Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2023-2024/III

Problem set 1

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Exercise sessions: 11.-12.1.2024 Hand-in due: Tue 16.1.2024 at 23:59

Topic: Sets, functions, real numbers

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 1 is Tue 16.1.2024 at 23:59.

Exercise 1 (Images and preimages of unions and intersections)**.**
Let X and Y be sets and $f \colon X \to Y$ a function.

(a) Show that for any $C, D \subset Y$, the preimages satisfy
$$f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D].$$
(b) Show that for any $C, D \subset Y$, the preimages satisfy
$$f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D].$$
(c) Show that for any $A, B \subset X$, the images satisfy
$$f[A \cup B] = f[A] \cup f[B].$$
(d) Give an example in which for the images of subsets $A, B \subset X$ we have
$$f[A \cap B] \neq f[A] \cap f[B].$$

Hint: *In parts (a)–(c) it is possible to argue by a chain of equivalent conditions*
$$x \in \textit{left hand side set} \iff \cdots \iff x \in \textit{right hand side set.}$$

Exercise 2 (Some geometric sums)**.**

(a) Consider the real-number sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ defined by
$$a_n = \sum_{j=0}^{n} \left(\frac{4}{5}\right)^j = 1 + \frac{4}{5} + \frac{16}{25} + \cdots + \frac{4^n}{5^n} \quad \text{and} \quad b_n = \sum_{j=0}^{n} \left(\frac{-1}{2}\right)^j.$$
Find simplified formulas for a_n and b_n. Find the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$, and prove directly using the definition of limits of real-number sequences that your answers indeed are those limits.
Hint: *To simplify, recall finite geometric sums.*

(b) Let $(a_n)_{n\in\mathbb{N}}$ be as in part (a). Does the set
$$A = \{a_n \mid n \in \mathbb{N}\} \subset \mathbb{R}$$
have a supremum, and if it does, what is $\sup(A)$? Does the set A have a maximum, and if it does, what is $\max(A)$?

(c) Let $(b_n)_{n\in\mathbb{N}}$ be as in part (a). Does the set
$$B = \{b_n \mid n \in \mathbb{N}\} \subset \mathbb{R}$$
have an infimum, and if it does, what is $\inf(B)$? Does the set B have a minimum, and if it does, what is $\min(B)$?

Exercise 3 (Calculating limits of sequences)**.**
Calculate the limits of the real-number sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, where

$$a_n = \frac{3n+4}{5n+6} \qquad \text{and} \qquad b_n = \frac{(1+2n)(1+3n)\,e^{-n}}{n^3\,e^{-2n} + n^2\,e^{-n}} \qquad \text{for } n \in \mathbb{N}.$$

You may use known properties of limits and limits that are well-known from earlier courses.

Exercise 4 (A variant of a formulation of the completeness axiom)**.**
Recall that one of the formulations of the completeness axiom of real numbers is:

(C2) Every increasing real number sequence $(a_n)_{n\in\mathbb{N}}$ which is bounded from above has a limit $\lim_{n\to\infty} a_n \in \mathbb{R}$.

Consider the statement

(C2') Every decreasing real number sequence $(b_n)_{n\in\mathbb{N}}$ which is bounded from below has a limit $\lim_{n\to\infty} b_n \in \mathbb{R}$.

Prove that (C2) implies (C2').

Hint: *Given a sequence $(b_n)_{n\in\mathbb{N}}$ as above, what can be said about the numbers $-b_n$ for $n \in \mathbb{N}$?*

Exercise 5 (Some points in the Cantor set)**.**
For a fixed $n \in \mathbb{N}$ and any $b_1, \ldots, b_n \in \{0,1\}$, consider the closed interval of length $\frac{1}{3^n}$ whose left endpoint is $\sum_{j=1}^n \frac{2\,b_j}{3^j} = \frac{2\,b_1}{3} + \frac{2\,b_2}{9} + \frac{2\,b_3}{27} + \cdots + \frac{2\,b_n}{3^n}$. Let $C_n \subset \mathbb{R}$ be the union of these 2^n intervals (see figure below)

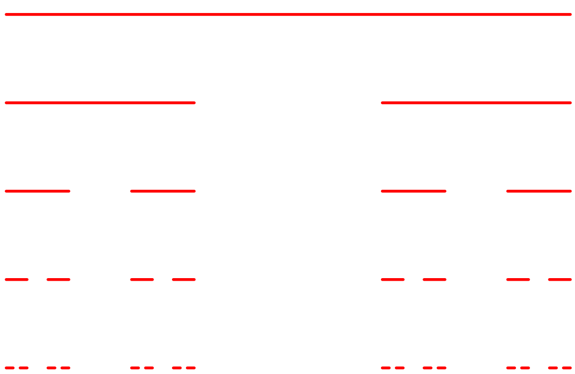
$$C_n = \bigcup_{b_1,\ldots,b_n\in\{0,1\}} \Big[\sum_{j=1}^n \frac{2\,b_j}{3^j} \, , \, \sum_{j=1}^n \frac{2\,b_j}{3^j} + \frac{1}{3^n} \Big].$$

The Cantor set $C \subset \mathbb{R}$ is the intersection of these sets over all $n \in \mathbb{N}$,

$$C = \bigcap_{n\in\mathbb{N}} C_n.$$

Show that $\frac{3}{4} \in C$ and $\frac{4}{5} \notin C$. (Or if $\frac{3}{4}$ and $\frac{4}{5}$ are too boring, use $\frac{11}{12}$ and $\frac{10}{11}$ instead.)

Hint: *To get started, it helps to note that $\frac{3}{4} = \sum_{k=0}^{\infty} \frac{2}{3^{1+2k}}$ (use geometric series), which can be written as an infinite series $\sum_{j=1}^{\infty} \frac{2\,b_j}{3^j}$ for a suitably chosen sequence $b_1, b_2, b_3, \ldots$ of zeros and ones.*



$[0,1]$

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

C_3

C_4