Aalto University **Problem set 3**
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021/III K Kytölä & D Adame-Carrillo

Exercise sessions: 28.-29.1.2021 **Hand-in due: Tue 2.2.2021 at 23:59**

Topic: Metric spaces, continuous functions

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 3 is Tue 2.2.2021 at 23:59.

Exercise 1 (Parallelogram law and norms not induced by inner products)**.**

(a) Prove that the norm in an inner product space V satisfies the *parallelogram law*

$$\|u+v\|^2 + \|u-v\|^2 = 2\,\|u\|^2 + 2\,\|v\|^2 \qquad \text{for all } u, v \in \mathsf{V}.$$

(b) Using part (a), prove that the norm of Example IV.25

$$\|(x,y)\|_\infty = \max\{|x|, |y|\}, \qquad \text{for } (x,y) \in \mathbb{R}^2,$$

is not induced by any inner product on $\mathbb{R}^2$.

Hint: *If the formula of part (a) does not hold for some pair of vectors $u, v \in \mathbb{R}^2$, then...*

Exercise 2 (An exotic metric on the real line)**.**

(a) Prove that the formula

$$\mathsf{d}(x,y) = \log\big(1 + |x-y|\big)$$

defines a metric on $\mathbb{R}$.

(b) Let $A = [0,1]$ and $B = (5,10]$. Determine the diameter $\mathrm{diam}(A)$ of the set A, and the distance $\mathrm{dist}(A,B)$ between the two sets, both with respect to the metric d of part (a).

✍ **Exercise 3** (A comparison of the Ecludean norm and the ℓ^1-norm)**.**
This exercise concerns the comparison of the norm $\|x\| = \|x\|_2 = \sqrt{\langle x, x\rangle}$ on the space $\mathbb{R}^d$ induced by the inner product $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$, and the norm $\|x\|_1 = \sum_{i=1}^d |x_i|$ of Exercise 4 of Problem set 2.

(a) Prove that for all $x \in \mathbb{R}^d$ we have

$$\|x\| \;\leq\; \|x\|_1 \;\leq\; \sqrt{d}\,\|x\|.$$

Hint: *(From Väisälä's book) The first inequality: Write $x = \sum x_j e_j$. The second inequality: Apply the Cauchy-Schwarz inequality to the vectors $(|x_1|, \ldots, |x_d|)$ and $(1, 1, \ldots, 1)$.*

(b) Give a concrete example of a vector $x \neq \overline{0}$, for which the latter inequality in part (a) becomes an equality. Such an example demonstrates that the coefficient $\sqrt{d}$ in the inequality is optimal (as small as possible).

✍ **Exercise 4.**

(a) Let X be a set, $J \neq \emptyset$ a nonempty index set, and $A_j \subset X$ subsets for each $j \in J$. Prove De Morgan's laws for arbitrary unions and intersections:

$$X \setminus \bigcup_{j \in J} A_j = \bigcap_{j \in J} (X \setminus A_j) \quad \text{and} \quad X \setminus \bigcap_{j \in J} A_j = \bigcup_{j \in J} (X \setminus A_j).$$

Hint: *Often the equality $A = B$ between two sets is most straightforward to prove in two stages: $A \subset B$ and $B \subset A$. In this problem the two stages can be combined, by directly establishing a chain of equivalent conditions $x \in A \Leftrightarrow \cdots \Leftrightarrow x \in B$.*

(b) Using part (a) and known properties of open sets, prove the following (stated as a Proposition in Chapter V of the lecture notes):

- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Hint: *The first proof could start with: "Let J be an index set, and $F_j \subset X$ closed sets, for $j \in J$."*

✍ **Exercise 5.** Let (X, d_X) and (Y, d_Y) be two metric spaces and let $K \geq 0$. Recall that a function $f \colon X \to Y$ is K-Lipschitz, if

$$\mathsf{d}_Y\big(f(a), f(b)\big) \leq K\, \mathsf{d}_X(a, b) \qquad \text{for all} \ \ a, b \in X.$$

Here, on the real line $\mathbb{R}$ and its subsets, we use the standard metric.

(a) Prove (with mean value theorem or algebraically), that the function

$$p \colon [-10, 10] \to \mathbb{R}, \qquad p(x) = x^3 \quad \text{for } x \in [-10, 10],$$

is K-Lipschitz for a suitably chosen $K \geq 0$.

(b) Prove that the function $q \colon [0, 1] \to [0, 1]$, $q(x) = \sqrt{x}$, is not K-Lipschitz for any $K \geq 0$ (although it is continuous).