Aalto University **Problem set 6**
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2023-2024/III K Kytölä & D Adame-Carrillo

(Exercise sessions: 15.-16.2.2024) Hand-in due: Tue 20.2.2024 at 23:59

Fill-in-the-blanks 1. Let (X, d) be a metric space, $A \subset X$ a compact subset, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A, and let $z \in X$. Assume that every convergent subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ has z as its limit, $\lim_{n\to\infty} x_{\varphi(n)} = z$. Complete the proof of the following claim.

Claim: The sequence $(x_n)_{n\in\mathbb{N}}$ converges to z.

Proof. We use a proof by contradiction and assume that $(x_n)_{n\in\mathbb{N}}$ does not converge to z.

In that case, we can find some $\varepsilon > 0$ such that we have $x_n \notin \mathcal{B}_\varepsilon(z)$ for

____ indices n.

Extracting such indices, we can form a subsequence $(x_{\vartheta(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{\vartheta(n)} \notin \mathcal{B}_\varepsilon(z)$ for every $n \in \mathbb{N}$.

Now by ____

the sequence $(x_{\vartheta(n)})_{n\in\mathbb{N}}$ in A must have some convergent subsequence $(x_{\vartheta(\psi(n))})_{n\in\mathbb{N}}$. This subsequence of a subsequence of $(x_n)_{n\in\mathbb{N}}$ is itself a subsequence of $(x_n)_{n\in\mathbb{N}}$, and since it is convergent, by assumption its limit must be

$$\lim_{n\to\infty} \underline{\hspace{5cm}} \; = \; z\,.$$

But since $x_{\vartheta(\psi(n))} \notin \mathcal{B}_\varepsilon(z)$ for every $n \in \mathbb{N}$, this is impossible. We have reached a contradiction, so the proof is complete. □

Fill-in-the-blanks 2. A *line segment* in $\mathbb{R}^n$ between points $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$ is the set

$$J = [\vec{x}_1, \vec{x}_2] = \{\vec{x}_1 + t(\vec{x}_2 - \vec{x}_1) \mid t \in [0,1]\} \subset \mathbb{R}^n.$$

A *broken line* in $\mathbb{R}^n$ through points $\vec{x}_1, \ldots, \vec{x}_\ell$ is the union of line segments

$$M = [\vec{x}_1, \vec{x}_2] \cup [\vec{x}_2, \vec{x}_3] \cup \cdots \cup [\vec{x}_{\ell-1}, \vec{x}_\ell] \subset \mathbb{R}^n.$$

A broken line M of this form is said to connect the points $\vec{x}_1$ and $\vec{x}_\ell$.

Complete the proof of the following claim.

Claim. If $D \subset \mathbb{R}^n$ is open and connected, then for all points $\vec{x}, \vec{y} \in D$ there exists a broken line $M \subset D$ connecting the points $\vec{x}$ and $\vec{y}$.

Proof. Assume $D \subset \mathbb{R}^n$ is open and connected, and $\vec{x}, \vec{y} \in D$.

Let $U \subset D$ be the set of all points $\vec{z} \in D$ to which the point $\vec{x}$ can be connected by some broken line. Below we will show that both $U \subset D$ and its complement $D \setminus U \subset D$ are open. Moreover, we clearly have $U \neq \emptyset$, since at least ________ $\in U$. With the connectedness of D these imply that $D \setminus U =$ ________ . In particular we can then conclude that $\vec{y} \in U$, i.e., the point $\vec{x}$ can be connected to $\vec{y}$ by a broken line, completing the proof.

Let us first show that $U \subset D$ is open. Let $\vec{z} \in U$. Then there exists a broken line $M \subset D$ connecting points ________ and $\vec{z}$. Since $D \subset \mathbb{R}^n$ is open and $\vec{z} \in D$, for some $r > 0$ we have $\mathcal{B}_r(\vec{z})$ __________ . For any $\vec{w} \in \mathcal{B}_r(\vec{z})$, the line segment $[\vec{z}, \vec{w}]$ is contained in the ball $\mathcal{B}_r(\vec{z})$. Therefore, by setting $M' = M \cup [\vec{z}, \vec{w}]$, we obtain a broken line $M' \subset$ ________ connecting points $\vec{x}$ and $\vec{w}$. This implies ________ $\in U$. We conclude that $\mathcal{B}_r(\vec{z}) \subset U$, showing that U is open.

Let us then show that $D \setminus U \subset D$ is open. Let $\vec{z} \in D \setminus U$. Again by openness of D, there exists an $r > 0$ such that ________________ . If $\mathcal{B}_r(\vec{z}) \subset D \setminus U$, then the openness of $D \setminus U$ follows. Assume the converse; that there exists $\vec{w} \in \mathcal{B}_r(\vec{z}) \cap U$. Then there exists a broken line $M \subset D$ connecting the points ________ and ________ . But since $[\vec{w}, \vec{z}] \subset \mathcal{B}_r(\vec{z})$, by setting $M' = M \cup \Big[$ ______ , ______ $\Big]$ we would obtain a broken line $M' \subset D$ connecting the points $\vec{x}$ and $\vec{z}$, which is a contradiction since ____________ . $\square$