Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2022-2023/III

Problem set 2

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(Exercise sessions: 19.-20.1.2023) Hand-in due: Tue 24.1.2023 at 23:59

Fill-in-the-blanks 1. Complete the following arguments, which exemplify certain expressions that *do not* yield norms. It is simplest to use the labels given in Chapter IV of the lecture notes for the conditions required of a norm.

- The formula $\|(x,y)\| = |x+y|$ does not define a norm for points $(x,y) \in \mathbb{R}^2$ in the plane $\mathbb{R}^2$. For example

  $$\text{condition } \big(\_\_\_\_\big)$$

  does not hold, since by choosing

  $$x = \_\_\_\_\_\_\_\_\_\_\_\_, \qquad y = \_\_\_\_\_\_\_\_\_\_\_\_$$

  we find

  ____________________, although ____________________,

  and this directly violates the above mentioned condition.
- The formula $\|f\| = |f(0)|$ does not define a norm in the space $C\big([-1,1]\big)$ of continuous functions on the interval $[-1,1]$. The reason:

  $$\text{condition } \big(\_\_\_\_\big)$$

  does not hold, because by choosing

  $$f(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad \text{for } t \in [-1,1],$$

  we find

  ____________________, although ____________________,

  and this directly violates the above mentioned condition.

  The two other conditions are nevertheless satisfied:

  Justification of condition $\big(\_\_\_\_\big)$:

  ____.

  Justification of condition $\big(\_\_\_\_\big)$:

  ____.

Fill-in-the-blanks 2. Complete the following arguments to show that subsequences of a convergent real-number sequence are convergent, and they have the same limit as the original sequence. Recall that a subsequence of a sequence $(x_n)_{n\in\mathbb{N}}$ is a sequence of the form $(x_{\varphi(n)})_{n\in\mathbb{N}}$, where $\varphi\colon \mathbb{N}\to\mathbb{N}$ is a strictly increasing function.

<u>Lemma</u>: If $\varphi\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then for all $n\in\mathbb{N}$ we have $\varphi(n)\geq n$.

<u>Proof of lemma</u>: Let $\varphi\colon \mathbb{N}\to\mathbb{N}$ be a strictly increasing function. We prove the claim by induction on n.

For the base case, $n = 1$, it suffices to note that

$$\underline{\hspace{2cm}} \;\geq\; \underline{\hspace{2cm}} \qquad \text{since } \varphi(1)\in\mathbb{N}.$$

For the induction step, assume that for a natural number $n\in\mathbb{N}$ we have $\varphi(n)\geq n$. We must prove the corresponding property for $n+1$. Since φ is strictly increasing, we have

$$\underline{\hspace{2cm}} \;>\; \underline{\hspace{2cm}},$$

which implies $\varphi(n+1)\geq\varphi(n)+1$, since the strict inequality above concerns natural numbers. To conclude the proof, we estimate

$$\varphi(n+1) \;\geq\; \varphi(n)+1 \;\geq\; \underline{\hspace{3cm}},$$

using ____ in the last step. □

<u>Theorem</u>: If $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers which converges to $\alpha\in\mathbb{R}$, then any subsequence of it also converges to α.

<u>Proof of theorem</u>: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} x_n = \alpha$, and let $(x_{\varphi(n)})_{n\in\mathbb{N}}$ be a subsequence of it. We must prove

$$\lim_{n\to\infty} \underline{\hspace{2.5cm}} \;=\; \alpha.$$

To prove this from the definition of limit, let $\varepsilon > 0$. From the assumption $\lim_{n\to\infty} x_n = \alpha$ we get that there exists an $N\in\mathbb{N}$

such that for all ____________ we have ________________________.

The same N will work for our present goal. So let $n\geq N$. Note that by the lemma we then get $\varphi(n)\geq n\geq N$. In particular from the above, we obtain $|x_{\varphi(n)}-\alpha|<\varepsilon$. This finishes the proof, directly by the definition of a limit. □