Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021-2022/III

Problem set 3

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Exercise sessions: 27.-28.1.2022 **Hand-in due: Tue 1.2.2022 at 23:59**

Topic: Metric spaces, continuous functions

Written solutions to the exercises marked with symbol ✍ are to be returned in MyCourses. Each exercise is graded on a scale 0–3. The deadline for returning solutions to problem set 3 is Tue 1.2.2022 at 23:59.

Exercise 1 (Parallelogram law and norms not induced by inner products).

(a) Prove that the norm in an inner product space V satisfies the *parallelogram law*

$$\|u+v\|^2 + \|u-v\|^2 = 2\,\|u\|^2 + 2\,\|v\|^2 \qquad \text{for all } u, v \in \mathsf{V}.$$

(b) Using part (a), prove that the norm of Example IV.25

$$\|(x,y)\|_\infty = \max\{|x|,|y|\}, \qquad \text{for } (x,y) \in \mathbb{R}^2,$$

is not induced by any inner product on $\mathbb{R}^2$.

Hint: *If the formula of part (a) does not hold for some pair of vectors $u, v \in \mathbb{R}^2$, then...*

Exercise 2 (An exotic metric on the real line).

(a) Prove that the formula

$$\mathsf{d}(x,y) = \log\big(1 + |x-y|\big)$$

defines a metric on $\mathbb{R}$.

(b) Let $A = [0,1]$ and $B = (5,10]$. Determine the diameter $\mathrm{diam}(A)$ of the set A, and the distance $\mathrm{dist}(A,B)$ between the two sets, both with respect to the metric d of part (a).

✍ **Exercise 3** (A comparison of the Euclidean norm and the ℓ^1-norm).
This exercise concerns the comparison of the norm $\|x\| = \|x\|_2 = \sqrt{\langle x,x\rangle}$ on the space $\mathbb{R}^d$ induced by the inner product $\langle x,y\rangle = \sum_{i=1}^d x_i y_i$, and the norm $\|x\|_1 = \sum_{i=1}^d |x_i|$ of Exercise 4 of Problem set 2.

(a) Prove that for all $x \in \mathbb{R}^d$ we have

$$\|x\| \;\le\; \|x\|_1 \;\le\; \sqrt{d}\,\|x\|.$$

Hint: *(From Väisälä's book) The first inequality: Write $x = \sum x_j e_j$. The second inequality: Apply the Cauchy-Schwarz inequality to the vectors $(|x_1|,\ldots,|x_d|)$ and $(1,1,\ldots,1)$.*

(b) Give a concrete example of a vector $x \neq \overline{0}$, for which the latter inequality in part (a) becomes an equality. Such an example demonstrates that the coefficient $\sqrt{d}$ in the inequality is optimal (as small as possible).

✍ **Exercise 4** (A product space metric and continuity via component functions)**.**
Let (X, d_X), (Y, d_Y), (Z, d_Z) be three metric spaces.

(a) Consider the Cartesian product $X \times Y$, and define

$$\mathsf{d}_1\colon (X \times Y) \times (X \times Y) \to [0, +\infty)$$

by the formula

$$\mathsf{d}_1\big((x_1, y_1), (x_2, y_2)\big) \;=\; \mathsf{d}_X(x_1, x_2) + \mathsf{d}_Y(y_1, y_2)$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$. Check that d_1 thus defined gives a metric on the product space $X \times Y$.

(b) Let $f\colon Z \to X$ and $g\colon Z \to Y$ be two functions. Consider the function

$$h\colon Z \to X \times Y \qquad \text{given by} \qquad h(z) = \big(f(z), g(z)\big) \;\; \text{for } z \in Z.$$

Prove that when the Cartesian product space $X \times Y$ is equipped with the metric d_1 of part (a) and when the functions f and g are continuous, then the function h is also continuous.

✍ **Exercise 5** (Some Lipschitz and non-Lipschitz functions)**.**
Let (X, d_X) and (Y, d_Y) be two metric spaces and let $K \geq 0$. Recall that a function $f\colon X \to Y$ is K-Lipschitz, if

$$\mathsf{d}_Y\big(f(a), f(b)\big) \leq \; K\, \mathsf{d}_X(a, b) \qquad \text{for all } \; a, b \in X.$$

Here, on the real line $\mathbb{R}$ and its subsets, we use the standard metric.

(a) Prove (with mean value theorem or algebraically), that the function

$$p\colon [-10, 10] \to \mathbb{R}, \qquad p(x) = x^3 \quad \text{for } x \in [-10, 10],$$

is K-Lipschitz for a suitably chosen $K \geq 0$.

(b) Prove that the function $q\colon [0,1] \to [0,1]$, $q(x) = \sqrt{x}$, is not K-Lipschitz for any $K \geq 0$ (although it is continuous).