Aalto University **Problem set 1**
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2023-2024/III K Kytölä & D Adame-Carrillo

(Exercise sessions: 11.-12.1.2024) Hand-in due: Tue 16.1.2024 at 23:59

Fill-in-the-blanks 1. Let $f: X \to Y$ and $g: Y \to Z$ be functions. Complete the following proofs about surjectivity of the function composition $g \circ f: X \to Z$.

Claim (a): If both f and g are surjective, then $g \circ f$ is also surjective.

Proof of (a). Assume that $f: X \to Y$ and $g: Y \to Z$ are surjective. To prove surjectivity of $g \circ f$, we must show that for every $z \in Z$ there exists $x \in$ ________ such that $(g \circ f)(x) = z$.

So let $z \in Z$. Then by surjectivity of ____________________ , there exists some $y \in Y$ such that $g(y) = z$. Fix some such y. Then by surjectivity of $f: X \to Y$, there exists some $x \in$ ________ such that $f(x) = y$. For such an x, we have

$$(g \circ f)(x) \overset{\text{def}}{=} g\big(f(x)\big) = g\big(\_\_\_\_\_\_\big) = \_\_\_\_\_\_ ,$$

and we have thus proven the surjectivity of $g \circ f: X \to Z$. □

Claim (b): If $g \circ f$ is surjective, then g is also surjective.

Proof of (b). Assume $g \circ f: X \to Z$ is surjective. To prove the surjectivity of $g: Y \to Z$, we must show that for every $z \in Z$ there exists a ____________________ such that $g\big(\_\_\_\_\_\_\_\_\big) = z$.

So let $z \in Z$. By surjectivity of $g \circ f: X \to Z$, there exists some $x \in X$ such that $(g \circ f)(x) = z$. Fix some such x. If we then set $y = f(x)$, then we have

$$g\big(\_\_\_\_\_\_\_\_\big) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = (g \circ f)(x) = z ,$$

and we have thus proven the surjectivity of $g: Y \to Z$. □

The following are not a part of this exercise, but they are important to understand.

(c) Assuming $g \circ f$ is surjective, is it possible to prove that f is also surjective?

What about the corresponding statements about injectivity?

(a') Assuming f and g are injective, is it possible to prove that $g \circ f$ is injective?
(b') Assuming $g \circ f$ is injective, is it possible to prove that g is injective?
(c') Assuming $g \circ f$ is injective, is it possible to prove that f is injective?

Fill-in-the-blanks 2. Complete the following proof of the squeeze theorem (sandwich principle, lemma of two policemen).

Claim: If three sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$ of real numbers satisfy $a_n \le b_n \le c_n$ starting from some index, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \beta \in \mathbb{R},$$

then the sequence $(b_n)_{n\in\mathbb{N}}$ also converges, and $\lim_{n\to\infty} b_n = \beta$.

Proof. Since the beginning of a sequence affects neither the convergence nor the limit of the sequence, we may assume that $a_n \le b_n \le c_n$ holds for all $n \in \mathbb{N}$. We will show that $\lim_{n\to\infty} b_n = \beta$.

Let $\varepsilon > 0$. We must show that $|b_n - \beta| < \varepsilon$ from some index on.

Idea: the expression $b_n - \beta$ has to be estimated from both directions; one relying on the sequence $(a_n)_{n\in\mathbb{N}}$, and the other on the sequence $(c_n)_{n\in\mathbb{N}}$. (Draw a figure!)

Since $\lim_{n\to\infty} a_n = \beta$ and $\varepsilon > 0$, there exists an $n'_\varepsilon \in \mathbb{N}$ such that

$$\underline{\qquad\qquad\qquad\qquad\qquad} \qquad \text{for all } n \ge n'_\varepsilon.$$

Since $\lim_{n\to\infty} c_n = \beta$ and $\varepsilon > 0$, there exists an $n''_\varepsilon \in \mathbb{N}$ such that

$$\underline{\qquad\qquad\qquad\qquad\qquad} \quad \text{for all} \quad \underline{\qquad\qquad\qquad\quad}\,.$$

Now choose

$$n_\varepsilon = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}\,.$$

With this choice, for any $n \ge n_\varepsilon$ we have $n \ge n'_\varepsilon$. Therefore we get

$$\beta - b_n \;\le\; \beta - a_n \;\le\; |\beta - a_n| \;<\; \underline{\qquad\qquad\quad}$$

(the leftmost inequality holds by virtue of the assumption $a_n \le b_n$). Similarly, for $n \ge n_\varepsilon$ we have $n \ge n''_\varepsilon$, so we get

$$b_n - \beta \;\le\; c_n - \beta \;\le\; \underline{\qquad\qquad\quad} \;<\; \underline{\qquad\qquad\quad}$$

(the leftmost inequality holds by virtue of the assumption $b_n \le c_n$). The above inequalities imply that

$$|b_n - \beta| < \underline{\qquad\quad}$$

for all $n \ge n_\varepsilon$. We have thus proved the claim. $\square$