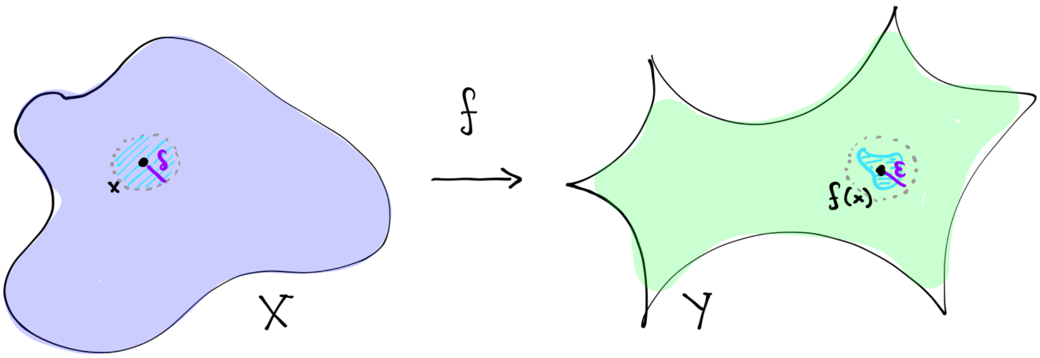

Metric spaces

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Contents

Foreword

These lecture notes are primarily intended for the core B.Sc. level mathematics course **MS-E1541** *Metric Spaces* at Aalto University.

The structure of these notes is largely based on an earlier version of the course taught by Pekka Alestalo, and parts of the textbook [**Väi99**]. The notes are still in an entirely preliminary and incomplete form, and I plan to frequently update them during the current course. You could help me — and perhaps more importantly the students who will use this material — by sending comments about mistakes, misprints, needs for clarification, etc., to me via the course's Zulip chat forum or by email (`kalle.kytola@aalto.fi`).

As textbooks for the course, [**Väi99**] matches the contents and level very well, and is extremely well written, but it only exists in Finnish and Swedish. As English textbooks we recommend for example parts of [**Car00**] or [**Rud76**] (both of these books cover significantly more material than the present course).

Symbols indicating material beyond core content:

Section symbols

♡ *stands for sections that are* **optional** *for the purposes of the present course; they are intended for those who prepare to study mathematics further or seek in depth understanding.*

Exercise symbols

✓ *stands for exercises which should be quick, easy, and* **routine***, in view of the preceeding definitions and/or basic results.*

⋱ *stands for exercises whose solution can be* **lengthy** *in comparison to regular ones, but not necessarily more difficult.*

♯ *stands for exercises that are more* **challenging** *than regular ones; they may require some creativity and a good command of various topics in basic mathematics, in addition to the newly introduced material.*

Lecture I

Foundations: set theory and logic

Study tips

This chapter is long... Most of the material should, however, be familiar to you already. This more extensive background chapter is intended to serve two purposes:

- If there are things you are not yet familiar with, you can still catch up now and follow the rest of the course.
- We fix some notations and terminology that will be used in the subsequent chapters. If you are already very familiar with set theory and logic, perhaps just quickly check the terms in **bold** and the related notation.

Among the topics of this chapter, the images and preimages of sets under functions (the last part of Section I.2) is perhaps the only one likely to not have been covered in prior basic courses. It may correspondingly deserve some more attention.

The students planning further mathematical studies are advised to take this opportunity to look into countable and uncountable infinite sets (Appendix A.1), although this topic is not strictly necessary for the present course.

Some perspectives on set theory before we start

Mathematics, as it has been practiced especially in the 20th and 21st centuries, is (almost entirely) founded on the notions of *sets* and their *elements*. To just quickly illustrate the expressive power of set theory we mention that, e.g., the notion of a function can be reduced to that of sets[1], and constructions of real numbers[2] and even of natural numbers[3] can be given based on sets. Such flexibility of set theory justifies taking it as the foundations of mathematics.

Historical reasons for looking for solid foundations of this type include the discovery of numerous apparent paradoxes, questions about the validity of certain arguments about numbers and functions, and even disagreements about what can be considered legitimate topics of mathematics in the first place. The late 19th century and early 20th century featured particularly notable controversies of this kind, and lead to the adoption of set theory as the (more or less) universally agreed on foundations. Later in this course, we will take a look at a few of those examples that had been

[1] A function *is* its "graph", a subset of the Cartesian product of the domain and codomain of the function. For more details, we refer to more serious treatments of set theory, e.g. [**Hal74**].

[2] One noteworthy construction of the real numbers is as Dedekind cuts of the set of rational numbers, see for example Wikipedia `https://en.wikipedia.org/wiki/Dedekind_cut`.

[3] Indeed, in a foundational crisis, mathematicians felt the need to *construct* natural numbers 1, 2, 3, . . . , instead of assuming their existence! Again we refer to serious treatments of set theory about the details.

considered controversial before the foundations were clarified: for example the Cantor set (Appendix B.3) and the Weierstrass function (Appendix **??**), and countable and uncountable infinities (Appendix A.1).

From a general perspective, being precise about the foundations probably became necessary when the level of abstraction increased sufficiently so that one could no longer rely on too informal or intuitive reasoning. The historical rationale therefore has its parallel in mathematical studies: there comes a point at which the level of abstraction requires the student to make peace with foundational questions. Given that mathematics is founded on set theory, a student of mathematics must become familiar and comfortable with set theoretical notation and reasoning. To be fair, in this course we use set theory in a rather boring way as a language which is necessary for precise definitions and statements. Set theory and foundations of mathematics themselves would lead to fascinating questions, but we will not pursue such directions here.

As a pragmatic motivation for the mathematics student, set theory forms the common language in which virtually all standard topics in university level mathematics are phrased: measure and integration, probability theory, group theory and algebra, differential geometry and algebraic geometry, real analysis, complex analysis, functional analysis, etc. etc.

The present course, specifically, falls almost entirely within what is often labeled as *point set topology* (as opposed to *algebraic topology*). This terminology is best understood by noting that the expression *an element of a set* is used exactly parallel to the expression *a point in space*.[4] To call a "space" a "point set" is to emphasize that as a mathematical object a space is first and foremost the set consisting of its points, and only as such potentially equipped with some further structure.

I.1. Basics of set theory

Sets and elements

Sets consist of elements — a **set** is the collection of all **elements** that belong to it. It is customary to denote sets by upper case letters and their elements by lower case letters, as we mostly do also below. However, this is merely a typographical practice often adopted because it can serve as a notational cue to the reader; it will be discarded when appropriate.[5]

We denote $a \in A$, if a is an element of a set A. If a is not an element of A, we denote $a \notin A$. The *binary relation* denoted by the symbol $\in$ (whose negation is denoted by the symbol $\notin$) is at the heart of set theory. In natural language, "$\in$" can

[4]We will in fact not precisely define the notions of *a point* and *a space* as such, but rather we use them as synonymous to *an element* and *a set* whenever we wish to draw attention to the fact that the set is also furnished with some additional structure. Various specific structures will be defined precisely later on: we will consider vector spaces, inner product spaces, normed spaces, metric spaces, topological spaces, … — but the general term *space* itself will not be used in a definite mathematical sense.

[5]Indeed, for example for sets whose elements themselves are sets, it becomes impossible to strictly adhere to such a typographical "rule".

be read as "to belong to"[6], for example

$$\begin{aligned} a \in A \quad &: \quad \text{(the element) } a \text{ belongs to (the set) } A, \qquad \text{(I.1)} \\ a \notin A \quad &: \quad \text{(the element) } a \text{ does not belong to (the set) } A. \end{aligned}$$

An often used informal (not exactly well-defined) example of a set is the set of all fruits, which has as elements apples, kiwis, mangos, etc. More mathematical examples of sets include

- the set $\mathbb{Z}$ of all integers, having $\ldots, -2, -1, 0, 1, 2, \ldots$ as its elements;
- the open interval $(-\pi, \pi)$ is the set whose elements are the real numbers x satisfying $-\pi < x < \pi$;
- the set of all continuous real-valued functions on the real line;
- the set of all subsets of the plane $\mathbb{R}^2$;
- ....

In particular, sets may have numbers as their elements (as is probably very familiar already). But sets may also have much more general types of elements: functions, other sets, etc. etc.[7]

Remark I.1 (What does it take to specify a set?)**.**
A set is known when its elements are known — i.e., when for every possible object we are able to decide whether the object is an element of the set or not.

In particular two sets A and B are equal, denoted $A = B$, if they have exactly the same elements.

A few more examples of sets are:

- the **empty set** $\emptyset$ is the set which does not contain any elements;
- a **singleton** is a set which contains a single element, the singleton consisting of an element a is denoted $\{a\}$.

Remark I.2 (Do not confuse a singleton with its element!)**.**
Note that

$$a \in \{a\}, \qquad \text{but} \qquad a \neq \{a\}.$$

Exercise (✓) I.1 (Empty set vs. the set consisting of the empty set, etc.)**.**
How many elements are in the following sets:

(a): $\emptyset$, (b): $\{\emptyset\}$, (c): $\{\{\emptyset\}\}$?

The simplest method of specifying a set is to list its elements:

- the notation $\{a_1, a_2, \ldots, a_n\}$ stands for the set consisting of $a_1, a_2, \ldots, a_n$.

[6]Suomeksi "∈" luetaan "kuuluu (joukkoon)".

[7]As an example, problems in nonparametric statistics of real-valued data are basically optimization problems over the set of all those functions from the set of all measurable subsets of the set of real numbers to the unit interval, with satisfy the three axioms of a probability measure. In traditional (parametric) statistics the optimization problem is resticted to a subset of the above set. Don't be fooled by apparent simplicity; the expressive power of set theory is vast, and actual mathematical applications make good use of that expressive power.

Note that we have, e.g., $\{9,1,1\} = \{1,9\}$ (this set has two elements: 1 and 9). A set is only the collection of elements belonging to it; the order of elements has no significance, and repetition is redundant.

Another common method of specifying a set is to use a logical condition as a criterion for whether an element belongs to the set:

- if A is a set and $P(x)$ is a logical proposition depending on a variable x, then the notation
$$\big\{x \in A \bigm| P(x)\big\} \tag{I.2}$$
stands for the set consisting of those elements x of the set A for which $P(x)$ is true.

Example I.3 (Examples of sets defined by a condition)**.**
In the following examples, various conditions are used to "extract" from the set $\mathbb{R}$ of all real numbers a subset of those that satisfy the condition in question:

$$\begin{aligned}\Big\{x \in \mathbb{R} \Bigm| \sin(x) = 0\Big\} &= \{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\} \\ \Big\{x \in \mathbb{R} \Bigm| x^4 = \frac{1}{3}\Big\} &= \Big\{\frac{-1}{\sqrt{\sqrt{3}}}, \frac{1}{\sqrt{\sqrt{3}}}\Big\} \\ \Big\{x \in \mathbb{R} \Bigm| x^2 < 0\Big\} &= \emptyset.\end{aligned}$$

We will also take some liberties to modify the above notational conventions in what we hope are reasonably self-explanatory ways, such as:

$$\begin{aligned}\Big\{a_j \Bigm| j \in \{1,2,\ldots,n\}\Big\} &= \{a_1, a_2, \ldots, a_n\}, \\ \big\{\sqrt{x} \bigm| x \in \mathbb{N}\big\} &= \Big\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots\Big\}, \qquad \text{etc.}\end{aligned}$$

Subsets

A set A is said to be a **subset** of another set B if every element of A is also an element of B; we then denote $A \subset B$. Being a subset is a binary relation among sets, denoted by the symbol $\subset$. Its negation is denoted by $\not\subset$, so $A \not\subset B$ means that there exists at least one element in the set A which is not an element of B.

Note the difference between the relations $\in$ and $\subset$.[8]

In natural language, the subset symbol $\subset$ could be read as[9]

$$A \subset B \qquad : \qquad \text{(the set) } A \text{ is contained in (the set) } B, \tag{I.3}$$

or simply as: A is a subset of B. For precise and unambiguous mathematical meaning, it is best to avoid mixing the natural language expressions "belongs to" and "is contained in", which stand for the relations $\in$ and $\subset$, respectively.

[8] If you have some experience in programming with *Python* or *Mathematica* (or other programming languages which encourage extensive use of nested lists), you are probably familiar with bugs which occur if you mix up a test of whether one list (perhaps containing just one element) is an element rather than a sublist of another list. A mathematical equivalent of such a programming error is mixing up the relations "$\in$" and "$\subset$". Avoid that bug!

[9] Suomeksi "$\subset$" voidaan lukea "sisältyy" — erotuksena relaatiosta "$\in$", joka luettiin "kuuluu". Toisinaan selkeyden vuoksi on kuitenkin turvallisinta käyttää ilmaisua "on osajoukko".

Evidently, the subset relation is **transitive**: if $A \subset B$ and $B \subset C$, then $A \subset C$.

Inverted relation symbols

When it is more appropriate to mention, e.g., a set before its element or a set before its subset, the inverted relation symbols $\ni$ and $\supset$ are used so that

$$\begin{array}{ccc} A \ni a & \text{means} & a \in A, \\ B \supset A & \text{means} & A \subset B. \end{array}$$

The relation $\supset$ can be read as "to contain", so that

$$B \supset A \qquad : \qquad \text{(the set) } B \text{ contains (the set) } A,$$

Remark I.4 (Equality of two sets)**.**
Two sets A and B are equal if they contain exactly the same elements, which occurs if and only if both are subsets of the other: $A \subset B$ and $B \subset A$.

While this may initially seem like a useless remark, it lends itself to a strategy of proof that is very commonly used. To *prove* that $A = B$, it is often practical to separately show $A \subset B$ and $A \supset B$, by first arguing that any element of A must necessarily be also an element of B, and then vice versa.

Familiar sets of numbers

The following examples of sets of numbers should be familiar:[10]

$$\text{the set of natural numbers} \qquad \mathbb{N} = \{1, 2, 3, 4, \ldots\} \tag{I.4}$$

$$\text{the set of integers} \qquad \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \tag{I.5}$$

$$\text{the set of rational numbers} \qquad \mathbb{Q} = \left\{ \frac{n}{m} \;\middle|\; n \in \mathbb{Z},\ m \in \mathbb{N} \right\} \tag{I.6}$$

$$\text{the set of real numbers}^{11} \qquad \mathbb{R}\,. \tag{I.7}$$

To give just a few examples, we have

$$\begin{array}{ccc} 2021 \in \mathbb{N}, & -42 \in \mathbb{Z}, & \sqrt{2} \in \mathbb{R}, \\ 0 \notin \mathbb{N}, & -\frac{5}{7} \in \mathbb{Q}, & \sqrt{2} \notin \mathbb{Q}. \end{array}$$

[10] Note that zero is not considered a natural number — in this course... Unfortunately, mathematical literature is very divided regarding conventions about this issue; different areas of mathematics and different authors adopt different conventions. As a rule of thumb, in analysis it is typical to follow our present convention $0 \notin \mathbb{N}$, whereas in algebra it is typical to follow the convention that zero is a natural number. In fact, if you ever attend my algebra courses, then the symbol $\mathbb{N}$ will be used in the different meaning... Sorry! In the present course, if we want to include zero, we use the notation $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ instead of $\mathbb{N}$.

The last example is classical: it was known to the ancient Greeks.[12] We provide it here as one of our first examples of a careful mathematical proof. It is also a great illustration of the method of proof by contradiction.

Proposition I.5 (Square root of two is irrational)**.**
We have $\sqrt{2} \notin \mathbb{Q}$.

Proof. Suppose, by contrary, that $\sqrt{2} \in \mathbb{Q}$. In that case we can write $\sqrt{2} = \frac{n}{m}$ with $n, m \in \mathbb{N}$. If the integers n and m have common factors, they may be cancelled, so we may assume that m and n here are chosen coprime (i.e., no integer greater than one divides both).

Multiplying by m, we obtain that $\sqrt{2}m = n$. Then squaring both sides we see that $2m^2 = n^2$. This shows that n^2 is even, which is only possible if n is even, i.e., $n = 2k$ for some $k \in \mathbb{N}$. But in this case we find $m^2 = \frac{1}{2}n^2 = \frac{1}{2}(2k)^2 = 2k^2$, which similarly implies that m^2 is even, and thus also m is even. Therefore 2 divides both n and m, which contradicts the choice of these coprime integers.

We conclude that $\sqrt{2}$ could not have been rational in the first place, so $\sqrt{2} \notin \mathbb{Q}$. $\square$

Exercise ($\sharp\ \ddots$**) I.2** (Irrationality of π)**.**
Prove that $\pi \notin \mathbb{Q}$.

From the above sets of numbers we obtain examples of subsets, for example $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{Z} \subset \mathbb{Q}$, and $\mathbb{Q} \subset \mathbb{R}$; or more concisely

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

By transitivity, further subset relations such as $\mathbb{N} \subset \mathbb{Q}$, $\mathbb{Z} \subset \mathbb{R}$, and $\mathbb{N} \subset \mathbb{R}$ are apparent from this.

Intervals

Intervals of various types are extremely frequently used subsets of the real line $\mathbb{R}$. For intervals between points $a, b \in \mathbb{R}$, $a \leq b$, we use the following notation and terminology:

$$\textbf{closed interval} \qquad [a,b] = \big\{ x \in \mathbb{R} \;\big|\; a \leq x \leq b \big\} \tag{I.8}$$

$$\text{(bounded) }\textbf{open interval} \qquad (a,b) = \big\{ x \in \mathbb{R} \;\big|\; a < x < b \big\} \tag{I.9}$$

$$(a,b] = \big\{ x \in \mathbb{R} \;\big|\; a < x \leq b \big\} \tag{I.10}$$

$$[a,b) = \big\{ x \in \mathbb{R} \;\big|\; a \leq x < b \big\}. \tag{I.11}$$

We also use the following notation for unbounded intervals,

$$(-\infty, b] = \big\{ x \in \mathbb{R} \;\big|\; x \leq b \big\}, \qquad [a, +\infty) = \big\{ x \in \mathbb{R} \;\big|\; a \leq x \big\}, \tag{I.12}$$

$$(-\infty, b) = \big\{ x \in \mathbb{R} \;\big|\; x < b \big\}, \qquad (a, +\infty) = \big\{ x \in \mathbb{R} \;\big|\; a < x \big\}, \tag{I.13}$$

[12]The (cult of) Pythagoreans held it that only rational numbers were of divine origin, and a (probably dubious) legend has it that they killed Hippasus for revealing the irrationality of the length ($\sqrt{2}$) of the diagonal of the unit square to the outside world. As for written records, a proof of this statement is contained in Euclid's "Elements" — almost surely the mathematical textbook that has endured the longest time in bestseller lists (few others are measured in millenia).

and sometimes $(-\infty, +\infty) = \mathbb{R}$ is used the whole real axis. In addition to the bounded open intervals (I.9), also the unbounded open intervals (I.13) are considered **open intervals**. By contrast, there are no unbounded closed intervals.

Operations with sets

New sets can be formed from old ones by set theoretic operations. For example, for two sets A and B, we define

$$\text{the } \textbf{union} \text{ of } A \text{ and } B: \quad A \cup B = \big\{x \;\big|\; x \in A \text{ or } x \in B\big\} \tag{I.14}$$
$$\text{the } \textbf{intersection} \text{ of } A \text{ and } B: \quad A \cap B = \big\{x \;\big|\; x \in A \text{ and } x \in B\big\} \tag{I.15}$$
$$\text{the } \textbf{set difference} \text{ of } A \text{ and } B: \quad A \setminus B = \big\{x \;\big|\; x \in A \text{ and } x \notin B\big\}. \tag{I.16}$$

If $B \subset A$, then $A \setminus B$ is also called the **complement** of B in A.

Example I.6 (A few examples of set operations)**.**
If we let

$$A = \{1, 2, 3, 4, 5, 6, \ldots\} \qquad \text{(natural numbers)}$$
$$B = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \qquad \text{(even integers)},$$

then the intersection, union, and set differences are

$$A \cap B = \{2, 4, 6, \ldots\}, \qquad A \setminus B = \{1, 3, 5, \ldots\},$$
$$A \cup B = \{\ldots, -6, -4, -2, 0, 1, 2, 3, 4, \ldots\}, \qquad B \setminus A = \{\ldots, -6, -4, -2, 0\}.$$

The set operations follow more or less obvious rules of calculation: for any sets A, B, C we have

$$\begin{cases} A \cup B = B \cup A \\ A \cap B = B \cap A \end{cases} \qquad \text{(commutativity)} \tag{I.17}$$
$$\begin{cases} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{cases} \qquad \text{(associativity)} \tag{I.18}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad \text{(distributivity)} \tag{I.19}$$
$$\begin{cases} A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \\ A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C). \end{cases} \qquad \text{(De Morgan's laws)} \tag{I.20}$$

The properties (I.17), at least, should be obvious from the definitions (I.14) and (I.15). We leave it as an exercise to verify most of the remaining properties, but as an example we prove the first De Morgan's law below.

Proof of the first De Morgan's law. The most straightforward proof strategy is the following: look at what condition characterizes an element of the set on the left hand side according to the definitions, and then successively rewrite equivalent versions of this condition until we reach the defining condition for elements on the set on the right hand side. The same proof strategy is convenient for all of the rules above. In the case of the first De Morgan's law, the appropriate chain of equivalent conditions is

$$\begin{aligned}
x \in A \setminus (B \cup C) \quad &\iff \quad x \in A \quad \text{and} \quad x \notin B \cup C \\
&\iff \quad x \in A \quad \text{and} \quad \big(x \notin B \ \text{ and } \ x \notin C\big) \\
&\iff \quad \big(x \in A \ \text{ and } \ x \notin B\big) \quad \text{and} \quad \big(x \in A \ \text{ and } \ x \notin C\big) \\
&\iff \quad x \in (A \setminus B) \cap (A \setminus C).
\end{aligned}$$

Thus an element x belongs to the set $A \setminus (B \cup C)$ if and only if it belongs to the set $(A \setminus B) \cap (A \setminus C)$. This is the asserted equality (I.20) of these two sets. □

Exercise I.3 (Proofs of set theoretic identities)**.**
Prove the remaining formulas among (I.17) – (I.20).

We can also form unions and intersections of more than two sets — in fact of arbitrary collections of sets. By an indexed collection $(A_j)_{j \in J}$ of sets, we mean that a set A_j is given for each index $j \in J$ (the set J is called the index set of the collection). The **union** of the collection is defined as

$$\bigcup_{j \in J} A_j = \Big\{ x \;\Big|\; x \in A_j \text{ for some } j \in J \Big\}. \tag{I.21}$$

The **intersection** is defined if the collection is non-empty, i.e. if $J \neq \emptyset$, and then it is

$$\bigcap_{j \in J} A_j = \Big\{ x \;\Big|\; x \in A_j \text{ for all } j \in J \Big\}. \tag{I.22}$$

Unions and intersections of two sets, (I.14) – (I.15), are recovered from (I.21) – (I.22) in the special case that the index set has two elements, for example $J = \{1, 2\}$.

When the index set is finite, in particular in the case $J = \{1, 2, \ldots, n\}$, as alternative notations we often use

$$\begin{aligned}
\bigcup_{j \in \{1,\ldots,n\}} A_j \;&=\; \bigcup_{j=1}^{n} A_j \;=\; A_1 \cup \cdots \cup A_n \\
\bigcap_{j \in \{1,\ldots,n\}} A_j \;&=\; \bigcap_{j=1}^{n} A_j \;=\; A_1 \cap \cdots \cap A_n.
\end{aligned}$$

When the index set is $J = \mathbb{N}$, it is conventional to use the alternative notations

$$\bigcup_{j \in \mathbb{N}} A_j = \bigcup_{j=1}^{\infty} A_j \qquad \text{and} \qquad \bigcap_{j \in \mathbb{N}} A_j = \bigcap_{j=1}^{\infty} A_j.$$

Example I.7 (Union and intersection of nested intervals)**.**
Consider the example $A_n = \big[\frac{-1}{n}, \frac{1}{n}\big]$ for $n \in \mathbb{N}$. Then the union and intersection of the collection $(A_n)_{n \in \mathbb{N}}$ are

$$\begin{aligned}
\bigcup_{n=1}^{\infty} \Big[\frac{-1}{n}, \frac{1}{n}\Big] &= [-1, 1] && \text{(closed interval)} \\
\bigcap_{n=1}^{\infty} \Big[\frac{-1}{n}, \frac{1}{n}\Big] &= \{0\} && \text{(singleton).}
\end{aligned}$$

The first of the above claims is easy, but let us justify the second claim in detail. Suppose that $x \in \bigcap_{n=1}^{\infty} \big[\frac{-1}{n}, \frac{1}{n}\big]$. According to the definition of an intersection, this means that $x \in \big[\frac{-1}{n}, \frac{1}{n}\big]$ for all $n \in \mathbb{N}$. At the very least x must then be a real number. But note that if $x \neq 0$, then by choosing a natural number $n_0 > \frac{1}{|x|}$, we have $|x| > \frac{1}{n_0}$, and so $x \notin \left[\frac{-1}{n_0}, \frac{1}{n_0}\right]$. This shows that a non-zero real number $x \neq 0$ can not belong to all of the sets $\left[\frac{-1}{n}, \frac{1}{n}\right]$, $n \in \mathbb{N}$, and thus does not belong to the intersection. On the other hand, $x = 0$ belongs to each of the sets: $0 \in \left[\frac{-1}{n}, \frac{1}{n}\right]$ for all $n \in \mathbb{N}$. By definition of the intersection, then, $0 \in \bigcap_{n=1}^{\infty} \left[\frac{-1}{n}, \frac{1}{n}\right]$. We have thus showed that the intersection consists of the single element 0, i.e., $\bigcap_{n=1}^{\infty} \left[\frac{-1}{n}, \frac{1}{n}\right] = \{0\}$ as claimed.

Example I.8 (Another union and intersection of nested intervals)**.**
Consider now the example $B_n = \left(0, \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Then the union and intersection of the collection $(B_n)_{n\in\mathbb{N}}$ are

$$\bigcup_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = (0,1) \qquad \text{(open interval)}$$
$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset \qquad \text{(empty set).}$$

The precise justifications of these claims are left as an exercise.

Exercise I.4 (Details of Example I.8)**.**
Provide careful reasoning to justify the claims made in Example I.8.

Example I.9 (Yet another union and intersection of nested intervals)**.**
Consider now the example $C_n = \left[0, 1 - \frac{1}{2n}\right]$ for $n \in \mathbb{N}$. Then the union and intersection of the collection $(C_n)_{n\in\mathbb{N}}$ are

$$\bigcup_{n=1}^{\infty} \left[0, 1 - \frac{1}{2n}\right] = [0,1) \qquad \text{(half-open interval)}$$
$$\bigcap_{n=1}^{\infty} \left[0, 1 - \frac{1}{2n}\right] = \left[0, \frac{1}{2}\right] \qquad \text{(closed interval).}$$

Note in particular that 1 does not belong to the union. The precise justifications are again left as an exercise.

Exercise I.5 (Details of Example I.9)**.**
Provide careful reasoning to justify the claims made in Example I.9.

Exercise I.6 (De Morgan laws for general unions and intersections)**.**
Let X be a set, $J \neq \emptyset$ a nonempty index set, and $A_j \subset X$ subsets for each $j \in J$. Prove De Morgan's laws for arbitrary unions and intersections:

$$X \setminus \bigcup_{j\in J} A_j = \bigcap_{j\in J} (X \setminus A_j) \qquad \text{and} \qquad X \setminus \bigcap_{j\in J} A_j = \bigcup_{j\in J} (X \setminus A_j).$$

Cartesian products

Another operation of set theory is forming Cartesian products of sets.

If A and B are sets, then their **Cartesian product** $A \times B$ is the set whose elements are **ordered pairs** (a, b) whose first member belongs to the former set, $a \in A$, and second member to the latter set, $b \in B$. In symbols, the Cartesian product is

$$A \times B = \left\{ (a,b) \;\middle|\; a \in A,\ b \in B \right\}. \tag{I.23}$$

A familiar example is the plane, which (as a set) is the Cartesian product of the real line $\mathbb{R}$ with itself:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \Big\{ (x, y) \;\Big|\; x \in \mathbb{R},\ y \in \mathbb{R} \Big\}.$$

Another familiar example is the rectangle, which (as a set) is the Cartesian product of two closed intervals

$$[\alpha, \beta] \times [\gamma, \delta] = \Big\{ (x, y) \;\Big|\; \alpha \leq x \leq \beta,\ \gamma \leq y \leq \delta \Big\}.$$

Cartesian products of three sets are defined similarly as ordered triples $A \times B \times C = \big\{ (a, b, c) \;\big|\; a \in A,\ b \in B,\ c \in C \big\}$, etc. The familiar n-dimensional space is an n-fold Cartesian product of the real line with itself

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} = \Big\{ (x_1, \ldots, x_n) \;\Big|\; x_1, \ldots, x_n \in \mathbb{R} \Big\}.$$

The elements of $\mathbb{R}^n$ are ordered n-tuples[13] of real numbers, consisting of the n coordinates of a point in this space.

One can consider Cartesian products of arbitrary collections of sets, but in this course the finite Cartesian products, such as the ones above, will be quite sufficient.

I.2. Functions

What is a function, precisely? According to the familiar idea, a function f associates a (unique) value $f(x)$ to each argument x. For a precise meaning, we must specify what is the set of acceptable arguments, and what is the set of possible values. The mathematical terms for these two sets are the **domain** of the function and the **codomain**[14] of the function, respectively.

The notations

$$f \colon X \to Y \qquad \text{or} \qquad X \xrightarrow{f} Y$$

indicate that f is a **function**, whose **domain** is a set X, and whose **codomain** is a set Y, i.e., that f assigns a unique value $f(x) \in Y$ to each $x \in X$. The notation

$$x \mapsto f(x) \qquad \text{(read: "x maps to $f(x)$")}$$

may be used to emphasize this assignment (but the domain X and the codomain Y must also be specified or should be otherwise clear from the context). We may view this as a mapping from the points x of the domain X to points of the codomain Y, and the term **mapping** is considered synonymous to the term **function**. Functions

[13] One uses the terms **pair**, **triple**, **quadruple**, **quintuple**, **sextuple**, . . . for ordered collections of two, three, four, five, six, . . . elements. For general n it has become conventional to refer to an ordered collection of n elements as an n**-tuple**.

[14] Sometimes the term **range** is used instead of *codomain*. Beware, however, as it is also common to use the term *range* to mean the set of actually attained values of the function, which may be smaller than the set of possible values that we are a priori willing to consider. The Finnish terms "määrittelyjoukko" and "arvojoukko" are often used for the domain and codomain, and the latter unfortunately suffers from exactly the same ambiguity of usage as the term *range*.

The terms *domain* and *codomain* are in my opinion not very illuminating, but they are well-established mathematical terminology. Were they not, I would advocate for using *source* and *target* instead. The unambiguous and descriptive Finnish terms "lähtö"/"lähtöjoukko" and "maali"/"maalijoukko" are used by [**Väi99**].

are often defined by giving a formula (but also any other way that ensures that a unique value is assigned to each argument is acceptable).

Example I.10 (An example of a function)**.**
We may define a function

$$r \colon \mathbb{R}^3 \to [0, +\infty)$$

by the formula

$$r((x, y, z)) = \sqrt{x^2 + y^2 + z^2} \qquad \text{for } (x, y, z) \in \mathbb{R}^3.$$

The domain of this function r is (by definition) $\mathbb{R}^3$, and the codomain is (by definition) $[0, +\infty)$. We must only make sure that for each $(x, y, z) \in \mathbb{R}^3$, the formula above is meaningful (as it is), and gives a value in the set $[0, +\infty)$ (as it does).

Often giving a name or a symbol for a function is purposeful: the letters $f, g, \ldots$ are most commonly used, r was chosen in Example I.10, and you have without a doubt seen also

$$\log \colon (0, \infty) \to \mathbb{R}, \qquad \sin \colon [0, 2\pi) \to [-1, 1], \qquad \text{etc.}$$

Occasionally, it is more meaningful to talk of functions $X \to Y$ (i.e., functions with domain X and codomain Y) without explicitly naming them. The "maps to" notation "$\mapsto$" may then be particularly convenient.

Example I.11 (A few more examples of functions)**.**

- The exponential function $\mathbb{R} \to (0, +\infty)$ is given by the formula $x \mapsto e^x$.
- The formula $\theta \mapsto (\cos(\theta), \sin(\theta))$ defines a function $(-\pi, +\pi] \to \mathbb{R}^2$.

Sometimes we simply consider established symbols as the names of functions, and indicate by a "dot" at which slot the argument is to be inserted to get the value.

Example I.12 (Yet a few more examples of functions)**.**

- The square root function $\sqrt{\cdot} \colon [0, +\infty) \to [0, +\infty)$ is given by $x \mapsto \sqrt{x}$.
- The absolute value function $|\cdot| \colon \mathbb{R} \to [0, +\infty)$ is given by $x \mapsto |x|$.

Example I.13 (The identity function)**.**
Let X be a set. Then the function

$$\mathrm{id}_X \colon X \to X \qquad\qquad \mathrm{id}_X(x) = x \quad \text{for } x \in X \tag{I.24}$$

is known as the **identity function** on X.

Note that functions defined by the same formula are not the same functions unless also their domains and codomains are the same!

Example I.14 (Different functions given by the same formula)**.**
The following functions are given by the rule $x \mapsto x^2$:

- $f_1 \colon \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = x^2$,
- $f_2 \colon \mathbb{R} \to [0, +\infty)$ given by $f_2(x) = x^2$,
- $f_3 \colon [0, +\infty) \to \mathbb{R}$ given by $f_3(x) = x^2$,
- $f_4 \colon (-\infty, 0] \to \mathbb{R}$ given by $f_4(x) = x^2$,
- $f_5 \colon (-\infty, 0] \to [0, +\infty)$ given by $f_5(x) = x^2$.

Nevertheless, each of the above is a different function: $f_1 \neq f_2$, $f_3 \neq f_5$, etc.

The examples we have used so far should appear unintimidating. The concept of a function is nevertheless very general: the domain and range of a function may be arbitrary sets. We will in particular encounter functions whose arguments are functions (i.e., the domain is some set of functions), functions whose values are sets (i.e., the codomain is some set whose elements are sets), etc. Be forewarned.

Surjective, injective, and bijective functions

A few properties of functions are relevant.

Definition I.15 (Surjectivity, injectivity, and bijectivity)**.**
Let $f\colon X \to Y$ be a function.

We say that f is **surjective** if for every $y \in Y$ there exists an $x \in X$ such that $y = f(x)$.

We say that f is **injective** if for any $x_1, x_2 \in X$ which are different, $x_1 \neq x_2$, also the corresponding function values are different, $f(x_1) \neq f(x_2)$.

We say that f is **bijective** if it is both surjective and injective.

Consider a function $f\colon X \to Y$ and a given $y \in Y$. If f is injective, there can be at most one $x \in X$ such that $f(x) = y$ (since for any $x' \neq x$ we have $f(x') \neq f(x) = y$). For a surjective function there always exists at least one such x. For bijective functions, therefore, there exists exactly one such x, i.e., the condition $y = f(x)$ gives rise to a well-defined mapping $y \mapsto x$, the **inverse function**

$$f^{-1}\colon Y \to X.$$

Exercise I.7 (Examples of injectivity and surjectivity)**.**
Consider the functions

$$f_1\colon \mathbb{R} \to \mathbb{R}, \qquad f_2\colon \mathbb{R} \to [0,+\infty), \qquad f_3\colon [0,+\infty) \to \mathbb{R},$$
$$f_4\colon (-\infty,0] \to \mathbb{R}, \qquad f_5\colon (-\infty,0] \to [0,+\infty),$$

from Example I.14, each given by the formula $x \mapsto x^2$. Check that:

(a) The functions f_2 and f_5 are surjective, whereas f_1, f_3, and f_4 are not.
(b) The functions f_3, f_4, and f_5 are injective, whereas f_1 and f_2 are not.
(c) The function f_5 is bijective, whereas f_1, f_2, f_3, and f_4 are not.
(d) The inverse function $f_5^{-1}\colon [0,+\infty) \to (-\infty,0]$ of f_5 is given by $f_5^{-1}(y) = -\sqrt{y}$, whereas f_1, f_2, f_3, and f_4 do not have inverse functions.

Example I.16 (The identity function is bijective)**.**
For any set X, the identity function $\mathrm{id}_X\colon X \to X$ of Example I.13 is bijective. It is its own inverse function[15]: $\mathrm{id}_X^{-1} = \mathrm{id}_X$.

[15] The property of the identity function that it is its own inverse may seem special, but there are also other functions satisfying this. Such functions are called *involutions*. We leave it to the reader to construct examples: the formulas $x \mapsto -x$, $y \mapsto \frac{1}{y}$, $z \mapsto \frac{-1}{z}$, $\beta \mapsto -\frac{1}{2}\log\big(\tanh(\beta)\big)$ may serve as an inspiration, but remember to specify the domains and codomains!

Compositions of functions

Suppose that

$$f\colon X \to Y \qquad \text{and} \qquad g\colon Y \to Z$$

are two functions such that the codomain of f is the same set as the domain of g (both are Y). Then we may form a new function,

$$g \circ f : \ X \to Z$$

called the **composition** of f and g by the rule that at a point $x \in X$ of its domain the value is

$$\big(g \circ f\big)(x) \ = \ g\big(f(x)\big) \in Z.$$

The composition is illustrated in the following diagram[16]:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$$x \xmapsto{\ f\ } f(x) \xmapsto{\ g\ } g\big(f(x)\big) \ .$$

Example I.17 (A composition of functions)**.**
Consider the functions

$$f\colon\ (0,2\pi) \to [-1,1) \qquad\qquad f(\theta) = \cos(\theta) \quad \text{for } \theta \in (0,2\pi)$$

$$g\colon\ [-1,1) \to \mathbb{R} \qquad\qquad g(u) = \frac{1}{(1-u)^2} \quad \text{for } u \in [-1,1).$$

Their composition is

$$g \circ f\colon\ (0,2\pi) \to \mathbb{R} \qquad\qquad \big(g \circ f\big)(\theta) = \frac{1}{\big(1-\cos(\theta)\big)^2} \quad \text{for } \theta \in (0,2\pi).$$

Example I.18 (Compositions with an inverse function)**.**
Suppose that

$$f\colon\ X \to Y$$

is bijective, and consider its inverse function

$$f^{-1}\colon\ Y \to X.$$

Then the composition of f and f^{-1} is (as you should verify from the definitions)

$$f^{-1} \circ f\colon\ X \to X \qquad\qquad \big(f^{-1} \circ f\big)(x) = x \quad \text{for } x \in X.$$

The composition of f^{-1} and f, on the other hand, is (as you should also verify)

$$f \circ f^{-1}\colon\ Y \to Y \qquad\qquad \big(f \circ f^{-1}\big)(y) = y \quad \text{for } y \in Y.$$

So both compositions are identity functions (see Example exa: the identity function) but on different sets:

$$f^{-1} \circ f \ = \ \mathrm{id}_X \qquad\qquad f \circ f^{-1} \ = \ \mathrm{id}_Y.$$

[16] This should also make it clear why the composition of f and g is denoted by $g \circ f$ in this order. We take an element $x \in X$, first apply f to it to get $f(x) \in Y$, and then apply g to the result to get $g\big(f(x)\big) \in Z$. The "strange" order in the composition is dictated by our convention that the function is written to the left of its argument. If we would denote for example $f(x) = x * f$ instead, then the composition it this order would be defined by the formula $x \mapsto (x * f) * g$ in which f and g appear in the same order as they are applied (assuming you read from left to right! — Hebrew, Persian, or Arabic readers probably have less qualms about the order of $g \circ f$ in the first place. . .).

Exercise (✓) I.8 (Composition of injective functions is injective)**.**
Show that if both $f\colon X \to Y$ and $g\colon Y \to Z$ are injective, then $g \circ f\colon X \to Z$ is also injective.

Exercise (✓) I.9 (Composition of surjective functions is surjective)**.**
Show that if both $f\colon X \to Y$ and $g\colon Y \to Z$ are surjective, then $g \circ f\colon X \to Z$ is also surjective.

Exercise I.10 (Injectivity of composition)**.**
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions such that $g \circ f\colon X \to Z$ is injective.

(a) Show that f is also injective.
(b) Construct an example which shows that g does not have to be injective.

Exercise I.11 (Surjectivity of composition)**.**
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions such that $g \circ f\colon X \to Z$ is surjective.

(a) Show that g is also surjective.
(b) Construct an example which shows that f does not have to be surjective.

The following exercise shows that the conditions in Example I.18 characterize the inverse function.

Exercise I.12 (Characterization of the inverse function)**.**
Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are two functions such that

$$g \circ f \;=\; \mathrm{id}_X \qquad\qquad f \circ g \;=\; \mathrm{id}_Y.$$

Prove that the both f and g are bijective, and we have $g = f^{-1}$ and $f = g^{-1}$.

Restricting the domain

Given a function $f\colon X \to Y$ and a subset $A \subset X$ of its domain, it is possible to consider a function defined only in the subset by otherwise the same formula. This function $A \to Y$ is denoted by $f|_A$ and is called the **restriction** of f to the subset A; it is given by

$$\begin{aligned} f|_A\colon\ & A \to Y \\ f|_A(a) = f(a) & \qquad \text{for } a \in A. \end{aligned} \tag{I.25}$$

Example I.19 (Examples of restrictions of functions)**.**
Consider again the functions

$$f_1\colon \mathbb{R} \to \mathbb{R}, \qquad f_2\colon \mathbb{R} \to [0,+\infty), \qquad f_3\colon [0,+\infty) \to \mathbb{R},$$
$$f_4\colon (-\infty,0] \to \mathbb{R}, \qquad f_5\colon (-\infty,0] \to [0,+\infty),$$

from Example I.14, each given by the formula $x \mapsto x^2$. Then we have

$$f_3 = f_1|_{[0,+\infty)}, \qquad f_4 = f_1|_{(-\infty,0]}, \qquad f_5 = f_2|_{(-\infty,0]}$$

Images and preimages under functions

Let $f\colon X \to Y$ be a function.

If $A \subset X$ is a subset of the domain of f, then the **image** of A under f is the subset

$$f[A] \;=\; \big\{ f(a) \;\big|\; a \in A \big\} \tag{I.26}$$

of the codomain of f consisting of those points $y \in Y$ which are obtained as the value at some point $a \in A \subset X$, i.e., $y = f(a)$.

Example I.20 (Some images).
Consider the function

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{given by} \qquad f(x) = x^2 \quad \text{for } x \in \mathbb{R}.$$

The image of the subset $A = \{-7, -5, 5, 7\} \subset \mathbb{R}$ is

$$f\big[\,\{-7, -5, 5, 7\}\,\big] \;=\; \{25, 49\}\,.$$

In particular the image of a subset may contain fewer elements than the subset itself.

Example I.21 (The image of a singleton).
The image of a singleton $\{x\} \subset X$ is $f\big[\,\{x\}\,\big] = \{f(x)\}$.

Exercise (✓) I.13 (Singleton image).
If the image of $A \subset X$ is a singleton $f[A] = \{y\}$, then what can be said about the behavior of the function f on the subset A?

Example I.22 (Surjectivity by images).
A function $f : X \to Y$ is surjective if and only if the image of the whole domain is the whole codomain, $f[X] = Y$.

If $B \subset Y$ is a subset of the codomain of f, then the **preimage** of B under f is the subset

$$f^{-1}[B] \;=\; \big\{x \in X \;\big|\; f(x) \in B\big\} \tag{I.27}$$

of the domain of f consisting of those points $x \in X$ at which the value belongs to the subset $B \subset Y$. The notation f^{-1} is used for both the inverse function and the preimage, but we try to consistently use square brackets in the latter case.[17] The inverse function only exists if f is a bijection, whereas preimages are always defined.

Example I.23 (Some preimages).
Consider the function

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{given by} \qquad f(x) = x^2 \quad \text{for } x \in \mathbb{R}.$$

The preimage of the (singleton) subset $A = \{5\} \subset \mathbb{R}$ is

$$f^{-1}\big[\,\{5\}\,\big] \;=\; \big\{-\sqrt{5},\ \sqrt{5}\,\big\}.$$

In particular the preimage of a singleton can contain more than one element.

The preimage of the (singleton) subset $A = \{-5\} \subset \mathbb{R}$ is the empty set,

$$f^{-1}\big[\,\{-5\}\,\big] \;=\; \emptyset.$$

In particular the preimage of a nonempty set can be empty.

[17] In most mathematical literature ordinary parentheses are used also for preimages, or parentheses are altogether omitted. You are in any case expected to figure out which one is which, since the inverse function is applied to *elements* of the codomain, whereas in the preimage is applied to *subsets* of the codomain.

Similarly, we try to consistently use square brackets for the images of subsets under a function. Elsewhere in literature ordinary parentheses or no parentheses at all are used. Again you are in any case expected to figure out which one is which, since the function is applied to *elements* of the domain, whereas in the image is applied to *subsets* of the domain.

Example I.24 (Bijectivity by preimages).
A function $f\colon X \to Y$ is bijective if and only if the preimage of every singleton $\{y\} \subset Y$ is a singleton.

Example I.25 (Injectivity by preimages).
A function $f\colon X \to Y$ is injective if and only if the preimage of every singleton $\{y\} \subset Y$ is either a singleton or the empty set.

Exercise I.14 (Images and preimages of unions and intersections).
Let X and Y be sets and $f\colon X \to Y$ a function.

(a) Show that for any $C, D \subset Y$, the preimages satisfy

$$f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D].$$

(b) Show that for any $C, D \subset Y$, the preimages satisfy

$$f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D].$$

(c) Show that for any $A, B \subset X$, the images satisfy

$$f[A \cup B] = f[A] \cup f[B].$$

(d) Give an example in which for the images of subsets $A, B \subset X$ we have

$$f[A \cap B] \neq f[A] \cap f[B].$$

Hint: *In parts (a)–(c) it is possible to argue by a chain of equivalent conditions*

$$x \in \textit{left hand side set} \quad \iff \quad \cdots \quad \iff \quad x \in \textit{right hand side set.}$$

A comparison of (b) and (d) above indicates that preimages behave better with set operations than images. Another example is the following.

Exercise I.15 (Images and preimages of complements).
Let X and Y be sets and $f\colon X \to Y$ a function.

(a) Suppose that $C \subset D \subset Y$. Show that the preimages satisfy

$$f^{-1}[D \setminus C] \;=\; f^{-1}[D] \setminus f^{-1}[C].$$

(b) Give an example in which for the images of subsets $A \subset B \subset X$ we have

$$f[B \setminus A] \;\neq\; f[B] \setminus f[A].$$

The formulas of Exercise I.14(a)–(c) generalize to arbitrary unions and intersections (more than two sets), and it is easy to modify the proofs to account for this general case. Specifically, if $f\colon X \to Y$ is a function and $B_j \subset Y$ for $j \in J$, then the preimages satisfy

$$f^{-1}\Big[\bigcup_{j\in J} B_j\Big] \;=\; \bigcup_{j\in J} f^{-1}[B_j], \qquad\qquad f^{-1}\Big[\bigcap_{j\in J} B_j\Big] \;=\; \bigcap_{j\in J} f^{-1}[B_j].$$

Likewise if $A_j \subset X$ for $j \in J$, then the images satisfy

$$f\Big[\bigcup_{j\in J} A_j\Big] \;=\; \bigcup_{j\in J} f[A_j].$$

I.3. Logic and related notation

Mathematics in general concerns with statements that can be logically proven to be true, starting from some specified assumptions. In other words, we want to deduce conclusions of interest from known premises. The rules for valid deduction are the subject of logic. Logic itself could be formalized, but for our purposes it suffices to introduce a little bit of notation and typical examples. The main message is that in this course and in all mathematics, we must have unambiguous definitions and statements, and we must provide valid arguments to justify the statements made.

Quantifiers

Statements involving free variables

Commonly, statements (predicates, logical formulas) involve one or more variables, and whether the statements are true or false depends on those variables. A statement may be thought of as a function of those variables, whose possible values are true and false.

Example I.26 (An example statement depending on a variable)**.**
The statement

"k is even"

is true for if the variable k has the value $k = -8$, but it is false if $k = 33$.

Those variables of which the truth value of a statement depends are called **free variables**; without specifying values for the free variables, the statement itself is not yet true nor false. A statement without free variables is called a **sentence** — it is then either true or false as it stands. Self-contained meaningful logical statements must be sentences!

In mathematical text we often indicate that we (temporarily) fix the values of (some) variables by expressions such as: "fix $n \in \mathbb{N}$" (after which the varible n is no longer thought of as free, but having some a priori arbitrary but fixed natural number value) or "let $\varepsilon > 0$" (after which the varible ε is no longer thought of as free, but fixed to some a priori arbitrary positive real value). Fixing a value for a variable in effect makes it a named constant.[18] Then statements involving the variables in question again have definite truth values.

Statements involving quantifiers

The following two example statements,

"for all real numbers x we have $x^2 \geq 0$"

and

"there exists a real number x such that $x^2 = -1$",

[18]This is the way a computer treats variables when running a program: at any time they are used, they must have assigned values or otherwise errors arise. The computer scientist or a programmer, on the other hand, may think of variables much like a mathematician or a logician.

both involve a variable x which has not been fixed, but the statements are in fact sentences. In particular they have definite truth values: the former sentence is true and the latter is false. The variable x in the above formulas is not free, because a *quantifier* applies to it: each statement above makes a claim about the *quantity* of values of x for which another statement, which involves x as a free variable, is true.

There are two quantifiers in ordinary logic. The **universal quantifier** denoted by the symbol $\forall$ means that the statement that follows it is true *for all values of the variable*; in natural language its symbol $\forall$ is read as "for all". The **existential quantifier** denoted by the symbol $\exists$ means that the statement that follows it is true *for at least one value of the variable*; in natural language its symbol $\exists$ is read as "there exists". Both quantifiers must be followed first by the variable which they quantify, and they should (in careful usage) also be followed by a specification of the set in which the variable is allowed to take values, and then another statement (possibly) involving the variable. For clarity we usually also separate the other statement (in which the variable appears as free) by a colon (:). For example the two example statements that we started with should be written as

$$\forall\, x \in \mathbb{R} : \quad x^2 \geq 0$$

and

$$\exists\, x \in \mathbb{R} : \quad x^2 = -1.$$

Example I.27 (Revisiting subsets)**.**
The meaning of the subset relation $A \subset B$ is, in concise logical notation with quantifiers,

$$\forall\, x \in A : \quad x \in B.$$

In other words, the statement above is true if $A \subset B$, and false if $A \not\subset B$.

Example I.28 (Revisiting unions and intersections)**.**
Let $(A_j)_{j \in J}$ be a collection of sets. Then the meaning of the statement that an element x belongs to the union of this collection, $x \in \bigcup_{j \in J} A_j$, can be expressed in concise logical notation with quantifiers as

$$\exists\, j \in J : \quad x \in A_j.$$

Similarly, if the collection is nonempty ($J \neq \emptyset$), then the meaning of the statement that an element x belongs to the intersection of this collection, $x \in \bigcap_{j \in J} A_j$, can be expressed as

$$\forall\, j \in J : \quad x \in A_j.$$

Note that the quantified variable appears as a free variable in the statement that follows, but that statement may itself have quantifiers for other variables. This is in fact very common in, e.g., definitions related to metric space topology (continuity, limits of sequences, etc.) — we will see plenty of examples later. For now, as just one example statement with more than one quantifier, consider

$$\forall\, \alpha \in \mathbb{R} : \quad \exists\, C > 0 : \quad \forall\, x \in \mathbb{R} : \quad \left| e^{3+5\alpha x - x^2} \right| \leq C$$

(this statement is true, by the way). Literally this statement reads "for all real numbers α there exists a positive number C such that for all real numbers x we have $|e^{3+5\alpha x-x^2}| \leq C$", and allowing for a bit more liberties of expression and interpretation could also be read "whatever the value of the real paramenter α, the expression $e^{3+5\alpha x-x^2}$ involving a real variable x is bounded in absolute value by some positive constant C". The use of logical notation makes the statements more concise and unambiguous. By contrast, in natural language one has to be quite careful to archieve an unambiguous meaning.[19]

Note that the order of quantifiers matters! For instance the statement

$$\forall\, x \in \mathbb{R} : \quad \exists\, C > 0 : \quad \forall\, \alpha \in \mathbb{R} : \quad \left| e^{3+5\alpha x - x^2} \right| \leq C$$

is false, although it "only" differs from the previous example (a true statement) by the order of quantifiers.

Exercise I.16 (Check the truth values of the above examples)**.**
Prove that the statement $\forall\, \alpha \in \mathbb{R} : \exists\, C > 0 : \forall\, x \in \mathbb{R} : |e^{3+5\alpha x - x^2}| \leq C$ is true and that the statement $\forall\, x \in \mathbb{R} : \exists\, C > 0 : \forall\, \alpha \in \mathbb{R} : |e^{3+5\alpha x - x^2}| \leq C$ is false.

Implications and equivalences

Implications

Suppose that P and Q are two statements (predicates) possibly involving free variables (the same variables in both). Then one can form a new statement out of them, called an **implication** and denoted by

$$P \implies Q,$$

which in natural language can be read in any of the following ways

$$P \text{ implies } Q\ ;$$
$$\text{if } P\text{, then } Q\ ;$$
$$P \text{ only if } Q\ .$$

The natural language descriptions (the most transparent of which is probably "if P, then Q") already should explain exactly how the truth value of an implication is determined: $P \Rightarrow Q$ is true if either P and Q are both true, or if P is false.[20]

Example I.29 (Revisiting subsets again)**.**
In concise logical notation, the meaning of the subset relation $A \subset B$ is the implication

$$x \in A \implies x \in B.$$

In other words, the above logical statement is true if $A \subset B$, and false if $A \not\subset B$.

A typical use of an implication is in theorem statements: P may represent the assumptions of a theorem (the premises) and Q the conclusion, so the theorem statement itself is the implication $P \Rightarrow Q$ ("if the assumptions hold, then also the conclusion holds"). An implication in a theorem statement should of course be true (the role of the *proof* of the theorem is exactly to verify the validity of the asserted implication).

[19]This does not mean that natural language is strictly worse. Among advantages of natural language are that it allows to emphasize certain aspects of the statement and to draw attention to interpretations (for example interpreting α as a parameter, C as a constant, and x as a variable in the above expression). None of this affects the validity of logical statements, but to claim that such things are irrelevant would be rather extreme...

[20]In particular, if P does not hold, the implication is claiming nothing whatsoever about the truth or falsity of Q. The only situation in which the implication is false is if P is true but Q is nevertheless false.

Note, however, that the implication $P \Rightarrow Q$ is merely a statement about the relationship of the truth values of the two statements P and Q — it does not signify any *causal* relationship.

Instead of $P \Rightarrow Q$ we also occasionally switch the places of P and Q and invert the arrow direction, i.e., we alternatively write

$$Q \iff P,$$

which in natural language can be read in any of the following ways

Q is implied by P ;

Q if P .

So we have seen altogether five different phrases in natural language for the same implication — the logical meaning of each is exactly the same, but they may be used to express a slightly different emphasis.

Equivalences

Two statements P and Q, possibly depending on some free variables (same in both), are said to be **logically equivalent** if they have the same truth value for all values of the free variables. This precisely amounts to requiring that both implications $P \Rightarrow Q$ and $P \Leftarrow Q$ hold (Q is true if P is true and vice versa). We denote **logical equivalence** by

$$P \iff Q,$$

and in natural language read it as either

P and Q are (logically) equivalent ;

P if and only if Q .

Example I.30 (Revisiting the equality of sets)**.**
The meaning of the equality of two sets $A = B$ is

$$x \in A \iff x \in B.$$

Remark I.31 (Warning: a convention used in definitions)**.**
When a new term is defined, we usually say that the term is used if some conditions are satisfied. In this context we actually mean that the term is used *if and only if* the conditions are satisfied! But it is conventional in definitions to only state the "if" part, leaving it implicit that the newly introduced term is *not* used unless the conditions are satisfied.

For example our definition of a bijective function was: "we say that f is bijective if it is both injective and surjective". We of course *also* meant that we *do not* call f bijective *unless* it is both injective and surjective.

So while the natural language word "if" in usual mathematical text signifies just one implication, in definitions it is conventionally used to signify equivalence.

Negations

If P is a logical statement (predicate), its **negation**

$$\texttt{not}\ P$$

is the logical statement which always has the opposite truth value: if P is true, then $\texttt{not}\ P$ is false and if P is false, then $\texttt{not}\ P$ is true. In natural language, $\texttt{not}$ is read as "not".

Clearly the statement $\texttt{not}\ (\texttt{not}\ P)$ is logically equivalent to P (a double negation).

We have already introduced specific symbols for the negations of a few common statements, e.g.,

$$\begin{aligned} \texttt{not}\ \big(a = b\big) &\qquad \text{is denoted} \qquad a \neq b \\ \texttt{not}\ \big(a \in A\big) &\qquad \text{is denoted} \qquad a \notin A \\ \texttt{not}\ \big(A \subset B\big) &\qquad \text{is denoted} \qquad A \not\subset B. \end{aligned}$$

Contrapositives

The implications

$$\begin{aligned} P &\implies Q \\ \texttt{not}\ Q &\implies \texttt{not}\ P \end{aligned} \tag{I.28}$$

are logically equivalent — you should make sure that you understand why![21] The second implication is known as the **contrapositive** of the first.

Let us take an example from Section I.2.

Example I.32 (Revisiting injectivity).
Recall definition of an injective function $f \colon X \to Y$ from Definition I.15. In logical symbols, injectivity means

$$\forall\, x_1, x_2 \in X : \qquad x_1 \neq x_2 \implies f(x_1) \neq f(x_2). \tag{I.29}$$

An equivalent form of this condition can be obtained by taking a contrapositive of the implication above. This is particularly helpful, since the contrapositive will involve negations of the propositions $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$, which are simply $x_1 = x_2$ and $f(x_1) = f(x_2)$, respectively. Using the contrapositive form, we find that injectivity equivalently means

$$\forall\, x_1, x_2 \in X : \qquad f(x_1) = f(x_2) \implies x_1 = x_2. \tag{I.30}$$

The original formulation (I.29) of injectivity can be read: "At any two different points, the function has different values". The formulation (I.30) obtained by contrapositive can be read: "Whenever the values at two points are the same, the points themselves must be the same". With a bith of thinking, one easily convinces oneself about the equivalence of these, but routine use of logical symbols clearly facilitates this.

The equivalence of the two implications in (I.28) in particular underlies the idea of a *proof by contrapositive* or an *indirect proof* — of which the *proof by contradiction* is essentially a special case.

To elaborate, suppose our goal is to prove that from assumption P, conclusion Q follows. Another way of doing that is to prove that if we assume that the desired conclusion Q does not hold, then it implies that the assumption P cannot hold, either. This is the proof by contrapositive.

[21] The most straightforward way to convince oneself about this is to write a truth table, i.e., to consider the truth value of both implications for all the four possibilities of the truth values of the propositions P and Q (P false and Q false, P false and Q true, P true and Q false, P true and Q true).

In particular, if there are no (explicit) assumptions, we can consider the assumptions represented by an identically true proposition P. Then this proof strategy is proof by contradiction. Indeed, by showing the contrapositive, we find that if the the desired conclusion Q would not hold, then the negation of the identically true assumption P would hold, which is absurd; a contradition. The proof of Proposition I.5 is a classic example.

Negations and quantifiers

Since typical definitions in metric space topology involve many logical quantifiers, it is essential to be comfortable working with them. In particular one has to be able to routinely form the negation of a statement involving a quantifier.

Consider a statement involving the universal quantifier, say

$$\forall\, x \in X : \quad P(x).$$

Its negation, $\texttt{not}\ \big(\forall\, x \in X : \ P(x)\big)$, is logically equivalent with

$$\exists\, x \in X : \quad \texttt{not}\ P(x);$$

you should make sure you understand why!

A common use of this observation is that to *disprove* the validity of a claim (for example a belief that we might have) that $P(x)$ holds for all $x \in X$, it is sufficient to find just one example of an $x \in X$ for which $P(x)$ is false. Such an example is called a **counterexample**, because it is sufficient to invalidate the whole original claim (which started with the universal quantifier).

And even if we are seeking to prove rather than to disprove a claim of this form, in an indirect proof (contrapositive) we would first form the negation.

Symmetrically, consider a statement involving the existential quantifier, say

$$\exists\, x \in X : \quad Q(x).$$

Its negation, $\texttt{not}\ \big(\exists\, x \in X : \ Q(x)\big)$, is logically equivalent with (think about it)

$$\forall\, x \in X : \quad \texttt{not}\ Q(x).$$

On the surface this does not appear as practical as the negation of a statement with a universal quantifier. But in indirect proofs, or when disproving a claim (say a false belief) of this form, we would have to form the negation. Another typical use of this appears when a statement involves many quantifiers; we may want to consider the negation to get rid of some universal quantifiers, but the existential quantifiers are affected as well.

Example I.33 (There are arbitrarily large natural numbers)**.**
The logical statement

$$\forall\, q \in \mathbb{Q} : \quad \exists\, n \in \mathbb{N} : \quad n > q \tag{I.31}$$

has the interpretation that there are arbitrarily large natural numbers ("for any rational number q there exists a larger natural number n").

The statement (I.31) is true: it is the so called **Archimedean property** of the rational numbers. Let us examine a proof of this statement using a proof by contradiction.

For a proof by contradiction, one would suppose that (I.31) is false, and show that this leads to a contradiction. If (I.31) is false, then its negation

$$\texttt{not}\ \Big(\forall\, q \in \mathbb{Q}: \quad \exists\, n \in \mathbb{N}: \quad n > q\Big).$$

is true. The original statement (I.31) begins with a universal quantifier ($\forall$), and its negation takes the form

$$\exists\, q \in \mathbb{Q}: \quad \texttt{not}\ \Big(\exists\, n \in \mathbb{N}: \quad n > q\Big).$$

In this form, after the first existential quantifier ($\exists$) we have the negation of a statement which begins with an existential quantifier ($\exists$), which we may further unravel. The negation of (I.31) is thus written as

$$\exists\, q \in \mathbb{Q}: \quad \forall\, n \in \mathbb{N}: \quad \texttt{not}\ \big(n > q\big).$$

The negation $\texttt{not}\ (n > q)$ is obviously $n \leq q$, so the negation of (I.31) has been rewritten as

$$\exists\, q \in \mathbb{Q}: \quad \forall\, n \in \mathbb{N}: \quad n \leq q,$$

which in natural language claims that there exists some rational number q such that all natural numbers n are bounded from above by it. This clearly already sounds absurd, but we leave it to the reader to now derive the contradiction.[22] The point here was to illustrate how we typically end up considering negations of statements with quantifiers, and how they can be systematically unraveled.

I.4. ♡ In depth topics about the set theory

The reader interested in deepening her understanding may wish to look into some of the following topics related to the subject of this lecture:

- The notion of cardinality as a way of comparing the sizes sets; in particular countable infinity as the smallest infinite cardinality and the uncountability of the set $\mathbb{R}$ of real numbers, Appendix A.1.

The notions of countable and uncountable cardinalities will be crucial in various advanced topics of mathematics, notably in measure theory and probability theory. Familiarity with these notions will be assumed in the corresponding courses.

[22] Hint: *If such a rational number $q \in \mathbb{Q}$ exists, write it as $q = \frac{m}{k}$ with $m \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then use for example the natural number $n = |m| + 1$ as a counterexample to the statement $\forall\, n \in \mathbb{N}: \ n \leq q$.*

Lecture II

Real numbers

There may not be much of a feeling of novelty to the real numbers, but we spend a lecture on this topic for two reasons: to appreciate some subtleties of real numbers themselves, and to get concrete examples of topological notions that will arise in more general context in the subsequent lectures.

We start by discussing the absolute value of real numbers and the related notion of distance on the real line, which is our first example of a metric, although the general definition will have to wait until Lecture V. This is the notion that underlies for example the convergence of sequences of real numbers. We look at basic properties of of convergence as an instructive example of the more general theory to be developed later in Lecture VIII. We also show that both rational and irrational numbers are dense on the real line.

We then briefly turn to foundational questions again: what exactly are real numbers? The standard rigorous mathematical approach is to start from their axiomatic properties, most of which are entirely unsurprising. We only mention the othe axioms in passing, but discuss the *completeness axiom* in more detail, since it is the most subtle and the most consequential for topology as well. It involves the notions of *supremum* (least upper bound) and *infimum* (greatest lower bound).

II.1. Absolute value, distances between numbers, and triangle inequality

At the core of topological considerations of numbers is the concept of distances on the real line. They are based on the absolute value — which itself is the distance of a given number to the origin.

The **absolute value** of a real number $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \tag{II.1}$$

It is easy to see that another expression for it is

$$|x| = \sqrt{x^2}, \tag{II.2}$$

where $\sqrt{\cdot}$ denotes the non-negative square root of a non-negative number. Clearly the absolute value satisfies

$$|x| = \big|-x\big|, \qquad |x| \geq 0, \qquad x \leq |x|, \qquad -x \leq |x|$$

for any $x \in \mathbb{R}$, and

$$\big|xy\big| \;=\; |x|\,|y|$$

for any $x, y \in \mathbb{R}$.

The **distance**[1] between two real numbers $x, y \in \mathbb{R}$ is the absolute value of their difference,

$$|x - y|. \tag{II.3}$$

The following **triangle inequality** is a simple and intuitive result, but as we will see later from Lecture V on, a rather straightforward generalization of it serves as the basis of a very fruitful general theory of metric spaces.

Proposition II.1 (Triangle inequality on $\mathbb{R}$)**.**
For all $x, y \in \mathbb{R}$ we have

$$\Big||x| - |y|\Big| \;\leq\; \big|x + y\big| \;\leq\; |x| + |y|. \tag{II.4}$$

Proof. Let $x, y \in \mathbb{R}$. There are two inequalities to prove in (II.4).

Let us start by proving the second inequality. Calculate the square of the absolute value of the sum, and estimate each term from above by its absolute value to get

$$\begin{aligned} \big|x+y\big|^2 = \big(x+y\big)^2 &= x^2 + 2xy + y^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 \\ &= \Big(|x| + |y|\Big)^2. \end{aligned}$$

Taking square roots ($\sqrt{\cdot}$ is an increasing function and thus respects the inequality), we obtain that

$$|x + y| \leq |x| + |y|.$$

To prove the first inequality, we use the second one already proved above. Applying the second inequality to the numbers $x + y$ and $-y$, on the other hand, gives

$$|x| \;=\; \big|(x + y) + (-y)\big| \;\leq\; |x + y| + |-y| \;=\; |x + y| + |y|,$$

and by subtracting $|y|$ from both sides, we get

$$|x| - |y| \;\leq\; |x + y|.$$

A similar application of the second inequality to the numbers $x + y$ and $-x$ (or just interchanging the roles of x and y above) yields

$$|y| - |x| \;\leq\; |x + y|.$$

Combining these gives the first asserted inequality

$$\Big||x| - |y|\Big| \;\leq\; |x + y|,$$

and the proof is complete. □

Another common form of the triangle inequality, which features the distance $|x - y|$ in exactly this form is

$$|x - y| \leq |x| + |y|. \tag{II.5}$$

It is obtained from the second inequality in (II.4) by plugging in x and $-y$, and noting $|-y| = |y|$.

There is a straightforward generalization of the triangle inequality to sums with finitely many many terms.

[1]Later, in Lecture V, we will see that this notion of a distance (II.3) is a special case of a *metric*, and it makes the real line a *metric space*. For now, however, we will work exclusively with the concrete case of real numbers, and we will not refer to the general notion.

Proposition II.2 (Triangle inequality for finite sums).
Let $x_1, \ldots, x_n \in \mathbb{R}$*. Then we have*

$$\big|x_1 + \cdots + x_n\big| \;\leq\; |x_1| + \cdots + |x_n|. \tag{II.6}$$

Exercise II.1 (Proof of Proposition II.2).
Prove Proposition II.2 by induction over $n \in \mathbb{N}$, using Proposition II.1.

Remark II.3 (Triangle inequality for infinite series).
Let $x_1, x_2, x_3, \ldots \in \mathbb{R}$. The natural generalization of (II.6) to infinite series would be

$$\Big|\sum_{j=1}^{\infty} x_j\Big| \overset{?}{\leq} \sum_{j=1}^{\infty} |x_j|.$$

With suitable interpretations, this generalization would indeed be valid. However, it immediately requires consideration of the convergence of the infinite series on both sides, and/or suitable interpretations in the case that one or both of the series diverge. For now, we will not try to use these types of generalizations, but we invite the reader to think about what can and cannot be said about it.

II.2. Sequences on the real line

Sequences of real numbers

A **sequence** (of real numbers) is an "infinite list"

$$(a_1, a_2, a_3, \ldots)$$

of real numbers $a_1, a_2, a_3, \ldots \in \mathbb{R}$. A precise definition of a sequence is a function

$$a \colon \mathbb{N} \to \mathbb{R},$$

and the "list" consists of the values of this function: $a_n = a(n)$ for $n \in \mathbb{N}$. We usually use notations such as

$$(a_1, a_2, a_3, \ldots) \;=\; (a_n)_{n=1}^{\infty} = (a_n)_{n \in \mathbb{N}}$$

for a sequence, and it is even common to use the lazy notation (a_n), although this fails to explicitly indicate that the indexing of the members of the sequence uses natural numbers.

Occasionally we also let the indexing start from something other than $n = 1$. When we do so, this should be quite clear from the context. The notation is then modified in obvious ways: e.g., $(a_0, a_1, a_2, \ldots) = (a_n)_{n=0}^{\infty}$ or $(a_{25}, a_{26}, a_{27}, \ldots) = (a_n)_{n=25}^{\infty}$; the precise interpretation is as functions $\{0, 1, 2, \ldots\} \to \mathbb{R}$ or $\{25, 26, 27, \ldots\} \to \mathbb{R}$. The definitions in the following subsections must then be adapted in obvious ways. Some properties of interest (particularly limits) in fact do not even depend on any finitely many initial members of the sequence, in which case the choice of the starting index is largely irrelevant.

Monotonicity properties of number sequences

The real axis has an order. Sequences which respect that order have some particularly nice properties.

Definition II.4 (Monotonicity properties of sequences of real numbers).
A real number sequence $(a_n)_{n\in\mathbb{N}}$ is

- **increasing**, if we have[2] $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$
- **strictly increasing**, if we have[3] $a_{n+1} > a_n$ for all $n \in \mathbb{N}$
- **decreasing**, if we have[4] $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$
- **strictly decreasing**, if we have[5] $a_{n+1} < a_n$ for all $n \in \mathbb{N}$
- **monotone**, if it is either increasing or decreasing.

Example II.5 (An increasing sequence).
The sequence $(1, 2, 4, 8, 16, 32, \ldots)$ is (strictly) increasing: its n:th member is given by the formula $a_n = 2^{n-1}$, and we have $a_{n+1} = 2^n > 2^{n-1} = a_n$ for all $n \in \mathbb{N}$.

Example II.6 (Another increasing sequence).
Let $(a_n)_{n\in\mathbb{N}}$ be defined by the formula $a_n = \frac{n}{n+1}$ for $n \in \mathbb{N}$. We claim that this sequence is increasing.

One way to verify this is to calculate the difference of consequtive members,

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0,$$

which gives $a_{n+1} > a_n$ (by adding a_n to both sides in the inequality).

Another (perhaps easier?) way is to calculate the ratio of consequtive terms

$$\frac{a_n}{a_{n+1}} = \frac{n/(n+1)}{(n+1)/(n+2)} = \frac{n(n+2)}{(n+1)^2} = \frac{n^2+2n}{n^2+2n+1} < 1,$$

which gives $a_n < a_{n+1}$ (by multiplying both sides by the positive number a_{n+1}).

Example II.7 (A decreasing sequence).
The sequence $(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots)$ is (strictly) decreasing: its n:th member is given by the formula $a_n = \frac{1}{n^2}$, and we have $a_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = a_n$ for all $n \in \mathbb{N}$.

Boundedness properties of number sequences

Definition II.8 (Boundedness properties of sequences of real numbers).
A real number sequence $(a_n)_{n\in\mathbb{N}}$ is

- **bounded from above**, if there exists some $u \in \mathbb{R}$ (called an **upper bound** for the sequence) such that $a_n \leq u$ for all $n \in \mathbb{N}$;
- **bounded from below**, if there exists some $\ell \in \mathbb{R}$ (called a **lower bound** for the sequence) such that $a_n \geq \ell$ for all $n \in \mathbb{N}$;
- **bounded**, if there exists some $r \in \mathbb{R}$ such that $|a_n| \leq r$ for all $n \in \mathbb{N}$.

Exercise II.2 (Boundedness from both above and below).
Prove that a real number sequence $(a_n)_{n\in\mathbb{N}}$ is bounded if and only if it is both bounded from above and bounded from below.

Example II.9 (A sequence bounded from below).
The sequence $(1, 2, 4, 8, 16, 32, \ldots)$ is bounded from below by, for example, the constant

[2] In other words, a sequence is increasing if $a_1 \leq a_2 \leq a_3 \leq \cdots$.
[3] In other words, a sequence is strictly increasing if $a_1 < a_2 < a_3 < \cdots$.
[4] In other words, a sequence is decreasing if $a_1 \geq a_2 \geq a_3 \geq \cdots$.
[5] In other words, a sequence is strictly decreasing if $a_1 > a_2 > a_3 > \cdots$.

$\ell = 1$.[6] To see this, note that the n:th member of the sequence is given by the formula $a_n = 2^{n-1}$, and we have $a_n = 2^{n-1} \geq 1 = \ell$ for all $n \in \mathbb{N}$.

This sequence is not bounded from above. Indeed for any $u \in \mathbb{R}$, we can take a natural number $n > \log_2(u) + 1$, and then $a_n = 2^{n-1} > 2^{\log_2(u)} = u$, so u does not work as an upper bound. Since $u \in \mathbb{R}$ was arbitrary, no upper bound exists for this sequence.

Example II.10 (A bounded sequence).
Let $(a_n)_{n\in\mathbb{N}}$ be defined by the formula $a_n = \frac{n}{n+1}$ for $n \in \mathbb{N}$. We claim that this sequence is bounded by, for example, the constant $r = 1$.[7] To see this, note that for any $n \in \mathbb{N}$ we have

$$|a_n| = \left|\frac{n}{n+1}\right| = \frac{n}{n+1} \leq 1 = r.$$

II.3. Limits of sequences on the real line

The definition of the limit of a sequence

Definition II.11 (Limit of a sequence of numbers).
A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a **limit** $\alpha \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists an index $n_\varepsilon \in \mathbb{N}$ such that $|a_n - \alpha| < \varepsilon$ whenever $n \geq n_\varepsilon$.

Remark II.12 (Meaning of convergence as a logical statement).
Definition II.11 is written in plain English (a natural language), as usual. It nevertheless has a precise logical meaning, and it is instructive to unravel the definition using logical symbols (a formal language). The meaning of the statement $a_n \to \alpha$ as $n \to \infty$ is:

$$\forall \varepsilon > 0 : \quad \exists n_\varepsilon \in \mathbb{N} : \qquad n \geq n_\varepsilon \implies |a_n - \alpha| < \varepsilon. \tag{II.7}$$

A few advantages of the formal statement (II.7) are that it is succinct, unambiguous, and understandable to mathematicians irrespective of whether their mother tongue is English, Finnish, Swedish, or some other natural language (being familiar with the logical symbols, you can easily "read it" in your native language).

The logical definition is crucial, because it gives the precise meaning to the word *limit* (and *converge*). But of course it is still also important to have an intuitive idea about the notion as well! So how should you think about Definition II.11? In Figure II.1 and its caption we attempt to illustrate and describe this.

Exercise II.3 (Non-zero limit implies members are eventually non-zero).
Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers which tends to a non-zero limit

$$\alpha = \lim_{n\to\infty} a_n \neq 0.$$

Show that there exists some $N \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq N$.

Hint: *We have* $|a_n| = |\alpha - (\alpha - a_n)| \geq |\alpha| - |\alpha - a_n|$ *by triangle inequality, Proposition II.1.*

Basic properties of limits of sequences

Having given the precise definition of limit, we are ready to state and prove some basic properties of limits.

[6] As a lower bound here we could equally well use $\ell = 0$ or $\ell = -7$ or indeed any number $\ell \leq 1$.
[7] As a bound here we could equally well use $r = 42$ or $r = 10^{23}$ or indeed any number $r \geq 1$.

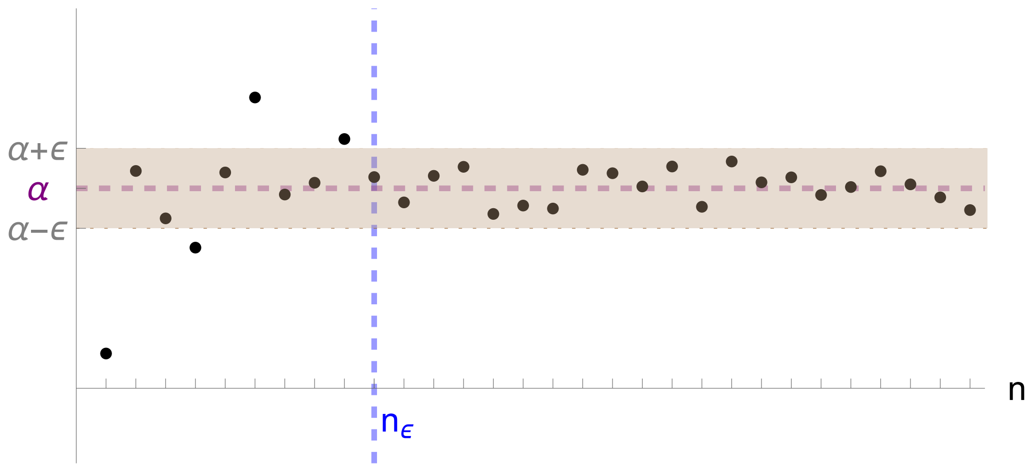


Figure II.1. An illustration of the limit of a number sequence. The limit of the sequence $(a_n)_{n\in\mathbb{N}}$ is $\alpha \in \mathbb{R}$ if for however small error $\varepsilon > 0$ we are willing to tolerate, the entire tail of the sequence starting from some index $n_\varepsilon \in \mathbb{N}$ lies within that error range from the value α; i.e., for all $n \geq n_\varepsilon$ we have $\alpha - \varepsilon < a_n < \alpha + \varepsilon$.

The first one says that the property of convergence implies that of boundedness.

Proposition II.13 (A convergent sequence of number is bounded)**.**
If a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers converges, then it is bounded.

Exercise II.4 (Proof of Proposition II.13)**.**
Prove Proposition II.13.

Exercise II.5 (Boundedness does not imply convergence)**.**
Show that there exists a bounded sequence which is not convergent. Conclude that the implication in the converse direction compared to Proposition II.13 does not hold.

Lemma II.14 (Preservation of inequalities)**.**
Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two convergent sequences of real numbers, with respective limits $\alpha = \lim_{n\to\infty} a_n$ and $\beta = \lim_{n\to\infty} b_n$. If we have

$$a_n \leq b_n \qquad \text{for all } n \in \mathbb{N},$$

then we also have the corresponding inequality

$$\alpha \;=\; \lim_{n\to\infty} a_n \;\leq\; \lim_{n\to\infty} b_n \;=\; \beta$$

for the limits.

Proof. Let $\varepsilon > 0$. Since $\frac{\varepsilon}{2} > 0$, it follows from the assumptions $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$ and the definition of limits that there exists numbers $n', n'' \in \mathbb{N}$ such that

$$\big|a_n - \alpha\big| < \frac{\varepsilon}{2} \quad \text{for } n \geq n', \qquad\qquad \big|b_n - \beta\big| < \frac{\varepsilon}{2} \quad \text{for } n \geq n''.$$

Consider now $n = \max\{n', n''\} \in \mathbb{N}$. Then $n \geq n'$ and $n \geq n''$, so both of the above hold. Using furthermore the assumption $a_n \leq b_n$, we find

$$\begin{aligned} 0 \;\leq\; b_n - a_n \;&=\; b_n - \beta + \beta - \alpha + \alpha - a_n \\ &\leq\; \big|b_n - \beta\big| + \beta - \alpha + \big|\alpha - a_n\big| \\ &<\; \frac{\varepsilon}{2} + \beta - \alpha + \frac{\varepsilon}{2} \\ &=\; \beta - \alpha + \varepsilon. \end{aligned}$$

Rearranging this inequality, we get $\alpha < \beta + \varepsilon$. Since this was shown to hold for an arbitrary $\varepsilon > 0$, it must be that $\alpha \leq \beta$.[8] □

Exercise II.6 (An improvement to Lemma II.14)**.**
Show that in Lemma II.14 it suffices to assume that the inequality holds for all sufficiently large n; more precicely that there exists $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$.

For sequences bounded from above or below, if a limit exists, it must also lie within the same bounds.

Corollary II.15 (Preservation of bounds)**.**
Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequences of real numbers, with limit $a = \lim_{n\to\infty} a_n$. If for some $u \in \mathbb{R}$ we have

$$a_n \leq u \qquad \text{for all } n \in \mathbb{N},$$

then we also have the corresponding inequality for the limit,

$$\alpha \;=\; \lim_{n\to\infty} a_n \;\leq\; u.$$

Similarly, if for some $\ell \in \mathbb{R}$ we have $a_n \geq \ell$ for all $n \in \mathbb{N}$, then we also have $\lim_{n\to\infty} a_n \geq \ell$.

Exercise II.7 (Proof of Corollary II.15)**.**
Prove the first part of Corollary II.15, by using a constant sequence $(b_n)_{n\in\mathbb{N}}$ in Lemma II.14. Prove the second part by a similar idea.

Exercise II.8 (An improvement to Corollary II.15)**.**
Formulate the following improvement precisely and prove it:
In Corollary II.15 it suffices to assume that the inequality holds for all sufficiently large n.
Hint: *Compare with Exercise II.6.*

Note that from Definition II.11 it is *not immediately obvious* that there could not exist several different numbers that satisfy what is required of a limit. In other words, if the condition (II.7) holds for $\alpha \in \mathbb{R}$, then could it also hold for some other

[8] This is a very common step in proofs, so let us look at it in detail now that we first use it. The claim is that

$$\forall\, \varepsilon > 0: \;\; \alpha < \beta + \varepsilon \qquad \Longrightarrow \qquad \alpha \leq \beta.$$

This is easy to prove by *contrapositive*: the claim is logically equivalent to (recall contrapositives and negations of statements with universal quantifier)

$$\alpha > \beta \qquad \Longrightarrow \qquad \exists\, \varepsilon > 0: \;\; \alpha \geq \beta + \varepsilon$$

In this formulation the claim is evident: if $\alpha > \beta$, then $\alpha - \beta > 0$ and we can use $\varepsilon = \alpha - \beta$ which indeed satisfies $\alpha = \beta + \varepsilon \geq \beta + \varepsilon$.

real number $\alpha' \in \mathbb{R}$ in place of α?[9] Fortunately the following result lifts any concerns about the possibility of such ambiguity.[10]

Corollary II.16 (Uniqueness of limits of real number sequences)**.**
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. If both $\alpha \in \mathbb{R}$ and $\alpha' \in \mathbb{R}$ are limits of this sequence, then $\alpha = \alpha'$.

Proof. Trivially, the inequality $a_n \leq a_n$ holds for all $n \in \mathbb{N}$. If we now suppose that both α and α' are limits of the sequence $(a_n)_{n\in\mathbb{N}}$, and we use the former limit for the left hand side and the latter for the right hand side, from Lemma II.14 we get that $\alpha \leq \alpha'$. Similarly using the former for the right hand side and the latter for the left hand side, we get $\alpha' \leq \alpha$. The combination of these gives $\alpha = \alpha'$. □

We next state a very practical result, which is in some sense an improved version of the preservation of inequalities — in it we do not even need to assume the existence of the limit, but the existence is a part of the conclusion! It states that if a sequence is "squeezed" in between two convergent sequences with same limit, then the sequence itself has to converge to this limit, too. This is often called *the squeeze theorem* (though we regard it more as a lemma). It also has various other affectionate nicknames. Some call it *the sandwich principle* — since it talks about a sequence "sandwiched" in between two others. Also *the lemma of two policemen* is descriptive: the idea being that one sequence "guards" the sequence of interest from above, preventing its escape to the upwards direction, and another sequence "guards" the sequence of interest from below, preventing its escape to the downwards direction.[11]

Lemma II.17 (Squeeze theorem)**.**
Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ be three sequences of real numbers. Suppose that for all $n \in \mathbb{N}$ we have

$$a_n \;\leq\; b_n \;\leq\; c_n.$$

Suppose also that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are convergent and have the same limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \beta \in \mathbb{R}.$$

Then the sequence $(b_n)_{n\in\mathbb{N}}$ is also convergent and its limit is the same,

$$\lim_{n\to\infty} b_n = \beta.$$

Exercise II.9 (Proof of Lemma II.17)**.**
Prove the squeeze theorem (Lemma II.17).

Example II.18 (An example of the squeeze theorem)**.**
Consider the sequence $(b_n)_{n\in\mathbb{N}}$ given by

$$b_n = 3 + 4^{-n}\,\sin(5n) \qquad \text{for } n \in \mathbb{N}.$$

[9] The notation $\lim_{n\to\infty} a_n$ *suggests* that the limit is uniquely determined, but we should actually *prove* that this is so, in order to be sure that the notation is unambiguously defined!

[10] It would not have been difficult to prove the uniqueness of limits more directly, immediately after the definition. In fact, in Chapter VIII we treat sequences and limits in a more general setup, and we give a more direct (as well as more general) proof of the uniqueness. Here, however, we think it instructive to give a proof based on the preservation of inequalities.

[11] Clearly the result must be important, given how imaginative nomenclature it has inspired!

Since $-1 \leq \sin(\theta) \leq +1$ for all $\theta \in \mathbb{R}$, we get

$$3 - 4^{-n} \ \leq \ b_n \ \leq \ 3 + 4^{-n} \qquad \text{for all } n \in \mathbb{N}.$$

We have $\lim_{n\to\infty}(3-4^{-n}) = 3$ and $\lim_{n\to\infty}(3+4^{-n}) = 3$, so we can use the sequences $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ defined by $a_n = 3-4^{-n}$ and $c_n = 3+4^{-n}$ in the squeeze theorem (Lemma II.17) to conclude that

$$\lim_{n\to\infty} b_n = 3.$$

Rules of calculation with limits

Given two real number sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, we often consider new sequences formed from these for example by[12]

$$\big(a_n + b_n\big)_{n\in\mathbb{N}}, \qquad \big(a_n\, b_n\big)_{n\in\mathbb{N}}, \qquad \big(a_n/b_n\big)_{n\in\mathbb{N}}.$$

Theorem II.19 (Rules of calculation with limits)**.**
Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two convergent sequences with respective limits $\alpha = \lim_{n\to\infty} a_n$ and $\beta = \lim_{n\to\infty} b_n$. Then we have

$$\lim_{n\to\infty}(a_n + b_n) \ = \ \alpha + \beta, \tag{II.8}$$

$$\lim_{n\to\infty}(a_n\, b_n) \ = \ \alpha\beta, \tag{II.9}$$

and if moreover $\beta \neq 0$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} \ = \ \frac{\alpha}{\beta}. \tag{II.10}$$

Before addressing the proof, we note the following. From (II.8) and (II.9) above, we get the following special cases by choosing $(b_n)_{n\in\mathbb{N}}$ to be a constant sequence $b_n = c$ for all $n \in \mathbb{N}$.

Corollary II.20 (Additive and multiplicative constants in limits)**.**
Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence with limit $\alpha = \lim_{n\to\infty} a_n$, and let $c \in \mathbb{R}$ be a constant. Then we have

$$\lim_{n\to\infty}(a_n + c) = \alpha + c, \qquad\qquad \lim_{n\to\infty}(c\, a_n) = c\alpha.$$

Proof of Theorem II.19. The proofs of all cases are quite similar, so we only do (II.9) and leave the other two as exercises.

So suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two convergent sequences with limits $\alpha = \lim_{n\to\infty} a_n$ and $\beta = \lim_{n\to\infty} b_n$. We must prove that the sequence $(a_n\, b_n)_{n\in\mathbb{N}}$ converges to $\alpha\beta$, and we

[12] The last of these is only well defined if $b_n \neq 0$ for all $n \in \mathbb{N}$ — otherwise we encounter a division by zero in a_n/b_n. But in fact, since we will be concerned with calculating limits as $n \to \infty$, we do not necessarily need to care if such a problem occurs just finitely many times along the sequence. If there are only finitely many indices $n \in \mathbb{N}$ for which $b_n = 0$, then we can still consider the sequence $(a_n/b_n)_{n=n_0}^{\infty}$ starting from a sufficiently large index n_0 such that $b_n \neq 0$ for all $n \geq n_0$. It is meaningful to consider the limit of this sequence, which we still write as $\lim_{n\to\infty} a_n/b_n$.

will do this directly using the Definition II.11 of limits. For this purpose, we first estimate

$$\begin{aligned}
& \big|a_n\, b_n - \alpha\beta\big| \\
= \; & \Big|a_n\, b_n - \alpha\, b_n + \alpha b_n - \alpha\beta\Big| && \text{(added and subtracted the same term)} \\
= \; & \Big|(a_n - \alpha)\, b_n + \alpha\,(b_n - \beta)\Big| && \text{(rearranged terms)} \\
\leq \; & \big|(a_n - \alpha)\, b_n\big| \; + \; \big|\alpha\,(b_n - \beta)\big| && \text{(triangle inequality)} \\
= \; & |b_n|\,\big|a_n - \alpha\big| \; + \; |\alpha|\,\big|b_n - \beta\big|.
\end{aligned}$$

By Proposition II.13, the convergent sequence $(b_n)_{n\in\mathbb{N}}$ is bounded, so there exists a constant $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Plugging this in the above estimate, we obtain[13]

$$\big|a_n\, b_n - \alpha\beta\big| \;\leq\; M\,\big|a_n - \alpha\big| \;+\; |\alpha|\,\big|b_n - \beta\big|. \tag{II.11}$$

Now let $\varepsilon > 0$. Because $\lim_{n\to\infty} a_n = \alpha$ and $\frac{\varepsilon}{3M}$ is a positive number, there exists an $n' \in \mathbb{N}$ such that we have

$$|a_n - \alpha| \;<\; \frac{\varepsilon}{3M} \qquad \text{for } n \geq n'.$$

Similarly, because $\lim_{n\to\infty} b_n = \beta$ and $\frac{\varepsilon}{3\,(|\alpha|+1)}$ is a positive number, there exists an $n'' \in \mathbb{N}$ such that we have

$$|b_n - \beta| \;<\; \frac{\varepsilon}{3\,(|\alpha|+1)} \qquad \text{for } n \geq n''.$$

Then if $n \geq \max\{n', n''\}$, we have both $n \geq n'$ and $n \geq n''$, so both of the above estimates hold, and from (II.11) we get

$$\begin{aligned}
\big|a_n\, b_n - \alpha\beta\big| \;&\leq\; M\, \underbrace{\big|a_n - \alpha\big|}_{\leq \varepsilon/3M} + |\alpha|\, \underbrace{\big|b_n - \beta\big|}_{\leq \varepsilon/3(|\alpha|+1)} \\
&\leq\; \underbrace{\frac{M}{3M}}_{\leq 1/3}\,\varepsilon \;+\; \underbrace{\frac{|\alpha|}{3\,(|\alpha|+1)}}_{\leq 1/3}\,\varepsilon \\
&\leq\; \frac{2}{3}\,\varepsilon \;<\; \varepsilon.
\end{aligned}$$

If we set $n_\varepsilon = \max\{n', n''\}$, then from the above we got $\big|a_n\, b_n - \alpha\beta\big| < \varepsilon$ for all $n \geq n_\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this by definition shows that $\lim_{n\to\infty}(a_n\, b_n) = \alpha\beta$, proving (II.9).

The proofs of (II.8) and (II.10) are left as exercises. □

Exercise II.10 (Completing the proof of Theorem II.19).
Prove (II.8) and (II.10).

Exercise II.11 (Examine properties of some sequences).
Let

$$a_n = \frac{n+1}{2n+1} \qquad \text{and} \qquad b_n = \sqrt[n]{3^n + 2^n} \qquad \text{for } n \in \mathbb{N}.$$

Find out whether the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are increasing, decreasing, bounded from above, bounded from below. Determine the limits of these sequence if they exist.

[13]After this, the idea is that assumed convergence of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ implies that both $|a_n - \alpha|$ and $|b_n - \beta|$ become small for n large. This should allow us to show that the expression (II.11) becomes small for large n. The rest of the proof is about making this idea precise.

II.4. Density of rational and irrational numbers

We will next look at an interesting property of the real axis: both the rational numbers and the irrational numbers are dense on $\mathbb{R}$. In particular, between any two rational numbers there is an irrational number, and between any two irrational number there is a rational number. The fact that the countably infinite (and thus set theoretically rather small) set of rational numbers is already dense in the (uncountably infinite) set of real numbers is a topological property of $\mathbb{R}$ known as *separability*.

Density of rational numbers

Let us show that between any two (different) real numbers, there exists a rational number.

Theorem II.21 (Density of rational numbers)**.**
Let $x, y \in \mathbb{R}$ with $x < y$. Then there exists a $q \in \mathbb{Q}$ such that $x < q < y$.

Proof. Since $y-x > 0$, we can choose an $n \in \mathbb{N}$ such that $n > \frac{1}{y-x}$. Now in the set $\big\{ \frac{m}{n} \;\big|\; m \in \mathbb{Z} \big\} \subset \mathbb{Q}$ of all integer multiples of $\frac{1}{n}$, the distance between consecutive ones is $\frac{1}{n}$. In particular since the interval $(x, y) \subset \mathbb{R}$ has length $y - x > \frac{1}{n}$, at least one such integer multiple $\frac{m}{n}$ must lie in this interval. In other words, the rational number $q = \frac{m}{n} \in \mathbb{Q}$ lies on the desired interval, $q \in (x, y)$. □

Although we only apparently showed that an open interval $(x, y) \subset \mathbb{R}$ of the real axis contains one rational number, it almost immediately follows that the interval in fact contains infinitely many rational numbers.

Corollary II.22 (An open interval contains infinitely many rational numbers)**.**
On any nonempty open interval $(x, y) \subset \mathbb{R}$, there are infinitely many rational numbers, i.e., $\#\big((x, y) \cap \mathbb{Q}\big) = \infty$.

Proof. By Theorem II.21, we may first choose one rational number $q_1 \in \mathbb{Q}$ such that $x < q_1 < y$. Then applying the same result to the interval (x, q_1), we can choose $q_2 \in \mathbb{Q}$ such that $x < q_2 < q_1 < y$. Inductively, once $q_1, \ldots, q_n \in \mathbb{Q}$ have been chosen so that

$$x < q_n < q_{n-1} < \cdots < q_2 < q_1 < y,$$

we choose a rational number $q_{n+1} \in \mathbb{Q}$ from the interval (x, q_n). This yields an infinite sequence $(q_n)_{n\in\mathbb{N}}$ of rational numbers on the interval (x, y). Finally note that by construction the sequence is strictly decreasing, so no two members of it are equal, and we have indeed found infinitely many different rational numbers on the interval. □

Density of irrational numbers

Similarly, between any two (different) real numbers, there exists an irrational number.

Theorem II.23 (Density of irrational numbers)**.**
Let $x, y \in \mathbb{R}$ with $x < y$. Then there exists a $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$.

Proof. Let us first use Corollary II.22 to choose two different rational numbers $q, r \in \mathbb{Q}$ so that $x < q < r < y$. It suffices to show that the subinterval $(q, r) \subset (x, y)$ contains an irrational number.

Let

$$z = q + \frac{r-q}{\sqrt{2}}.$$

Clearly we have $z > q$, since $r - q > 0$ and $\frac{1}{\sqrt{2}} > 0$. On the other hand $\frac{1}{\sqrt{2}} < 1$, so we also have

$$z = q + \frac{r-q}{\sqrt{2}} < q + 1 \cdot (r - q) = r.$$

From these we get that the number z lies in the subinterval of interest, $x < q < z < r < y$. It remains to show that z is irrational. From the definition of z, we can solve

$$\sqrt{2} = \frac{r-q}{z-q}.$$

Now argue by contradiction: if z were rational, then the right hand side above would also be rational (recall that q, r were chosen rational), so $\sqrt{2}$ would be rational, contradicting Proposition I.5. We conclude the desired irrationality $z \in \mathbb{R} \setminus \mathbb{Q}$. $\square$

Again from knowing that an open interval $(x, y) \subset \mathbb{R}$ of the real axis contains one irrational number, we easily get that the interval in fact contains infinitely many irrational numbers.

Corollary II.24 (An open interval contains infinitely many irrational numbers)**.** *On any nonempty open interval $(x, y) \subset \mathbb{R}$, there are infinitely many irrational numbers, i.e., $\#\big((x, y) \setminus \mathbb{Q}\big) = \infty$.*

Exercise II.12 (Proof of Corollary II.24)**.**
Prove Corollary II.24, using Theorem II.23 and an argument similar to Corollary II.22.

II.5. Axioms of the real numbers

So far we have talked about real numbers, but we have not in fact precisely said what they are! If one looks closely, this actually turns out not to be an entirely trivial philosophical matter. . .

As with many other mathematical structures, the standard rigorous approach is to take certain properties of real numbers as axioms (statements that are simply accepted as true), and require that any other statement made about real numbers is logically deduced from these axioms.[14] [15] We want the axioms to be as modest

[14] In practice, it would be very tedious to give full proofs of all properties of real numbers that we use, as this would include much of what you have learned about mathematics since kindergarten. . . We will therefore not ask you to provide proofs of totally commonplace statements about the real numbers — but instead to just realize that they must in principle be logical consequences of the axioms. In Appendix B.1 all of the axioms of $\mathbb{R}$ are at least stated, and derivations of other properties from them are exemplified.

[15] In addition to giving the axioms, it would be desirable to provide a (set theoretic) construction of real numbers, for which the axioms can be shown to be true. Suffice to say that there are such constructions, but we will not discuss them in detail here. Any such construction of course will admit something else as a starting point. An apparently benign starting point is to assume "uncontroversial" axioms of natural numbers (but in view of Gödel's incompleteness theorem this is in fact not without its own issues!).

as possible, so that the starting point is uncontroversial. And since everything else is deduced from the starting point by logical reasoning, also everything else will be uncontroversial — which of course is the point of mathematics!

The set of real numbers is denoted by $\mathbb{R}$, and it is equipped with two binary operations:

$$\begin{aligned} \text{addition “}+\text{”:} \quad & \mathbb{R}\times\mathbb{R}\to\mathbb{R} \\ & (x,y)\mapsto x+y, \\ \text{multiplication “}\cdot\text{”:} \quad & \mathbb{R}\times\mathbb{R}\to\mathbb{R} \\ & (x,y)\mapsto x\cdot y \\ & \text{(omitted in usual notation, so } x\cdot y = xy), \end{aligned}$$

and a binary relation

$$\text{less than “}<\text{”}.$$

The axioms of real numbers fall into three types: the *field axioms* concern only the operations of addition and multiplication, the *order axioms* concern the order relation $<$ and how it behaves under addition and multiplication, and finally the *completeness axiom* which is in many ways the most subtle.

On the field axioms and order axioms

The field axioms and order axioms concern properties of the real numbers that are without a doubt familiar already. We list these axioms and briefly discuss them in Appendix B.1. For a broader mathematical perspective, it is worthwhile to take a look and to appreciate also these familiar properties of addition, multiplication, and order.

Here we content ourselves to mentioning that there are various other *fields* regularly used in mathematics — for example:

- the field of rational numbers $\mathbb{Q}$;
- the field of complex numbers $\mathbb{C}$;
- finite fields, the simplest example of which is the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of p elements, where p is a prime number;
- algebraic number fields, a simple example of which is the field $\mathbb{Q}(\sqrt{33})$ of rational numbers adjoined with a square root of 33;
- the field $\mathbb{Q}_p$ of p-adic numbers, where p is a prime number;
- the field $\mathbb{K}(q)$ of rational functions in a single variable q over another field $\mathbb{K}$;
- $\cdots$

All of these satisfy the same *field axioms* as $\mathbb{R}$, so the rules governing addition and multiplication with all their consequences are similar.

From the natural numbers, it is not difficult to construct the rational numbers. From the rational numbers, the real numbers can be constructed for example using what is known as Dedekind cuts. There are many other constructions as well. Importantly, however, a sort of uniqueness property of real numbers holds: any two constructions (in which the axioms hold) yield results that are in all essential ways the same (isomorphic). So we do not need to care about the chosen construction — the axioms of real numbers themselves are indeed a good starting point.

The field $\mathbb{Q}$ of rational numbers (as well as some but not all algebraic number fields) moreover has an order relation compatible with the addition and multiplication, in that it satisfies also the same *order axioms* as $\mathbb{R}$. The "only" difference between the real numbers and the rational numbers must therefore stem from the *completeness axiom*, which we discuss in more detail below.

Completeness axiom

We will next address to the one really subtle axiom of the real numbers: the *completeness axiom*. But before stating it, we first need to introduce a few notions that will make an appearance here.

Supremum and infimum

Let us start from notions of boundedness for subsets of the real line. If you compare the following with Definition II.8, a common theme becomes evident.

Definition II.25 (Upper and lower bounds)**.**
Let $A \subset \mathbb{R}$ be a subset.

A real number $u \in \mathbb{R}$ is an **upper bound** for A if for all $a \in A$ we have $a \leq u$.

A real number $\ell \in \mathbb{R}$ is a **lower bound** for A if for all $a \in A$ we have $a \geq \ell$.

Example II.26 (Examples of upper and lower bounds)**.**

- Consider the open interval $(-\pi, +\pi) \subset \mathbb{R}$. The number π is an upper bound for the set $(-\pi, +\pi)$. But also any number $u > \pi$ is an upper bound for $(-\pi, +\pi)$. Likewise, any number $\ell \leq -\pi$ is a lower bound for the set $(-\pi, +\pi)$.
- Consider the set $\mathbb{N} \subset \mathbb{R}$ of natural numbers. The number 1 is a lower bound for $\mathbb{N}$, and so is any number $\ell < 1$. The set $\mathbb{N}$ does not have any upper bounds: for any $u \in \mathbb{R}$ there exists a natural number $n \in \mathbb{N}$ such that $n > u$.
- The sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ do not have any upper or lower bounds.
- Consider the empty set $\emptyset \subset \mathbb{R}$. Any real number $u \in \mathbb{R}$ is an upper bound for $\emptyset$.[16] Likewise, any real number $\ell \in \mathbb{R}$ is a lower bound for $\emptyset$.

From these examples we learn that upper and lower bounds for arbitrary sets do not need to exist. Moreover, when they exist, they are not unique.

Definition II.27 (Sets bounded from above and below)**.**
A subset $A \subset \mathbb{R}$ is **bounded from above** if it has an upper bound, and **bounded from below** if it has a lower bound.

The following definition introduces the key notion in the completeness axiom: the least upper bound or supremum. The supremum and its counterpart infimum also feature in a number of contexts in real analysis — some applications are discussed in Appendix B.4.

[16] Indeed, there are no elements $a \in \emptyset$ for which we would get a nontrivial condition that u needs to satisfy to qualify as an upper bound for the empty set.

Definition II.28 (Supremum and infimum).

Let $A \subset \mathbb{R}$ be a nonempty subset ($A \neq \emptyset$) which is bounded from above. A number $u_0 \in \mathbb{R}$ is called the **least upper bound** or **supremum** of A, if u_0 is an upper bound for A (i.e. $\forall a \in A: \ a \leq u_0$) and for all upper bounds u of A we have $u_0 \leq u$. We then denote $u_0 = \sup A$.

Let $B \subset \mathbb{R}$ be a nonempty subset ($B \neq \emptyset$) which is bounded from below. A number $\ell_0 \in \mathbb{R}$ is called the **greatest lower bound** or **infimum** of B, if ℓ_0 is a lower bound for B (i.e. $\forall b \in B: \ b \geq \ell_0$) and for all lower bounds ℓ of B we have $\ell_0 \geq \ell$. We then denote $\ell_0 = \inf B$.

Implicit in the notations $\sup A$ and $\inf A$ (and in our use of the definite article above) is that the least upper bound and the greatest lower bound are uniquely determined by the above properties. This is indeed not difficult to check, and we leave it as an exercise to the reader.

The notions of supremum and infimum are crucial in this course as well as mathematics more generally, so let us give a more practical characterization.

Lemma II.29 (Characterization of supremum).

A number $u_0 \in \mathbb{R}$ is the supremum of a nonempty subset $A \subset \mathbb{R}$ which is bounded from above, if and only if the following two conditions hold:

- *For all $a \in A$ we have $a \leq u_0$;*[17]
- *For all $\varepsilon > 0$ there exists some $a \in A$ such that $a > u_0 - \varepsilon$.*[18]

Exercise II.13 (Proof of Lemma II.29).

Prove Lemma II.29.

Hint: *Compare the defining properties of supremum in Definition II.28 with the interpretations of the two conditions in the lemma, given in the footnotes.*

Exercise II.14 (Characterization of infimum).

Formulate and prove a the counterpart of Lemma II.29 for infimum, i.e., two conditions characterizing the infimum of a non-empty set $B \subset \mathbb{R}$ which is bounded from below.

Supremum and infimum are generalizations of the notions of maximum (i.e., the largest element) and minimum (i.e., the smallest element). For comparison, let us recall also these notions.

Definition II.30 (Maximum and minimum).

Let $A \subset \mathbb{R}$ be a subset.

An element $a' \in A$ is the **maximum** (i.e. the **largest element**) of A, if for all $a \in A$ we have $a \leq a'$. We then denote $a' = \max A$.

An element $a'' \in A$ is the **minimum** (i.e. the **smallest element**) of A, if for all $a \in A$ we have $a \geq a''$. We then denote $a'' = \min A$.

The crucial difference between the maximum and the supremum (or an upper bound more generally) is that the maximum of A is required to be an element of the set A

[17]This condition simply says that u_0 is an upper bound for A.

[18]This condition says that any smaller number $u_0 - \varepsilon$ cannot be an upper bound for A.

itself. A similar remark applies to the difference between the minimum and the infimum.

Note, however, that not every subset $A \subset \mathbb{R}$ has a maximum (resp. minimum) — even if we assume A to be non-empty and bounded from above (resp. from below).

Example II.31 (Open intervals have no maximum and minimum)**.**
The open interval $(-\pi, \pi)$ has no largest element and no smallest element ($\nexists \max(-\pi,\pi)$, $\nexists \min(-\pi,\pi)$). In fact, open intervals never have a maximum or minimum.

Example II.32 (A bounded nonempty set)**.**
The set $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ is bounded and non-empty. The largest element of this set is $\max A = 1$. The set A has no smallest element ($\nexists \min A$). The least upper bound of this set is $\sup A = 1$ and the greatest lower bound of it is $\inf A = 0$. Note that in this case the infimum is not an element of the set, $0 \notin A$.

Exercise II.15 (Examine properties of some sets)**.**
Consider the sets

$$A_1 = [0, \sqrt{2}] \cap \mathbb{Q}, \qquad A_2 = \left\{e^{-x} \mid x \in \mathbb{R}\right\}, \qquad A_3 = \left\{\frac{7n(-1)^n + 4}{6n} \;\middle|\; n \in \mathbb{N}\right\}.$$

Answer the following questions about each of the sets A_j, $j \in \{1,2,3\}$.

- Is the set A_j bounded from above?
- Is it bounded from below?
- Does it have a maximum, and if it does, what is $\max A_j$?
- Does it have a supremum, and if it does, what is $\sup A_j$?
- Does it have a minimum, and if it does, what is $\min A_j$?
- Does it have an infimum, and if it does, what is $\inf A_j$?

Exercise II.16 (The relationship between supremum and maximum)**.**
Let $A \subset \mathbb{R}$.

(a) Prove that if $\max A$ exists, then we have $\sup A = \max A$.
(b) Prove that if $\sup A \in A$ (so in particular the supremum exists), then the maximum exists and we have $\max A = \sup A$.

Exercise II.17 (The relationship between infimum and minimum)**.**
Formulate and prove an analogous result for the infimum and minimum.

Formulations of the completeness axiom

There are various equivalent ways of formulating the completeness axiom of $\mathbb{R}$. We first state three formulations, and later discuss their equivalence.

The **completeness axiom** of $\mathbb{R}$ refers to any of the following (equivalent) statements:

(C1): Every non-empty subset $A \subset \mathbb{R}$ which is bounded from above has a least upper bound $\sup A \in \mathbb{R}$.
(C2): Every increasing real number sequence $(a_n)_{n\in\mathbb{N}}$ which is bounded from above has a limit $\lim_{n\to\infty} a_n \in \mathbb{R}$.
(C3): Every collection $(I_n)_{n\in\mathbb{N}}$ of closed intervals $I_n \subset \mathbb{R}$, which is nested in the sense that $I_{n+1} \subset I_n$ for every $n \in \mathbb{N}$, has a nonempty intersection

$$\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset.$$

Regarding the last formulation (C3), note that in Example I.8 we saw that the intersection of nested *open intervals* can be empty. It is therefore important that in (C3) the intervals are required to be *closed*.

Furthermore, it is easy to see that the first two formulations above are equivalent with the following two, respectively:

- **(C1'):** Every non-empty subset $A \subset \mathbb{R}$ which is bounded from below has a greatest lower bound $\inf A \in \mathbb{R}$.
- **(C2'):** Every decreasing real number sequence $(a_n)_{n\in\mathbb{N}}$ which is bounded from below has a limit $\lim_{n\to\infty} a_n \in \mathbb{R}$.

Exercise II.18 (Equivalence of (C1) and (C1')).
Prove that (C1) and (C1') are logically equivalent.

Exercise II.19 (Equivalence of (C2) and (C2')).
Prove that (C2) and (C2') are logically equivalent.

The equivalence of the three formulations of the completeness axiom is proven in Appendix B.2.

As an example application of the formulation (C2) of the completeness axiom, we discuss the existence of real numbers given by decimal expansions.

Example II.33 (Decimal expansions).
Let $(d_1, d_2, d_3, \ldots) = (d_k)_{k\in\mathbb{N}}$ be a sequence of digits $d_k \in \{0, 1, 2, \ldots, 8, 9\}$. The precise definition of the **decimal number**

$$0.d_1d_2d_3\ldots$$

is the sum of the series

$$\sum_{k=1}^{\infty} d_k\, 10^{-k}. \tag{II.12}$$

In particular, the existence of this decimal number requires the series to be convergent.

The partial sums of the series

$$a_n = \sum_{k=1}^{n} d_k\, 10^{-k}$$

clearly form an increasing sequence $(a_n)_{n\in\mathbb{N}}$, because

$$a_{n+1} - a_n \;=\; \sum_{k=1}^{n+1} d_k\, 10^{-k} - \sum_{k=1}^{n} d_k\, 10^{-k} \;=\; d_{n+1}\, 10^{-n-1} \;\geq\; 0.$$

The sequence of the partial sums is also bounded from above, because

$$\begin{aligned} a_n &= \sum_{k=1}^{n} d_k\, 10^{-k} && \\ &\leq \sum_{k=1}^{n} 9 \cdot 10^{-k} && \text{(since } d_k \leq 9\text{)} \\ &= \frac{9}{10} \sum_{\ell=0}^{n-1} 10^{-\ell} && \text{(change summation index to } \ell = k-1\text{)} \\ &= \frac{9}{10}\, \frac{1-10^{-n}}{1-\frac{1}{10}} && \text{(finite geometric sum)} \\ &\leq \frac{9}{10}\, \frac{1}{9/10} \;=\; 1. && \end{aligned}$$

According to the formulation (C2) of the completeness axiom, the sequence $(a_n)_{n\in\mathbb{N}}$ of partial sums converges, because it is increasing and bounded from above. This means that the series (II.12) converges.

We conclude that every decimal expansion indeed represents some real number.

Conventions to extend the notion of supremum and infimum

In Definition II.28 we considered the supremum only for subsets A which are nonempty and bounded from above. If A fails one of these properties, we will use the following conventions:

- If a set $A \subset \mathbb{R}$ is not bounded from above (and thus has no upper bounds), we interpret the least upper bound as the symbol $+\infty$ (not a number),
$$\sup A \;=\; +\infty.$$
- For the empty set $\emptyset \subset \mathbb{R}$ (for which any real number is an upper bound), we interpret the least upper bound as the symbol $-\infty$ (not a number),
$$\sup \emptyset \;=\; -\infty.$$

Symmetrically for the infimum we use the conventions:

- If a set $A \subset \mathbb{R}$ is not bounded from below (and thus has no lower bounds), we interpret the greatest lower bound as the symbol $-\infty$ (not a number),
$$\inf A \;=\; -\infty.$$
- For the empty set $\emptyset \subset \mathbb{R}$ (for which any real number is a lower bound), we interpret the greatest lower bound as the symbol $+\infty$ (not a number),
$$\inf \emptyset \;=\; +\infty.$$

II.6. ♡ In depth topics about the real numbers

The reader interested in deepening her understanding may wish to look into some of the following topics related to the subject of this lecture:

- The more familiar axioms of the real numbers, Appendix B.1.
- The proof of equivalence of the different formulations of the completeness axiom, Appendix B.2.

- The Cantor set — a fractal subset of the unit interval with interesting set theoretic, topological, and measure theoretic properties, Appendix B.3.
- Applications of supremum and infimum, Appendix B.4.

Lecture III

Sequences and functions on the real line

The main theme of this section is continuous functions of a real variable — certainly to some extent a familiar topic already. We will in particular verify that polynomials and rational functions are continuous. We will examine continuous functions on closed intervals, and show that they always have a maximum and a minimum — a crucial existence result for optimization tasks[1], for example — and especially that they are bounded. We also look into the intermediate value theorem (Bolzano's theorem), according to which continuous functions on intervals "can not skip values".

One of the main objectives is to precisely define the notion of continuity, and to rigorously prove the above (probably familiar) facts. Another goal is to start drawing attention to the underlying topological reasons behind such important results: later in the course it is possible to appreciate the role of compactness (of closed intervals) and connectedness (of intervals) in the results.

As a tool we will still use real number sequences, and in particular their judiciously chosen subsequences. A key role is played by the fundamental fact that from a bounded sequence it is always possible to pick some convergent subsequence.[2]

III.1. Real number sequences

Subsequences

Consider a sequence of real numbers,

$$(x_1, x_2, x_3, x_4, x_5, \ldots).$$

The idea of a subsequence is that we leave out some members from the original sequence. For example if we leave out the first member of the above sequence, we obtain the (sub)sequence

$$(x_2, x_3, x_4, x_5, x_6, \ldots),$$

and if we leave out all members of even indices, we obtain the (sub)sequence

$$(x_1, x_3, x_5, x_7, x_9, \ldots).$$

We anyway require that infinitely many members remain after some leaving out some. Also the convention is that the remaining members are relabeled by the

[1]Mathematically, optimization is formulated as the task of finding the maximum (or minimum) of some objective function: e.g., "profit", "accuracy", "efficiency", ... (or "cost", "error", "risk", ...). If the maximum (or minimum) does not exist, the problem is not even well posed and it is meaningless to look for the optimum!

[2]This fact that makes use of the completeness axiom of the real number field, and will itself in later chapters be recognized as a compactness statement.

original index set $\mathbb{N}$ again[3] — this ensures that the subsequence itself will in fact be a sequence (in the sense of Section II.2).

The precise definition of a subsequence is the following.

Definition III.1 (Subsequence).
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence, and $\varphi\colon \mathbb{N} \to \mathbb{N}$ a strictly increasing function. Then the sequence

$$\big(x_{\varphi(n)}\big)_{n\in\mathbb{N}} = (x_{\varphi(1)}, x_{\varphi(2)}, x_{\varphi(3)}, \ldots)$$

is called a **subsequence** of the sequence $(x_n)_{n\in\mathbb{N}}$.

Another common notation for a subsequence is

$$\big(x_{n_k}\big)_{k\in\mathbb{N}} \;=\; (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)\;,$$

where k is used for the new indices, and from the original sequence we have picked members with indices $n_1 < n_2 < n_3 < \cdots$, corresponding to $n_k = \varphi(k)$ in the above definition.

It is easy to show (by induction) that for a strictly increasing function $\varphi\colon \mathbb{N} \to \mathbb{N}$ we have

$$\varphi(n) \;\geq\; n \qquad \text{for all } n \in \mathbb{N},$$

which implies that a member in a subsequence corresponds to a member in the original sequence which is at least as far in the tail of the original sequence (its index in the original sequence is at least as large as its index in the subsequence). This observation makes the proof of the following lemma very easy.

Lemma III.2 (All subsequences of a convergent sequence have the same limit).
Suppose that $(x_n)_{n\in\mathbb{N}}$ is a real number sequence which converges to a limit $\lim_{n\to\infty} x_n = x$. Then any subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ also converges to the same limit, $\lim_{n\to\infty} x_{\varphi(n)} = x$.

Exercise III.1 (Proof of Lemma III.2).
Prove Lemma III.2.

Example III.3 (The alternating sign sequence).
Consider the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = (-1)^n$, i.e.,

$$(a_n)_{n\in\mathbb{N}} = (-1, +1, -1, +1, -1, \ldots).$$

The subsequence of even members, obtained with $\varphi_{\text{even}}(n) = 2n$, is

$$(a_{2n})_{n\in\mathbb{N}} = (+1, +1, +1, +1, \ldots),$$

and it obviously converges to $+1$.

The subsequence of odd members, obtained with $\varphi_{\text{odd}}(n) = 2n - 1$, is

$$(a_{2n-1})_{n\in\mathbb{N}} = (-1, -1, -1, -1, \ldots),$$

and it obviously converges to -1.

The fact that these two subsequences have different limits implies (by Lemma III.2) that the original sequence $(a_n)_{n\in\mathbb{N}}$ does not converge. (Of course this could also be proven directly!)

[3]It is possible to consider index sets other than $\mathbb{N}$, but even in such cases we require that the indices are consecutive integers. For simplicity, let us focus on the case when the index set is $\mathbb{N}$.

Example III.4 (A sequence without convergent subsequences).

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = -n$, i.e.,

$$(a_n)_{n\in\mathbb{N}} = (-1, -2, -3, -4, \ldots).$$

It is easy to see that this sequence has no convergent subsequences (in fact it has no bounded subsequences, so this follows from Proposition II.13).

Existence of monotone subsequences

It turns out that from any real number sequence, it is possible to pick a subsequence which is monotone (either increasing or decreasing). This observation is somewhat interesting on its own right, but its real value is as a lemma towards the next very important result, Theorem III.6.

Lemma III.5 (Any real number sequence has a monotone subsequence).

Let $(x_n)_{n\in\mathbb{N}}$ be a real number sequence. Then there exists a subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which is monotone.

Proof. Define the set

$$J \;:=\; \Big\{ n \in \mathbb{N} \;\Big|\; x_m \leq x_n \text{ for all } m > n \Big\}$$

of those indices, whose corresponding value is never exceeded later in the sequence. We consider two cases: the set J is either finite or infinite.

Suppose first that the set J is infinite. In this case, by enumerating the elements of J in an increasing order

$$J = \{n_1, n_2, n_3, \ldots\} \qquad \text{with} \qquad n_1 < n_2 < n_3 < \cdots,$$

we obtain indices such that the corresponding subsequence

$$(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$$

is decreasing: indeed we have $x_{n_{k+1}} \leq x_{n_k}$ for all $k \in \mathbb{N}$ because $n_k \in J$ and $n_{k+1} > n_k$. In this case we have thus found a monotone (decreasing) subsequence.

Suppose then that the set J is finite. In this case there exists[4] some $n_1 \in \mathbb{N}$ such that all of the elements of J are smaller than n_1. By the definition of the set J, this implies that starting from the index n_1, every member of the sequence is later exceeded; or more formally

$$\forall\, n \geq n_1 : \quad \exists\, m > n : \quad x_m > x_n.$$

We can use this property to recursively find members which form a good subsequence. When the first k indices $n_1 < \cdots < n_k$ have been chosen, the property above implies that there exists some $n_{k+1} > n_k$ such that $x_{n_{k+1}} > x_{n_k}$, which we will use as the $k+1$:st index. The resulting subsequence $(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ will be strictly increasing by construction: $x_{n_1} < x_{n_2} < x_{n_3} < \cdots$. In this case, too, we have found a monotone (increasing) subsequence. The proof is thus complete. □

Bounded sequences have convergent subsequences

The following result is crucially important in real analysis. We will see some of its familiar but powerful consequences already in this lecture.

Theorem III.6 (Bounded real number sequences have convergent subsequences).

Let $(x_n)_{n\in\mathbb{N}}$ be a real number sequence which is bounded, i.e., for some $r \in \mathbb{R}$ we have $|x_n| \leq r$ for all $n \in \mathbb{N}$. Then there exists a subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which is convergent.

[4] If $J \neq \emptyset$, then $n_1 = \max J + 1$ works, and if $J = \emptyset$ then $n_1 = 1$ works.

Proof. By Lemma III.5, the sequence $(x_n)_{n\in\mathbb{N}}$ has a monotone subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$. But this subsequence is clearly also bounded (by the same constant r as the original sequence). As a bounded monotone sequence of real numbers, $(x_{\varphi(n)})_{n\in\mathbb{N}}$ converges by the completeness axiom — formulation (C2) or (C2'). We conclude that $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. □

III.2. Functions of real variable

We will now consider real-valued functions of a real variable. Being a function of a real variable means that we take the domain of the function to be a subset $A \subset \mathbb{R}$ of the real axis. Real-valuedness means that we (can) take the codomain of the functions to be $\mathbb{R}$. Therefore we are interested in functions

$$f\colon A \to \mathbb{R} \qquad \text{where } A \subset \mathbb{R}.$$

Important special cases of the choice of the domain include, e.g., closed intervals $A = [a, b] \subset \mathbb{R}$ and the whole real axis $A = \mathbb{R}$.

Continuity of a function of real variable

The intuitive idea of continuous functions should be familiar from calculus courses. As a precise definition, we take the following.[5]

Definition III.7 (Continuity of a real-valued function of a real variable)**.**
Let $A \subset \mathbb{R}$, and let $f\colon A \to \mathbb{R}$ be a function. We say that f is **continuous** if the following holds: whenever $(a_n)_{n\in\mathbb{N}}$ is a sequence in A which is convergent and its limit is in the domain, $\lim_{n\to\infty} a_n \in A$, then we have

$$\lim_{n\to\infty} f(a_n) = f\Big(\lim_{n\to\infty} a_n\Big).$$

To elaborate — given a sequence $(a_n)_{n\in\mathbb{N}}$ in the domain A of the function $f\colon A \to \mathbb{R}$, we map the members $a_n \in A$ of the sequence to real numbers $f(a_n) \in \mathbb{R}$, and form a real number sequence $\big(f(a_n)\big)_{n\in\mathbb{N}}$ of these images under f, i.e., of the corresponding function values. Continuity of f (Definition III.7) is the requirement that if the original sequence is convergent and its limit is in the domain A, then also the image sequence is convergent and its limit is the image of the limit of the original sequence. In somewhat imprecise terms, "we are allowed to interchange the order of (i) taking the limit and (ii) applying the function".[6]

Certainly a function which abruptly jumps from one value to another should be the simplest example of a function that is not continuous; see Figure III.1. In the next exercise you check the discontinuity of such a step function directly from the definition.

[5] Later, in Chapter VI, we define continuity in a more general context. The general definition we take will be shown to be equivalent to the definition used here, in the special case of real-valued functions of a real variable.

[6] Note, however, that we still have to assume convergence of the original sequence to a limit in the domain.

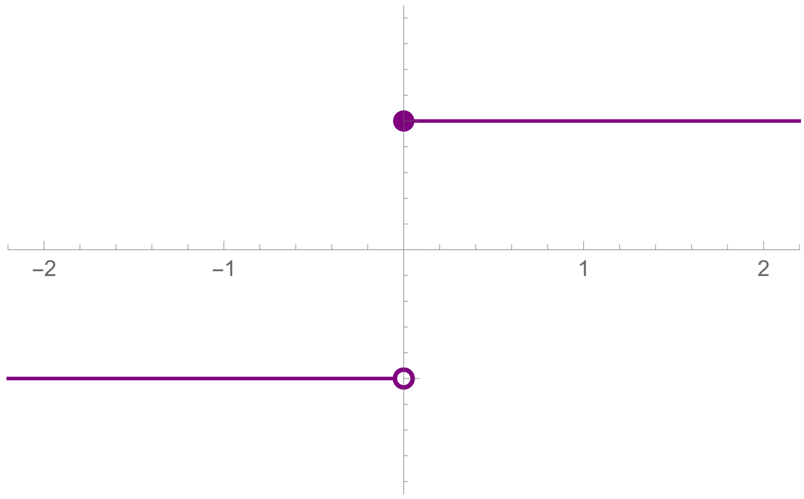


FIGURE III.1. The step function of Exercise III.2 is a basic example of a function that is not continuous.

Exercise III.2 (Step function is discontinuous)**.**
Consider the function

$$f\colon\ \mathbb{R} \to \mathbb{R}, \qquad f(x) := \begin{cases} -1 & \text{for } x < 0 \\ +1 & \text{for } x \geq 0. \end{cases}$$

Show that it is not a continuous function $\mathbb{R} \to \mathbb{R}$.
Hint: *By Definition III.7 it suffices to find* some *convergent sequence* $(x_n)_{n\in\mathbb{N}}$ *on* $\mathbb{R}$ *such that* $\lim_n f(x_n) \neq f\big(\lim_n x_n\big)$.

Example III.8 (Another discontinuous function)**.**
Consider the function

$$f\colon\ [0,+\infty) \to \mathbb{R}, \qquad f(x) := \begin{cases} -1 & \text{for } x = 0 \\ \cos(\pi + 2\pi/x) & \text{for } x > 0. \end{cases}$$

Is this function continuous or not? (For a plot, see Figure III.2.)

Intuitively it seems that any potential problem should occur at $x = 0$, if at all. So let us try out some sequences $(x_n)_{n\in\mathbb{N}}$ which tend to 0.

First try the sequence with members $x_n = \frac{1}{n}$. This sequence has limit

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

The values of f at members of the sequence are

$$f(x_n) = \cos\Big(\pi + \frac{2\pi}{1/n}\Big) = \cos(\pi + 2\pi n) = -1 \qquad \text{for } n \in \mathbb{N}.$$

Therefore we find

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (-1) = -1 = f(0) = f\big(\lim_{n\to\infty} x_n\big).$$

This is as required in the definition of continuity. (Note that $x_n = \frac{1}{n} \in [0,+\infty)$ for all $n \in \mathbb{N}$, so this is a sequence in the domain of definition of f, and also the limit $\lim_n x_n = 0$ remains in the domain, $0 \in [0,+\infty)$.)

If you were to try for example the sequence with members $x_n = 2^{-n}$, which also tends to zero, you would again find $\lim f(x_n) = f\big(\lim x_n\big)$.

But Definition III.7 requires a similar conclusion for *all* sequences (which are convergent in the domain of f). So a single counterexample will be sufficient to show discontinuity! For

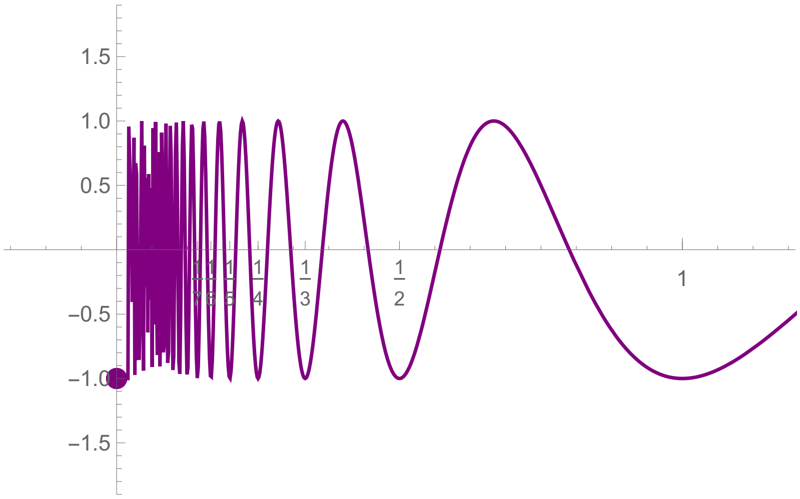


FIGURE III.2. The function of Example III.8.

such a counterexample, take the sequence with members $x_n = \frac{4}{4n-1}$. One again notes that $x_n \in [0, +\infty)$ for all $n \in \mathbb{N}$, and $\lim x_n = 0$. Now the function values along this sequence are

$$f(x_n) = \cos\Big(\pi + \frac{2\pi}{4/(4n-1)}\Big) = \cos\Big(\pi + \frac{\pi}{2}(4n-1)\Big) = \cos\Big(\frac{\pi}{2} + 2\pi n\Big) = 0$$

for $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 \neq -1 = f(0) = f\big(\lim_{n\to\infty} x_n\big).$$

This shows that f is not continuous.

Exercise III.3 (Indicator function of rational numbers)**.**
Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad f(x) := \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Is it continuous?

Note that continuity depends on the domain of definition of the function: a discontinuous function restricted to a smaller set can become continuous, as you will see in the next example.

Exercise III.4 (The restriction of a discontinuous function can be continuous)**.**
Consider the step funtion $f\colon \mathbb{R} \to \mathbb{R}$ of Exercise III.2, and let $\widetilde{f} = f|_{\mathbb{R}\setminus\{0\}}$ be its restriction to the set $\mathbb{R} \setminus \{0\}$ non-zero real numbers. Show that $\widetilde{f}\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous.

In the converse direction things are stable: a function never loses its continuity when restricting the domain.

Exercise III.5 (Restrictions of continuous functions are continuous)**.**
Suppose $f\colon A \to \mathbb{R}$ is a continuous function and $\widetilde{A} \subset A$ is a subset. Show that the restriction $\widetilde{f} = f|_{\widetilde{A}}$ is a continuous function $\widetilde{f}\colon \widetilde{A} \to \mathbb{R}$.

Operations on continuous functions

Because the definition of continuity relies on limits of real number sequences, our earlier results about limits easily yield some fundamental results about continuous functions. Particularly important is the fact that the following pointwise operations preserve continuity.

Definition III.9 (Pointwise operations on real-valued functions of real variable)**.**
Let $A \subset \mathbb{R}$, and let $f, g \colon A \to \mathbb{R}$ be two real-valued functions on A.

The **pointwise sum** of f and g is the function

$$f + g\colon\ A \to \mathbb{R} \qquad \big(f+g\big)(x) := f(x) + g(x) \quad \text{for } x \in A.$$

The **pointwise product** of f and g is the function

$$fg\colon\ A \to \mathbb{R} \qquad \big(fg\big)(x) := f(x)\,g(x) \quad \text{for } x \in A.$$

The **pointwise quotient** of f and g is the function

$$f/g\colon\ A' \to \mathbb{R} \qquad \big(f/g\big)(x) := \frac{f(x)}{g(x)} \quad \text{for } x \in A',$$

where $A' := \big\{x \in A \;\big|\; g(x) \neq 0\big\}$.

Note that the pointwise quotient f/g is only defined on the subset $A' \subset A$ of the domain, where the function g is non-vanishing — in order to avoid ill-defined division by zero.

Scalar multiplication of a function $f\colon A \to \mathbb{R}$ is a special case of the pointwise product: it is natural to interpret a real number $c \in \mathbb{R}$ also as the constant function $x \mapsto c$ on A, so $cf\colon A \to \mathbb{R}$ is defined by $(cf)(x) := c\,f(x)$ for $x \in A$.

Proposition III.10 (Continuity preserving operations)**.**
Let $A \subset \mathbb{R}$, and let $f, g\colon A \to \mathbb{R}$ be two continuous real-valued functions on A. Then also

(i) *the pointwise sum function $f + g\colon A \to \mathbb{R}$ is continuous,*
(ii) *the pointwise product function $fg\colon A \to \mathbb{R}$ is continuous,*
(iii) *the pointwise quotient function $f/g\colon A' \to \mathbb{R}$ is continuous on the subset $A' = \big\{x \in A \;\big|\; g(x) \neq 0\big\}$.*

Proof. The proofs of all cases are essentially similar, so let us provide the details only for (ii), and leave (i) and (iii) as exercises.

proof of (ii): Assume $f, g\colon A \to \mathbb{R}$ are continuous. We must prove that $fg\colon A \to \mathbb{R}$ is continuous, and we will do this directly using Definition III.7 of continuity. So suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $x_n \in A$ for all $n \in \mathbb{N}$, and $(x_n)_{n\in\mathbb{N}}$ converges to a limit in A. Let us denote the limit by $x = \lim_{n\to\infty} x_n \in A$. By definition of continuity, we must consider the sequence $\big((fg)(x_n)\big)_{n\in\mathbb{N}}$ of values of the pointwise product function fg and show that it converges to the limit $(fg)(x)$. By definition of pointwise products, the values are $(fg)(x_n) = f(x_n)\,g(x_n)$. By assumptions of continuity of f and g we know that $\lim_{n\to\infty} f(x_n) = f(x)$ and $\lim_{n\to\infty} g(x_n) = g(x)$, so by (II.9) of Theorem II.19 we have

$$\lim_{n\to\infty}\Big(f(x_n)\,g(x_n)\Big) = \Big(\lim_{n\to\infty} f(x_n)\Big)\Big(\lim_{n\to\infty} g(x_n)\Big) = f(x)\,g(x).$$

By definition of the pointwise products the right hand side above is $(fg)(x)$, so we have shown

$$\lim_{n\to\infty}\Big((fg)(x_n)\Big) = (fg)(x),$$

which is what was needed to prove continuity. □

Exercise III.6 (Proof of Proposition III.10 (i) and (iii)).
Prove parts (i) and (iii) of the above proposition.

Exercise III.7 (Continuity of the tangent trigonometric function).
Is the function

$$\tan\colon \mathbb{R}\setminus\Big\{\frac{\pi}{2}+n\pi\;\Big|\;n\in\mathbb{Z}\Big\}\to\mathbb{R}$$

continuous?
Hint: *The continuity of the trigonometric functions* $\sin\colon\mathbb{R}\to\mathbb{R}$ *and* $\cos\colon\mathbb{R}\to\mathbb{R}$ *can be considered known here (a precise justification will be given in Chapter IX). Recall that* $\tan(x)=\frac{\sin(x)}{\cos(x)}$.

Corollary III.11 (Polynomials and rational functions are continuous).
Let $a_0,a_1,a_2,\ldots,a_n\in\mathbb{R}$ *and consider the polynomial function*

$$P(x) := a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \qquad \textit{for } x\in\mathbb{R}.$$

Then $P\colon\mathbb{R}\to\mathbb{R}$ *is a continuous function. Let* $a_0,a_1,a_2,\ldots,a_n\in\mathbb{R}$ *and consider the polynomial function*

Let also $b_0,b_1,b_2,\ldots,b_m\in\mathbb{R}$ *and*

$$Q(x) := b_0 + b_1x + b_2x^2 + \cdots + b_mx^m \qquad \textit{for } x\in\mathbb{R}.$$

Consider the rational function

$$R(x) = \frac{P(x)}{Q(x)} \qquad\qquad \textit{for } x\in A,$$

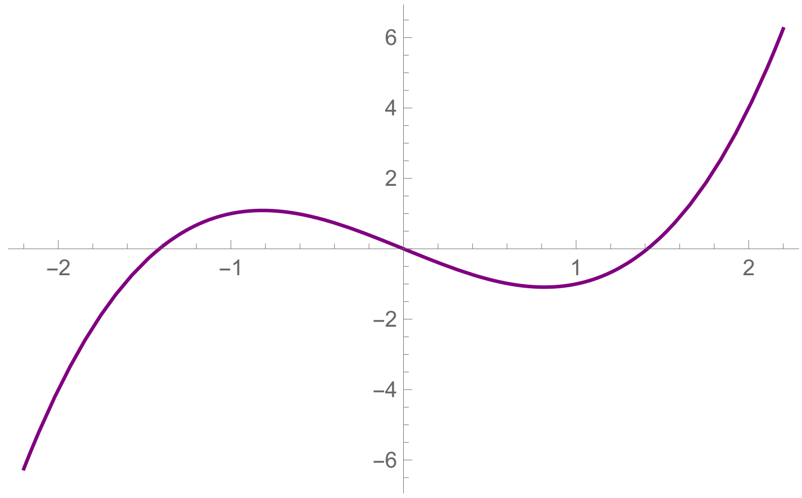
where $A=\big\{x\in\mathbb{R}\;\big|\;Q(x)\neq 0\big\}$. *Then* $R\colon A\to\mathbb{R}$ *is a continuous function.*

Proof. Let us first observe that the continuity of the identity function $x\mapsto x$ (as a function $\mathbb{R}\to\mathbb{R}$) is trivial from the definition. The function $x\mapsto x^2$ is a pointwise product of identity functions, and therefore also continuous by Proposition III.10(ii). Continuing to take pointwise products with the identity function, we find that the monomial functions $x\mapsto x^k$ are continuous for all $k\in\mathbb{N}$ (easy induction).

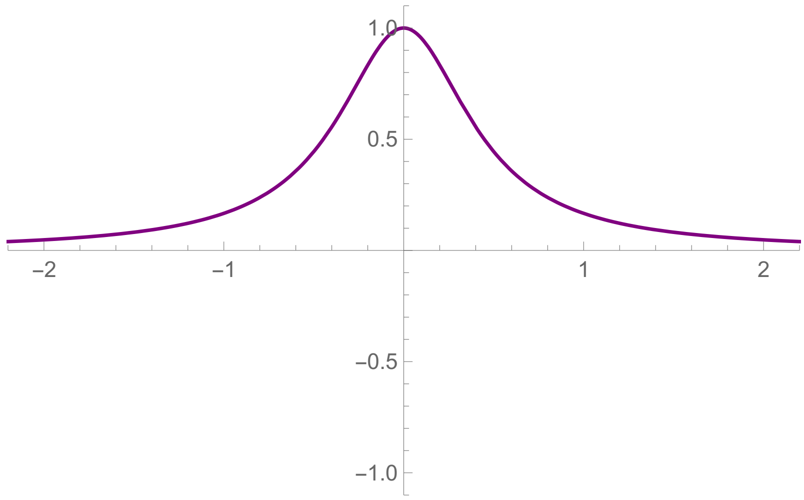
Then note that for any $c\in\mathbb{R}$ the continuity of the constant function $x\mapsto c$ (as a function $\mathbb{R}\to\mathbb{R}$) is also obvious from the definition. Taking $c=a_k\in\mathbb{R}$ and further taking pointwise product with a monomial function, we find that $x\mapsto a_kx^k$ is continuous, for any $k\in\mathbb{Z}_{\geq 0}$ (for the case $k=0$ one does not even need a product with monomial function). The polynomial $P(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ is a finite sum of terms of this type, so continuous by Proposition III.10(i), inductively applied.

Since polynomial functions given by $P(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ and $Q(x)=b_0+b_1x+b_2x^2+\cdots+b_mx^m$ are continuous, we get from Proposition III.10(iii) that the rational function given by the pointwise quotient $R(x)=\frac{P(x)}{Q(x)}$ is continuous on the subset $A=\big\{x\in\mathbb{R}\;\big|\;Q(x)\neq 0\big\}\subset\mathbb{R}$ where the denominator is non-vanishing. □

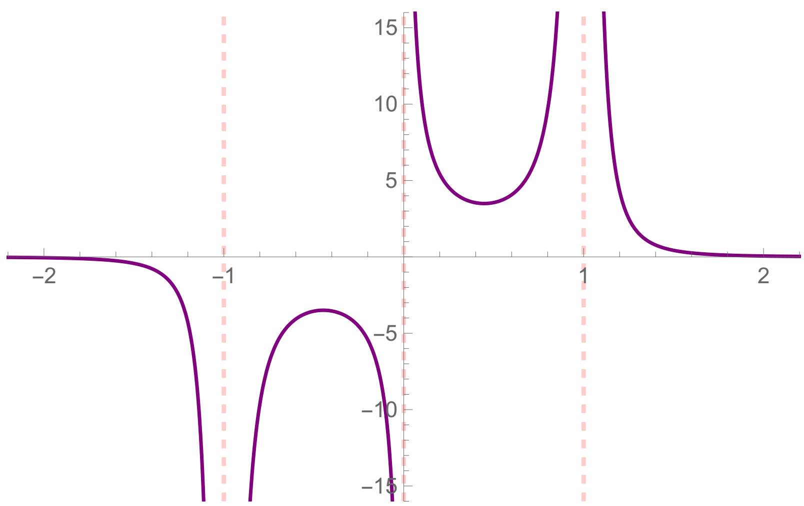
Later in the course we will develop more powerful tools for verifying the continuity of functions. But knowing the continuity of polynomial and rational functions at least assures us that some nontrivial continuous functions exist; see Figure III.3 for a few concrete examples.



(a) The polynomial $x \mapsto x^3 - 2x$ defines a function $\mathbb{R} \to \mathbb{R}$.



(b) The function $x \mapsto \frac{1}{1+5x^2}$ is a rational function function $\mathbb{R} \to \mathbb{R}$.



(c) The rational function $x \mapsto \frac{1}{x(x^2-1)^2}$ has poles at $x = 0$ and $x = \pm 1$, and defines a (continuous) function $\mathbb{R} \setminus \{-1, 0, +1\} \to \mathbb{R}$.

FIGURE III.3. Polynomials and rational functions are continuous.

III.3. Continuous functions on a closed interval

Let us now address specifically the case when the domain of the function is closed interval on the real axis, $A = [a, b] \subset \mathbb{R}$. This case appears very frequently in applications. We address some key properties of continuous functions

$$f : [a, b] \to \mathbb{R}$$

on a closed interval; later in the course we will understand them from more general perspectives.

Let us recall a few key concepts: a function $f : A \to \mathbb{R}$ is **bounded** if there exists a $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in A$. A function $f : A \to \mathbb{R}$ has a **maximum** if there exists a point $x_{\max} \in A$ such that $f(x) \leq f(x_{\max})$ for all $x \in A$, and it has a **minimum** if there exists a point $x_{\min} \in A$ such that $f(x) \geq f(x_{\min})$ for all $x \in A$.

Remark III.12 (Reinterpretation in terms of the range of the function).
Consider the image $f[A] = \{f(x) \mid x \in A\} \subset \mathbb{R}$ of the whole domain A (this is often called the range of f). It is straightforward to verify that the function f is bounded if and only if the set $f[A] \subset \mathbb{R}$ is bounded, and the function f has a maximum (resp. minimum) if and only if the set $f[A]$ has a maximum (resp. minimum).

The following important properties are related to compactness (see Chapter XI) of the closed interval.

Theorem III.13 (Extrema of continuous functions on a closed interval).
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then,

(i) *f is bounded,*
(ii) *f has a maximum and a minimum.*

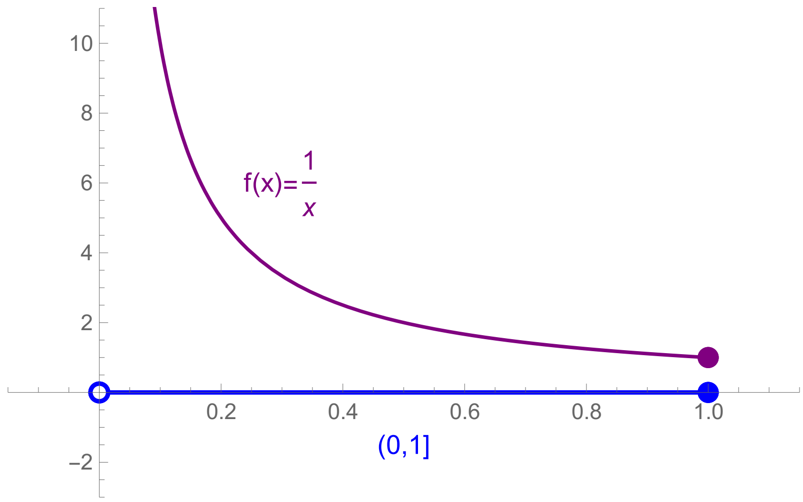
Before giving the proof, let us note that the conclusions of Theorem III.13 rely crucially on the assumption that the interval is closed. The examples in Figure III.4 illustrate what can go wrong on non-closed intervals.

Exercise III.8 (Continuous functions on non-closed intervals).

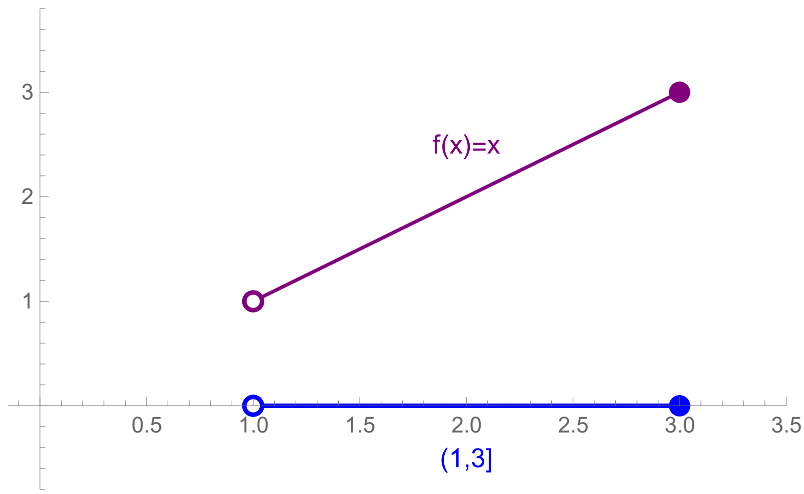
(a) Prove that the function of Figure III.4(a) is continuous but not bounded.
(b) Prove that the function of Figure III.4(b) is continuous but has no minimum.
(c) Find an example of a continuous function on an interval, which has no maximum.

Proof of Theorem III.13 Let us prove the two assertions separately.

proof of (i): Suppose, by way of contradiction, that f is not bounded. Then for every $n \in \mathbb{N}$, it is possible to choose some $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. Let us form a sequence $(x_n)_{n \in \mathbb{N}}$ using such choices. This sequence is bounded, since $a \leq x_n \leq b$ for all $n \in \mathbb{N}$. Therefore, by Theorem III.6, it has a convergent subsequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$. If we denote the limit of such a convergent subsequence by $x = \lim_{n \to \infty} x_{\varphi(n)}$, then by the preservation of bounds we have $a \leq x \leq b$ (see Corollary II.15). By continuity, then, $\big(f(x_{\varphi(n)})\big)_{n \in \mathbb{N}}$ converges to $f(x)$. But by the choice of x_n we have $|f(x_{\varphi(n)})| \geq \varphi(n) \geq n$ for all $n \in \mathbb{N}$, which means that the sequence $\big(f(x_{\varphi(n)})\big)_{n \in \mathbb{N}}$ is not bounded and so cannot be convergent (see Proposition II.13). This is a contradiction. We conclude that f had to be bounded.



(a) The function $f\colon (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$ is continuous (a rational function) but not bounded.



(b) The function $f\colon (1,3] \to \mathbb{R}$ given by $f(x) = x$ is continuous (a polynomial function) but has no minimum.

FIGURE III.4. Continuous functions on general intervals do not need to be bounded and do not need to have minima and maxima.

proof of (ii): Let us only prove that f has a maximum — the existence of minimum can be concluded similarly (or by considering the maximum of the continuous function $-f$).

By (i) f is bounded, so the supremum of its values is finite,

$$C \;:=\; \sup \big\{ f(x) \;\big|\; x \in [a,b] \big\} \;\in\; \mathbb{R}.$$

For all $n \in \mathbb{N}$ there then exists some $x_n \in [a,b]$ such that $f(x_n) \geq C - \frac{1}{n}$. Let us form a sequence $(x_n)_{n\in\mathbb{N}}$ using such choices. As in part (i), this sequence is bounded and therefore has a convergent subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ whose limit $x = \lim_{n\to\infty} x_{\varphi(n)}$ is also on the interval $[a,b]$. By the choice of C (supremum of the values) and of x_n we have

$$C \;\geq\; f(x_{\varphi(n)}) \;\geq\; C - \frac{1}{\varphi(n)} \;\geq C - \frac{1}{n}.$$

The squeeze theorem (Lemma II.17) thus gives $\lim_{n\to\infty} f(x_{\varphi(n)}) = C$. On the other hand, by continuity we have

$$f(x) = f\big(\lim_{n\to\infty} x_{\varphi(n)}\big) = \lim_{n\to\infty} f(x_{\varphi(n)}) = C.$$

We conclude that

$$f(x) = \sup\big\{f(x) \bigm| x \in [a,b]\big\},$$

which implies that the maximum of $f\colon [a,b] \to \mathbb{R}$ is attained at $x \in [a,b]$. □

The following property is related to connectedness (see Chapter XII) of the closed interval.

Theorem III.14 (Intermediate value theorem / Bolzano's theorem)**.**
Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Assume that $f(a)f(b) < 0$.[7] *Then there exists a point $z \in (a,b)$ such that $f(z) = 0$.*

Proof. We may assume that $f(a) < 0$ and $f(b) > 0$; the other case is similar (or is obtained from this one by considering the continuous function $-f$ with the same zeroes as f).

Consider the subset

$$N = \Big\{x \in [a,b] \Bigm| f(x) < 0\Big\}$$

of the interval $[a,b]$ where the function f is negative. Since $f(a) < 0$, we have at least $a \in N$, so this subset is non-empty, $N \neq \emptyset$. The subset is also bounded from above, because the right endpoint b of the interval is an upper bound for N. So let

$$z = \sup N.$$

The earlier observations yield $z \geq a$ and $z \leq b$, so we have $z \in [a,b]$. By a characterization of the supremum (Lemma II.29), there exists a sequence $(x_n^-)_{n\in\mathbb{N}}$ in the set N such that $\lim_{n\to\infty} x_n^- = z$. By continuity of f and by preservation of inequalities, we have

$$f(z) = f\big(\lim_{n\to\infty} x_n^-\big) = \lim_{n\to\infty} \underbrace{f(x_n^-)}_{<0} \leq 0.$$

In particular we have $z \neq b$, since $f(b) > 0$.

Then consider (for example) the sequence $(x_n^+)_{n\in\mathbb{N}}$ with $x_n^+ = z + \frac{1}{n}$, which also has the property $\lim_{n\to\infty} x_n^+ = z$ and $f(x_n^+) \geq 0$ since $x_n^+ \notin N$ (note that for large enough n we have $x_n^+ = z + \frac{1}{n} \leq b$, so the members of the sequence are in the domain $[a,b]$ of f). By continuity of f and by preservation of bounds, we have

$$f(z) = f\big(\lim_{n\to\infty} x_n^+\big) = \lim_{n\to\infty} \underbrace{f(x_n^+)}_{\geq 0} \geq 0.$$

In particular we have $z \neq a$, since $f(a) < 0$.

Therefore we have $z \in (a,b)$, and the above two inequalities combined yield $f(z) = 0$. □

Corollary III.15 (Continuous images of intervals)**.**
If $I \subset \mathbb{R}$ is an interval and $f\colon I \to \mathbb{R}$ is continuous, then the image $f[I]$ is also an interval.

Proof. Let $f\colon I \to \mathbb{R}$ be a continuous function defined on an interval $I \subset \mathbb{R}$. To show that the image $f[I]$ is an interval, it suffices to show that for any two values $\alpha, \beta \in f[I]$ in the image with $\alpha < \beta$, any value $\gamma \in (\alpha, \beta)$ in between them also belongs to the image, $\gamma \in f[I]$.

So suppose $\alpha, \beta \in f[I]$ with $\alpha < \beta$, and let $\gamma \in (\alpha,\beta)$. By definition of images, there exists some $a, b \in I$ such that $f(a) = \alpha$ and $f(b) = \beta$. We have either $a < b$ or $a > b$. Both

[7] This is just a concise way of saying that $f(a)$ and $f(b)$ are non-zero and have opposite signs: either $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$.

cases are handled similarly, so let us only present the details in the case $a < b$. Then by Exercise III.5 and Proposition III.10, the rule $x \mapsto f(x) - \gamma$ defines a continuous function on the closed subinterval $[a, b] \subset I$. Moreover, its values at the endpoints of $[a, b]$ have opposite signs: $f(a) - \gamma = \alpha - \gamma < 0$ and $f(b) - \gamma = \beta - \gamma > 0$. By Theorem III.14, then, there exists a point $z \in (a, b) \subset I$ such that $f(z) - \gamma = 0$, i.e., $f(z) = \gamma$. This shows $\gamma \in f[I]$. □

Example III.16 (The image of an interval under the sine function).
Consider the sine function

$$f_1 \colon (0, \pi) \to \mathbb{R} \qquad\qquad f_1(x) = \sin(x) \quad \text{for } x \in (0, \pi).$$

This function is continuous.[8] The domain of definition is the open interval $(0, \pi)$, and its image under f_1 is the half-open interval $f_1\big[(0, \pi)\big] = (0, 1]$.

If instead we considered the sine function on the longer interval

$$f_2 \colon (0, 2\pi) \to \mathbb{R} \qquad\qquad f_2(x) = \sin(x) \quad \text{for } x \in (0, 2\pi),$$

then the image would be the closed interval $f_2\big[(0, 2\pi)\big] = [-1, 1]$.

The examples above show in particular that under a continuous function, the image of an open interval can be a closed or a half-open interval. Of course the image of an open interval can also be an open interval.[9]

However, by the next exercise the image of a closed interval under a continuous function is necessarily always a closed interval![10]

Exercise III.9 (The continuous image of a closed interval is a closed interval).
Using the combination of Corollary III.15 and Theorem III.13, prove that if $f \colon [a, b] \to \mathbb{R}$ is continuous, then $f\big[[a, b]\big] \subset \mathbb{R}$ is a closed interval.

The following property is often used in non-degeneracy statements of formulas given by integrals; especially certain function space norms in the next chapter.

Exercise III.10 (Integrals of non-zero nonnegative functions are positive).
Let $[a, b]$ be a closed interval which is nondegenerate, $a < b$. Let $f \colon [a, b] \to \mathbb{R}$ be a continuous function.

(a) Show that if there exists an $x_0 \in [a, b]$ such that $f(x_0) > 0$, then there exists some $\delta > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$.
Hint: *Argue by contradiction: if this property is not true, you can construct a sequence which violates the assumed continuity of f.*

(b) Improve the conclusion of part (a) as follows. Show that if there exists an $x_0 \in [a, b]$ such that $f(x_0) > 0$, then there exists some $c > 0$ and some $\delta' > 0$ such that $f(x) > c$ for all $x \in (x_0 - \delta', x_0 + \delta') \cap [a, b]$.
Hint: *You can let $c = \frac{1}{2} f(x_0)$, and apply part (a) to the function defined by $\tilde{f}(x) = f(x) - c$.*

(c) Show that if f is non-negative, i.e., $f(x) \geq 0$ for all $x \in [a, b]$, and is not the constant function 0, then its integral is positive

$$\int_a^b f(x)\, \mathrm{d}x \; > \; 0. \tag{III.1}$$

[8]The appropriate tools for the precise justification of the continuity of trigonometric functions are developed in Chapter IX; for now we just accept the continuity statement.

[9]Consider for example the identity function $\mathrm{id}_{(a,b)} \colon (a, b) \to (a, b)$.

[10]The attentive reader may worry that constant functions are continuous and their images are singletons. This is indeed correct, but it does not contradict the assertion — a singleton is a special type of a closed interval, corresponding to the degenerate case when the interval endpoints coincide. Namely, when $a = b$, we have $[a, b] := \big\{x \in \mathbb{R} \,\big|\, a \leq x \leq b\big\} = \{a\}$.

Hint: *Use part (b) and a lower bound for the integral based on what you know of the values of f.*

Lecture IV

Normed spaces and inner product spaces

In this section we start to study more general spaces than just the familiar real axis $\mathbb{R}$. We will consider vector spaces equipped with some structure that in particular suffices to induce a topology. Specifically, we consider vector spaces equipped with an inner product or at least a norm.

The familiar d-dimensional Euclidean space $\mathbb{R}^d$ has an inner product and a norm induced by it, but even this finite-dimensional vector space $\mathbb{R}^d$ could be equipped with some slightly different norms — and for some applications it is meaningful to do so. There are also rather straightforward infinite-dimensional counterparts of these finite-dimensional normed spaces. Perhaps the most important applications of the structures introduced here are, however, various spaces of functions. We will already introduce a few function space examples that we keep returning to throughout the rest of this course, and that you will be very likely to encounter in whatever field of mathematics you end up using in the future. But the number of relevant examples of such function spaces is so vast that we are in fact only scratching the surface of a deep topic: in courses on Hilbert spaces, Banach spaces, Measure theory, Probability theory, Partial differential equations, etc. you will encounter many more. The basic structures introduced here will nevertheless be present in almost all of those![1]

IV.1. Vector spaces

The topic of vector spaces by itself belongs to algebra or linear algebra. But because vector spaces with some additional structure feature so crucially in analysis as well, we review the notion here. However, for concreteness, we only define *real vector spaces*, i.e., vector spaces over the field $\mathbb{R}$ of real numbers. Algebraists would typically study vector spaces over arbitrary fields.[2] The term **vector** will be used for elements of the vector space, and the term **scalar** for elements of the field over which the vector space is defined — in our case for real numbers.

[1] If one deliberately tries to find vector spaces in analysis that are *not* either inner product spaces or at least normed spaces, then among the easiest relevant examples are certain spaces of distributions. Even they have a topology, and while the topology does not come from a norm, it is not too far from that — in typical cases it comes from a collection of seminorms.

[2] Actually also *complex vector spaces*, i.e., vector spaces over the field $\mathbb{C}$ of complex numbers, appear frequently in analysis, probability, etc. — and the most important additional structures they can be equipped with are again (complex) inner products and norms. Complex inner product spaces and complex normed spaces would require only relatively minor modifications to the treatment in this chapter. We choose to focus on just the real case for the sake of clarity. We trust that when you will need to work with the complex case, you will be well prepared for that as well.

Axioms of real vector spaces

For brevity, the term vector space will from here on always mean a real vector space — defined precisely by the following axiomatic properties.

Definition IV.1 (Vector space)**.**
A **(real) vector space** is a set V equipped with two operations,

vector addition	scalar multiplication
$\mathsf{V} \times \mathsf{V} \to \mathsf{V}$	$\mathbb{R} \times \mathsf{V} \to \mathsf{V}$
$(\vec{u}, \vec{v}) \mapsto \vec{u} + \vec{v}$	$(c, \vec{v}) \mapsto c\,\vec{v}$,

such that the following properties hold:

Commutativity of vector addition:
$$\forall\, \vec{u}, \vec{v} \in \mathsf{V} : \quad \vec{u} + \vec{v} = \vec{v} + \vec{u} \tag{IV.1}$$
Associativity of vector addition:
$$\forall\, \vec{u}, \vec{v}, \vec{w} \in \mathsf{V} : \quad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \tag{IV.2}$$
Neutral element for vector addition:
$$\exists\, \vec{0} \in \mathsf{V} : \quad \forall\, \vec{v} \in \mathsf{V} : \quad \vec{v} + \vec{0} = \vec{v} \tag{IV.3}$$
Opposite vectors:
$$\forall\, \vec{v} \in \mathsf{V} : \quad \exists\, (-\vec{v}) \in \mathsf{V} : \quad \vec{v} + (-\vec{v}) = \vec{0} \tag{IV.4}$$
Compatibility of scalar multiplication:
$$\forall\, c, d \in \mathbb{R} : \quad \forall\, \vec{v} \in \mathsf{V} : \quad c\,(d\,\vec{v}) = (cd)\,\vec{v} \tag{IV.5}$$
Unit scalar multiplication:
$$\forall\, \vec{v} \in \mathsf{V} : \quad 1\,\vec{v} = \vec{v} \tag{IV.6}$$
Distributivity of vector addition:
$$\forall\, c \in \mathbb{R} : \quad \forall\, \vec{u}, \vec{v} \in \mathsf{V} : \quad c\,(\vec{u} + \vec{v}) = c\,\vec{u} + c\,\vec{v} \tag{IV.7}$$
Distributivity of scalar addition:
$$\forall\, c, d \in \mathbb{R} : \quad \forall\, \vec{v} \in \mathsf{V} : \quad (c + d)\,\vec{v} = c\,\vec{v} + d\,\vec{v}. \tag{IV.8}$$

These axioms are probably every bit as unsurprising as the field axioms of the real numbers given in Appendix B.1. But again, from such a simple starting point, a very fruitful theory emerges.

In the above definition, we used arrows on top of vectors, e.g. $\vec{u}, \vec{v}, \vec{w} \in \mathsf{V}$. In the long run, that becomes mainly an additional nuisance, so we will soon adopt the convention to label elements of the vector space without such decorations[3], e.g. by $u, v, w \in \mathsf{V}$.

First examples of real vector spaces

The most obvious example of a real vector space is the following.

[3]Also the zero vector $\vec{0}$ is commonly denoted by simply 0, and the reader is assumed to figure out from the context whether 0 stands for the scalar zero $0 \in \mathbb{R}$ or the vector zero $0 \in \mathsf{V}$ (two very different objects).

Example IV.2 (The standard d-dimensional space $\mathbb{R}^d$)**.**
The set

$$\mathbb{R}^d = \Big\{ (x_1, \ldots, x_d) \;\Big|\; x_1, \ldots, x_d \in \mathbb{R} \Big\} \tag{IV.9}$$

is a real vector space, when equipped with the coordinatewise vector addition

$$\begin{aligned}(x_1, \ldots, x_d) + (y_1, \ldots, y_d) \;&=\; (x_1 + y_1, \ldots, x_d + y_d) \\ &\qquad \text{for } (x_1, \ldots, x_d),\ (y_1, \ldots, y_d) \in \mathbb{R}^d\end{aligned}$$

and coordinatewise scalar multiplication

$$\begin{aligned}c\,(x_1, \ldots, x_d) \;&=\; (c\,x_1, \ldots, c\,x_d) \\ &\qquad \text{for } c \in \mathbb{R} \text{ and } (x_1, \ldots, x_d) \in \mathbb{R}^d\end{aligned}$$

The zero vector is $\vec{0} = (0, 0, \ldots, 0) \in \mathbb{R}^d$, and the opposite vector to $(x_1, \ldots, x_d) \in \mathbb{R}^d$ is $(-x_1, \ldots, -x_d)$. Verifying the properties (IV.1) – (IV.8) for this case is straightforward.

The previous example has a straightforward infinite-dimensional counterpart as well.

Example IV.3 (The space of real number sequences)**.**
The set[4]

$$\mathbb{R}^{\mathbb{N}} \;=\; \Big\{ (x_1, x_2, x_3, \ldots) \;\Big|\; x_1, x_2, x_3, \ldots \in \mathbb{R} \Big\} \tag{IV.10}$$

of all real number sequences is a real vector space, when equipped with the coordinatewise vector addition

$$\begin{aligned}(x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots) \;&=\; (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3, \ldots,) \\ &\qquad \text{for } (x_1, x_2, x_3, \ldots),\ (y_1, y_2, y_3, \ldots) \in \mathbb{R}^{\mathbb{N}}\end{aligned}$$

and coordinatewise scalar multiplication

$$\begin{aligned}c\,(x_1, x_2, x_3, \ldots) \;&=\; (c\,x_1,\ c\,x_2,\ c\,x_3, \ldots) \\ &\qquad \text{for } c \in \mathbb{R} \text{ and } (x_1, x_2, x_3, \ldots) \in \mathbb{R}^{\mathbb{N}}.\end{aligned}$$

The zero vector is the sequence of zeroes $\vec{0} = (0, 0, 0, \ldots)$, and the opposite vector to a sequence $(x_1, x_2, x_3, \ldots)$ is the sequence $(-x_1, -x_2, -x_3, \ldots)$. Verifying that the properties (IV.1) – (IV.8) is again straightforward.

Example IV.4 (Space of polynomials)**.**
Consider the space

$$\mathbb{R}[x] \;=\; \left\{ \sum_{k=0}^{d} c_k x^k \;\middle|\; d \in \mathbb{N}_0,\ c_0, c_1, \ldots, c_d \in \mathbb{R} \right\} \tag{IV.11}$$

of all polynomials in variable x, with real coefficients. The addition and scalar multiplication are defined coefficientwise, as usual for polynomials. It is easy to verify properties (IV.1) – (IV.8), so the space $\mathbb{R}[x]$ of real polynomials in one variable is a real vector space.[5]

Example IV.5 (Spaces of real valued functions)**.**
Let X be any set. Consider the space

$$\mathbb{R}^X \;=\; \Big\{ \text{ functions } f \colon X \to \mathbb{R} \Big\} \tag{IV.12}$$

[4] If X and Y are sets, the notation Y^X is often used for the set of all functions $X \to Y$. The notation $\mathbb{R}^{\mathbb{N}}$ correspondingly stands for all functions $\mathbb{N} \to \mathbb{R}$ — which in view of the precise definition of sequences is exactly the set of all real number sequences.

[5] In fact, besides just addition and scalar multiplication of polynomials, we can also define multiplication of polynomials as usual. So the space $\mathbb{R}[x]$ has even further algebraic structure than just that of a vector space over $\mathbb{R}$ — it is what is known as an algebra over $\mathbb{R}$. We refer to courses on algebra for details.

of all functions whose domain is X and codomain is $\mathbb{R}$. We equip it with the *pointwise addition* and *pointwise scalar multiplication* as follows. If $f, g \in \mathbb{R}^X$, i.e., if $f, g \colon X \to \mathbb{R}$ are two functions, then we define the new function $f + g \colon X \to \mathbb{R}$ by the formula

$$\big(f+g\big)(x) \;=\; f(x) + g(x) \qquad \text{for } x \in X. \tag{IV.13}$$

Likewise, if $c \in \mathbb{R}$ is a scalar and $f \in \mathbb{R}^X$, i.e., $f \colon X \to \mathbb{R}$ is a function, then we define the new function $cf \colon X \to \mathbb{R}$ by the formula

$$\big(cf\big)(x) \;=\; c\, f(x) \qquad \text{for } x \in X. \tag{IV.14}$$

It is straightforward to check that the properties (IV.1) – (IV.8) hold, so the space $\mathbb{R}^X$ of real valued functions on an arbitrary set X is a real vector space.[6]

In fact, Example IV.3 is readily a special case of the above, corresponding to the choice of domain $X = \mathbb{N}$: functions $\mathbb{N} \to \mathbb{R}$ are exactly real number sequences.

Also Example IV.2 can be seen as a special case of this: $\mathbb{R}^d$ can be interpreted as the set of real-valued functions on the finite set $\{1, 2, \ldots, d\}$. A vector $(x_1, \ldots, x_d) \in \mathbb{R}$ corresponds to the function $f \colon \{1, 2, \ldots, d\} \to \mathbb{R}$ such that $f(j) = x_j$ for $j \in \{1, 2, \ldots, d\}$. This way we can identify the spaces $\mathbb{R}^d$ and $\mathbb{R}^{\{1,2,\ldots,d\}}$.

Vector subspaces

We saw a number of examples of vector spaces above, but other important examples are often constructed as vector subspaces of other vector spaces. The precise definition is the following.

Definition IV.6 (Vector subspace)**.**
Let V be a vector space. A subset $\mathsf{W} \subset \mathsf{V}$ is called a **vector subspace** of V, if the following conditions hold:

- We have $\vec{0} \in \mathsf{W}$;
- If $\vec{u}, \vec{v} \in \mathsf{W}$, then we also have $\vec{u} + \vec{v} \in \mathsf{W}$.
- If $c \in \mathbb{R}$ and $\vec{v} \in \mathsf{W}$, then we also have $c\,\vec{v} \in \mathsf{W}$.

Example IV.7 (The subspace spanned by finitely many vectors)**.**
If V is a vector space and $\vec{v}_1, \ldots, \vec{v}_n \in \mathsf{V}$ are vectors in it, then the set

$$\mathsf{W} \;=\; \left\{ c_1\,\vec{v}_1 + \cdots + c_n\,\vec{v}_n \;\middle|\; c_1, \ldots, c_n \in \mathbb{R} \right\} \tag{IV.15}$$

of all linear combinations of them satisfies the properties in Definition IV.6, i.e. it is a vector subspace $\mathsf{W} \subset \mathsf{V}$. It is called the linear span of the vectors $\vec{v}_1, \ldots, \vec{v}_n$.[7]

Function spaces are often defined as subspaces of the space of all functions with a given domain and a codomain: instead of considering the space of arbitrary functions, we usually care about sufficiently well-behaved functions. The following typical example is very important.

[6] In fact, besides just pointwise addition and scalar multiplication of real valued functions, it is possible to also define pointwise multiplication of such functions. So the space $\mathbb{R}^X$ also has even more algebraic structure than just that of a vector space over $\mathbb{R}$ — it is also an algebra over $\mathbb{R}$.

[7] More generally, if $S \subset \mathsf{V}$ is any subset, then the subset consisting of all *finite* linear combinations of elements of S is still a vector subspace. This is denoted $\mathrm{span}(S) \subset \mathsf{V}$, and called the linear span of the subset $S \subset \mathsf{V}$ — it is the smallest vector subspace of V, which contains all vectors of S. Checking this is not difficult, but it is better suited for a course of linear algebra.

Example IV.8 (The space of continuous functions on an interval).
Let $[a, b] \subset \mathbb{R}$ be a closed interval. Consider the set

$$\mathcal{C}\big([a, b]\big) = \Big\{ f : [a, b] \to \mathbb{R} \text{ continuous function} \Big\}. \tag{IV.16}$$

This set of continuous real-valued functions on $[a, b]$ is obviously a subset of the space $\mathbb{R}^{[a,b]}$ of *all* real valued functions on $[a, b]$ from Example IV.5. Since the zero-function (constant) is continuous, and since pointwise sums and scalar multiples of continuous functions are continuous (by Proposition III.10), we see that $\mathcal{C}([a, b])$ is a vector subspace in $\mathbb{R}^{[a,b]}$.

Let us finally give a few examples of subspaces of the sequence space of Example IV.3.

Example IV.9 (The space of bounded sequences).
UNDER CONSTRUCTION!

Example IV.10 (The space of convergent sequences).
UNDER CONSTRUCTION!

Example IV.11 (The space of sequences tending to zero).
UNDER CONSTRUCTION!

IV.2. Inner product spaces

A vector space, by itself, is a purely algebraic notion. Applying the vector addition inductively, one can define finite sums of vectors, but it would be meaningless to form infinite series, for example. Infinite series, as well as for example continuity of functions, require some additional structure — in fact something that gives rise to a topology on the vector space.

Perhaps the most familiar of such additional structures is the notion of an inner product, from which a notion of length of vectors as well as a notion of angle between vectors can be obtained.[8]

Axioms of inner product spaces

Definition IV.12 (Inner product space).
Let V be a real vector space. A binary operation

$$\begin{aligned} \mathsf{V} \times \mathsf{V} &\to \mathbb{R} \\ (\vec{x}, \vec{y}) &\mapsto \langle \vec{x}, \vec{y} \rangle \end{aligned} \tag{IV.17}$$

is called an **inner product**, if it satisfies the following conditions

(IP1) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$ for all $\vec{x}, \vec{y} \in \mathsf{V}$; **(symmetricity)**
(IP2) $\langle c\,\vec{x}, \vec{y} \rangle = c\,\langle \vec{x}, \vec{y} \rangle$ for all $c \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathsf{V}$; **(linearity)**[9]

[8]Arguably the notion of angles is more geometric than topological, so inner products provide us with geometric as well as topological structure.

[9]The linearity of an inner product with respect to its first argument includes two conditions: that it respects both scalar multiplication (IP2) and vector addition (IP3). These can be expressed concisely together in the property that $\langle c_1 u_1 + c_2 u_2, v \rangle = c_1 \langle u_1, v \rangle + c_2 \langle u_2, v \rangle$ for all $u_1, u_2, v \in \mathsf{V}$ and $c_1, c_2 \in \mathbb{R}$. Moreover, if we combine this with symmetricity (IP1), we also get

(IP3) $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$ for all $\vec{x}, \vec{y}, \vec{z} \in \mathsf{V}$; (**linearity**)
(IP4) $\langle \vec{x}, \vec{x} \rangle \geq 0$ for all $\vec{x} \in \mathsf{V}$; (**positive semi-definiteness**)
(IP5) $\langle \vec{x}, \vec{x} \rangle = 0$ only if $\vec{x} = \vec{0} \in \mathsf{V}$. (**non-degeneracy**)[10]

A vector space V equipped with an inner product is called an **inner product space**.

To concisely and unambiguously mention both the vector space V and the inner product $\langle \cdot, \cdot \rangle$ on it, we may refer to the *pair* $(\mathsf{V}, \langle \cdot, \cdot \rangle)$ as an inner product space.

Examples of inner product spaces

Example IV.13 (Euclidean spaces)**.**
In the vector space $\mathbb{R}^d$ of Example IV.2, the formula

$$\langle \vec{x}, \vec{y} \rangle \;=\; \sum_{j=1}^{d} x_j y_j \qquad \text{for } \; \vec{x} = (x_1, \ldots, x_d), \; \vec{y} = (x_1, \ldots, y_d) \in \mathbb{R}^d \tag{IV.18}$$

defines the usual inner product — verifying the properties is straighforward. The corresponding inner product space is called the d-dimensional **Euclidean space**.

Let us verify the properties (IP1) – (IP5) in this familiar case.

(IP1): For any $\vec{x} = (x_1, \ldots, x_d), \vec{y} = (x_1, \ldots, y_d) \in \mathbb{R}^d$ we have

$$\langle \vec{x}, \vec{y} \rangle \;=\; \sum_{j=1}^{d} x_j y_j \;=\; \sum_{j=1}^{d} y_j x_j \;=\; \langle \vec{y}, \vec{x} \rangle.$$

(IP2): For any $\vec{x} = (x_1, \ldots, x_d), \vec{y} = (x_1, \ldots, y_d) \in \mathbb{R}^d$ and $c \in \mathbb{R}$, we have first of all $c\,\vec{x} = (cx_1, \ldots, cx_d)$ and then

$$\langle c\,\vec{x}, \vec{y} \rangle \;=\; \sum_{j=1}^{d} (c\,x_j)\, y_j \;=\; c \sum_{j=1}^{d} x_j y_j \;=\; c\,\langle \vec{x}, \vec{y} \rangle.$$

(IP3): For any $\vec{x} = (x_1, \ldots, x_d), \vec{y} = (x_1, \ldots, y_d), \vec{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$, we have first of all $\vec{x} + \vec{y} = (x_1 + y_1, \ldots, x_d + y_d)$ and then

$$\begin{aligned} \langle \vec{x} + \vec{y},\; \vec{z} \rangle \;=\; \sum_{j=1}^{d} (x_j + y_j)\, z_j \;&=\; \sum_{j=1}^{d} \big(x_j z_j + y_j z_j \big) \\ &=\; \sum_{j=1}^{d} x_j z_j + \sum_{j=1}^{d} y_j z_j \;=\; \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle. \end{aligned}$$

$\langle u, d_1 v_1 + d_2 v_2 \rangle = d_1 \langle u, v_1 \rangle + d_2 \langle u, v_2 \rangle$ for all $u, v_1, v_2 \in \mathsf{V}$ and $d_1, d_2 \in \mathbb{R}$. This yields **bilinearity** of the inner product: an inner product is linear in both of its arguments.

[10] In fact we have $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$. In (IP5) we only require the "only if" implication in order to minimize what needs to be checked in applications. The other direction follows from already required properties. Namely, in any vector space we have $\vec{0} = 0\,\vec{0}$ (you can derive this from the properties of Definition IV.1 as an algebraic exercise), so using (IP2) we get that all inner products with the zero vector $\vec{0}$ vanish:

$$\langle \vec{0}, \vec{v} \rangle \;=\; \langle 0\,\vec{0}, \vec{v} \rangle \;=\; 0\,\langle \vec{0}, \vec{v} \rangle \;=\; 0.$$

In particular, the zero vector has zero inner product with itself, giving the "if" implication.

(IP4): For any $\vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have, by nonnegativity of squares of real numbers,

$$\langle \vec{x}, \vec{x} \rangle \;=\; \sum_{j=1}^{d} x_j x_j \;=\; \sum_{j=1}^{d} x_j^2 \;\geq\; 0,$$

(IP5): From the formula above, we see that $\langle \vec{x}, \vec{x} \rangle$ can only be zero for $\vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, if for each $j \in \{1, \ldots, d\}$ we have $x_j^2 = 0$. This only occurs if each component x_j is zero and the vector is $\vec{x} = (0, \ldots, 0) = \vec{0}$.

Example IV.14 (An inner product in a function space).
Consider the vector space $\mathcal{C}([a,b])$ of continuous real-valued functions on a closed interval $[a,b] \subset \mathbb{R}$, which is nondegenerate, $a < b$. Define $\langle \cdot, \cdot \rangle \colon \mathcal{C}([a,b]) \times \mathcal{C}([a,b]) \to \mathbb{R}$ by

$$\langle f, g \rangle \;=\; \int_a^b f(t) g(t)\, \mathrm{d}t \qquad \text{for } \; f, g \in \mathcal{C}\big([a,b]\big). \tag{IV.19}$$

This is called the L^2-inner product of functions. Let us verify that this is indeed an inner product on $\mathcal{C}([a,b])$. Note that the verifications are very similar to those in Example IV.13; only in (IP5) an essentially new observation is used.

(IP1): For any $f, g \in \mathcal{C}([a,b])$ we have

$$\langle f, g \rangle \;=\; \int_a^b f(t) g(t)\, \mathrm{d}t \;=\; \int_a^b g(t) f(t)\, \mathrm{d}t \;=\; \langle g, f \rangle.$$

(IP2): For any $f, g \in \mathcal{C}([a,b])$ and $c \in \mathbb{R}$, recalling the definition $(cf)(t) = c\, f(t)$ of pointwise scalar multiplication of functions, we get

$$\langle c\, f, g \rangle \;=\; \int_a^b \big(cf\big)(t)\; g(t)\; \mathrm{d}t \;=\; \int_a^b c\, f(t)\, g(t)\; \mathrm{d}t \;=\; c \int_a^b f(t) g(t)\, \mathrm{d}t \;=\; c\, \langle f, g \rangle.$$

(IP3): For any $f, g, h \in \mathcal{C}([a,b])$, recalling the definition $(f+g)(t) = f(t) + g(t)$ of pointwise addition of functions, we get

$$\begin{aligned}
\langle f + g, h \rangle \;&=\; \int_a^b \big(f + g\big)(t)\; h(t)\; \mathrm{d}t \\
&=\; \int_a^b \Big(f(t) + g(t)\Big)\; h(t)\; \mathrm{d}t \\
&=\; \int_a^b f(t) h(t)\, \mathrm{d}t + \int_a^b g(t) h(t)\, \mathrm{d}t \;=\; \langle f, h \rangle + \langle g, h \rangle.
\end{aligned}$$

(IP4): For any $f \in \mathcal{C}([a,b])$, we have

$$\langle f, f \rangle \;=\; \int_a^b \underbrace{f(t) f(t)}_{= f(t)^2 \geq 0}\; \mathrm{d}t \;\geq\; 0.$$

(IP5): From the formula above, we see that $\langle f, f \rangle$ can only be zero for $f \in \mathcal{C}([a,b])$ if $\int_a^b f(t)^2\, \mathrm{d}t = 0$. Since $t \mapsto f(t)^2$ is a continuous nonnegative function on $[a,b]$, by the contrapositive of Exercise III.10, the vanishing of this integral implies that $f(t)^2 = 0$ for all $t \in [a,b]$. But this implies that $f(t) = 0$ for all $t \in [a,b]$, i.e., that f is the zero function.

The combination of the following two exercises amount to a complete characterization of inner products in finite-dimensional vector spaces.

Exercise IV.1 (Constructing inner products on $\mathbb{R}^d$).

Let $A \in \mathbb{R}^{d\times d}$ be a symmetric[11] positive definite[12] $d \times d$ matrix. Show that the formula $\langle x, y\rangle = x^\top A\, y$ for $x, y \in \mathbb{R}^d$ defines an inner product on $\mathbb{R}^d$.

Exercise IV.2 (Characterizing inner products on $\mathbb{R}^d$).

Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathbb{R}^d$ (not necessarily the standard Euclidean inner product, but any inner product that satisfies the requirements in Definition IV.12). Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the standard basis vectors of $\mathbb{R}^d$. For all $i, j \in \{1, \ldots, d\}$, define $a_{i,j} = \langle \mathbf{e}_i, \mathbf{e}_j\rangle$, and let $A \in \mathbb{R}^{d\times d}$ denote the matrix with these entries $a_{i,j}$.

(a) Show that with this definition, the inner product of any two vectors $x, y \in \mathbb{R}^d$ can be written as $\langle x, y\rangle = x^\top A\, y$.
(b) Prove that the matrix A is symmetric and positive definite.

The norm induced by an inner product

An inner product in particular yields a notion of lenght of vectors, i.e., a norm. In an inner product space V, we define the **norm** $\|v\|$ of $v \in \mathsf{V}$ as

$$\|v\| \;=\; \sqrt{\langle v, v\rangle}. \tag{IV.20}$$

Example IV.15 (The Euclidean norm).

The d-dimensional Euclidean space is the vector space $\mathbb{R}^d$ equipped with the inner product of Example IV.13. The norm of $\vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ in this space takes the form

$$\|\vec{x}\| \;=\; \sqrt{\langle \vec{x}, \vec{x}\rangle} \;=\; \sqrt{x_1^2 + \cdots + x_d^2}\,. \tag{IV.21}$$

Example IV.16 (The L^2-norm on the space of continuous functions on a closed interval).

Recall from Example IV.14 that the space $\mathcal{C}([a, b])$ of continuous real-valued functions on a nondegenerate closed interval $[a, b] \subset \mathbb{R}$ is an inner product space with inner product (IV.19). We denote the norm induced by it by $\|\cdot\|_2$. Explicitly for a continuous function $f\colon [a, b] \to \mathbb{R}$, this norm reads

$$\|f\|_2 \;=\; \sqrt{\langle f, f\rangle} \;=\; \left(\int_a^b f(t)^2 \,\mathrm{d}t\right)^{1/2}.$$

Note, however, that we will soon give a more general meaning to the term *norm* — formula (IV.20) is a special instance, which is only meaningful in inner product spaces (indeed, the right hand side features the inner product explicitly). To emphasize that (IV.20) is not just any norm, we also call it *the norm induced by the inner product* $\langle\cdot,\cdot\rangle$.

For perspective, we mention already that the norm, by giving the lengths $\|v\|$ of vectors $v \in \mathsf{V}$, can be used also to define distances $\|u - v\|$ between vectors $u, v \in \mathsf{V}$. It thus equips the space V with topological structure. We return to this in more detail in Lecture V.

A crucial property of inner products and the norms induced by them is the following inequality.

---

[11] A square matrix A is symmetric if it is its own transpose, $A^\top = A$.

[12] A square matrix A is said to be **positive semi-definite** if $x^\top A\, x \geq 0$ for all vectors x, and **positive definite** if $x^\top A\, x > 0$ for all nonzero vectors x.

Theorem IV.17 (Cauchy-Schwarz inequality in inner product spaces)**.**
Let V *be an inner product space. Then for any* $u, v \in \mathsf{V}$ *we have*

$$\big|\langle u, v\rangle\big| \;\leq\; \|u\|\,\|v\|. \tag{IV.22}$$

Proof. If either $u = 0$ or $v = 0$, then both sides of (IV.22) are zero and the inequality holds trivially. We may therefore assume that $u, v \in \mathsf{V} \setminus \{0\}$.

Observe that for any $t \in \mathbb{R}$, by (IP4) for the vector $u + t\,v \in \mathsf{V}$ we have $0 \leq \langle u + t\,v,\ u + t\,v\rangle$. Using bilinearity of the inner product, we can write this as

$$\begin{aligned} 0 &\leq \langle u + t\,v,\ u + t\,v\rangle \\ &= \langle u, u\rangle + \langle u, t\,v\rangle + \langle t\,v, u\rangle + \langle t\,v, t\,v\rangle \\ &= \|u\|^2 + 2t\,\langle u, v\rangle + t^2\,\|v\|^2. \end{aligned}$$

This last expression is a polynomial $at^2 + bt + c$, with coefficients $a = \|v\|^2$, $b = 2\,\langle u, v\rangle$, and $c = \|u\|^2$. The fact that polynomial is non-negative for all $t \in \mathbb{R}$ means that it has either no real roots or just one real root, so its discriminant is non-positive, $b^2 - 4ac \leq 0$. By plugging in the coefficients a, b, c, this reads

$$4\big(\langle u, v\rangle\big)^2 - 4\,\|u\|^2\,\|v\|^2 \;\leq\; 0.$$

Dividing by 4 and rearranging we get $\big(\langle u, v\rangle\big)^2 \leq \|u\|^2\,\|v\|^2$. Then by taking square roots we get (IV.22). □

Exercise IV.3 (When does equality occur in the Cauchy-Schwarz inequality?)**.**

(a) Show that if $u, v \in \mathsf{V}$ are such that $u = c\,v$ for some $c \in \mathbb{R}$, then equality occurs in the Cauchy-Schwarz inequality (IV.22) for these vectors.
(b) By carefully examining the steps of the proof of Theorem IV.17, show that for two vectors $u, v \in \mathsf{V}$, equality can occur in the Cauchy-Schwarz inequality only if one of them is a real scalar multiple of the other.

The Cauchy-Schwarz inequality is a tool used in numerous circumstances, but already the following analogue of Proposition II.1 indicates its importance.

Corollary IV.18 (Triangle inequality in inner product spaces)**.**
Let V *be an inner product space. Then for any* $u, v \in \mathsf{V}$ *we have*

$$\|u + v\| \;\leq\; \|u\| + \|v\|. \tag{IV.23}$$

Proof. The proof is very similar to that of Proposition II.1. We first expand the square of $\|u + v\|$ using bilinearity, and then estimate it using (IV.22) as follows

$$\begin{aligned} \|u + v\|^2 &= \langle u + v,\ u + v\rangle && \text{(definition)} \\ &= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle && \text{(bilinearity)} \\ &= \|u\|^2 + 2\,\langle u, v\rangle + \|v\|^2 && \text{(symmetricity)} \\ &\leq \|u\|^2 + 2\,\big|\langle u, v\rangle\big| + \|v\|^2 && \text{(taking absolute value)} \\ &\leq \|u\|^2 + 2\,\|u\|\,\|v\| + \|v\|^2 && \text{(Cauchy-Schwarz)} \\ &= \big(\|u\| + \|v\|\big)^2. && \text{(binomial formula)} \end{aligned}$$

Taking the square roots, we get (IV.23). □

Exercise IV.4 (Parallelogram rule)**.**
Let V be an inner product space. Show that for any two vectors $u, v \in \mathsf{V}$ we have the following parallelogram rule

$$\|u + v\|^2 + \|u - v\|^2 \;=\; 2\,\|u\|^2 + 2\,\|v\|^2. \tag{IV.24}$$

Let us also indicate some direct applications of the Cauchy-Schwarz inequality.

Exercise IV.5 (A sum bound based on Cauchy-Schwarz inequality)**.**
Let $x_1, \ldots, x_d \in \mathbb{R}$. Prove the following inequality

$$|x_1| + \cdots + |x_d| \;\leq\; \sqrt{d}\,\sqrt{x_1^2 + \cdots + x_d^2}.$$

Hint: *Consider the Euclidean space $\mathbb{R}^d$, and apply the Cauchy-Schwarz inequality with $u = (1, 1, \ldots, 1)$ and a suitably chosen v.*

Example IV.19 (Cauchy-Schwarz inequality in a function space)**.**
Recall from Examples IV.14 and IV.16 that the space $\mathcal{C}([a, b])$ of continuous real-valued functions on a nondegenerate closed interval $[a, b] \subset \mathbb{R}$ is an inner product space. The Cauchy-Schwarz inequality in the inner product space $\mathcal{C}([a, b])$ amounts to the following integral inequality

$$\left| \int_a^b f(t)\, g(t)\, \mathrm{d}t \right| \;\leq\; \left(\int_a^b f(t)^2\, \mathrm{d}t \right)^{1/2} \left(\int_a^b g(t)^2\, \mathrm{d}t \right)^{1/2}$$

for all continuous functions $f, g \colon [a, b] \to \mathbb{R}$.

As a special case, if $h \in \mathcal{C}([a, b])$ and we take $f = |h|$ and $g = 1$, we get

$$\int_a^b |h(t)|\, \mathrm{d}t \;\leq\; \sqrt{b - a} \left(\int_a^b h(t)^2\, \mathrm{d}t \right)^{1/2}.$$

Angles and orthogonality

Let V be an inner product space.

Suppose that $u, v \in \mathsf{V} \setminus \{0\}$ are two non-zero vectors. Then by the Cauchy-Schwarz inequality (Theorem IV.17) we have

$$-1 \;\leq\; \frac{\langle u, v \rangle}{\|u\|\,\|v\|} \;\leq\; 1.$$

By bijectivity of $\cos \colon [0, \pi] \to [-1, 1]$, there therefore exists a unique $\theta \in [0, \pi]$ such that $\frac{\langle u,v \rangle}{\|u\|\,\|v\|} = \cos(\theta)$, i.e., that

$$\langle u, v \rangle \;=\; \cos(\theta)\, \|u\|\, \|v\|. \tag{IV.25}$$

This $\theta \in [0, \pi]$ is called the **angle** between the vectors u and v. In the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ this of course coincides with the familiar notion of (undirected[13]) angle.

The case of the right angle $\theta = \frac{\pi}{2}$ is special. Since $\cos\left(\frac{\pi}{2}\right) = 0$, it occurs when the inner product between two vectors vanishes. Correspondingly, we make the following definitions.

Definition IV.20 (Orthogonality and orthonormality)**.**
Two vectors $u, v \in \mathsf{V}$ are said to be **orthogonal** if their inner product vanishes, $\langle u, v \rangle = 0$.

[13]In the plane $\mathbb{R}^2$ there is a convention about what is the positive orientation, and it is natural to define also the directed angle from $u \in \mathbb{R}^2 \setminus \{0\}$ to $v \in \mathbb{R}^2 \setminus \{0\}$, which is a number $\phi \in (-\pi, \pi]$. But the sign of this directed angle changes if we interchange u and v. In a general inner product space, two linearly independent vectors u, v span a plane, but this plane may be given either one of two possible orientations. The undirected angle does not depend on the choice of the orientation.

A collection $S \subset \mathsf{V}$ is said to be **orthogonal** if any two different vectors $u, v \in S$, $u \neq v$, are orthogonal.

A collection $S \subset \mathsf{V}$ is said to be **orthonormal** if it is an orthogonal collection and all $v \in S$ have unit norm, $\|v\| = 1$.

The cases of angles $\theta = 0$ and $\theta = \pi$ are also special: they correspond to vectors that lie on the same line through the origin.

Lemma IV.21 (Collinear vectors)**.**
Let $u, v \in \mathsf{V} \setminus \{0\}$ be two non-zero vectors in an inner product space V, and let θ be the angle between them. Then we have

- *$\theta = 0$ if and only if $u = c\,v$ for some $c > 0$;*
- *$\theta = \pi$ if and only if $u = c\,v$ for some $c < 0$.*

Exercise IV.6 (Proof of Lemma IV.21)**.**
Prove the two statements made in Lemma IV.21.
Hint: *The "if" directions of both statements are easy. For the "only if" statements, use Exercise IV.3.*

Exercise IV.7 (The dihedral angle of a regular tetrahedron)**.**
Calculate the dihedral angle (the angle between two adjacent faces) of a regular tetrahedron using the following idea:

Place the vertices of the tetrahedron in the 4-dimensional space $\mathbb{R}^4$ at points $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ ja $\mathbf{e}_4$ (standard basis vectors).

(i) Verify that the lengths of the edges $\|\mathbf{e}_i - \mathbf{e}_j\|$, for $i \neq j$, are all equal.
(ii) By symmetry, the angle in question is the same as (for example) the angle between the vectors
$$\mathbf{u} = \mathbf{e}_4 - \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) \qquad \text{and} \qquad \mathbf{v} = \mathbf{e}_3 - \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2),$$
so it is easily calculated.

Hint: *Draw a figure! Answer:* $\arccos(1/3)$.

There are various function spaces that have inner products, and in many cases appropriate orthogonal polynomials in such spaces play an important role. To get some flavor of this, the following exercise deals with the first few[14] Legendre polynomials.

Exercise IV.8 (Some Legendre polynomials)**.**
Consider the three first *Legendre polynomials*
$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= (3x^2 - 1)/2. \end{aligned}$$

(a) Show that these polynomials are pairwise *orthogonal* with respect to the inner product
$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, \mathrm{d}x,$$
i.e., that we have $\langle P_i, P_j \rangle = 0$ for $i \neq j$.

[14]If you are more ambitious, you can try to continue and find polynomials of higher degree which satisfy similar orthogonality.

(b) Determine coefficients $c_0, c_1, c_2 \in \mathbb{R}$ such that $\|c_k P_k\| = 1$ for all $k \in \{0, 1, 2\}$, where as norm we use the one arising from the inner product in part (a).

Also the idea of Fourier series is based on orthogonality of trigonometric functions in a suitable function space.

Exercise IV.9 (Ideas behind Fourier series)**.**

(a) Consider a countable orthonormal collection $(e_n)_{n\in\mathbb{N}}$ in an inner product space V. Assume that a vector $v \in \mathsf{V}$ can be expressed, for some $m \in \mathbb{N}$, as a (finite) linear combination

$$v = \alpha_1 \, e_1 + \cdots + \alpha_m \, e_m \qquad \text{with some coefficients } \alpha_n \in \mathbb{R}.$$

Calculate the inner products between the vector v and the vectors e_n from the orthonormal collection, and deduce a formula for the coefficients α_n.

(b) Define functions $c_0, s_1, c_1, s_2, c_2, s_3, c_3, \ldots$ of a real variable x by the formulas

$$\begin{aligned} c_0(x) &= \frac{1}{\sqrt{2}}, \\ s_n(x) &= \sin(nx) \qquad (\text{for } n \in \mathbb{N}), \\ c_n(x) &= \cos(nx) \qquad (\text{for } n \in \mathbb{N}). \end{aligned}$$

Prove that the (countably infinite) collection of these functions is orthonormal in the space $C([-\pi, \pi])$ of continuous functions on $[-\pi, \pi]$, with respect to the rescaled inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x.$$

(c) The idea of Fourier series is founded on the following result: Every continuously differentiable 2π-periodic function $f \colon \mathbb{R} \to \mathbb{R}$ can be represented as

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \big(\alpha_n \cos(nx) + \beta_n \sin(nx)\big).$$

Consider for simplicity the case of a function f, for which the above series contains only finitely many terms, i.e., for some $m \in \mathbb{N}$ we have

$$f(x) = \alpha_0 + \sum_{n=1}^{m} \big(\alpha_n \cos(nx) + \beta_n \sin(nx)\big).$$

Using parts (a) and (b), derive the following formula for the coefficients

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x \qquad (n \in \mathbb{N}).$$

IV.3. Normed spaces

Axioms of normed spaces

Definition IV.22 (Normed space)**.**

Let V be a real vector space. A function

$$\begin{aligned} \mathsf{V} &\to [0, \infty) \\ \vec{x} &\mapsto \|\vec{x}\| \end{aligned} \tag{IV.26}$$

is called a **norm**, if it satisfies the following conditions

(N1) $\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$ for all $\vec{x}, \vec{y} \in \mathsf{V}$; (**triangle inequality**)
(N2) $\|c\,\vec{x}\| = |c| \, \|\vec{x}\|$ for all $c \in \mathbb{R}$ and $\vec{x} \in \mathsf{V}$; (**homogeneity**)

(N3) $\|\vec{x}\| = 0$ only if $\vec{x} = \vec{0} \in \mathsf{V}$. **(non-degeneracy)**[15]

A vector space V equipped with a norm is called a **normed space**.

To concisely and unambiguously mention both the vector space V and the norm $\|\cdot\|$ on it, we may refer to the *pair* $(\mathsf{V}, \|\cdot\|)$ as a normed space.

By the following two remarks, we in fact immediately get some examples of norms and normed spaces from Section IV.2.

Remark IV.23 (The norm induced by an inner product is a norm).
In an inner product space, we used (IV.20) to define a norm using the inner product, and this gives rise to a norm also in the sense of Definition IV.22. Indeed, for the norm $\|v\| = \sqrt{\langle v, v\rangle}$ induced by the inner product, we already saw in Corollary IV.18 that (N1) holds. The other two properties are easy. From (IP2) and (IP1) we get $\langle c\,v, c\,v\rangle = c^2\,\langle v, v\rangle$, so

$$\|c\,v\| = \sqrt{c^2\,\langle v, v\rangle} = |c|\,\sqrt{\langle v, v\rangle} = |c|\,\|v\|,$$

establishing (N2). The property (N3) follows directly from (IP5).

Remark IV.24 (All inner product spaces are normed spaces).
It follows from the remark above that any inner product space becomes a normed space, when it is equipped with the norm induced by the inner product as in (IV.20).

In particular, the Euclidean space $\mathbb{R}^d$ of Example IV.13 and the space $\mathcal{C}([a,b])$ of continuous functions on a closed interval with the L^2-inner product of Example IV.14 can both be viewed as normed spaces as well.

Examples of normed spaces

By Remark IV.24, any inner product space is an example of a normed space. However, the converse is not true: there are normed spaces that are not inner product spaces. The concept of a normed space is genuinely more general. Let us now look at some such examples.

One can in fact even equip the standard d-dimensional vector space $\mathbb{R}^d$ with other norms besides the Euclidean norm on the inner product space of Example IV.13. First of all, we can use other inner products as indicated in Exercises IV.1 and IV.2, and they induced other norms on $\mathbb{R}^d$. But to focus on norms that are not induced by any inner product — here are some examples.

Example IV.25 (The ℓ^∞-norm on $\mathbb{R}^d$).
Define, for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$\|x\|_\infty \;=\; \max\big\{|x_1|, \ldots, |x_d|\big\} = \max_{j=1,\ldots,d} |x_j|. \tag{IV.27}$$

We call this the ℓ^∞**-norm** on $\mathbb{R}^d$ — but let us verify that this indeed is a norm!

(N1): Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ are two vectors in $\mathbb{R}^d$. By definition, their vector sum is $x+y = (x_1+y_1, \ldots, x_d+y_d)$. For any $j \in \{1, \ldots, d\}$ we have $|x_j+y_j| \leq |x_j|+|y_j|$

[15] We in fact have $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$. In (N3) we only require the "only if" implication to minimize work to check these properties in examples. The other direction follows from already required properties. Namely, we have $\vec{0} = 0\,\vec{0}$, so by (N2) we get

$$\|\vec{0}\| \;=\; \|0\,\vec{0}\| \;=\; |0|\,\|\vec{0}\| \;=\; 0.$$

by the triangle inequality of real numbers (Proposition II.1), and we may further estimate

$$|x_j + y_j| \;\leq\; |x_j| + |y_j| \;\leq\; \max_{j=1,\ldots,d} |x_j| + \max_{j=1,\ldots,d} |y_j|.$$

Since this estimate holds for each j, we get

$$\max_{j=1,\ldots,d} |x_j + y_j| \;\leq\; \max_{j=1,\ldots,d} |x_j| + \max_{j=1,\ldots,d} |y_j|,$$

which is exactly property (N1) for the ℓ^∞ norm, $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$.

(N2): Let $c \in \mathbb{R}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. By definition of scalar multiplication, we have $c\,x = (c\,x_1, \ldots, c\,x_d)$. For any $j \in \{1, \ldots, d\}$ we have

$$|c\,x_j| \;=\; |c|\,|x_j| \;\leq\; |c| \max_{j=1,\ldots,d} |x_j|.$$

Since this estimate holds for each j, we get

$$\max_{j=1,\ldots,d} |c\,x_j| \;\leq\; |c| \max_{j=1,\ldots,d} |x_j|,$$

and by considering the index j for which the maximal x_j is attained, we see that equality in fact holds here. The equality is exactly property (N2) for the ℓ^∞ norm, $\|c\,x\|_\infty = |c|\,\|x\|_\infty$.

(N3): If for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have $\|x\|_\infty = 0$, i.e., $\max\{|x_1|, \ldots, |x_d|\} = 0$, then for each j we must have $x_j = 0$, which implies that $x = (0, \ldots, 0)$. This establishes (N3).

Exercise IV.10 (The ℓ^1-norm on $\mathbb{R}^d$)**.**
Define, for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$\|x\|_1 \;=\; |x_1| + \cdots + |x_d| \;=\; \sum_{j=1}^{d} |x_j|. \tag{IV.28}$$

We call this the **ℓ^1-norm** on $\mathbb{R}^d$. Prove that it satisfies (N1), (N2), (N3).

Exercise IV.11 (The "unit circles" in $\mathbb{R}^2$ with different norms)**.**
Consider the two-dimensional vector space $\mathbb{R}^2$ equipped with the norms of Example IV.25 and Exercise IV.10.

(a) Draw the "unit circle" with respect to the ℓ^∞-norm,

$$\Big\{ (x, y) \in \mathbb{R}^2 \;\Big|\; \|(x, y)\|_\infty = 1 \Big\}.$$

(b) Draw the "unit circle" with respect to the ℓ^1-norm,

$$\Big\{ (x, y) \in \mathbb{R}^2 \;\Big|\; \|(x, y)\|_1 = 1 \Big\}.$$

Exercise IV.12 (The ℓ^∞-norm is not induced by an inner product)**.**
Consider the norm $\|\cdot\|_\infty$ of Example IV.25 on $\mathbb{R}^d$ with $d \geq 2$. Find two vectors $u, v \in \mathbb{R}^d$ such that $\|u + v\|_\infty^2 + \|u - v\|_\infty^2 \;\neq\; 2\,\|u\|_\infty^2 + 2\,\|v\|_\infty^2$. Use Exercise IV.4 to conclude that the norm $\|\cdot\|_\infty$ is not induced by any inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$.

Exercise IV.13 (The ℓ^1-norm is not induced by an inner product)**.**
Consider the norm $\|\cdot\|_1$ of Exercise IV.10 on $\mathbb{R}^d$ with $d \geq 2$. Show that the norm $\|\cdot\|_1$ is not induced by any inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$.

So neither of the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ on $\mathbb{R}^d$ comes from an inner product (when $d \geq 2$). By the following exercises, both are comparable up to constant multiples depending on the dimension d, and moreover both are similarly comparable to the Euclidean norm induced by the standard inner product.

Exercise IV.14 (A comparison of the ℓ^1-norm and the ℓ^∞-norm)**.**
Let $\|\cdot\|_\infty$ and $\|\cdot\|_1$ denote the norms of Example IV.25 and Exercise IV.10 on $\mathbb{R}^d$. Prove that for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have

$$\|x\|_\infty \;\leq\; \|x\|_1 \;\leq\; d\,\|x\|_\infty. \tag{IV.29}$$

Exercise IV.15 (A comparison of the ℓ^1-norm and the Euclidean norm)**.**
Let $\|\cdot\|$ denote the ordinary Euclidean norm on $\mathbb{R}^d$, induced by the standard inner product, and let $\|\cdot\|_1$ denote the ℓ^1-norm of Exercise IV.10. Prove that for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have

$$\|x\| \;\leq\; \|x\|_1 \;\leq\; \sqrt{d}\,\|x\|.$$

Hint: *Recall Exercise IV.5.*

Exercise IV.16 (A comparison of the ℓ^∞-norm and the Euclidean norm)**.**
Show that there exists a constant $C_d > 0$ such that for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have

$$\frac{1}{C_d}\,\|x\| \;\leq\; \|x\|_\infty \;\leq\; C_d\,\|x\|.$$

Both of the above are examples of ℓ^p-norms.

Example IV.26 (The ℓ^p-norm on $\mathbb{R}^d$)**.**
Let $p \in [1, \infty)$ Define, for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$\|x\|_p \;=\; \Big(\sum_{j=1}^{d} |x_j|^p\Big)^{1/p}. \tag{IV.30}$$

We call this the ℓ^p**-norm** on $\mathbb{R}^d$. It satisfies (N1), (N2), (N3), but verifying (N1) requires slightly more work than in the cases above. This is a generalization which includes, at specific values of $p \geq 1$, some cases we have seen already:

- In the case $p = 1$, from (IV.30) we clearly recover the ℓ^1-norm (IV.28) of Exercise IV.10.
- In the case $p = 2$, from (IV.30) we recover the Euclidean norm of Example IV.15,

$$\|x\|_2 \;=\; \Big(\sum_{j=1}^{d} x_j^2\Big)^{1/2} \;=\; \sqrt{\langle x, x\rangle}\;.$$

- Exercise IV.17 will show that in the limit $p \to \infty$, we recover the ℓ^∞-norm of Example IV.25.

The case $p = 2$ yields a norm induced by the inner product of Example IV.13, but for $d \geq 2$ no other ℓ^p-norm comes from an inner product.

Exercise IV.17 (The ℓ^p-norm as $p \to \infty$)**.**
UNDER CONSTRUCTION!

Also many important function spaces are equipped with norms that are not induced by any inner product. Let us give some such examples. The next one, in particular, is very frequently used in applications.

Example IV.27 (The supremum norm on the space of continuous functions)**.**
Consider again the vector space $\mathcal{C}([a, b])$ of continuous real-valued functions on a closed

interval $[a,b] \subset \mathbb{R}$. For $f \in \mathcal{C}([a,b])$, let us define[16]

$$\begin{aligned} \|f\|_\infty \;&=\; \sup\Big\{ |f(t)| \;\Big|\; t \in [a,b] \Big\} \\ &=\; \sup_{t\in[a,b]} |f(t)|. \end{aligned} \tag{IV.31}$$

Note that since the continuous function f on a closed interval $[a,b]$ is bounded (Theorem III.13), the supremum here is finite, $\|f\|_\infty \in [0,\infty)$. In fact, since the continuous function $t \mapsto |f(t)|$ on the closed interval $[a,b]$ attains its maximum, we in fact have $\|f\|_\infty = \max\big\{ |f(t)| \;\big|\; t \in [a,b] \big\}$.

We call $\|\cdot\|_\infty$ the the **supremum norm**, the **sup-norm**, or the **uniform norm** — but let us verify that it indeed is a norm.

(N1): Let $f, g \in \mathcal{C}([a,b])$. Recall that addition $f+g$ in $\mathcal{C}([a,b])$ is defined pointwise by $(f+g)(t) = f(t) + g(t)$. For any $t \in [a,b]$, using the triangle inquality of real numbers (Proposition II.1), we get

$$\begin{aligned} \big|(f+g)(t)\big| \;&=\; \big|f(t)+g(t)\big| \\ &\le\; |f(t)| + |g(t)| \;\le\; \sup_{t\in[a,b]} |f(t)| + \sup_{t\in[a,b]} |g(t)| \;=\; \|f\|_\infty + \|g\|_\infty. \end{aligned}$$

Since this holds for any $t \in [a,b]$, we also get

$$\|f+g\|_\infty \;=\; \sup_{t\in[a,b]} \big|(f+g)(t)\big| \;\le\; \|f\|_\infty + \|g\|_\infty,$$

which is exactly the triangle inequality (N1) for the norm $\|\cdot\|_\infty$.

(N2): Let $f \in \mathcal{C}([a,b])$ and $c \in \mathbb{R}$. Recall that scalar multiplication cf in $\mathcal{C}([a,b])$ is defined pointwise by $(cf)(t) = c\,f(t)$. For $t \in [a,b]$, we have

$$\big|(cf)(t)\big| \;=\; \big|c\,f(t)\big| \;=\; |c|\,|f(t)| \;\le\; |c| \sup_{t\in[a,b]} |f(t)| \;=\; |c|\,\|f\|_\infty,$$

and since this holds for all $t \in [a,b]$, we get

$$\|cf\|_\infty \;=\; \sup_{t\in[a,b]} \big|(cf)(t)\big| \;\le\; |c|\,\|f\|_\infty.$$

This inequality is only a half of the desired equality (IP2), but we can use it as follows. If $c = 0$, then $cf = 0$ is the constant function zero, so clearly $\|cf\|_\infty = \|0\|_\infty = 0 = |0|\,\|f\|_\infty$ as desired. Assume therefore that $c \neq 0$. Then by using the already established inequality with scalar $\frac{1}{c}$ and function cf, we get have

$$\|f\|_\infty \;=\; \Big\|\frac{1}{c}\,(cf)\Big\|_\infty \;\le\; \frac{1}{|c|}\,\|cf\|_\infty,$$

which upon rearranging yields the inequality $\|cf\|_\infty \ge |c|\,\|f\|_\infty$ in the other direction. The equality (IP2) follows.

(N3): If $c \in \mathcal{C}([a,b])$ satisfies $\|f\|_\infty = 0$, then $\sup_{t\in[a,b]} |f(t)| = 0$ so we cannot have $|f(t)| > 0$ for any t, and f has to be the constant function zero.

Exercise (♯) IV.18 (Yet another norm on the space of continuous functions)**.**
Consider again the vector space $\mathcal{C}([a,b])$ of continuous real-valued functions on a nondegenerate closed interval $[a,b] \subset \mathbb{R}$. For $f \in \mathcal{C}([a,b])$, let us define

$$\|f\|_1 \;=\; \int_a^b |f(t)|\,\mathrm{d}t. \tag{IV.32}$$

Verify that $\|\cdot\|_1$ is a norm on $\mathcal{C}([a,b])$. We call it the L^1**-norm**.

Contrary to the norm $\|\cdot\|_2$ of Example **??** on $\mathcal{C}([a,b])$, neither of the above norms is induced by an inner product.

[16]The second line of the formula here means exactly the same as the first line, but it is a more concise notation and as such often very convenient.

Exercise IV.19 (The sup-norm does not come from an inner product)**.**
Prove that if the interval $[a, b]$ is non-degenerate, $a < b$, then the sup-norm $\|\cdot\|_\infty$ on $\mathcal{C}([a, b])$ is not induced by any inner product.

Exercise IV.20 (The L^1-norm does not come from an inner product)**.**
Prove that if the interval $[a, b]$ is non-degenerate, $a < b$, then the L^1-norm $\|\cdot\|_1$ on $\mathcal{C}([a, b])$ is not induced by any inner product.

Exercise IV.21 (A non-norm)**.**
UNDER CONSTRUCTION!

Exercise IV.22 (Another non-norm)**.**
UNDER CONSTRUCTION!

Lecture V

Metric spaces

In this lecture we reach the level of generality at which we will work on almost exclusively from here on: we introduce the notion of a *metric space*. The spaces we have considered so far — the real line, the Euclidean space, more general inner product spaces and normed spaces — all turn out to be at least metric spaces. But there is a vast amount of other relevant examples of metric spaces, too! What is important is that fruitful general theory for metric spaces can be developed, which can then be applied to the all of the specific cases.

The development of the theory will be mainly postponed to the subsequent lectures. The goal of this lecture is to just define metric spaces, give sufficiently many interesting examples of them, and introduce a few first concepts that make sense in general metric spaces. The concepts of open and closed sets in metric spaces turn out to be the most important for the development of the theory, although their importance may not yet be evident at the first sight.

V.1. Axioms of metric spaces

Intuitively, a metric space is a space where there is a reasonable notion of *distance* between points. The distances are encoded in a function called a metric. The only requirements, informally described, are: *(symmetricity)* that the distance between two points is the same both ways, *(triangle inequality)* that the distance from one point to another cannot be reduced by taking a "shortcut" through a third point, and *(separation)* that the distance between two different points is always strictly positive, while the distance from a point to itself is zero.

The precise definition is the following.

Definition V.1 (Metric space)**.**
Let X be a set. A function

$$\mathsf{d}\colon X \times X \to [0, \infty) \tag{V.1}$$

is a **metric** on X if the following conditions hold:

(M-s): For any $x, y \in X$ we have ("symmetricity")

$$\mathsf{d}(x,y) \;=\; \mathsf{d}(y,x). \tag{M-s}$$

(M-Δ): For any $x, y, z \in X$ we have ("triangle inequality")

$$\mathsf{d}(x,z) \;\le\; \mathsf{d}(x,y) + \mathsf{d}(y,z). \tag{M-Δ}$$

(M-0): We have ("separation")

$$\mathsf{d}(x,y) = 0 \;\text{ if and only if }\; x = y. \tag{M-0}$$

A set X equipped with a metric d is called a **metric space**.

To concisely and unambiguously mention both the set X and the metric d on it, we may refer to the *pair* (X, d) as a metric space.

V.2. Examples of metric spaces

Metric spaces are very general, and most spaces that we have seens so far are in fact also metric spaces. Let us start from something familiar and safe.

Example V.2 (The real line is a metric space).
On the real line $\mathbb{R}$, we use the familiar notion of distances between points, encoded in the metric on $\mathbb{R}$ given by the formula

$$\mathsf{d}_{\mathbb{R}}(x, y) \;=\; |y - x| \qquad \text{for } x, y \in \mathbb{R}. \tag{V.2}$$

Checking the properties (M-s), (M-Δ), (M-0) directly is not difficult (it is a good exercise!), but the properties also follow from the more general Remark V.5 below.

We call (V.2) the **standard metric on** $\mathbb{R}$. Unless otherwise mentioned, we consider the real axis $\mathbb{R}$ equipped with this metric.

Although the above metric on $\mathbb{R}$ is generally the reasonable choice that we will use without explicit mention, let us take this opportunity to point out that we *could* equip the set $\mathbb{R}$ also with some other more exotic metrics.

Exercise V.1 (Another metric on the real line).
Prove that the formula

$$\mathsf{d}(x, y) \;=\; \log\Big(1 + |x - y|\Big)$$

also defines a metric on $\mathbb{R}$.

How creative can you get with metrics on the line?

Exercise V.2 (Yet other metrics on the real line).
Find further examples of metrics on $\mathbb{R}$, besides those in Example V.2 and Exercise V.1.
Hint: *First try to come up with some creative new formula! If that is too hard, a simple solution is to "change the units of distances" by multiplying a given metric*[1] *by some positive constant. Another idea that always works is to "allow for teleportation at fixed cost"*[2] *— a precise sense of which we will discuss in Lecture VII. Yet another idea is to define a simple* $\{0,1\}$*-valued metric (see Example V.7).*

Ok, all of this is to illustrate that the same set can be equipped with many possible metrics. But on $\mathbb{R}$ the standard metric (V.2) is the one to keep in mind!

Let us then look at the next most familiar case.

[1] Note that you can apply the "change of units" to for example the the standard metric or the more exotic metric on Exercise V.1.

[2] Also the "teleportation" modification can be done to either the standard metric or some more exotic metric to start with.

Example V.3 (The Euclidean spaces are metric spaces).

In the d-dimensional Euclidean space $\mathbb{R}^d$, the usual notion of distances comes from the Euclidean norm $\|\cdot\|_2$ of (IV.21); the corresponding metric is given by the familiar formula

$$\mathsf{d}_{\mathbb{R}^d}(\vec{x},\vec{y}) \;=\; \|\vec{y}-\vec{x}\|_2 \;=\; \sqrt{(y_1-x_1)^2+\cdots+(y_d-x_d)^2} \qquad \text{(V.3)}$$
$$\text{for } \vec{x}=(x_1,\ldots,x_d), \vec{y}=(y_1,\ldots,y_d)\in\mathbb{R}^d.$$

One can check the properties (M-s), (M-Δ), (M-0) directly, but they also follow from the more general Remark V.5 below.

When we refer to $\mathbb{R}^d$ as a Euclidean space, we consider it equipped with this metric (V.3).

As for the real line, the Euclidean metric above is the usual reasonable choice, but there are other options as well. Let us give just one such example for now.

Example V.4 (The Manhattan metric in the plane).

In the plane $\mathbb{R}^2$, the formula

$$\mathsf{d}\big((x,y),\,(x',y')\big) \;=\; |x'-x|+|y'-y| \qquad \text{for } (x,y),\,(x',y')\in\mathbb{R}^2 \qquad \text{(V.4)}$$

defines a metric — the required properties again follow using Remark V.5 for the norm $\|\cdot\|_1$ of Exercise IV.10. The metric (V.4) is sometimes referred to the *Manhattan metric*, because the distance between two points is obtained as if one had to separately move along horizontal "streets" (the term $|x'-x|$) and vertical "avenues" (the term $|y'-y|$).

In the examples above we did not do detailed checks of the properties (M-s), (M-Δ), (M-0). The reason is that they can all be handled at once with the following general observation, which in fact immediately gives also many further examples of metric spaces.

Remark V.5 (All normed spaces are metric spaces).

Suppose that $(\mathsf{V},\|\cdot\|)$ is a normed space. Then the **metric induced by the norm** $\|\cdot\|$ is

$$\mathsf{d}_{\mathsf{V}}(\vec{u},\vec{v}) \;=\; \|\vec{v}-\vec{u}\| \qquad \text{for } \vec{u},\vec{v}\in\mathsf{V}. \qquad \text{(V.5)}$$

Let us check that (V.5) is indeed a metric on V in the sense of Definition V.1.

(M-s): Let $\vec{u},\vec{v}\in\mathsf{V}$. Noting that $\vec{u}-\vec{v}=(-1)\,(\vec{v}-\vec{u})$ and using the homogeneity property (N2) of norms, we straightforwardly get the desired symmetricity of d_{V}

$$\mathsf{d}_{\mathsf{V}}(\vec{v},\vec{u}) \;=\; \|\vec{u}-\vec{v}\| \;=\; \big\|(-1)\,(\vec{v}-\vec{u})\big\| \;=\; |-1|\,\|\vec{v}-\vec{u}\| \;=\; \|\vec{v}-\vec{u}\| \;=\; \mathsf{d}_{\mathsf{V}}(\vec{u},\vec{v}).$$

(M-Δ): Let $\vec{u},\vec{v},\vec{w}\in\mathsf{V}$. Then using the triangle inequality (N1) for the norm $\|\cdot\|$, we get the desired triangle inequality of d_{V} as follows

$$\begin{aligned}
\mathsf{d}_{\mathsf{V}}(\vec{u},\vec{w}) \;&=\; \|\vec{w}-\vec{u}\| && \text{(definition of } \mathsf{d}_{\mathsf{V}})\\
&=\; \big\|(\vec{w}-\vec{v})+(\vec{v}-\vec{u})\big\| && \text{(added and subtracted } \vec{v})\\
&\le\; \|\vec{w}-\vec{v}\|+\|\vec{v}-\vec{u}\| && \text{(N1)}\\
&=\; \mathsf{d}_{\mathsf{V}}(\vec{v},\vec{w})+\mathsf{d}_{\mathsf{V}}(\vec{u},\vec{v})\,. && \text{(definition of } \mathsf{d}_{\mathsf{V}})
\end{aligned}$$

(M-0): Suppose that $\vec{u},\vec{v}\in\mathsf{V}$ are such that $\mathsf{d}_{\mathsf{V}}(\vec{u},\vec{v})=0$. By definition of d_{V}, this means that $\|\vec{v}-\vec{u}\|=0$. By property (N3) of the norm $\|\cdot\|$, this implies that $\vec{v}-\vec{u}=0$. Adding $\vec{u}$ to both sides we find $\vec{v}=\vec{u}$, as desired. The converse implication is easy: we have $\mathsf{d}_{\mathsf{V}}(\vec{v},\vec{v})=\|\vec{v}-\vec{v}\|=\|\vec{0}\|=0$ for any $\vec{v}\in\mathsf{V}$.

The verification of the properties of metric in Examples V.2, V.3, and V.4 now follow directly by applying Remark V.5 to the normed spaces $(\mathbb{R},|\cdot|)$, $(\mathbb{R}^d,\|\cdot\|_2)$, and $(\mathbb{R}^2,\|\cdot\|_1)$, respectively.

We also get lots of other examples — just check again every example of a normed space from Lecture IV (including all examples of inner product spaces, in view of Remark IV.24). Let us just take one example explicitly.

Example V.6 (A metric on the space of continuous function on a closed interval)**.**
The space $\mathcal{C}([a,b])$ on continuous real-valued functions on a closed interval $[a,b]$, equipped with the supremum norm $\|\cdot\|_\infty$ of Example IV.27, is a normed space — and thus in particular a metric space by Remark V.5. The metric (V.5) induced by the supremum norm (IV.31) takes the form

$$\mathsf{d}(f,g) \;=\; \|g-f\|_\infty \;=\; \sup_{t\in[a,b]} \big|g(t)-f(t)\big| \qquad \text{for } f,g \in \mathcal{C}([a,b]).$$

Let us finally mention an extremely simple metric you could use on any set.

Example V.7 (The 0/1 metric)**.**
Let X be any set. The formula

$$\mathsf{d}_{0/1}(x,y) \;=\; \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x\neq y \end{cases} \qquad \text{for } x,y\in X \tag{V.6}$$

is easily seen to define a metric on X. It is called the **0/1-metric** on X. It equips the set X with a structure that in some cases (discrete spaces) is reasonable, but in other cases is bizarre — there are no points "near each other", the distance between all pairs of points is the same!

Nevertheless, $(X,\mathsf{d}_{0/1})$ a valid example of a metric space, it has some relevant applications, and it often provides simple and illuminating counterexamples.

Metric inherited to subsets

Let (X,d) be a metric space, and $X'\subset X$ any subset. Then the subset naturally inherits a metric from the whole space — since d gives distances between all pairs of points in X, it certainly also gives distances between all pairs of points in the subset $X'\subset X$. Formally, the **inherited metric** $\mathsf{d}'\colon X'\times X'\to[0,\infty)$ on X' is given by

$$\mathsf{d}'(x',y') = \mathsf{d}(x',y') \qquad \text{for } x',y'\in X'\subset X, \tag{V.7}$$

i.e., the restriction of $\mathsf{d}\colon X\times X\to[0,\infty)$ to the subset $X'\times X'\subset X\times X$. The properties (M-s), (M-s), and (M-s) for d' directly follow as special cases of the corresponding properties of d.

We now get lots of new examples of metric spaces!

Example V.8 (Subsets of the real line are metric spaces)**.**
Since the real line $\mathbb{R}$ is a metric space (Example V.2), all its subsets also become metric spaces with the metrics they inherit by (V.7).

In particular, any of the following are metric spaces:

- intervals such as $[0,1]\subset\mathbb{R}$, $(-\infty,0)\subset\mathbb{R}$, $(-\pi,\pi]\subset\mathbb{R}$;
- the set of rational numbers $\mathbb{Q}\subset\mathbb{R}$;
- the set of irrational numbers $\mathbb{R}\setminus\mathbb{Q}\subset\mathbb{R}$;
- the Cantor set $C\subset\mathbb{R}$ (Appendix B.3);
- etc. etc.

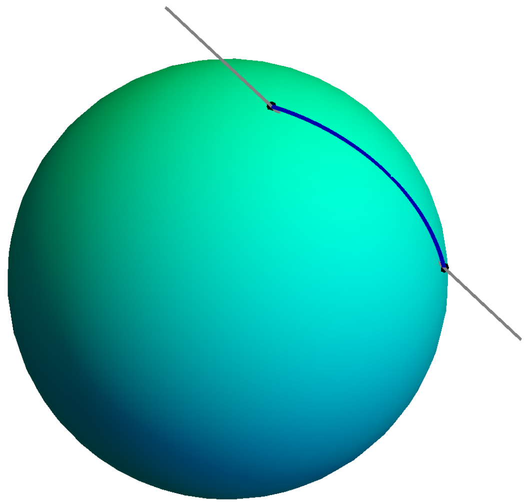


FIGURE V.1. The unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ inherits a metric from $\mathbb{R}^3$, but it also has another natural metric corresponding to the minimal length of paths along the surface.

Example V.9 (Subsets of the plane are metric spaces)**.**
Since the Euclidean plane $\mathbb{R}^2$ is a metric space (Example V.3), all its subsets also become metric spaces with the metrics they inherit by (V.7).

In particular, all lines, circles, curves, rectangles, triangles, disks, half-spaces, cones, the Koch snowflake, the Sierpinski carpet, ... in the plane $\mathbb{R}^2$ are metric spaces.

You have to admit that the number of relevant examples of metric spaces is vast.

Since we happen to live on this (approximately) spherical planet called Earth, and at least traditionally people have had to care about distances between places on its surface, let us still address the question of metrics on the sphere.

Example V.10 (The unit sphere in $\mathbb{R}^3$)**.**
Consider the unit sphere

$$\mathbb{S}^2 = \Big\{ (x, y, z) \in \mathbb{R}^3 \;\Big|\; x^2 + y^2 + z^2 = 1 \Big\} \subset \mathbb{R}^3$$

in the 3-dimensional space $\mathbb{R}^3$. Since the Euclidean space $\mathbb{R}^3$ is a metric space (Example V.3), the sphere $\mathbb{S}^2$ also inherits a metric as its subset. Note, however, that in this metric inherited from $\mathbb{R}^3$, the distances between points on the sphere are their shortest distances in the 3-dimensional space (i.e., along straight lines) — see the gray line in Figure V.1.

This notion of distances on $\mathbb{S}^2$ is relevant for example for neutrinos passing through the Earth (but we only use neutrino detectors relatively close to the surface of the earth).

Example V.11 (Another metric on the unit sphere in $\mathbb{R}^3$)**.**
For the purposes of, say, aviation or nautical seafaring, the natural metric on the sphere

$$\mathbb{S}^2 \; = \; \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2+z^2 = 1 \right\} \; \subset \; \mathbb{R}^3$$

would be given by shortest distances along the surface of the sphere — see the blue path in Figure V.1. The easiest precise definition of this metric is in terms of the infimum of the arclengths of paths in $\mathbb{S}^2$ connecting the two points, see Appendix B.4, but we leave the details to the interested reader.

V.3. First notions in metric spaces

Let us now introduce some notions that are immediately meaningful in metric spaces. Let (X, d) be a metric space throughout.

Balls

For $z \in X$ and $r > 0$, the set

$$\mathcal{B}_r(z) \; = \; \left\{ x \in X \;\middle|\; \mathsf{d}(z,x) < r \right\} \tag{V.8}$$

is called the **open ball** of radius r centered at z. Note that while in Euclidean spaces these are balls in an ordinary sense, in general such "balls" can be quite different. Note that by (M-0), the center always belongs to the ball, $z \in \mathcal{B}_r(z)$ for any $r > 0$.

Similarly, the **closed ball** of radius $r \geq 0$ centered at $z \in X$ is

$$\overline{\mathcal{B}}_r(z) \; = \; \left\{ x \in X \;\middle|\; \mathsf{d}(z,x) \leq r \right\}; \tag{V.9}$$

the only difference to (V.8) being that also points at exactly distance r from z are included. In particular the open ball is a subset in the closed ball of the same center and radius, $\mathcal{B}_r(z) \subset \overline{\mathcal{B}}_r(z)$. By (M-0) we have $z \in \overline{\mathcal{B}}_r(z)$ for any $r \geq 0$.

Furthermore the set

$$\left\{ x \in X \;\middle|\; \mathsf{d}(z,x) = r \right\} \; = \; \overline{\mathcal{B}}_r(z) \setminus \mathcal{B}_r(z) \tag{V.10}$$

of points exactly at distance $r > 0$ from z could be called a *sphere* (compare with Example V.10).

Example V.12 (Concrete examples of balls)**.**

- open/closed balls on the real axis are open/closed intervals;
- open/closed balls w.r.t. discrete 0/1 metric;
- balls in the function space $\mathcal{C}([a,b])$.

UNDER CONSTRUCTION!

For any $z \in X$ and $0 < r_1 < r_2$, directly from (V.8) we see $\mathcal{B}_{r_1}(z) \subset \mathcal{B}_{r_2}(z)$ — balls of smaller radius are contained in balls of larger radius with the same center.

Diameter and distance

For $A \subset X$ a subset, we define the **diameter** as

$$\operatorname{diam}(A) \;=\; \sup\Big\{ \mathsf{d}(a_1, a_2) \;\Big|\; a_1, a_2 \in A \Big\}. \tag{V.11}$$

If $\operatorname{diam}(A) < \infty$, we say that the set A is **bounded**.

Example V.13 (Balls are bounded).
Let $z \in X$ and $r > 0$. What is the diameter of $A = \mathcal{B}_r(z)$?

If $x_1, x_2 \in \mathcal{B}_r(z)$, then by definition $\mathsf{d}(z, x_1) < r$ and $\mathsf{d}(z, x_2) < r$. By triangle inequality (M-Δ) we get

$$\mathsf{d}(x_1, x_2) \;\leq\; \mathsf{d}(x_1, z) + \mathsf{d}(z, x_2) < r + r = 2r.$$

In view of the definition (V.11) this shows that

$$\operatorname{diam}\big(\mathcal{B}_r(z)\big) \leq 2r.$$

In particular this shows that any open ball $\mathcal{B}_r(z)$ is bounded.

One should not, however, expect the diameter of a ball of radius r to be exactly $2r$, in general. For example in a space with 0/1 metric, if $0 < r \leq 1$, we have $\mathcal{B}_r(z) = \{z\}$ and $\operatorname{diam}\big(\mathcal{B}_r(z)\big) = \operatorname{diam}(\{z\}) = 0$.

Lemma V.14 (Sets contained in balls are bounded).
If $A \subset \mathcal{B}_z(r)$ for some $z \in X$ and $r > 0$, then the set A is bounded.

Proof. If $A \subset A' \subset X$, then from definition (V.11) it is clear that $\operatorname{diam}(A) \leq \operatorname{diam}(A')$. In particular assuming $A \subset \mathcal{B}_z(r)$, we get $\operatorname{diam}(A) \leq \operatorname{diam}(\mathcal{B}_z(r)) \leq 2r$ by the example above. This shows that $\operatorname{diam}(A) < \infty$ and establishes boundedness of A. □

Also the following converse implication holds, and it is frequently useful.

Exercise V.3 (Bounded sets are contained in some balls).
Suppose that $z \in X$. Show that if $A \subset X$ is bounded, then there exists some $r > 0$ (sufficiently large) such that $A \subset \mathcal{B}_r(z)$.
Hint: *The case $A = \emptyset$ is trivial (why?). If $A \neq \emptyset$, pick some $a \in A$, and choose (for example) $r = \operatorname{diam}(A) + \mathsf{d}(z, a) + 1$. Use the triangle inequality to conclude.*

The **distance** between non-empty subsets $A, B \subset X$ is defined as

$$\operatorname{dist}(A, B) \;=\; \inf\Big\{ \mathsf{d}(a, b) \;\Big|\; a \in A,\; b \in B \Big\}. \tag{V.12}$$

Obviously for any $a, b \in X$, the distance between the corresponding singletons is just $\operatorname{dist}(\{a\}, \{b\}) = \mathsf{d}(a, b)$, so in this way the distance is a more general notion than the metric itself.[3]

We always have $\operatorname{dist}(A, B) \geq 0$ for non-empty subsets $A, B \subset X$, since $\mathsf{d}(a, b) \geq 0$ for all $a \in A$, $b \in B$. If $A \cap B \neq \emptyset$, then the existence of a point $x \in A \cap B$ and the property $\mathsf{d}(x, x) = 0$ implies that $\operatorname{dist}(A, B) = 0$. But it is possible to have vanishing distance even between two disjoint sets!

[3] Despite some parallels of properties and interpretation, the distance is *not* a metric on the set $\mathscr{P}(X)$ of all subsets of X — it is nonnegative and symmetric, but generally fails the triangle inequality and nondegeneracy, as shown by the examples right afterwards.

Example V.15 (Disjoint sets at zero distance).
Consider the real line $\mathbb{R}$ with its usual metric $\mathsf{d}_{\mathbb{R}}$ (Example V.2). Let $A = (-\infty, 0] \subset \mathbb{R}$ and $B = (0, 1]$. The sets A and B are disjoint, $A \cap B = \emptyset$. We nevertheless have $\mathrm{dist}(A, B) = 0$ (think about the precise justification!).

The general notion of course captures also familiar problems such as the following.

Exercise V.4 (Distance from a circle to a line).
Consider the Euclidean plane $\mathbb{R}^2$ (Example V.3). Let

$$
\begin{aligned}
C &= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1 \right\} \subset \mathbb{R}^2, \\
L &= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y = 2 - x \right\} \subset \mathbb{R}^2.
\end{aligned}
$$

Calculate the distance between the circle C and the line L, and argue precisely that your answer $\mathrm{dist}(C, L) = \sqrt{2} - 1$ indeed is the infimum in the definition (V.12).

V.4. Open and closed sets

The notion of an open subset in a metric space, and the related notion of a closed subset, turn out to lead to an extremely rich theory. All general topology, ultimately, is about open sets. This is probably hard to believe at first sight, but in the chapters that follow we hope you get to appreciate that this simple-looking concept is in fact very profound.

Let (X, d) be a metric space, throughout.

Open sets

Definition V.16 (Open set).
A subset $U \subset X$ in a metric space X is **open** if for all $x \in U$ there exists an $r_x > 0$ such that $\mathcal{B}_{r_x}(x) \subset U$.

In other words, all points of an open set must have a little bit of room around them within the set itself: a small ball around the point is contained in the set.

The following two examples are trivial — but important!

Example V.17 (The empty set is open).
The empty set $\emptyset$ is a subset in any metric space X. It is obvious (by careful thinking[4]) that $\emptyset \subset X$ is open in the sense of Definition V.16.

Example V.18 (The whole space is open).
In any metric space X, the whole space X is a subset. It is obvious (by careful thinking[5]) that $X \subset X$ is open in the sense of Definition V.16.

[4]There are no points in the empty set $\emptyset$, so Definition V.16 contains no requirements (there is only one requirement for every point x of the set), and in particular there is no required property that would not be satisfied! So $\emptyset$ is open.

[5]Let $x \in X$. Take any $r_x > 0$, for example $r_x = 1$. The open ball $\mathcal{B}_{r_x}(x)$ is *by definition* (V.8) contained in the space X, i.e., $\mathcal{B}_{r_x}(x) \subset X$. In view of Definition V.16, this shows that X is open.

Ok, let us have our first nontrivial example.[6]

Example V.19 (Open balls are open sets)**.**
Suppose $z \in X$ and $r > 0$. We claim that the open ball $\mathcal{B}_r(z) \subset X$ is an open subset.

We check this directly from Definition V.16. So let $x \in \mathcal{B}_r(z)$. By definition (V.8) this means that $\mathsf{d}(z,x) < r$. Now let $r_x = r - \mathsf{d}(z,x)$ and note that $r_x > 0$ because $\mathsf{d}(z,x) < r$. Now we only need to check that $\mathcal{B}_{r_x}(x) \subset \mathcal{B}_r(z)$. So let $y \in \mathcal{B}_{r_x}(x)$, which by definition means $\mathsf{d}(x,y) < r_x$. Using triangle inequality (M-Δ), we now get

$$\mathsf{d}(z,y) \;\leq\; \mathsf{d}(z,x) + \mathsf{d}(x,y) \;<\; \mathsf{d}(z,x) + r_x \;=\; \mathsf{d}(z,x) + \big(r - \mathsf{d}(z,x)\big) \;=\; r,$$

which shows that $y \in \mathcal{B}_r(z)$. Since $y \in \mathcal{B}_{r_x}(x)$ was arbitrary, we conclude $\mathcal{B}_{r_x}(x) \subset \mathcal{B}_r(z)$, which is what we needed to check.

Exercise (✓) V.5 (Punctured space is open)**.**
Let $z \in X$. The complement $X \setminus \{z\}$ of the singleton $\{z\}$ is called a **punctured space**. Show that the punctured space $X \setminus \{z\} \subset X$ is open.

Exercise V.6 (Complement of a closed ball is open)**.**
Let $z \in X$ and $r \geq 0$. Show that the complement $X \setminus \overline{\mathcal{B}}_r(z)$ of the closed ball $\overline{\mathcal{B}}_r(z)$ is open.

The following proposition shows how we can construct new open sets out of given ones. Again despite its simplicity, it will turn out unbelievably consequential.

Proposition V.20 (Unions and intersections of open sets)**.**

(a) *The union of any collection of open sets in X is also open.*
(b) *The intersection of any finite collection of open sets in X is also open.*

Proof. We prove statements (a) and (b) separately.

proof of (a): Let $(U_j)_{j \in J}$ be a collection of open sets $U_j \subset X$. Denote the union of this collection by

$$U \;=\; \bigcup_{j \in J} U_j.$$

We prove that $U \subset X$ is open, directly from Definition V.16. So let $x \in U = \bigcup_{j \in J} U_j$. By definition of the union, there exists some $j_0 \in J$ such that $x \in U_{j_0}$. Then since $U_{j_0} \subset X$ was assumed open, there exists some $r > 0$ such that $\mathcal{B}_r(x) \subset U_{j_0}$. But now

$$\mathcal{B}_r(x) \;\subset\; U_{j_0} \;\subset\; \bigcup_{j \in J} U_j \;=\; U.$$

The existence of such an $r > 0$ for any $x \in U$ by definition shows that U is open.

proof of (b): Let $U_1, U_2, \ldots, U_m \subset X$ be open sets (finitely many!). Consider their intersection

$$V = U_1 \cap U_2 \cap \cdots \cap U_m.$$

We prove that $V \subset X$ is open, directly from Definition V.16. So let $x \in V = \bigcap_{j=1}^m U_j$. By definition of the intersection, for all $j \in \{1, \ldots, m\}$ we have $x \in U_j$. Then since each $U_j \subset X$ was assumed open, there exists $r_j > 0$ such that $\mathcal{B}_{r_j}(x) \subset U_j$. Now let $r = \min\{r_1, \ldots, r_m\}$. We have $r > 0$, as the minimum of these finitely many positive numbers. For each $j \in \{1, \ldots, m\}$, since $r \leq r_j$, we have $\mathcal{B}_r(x) \subset \mathcal{B}_{r_j}(x) \subset U_j$. By definition

[6] If the phrase "open balls are open sets" sounds tautological, remember that we have given a precise meaning to both the term *open ball* (V.8) and the term *open set* (Definition V.16), and it is the precise definitions that carry logical content, not the words we have chosen to use... The terminology has actually been chosen *because of* the properties that we are just about to establish!

of intersection, this implies $\mathcal{B}_r(x) \subset \bigcap_{j=1}^{m} U_j = V$. The existence of such an $r > 0$ for any $x \in U$ by definition shows that V is open. □

Example V.21 (Discrete topology).
UNDER CONSTRUCTION!

Open sets in subspaces of metric spaces

The following simple observation is the foundation of subspace topology.

Lemma V.22 (Open sets in a subspace of a metric space).
Let (X, d) be a metric space, and let $X' \subset X$ be a subset, which we equip with the inherited metric d' given by (**??**)*.*

Then a subset $U' \subset X'$ is open if and only if there exists an open set $U \subset X$ such that $U' = U \cap X'$.

Proof. We prove the "if" and "only if" implications separately.

proof of "only if": UNDER CONSTRUCTION!

proof of "if": UNDER CONSTRUCTION! □

Closed sets

The definition of a closed set directly refers to that of an open set (Definition V.16).

Definition V.23 (Closed set).
A subset $A \subset X$ in a metric space X is **closed** if its complement $X \setminus A \subset X$ is open.

Let us again start by two trivial but important examples.

Example V.24 (The empty set is closed).
The empty set $\emptyset$ is a subset in any metric space X. It is closed, since its complement $X \setminus \emptyset = X$ is the whole space, which is open by Example V.18.

Example V.25 (The whole space is closed).
In a metric space X, the whole space is a subset $X \subset X$. It is closed, since its complement $X \setminus X = \emptyset$ is the empty set, which is open by Example V.17.

The first nontrivial example is the following.

Example V.26 (Closed balls are closed sets).
Let $z \in X$ and $r \geq 0$. The closed ball $\overline{\mathcal{B}}_r(z) \subset X$ is closed, since by Exercise V.6 its complement $X \setminus \overline{\mathcal{B}}_r(z) \subset X$ is open.

Example V.27 (Singletons are closed sets).
Let $z \in X$. The singleton $\{z\} \subset X$ is closed, since by Exercise V.5 its complement $X \setminus \{z\} \subset X$ is open.

As an exercise, you can prove the following directly from definition.

Exercise V.7 (Finite sets are closed).
Let $A = \{z_1, \ldots, z_m\} \subset X$ be a finite subset in a metric space X. Prove that $A \subset X$ is closed.

Proposition V.28 (Unions and intersections of closed sets).

(a) *The intersection of any collection of closed sets in X is also closed.*
(b) *The union of any finite collection of closed sets in X is also closed.*

Exercise V.8 (Proof of Proposition V.28).
Prove Proposition V.28.
Hint: *Use Exercise I.6 and the corresponding result for open sets.*

You can now consider Exercise V.7 again, in view of Proposition V.28 and Example V.27. The right tools make things significantly easier.

Warning: clopen sets!

You should *not* think of open and closed as opposites![7] They are mathematical terms whose meaning was given in Definitions V.16 and V.23.

There are, in particular, sets which are both open and closed.

Remark V.29 (Clopen sets).
According to Examples V.17 and V.24, the empty set $\emptyset \subset X$ is both open and closed.

According to Examples V.18 and V.25, the whole space $X \subset X$ is both open and closed.

Depending on the metric space, there can exist also other subsets which are both open and closed.[8] Such sets are sometimes called **clopen** (although this notion is not needed often).

Example with discrete topology... UNDER CONSTRUCTION!

There are also sets which are neither open nor closed.

Example V.30 (The set of rational numbers is neither open nor closed on the real line).
Consider the real line $\mathbb{R}$ with its usual metric, and consider the subset $\mathbb{Q} \subset \mathbb{R}$ of rational numbers in it. We claim that $\mathbb{Q}$ is neither open nor closed.

To see that $\mathbb{Q}$ is not open, take a rational number $q \in \mathbb{Q}$, for example $q = 0$. Now for an arbitrary $r > 0$, the open ball $\mathcal{B}_r(q) = (q - r, q + r)$ is a non-empty open interval, and thus by Theorem II.23 contains some irrational numbers and therefore we have $\mathcal{B}_r(q) \not\subset \mathbb{Q}$. This shows that $\mathbb{Q}$ is not open.

To see that $\mathbb{Q}$ is not closed, we must by definition show that its complement $\mathbb{R} \setminus \mathbb{Q}$ is not open. So take an irrational number $z \in \mathbb{R} \setminus \mathbb{Q}$, for example $z = \sqrt{2}$. Now for an arbitrary $r > 0$, the open ball $\mathcal{B}_r(z) = (z - r, z + r)$ is a non-empty open interval, and thus by Theorem II.21

[7] Sometimes this warning is summarized as "sets are not doors". A door, arguably, is always either open or closed — whereas subsets in a metric space can be both open and closed simultaneously, or neither open or closed.

[8] This in fact amounts to the disconnectedness of the space — which we will define and study in Lecture XII.

contains some rational numbers and therefore we have $\mathcal{B}_r(z) \not\subset \mathbb{R} \setminus \mathbb{Q}$. This shows that $\mathbb{R} \setminus \mathbb{Q}$ is not open, i.e., that $\mathbb{Q}$ is not closed.

Example V.31 (A half-open interval is neither open nor closed on the real line).

To reiterate, *open* and *closed* mean exactly what they were defined to mean, they are are *not* the opposites of each other!

V.5. ♡ Interior, exterior, and boundary

The following notions are frequently used in calculus, and various areas of analysis and its applications. They are, however, not strictly necessary for the theory covered in this course. The definitions below should be easy to grasp once one has a feeling for what metric spaces are, but if the amount of definitions up until now in this chapter seems overwhelming already, then one can postpone studying these concepts until they are needed in other courses.

Given a subset A of a metric space X, an often useful idea is to classify the points x of the space X according to how they lie in relation to the subset A: whether they are well inside, well outside, or neither really. The precise definitions and terminology are as follows.

Definition V.32 (Interior and exterior).
A point $x \in X$ is said to be an **interior point** of $A \subset X$ if for some $r > 0$ we have $\mathcal{B}_r(x) \subset A$. A point x is said to be an **exterior point** of A if for some $r > 0$ we have $\mathcal{B}_r(x) \subset X \setminus A$. The set of all interior points of A is denoted by A°, and the set of all exterior points by $\mathrm{ext}(A)$.

From the definition it is clear that $A^\circ \subset A$ and $\mathrm{ext}(A) \subset X \setminus A$. In particular the interior and the exterior are mutually disjoint, $A^\circ \cap \mathrm{ext}(A) = \emptyset$.

Definition V.33 (Boundary).
The **boundary** of subset $A \subset X$ is defined as

$$\partial A = X \setminus \big(A^\circ \cup \mathrm{ext}(A)\big)$$

The points $x \in \partial A$ are called **boundary points** of $A \subset X$.

In other words, a point x is a boundary point of A if it is neither an interior point nor an exterior point. The following characterization is therefore obvious (consider the contrapositive).

Lemma V.34 (A direct characterization of boundary points).
A point $x \in X$ is a boundary point of $A \subset X$ if and only if for all $r > 0$ the ball $\mathcal{B}_r(x)$ contains both a point of A and a point of its complement, i.e., $\mathcal{B}_r(x) \cap A \neq \emptyset$ and $\mathcal{B}_r(x) \cap (X \setminus A) \neq \emptyset$.

In summary we have defined the interior, the exterior, and the boundary of A so that they form a partition

$$X = A^\circ \cup \mathrm{ext}(A) \cup \partial A.$$

of the space to three mutually disjoint subsets.

Examples

Example V.35 (The interior of an interval).
UNDER CONSTRUCTION!

Example V.36 (The boundary of an interval).
UNDER CONSTRUCTION!

Proposition V.37 (The interior and exterior are open).
Let $A \subset X$. Then both A° and $\mathrm{ext}(A)$ are open sets in X.

Proof. UNDER CONSTRUCTION! □

Corollary V.38 (The boundary is closed).
Let $A \subset X$. Then the boundary $\overline{A}$ is a closed set in X.

Proof. By Proposition V.37, both A° and $\mathrm{ext}(A)$ are open, so their union $A^\circ \cup \mathrm{ext}(A)$ is also open by Proposition V.20(a). The boundary $\partial A = X \setminus \big(A^\circ \cup \mathrm{ext}(A)\big)$ is by definition the complement of this open set, so it is closed (by definition). □

V.6. ♡ Closure

At this stage we will define the closure of a subset in terms of the above notions.

Definition V.39 (Closure).
The **closure** of a subset $A \subset X$ is $\overline{A} = A \cup \partial A$.

UNDER CONSTRUCTION!

Appendix A

Further topics in set theory

A.1. ♡ Cardinalities of sets

An observation about finite sets

Let us denote the number of elements of a set A by $\#A$.

Example A.1 (Number of elements in a few example sets)**.**

- For the empty set, we have $\#\emptyset = 0$.
- For a singleton, we have $\#\{a\} = 1$.
- We have $\#\{0, 1, 2, \ldots, 8, 9\} = 10$.
- Let $c_0, c_1, \ldots, c_{d-1}, c_d \in \mathbb{R}$ with $c_d \neq 0$, so that $p(x) = c_d\, x^d + c_{d-1}\, x^{d-1} + \cdots + c_1\, x + c_0$ is a polynomial of degree $d \in \mathbb{N}_0$. Then for the set of real zeroes of p we have
$$\#\left\{x \in \mathbb{R} \;\middle|\; p(x) = 0\right\} \;\leq\; d.$$

If a set A has infinitely many different elements, we denote $\#A = \infty$.

Example A.2 (Some infinite sets)**.**
The sets of natural numbers, integers, rational numbers, and real numbers are infinite
$$\#\mathbb{N} = \infty, \qquad \#\mathbb{Z} = \infty, \qquad \#\mathbb{Q} = \infty, \qquad \#\mathbb{R} = \infty.$$
Also nontrivial intervals are infinite, for example $\#(\pi, \pi] = \infty$.

Moreover, as we will verify later, a non-empty open interval $(a, b) \subset \mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers
$$\#\Big((a, b) \cap \mathbb{Q}\Big) \;=\; \infty, \qquad \#\Big((a, b) \setminus \mathbb{Q}\Big) \;=\; \infty.$$

A set A is called **finite** if $\#A \in \{0, 1, 2, \ldots\}$, and **infinite** if $\#A = \infty$. It turns out that among infinite sets, some are nevertheless larger than others. The notion of *cardinality* captures this. To motivate the definition, let us nevertheless begin by some observations about sizes of finite sets.

Observation A.3 (Comparing sizes of sets using surjective functions)**.**
If A, B are two finite sets and $B \neq \emptyset$, then the following are equivalent:[1]

- $\#A \geq \#B$
- there exists a surjective function $A \to B$.[2]

[1]For the empty set $B = \emptyset$, the comparison of sizes has to be handled separately (the empty set is smaller than any other set), since from a nonempty set $A \neq \emptyset$ there does not exist any functions $A \to \emptyset$ — let alone surjective functions.

[2]The idea is the following. Suppose that A is the set of all students of this course, and B is the set of all exercise groups. Every student is assigned to exactly one exercise group, so that the assignment defines a function $A \to B$. The function is surjective if every exercise group has at least one student. The gist of this observation is that in such a case we can conclude that there are

Cardinality comparison and equal cardinalities

In view of the above observation, it appears meaningful to consider a set A at least as large as a set $B \neq \emptyset$ if there exists a surjective function $A \to B$. In this case we denote $A \succeq B$ — or interchangeably $B \preceq A$. This is the comparison of **cardinalities** of sets. We may observe the following properties, which match with our intuition about sizes of sets:

- If $A \succeq B$ and $B \succeq C$, then $A \succeq C$.[3]
- If $B \subset A$ is a nonempty subset, then $A \succeq B$.[4]

Example A.4 (Cardinality comparisons of some infinite sets)**.**
Because of the subset relations $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, we have the cardinality comparisons $\mathbb{N} \preceq \mathbb{Z} \preceq \mathbb{Q} \preceq \mathbb{R}$.

We say that two nonempty sets A and B have **equal cardinalities** if $A \succeq B$ and $A \preceq B$. By Observation A.3, (nonempty) finite sets have equal cardinalities if and only if they have the same number of elements. Let us then look at some examples with infinite sets.

Example A.5 (Equal cardinality for a set and its proper subsets)**.**
Consider the set $\mathbb{N}$ of natural numbers, and the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ of nonnegative integers,

$$\begin{aligned} \mathbb{N} &= \{1, 2, 3, 4, \ldots\} \\ \mathbb{N}_0 &= \{0, 1, 2, 3, 4 \ldots\}. \end{aligned}$$

Because of the subset relation $\mathbb{N} \subset \mathbb{N}_0$, we have $\mathbb{N} \preceq \mathbb{N}_0$. On the other hand, we also have a surjective function $f \colon \mathbb{N} \to \mathbb{N}_0$ given by $f(n) = n - 1$ for $n \in \mathbb{N}$, so that $\mathbb{N} \succeq \mathbb{N}_0$ holds, too. Therefore the sets $\mathbb{N}$ and $\mathbb{N}_0$ have equal cardinalities.

In particular an infinite set can have a proper subset with equal cardinality: the set $\mathbb{N}_0$ has one extra element compared to $\mathbb{N}$ (namely zero), but this does not affect the "size" of the set; these two infinite sets have equal cardinalities.

Example A.6 (Another equal cardinality for a set and its proper subsets)**.**
Consider the set $\mathbb{N}$ of natural numbers and the set $A = \{2, 4, 6, \ldots\}$ of even natural numbers. Because of the subset relation $A \subset \mathbb{N}$, we have $A \preceq \mathbb{N}$. On the other hand, we also have a surjective function $f \colon A \to \mathbb{N}$ given by $f(m) = \frac{1}{2}m$ for $m \in A$, so that $A \succeq \mathbb{N}$ holds, too. Therefore the sets $\mathbb{N}$ and A have equal cardinalities.

The set $\mathbb{N}_0$ has infinitely many extra elements compared to A (namely the odd natural numbers), but this does not affect the "size" of the set; these two infinite sets have equal cardinalities.

---

at least as many students as there are exercise classes (and conversely: an assignment that leaves no exercise group empty exists if there are at least as many students as there are exercise classes).

[3]Recall Exercise I.9: if $f \colon A \to B$ and $g \colon B \to C$ are surjective, then also the composition $g \circ f \colon A \to C$ is surjective.

[4]Indeed, choose some element $b_0 \in B$ of the nonempty set B, and define a function f from A to the subset $B \subset A$ by

$$f(a) = \begin{cases} a & \text{if } a \in B \subset A \\ b_0 & \text{if } a \in A \setminus B. \end{cases}$$

Then f is surjective.

Exercise A.1 (The sets of natural numbers and integers have equal cardinalities).
Prove that we have $\mathbb{N} \preceq \mathbb{Z}$ and $\mathbb{N} \succeq \mathbb{Z}$.
Hint: *One direction is obvious; for the other, construct some surjectve function* $\mathbb{N} \to \mathbb{Z}$.

Countable and uncountable infinite sets

These examples show that many common infinite sets have the same cardinality as the set of natural numbers. This motivates the following definition.

Definition A.7 (Countable and uncountable infinite sets).
An infinite set A is **countably infinite** if $A \preceq \mathbb{N}$, i.e., if there exists a surjective function $f : \mathbb{N} \to A$; otherwise A is **uncountably infinite**.

Remark A.8 (Enumeration of elements of a countably infinite set).
If A is a countably infinite set, then it is possible to list or enumerate all elements of A as follows. If $f : \mathbb{N} \to A$ is a surjective function, then we may form a sequence

$$(a_1, a_2, a_3, \ldots)$$

with $a_n = f(n)$ for $n \in \mathbb{N}$. By surjectivity, each element of A appears at least once in this sequence.

It is not possible to enumerate elements of uncountably infinite sets in this way!

In Examples A.5 and A.6 and Exercise A.1 we have seen some examples of countably infinite sets. The most important example of an uncountably infinite set is the set of real numbers.

Theorem A.9 (The set of real numbers is uncountably infinite).
The set $\mathbb{R}$ *of real numbers is uncountably infinite.*

The proof is given in the next subsection.

By contrast, and perhaps surprisingly, the set of rational numbers is in fact countable. In this sense the set $\mathbb{Q}$ of rational numbers is much smaller than the set $\mathbb{R}$ of real numbers.

Theorem A.10 (The set of rational numbers is countably infinite).
The set $\mathbb{Q}$ *of rational numbers is countably infinite.*

Countabitity of the set of rational numbers

The key to the proof of Theorem A.10 is the following frequently useful lemma.

Lemma A.11 (The Cartesian product of two countable sets is countable).
Suppose that A *and* B *are two countably infinite sets. Then their Cartesian product* $A \times B$ *is also countably infinite.*

It is still clearest to prove this lemma by first explicitly addressing the following special case.

Lemma A.12 (The set of pairs of natural numbers is countable)**.**
The set $\mathbb{N} \times \mathbb{N} = \{(n, m) \mid n \in \mathbb{N},\ m \in \mathbb{N}\}$ *of pairs of natural numbers is countably infinite.*

Proof. We must show that there exists a surjective function $t\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. This can be done in various ways; the enumeration illustrated in Figure A.1 is easy to visualize, but let us choose a different construction. Namely, the function $t\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ given by

$$t(n) = \begin{cases} (k, \ell) & \text{if } n = 2^{k-1}\, 3^{\ell-1} \text{ for } k, \ell \in \mathbb{N} \\ (1, 1) & \text{if } n \text{ contains other prime factors besides 2 and 3} \end{cases}$$

is well-defined because of the unique prime factorization of natural numbers, and it is surjective, since any $(k, \ell) \in \mathbb{N} \times \mathbb{N}$ is obtained as a value at $n = 2^{k-1}\, 3^{\ell-1} \in \mathbb{N}$. □

Proof of Lemma A.11. Assume that A and B are countable. To prove countability of $A \times B$, we must exhibit a surjective function $\mathbb{N} \to A \times B$.

By countability of A and B, there exists surjective functions

$$g_1\colon\ \mathbb{N} \to A \qquad\qquad g_2\colon\ \mathbb{N} \to B.$$

We may combine them into a function $g\colon \mathbb{N} \times \mathbb{N} \to A \times B$ defined by

$$g(n, m) = \big(g_1(n),\, g_2(m)\big),$$

which is clearly surjective.[5] From Lemma A.12 we get the existence of a surjective function $t\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Now the composition $g \circ t\colon \mathbb{N} \to A \times B$,

$$\begin{tikzcd} \mathbb{N} \arrow[r, "t"] \arrow[rr, bend right, "g \circ t"'] & \mathbb{N} \times \mathbb{N} \arrow[r, "g"] & A \times B, \end{tikzcd}$$

is surjective. Countability of $A \times B$ follows. □

Proof of Theorem A.10. To show countability of the set $\mathbb{Q}$ of rational numbers, we must exhibit a surjective function $\mathbb{N} \to \mathbb{Q}$.

By definition, rational numbers $q \in \mathbb{Q}$ are of the form $q = \frac{m}{n}$, with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. In other words, the function

$$r\colon \mathbb{Z} \times \mathbb{N} \to \mathbb{Q} \qquad \text{given by} \qquad r(m, n) = \frac{m}{n} \qquad \text{for } (m, n) \in \mathbb{Z} \times \mathbb{N}$$

is surjective.

The set $\mathbb{Z}$ of integers is countable by Exercise A.1, and the set $\mathbb{N}$ of natural numbers is obviously countable. By Lemma A.11 the Cartesian product $\mathbb{Z} \times \mathbb{N}$ is therefore countable, so there exists a surjective function

$$s\colon \mathbb{N} \to \mathbb{Z} \times \mathbb{N}.$$

Now the composition $r \circ s\colon \mathbb{N} \to \mathbb{Q}$,

$$\begin{tikzcd} \mathbb{N} \arrow[r, "s"] \arrow[rr, bend right, "h \circ s"'] & \mathbb{Z} \times \mathbb{N} \arrow[r, "r"] & \mathbb{Q}, \end{tikzcd}$$

is surjective. Countability of $\mathbb{Q}$ follows. □

[5] Indeed, suppose that $(a, b) \in A \times B$. By surjectivity of $g_1\colon \mathbb{N} \to A$ there exists an $n \in \mathbb{N}$ such that $g_1(n) = a$ and by surjectivity of $g_2\colon \mathbb{N} \to B$ there exists an $m \in \mathbb{N}$ such that $g_1(m) = b$. Therefore we have $g(n, m) = \big(g_1(n),\, g_2(m)\big) = (a, b)$, showing surjectivity of g.

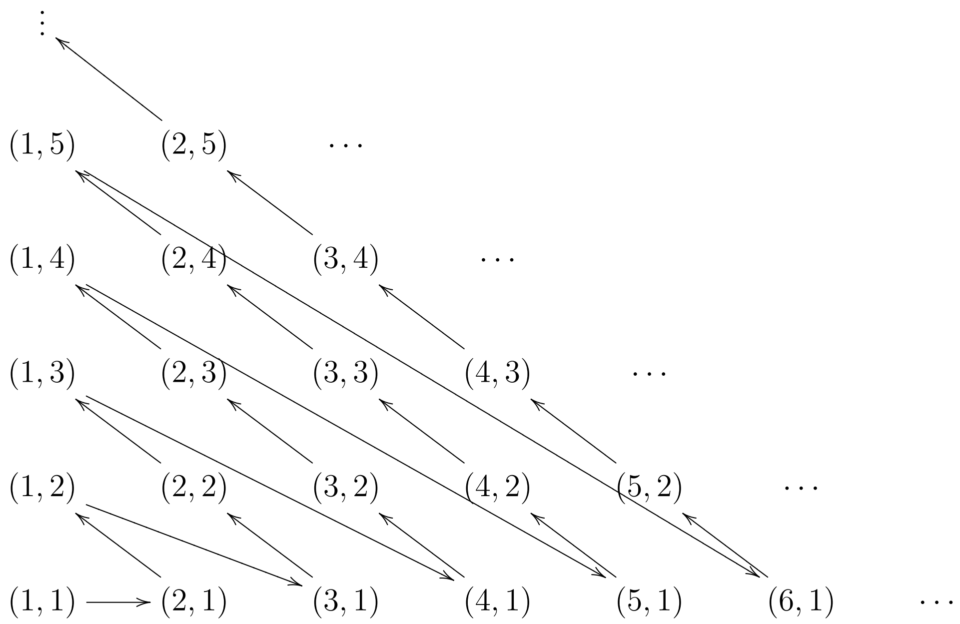


FIGURE A.1. The arrows indicate an enumeration of $\mathbb{N} \times \mathbb{N}$:

$$\big((1,1),\ (2,1),\ (1,2),\ (3,1),\ (2,2),\ (1,3),\ (4,1),\ (3,2),\ (2,3),\ (1,4),\ \ldots\big).$$

The idea is to observe that for pairs $(n, m) \in \mathbb{N} \times \mathbb{N}$, the sum $n + m$ takes values $2, 3, 4, \ldots$, and for each value of the sum it is straightforward to list the are finitely many pairs (n, m) in increasing order of the corresponding m (for example). This enumeration gives another construction of a surjective function $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

♡ *Uncountability of the set of real numbers*

Let us now provide the proof that the set $\mathbb{R}$ of real numbers is uncountably infinite. The argument is known as Cantor's diagonal argument. Here we use it to show the impossibility of enumerating the real numbers (even on the unit interval), but variants of the diagonal argument are also used frequently constructively; we will even see some examples later in this course.

Proof of Theorem A.9. In order to prove that $\mathbb{R}$ is not countable, it suffices to prove that the subset $[0, 1) \subset \mathbb{R}$ is not countable.

We do the proof by contradiction: we assume that $[0, 1)$ is countable, and derive a contradiction from this assumption. If $[0, 1)$ were countable, then there would exist a surjective function $f : \mathbb{N} \to [0, 1)$. For each $n \in \mathbb{N}$, we could write the decimal expansion of the number $f(n) \in [0, 1)$:

$$f(n) = 0.\, d_1^{(n)}\, d_2^{(n)}\, d_3^{(n)} \ldots \qquad \text{or more precisely} \qquad f(n) = \sum_{j=1}^{\infty} d_j^{(n)}\, 10^{-j},$$

where $d_j^{(n)} \in \{0, 1, 2, \dots, 8, 9\}$ for every $j \in \mathbb{N}$. In order to show that f could not have been surjective, we now examine these decimal expansions

$$\begin{aligned}
f(1) &= 0.\, \underline{d}_1^{(1)}\, d_2^{(1)}\, d_3^{(1)}\, d_4^{(1)} \dots \\
f(2) &= 0.\, d_1^{(2)}\, \underline{d}_2^{(2)}\, d_3^{(2)}\, d_4^{(2)} \dots \\
f(3) &= 0.\, d_1^{(3)}\, d_2^{(3)}\, \underline{d}_3^{(3)}\, d_4^{(3)} \dots \\
f(4) &= 0.\, d_1^{(4)}\, d_2^{(4)}\, d_3^{(4)}\, \underline{d}_4^{(4)} \dots \\
\vdots \; &= \qquad \ddots
\end{aligned}$$

in particular along the diagonal marked by the underlined digits $\underline{d}_k^{(k)}$, $k \in \mathbb{N}$. Let us construct a number $x \in [0, 1)$ whose first decimal digit is different from the first digit of $f(1)$, second digit is different from the second digit of $f(2)$, etc. Specifically, we choose the k:th digit as

$$d_k := \begin{cases} 8 & \text{if } d_k^{(k)} = 3, \\ 3 & \text{if } d_k^{(k)} \neq 3, \end{cases}$$

which guarantees that $d_k \neq d_k^{(k)}$ for each $k \in \mathbb{N}$. Now consider the number

$$x = 0.\, d_1 d_2 d_3 \dots \qquad \text{or more precisely} \qquad x = \sum_{j=1}^{\infty} d_j\, 10^{-j}.$$

We have $0 \leq x < 1$, (the strict second inequality uses the fact that 9 does not appear as a digit in x, so x is not $0.9999 \dots = 1$). It is straightforward to check that for every $k \in \mathbb{N}$, we have $x \neq f(k)$, since the k:th digits of x and $f(k)$ are different. But the existence of such an $x \in [0, 1)$ contradicts the surjectivity of f, and finishes the proof. $\square$

Appendix B

Further topics about real numbers

B.1. ♡ Field axioms and order axioms of the real numbers

Here we take a somewhat closer look into the *field axioms* and *order axioms* of the real numbers. The properties addressed in these axioms are undoubtedly more familiar than those in the completeness axiom. A closer look at nevertheless offers an instructive perspective especially into the similarities and differences of the real numbers to other fields used in mathematics (rational numbers, complex numbers, algebraic number fields, finite fields, . . .).

♡ Field axioms

The field axioms concern two operations of calculation with numbers: addition and multiplication, and we of course use the standard notational convention that multiplication takes precedence over addition.

The field axioms of real numbers state that the set $\mathbb{R}$ equipped with the operations $+$ and $\cdot$ is a field:

$$\begin{aligned}
&\text{Commutativity of addition:} \\
\forall x, y \in \mathbb{R} :\quad & x + y = y + x && \text{(B.1)} \\
&\text{Associativity of addition:} \\
\forall x, y, z \in \mathbb{R} :\quad & x + (y + z) = (x + y) + z && \text{(B.2)} \\
&\text{Neutral element for addition:} \\
\exists 0 \in \mathbb{R} :\quad \forall x \in \mathbb{R} :\quad & x + 0 = x && \text{(B.3)} \\
&\text{Opposite elements:} \\
\forall x \in \mathbb{R} :\quad \exists (-x) \in \mathbb{R} :\quad & x + (-x) = 0 && \text{(B.4)} \\
&\text{Commutativity of multiplication:} \\
\forall x, y \in \mathbb{R} :\quad & x \cdot y = y \cdot x && \text{(B.5)} \\
&\text{Associativity of multiplication:} \\
\forall x, y, z \in \mathbb{R} :\quad & x \cdot (y \cdot z) = (x \cdot y) \cdot z && \text{(B.6)} \\
&\text{Neutral element of multiplication:} \\
\exists 1 \in \mathbb{R} \setminus \{0\} :\quad \forall x \in \mathbb{R} :\quad & x \cdot 1 = x && \text{(B.7)} \\
&\text{Inverse elements:} \\
\forall x \in \mathbb{R} \setminus \{0\} :\quad \exists x^{-1} \in \mathbb{R} :\quad & x \cdot x^{-1} = 1 && \text{(B.8)} \\
&\text{Distributivity of addition over multiplication:} \\
\forall x, y, z \in \mathbb{R} :\quad & x \cdot (y + z) = x \cdot y + x \cdot z && \text{(B.9)}
\end{aligned}$$

Hopefully the reader finds these uncontroversial enough to be admitted as axioms.

Note that the axioms explicitly state the existence of two real numbers, 0 ("zero", the neutral element for addition) and 1 ("one", the neutral element for multiplication), and they require that these two are different, $1 \neq 0$ (since $1 \in \mathbb{R} \setminus \{0\}$). Moreover, zero and one are uniquely determined by the properties required of them in the axioms.

Example B.1 (Uniqueness of zero).

Claim: If both $0 \in \mathbb{R}$ and $0' \in \mathbb{R}$ satisfy the property of being neutral elements for addition, then we have $0 = 0'$.

Proof: Suppose that $x + 0 = x$ and $x + 0' = x$ for all $x \in \mathbb{R}$. Then we get

$$0 \overset{\text{(B.3)'}}{=} 0 + 0' \overset{\text{(B.1)}}{=} 0' + 0 \overset{\text{(B.3)}}{=} 0' ,$$

where we first used the neutral element property of $0'$, then commutativity of addition, and finally the neutral element property of 0. □

Exercise (✓) B.1 (Uniqueness of one).

Prove from the field axioms of real numbers that if both $1 \in \mathbb{R}$ and $1' \in \mathbb{R}$ satisfy the property of being neutral elements for multiplication, then we have $1 = 1'$.

Exercise B.2 (Uniqueness of opposite elements and inverse elemens).

Prove that for any $x \in \mathbb{R}$, the opposite element $(-x)$ is unique, and that for any $x \in \mathbb{R} \setminus \{0\}$, the inverse element x^{-1} is unique.

The following example indicates how yet a few other familiar facts about the real numbers are proved starting from the field axioms.

Example B.2 (Natural numbers from the field axioms).

Since we know that there exists a real number $1 \in \mathbb{R}$, we can use addition to define a new number $2 := 1 + 1 \in \mathbb{R}$. Similarly we define $3 := 2 + 1$, $4 := 3 + 1$, $5 := 4 + 1$, $6 := 5 + 1$, etc. These satisfy the usual properties. As an example, let us verify the following.

Claim: We have $2 + 2 = 4$.

Proof: Calculate, using the definition of 2, the associativity of addition, the definition of 3, and the definition of 4:

$$2 + 2 \overset{\text{(def of 2)}}{=} 2 + (1 + 1) \overset{\text{(B.2)}}{=} (2 + 1) + 1 \overset{\text{(def of 3)}}{=} 3 + 1 \overset{\text{(def of 4)}}{=} 4 .$$

This proves the claim. □

Exercise B.3 (Other exciting properties of natural numbers).

Prove some other exciting properties of these numbers, for example $2 + 3 = 5$ and $2 \cdot 3 = 6$.

Example B.3 (Yet another consequence of the field axioms).

We next verify a familiar property of real numbers which involves this number $2 := 1 + 1$ and an arbitrary real number x.

Claim: For any $x \in \mathbb{R}$ we have $x + x = 2x$.

Proof: Using the neutral element for multiplication (twice), distributivity, and definition of 2, we get the following expression for $x + x$:

$$x + x \overset{\text{(B.7)}}{=} 1 \cdot x \, + \, 1 \cdot x \overset{\text{(B.9)}}{=} (1 + 1) \cdot x \overset{\text{(def of 2)}}{=} 2 \cdot x.$$

The claim follows. □

Exercise B.4 (Multiplication by zero)**.**
Prove that for any $x \in \mathbb{R}$ we have $0 \cdot x = 0$.

Ok, you get the point! The familiar facts about the real numbers that you knew since kindergarten can be deduced from the axioms. In order that we get somewhere in this course, we will not actually require you to provide detailed proofs of totally commonplace statements such as $0 \cdot x = 0$ or $2 \cdot 2 = 4$ or $(-1)^2 = 1$ or $(x-y)^2 = x^2 - 2xy + y^2$ — as long as you realize that *in principle* they should be obtained as logical consequences of the axioms (and familiar notational conventions and definitions).[1]

Other fields

There are many other fields besides $\mathbb{R}$, and they all satisfy the field axioms. In this course we do not use other fields than $\mathbb{R}$ and $\mathbb{Q}$, but for perspective we mention a few prominent examples of fields:

- the field of rational numbers $\mathbb{Q}$;
- the field $\mathbb{Q}(\sqrt{5})$ of rational numbers adjoined with a square root of 5;
- the field of complex numbers $\mathbb{C}$;
- the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of p elements, where p is a prime number;
- the finite field $\mathbb{F}_{p^k}$ of p^k elements, where p is a prime number and $k \in \mathbb{N}$;
- the field $\mathbb{Q}_p$ of p-adic numbers, where p is a prime number;
- the field $\mathbb{K}(q)$ of rational functions in a single variable q over another field $\mathbb{K}$;
- $\cdots$

In particular all (familiar) consequences of the field axioms hold for these as well: for example if $z, w \in \mathbb{C}$ are complex numbers, then $(z + w) = z^2 + 2zw + w^2$, where $2 := 1 + 1 \in \mathbb{C}$. Note, however, that for instance in the two element field $\mathbb{F}_2$, we have $2 = 0$. So you should not try to derive the property $2 \neq 0$ of real numbers from the field axioms (you would fail, it is not a logical consequence of them!).

♡ Order axioms

Since there are other fields besides just the real numbers, the real numbers must have some further specific properties that distinguish them. Having an order relation $<$ turns out to be quite special already. To work with the order relation $<$, we adopt the usual additional notations such as: $y > x$ means $x < y$, $x \leq y$ means that either $x < y$ or $x = y$, etc.

[1] Nowadays there are, by the way, formally verified proofs of a vast amount of basic statements about real numbers and even metric space topology; certainly including all statements in this section. What this means is that the statements have been phrased in an entirely formal language, proofs have also been given in the formal language, and a computer has verified that the proofs adhere to the rules of logic and actually demonstrate the statements. See, for example, `https://leanprover-community.github.io/`. I slightly doubt, however, that all of that makes for much more pleasant reading than these lecture notes... (If you think otherwise, I will accept your formally verified solutions to the exam problems!) Moreover, producing the fully formal proofs is so tedious that already in some parts of this undergraduate course we actually get deeper into mathematics than where hundreds of mathematicians and computer scientists have gotten in the past decades with formally verified mathematics. A noble pursuit, nevertheless!

The order axioms of the real number field are:

$$
\begin{aligned}
&\text{Alternatives:} && \\
&\forall x, y \in \mathbb{R}: && \text{one and only one of the three relations} \\
& && x < y,\ x = y,\ x > y \text{ is true} \qquad \text{(B.10)} \\
&\text{Transitivity:} && \\
&\forall x, y, z \in \mathbb{R}: && x < y \text{ and } y < z \implies x < z \qquad \text{(B.11)} \\
&\text{Compatibility with addition:} && \\
&\forall x, y, z \in \mathbb{R}: && x < y \implies x + z < y + z \qquad \text{(B.12)} \\
&\text{Compatibility with multiplication:} && \\
&\forall x, y \in \mathbb{R}: && x > 0 \text{ and } y > 0 \implies x \cdot y > 0 \qquad \text{(B.13)}
\end{aligned}
$$

From these and the field axioms together, one can derive consequences.

Example B.4 (Some consequences of the order axioms)**.**
Claim: We have $0 < 1$.

Proof: Since we know that $0 \neq 1$, there are two mutually exclusive alternatives left: $0 < 1$ or $0 > 1$. Let us prove that the second one is impossible. So assume that $0 > 1$. By adding -1 and using compatibility with addition, we get (after simplifying by neutral element of addition and opposite element properties) that $-1 > 0$. Then by compatibility with multiplication we get $(-1)(-1) > 0$. But from the field axioms one can show that $(-1)(-1) = 1$, so this simplifies to $1 > 0$. In this case we have both $1 > 0$ and $0 > 1$, which is not allowed by the alternatives. So we have to discard the possibility that $0 > 1$, and we conclude that $0 < 1$ holds. □

As the example of the field of two elements shows, the property $2 \neq 0$ of real numbers cannot be derived from the field axioms alone. Having also the order axioms, we can now prove it.

Claim: We have $2 \neq 0$ in $\mathbb{R}$.

Proof: We already know that $1 > 0$. Adding 1 and using compatibility with addition, we find $2 > 1$. Transitivity then implies $2 > 0$. This rules out the possibility that $2 = 0$ by the axiom that exactly one of the alternatives $2 > 0$, $2 = 0$, and $2 < 0$, is true. □

Exercise B.5 (Product of two negative numbers)**.**
Prove that the product of any two negative real numbers is positive.

Ok, you get the point again. All the familiar properties you already knew in kindergarten are logical consequences of the axioms.

Note that the rational number field $\mathbb{Q}$ also satisfies the order axioms. Therefore if $\mathbb{R} \neq \mathbb{Q}$, there must be yet something else that distinguishes the real numbers. It is going to be the most profound of the axioms of real numbers.

B.2. ♡ Equivalence of the formulations of the completeness axiom

Recall that in Lecture II, any of the following statements was called the **completeness axiom** of $\mathbb{R}$:

- **(C1):** Every non-empty subset $A \subset \mathbb{R}$ which is bounded from above has a least upper bound $\sup A \in \mathbb{R}$.

(C2): Every increasing real number sequence $(a_n)_{n\in\mathbb{N}}$ which is bounded from above has a limit $\lim_{n\to\infty} a_n \in \mathbb{R}$.

(C3): Every collection $(I_n)_{n\in\mathbb{N}}$ of closed intervals $I_n \subset \mathbb{R}$, which is nested in the sense that $I_{n+1} \subset I_n$ for every $n \in \mathbb{N}$, has a nonempty intersection

$$\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset.$$

Let us now prove the equivalence of (C1), (C2), and (C3). We prove separately the implications

$$\text{(C1)} \implies \text{(C2)}, \qquad \text{(C2)} \implies \text{(C3)}, \qquad \text{(C3)} \implies \text{(C1)}.$$

The equivalence of all three follows by combining these implications.

Proof of (C1) ⇒ (C2). Assume (C1). Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers which is increasing and bounded from above. We must prove that this sequence converges.

Since the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded from above, the set

$$A = \{a_n \mid n \in \mathbb{N}\}$$

of its values is a subset in $\mathbb{R}$ which is bounded from above. By assumption (C1), then, this subset has a least upper bound $t_0 := \sup\{a_n \mid n \in \mathbb{N}\} \in \mathbb{R}$. We will prove that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to t_0.

Let $\varepsilon > 0$. Since $t_0 - \varepsilon$ is not an upper bound for $\{a_n \mid n \in \mathbb{N}\}$, there exists some $n_\varepsilon \in \mathbb{N}$ such that $a_{n_\varepsilon} > t_0 - \varepsilon$. Since the sequence is increasing, for all $n \geq n_\varepsilon$ we must then also have $a_n > t_0 - \varepsilon$. On the other hand, since t_0 is an upper bound for $\{a_n \mid n \in \mathbb{N}\}$, we have $a_n \leq t_0$ for all $n \in \mathbb{N}$. For $n \geq n_\varepsilon$ we have thus obtained

$$t_0 - \varepsilon \;<\; a_n \;\leq\; t_0 \;<\; t_0 + \varepsilon,$$

which implies $|a_n - t_0| < \varepsilon$. Since such an $n_\varepsilon \in \mathbb{N}$ was found for an arbitrary $\varepsilon > 0$, we have by definition of limits shown that $\lim_{n\to\infty} a_n = t_0$.

Property (C2) is thus established. □

Proof of (C2) ⇒ (C3). Assume (C2). Let $(I_n)_{n\in\mathbb{N}}$ be a collection of closed intervals $I_n = [a_n, b_n] \subset \mathbb{R}$ such that $I_{n+1} \subset I_n$ for every $n \in \mathbb{N}$. We must prove that $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$.

The nesting condition

$$[a_{n+1}, b_{n+1}] \;=\; I_{n+1} \;\subset\; I_n \;=\; [a_n, b_n]$$

simply amounts to the following inequalities among the endpoints of the intervals:

$$a_n \;\leq\; a_{n+1} \;\leq\; b_{n+1} \;\leq\; b_n.$$

Therefore the sequence $(a_n)_{n\in\mathbb{N}}$ of the left endpoints of the intervals is increasing. This sequence is also bounded from above, because for any $n \in \mathbb{N}$ we have $a_n \leq b_n \leq b_1$ (the second inequality follows from the fact that the sequence of right endpoints is decreasing). By assumption (C2), the sequence $(a_n)_{n\in\mathbb{N}}$ therefore has a limit $\alpha = \lim_{n\to\infty} a_n$. We will show that $\alpha \in \bigcap_{n\in\mathbb{N}} I_n$, and it will follow that the intersection is nonempty.

To show that $\alpha \in \bigcap_{m\in\mathbb{N}} I_m$, by definition of intersection we must show that $\alpha \in I_m$ for all $m \in \mathbb{N}$. Since $I_m = [a_m, b_m]$, this amounts to proving that

$$a_m \overset{?}{\leq} \alpha \overset{?}{\leq} b_m.$$

The first inequality above is clear: for $n \geq m$ we have $a_n \geq a_m$ (the left endopints form an increasing sequence), so by preservation of constant inequalities (Corollary II.15) we have $\alpha = \lim_{n\to\infty} a_n \geq a_m$. To prove the other inequality, we argue by contradiction. Suppose that it is not true, i.e., that $\alpha > b_m$. By definition of the limit $\alpha = \lim_{n\to\infty} a_n$, for the positive number $\varepsilon = \alpha - b_m > 0$, there exists an $n_\varepsilon \in \mathbb{N}$ so that for $n \geq n_\varepsilon$ we have $|a_n - \alpha| < \varepsilon = \alpha - b_m$, which implies

$$a_n > \alpha - \varepsilon = \alpha - (\alpha - b_m) = b_m.$$

In particular for $n \geq \max\{n_\varepsilon, m\}$, since the sequence of right endpoints is decreasing, we would get $a_n > b_m \geq b_n$. This, however, is a contradiction with the ordering $a_n \leq b_n$ of the left and right endpoints of $I_n = [a_n, b_n]$. With this contradiction, we have also established the other claimed inequality. We have thus concluded $\alpha \in I_m$ for all $m \in \mathbb{N}$, which shows that the intersection $\bigcap_{m\in\mathbb{N}} I_m$ is not empty, establishing (C3). □

Proof of (C3) ⇒ (C1). Assume (C3). Let $A \subset \mathbb{R}$ be a nonempty subset which is bounded from above. We must prove that A has a least upper bound.

Since A is bounded from above, it has an upper bound, and since A is moreover nonempty, it is possible to choose an upper bound $b_0 \in \mathbb{R}$ so that the number $a_0 = b_0 - 1$ is not an upper bound for A.

Consider the midpoint $c_0 = \frac{a_0+b_0}{2}$. If c_0 is an upper bound for A, we set $a_1 = a_0$ and $b_1 = c_0$. Otherwise we set $a_1 = c_0$ and $b_1 = b_0$. With such choices, we are guaranteed that b_1 is an upper bound for A, while a_1 is not.

Continue inductively. When b_n is an upper bound for A and a_n is not, consider the midpoint $c_n = \frac{a_n+b_n}{2}$. Depending on whether c_n is an upper bound for A or not, set $a_{n+1} = a_n$ and $b_{n+1} = c_n$, or $a_{n+1} = c_n$ and $b_{n+1} = b_n$. By construction we get a decreasing sequence $(b_n)_{n\in\mathbb{N}}$ of upper bounds for A, and an increasing sequence $(a_n)_{n\in\mathbb{N}}$ of numbers that are not upper bounds for A, and moreover $a_n < b_n$ for all $n \in \mathbb{N}$. The distances are halved at every step, so $b_n - a_n = 2^{-n}$.

From the above properties we get that the closed intervals $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ are nested, $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. By the assumed property (C3), the intersection $\bigcap_{n\in\mathbb{N}} I_n$ is then nonempty. On the other hand, the intersection cannot contain two distinct points: if $x, y \in \bigcap_{n\in\mathbb{N}} I_n$, then for all $n \in \mathbb{N}$ we have $x, y \in I_n = [a_n, b_n]$, which implies that $|x - y| \leq b_n - a_n = 2^{-n} \to 0$, so $|x - y| = 0$ and $x = y$. We conclude that the intersection contains exactly one point ξ, i.e., it is a singleton $\bigcap_{n\in\mathbb{N}} I_n = \{\xi\}$. From the same estimate we also get that

$$\xi - 2^{-n} \ \leq \ a_n \ \leq \ \xi \ \leq \ b_n \ \leq \ \xi + 2^{-n}.$$

By the squeeze theorem (Lemma II.17), this implies $\lim_{n\to\infty} a_n = \xi$ and $\lim_{n\to\infty} b_n = \xi$.

Our goal was to show that the set A has a least upper bound. We will show that ξ is it.

Let us first verify that ξ is an upper bound for A. So let $x \in A$. By construction each b_n is an upper bound for A, so $b_n \geq x$ for all $n \in \mathbb{N}$. Since we have $\xi = \lim_{n\to\infty} b_n$, from the preservation of bounds (Corollary II.15) we get that $\xi \geq x$. This was true for an arbitrary element $x \in A$, so indeed ξ is an upper bound for A.

Let us then show that no smaller number $\xi' < \xi$ is an upper bound for A. Since $\lim_{n\to\infty} a_n = \xi$, for any $\xi' < \xi$ we have $a_n > \xi'$ for large enough n (take $\varepsilon = \xi - \xi' > 0$ in the definition of limit), and since a_n was not an upper bound for A, the smaller number $\xi' < a_n$ can not be either.

This proves that $\sup A = \xi$ exists, establishing (C1). □

B.3. ♡ Cantor set

The Cantor set, or Cantor's $\frac{1}{3}$-set, is a subset $C \subset [0, 1]$, which has interesting properties from a topological, set theoretic, and measure theoretic points of view.

Informally, the Cantor set C is constructed by taking the unit interval $[0, 1]$, removing its middle third, then removing the middle third of both of the remaining thirds, and successively always removing the middle third of each remaining interval. Figure B.1 illustrates this iterative construction.

$$C_0 = [0,1]$$

$$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$$

$$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$$

$$C_3 \;:\; \text{union of 8 intervals of length } \frac{1}{27}$$

$$C_4 \;:\; \text{union of 16 intervals of length } \frac{1}{81}$$

$$C_5 \;:\; \text{union of 32 intervals of length } \frac{1}{243}$$

$$\vdots$$

FIGURE B.1. The Cantor set $C = \bigcap_{n\in\mathbb{N}} C_n \subset [0,1]$ is constructed iteratively so that from stage n approximation C_n one removes the middle thirds of the remaining intervals to obtain stage $n+1$ approximation C_{n+1}.

More formally, the Cantor set C is defined, using a sequence $(C_n)_{n\in\mathbb{N}}$ of nested approximating sets, as the intersection

$$C = \bigcap_{n\in\mathbb{N}} C_n, \tag{B.14}$$

where the sets C_n, $n \in \mathbb{N}$, are as follows. It makes sense to think of the unit interval $C_0 = [0,1]$ as the zeroth stage approximation. The first stage approximation is the set $C_1 = \big[0,\frac{1}{3}\big] \cup \big[\frac{2}{3},1\big]$; a union of 2 closed intervals of length $\frac{1}{3}$. The second stage approximation is $C_2 = \big[0,\frac{1}{9}\big] \cup \big[\frac{2}{9},\frac{1}{3}\big] \cup \big[\frac{2}{3},\frac{7}{9}\big] \cup \big[\frac{8}{9},1\big]$; a union of 4 closed intervals of length $\frac{1}{9}$. The n:th stage approximation is the union of 2^n closed intervals of length 3^{-n}, specifically

$$C_n = \bigcup_{b_1,\ldots,b_n\in\{0,1\}} I^{(n)}_{b_1,\ldots,b_n} \tag{B.15}$$

where

$$I^{(n)}_{b_1,\ldots,b_n} = \Big[2\sum_{j=1}^{n} b_j\, 3^{-j}\,,\;\; 2\sum_{j=1}^{n} b_j\, 3^{-j} + 3^{-n}\Big]. \tag{B.16}$$

The interval (B.16) is exactly obtained by keeping the left or right third, corresponding to $b_n = 0$ and $b_n = 1$, respectively, of the $n-1$:st stage interval $I^{(n-1)}_{b_1,\ldots,b_{n-1}}$. The middle third of $I^{(n-1)}_{b_1,\ldots,b_{n-1}}$ has been removed in stage n. In particular, we have the nesting property $C_n \subset C_{n-1}$ for all $n \in \mathbb{N}$, i.e.,

$$C_0 \supset C_1 \supset C_2 \supset C_3 \supset \cdots .$$

Let us now make some observations regarding the set C defined as the intersection (B.14).

The Cantor set is nonempty

The first observation is that the set C is non-empty,

$$C \neq \emptyset.$$

There are various ways to see this; the easiest is to just note that $0 \in C_n$ and $1 \in C_n$ for every $n \in \mathbb{N}$, so in the intersection we at least have $0 \in C$ and $1 \in C$. In other words, at least the two endpoints of the unit interval which constitutes the zeroth stage approximation C_0 belong to the Cantor set C. By an essentially similar reasoning one can see that all of the 2^{n+1} endpoints of the 2^n intervals of stage n remain in the intersection — we leave the detailed verification of this as an exercise. When all different stages $n \in \mathbb{N}$ are considered, such approximating interval endpoints give countably infinitely many different points in C — we again leave the details as an exercise, because it turns out that there are vastly more points in C, and they admit a unified description.

The Cantor set is uncountably infinite

The unified description of points of C is by infinite binary sequences $(b_1, b_2, b_3, \ldots)$. First, suppose that an infinite binary sequence $(b_n)_{n\in\mathbb{N}}$ is given, with $b_n \in \{0, 1\}$ for $n \in \mathbb{N}$. Note the nesting of the intervals,

$$I^{(1)}_{b_1} \supset I^{(2)}_{b_1,b_2} \supset I^{(3)}_{b_1,b_2,b_3} \supset \cdots \supset I^{(n)}_{b_1,\ldots,b_n} \supset I^{(n+1)}_{b_1,\ldots,b_n,b_{n+1}} \supset \cdots$$

By the formulation (C3) of the completeness axiom, the intersection of these nested closed intervals is nonempty, and since the lengths 3^{-n} of the intervals are tending to 0 as $n \to \infty$, it is not difficult to see that the intersection cannot contain more than one point. Therefore there exists some $x \in \mathbb{R}$ (which of course depends on the chosen binary sequence $(b_n)_{n\in\mathbb{N}}$) such that

$$\bigcap_{n\in\mathbb{N}} I^{(n)}_{b_1,\ldots,b_n} = \{x\}.$$

Since $x \in I^{(n)}_{b_1,\ldots,b_n} \subset C_n$ for every $n \in \mathbb{N}$, we then have also

$$x \in \bigcap_{n\in\mathbb{N}} C_n = C.$$

This shows that to any infinite binary sequence $(b_n)_{n\in\mathbb{N}}$, we can associate a point x in the Cantor set C. We will observe also that if $(b_n)_{n\in\mathbb{N}}$ and $(b'_n)_{n\in\mathbb{N}}$ are two different binary sequences, then the corresponding points x and x' are different. Indeed then for some $k \in \mathbb{N}$ we have $b_k \neq b'_k$, and the corresponding k:th stage intervals are disjoint, $I^{(k)}_{b_1,\ldots,b_k} \cap I^{(k)}_{b'_1,\ldots,b'_k} = \emptyset$. But since the corresponding points belong to these disjoint intervals, $x \in I^{(k)}_{b_1,\ldots,b_k}$ and $x' \in I^{(k)}_{b'_1,\ldots,b'_k}$, we must have $x \neq x'$.

Exercise (♯) B.6 (The point in Cantor set corresponding to a binary sequence).

Show that the point x, corresponding to the binary sequence $(b_n)_{n\in\mathbb{N}}$, is given by the formula

$$x = 2\sum_{j=1}^{\infty} b_j\, 3^{-j}. \tag{B.17}$$

Use this to give a different proof that different binary sequences yield different points in the Cantor set.

Remark B.5 (Describing the Cantor set by ternary numbers)**.**

Consider the analogue of decimal expansions $\sum_{j=1}^{\infty} d_j\, 10^{-j}$ where $d_1, d_2, \ldots \in \{0, 1, \ldots, 8, 9\}$ (see Example II.33), but in base three instead of base ten. These are numbers of the form $\sum_{j=1}^{\infty} t_j\, 3^{-j}$ where the "digits" are $t_1, t_2, \ldots \in \{0, 1, 2\}$.

From the expression (B.17) we see that the points in the Cantor set correspond to base three "decimal expansions" (ternary expansions) which only ever use digits 0 and 2; never 1.

Conversely, we will show that each point of the Cantor set corresponds to some binary sequence as above. Suppose that $x \in C$. By the definition of intersection, we then have $x \in C_n$ for every $n \in \mathbb{N}$. Since C_n is the union of disjoint intervals (B.15), there must exist exactly one choice of $b_1^{(n)}, \ldots, b_n^{(n)} \in \{0, 1\}$ such that $x \in I^{(n)}_{b_1^{(n)}, \ldots, b_n^{(n)}}$. Let us now define an infinite sequence $(b_n)_{n\in\mathbb{N}}$ by setting $b_n = b_n^{(n)}$ where $b_n^{(n)}$ came from the above choice of the n:th stage interval. But in fact $b_n^{(m)}$ will not depend on which stage $m \geq n$ we were considering[2]; we have $b_n^{(m)} = b_n$ for all $m \geq n$. Therefore we have $x \in I^{(n)}_{b_1, \ldots, b_n}$ for all $n \in \mathbb{N}$, i.e., $x \in \bigcap_{n\in\mathbb{N}} I^{(n)}_{b_1, \ldots, b_n}$. But this intersection of nested intervals is exactly the singleton that we used to define the point in C which corresponds to the binary sequence $(b_n)_{n\in\mathbb{N}}$, so we have now shown that any point $x \in C$ is of this form.

On measure theoretic properties of the Cantor set

While this is not a course in measure theory, it is interesting to note that the Cantor set C, which is set theoretically large in that it is uncountably infinite, is nevertheless measure theoretically as small as possible: it has *Lebesgue measure zero.* The Lebesgue measure on $\mathbb{R}$ is the good mathematical generalization of length. It is not difficult to convince oneself that the "length" of the Cantor set C should indeed be zero, by the following argument. For any $n \in \mathbb{N}$, we have $C \subset C_n$, i.e., The Cantor set C is contained in the union of 2^n intervals of length 3^{-n} each. Therefore it is reasonable to say that its length is at most $2^n\, 3^{-n} = \left(\frac{2}{3}\right)^n$. But $n \in \mathbb{N}$ was arbitrary here, and as $n \to \infty$, this upper bound on the length is tending to 0.

Another measure theoretic point of view to the Cantor set is its fractal dimension. In arbitrary metric spaces one can define Hausdorff measures of arbitrary dimensions $d \in [0, +\infty)$, and the Hausdorff dimension of a set is defined as the infimum of dimensions d such that the d-dimensional Hausdorff measure vanishes. The set C has Hausdorff dimension $\frac{\log 2}{\log 3} \approx 0.6309$, which is strictly less than the dimension 1 of any nonempty open subset of the real line, but strictly larger than the dimension 0 of countable sets such as $\mathbb{Q}$.

[2] Suppose $m \geq n$. The numbers $b_1^{(n)}, \ldots, b_n^{(n)} \in \{0, 1\}$ are chosen so that $x \in I^{(n)}_{b_1^{(n)}, \ldots, b_n^{(n)}}$. Likewise, the numbers $b_1^{(m)}, \ldots, b_m^{(m)} \in \{0, 1\}$ are chosen so that $x \in I^{(m)}_{b_1^{(m)}, \ldots, b_m^{(m)}}$. But we have the nesting property: an m:th stage interval is contained in exactly one n:th stage interval

$$I^{(m)}_{b_1^{(m)}, \ldots, b_m^{(m)}} \subset I^{(n)}_{b_1^{(m)}, \ldots, b_n^{(m)}}.$$

By disjointness of the n:th stage intervals, we must have $b_1^{(m)} = b_1^{(n)}, \ldots, b_n^{(m)} = b_n^{(n)}$.

On topological properties of the Cantor set

We mention without entering details a few interesting topological properties of the Cantor set $C \subset \mathbb{R}$. These require notions from later lectures, and are perhaps best treated as additional exercises at the suitable point in the course.

After Lecture XI you can do the following.

Exercise B.7 (The Cantor set is compact)**.**
Prove that the Cantor set C is compact.

After the definitions in Lecture V, it is possible to do the following, with concepts from Lecture XI this probably becomes easier.

Exercise (♯) B.8 (The Cantor set is nowhere dense)**.**
Prove that the Cantor set C has the property that any nonempty open set $V \subset \mathbb{R}$ contains a nonempty open set $U \subset V \subset \mathbb{R}$ such that $U \cap C = \emptyset$.

The set C is in this topological sense small; smaller even than the countable but dense set $\mathbb{Q}$ of rational numbers, which it intersets every nonempty open set $U \subset \mathbb{R}$ (which can be seen from Theorem II.21).

After Lecture XII you can do the following.

Exercise (♯) B.9 (The Cantor set is totally disconnected)**.**
Prove that all connected components of the Cantor set C are singletons.

B.4. ♡ Applications of supremum and infimum

The notions of supremum and infimum may seem a bit abstract. Indirectly, via a different formulation of the completeness axiom, we saw that supremum is important for instance for the existence of real numbers with given decimal expansions (Example II.33). But it is good to realize that supremum is also quite directly used in many common constructions in analysis. We describe a few examples here, but the intention is not to elaborate on the details in full.

♡ Arc length of a curve

One relatively simple, intuitive, and commonly used application of supremum is the length of a curve; also called *arc length*. Here we describe the definition in the context of curves in a metric space (X, d), which is introduced Lecture V. Also the notion of continuity from Lecture VI is briefly invoked.

It is also possible to treat this as an example without the general definitions, however. The reader who so prefers may consider curves in the familiar d-dimensional space $X = \mathbb{R}^d$, and interpret the distances between two points $\vec{x} = (x_1, \ldots, x_d)$ and $\vec{y} = (y_1, \ldots, y_d)$ in $\mathbb{R}^d$ as given by the usual formula

$$\mathsf{d}(\vec{x}, \vec{y}) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_d - x_d)^2}.$$

A (parametrized) curve in X is a continuous function from the unit interval $[0,1]$ to the space X,

$$\gamma\colon [0,1] \to X.$$

For a given subdivision

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$$

of the unit interval, it is natural to interpret the expression

$$\sum_{j=1}^{n} \mathsf{d}\big(\gamma(t_{j-1}),\gamma(t_j)\big) \;=\; \mathsf{d}\big(\gamma(t_0),\gamma(t_1)\big) + \cdots + \mathsf{d}\big(\gamma(t_{n-1}),\gamma(t_n)\big)$$

as an approximation of the length of the curve γ. Under refinement of the subdivision, the approximate length never decreases. Therefore the natural notion of the length of the curve, as obtained in the limit of refinements of subdivisions, is the supremum of all the approximating expressions: the **length** of the curve γ is *defined* as

$$L(\gamma) \;:=\; \sup\left\{ \sum_{j=1}^{n} \mathsf{d}\big(\gamma(t_{j-1}),\gamma(t_j)\big) \;\middle|\; n\in\mathbb{N},\; 0=t_0<t_1<\cdots<t_n=1 \right\}. \qquad \text{(B.18)}$$

In $\mathbb{R}^d$, for differentiable (or at least piecewise differentiable) curves γ, one can show that this definition of length coincides with the integral $L(\gamma) = \int_0^1 \|\gamma'(t)\|\,\mathrm{d}t$, but the integral is in fact more complicated to define precisely (see below) and it exists less generally than (B.18): it cannot be defined in general metric spaces, and even in $\mathbb{R}^d$ it requires differentiability assumptions, which depend on the chosen parametrization, etc. The staightforward definition (B.18) using supremum is clearly good!

♡ Infinite sums with non-negative terms

If we have an arbitrary collection $(s_i)_{i\in I}$ of nonnegative terms $s_i \geq 0$, then there is a quick and rather simple way to give a meaning to the sum

$$\sum_{i\in I} s_i.$$

This is rather remarkable, since we can allow the index set I to be arbitrarily large, meaning that the number of terms in the sum can be finite, countably infinite, or even uncountably infinite.

Of course a partial sum that includes only finitely many terms $s_{i_1},\ldots,s_{i_k}$, with $i_1,\ldots,i_k \in I$ distinct, is trivially defined as

$$s_{i_1} + \cdots + s_{i_k}.$$

The idea is that the other terms omitted are non-negative, so the "true sum" must be at least as large. In other words the "true sum" should be an upper bound for the set of all finite partial sums. The definition

$$\sum_{i\in I} s_i \;:=\; \sup\left\{ s_{i_1} + \cdots + s_{i_k} \;\middle|\; k\in\mathbb{N}_0,\; i_1,\ldots,i_k \in I \text{ distinct} \right\}$$

declares the "true sum" as the least upper bound for the set of all finite partial sums — i.e., it requires that the gap in between the finite partial sums and the "true sum" can be made arbitrarily small by including enough terms.

The set of all finite partial sums is nonempty (we always have the possibility of including $k = 0$ terms in the partial sum), but it may or may nor be bounded, so we obviously have to allow the possibility $\sum_{i \in I} s_i = +\infty$ (in keeping with the convention of supremum of unbounded sets) if arbitrarily large finite partial sums exist.

This definition also has the advantage that it is clear that the sum does not depend on the "order of elements" (in fact we did not even require the index set I to be ordered!).

But to be fair, this definition crucially relies on the terms being nonnegative. It does not admit easy generalizations to infinite sums with terms of both signs.[3]

Exercise B.10 (♡ Uncountably many strictly positive terms yields infinite sum).
Let $(s_i)_{i \in I}$ be a collection of nonnegative terms $s_i \geq 0$. Consider the sum $\sum_{i \in I} s_i$.

(a) Suppose that $\sum_{i \in I} s_i < +\infty$. Show that for any $m \in \mathbb{N}$, there can only exist finitely many indices $i \in I$ such that $s_i \geq \frac{1}{m}$.
(b) Suppose again that $\sum_{i \in I} s_i < +\infty$. Show that there can only exist countably many indices $i \in I$ such that $s_i > 0$.
Hint: *Consider all different $m \in \mathbb{N}$ in (a). Recall that countable unions of finite sets are countable.*
(c) Now prove that if there are uncountably infinitely many indices i corresponding to a strictly positive term $s_i > 0$, then we have $\sum_{i \in I} s_i = +\infty$.
Hint: *Contrapositive of (b).*

♡ Riemann integration

Let

$$f \colon [a, b] \to \mathbb{R}$$

be a real valued function defined on a closed interval $[a, b]$. Assume also that f is bounded in the sense that there exists some $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.

Consider a finite subdivision of the interval $[a, b]$, consisting of points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

On the subdivision interval $[x_{j-1}, x_j]$ for $j \in \{1, \ldots, n\}$, the values of the function have the greatest lower bound and least upper bound

$$s_j = \inf \left\{ f(x) \;\middle|\; x_{j-1} \leq x \leq x_j \right\} \qquad \text{and} \qquad t_j = \sup \left\{ f(x) \;\middle|\; x_{j-1} \leq x \leq x_j \right\}.$$

Using these, we define the lower and upper Riemann sums

$$S = \sum_{j=1}^{n} s_j \left(x_j - x_{j-1} \right) \qquad \text{and} \qquad T = \sum_{j=1}^{n} t_j \left(x_j - x_{j-1} \right)$$

associated with the subdivision $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

It is not particularly difficult to show that the lower and upper Riemann sums satisfy the following:

[3] Somewhat more general sums (and integrals) are studied in measure theory.

- If a subdivision $a = x'_0 < x'_1 < x'_2 < \cdots < x'_m = b$ is a refinement of the above subdivision (i.e. $\{x_0, x_1, \ldots, x_n\} \subset \{x'_0, x'_1, \ldots, x'_m\}$) and S' and T' are the associated lower and upper Riemann sums, then $S \leq S'$ and $T' \leq T$.
- If $\tilde{S}$ and $\tilde{T}$ are the lower and upper Riemann sums associated with any subdivision, then $\tilde{S} \leq T$ (and symmetrically $S \leq \tilde{T}$).

Exercise B.11 (♡ Proof of the properties of upper and lower Riemann sums)**.**
Prove the two claims above.

Since increasingly fine subdivisions lead to larger lower Riemann sums, it is natural to define the lower integral $I_-(f)$ as the least upper bound for the set of all lower Riemann sums for all subdivisions; or explicitly

$$I_-(f) := \sup \left\{ \sum_{j=1}^{n} \Big(\inf_{x \in [x_{j-1}, x_j]} f(x) \Big) \left(x_j - x_{j-1}\right) \,\middle|\, a = x_0 < x_1 < \ldots < x_n = b \right\}.$$

Similarly, since increasingly fine subdivisions lead to smaller upper Riemann sums, it is natural to define the upper integral $I_+(f)$ as the greatest lower bound for the set of all upper Riemann sums for all subdivisions; or explicitly

$$I_+(f) := \inf \left\{ \sum_{j=1}^{n} \Big(\sup_{x \in [x_{j-1}, x_j]} f(x) \Big) \left(x_j - x_{j-1}\right) \,\middle|\, a = x_0 < x_1 < \ldots < x_n = b \right\}.$$

From the earlier observations one straightforwardly gets $I_-(f) \leq I_+(f)$.

Definition B.6 (Riemann integral)**.**
A function $f \colon [a, b] \to \mathbb{R}$ is said to be **(Riemann) integrable**, if $I_-(f) = I_+(f)$. In this case we define its **(Riemann) integral** as

$$\int_a^b f(x)\,\mathrm{d}x \;:=\; I_-(f) \;=\; I_+(f).$$

From characterizations of supremum and infimum, it is quite straightforweard to prove that

Theorem B.7 (Characterization of Riemann integrability)**.**
A function $f \colon [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a subdivision such that the associated upper and lower Riemann sums satisfy $T - S < \varepsilon$.

Exercise B.12 (♡ Proof of Theorem B.7)**.**
Prove the above theorem.

To properly define the familiar integral, therefore, we end up using the notions of supremum and infimum — in fact many times.

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References

Bibliography

[Car00] N. L. Carothers. *Real analysis*. Cambridge University Press, 2000.
[Hal74] Paul R. Halmos. *Naive set theory*. Undergraduate Texts in Mathematics. Springer, 1974.
[Rud76] W. Rudin. *Principles of mathematical analysis*. McGraw-Hill, 1976.
[Väi99] Jussi Väisälä. *Topologia I*. Limes ry., 1999.