

(Exercise sessions: 25.-26.1.2024) Hand-in due: Tue 23.1.2024 at 23:59

Fill-in-the-blanks 1. Let (X, d) be a metric space and $(V, \|\cdot\|)$ a normed space. On V , we use the metric induced by the norm, $d_V(u, v) = \|v - u\|$ for $u, v \in V$. Complete the proof of the following statement.

Claim. If $f, g: X \rightarrow V$ are two continuous functions, then the function $f + g: X \rightarrow V$ is also continuous.

Remark: The function $f + g: X \rightarrow V$ is the pointwise sum of the vector valued functions f and g , defined by the formula $(f + g)(x) = f(x) + g(x)$ (the right hand side is vector addition in vector space V).

Proof. First fix $x \in X$. Let us show that $f + g$ is continuous at the point x . Let $\varepsilon > 0$. Then also $\frac{\varepsilon}{2} > 0$, and since by assumption f is continuous at x , there exists a

_____ such that we have $\|f(x') - f(x)\| < \frac{\varepsilon}{2}$ whenever $x' \in X$ and

_____.
 Similarly, since g is also continuous at x by assumption, there exists a _____ such that _____.

Now let $\delta = ______ > 0$.

Then whenever $x' \in X$ and $d(x, x') < \delta$, we have both

$$d(x, x') < ______ \quad \text{and} \quad d(x, x') < ______, \quad \text{so}$$

$$\|(f + g)(x') - (f + g)(x)\| = ______$$

$$\leq ______$$

$$< ______ = \varepsilon.$$

This implies that $f + g$ is continuous at x . Since $x \in X$ was arbitrary, $f + g$ is therefore a continuous function. \square

Fill-in-the-blanks 2. Consider the space $\mathcal{C}([-1, 1])$ of continuous real-valued functions on the closed interval $[-1, 1]$, and its subspace

$$\mathcal{C}^1([-1, 1]) = \left\{ x \in \mathcal{C}([-1, 1]) \mid x' \text{ exists and is continuous } [-1, 1] \rightarrow \mathbb{R} \right\}$$

of continuously differentiable functions. Equip $\mathcal{C}([-1, 1])$ with the sup-norm and its induced metric, and equip $\mathcal{C}^1([-1, 1]) \subset \mathcal{C}([-1, 1])$ with the metric it inherits as a subset. Taking the derivative is a function $D: \mathcal{C}^1([-1, 1]) \rightarrow \mathcal{C}([-1, 1])$; concretely, $D(x) = x'$ for $x \in \mathcal{C}^1([-1, 1])$. Complete the proof of the following claim:

Claim. The function $D: \mathcal{C}^1([-1, 1]) \rightarrow \mathcal{C}([-1, 1])$ is not Lipschitz.

Proof. Let us argue by contradiction: suppose instead that D is M -Lipschitz for some $M \geq 0$. With such an M fixed, define a function $x: [-1, 1] \rightarrow \mathbb{R}$ by

$$x(t) = \sin((M+1)t) \quad \text{for } t \in [-1, 1].$$

We have

$$|x(t)| = |\sin((M+1)t)| \leq \text{_____} \quad \text{for all } t \in [-1, 1],$$

so for the sup-norm of x we get

$$\|x\|_\infty = \sup_{t \in [-1, 1]} \text{_____} \leq \text{_____}.$$

The function x is continuously differentiable, with derivative given by

$$x'(t) = \text{_____} \quad \text{for } t \in [-1, 1].$$

In particular we get the following inequality for the sup-norm of the derivative x' :

$$\|x'\|_\infty = \sup_{t \in [-1, 1]} |x'(t)| \text{ _____ } |x'(0)| = \text{_____}.$$

Now let us consider the distance of D evaluated at the two arguments $x \in \mathcal{C}^1([-1, 1])$ and $0 \in \mathcal{C}^1([-1, 1])$. The previous observation yields

$$d(D(0), D(x)) = \|D(x) - D(0)\|_\infty = \|x' - 0\|_\infty \geq \text{_____}.$$

This, however, is a contradiction with the assumed M -Lipschitz property of D , because that property says

$$d(D(0), D(x)) \leq M d(0, x) = M \|x - 0\|_\infty \leq \text{_____},$$

where the last inequality follows by an earlier observation. The contradiction shows that D cannot be M -Lipschitz for any $M \geq 0$, which completes the proof of the claim. \square