Aalto University **Problem set 6**
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021/III K Kytölä & D Adame-Carrillo

(Exercise sessions: 18.-19.2.2021) Hand-in due: Tue 23.2.2021 at 23:59

Fill-in-the-blanks 1. Let (X, d) be a metric space. Given a subset $A \subset X$, a point $x \in X$ is said to be

- an *interior point* of A, if for some $r > 0$ we have $\mathcal{B}_r(x) \subset A$
- an *exterior point* of A, if for some $r > 0$ we have $\mathcal{B}_r(x) \subset X \setminus A$
- a *boundary point* of A, if x is neither an interior point of A nor and exterior point of A.

The set of all interior points of A is denoted A°, the set of all exterior points $\mathrm{ext}(A)$, and the set of all boundary points ∂A. These form a partition $X = A^\circ \cup \mathrm{ext}(A) \cup \partial A$ of the whole space X to three disjoint subsets, of which A° and $\mathrm{ext}(A)$ are open in X. The *closure* of A is defined as $\overline{A} = X \setminus \mathrm{ext}(A)$, and as the complement of the open set $\mathrm{ext}(A)$, it is closed.

Complete the proofs of the following claims.

Claim (i). $A \subset \overline{A}$.

Proof. By considering complements, the claim can be equivalently formulated as $X \setminus A \supset X \setminus \overline{A}$. Directly from the definition of closure we get that $X \setminus \overline{A} =$ __________ . Therefore, let $x \in \mathrm{ext}(A)$. Then for some $r > 0$ we have $\mathcal{B}_r(x) \subset X \setminus A$. In particular we get $x \in \mathcal{B}_r(x) \subset X \setminus A$. This proves the claim $X \setminus \overline{A} \subset X \setminus A$. □

Claim (ii). If $F \subset X$ is closed and $A \subset F$, then $\overline{A} \subset F$.

Proof. Let $F \subset X$ be a closed subset such that $A \subset F$. Then $X \setminus F$ is open and $X \setminus A$ ______ $X \setminus F$. If $x \in X \setminus F$, then by openness for some $r > 0$ we have $\mathcal{B}_r(x) \subset X \setminus F \subset X \setminus A$, which by definition shows that $x \in$ __________ . We therefore have $X \setminus F \subset \mathrm{ext}(A)$. For the complements we get $F \supset X \setminus \mathrm{ext}(A) = \overline{A}$ as claimed. □

Claim (iii). $\overline{A}$ is the smallest closed set which contains A.

Proof. The closure $\overline{A}$ is a closed set, and by part __________ , it contains A. By part __________ , on the other hand, every closed subset containing A itself contains at least $\overline{A}$. □

Fill-in-the-blanks 2. A *line segment* in $\mathbb{R}^n$ between points $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$ is the set

$$J = [\vec{x}_1, \vec{x}_2] = \{\vec{x}_1 + t(\vec{x}_2 - \vec{x}_1) \mid t \in [0,1]\} \subset \mathbb{R}^n.$$

A *broken line* in $\mathbb{R}^n$ through points $\vec{x}_1, \ldots, \vec{x}_\ell$ is the union of line segments

$$M = [\vec{x}_1, \vec{x}_2] \cup [\vec{x}_2, \vec{x}_3] \cup \cdots \cup [\vec{x}_{\ell-1}, \vec{x}_\ell] \subset \mathbb{R}^n.$$

A broken line M of this form is said to connect the points $\vec{x}_1$ and $\vec{x}_\ell$.

Complete the proof of the following claim.

Claim. If $D \subset \mathbb{R}^n$ is open and connected, then for all points $\vec{x}, \vec{y} \in D$ there exists a broken line $M \subset D$ connecting the points $\vec{x}$ and $\vec{y}$.

Proof. Assume $D \subset \mathbb{R}^n$ is open and connected, and $\vec{x}, \vec{y} \in D$.

Let $U \subset D$ be the set of all points $\vec{z} \in D$ to which the point $\vec{x}$ can be connected by some broken line. Below we will show that both $U \subset D$ and its complement $D \setminus U \subset D$ are open. Moreover, we clearly have $U \neq \emptyset$, since at least ________ $\in U$. With the connectedness of D these imply that $D \setminus U =$ ________ . In particular $\vec{y} \in U$, i.e., the point $\vec{x}$ can be connected to $\vec{y}$ by a broken line, completing the proof.

Let us first show that $U \subset D$ is open. Let $\vec{z} \in U$. Then there exists a broken line $M \subset D$ connecting points ________ and $\vec{z}$. Since $D \subset \mathbb{R}^n$ is open and $\vec{z} \in D$, for some $r > 0$ we have $\mathcal{B}_r(\vec{z})$ __________ . For any $\vec{w} \in \mathcal{B}_r(\vec{z})$, the line segment $[\vec{z}, \vec{w}]$ is contained in the ball $\mathcal{B}_r(\vec{z})$. Therefore, by setting $M' = M \cup [\vec{z}, \vec{w}]$, we obtain a broken line $M' \subset$ ________ connecting points $\vec{x}$ and $\vec{w}$. This implies ________ $\in U$. We conclude that $\mathcal{B}_r(\vec{z}) \subset U$, showing that U is open.

Let us then show that $D \setminus U \subset D$ is open. Let $\vec{z} \in D \setminus U$. Again by openness of D, there exists an $r > 0$ such that ________________ . If $\mathcal{B}_r(\vec{z}) \subset D \setminus U$, then the openness of $D \setminus U$ follows. Assume the converse; that there exists $\vec{w} \in \mathcal{B}_r(\vec{z}) \cap U$. Then there exists a broken line $M \subset D$ connecting the points ________ and ________ . But since $[\vec{w}, \vec{z}] \subset \mathcal{B}_r(\vec{z})$, by setting $M' = M \cup \Big[$ ______ , ______ $\Big]$ we would obtain a broken line $M' \subset D$ connecting the points $\vec{x}$ and $\vec{z}$, which is a contradiction since ____________ . $\square$