(Exercise sessions: 25.-26.1.2024) Hand-in due: Tue 30.1.2024 at 23:59

Fill-in-the-blanks 1. Let (X, d) be a metric space and $(\mathsf{V}, \|\cdot\|)$ a normed space. On V, we use the metric induced by the norm, $\mathsf{d}_{\mathsf{V}}(u, v) = \|v - u\|$ for $u, v \in \mathsf{V}$. Complete the proof of the following statement.

Claim. If $f, g \colon X \to \mathsf{V}$ are two continuous functions, then the function $f + g \colon X \to \mathsf{V}$ is also continuous.

Remark: The function $f + g \colon X \to \mathsf{V}$ is the pointwise sum of the vector valued functions f and g, defined by the formula $(f + g)(x) = f(x) + g(x)$ (the right hand side is vector addition in vector space V).

Proof. First fix $x \in X$. Let us show that $f + g$ is continuous at the point x. Let $\varepsilon > 0$. Then also $\frac{\varepsilon}{2} > 0$, and since by assumption f is continuous at x, there exists a

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such that we have $\|f(x') - f(x)\| < \frac{\varepsilon}{2}$ whenever $x' \in X$ and

____________________ .

Similarly, since g is also continuous at x by assumption, there exists a

______________ such that __________________________________ .

Now let $\delta =$ ________________________ > 0.

Then whenever $x' \in X$ and $\mathsf{d}(x, x') < \delta$, we have both

$\mathsf{d}(x, x') <$ ____________ and $\mathsf{d}(x, x') <$ ____________ , so

$\|(f + g)(x') - (f + g)(x)\| =$ ________________________________

$\leq$ ________________________________

$<$ __________________________ $= \varepsilon.$

This implies that $f + g$ is continuous at x. Since $x \in X$ was arbitrary, $f + g$ is therefore a continuous function. $\square$

Fill-in-the-blanks 2. Consider the space $\mathcal{C}([-1,1])$ of continuous real-valued functions on the closed interval $[-1,1]$, and its subspace

$$\mathcal{C}^1([-1,1]) \;=\; \Big\{ x \in \mathcal{C}([-1,1]) \;\Big|\; x' \text{ exists and is continuous } [-1,1] \to \mathbb{R} \Big\}$$

of continuously differentiable functions. Equip $\mathcal{C}([-1,1])$ with the sup-norm and its induced metric, and equip $\mathcal{C}^1([-1,1]) \subset \mathcal{C}([-1,1])$ with the metric it inherits as a subset. Taking the derivative is a function $D \colon \mathcal{C}^1([-1,1]) \to \mathcal{C}([-1,1])$; concretely, $D(x) = x'$ for $x \in \mathcal{C}^1([-1,1])$. Complete the proof of the following claim:

Claim. The function $D \colon \mathcal{C}^1([-1,1]) \to \mathcal{C}([-1,1])$ is not Lipschitz.

Proof. Let us argue by contradiction: suppose instead that D is M-Lipschitz for some $M \geq 0$. With such an M fixed, define a function $x \colon [-1,1] \to \mathbb{R}$ by

$$x(t) \;=\; \sin\big((M+1)\,t\big) \qquad \text{for } t \in [-1,1].$$

We have

$$|x(t)| \;=\; \big|\sin\big((M+1)\,t\big)\big| \;\leq\; \underline{\hspace{2cm}} \qquad \text{for all } t \in [-1,1],$$

so for the sup-norm of x we get

$$\|x\|_\infty \;=\; \sup_{t\in[-1,1]} \underline{\hspace{4cm}} \;\leq\; \underline{\hspace{1.5cm}}.$$

The function x is continuously differentiable, with derivative given by

$$x'(t) = \underline{\hspace{7cm}} \qquad \text{for } t \in [-1,1].$$

In particular we get the following inequality for the sup-norm of the derivative x':

$$\|x'\|_\infty \;=\; \sup_{t\in[-1,1]} |x'(t)| \;\underline{\hspace{1cm}}\; |x'(0)| \;=\; \underline{\hspace{2cm}}\,.$$

Now let us consider the distance of D evaluated at the two arguments $x \in \mathcal{C}^1([-1,1])$ and $0 \in \mathcal{C}^1([-1,1])$. The previous observation yields

$$\mathsf{d}\big(D(0), D(x)\big) \;=\; \|D(x) - D(0)\|_\infty \;=\; \|x' - 0\|_\infty \;\geq\; \underline{\hspace{2.5cm}}\,.$$

This, however, is a contradiction with the assumed M-Lipschitz property of D, because that property says

$$\mathsf{d}\big(D(0), D(x)\big) \;\leq\; M\,\mathsf{d}(0,x) \;=\; M\,\|x - 0\|_\infty \;\leq\; \underline{\hspace{2cm}}\,,$$

where the last inequality follows by an earlier observation. The contradiction shows that D cannot be M-Lipschitz for any $M \geq 0$, which completes the proof of the claim. $\square$