“Galois to Automorphic” Direction of Categorical Geometric Langlands

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Contents

In a nutshell :

- *geometric* means instead of a number field F we have the field of rational functions $K := K(X)$ of some smooth, projective curve over some base field k.
- *categorical* means that the focus is not on cusp forms but instead their categorification as sheaves on Bun_G.

1 Some foundational constructions (15m)

We fix a base field k of characteristic zero and a smooth projective curve X over k. At some point, this will break can we need to use $k = \mathbb{C}$. I now describe some foundational constructions which is needed to dscribe the Galois and automorphic side of categorical geometric Langlands.

Idea 1 : Functor of points

Define the category of affine schemes Aff_k as the opposite of the category of (commutative) k-algebras. For each k-algebra A, we use $\mathrm{Spec}\, A$ to denote the corresponding object in Aff_k. The arrows are reversed because philosophically, we are viewing each A as the ring of functions on $\mathrm{Spec}\, A$.

> **Definition**

> The category of *presheaves over* k is defined as
>
> $$\mathrm{PSh}_k := \mathrm{Fun}(\mathrm{Aff}_k, \mathbf{Set}^{\mathrm{op}})$$
>
> We denote the Yoneda embedding with Spec so that given a k-algebra, $\mathrm{Spec}\, A \in \mathrm{PSh}_k$. We refer to A as the ring of functions of $\mathrm{Spec}\, A$.

The idea behind this definition is that every "space" in algebraic geometry should be determined by how affine schemes map into them. For example, the affine line $\mathbb{A}^1$ is defined as the forgetful functor $\mathrm{Aff}_k \to \mathbf{Set}^{\mathrm{op}}$, sending each affine scheme $\mathrm{Spec}\, A$ to its ring of functions A.

The category PSh_k is nice in the sense that it shares many categorical properties with $\mathbf{Set}$ so that any construction you can think of in $\mathbf{Set}$ can be done in PSh_k. As an example, we introduce the formal disk.

Intuitively, the formal disk in $\mathbb{A}^1$ around 0 is the space of points in $\mathbb{A}^1$ infinitesimally close to 0. Although the idea of "infinitesimally close" is ill-defined in analysis, the notion of nilpotence works for algebraic geometry. We first define the quotient of $\mathbb{A}^1$ by the relation of two points being infinitesimally close.

> **Definition – de Rham space**
>
> The *de Rham space of* $\mathbb{A}^1$ is defined as the presheaf such that for any affine scheme $\mathrm{Spec}\, A$,
>
> $$\mathbb{A}^1_{\mathrm{dR}}(A) := \mathbb{A}^1(A_{\mathrm{red}})$$

For any affine S, we have a closed embedding $S_{\mathrm{red}} \to S$. Restricting along this embedding turns points $S \to \mathbb{A}^1$ into points $S \to \mathbb{A}^1_{\mathrm{dR}}$. This defines a projection

$$\pi : \mathbb{A}^1 \to \mathbb{A}^1_{\mathrm{dR}}$$

Here's an example of two points which get identified under this projection. Consider $t, -t \in k[t]/(t^2) = \mathbb{A}^1(k[t]/(t^2))$, which represents two tangent vectors at 0. Then $\pi(t) = 0 = \pi(-t) \in \mathbb{A}^1((k[t]/(t^2))_{\mathrm{red}})$. This makes sense because the geometric intuition is that tangent vectors are in the first order neighbourhood of 0. We now define the formal disk.

> **Definition**
>
> The *formal disk* is defined by the pullback diagram :
>
> $$\begin{tikzcd} \mathbb{A}^1 \arrow[d, "\pi"'] & D^\wedge \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ \mathbb{A}^1_{\mathrm{dR}} & \mathrm{Spec}\, k \arrow[l, "0"] \end{tikzcd}$$

Fiber products in PSh_k are very concrete. Simply take fiber products "pointwise". This means that for each affine A, we have the pullback square :

$$\begin{tikzcd} \mathbb{A}^1(A) \arrow[d, "\pi"'] & D^\wedge(A) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ \mathbb{A}^1_{\mathrm{dR}}(A) & (\mathrm{Spec}\, k)(A) \arrow[l, "0"] \end{tikzcd}$$

This is easily computed : a map $f : \operatorname{Spec} A \to \mathbb{A}^1$ is exactly a function $f \in A$ and $(\operatorname{Spec} k)(A)$ is singleton so we are looking at exactly $f \in A$ which become zero in $(\mathbb{A}^1_{\mathrm{dR}})(A) = A_{\mathrm{red}}$. In other words, $D^\wedge(A) = N_A$ the nilradial of A.

Here is another definition of $D^\wedge$ which shows how nice PSh_k is. We have a sequence of close embeddings into $\mathbb{A}^1$:

$$\operatorname{Spec} k[t]/(t) \longrightarrow \cdots \longrightarrow \operatorname{Spec} k[t]/(t^{n+1}) \longrightarrow \cdots \longrightarrow \mathbb{A}^1$$

In PSh_k we can take the "union" i.e. colimit $\operatorname{Spf} k[\![t]\!] := \varinjlim_n \operatorname{Spec} k[t]/(t^{n+1})$ of this system of closed embeddings. What does this look like? Again, we do colimits "pointwise", that is to say, for each affine $\operatorname{Spec} A$, the above corresponds to subset of A

$$\{f \in A \mid f = 0\} \xrightarrow{\subseteq} \cdots \xrightarrow{\subseteq} \left\{f \in A \mid f^{n+1} = 0\right\} \xrightarrow{\subseteq} \cdots \xrightarrow{\subseteq} A$$

We simply take the colimit, which is just the union. We've recovered $D^\wedge(A)$! What we've just shown is the following isomorphism in PSh_k :

$$\operatorname{Spf} k[\![t]\!] \simeq D^\wedge$$

Idea 2 : Quasi-coherent sheaves on presheaves

We will need to talk about sheaves on spaces more general than schemes so let me just say a few words about quasi-coherent sheaves.

There is actually a sensible definition of quasi-coherent sheaves on *any* presheaf. (For talk, I won't bore you with the technical definition and refer to the notes here.)

> **Definition**
> Let $X \in \mathrm{PSh}_k$. A quasi-coherent sheaf $\mathcal{F}$ on X consists of the following data :
>
> – for each $x : \operatorname{Spec} A \to X$, an A-module $\mathcal{F}_x$. This is called the fiber of $\mathcal{F}$ at x.
>
> – for each f below on the left we have a f^* :
>
> $$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "y"] \arrow[d, "f"'] & X \\ \operatorname{Spec} A \arrow[ur, "x"'] & \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathcal{F}_y & \mathcal{F}_x \arrow[l, "f^*"'] \end{tikzcd} \quad \text{in } A\mathbf{Mod}$$
>
> Furthermore, we have *quasi-coherence*, meaning $f^* : \mathcal{F}_x \to \mathcal{F}_y$ induces $\mathcal{F}_x \otimes_A B \simeq \mathcal{F}_y$.
>
> – for each f, g below on the left we have

$$
\begin{tikzcd}
\mathrm{Spec}\,C \arrow[d, "g"'] \arrow[dr, "z"] & & & \mathcal{F}_z & \\
\mathrm{Spec}\,B \arrow[r, "y"] \arrow[d, "f"'] & X & \rightsquigarrow & & \mathcal{F}_y \arrow[ul, "g^*"'] \\
\mathrm{Spec}\,A \arrow[ur, "x"'] & & & \mathcal{F}_x \arrow[uu, "(fg)^*"] \arrow[ur, "f^*"'] &
\end{tikzcd}
$$

Morphisms of quasi-coherent sheaves are defined fiberwise. The collection of quasi-coherent sheaves on X naturally forms an abelian category $\mathrm{QCoh}\,X$.

The above definition is sensible in the sense that :

1. $\mathrm{QCoh}(\mathrm{Spec}\,A) \simeq A\mathbf{Mod}$ for any affine $\mathrm{Spec}\,A$.
2. for a morphism $f : X \to Y$ of presheaves, one can construct a functor $f^* : \mathrm{QCoh}\,Y \to \mathrm{QCoh}\,X$
3. For X a scheme and $\mathcal{U} = \coprod_{i \in I} U_i$ a Zariski cover by affine opens, pulling back along $j : \mathcal{U} \to X$ gives an equivalence
$$\mathrm{QCoh}\,X \xrightarrow[\sim]{j^*} \mathrm{QCoh}^*(X,\mathcal{U})$$
where the latter is descent data of modules for the cover $\mathcal{U}$. This recovers the usual definition of quasi-coherent sheaves on a scheme.

Idea 3 : de Rham space

We've seen the de Rham space construction when we defined the formal disk. I need to explain the de Rham space a bit more because this is how we're gonna get both local systems and D-modules.

One can see that the de Rham space construction applies to all presheaves. The general definition is identical. This in particular applies to our smooth projective curve X. Recall I said for motivation that X_{dR} looks like X where one identifies all infinitesimally close points. Making an analogy to sets, whenever we have an equivalence relation $R \rightrightarrows X$, we can form the quotient $X \to X/R$ and recover the equivalence relation as $R \simeq X \times_{X/R} X$. Again, by the amazing categorical properties of PSh_k, we can play this game with the projection $\pi : X \to X_{\mathrm{dR}}$. We make the following definition.

Definition – Infinitesimal Groupoid of X

The *infinitesimal groupoid of X* is defined by the fiber product diagram :

$$
\begin{tikzcd}
\widehat{\Delta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
X \arrow[r] & X_{\mathrm{dR}}
\end{tikzcd}
$$

One can then show the following :

1. by a similar argument to how we showed $D^\wedge \simeq \operatorname{Spf} k[\![t]\!]$, one can show that the union of the n-order neighbourhoods of the diagonal of X recovers the infinitesimal groupoid.

$$\varinjlim_n \Delta^{(n)} \simeq \widehat{\Delta}$$

2. We have a coequalizer diagram in PSh_k

$$\widehat{\Delta} \rightrightarrows X \to X_{\mathrm{dR}}$$

   This affirms our intuition that X_{dR} is X with infinitesimally close points identified. [1]

3. Quasi-coherent sheaves on X_{dR} are the same as quasi-coherent sheaves on X which "respect the equivalence relation $\widehat{\Delta}$" in the sense that such $\mathcal{F} \in \mathrm{QCoh}\, X$ are equipped with isomorphisms $\tau_{x,y} : \mathcal{F}_x \simeq \mathcal{F}_y$ for any pair (x, y) of infinitesimally close points and that these isomorphisms needs to be "reflexive and transitive". The formal way of saying this is quasi-coherent sheaves on X equipped with descent data w.r.t. the groupoid $\widehat{\Delta}$.

   In fact, one can show that descent data w.r.t. $\widehat{\Delta}$ on a quasi-coherent sheaf $\mathcal{F}$ on X is equivalent to giving a flat connection $\Delta : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/k}$. In particular, this gives the definition

   > **Definition – D-modules on a presheaf**
   > Let X be a presheaf. Then the category of D-modules on X is defined as
   >
   > $$D\mathbf{Mod}(X) := \mathrm{QCoh}(X_{\mathrm{dR}})$$

4. More generally than quasi-coherent sheaves, via pullback along $\pi : X \to X_{\mathrm{dR}}$, the category of presheaves over X_{dR} is equivalent to the category of presheaves over X which "respect the equivalence relation $\widehat{\Delta}$". [2]

Points (3) and (4) combined say that objects living over X_{dR} should be thought of as objects on X equipped with a connection. In particular, a quasi-coherent sheaf $\mathcal{F}$ on X_{dR} whose underlying quasi-coherent sheaf on X is a vector bundle of rank n should be thought of as a n-dimensional representation of the fundamental group of X. This is literally true when $k = \mathbb{C}$ via the equivalence between

1. Vector bundles on X equipped with a flat connection. This gives an isomorphism of fibers when points are "infinitesimally close".

2. Locally constant sheaves on X valued in (finite dimensional) $\mathbb{C}$-vector spaces. This gives isomorphism of fibers when points are related by being in a common open that is small enough.

3. Functors from the fundamental groupoid of X into the category of (finite dimensional) vector spaces over $\mathbb{C}$. This gives isomorphism of fibers when points are related by a path-up-to-homotopy.

For the Galois side of geometric Langlands, we will want group morphisms $\pi_1(X, x) \to G^L$ where G^L is the Langlands dual of a reductive group G over k. The above analogies then suggest that the correct formulation

[1] This actually uses formal smoothness of X.

[2] Again, the formal way of saying this is "equipped with descent data w.r.t the groupoid $\widehat{\Delta}$".

of the Galois side is G^L *bundles on X equipped with a connection*. [1] This brings us to the question : how to define G^L bundles on something like X_{dR} ? A more important question is : can we also make the collection of G^L bundles on X_{dR} into an object of algebraic geometry i.e. a presheaf over k?

Idea 4 : Stack quotients by group actions

Let me first say that the standard definition of G bundles on a scheme X generalise to the setting where G is any group object in PSh_k and X is any presheaf. [2] I leave the definition to the notes.

> **Definition**
>
> Let G be a group object in PSh_k and X a general presheaf. Then a G bundle on X is a morphism $P \to X$ where
>
> 1. P is equipped with a right G-action, X the trivial action, and $P \to X$ is G-equivariant
> 2. for all points $x : S \to X$ where S is affine, there exists an fpqc affine $T \to S$ such that $T \times_X P$ is isomorphic to $T \times G$ as G-actions.

And now we try to define the *classifying space of G bundles*. This is related to taking quotients by group actions so I will describe the general procedure. If I had more time, I would do a more detailed motivation for stacks. but we're short on time here so let me just give you the answer people came up with and some intuition to make it believable.

> **Definition – Quotient stack**
>
> Let X be a presheaf and G a group presheaf. Then the *(fpqc) quotient stack* X / G is defined as the "functor" that sends each affine S to the *groupoid* of diagrams
>
> $$\begin{tikzcd} P \arrow[r, "G\text{ equiv}"] \arrow[d, "\text{G bundle}"'] & X \\ S \end{tikzcd}$$

Intuition for this definition :

1. Analogy with group actions on sets : Assume momentarily that X and G and S are sets. Then a map $S \to X$ is equivalent to a diagram

$$\begin{tikzcd} S \times G \arrow[r, "G\text{ equiv}"] \arrow[d, "\text{G bundle}"'] & X \\ S \end{tikzcd}$$

   Intuitively, "the points in S maps to points in X which gives G orbits in the obvious way". The set of such diagrams forms a groupoid. Intuitively, a path moves the S-family of points in X along their individual orbits.

[1] A precise formulation of this without using de Rham spaces can be found in [BB-93].
[2] Another name for G bundles is G *torsors*.

$$\begin{tikzcd} S \times G \arrow[r, "x"] \arrow[d] \arrow[dr, "g"'] & X \\ S & S \times G \arrow[l] \arrow[u, "y=g.x"'] \end{tikzcd}$$

We see that the path connected components of this groupoid bijects with the set of S-families of G orbits in X, i.e. maps $S \to X/G$.

Now back to algebraic geometry, the idea is the same except we allow the S-family of copies of G to "vary non-trivially across S". This is exactly what a G bundle on S is.

2. There is a map $X \to X/G$. For each point $x : S \to X$ where S is affine, note that this is equivalent to a diagram of the above form. This says "points of X give rise to G-orbits in X". So we send $x \in X(S)$ to the point in $(X/G)(S)$ which is the trivial G bundle on S.

3. Maps into X/G glue. This is a natural generalisation of the fact that maps into a fixed scheme glue which reconciles with the fact that points of X/G now have *possibly non-trivial "equalities"*. [1] The fancy way of saying this is that X/G satisfies (fpqc) descent. "Groupoid valued presheaves which satisfy descent" are what people mean by *stacks*.

4. Taking $X = \operatorname{Spec} k$ equipped with the trivial action gives the *classifying stack of G bundles*, usually denoted with BG. [2]

5. $\operatorname{QCoh} X/G$ is equivalent to the category $\operatorname{QCoh}^*(X, X \times G)$ of quasi-coherent sheaves on X equipped with equivariance w.r.t. the action of G. [3] In particular, $\operatorname{QCoh}(BG) \simeq \operatorname{Rep}_k G$.

Now that I've given some reasons for why we want to use stacks, one can hopefully see that technically speaking we now need to redo the foundations, replacing presheaves with *prestacks*. This is what [Lurie-HTT] does for us. Modulo 500+ pages of technicalities, the definition is the same except sets are replaced with sets with paths and paths between paths and so on, A.K.A. *∞-groupoids*. [4]

> **Definition**
>
> The ∞-category of *prestacks over k* is defined as the functor category
>
> $$\mathrm{PStk}_k := \mathrm{Fun}(\mathrm{Aff}_k, \infty\mathrm{Grpd}^{\mathrm{op}})$$
>
> where $\infty\mathrm{Grpd}$ is the ∞-category of ∞-groupoids.
>
> There is a full subcategory of prestacks satisfying (fpqc) descent. This is called the category of *stacks over k*.
>
> $$\mathrm{Stk}_k \subseteq \mathrm{PStk}_k$$

[1] If anyone asks, this also explains why we want groupoids. If we took isomorphism classes instead, then maps don't glue. Give the $\mathbb{BG}_m$ and $\mathbb{P}^1$ example.

[2] Fingers crossed no one asks about the universal property of stack quotient. I can't find reference in [Lurie-HTT].

[3] Formally, equivariance w.r.t. the action groupoid $G \times X \rightrightarrows X$.

[4] If anyone asks : technically, we only need 1-groupoids and this is what [BD] does. But all the latest papers, e.g. by Gaitsgory, are done in the language of higher categories as developed by Lurie.

2 Automorphic and Galois side (10min)

Now I can give you an impression of the Galois and automorphic side. Reminder, X is a smooth projective curve over a field k of characteristic zero. We now let $k = \mathbb{C}$ and introduce a reductive group G over k and let G^L denote the Langlands dual. [1]

> **Definition – Galois side**
> The *stack of G^L local systems on X* is defined as
>
> $$\mathrm{LocSys}_{G^L} := \underline{\mathrm{Hom}}(X_{\mathrm{dR}}, BG^L)$$
>
> [a]
>
> [a] The $\underline{\mathrm{Hom}}$ here is internal hom in PStk_k.

I've already mentioned the intuition behind this but let me just say it again : k valued points of this LocSys_{G^L} are maps $X_{\mathrm{dR}} \to BG^L$, which are G^L bundles on X equipped with a connection.

Now I describe the automorphic side.

> **Definition – Automorphic side**
> The *stack of G bundles on X* is defined as
>
> $$\mathrm{Bun}_G X := \underline{\mathrm{Hom}}(X, BG)$$
>
> The automorphic side the categorical geometric Langlands is defined as the category
>
> $$D\mathbf{Mod}(\mathrm{Bun}_G X)$$

The above is the categorical geometric analogue of the space of (unramified) cusp forms. Here are some questions you might have :

1. Question : Why $\mathrm{Bun}_G X$? Short answer : geometric intepretation of adeles. Long answer : let's look at the example of $G = \mathrm{GL}_n$ and $X = \mathbb{P}^1$. Then $\pi_0 \mathrm{Bun}_G X(k)$ is the set of isomorphism classes of GL_n bundles on $\mathbb{P}^1$, equivalently rank n vector bundles on $\mathbb{P}^1$. There's an easy classification : take the standard cover $U_0 + U_1 \to \mathbb{P}^1$ with $U_i = \mathbb{A}^1$. Then descent for rank n vector bundles says that a rank n vector bundle E on $\mathbb{P}^1$ is equivalently a pair of rank n vector bundles E_i on U_i equipped with an isomorphism $\alpha : E_0 \to E_1$ over $U_0 \cap U_1$. But vector bundles on $\mathbb{A}^1$ can all be trivialised, giving us an isomorphism of descent data :

$$\begin{tikzcd}
E_0 & E_0|_{U_{01}} \arrow[l] \arrow[r, "\alpha"] & E_1|_{U_{01}} \arrow[r] & E_1 \\
\mathcal{O}_{U_0}^{\oplus n} \arrow[u, "s_0"] & \mathcal{O}_{U_{01}}^{\oplus n} \arrow[l] \arrow[u, "s_0"] \arrow[r, "s_1^{-1}\alpha s_0"'] & \mathcal{O}_{U_{01}}^{\oplus n} \arrow[u, "s_1"] \arrow[r] & \mathcal{O}_{U_1}^{\oplus n} \arrow[u, "s_1"]
\end{tikzcd}$$

[1] which now also leaves over k because $k = \mathbb{C}$ is algebraically closed?

So for each line bundle E on $\mathbb{P}^1$ with trivialisations s_i on U_i, we obtain an element $s_1^{-1}\alpha s_0 \in \mathrm{GL}_n(U_{01})$. The trivialisations s_i form a torsor for $\mathrm{GL}_n(U_i)$. Thus, taking isomorphism classes gives us

$$\pi_0 \operatorname{Bun}_{\mathrm{GL}_n} X(k) \simeq \mathrm{GL}_n(U_0)\backslash \mathrm{GL}_n(U_{01})/ \mathrm{GL}_n(U_1)$$

Now we show how adeles appear.

(a) For a fixed open U of the curve with reduced complement Z. The idea is to replace the cover U_0, U_1 by $U, \coprod_{x\in Z} D_x$. [1] One has

$$\begin{aligned}
&\{\text{isomorphism class of rank } n \text{ vector bundles on } X \text{ which trivialise on } U\} \\
\simeq\ & \mathrm{GL}_n(U)\backslash \mathrm{GL}_n(U \cap \coprod_{x\in S} D_x)/ \mathrm{GL}_n(\coprod_{x\in Z} D_x) \\
\simeq\ & \mathrm{GL}_n(U)\backslash \mathrm{GL}_n(\prod_{x\in U} \mathcal{O}_x^\wedge \times \prod_{x\in Z} K_x^\wedge)/ \prod_{x\in Z} \mathrm{GL}_n(\mathcal{O}_x^\wedge)
\end{aligned}$$

(b) Now take the union across all opens U. $\mathrm{GL}_n(U)$ becomes $\mathrm{GL}_n(K)$ [2], middle term becomes $\mathrm{GL}_n(\mathbb{A}_X)$ and the right hand term $\mathrm{GL}_n(\mathbb{O}_X)$. We obtain what's called the *Weil uniformisation*.

$$\pi_0 \operatorname{Bun}_{\mathrm{GL}_n} X(k) \simeq \mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_X)/ \mathrm{GL}_n(\mathbb{O}_X)$$

2. Question : why D-modules on $\operatorname{Bun}_G X$? Short answer : perverse sheaves categorify $\overline{\mathbb{Q}_\ell}$-valued functions. [3] Over $k = \mathbb{C}$, D-modules (with certain conditions) equivalent to perverse sheaves via Riemann–Hilbert correspondence.

3 Definition of Hecke Eigensheaves (10min)

Finally, we are ready to describe the Galois to automorphic side of categorical geometric Langlands. This requires defining the Galois and automorphic Hecke actions of $\operatorname{Rep}_k G^L$ on $D\mathbf{Mod}(\operatorname{Bun}_G)$. We begin with the geometric analogue of the local spherical Hecke algebras from arithmetic Langlands. Let $x \in X(k)$.

Definition

The *category of spherical D-modules at x* is defined as

$$\mathrm{Sph}_{G,x} := D\mathbf{Mod}(G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge)/G(\mathcal{O}_x^\wedge))$$

where $G(\mathcal{O}_x^\wedge), G(K_x^\wedge)$ are the *arc and loop group at x*, suitably defined as presheaves over k. We will call the (fpqc) double quotient

$$\mathrm{Hecke}_x^{\mathrm{Loc}} := G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge)/G(\mathcal{O}_x^\wedge)$$

the *local Hecke correspondence at x*. Using the same "transition map" argument we did for vector bundles on $\mathbb{P}^1$, one can show that $\mathrm{Hecke}_x^{\mathrm{Loc}}$ fits in a cartesian square

[1] If anyone asks, this works by fpqc descent and the fact that X is finite type.

[2] GL_n separated so maps $\mathrm{GL}_n(U) \subseteq \mathrm{GL}_n(V)$ for $V \subseteq U$

[3] Precise statements can be found in [KW, Chapter III, Theorem 12.1]

$$\begin{tikzcd} \mathrm{Bun}_G D_x & \mathrm{Hecke}_x^{\mathrm{Loc}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ \mathrm{Bun}_G D_x^\circ \arrow[from=u] & \mathrm{Bun}_G D_x \arrow[l] \end{tikzcd}$$

This allows one to define a symmetric monoidal tensor product on $\mathrm{Sph}_{G,x}$ via convolution. Essentially, matrix multiplication. [a]

[a] Actually, I still haven't proven this. [BZN-15] talks about this but doesn't give reference.

We will define an action of $\mathrm{Sph}_{G,x}$ on $D\mathbf{Mod}(\mathrm{Bun}_G)$. But to do this, we need to know how to make D-modules on Bun_G using a point $x \in X(k)$. We have the following

Definition – Full level structure at x

Define the *stack of G bundles on X equipped with full level structure at x* by the cartesian diagram

$$\begin{tikzcd} \mathrm{Bun}_G X & \mathrm{Bun}_{G,x}^{\mathrm{lvl}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ \mathrm{Bun}_G D_x \arrow[from=u] & \mathrm{Spec}\, k \arrow[l, "\mathrm{triv}"'] \end{tikzcd}$$

There is an action of $G(K_x^\wedge)$ on Bun_G^x

$$\mathrm{ReGlue} : \mathrm{Bun}_{G,x}^{\mathrm{lvl}} \times G(K_x^\wedge) \to \mathrm{Bun}_{G,x}^{\mathrm{lvl}}$$

given intuitively by the following : For $g \in G(K_x^\wedge)$, P a G bundle on X and $s : \mathrm{Triv}|_{D_x} \to P|_{D_x}$, make the following descent data :

$$\mathrm{Triv}|_{D_x} \longleftarrow \mathrm{Triv}|_{D_x^\circ} \xrightarrow{sg} P|_{D_x^\circ} \longrightarrow P|_{X\setminus\{x\}}$$

This then $\mathrm{ReGlue}(g, P, s)$ is defined as the G bundle on X obtained by gluing the above descent data. [a]

[a] If anyone asks : Technically speaking, we must be able to do this gluing in families. Although fpqc descent is not enough to glue over non locally finite type spaces, the local finite type condition of X ensures that we only need to know how to glue in finite type situations. This is then covered by fpqc descent. Alternatively, if one doesn't care about being minimal, one can use the Beauville–Laszlo theorem.

There is a map

$$\mathrm{Bun}_{G,x}^{\mathrm{lvl}} \to \mathrm{Bun}_G$$

obtained by forgeting the full level structure at x. This morphism is equivariant w.r.t. the action of $G(\mathcal{O}_x^\wedge)$ on $\mathrm{Bun}_{G,x}^{\mathrm{lvl}}$ and in fact is a $G(\mathcal{O}_x^\wedge)$ bundle. We obtain

$$\mathrm{Bun}_G \simeq G(\mathcal{O}_x^\wedge)\backslash \mathrm{Bun}_{G,x}^{\mathrm{lvl}}$$

Thus, D-modules on Bun_G can be obtained from arc-group-equivariant D-modules on $\mathrm{Bun}^{\mathrm{lvl}}_{G,x}$. This gives us a way of defining the local Hecke action by *convolution*.

Definition – Local Hecke action
Consider the diagram

$$\begin{tikzcd} G(K_x^\wedge) & G(K_x^\wedge) \times \mathrm{Bun}^{\mathrm{lvl}}_{G,x} \arrow[l, "p"'] \arrow[r, "p_1"'] \arrow[d, "p_0"'] & \mathrm{Bun}^{\mathrm{lvl}}_{G,x} \\ & \mathrm{Bun}^{\mathrm{lvl}}_{G,x} & \end{tikzcd}$$

We can take a lot of (fpqc) quotients by the arc group and get

$$\begin{tikzcd} G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge)/G(\mathcal{O}_x^\wedge) & G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge) \times^{G(\mathcal{O}_x^\wedge)} \mathrm{Bun}^{\mathrm{lvl}}_{G,x} \arrow[l, "p"'] \arrow[r, "p_1"'] \arrow[d, "p_0"'] & G(\mathcal{O}_x^\wedge)\backslash \mathrm{Bun}^{\mathrm{lvl}}_{G,x} \simeq \mathrm{Bun}_G \\ & G(\mathcal{O}_x^\wedge)\backslash \mathrm{Bun}^{\mathrm{lvl}}_{G,x} \arrow[d, phantom, "\wr"] & \\ & \mathrm{Bun}_G & \end{tikzcd}$$

The *local Hecke action* is defined as[a]

$$\begin{aligned} \mathrm{Sph}_{G,x} \times D\mathbf{Mod}(\mathrm{Bun}_G) &\to D\mathbf{Mod}(\mathrm{Bun}_G) \\ \mathcal{S}, \mathcal{F} &\mapsto H_x(\mathcal{S}, \mathcal{F}) := (p_1)_* \left(p^! S \boxtimes p_0^! \mathcal{F}\right) \end{aligned}$$

[a] if anyone asks : give nice interpretation of middle term as Hecke correspondence at x.

In order to make the definition of Hecke eigensheaf, we need to bundle of the local Hecke actions together into a single action over X. This is roughly done in the following steps :

1. The arc and loop groups relativises to group presheaves over X

$$\begin{tikzcd} G(\mathcal{O}_X^\wedge) \arrow[rr] \arrow[dr] & & G(K_X^\wedge) \arrow[dl] \\ & X & \end{tikzcd}$$

   The local Hecke correspondences also relativises

$$\mathrm{Hecke} := G(\mathcal{O}_X^\wedge)\backslash G(K_X^\wedge)/G(\mathcal{O}_X^\wedge) \to X$$

2. The family $\mathrm{Bun}^{\mathrm{lvl}}_{G,x} \to \mathrm{Bun}_G$ relativises to

$$\mathrm{Bun}^{\mathrm{lvl}}_G \to X \times \mathrm{Bun}_G$$

and the action so does the action of the loop group

$$G(K^{\wedge}_X) \times_X \mathrm{Bun}^{\mathrm{lvl}}_G \to \mathrm{Bun}^{\mathrm{lvl}}_G$$

3. The convolution diagram relativises :

$$\begin{tikzcd} G(\mathcal{O}^{\wedge}_X)\backslash G(K^{\wedge}_X)/G(\mathcal{O}^{\wedge}_X) & G(\mathcal{O}^{\wedge}_X)\backslash G(K^{\wedge}_X) \times^{G(\mathcal{O}^{\wedge}_X)} \mathrm{Bun}^{\mathrm{lvl}}_{G,X} \arrow[l, "p"'] \arrow[r, "p_1"'] \arrow[d, "p_0"'] & G(\mathcal{O}^{\wedge}_X)\backslash \mathrm{Bun}^{\mathrm{lvl}}_{G,X} \simeq X \times \mathrm{Bun}_G \\ & G(\mathcal{O}^{\wedge}_X)\backslash \mathrm{Bun}^{\mathrm{lvl}}_{G,X} \arrow[d] & \\ & \mathrm{Bun}_G & \end{tikzcd}$$

This defines the *global Hecke action* : [1]

$$\begin{aligned} \mathrm{Sph}_{G,X} \times D\mathbf{Mod}(\mathrm{Bun}_G) &\to D\mathbf{Mod}(X \times \mathrm{Bun}_G) \\ \mathcal{S}, \mathcal{F} &\mapsto H_x(\mathcal{S}, \mathcal{F}) := (p_1)_* \left(p^! S \boxtimes p_0^! \mathcal{F}\right) \end{aligned}$$

The above was all on the automorphic side. Finally, we make contact with the Galois side.

> **Proposition – Geometric Satake**
> Consider the category of spherical D-modules
>
> $$\mathrm{Sph}_G := D\mathbf{Mod}(\mathrm{Hecke}^{\mathrm{Loc}})$$
>
> where $\mathrm{Hecke}^{\mathrm{Loc}} := G[\![t]\!]\backslash G((t))/G[\![t]\!]$. This is the "coordinate" version of $\mathrm{Sph}_{G,x}$ we have seen before. [a] Then the convolution product on Sph_G makes Sph_G into a rigid tensor category. There exists a (faithful exact) tensor functor $F : \mathrm{Sph}_G \to \mathbf{Vec}_k$ which induces an equivalence of tensor categories
>
> $$\mathrm{Sph}_G \simeq \mathrm{Rep}_k\, G^L, \mathcal{S}_V \leftarrow\!\shortmid V$$
>
> via the Tannakian formalism. [b]
>
> [a] If anyone asks : given a choice $\mathbb{C}[\![t]\!] \cong \mathcal{O}^{\wedge}_x$, we have $\mathrm{Sph}_G \simeq \mathrm{Sph}_{G,x}$.
> [b] If anyone asks : roughly speaking, this recovers the classical Satake isomorphism by taking Grothendieck rings on both sides. RHS has basis by dominant coweights $X_\bullet(T)^+$. See [Zhu16, Section 5.6].

The fibers of $\mathrm{Hecke} \to X$ are the local Hecke correspondences $\mathrm{Hecke}^{\mathrm{Loc}}_x$ at each point $x \in X(k)$ and given a choice of uniformizer at x, we have $\mathrm{Sph}_G \simeq \mathrm{Sph}_{G,x}$ as tensor categories. We take on faith that this idea

[1] If anyone asks : give geometric interpretation of middle term as Hecke correspondence.

allows one to take $\mathcal{S} \in \mathrm{Sph}_G$ and extend it across $\mathrm{Hecke}^{\mathrm{Loc}}$ in the X-direction, [1] giving a functor

$$\begin{aligned} \mathrm{Sph}_G &\to \mathrm{Sph}_{G,X} \\ \mathcal{S} &\mapsto \tilde{\mathcal{S}} \end{aligned}$$

This gives us *two* actions of $\mathrm{Sph}_G \simeq \mathrm{Rep}_k G^L$ on $D\mathbf{Mod}(\mathrm{Bun}_G)$.

1. On the automorphic side : we let Sph_G act through all points at once via $\mathrm{Sph}_{G,X}$

$$\begin{aligned} \mathrm{Sph}_G \times D\mathbf{Mod}(\mathrm{Bun}_G) &\to D\mathbf{Mod}(X \times \mathrm{Bun}_G) \\ \mathcal{S}, \mathcal{F} &\mapsto H(\tilde{\mathcal{S}}, \mathcal{F}) \end{aligned}$$

2. On the Galois side : given a G^L local system σ on X, we can pullback representations of G^L along σ :

$$X_{\mathrm{dR}} \xrightarrow[\sigma]{} BG^L \qquad \rightsquigarrow \qquad D\mathbf{Mod}(X) \xleftarrow{\sigma^*} \mathrm{Rep}_k G^L$$

   This sends a representation V to the associated bundle V_σ which has a connection via σ. So we have the action :

$$\begin{aligned} \mathrm{Rep}_k G^L \times D\mathbf{Mod}(\mathrm{Bun}_G) &\to D\mathbf{Mod}(X \times \mathrm{Bun}_G) \\ V, \mathcal{F} &\mapsto V_\sigma \boxtimes \mathcal{F} \end{aligned}$$

Asking for these two actions to coincide is (roughly) the definition of a Hecke eigensheaf.

> **Definition**
> Let $\sigma \in \mathrm{LocSys}_{G^L} X(k)$ be a G^L local system on X. Then a *Hecke eigensheaf with eigenvalue* σ is a $\mathcal{F} \in D\mathbf{Mod}(\mathrm{Bun}_G)$ equipped with isomorphisms for each $V \in \mathrm{Rep}_k G^L$,[a]
>
> $$H(\tilde{\mathcal{S}_V}, \mathcal{F}) \simeq V_\sigma \boxtimes \mathcal{F}$$
>
> [a]Technically, there are more conditions. See [Gomez, Section 4.3].

4 How to make Hecke Eigensheaves? (10min)

Main question : given a $G^\vee$ local system E on X, how to make a Hecke eigensheaf $\mathcal{F}$ on X with eigenvalue E?

- table of Hecke paradigm / BB localisation comparing classical case and infinite dim case
- Localisation functor

$$(\mathfrak{g} \otimes K_x^\wedge, G(\mathcal{O}_x^\wedge))\mathbf{Mod} \to D\mathbf{Mod}(\mathrm{Bun}_G)$$

[1] An account of this is given in [Gomez, Section 4.2]. However, it is somewhat unsatisfactory because the choice of a point $x \in X(k)$ is used whereas the global Hecke action doesn't mention any point in particular. A fix should be $\mathrm{Hecke} \simeq X^\wedge \times^{\mathrm{Aut}\, D^\wedge} \mathrm{Hecke}^{\mathrm{Loc}}$ where $X^\wedge$ is the moduli space of formal parameters of X and $\mathrm{Aut}\, D^\wedge$ is the pro-algebraic group of automorphisms of the formal disk preserving 0.

– Question : what is the deal with the quantized Hitchin system? Answer : NOT CLEAR. I know Hitch is commutative subalgebra of $\Gamma(\mathrm{Bun}_G, D^{\mathrm{crit}})$ and that $\mathrm{Spec}\,\mathrm{Hitch}$ closed embeds into $\mathrm{LocSys}_{G^\vee}$ but how does this line up with localisation?

  TODO : reread 2nd overview by Gaitsgory and intro in [FG-06].

– Question : how does $\mathrm{Spec}\,\mathrm{Hitch}$ relate to local system? Answer : Feigin–Frenkel isomorphism. Analogous to Harish-Chandra isomorphism

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