Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2023-2024/III

**Problem set 3**

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**Exercise sessions: 25.-26.1.2024 Hand-in due: Tue 30.1.2024 at 23:59**

*Topic: Metric spaces, continuous functions*

*Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 3 is Tue 30.1.2024 at 23:59.*

*Remark:* Generally, continuity here refers to the $\varepsilon$-$\delta$ definition of Chapter VI. You are, however, allowed to use the fact that for real functions, this is logically equivalent to the definition of continuity by sequences, which was used in Chapter III. (The equivalence will be proven later in the course.)

**Exercise 1** (An exotic metric on the real line)**.**

(a) Prove that the formula

$$\mathsf{d}(x,y) = \log\big(1 + |x-y|\big)$$

defines a metric on $\mathbb{R}$.

(b) Let $A = [0,1]$ and $B = (5,10]$. Determine the diameter $\mathrm{diam}(A)$ of the set $A$, and the distance $\mathrm{dist}(A,B)$ between the two sets, both with respect to the metric $\mathsf{d}$ of part (a).

**Exercise 2** (The minimum map on a function space)**.**
Let $[a,b] \subset \mathbb{R}$ be a closed interval and consider the space $\mathcal{C}([a,b])$ of continuous real-valued functions on this interval equipped with the supremum-norm $\|\cdot\|_\infty$ and the metric induced by it. For any $f \in \mathcal{C}([a,b])$, the minimum $\min_{x\in[a,b]} f(x)$ exists (by Theorem III.14), and we can thus define a function

$$\mathrm{Min}\colon \mathcal{C}([a,b]) \to \mathbb{R}$$

on this function space by the formula

$$\mathrm{Min}(f) := \min_{x\in[a,b]} f(x) \qquad \text{for } f \in \mathcal{C}([a,b]).$$

Show that $\mathrm{Min}\colon \mathcal{C}([a,b]) \to \mathbb{R}$ is 1-Lipschitz, and conclude that it is in particular continuous.

✍ **Exercise 3** (A product space metric and continuity via component functions)**.**
Let $(X, \mathsf{d}_X)$, $(Y, \mathsf{d}_Y)$, $(Z, \mathsf{d}_Z)$ be three metric spaces.

(a) Consider the Cartesian product $X \times Y$, and define

$$\mathsf{d}_1 \colon (X \times Y) \times (X \times Y) \to [0, +\infty)$$

by the formula

$$\mathsf{d}_1\big((x_1, y_1), (x_2, y_2)\big) = \mathsf{d}_X(x_1, x_2) + \mathsf{d}_Y(y_1, y_2)$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$. Check that $\mathsf{d}_1$ thus defined gives a metric on the product space $X \times Y$.

(b) Let $f \colon Z \to X$ and $g \colon Z \to Y$ be two functions. Consider the function

$$h \colon Z \to X \times Y \quad \text{given by} \quad h(z) = \big(f(z), g(z)\big) \ \text{ for } z \in Z.$$

Prove that when the Cartesian product space $X \times Y$ is equipped with the metric $\mathsf{d}_1$ of part (a) and when the functions $f$ and $g$ are continuous, then the function $h$ is also continuous.

✍ **Exercise 4** (Some Lipschitz and non-Lipschitz functions)**.**
Let $(X, \mathsf{d}_X)$ and $(Y, \mathsf{d}_Y)$ be two metric spaces and let $K \geq 0$. Recall that a function $f \colon X \to Y$ is $K$-Lipschitz, if

$$\mathsf{d}_Y\big(f(a), f(b)\big) \leq K\, \mathsf{d}_X(a, b) \qquad \text{for all } \ a, b \in X.$$

Here, on the real line $\mathbb{R}$ and its subsets, we use the standard metric.

(a) Prove (with mean value theorem or algebraically), that the function

$$p \colon [-10, 10] \to \mathbb{R}, \qquad p(x) = x^3 \quad \text{for } x \in [-10, 10],$$

is $K$-Lipschitz for a suitably chosen $K \geq 0$.

(b) Prove that the function $q \colon [0, 1] \to [0, 1]$, $q(x) = \sqrt{x}$, is not $K$-Lipschitz for any $K \geq 0$ (although it is continuous).

✍ **Exercise 5** (The interior, boundary, and closure of a set in the plane)**.**
Consider the subset

$$A = \Big\{ (x, y) \in \mathbb{R}^2 \,\Big|\, x \geq y^2 \Big\}$$

of the Euclidean plane $\mathbb{R}^2$.

(a) Draw a picture of the subset $A \subset \mathbb{R}^2$.
(b) What is the interior $A^\circ$?
(c) What is the boundary $\partial A$?
(d) What is the closure $\overline{A}$?

Justify your answers directly using the definition of the interior, boundary, and closure of subsets in metric spaces.

Hint: *After drawing the picture, estimating some relevant distances in each coordinate direction separately may be a good way to get started.*

Hint: *It may help to note that the function $z \mapsto z^2$ is continuous $\mathbb{R} \to \mathbb{R}$ (it is a polynomial function).*