.

Hint: *Construct a sequence (f_n) for which $\|f_n\|_\infty \le 1$ for all $n \in \mathbb{N}$ and*

$$\|f_k - f_n\|_\infty \ge 1$$

whenever $k \neq n$. Such a sequence cannot have a convergent subsequence (why?).

Remark: While in Euclidean spaces $\mathbb{R}^d$, compactness of a subset is equivalent to its closedness and boundedness (Bolzano-Weierstrass theorem / Heine-Borel theorem), this exercise shows that the same is not true in general metric spaces! Infinite-dimensional normed spaces (like $\mathcal{C}([0,1])$ in this exercise) are typical counterexamples.

✍ **Exercise 4** (The union of two compact sets is compact).
Prove that for any two compact subsets $A, B \subset X$ of a metric space X, the union $A \cup B$ is compact, . . .

(a) . . . directly from the definition of (sequential) compactness;
(b) . . . by using the characterization of compactness by open covers.

Hint: *The proofs should begin as follows:*
a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $A \cup B$. . . .
b) Let $(U_j)_{j\in J}$ be an open cover of $A \cup B$. . . .

✍ **Exercise 5** (Boundary behavior of a homeomorphism of the disk).
Let

$$D = \left\{v \in \mathbb{R}^2 \;\middle|\; \|v\|_2 < 1\right\}$$

be the open unit disk in the Euclidean plane $\mathbb{R}^2$ (i.e., D is the open ball of radius 1 centered at the origin). Let $f\colon D \to D$ be a homeomorphism of D to itself.

Show that if $(v_n)_{n\in\mathbb{N}}$ is a sequence in D such that

$$\lim_{n\to\infty} \|v_n\|_2 = 1,$$

then we also have

$$\lim_{n\to\infty} \|f(v_n)\|_2 = 1.$$

Hint: *Beware of mistakes based on intuitively plausible but incorrect ideas! For example, we are not assuming that $\lim_{n\to\infty} v_n$ exists in $\mathbb{R}^2$, and even if we were, it would not follow that $\lim_{n\to\infty} f(v_n)$ would exist in $\mathbb{R}^2$ (thinking of counterexamples to this may be instructive). The good arguments here use compactness in an essential way.*