Problem set 3

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Exercise sessions: 27.-28.1.2022 Hand-in due: Tue 1.2.2022 at 23:59

Topic: Metric spaces, continuous functions

Written solutions to the exercises marked with symbol \triangle are to be returned in My-Courses. Each exercise is graded on a scale 0-3. The deadline for returning solutions to problem set 3 is Tue 1.2.2022 at 23:59.

Exercise 1 (Parallelogram law and norms not induced by inner products).

(a) Prove that the norm in an inner product space ${\sf V}$ satisfies the parallelogram law

$$||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$$
 for all $u, v \in V$.

(b) Using part (a), prove that the norm of Example IV.25

$$||(x,y)||_{\infty} = \max\{|x|,|y|\}, \quad \text{for } (x,y) \in \mathbb{R}^2,$$

is not induced by any inner product on \mathbb{R}^2 .

<u>Hint</u>: If the formula of part (a) does not hold for some pair of vectors $u, v \in \mathbb{R}^2$, then...

Exercise 2 (An exotic metric on the real line).

(a) Prove that the formula

$$d(x,y) = \log(1 + |x - y|)$$

defines a metric on \mathbb{R} .

- (b) Let A = [0, 1] and B = (5, 10]. Determine the diameter diam(A) of the set A, and the distance dist(A, B) between the two sets, both with respect to the metric d of part (a).
- Exercise 3 (A comparison of the Euclidean norm and the ℓ^1 -norm). This exercise concerns the comparison of the norm $||x|| = ||x||_2 = \sqrt{\langle x, x \rangle}$ on the space \mathbb{R}^d induced by the inner product $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$, and the norm $||x||_1 = \sum_{i=1}^d |x_i|$ of Exercise 4 of Problem set 2.
 - (a) Prove that for all $x \in \mathbb{R}^d$ we have

$$||x|| \le ||x||_1 \le \sqrt{d} ||x||.$$

<u>Hint</u>: (From Väisälä's book) The first inequality: Write $x = \sum x_j e_j$. The second inequality: Apply the Cauchy-Schwarz inequality to the vectors $(|x_1|, \ldots, |x_d|)$ and $(1, 1, \ldots, 1)$.

(b) Give a concrete example of a vector $x \neq \overline{0}$, for which the latter inequality in part (a) becomes an equality. Such an example demonstrates that the coefficient \sqrt{d} in the inequality is optimal (as small as possible).

- **Exercise 4** (A product space metric and continuity via component functions). Let (X, d_X) , (Y, d_Y) , (Z, d_Z) be three metric spaces.
 - (a) Consider the Cartesian product $X \times Y$, and define

$$d_1 \colon (X \times Y) \times (X \times Y) \to [0, +\infty)$$

by the formula

$$\mathsf{d}_1\big((x_1,y_1),\,(x_2,y_2)\big) \;=\; \mathsf{d}_X(x_1,x_2) + \mathsf{d}_Y(y_1,y_2)$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$. Check that d_1 thus defined gives a metric on the product space $X \times Y$.

(b) Let $f: Z \to X$ and $g: Z \to X$ be two functions. Consider the function

$$h: Z \to X \times Y$$
 given by $h(z) = (f(z), g(z))$ for $z \in Z$.

Prove that when the Cartesian product space $X \times Y$ is equipped with the metric d_1 of part (a) and when the functions f and g are continuous, then the function h is also continuous.

Exercise 5 (Some Lipschitz and non-Lipschitz functions). Let (X, d_X) and (Y, d_Y) be two metric spaces and let $K \geq 0$. Recall that a function $f: X \to Y$ is K-Lipschitz, if

$$d_Y(f(a), f(b)) \le K d_X(a, b)$$
 for all $a, b \in X$.

Here, on the real line \mathbb{R} and its subsets, we use the standard metric.

(a) Prove (with mean value theorem or algebraically), that the function

$$p: [-10, 10] \to \mathbb{R}, \qquad p(x) = x^3 \quad \text{ for } x \in [-10, 10],$$

is K-Lipschitz for a suitably chosen $K \geq 0$.

(b) Prove that the function $q: [0,1] \to [0,1], \ q(x) = \sqrt{x}$, is not K-Lipschitz for any $K \ge 0$ (although it is continuous).