The "Galois to automorphic" direction of categorical geometric Langlands

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Abstract

In this talk, I give a statement of the "Galois to automorphic" direction of categorical geometric Langlands. I will describe the Galois and automorphic side, the Hecke action on both sides, and the definition of Hecke eigensheaves. On the way, I hope to give motivation for the various objects at play : the stack of G^L local systems on the fixed curve X, the stack of G bundles on X, D-modules, arc groups, loop groups, the affine Grassmannian, and geometric Satake.

Contents

In a nutshell :

- *geometric* means instead of a number field F we have the field of rational functions $K := K(X)$ of some smooth, projective curve over some base field k, which we will assume to be $\mathbb{C}$ in this talk.
- *categorical* means that the focus is not on cusp forms but instead their categorification as sheaves on Bun_G.

1 Galois side

I begin by giving an impression of the Galois side. Let G be a reductive group over k and let G^L denote the Langlands dual. [1]

In geometric Langlands, the L-parameters will be group morphisms $\pi_1(X, x) \to G^L$. The definition of the Galois side is as follows.

[1]Because we are working with $k = \mathbb{C}$, G^L is also over k with root datum transpose to that of G.

> **Definition – Galois side**
> The *stack of G^L local systems on X* is defined as
> $$\mathrm{LocSys}_{G^L} := \underline{\mathrm{Hom}}(X_{\mathrm{dR}}, BG^L)$$

Let me give some comments on what this has to do with morphisms $\pi_1(X,x) \to G^L$.

1. (over de Rham space = connection) X_{dR} here is the *de Rham space* of X. It is the quotient of X by the "equivalence relation"
$$\widehat{\Delta} \rightrightarrows X \times X$$
which says $(x,y) \in \widehat{\Delta}$ iff x, y are "*infinitesimally close*". $\widehat{\Delta}$ is called the *infinitesimal groupoid* and one can show that
$$\varinjlim_n \Delta^{(n)} \simeq \widehat{\Delta}$$
where $\Delta^{(n)}$ is the n-order neighbourhood of the diagonal of X. One can also show that pulling back along the quotient map $\pi : X \to X_{\mathrm{dR}}$ induces an equivalence
$$\mathrm{QCoh}\, X_{\mathrm{dR}} \xrightarrow[\sim]{\pi^*} \mathrm{QCoh}^*(X, \widehat{\Delta})$$
where the latter consists of $\mathcal{F} \in \mathrm{QCoh}\, X$ which "respect the equivalence relation $\widehat{\Delta}$" in the sense that such $\mathcal{F} \in \mathrm{QCoh}\, X$ are equipped with isomorphisms $\tau_{x,y} : \mathcal{F}_x \simeq \mathcal{F}_y$ for any pair (x,y) of infinitesimally close points and that these isomorphisms needs to be "reflexive and transitive". The formal way of saying this is quasi-coherent sheaves on X equipped with descent data w.r.t. the groupoid $\widehat{\Delta}$. It turns out that descent data w.r.t $\widehat{\Delta}$ is equivalent to giving a flat connection on $\mathcal{F}$. [BDSeminar, Theorem 0.4] This gives us a definition / theorem.

 > **Definition**
 > The *category of D-modules on X* is defined to be
 > $$D\mathbf{Mod}\, X := \mathrm{QCoh}\, X_{\mathrm{dR}}$$

 Why is this relevant? It's relevant because over $\mathbb{C}$ there are three equivalent ways of thinking about local systems :

$$\begin{tikzcd}
\text{(finite) representations of fundamental group(oid) of } X \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \text{locally const (finite rank) sheaves on } X \arrow[dl, "\sim"] \\
& \text{vector bundles on } X \text{ equipped with flat connection} &
\end{tikzcd}$$

 So this tells us that a morphism $\pi_1(X,x) \to G^L$ can be thought of as a G^L bundle on X equipped with a connection. Using the de Rham space, this is precisely a G^L bundle over X_{dR}. which brings me to the second comment.

2. BG^L here is the *classifying stack of G^L bundles.* [1] It does what you expect it to do : a map $S \to BG^L$ from a scheme S is precisely a diagram of the form

$$\begin{tikzcd} P \arrow[r, "G^L \text{ equiv}"] \arrow[d, "G^L \text{ bundle}"'] & \mathrm{pt} \\ S \end{tikzcd}$$

Unfortunately, we're short on time for a good motivation for stacks. One thing we need to know is : there is a point $\pi : \mathrm{pt} \to BG^L$ corresponding to the trivial G^L bundle on pt and pullback of quasi-coherent sheaves along π^* induces

$$\pi^* : \mathrm{QCoh}\, BG^L \xrightarrow{\sim} \mathrm{Rep}_k\, G^L$$

3. (Technical remark. Only mention if anyone asks.) The $\underline{\mathrm{Hom}}$ here is internal hom in PStk_k, the ∞-category of prestacks over k. This is the correct setting which reconciles the two conflicting goals : taking quotients and wanting descent.

2 Automorphic side

Now I describe the automorphic side.

> **Definition – Automorphic side**
> The *stack of G bundles on X* is defined as
>
> $$\mathrm{Bun}_G\, X := \underline{\mathrm{Hom}}(X, BG)$$
>
> The automorphic side the categorical geometric Langlands is defined as the category
>
> $$D\mathbf{Mod}(\mathrm{Bun}_G\, X)$$
>
> I might only write Bun_G sometimes.

The above is the categorical geometric analogue of the space of (unramified) cusp forms. Here are some questions you might have :

1. Question : Why $\mathrm{Bun}_G\, X$? Short answer : geometric interpretation of double coset space of adeles. Long answer : By "geometric interpretation" I mean that the path connected components of the groupoid $\mathrm{Bun}_G(k)$ gives back the double coset space of adeles. First, let me show you why double cosets appear by looking at the example of $G = \mathrm{GL}_1$ and $X = \mathbb{P}^1$. Then $\pi_0\, \mathrm{Bun}_G\, X(k)$ is the set of isomorphism classes of GL_1 bundles on $\mathbb{P}^1$, equivalently line bundles on $\mathbb{P}^1$. There's an easy classification : take the standard cover $U_0 + U_1 \to \mathbb{P}^1$ with $U_i = \mathbb{A}^1$. Then fpqc descent says that a a line bundle L on $\mathbb{P}^1$ is equivalently a pair of line bundles L_i on U_i equipped with an isomorphism $\alpha : L_0 \to E_1$ over $U_0 \cap U_1$. But vector bundles on $\mathbb{A}^1$ can all be trivialised, giving us an isomorphism of descent data :

[1] If anyone asks : give intuition of (fpqc) stack quotient.

$$
\begin{tikzcd}
L & \rightsquigarrow & L_0 & L_0|_{U_{01}} \arrow[l] \arrow[r, "\alpha"] & L_1|_{U_{01}} \arrow[r] & L_1 \\
 & & \mathcal{O}_{U_0} \arrow[u, "s_0"] & \mathcal{O}_{U_{01}} \arrow[l] \arrow[u, "s_0"] \arrow[r, "s_1^{-1}\alpha s_0"'] & \mathcal{O}_{U_{01}} \arrow[u, "s_1"] \arrow[r] & \mathcal{O}_{U_1} \arrow[u, "s_1"]
\end{tikzcd}
$$

So each line bundle L on $\mathbb{P}^1$ with trivialisations s_i on U_i is determined up to isomorphism by the element $s_1^{-1}\alpha s_0 \in \mathrm{GL}_n(U_{01})$. In other words we have the isomorphism below :

$$
\begin{tikzcd}
\pi_0 \left\{\text{line bundles on } \mathbb{P}^1 \text{ with } s_0, s_1\right\} \arrow[d] & \mathrm{GL}_1(U_{01}) \arrow[l, "\sim"'] \arrow[d] \\
\pi_0(\mathrm{Bun}_{\mathrm{GL}_1} \mathbb{P}^1)(k) & GL_1(U_0)\backslash GL_1(U_{01})/ \mathrm{GL}_1(U_1) \arrow[l, "\sim"]
\end{tikzcd}
$$

The set of trivialisations s_i form torsors for $\mathrm{GL}_1(U_i)$ respectively. Thus, forgetting the trivialisations gives us the bottom bijection.

Now we show how adeles appear.

(a) For a fixed open U of the curve. The idea is to replace the cover U_0, U_1 by $U, \coprod_{x\in X} D_x$ where D_x is the disk around $x \in X(k)$. [1] One has

$$
\begin{aligned}
&\{\text{isomorphism class of line bundles on } X \text{ which trivialise on } U\} \\
&\simeq \mathrm{GL}_1(U)\backslash \mathrm{GL}_1(U \cap \coprod_{x\in X} D_x)/ \mathrm{GL}_1(\coprod_{x\in X} D_x) \\
&\simeq \mathrm{GL}_1(U)\backslash \mathrm{GL}_1(\prod_{x\in U} \mathcal{O}_x^\wedge \times \prod_{x\notin U} K_x^\wedge)/ \mathrm{GL}_1(\prod_{x\in X} \mathcal{O}_x^\wedge)
\end{aligned}
$$

(b) Now take the union across all opens U. $\mathrm{GL}_1(U)$ becomes $\mathrm{GL}_1(K)$ [2], middle term becomes $\mathrm{GL}_1(\mathbb{A})$ and the right hand term $\mathrm{GL}_1(\mathcal{O})$ We obtain what's called the *Weil uniformisation*.

$$
\pi_0 \mathrm{Bun}_{\mathrm{GL}_1} X(k) \simeq \mathrm{GL}_1(K)\backslash \mathrm{GL}_1(\mathbb{A})/ \mathrm{GL}_1(\mathbb{O}_X)
$$

So we see that Bun_G geometricises the adelic double coset.

2. Question : why D-modules on $\mathrm{Bun}_G X$? Short answer : when $k = \mathbb{F}_q$, perverse sheaves categorify functions and when $k = \mathbb{C}$, D-modules (with certain conditions) equivalent to perverse sheaves via Riemann–Hilbert correspondence.

   Long answer : I've already touched on the Riemann–Hilbert correspondence. It is a generalisation of the equivalence between locally constant sheaves and vector bundles equipped with flat connections. So let me describe how perverse sheaves categorify functions.

[1] If anyone asks, this works by fpqc descent and the fact that X is finite type.

[2] GL_1 separated so maps $\mathrm{GL}_1(U) \subseteq \mathrm{GL}_1(V)$ for $V \subseteq U$

The setting is $k = \mathbb{F}_q$ and for simplicity again let $G = \mathrm{GL}_1$. We fix an algebraic closure $\mathbb{F}_q \subseteq \overline{\mathbb{F}}$. Let $\mathcal{F} \in \mathrm{Perv}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$. I won't tell you what this is, but all you need to know is that given a $x : \mathrm{Spec}\,\mathbb{F}_q \to \mathrm{Bun}_G$, we can get a $\overline{\mathbb{Q}}_\ell$-vector space $\mathcal{F}_{\overline{x}}$ by pulling back.

$$\mathrm{Frob}_x^g \circlearrowright \mathcal{F}_{\overline{x}} \leftsquigarrow \mathcal{F}_x \leftsquigarrow \mathcal{F}$$

$$\mathrm{Frob}_x^g \circlearrowright \overline{x} \longrightarrow x \longrightarrow \mathrm{Bun}_1$$

Intuitively, the Galois group $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ acts on $\overline{x} \to x$, and thus also on $\mathcal{F}_{\overline{x}}$. We can then take the trace of the (geometric) Frobenius. All together, we obtain a ring map

$$\begin{gathered} K(\mathrm{Perv}(\mathrm{Bun}_1, \overline{\mathbb{Q}_\ell})) \to \overline{\mathbb{Q}_\ell}\,[\mathrm{GL}_1(K)\backslash \mathrm{GL}_1 \mathbb{A}/ \mathrm{GL}_1 \mathcal{O}] \\ \mathcal{F} \mapsto (x \mapsto \mathrm{Tr}(\mathrm{Frob}_x^g, \mathcal{F}_{\overline{x}})) \end{gathered}$$

This is roughly what people mean by "perverse sheaves categorify functions". [1] In [Dri83], Drinfeld's proof of geometric Langlands for GL_2 constructs Hecke eigenfunctions by first constructing the perverse sheaf, then takes trace of Frobenius.

3 Hecke action from automorphic side

To give a statement of the "Galois to automorphic" direction of categorical geometric Langlands, I need to describe the two Hecke actions on $D\mathbf{Mod}(\mathrm{Bun}_G)$. We begin with the automorphic side by giving the geometric analogue of the local spherical Hecke algebras from arithmetic Langlands. Let $x \in X(k)$.

> **Definition**
> The *spherical Hecke category at x* is defined as
>
> $$\mathrm{Sph}_{G,x} := D\mathbf{Mod}(G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge)/G(\mathcal{O}_x^\wedge))$$
>
> where $G(\mathcal{O}_x^\wedge), G(K_x^\wedge)$ are the *arc and loop group at x*.

The arc and loop groups are respectively schemes and ind-affine schemes over k. [Zhu16, Proposition 1.3.2] Their k-valued points recover the honest-to-god groups $G(\mathcal{O}_x^\wedge), G(K_x^\wedge)$. The space underlying $\mathrm{Sph}_{G,x}$ is important so we'll give it a name.

> **Definition**
> We will call the (fpqc) double quotient
>
> $$\mathrm{Hecke}_x^{\mathrm{Loc}} := G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge)/G(\mathcal{O}_x^\wedge)$$
>
> the *local Hecke correspondence at x*. The multiplication on the loop group induces a diagram involving $\mathrm{Hecke}_x^{\mathrm{Loc}}$

[1] A precise statement can be found in [KW, Chapter III, Theorem 12.1].

$$\begin{tikzcd} G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge)/G(\mathcal{O}_x^\wedge) & G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge) \times^{G(\mathcal{O}_x^\wedge)} G(K_x^\wedge)/G(\mathcal{O}_x^\wedge) \arrow[l, "p_0"'] \arrow[r, "m"] \arrow[d, "p_1"'] & G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge)/G(\mathcal{O}_x^\wedge) \\ & G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge)/G(\mathcal{O}_x^\wedge) & \end{tikzcd}$$

This allows one to define a monoidal product _ $*$ _ on $\mathrm{Sph}_{G,x}$ via convolution.

The monoidal structure on $\mathrm{Sph}_{G,x}$ is a categorification of the algebra structure on the local Hecke algebras from arithmetic Langlands. In arithmetic Langlands, the local Hecke algebras act on the space of (unramified) cusp forms by Hecke operators. In analogy to this, we now define an action of $\mathrm{Sph}_{G,x}$ on $D\mathbf{Mod}(\mathrm{Bun}_G)$.

$$H_x : \mathrm{Sph}_{G,x} \times D\mathbf{Mod}(\mathrm{Bun}_G) \to D\mathbf{Mod}(\mathrm{Bun}_G)$$

But to do this, we need to know how to make D-modules on Bun_G using a point $x \in X(k)$. We have the following

Definition – Full level structure at x

Define the *stack of G bundles on X equipped with full level structure at x* by the cartesian diagram

$$\begin{tikzcd} \mathrm{Bun}_G X \arrow[d] & \mathrm{Bun}_{G,x}^{\mathrm{lvl}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ \mathrm{Bun}_G D_x & \mathrm{Spec}\, k \arrow[l, "\mathrm{triv}"] \end{tikzcd}$$

There is a map

$$\mathrm{Bun}_{G,x}^{\mathrm{lvl}} \to \mathrm{Bun}_G$$

obtained by forgeting the full level structure at x. This morphism is equivariant w.r.t. the action of $G(\mathcal{O}_x^\wedge)$ on $\mathrm{Bun}_{G,x}^{\mathrm{lvl}}$ and in fact is a $G(\mathcal{O}_x^\wedge)$ bundle. We obtain

$$\mathrm{Bun}_G \simeq G(\mathcal{O}_x^\wedge)\backslash \mathrm{Bun}_{G,x}^{\mathrm{lvl}}$$

Thus, D-modules on Bun_G can be obtained from arc-group-equivariant D-modules on $\mathrm{Bun}_{G,x}^{\mathrm{lvl}}$. The action of $\mathrm{Sph}_{G,x}$ will also be defined by convolution. With that idea, we see that we need to extend the action of the arc group to an action of the loop group.

Definition – Regluing

There is an action of $G(K_x^\wedge)$ on Bun_G^x

$$\mathrm{ReGlue} : \mathrm{Bun}_{G,x}^{\mathrm{lvl}} \times G(K_x^\wedge) \to \mathrm{Bun}_{G,x}^{\mathrm{lvl}}$$

given intuitively by the following : For $g \in G(K_x^\wedge)$, P a G bundle on X and $s : \mathrm{Triv}|_{D_x} \to P|_{D_x}$, make the following descent data :

$$\mathrm{Triv}|_{D_x} \longleftarrow \mathrm{Triv}|_{D_x^\circ} \xrightarrow{sg} P|_{D_x^\circ} \longrightarrow P|_{X\setminus\{x\}}$$

This then $\mathrm{ReGlue}(g, P, s)$ is defined as the G bundle on X obtained by gluing the above descent data. [a]

[a] If anyone asks : Technically speaking, we must be able to do this gluing in families. Although fpqc descent is not enough to glue over non locally finite type spaces, the local finite type condition of X ensures that we only need to know how to glue in finite type situations. This is then covered by fpqc descent. Alternatively, if one doesn't care about being minimal, one can use the Beauville–Laszlo theorem.

We can now define the local Hecke action by convolution. (Maybe skip the diagram.)

Definition – Local Hecke action

Consider the diagram

$$\begin{tikzcd}
G(K_x^\wedge) & G(K_x^\wedge) \times \mathrm{Bun}_{G,x}^{\mathrm{lvl}} \arrow[l, "p_0"'] \arrow[r, "\mathrm{ReGlue}"] \arrow[d, "p_1"'] & \mathrm{Bun}_{G,x}^{\mathrm{lvl}} \\
& \mathrm{Bun}_{G,x}^{\mathrm{lvl}} &
\end{tikzcd}$$

We can take a lot of (fpqc) quotients by the arc group and get

$$\begin{tikzcd}
G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge)/G(\mathcal{O}_x^\wedge) & G(\mathcal{O}_x^\wedge)\backslash G(K_x^\wedge) \times^{G(\mathcal{O}_x^\wedge)} \mathrm{Bun}_{G,x}^{\mathrm{lvl}} \arrow[l, "p_0"'] \arrow[r, "\mathrm{ReGlue}"] \arrow[d, "p_1"'] & G(\mathcal{O}_x^\wedge)\backslash \mathrm{Bun}_{G,x}^{\mathrm{lvl}} & \simeq & \mathrm{Bun}_G \\
& G(\mathcal{O}_x^\wedge)\backslash \mathrm{Bun}_{G,x}^{\mathrm{lvl}} \arrow[d, phantom, "\wr"] & & & \\
& \mathrm{Bun}_G & & &
\end{tikzcd}$$

The *local Hecke action* is defined as[a]

$$\begin{aligned}
\mathrm{Sph}_{G,x} \times D\mathbf{Mod}(\mathrm{Bun}_G) &\to D\mathbf{Mod}(\mathrm{Bun}_G) \\
\mathcal{S}, \mathcal{F} \mapsto H_x(\mathcal{S}, \mathcal{F}) &:= (\mathrm{ReGlue})_* \left(p_0^! S \otimes p_1^! \mathcal{F}\right)
\end{aligned}$$

[a] if anyone asks : give nice interpretation of middle term as Hecke correspondence at x.

In order to make the definition of Hecke eigensheaf, we need to bundle of the local Hecke actions together into a single action over X. This is roughly done in the following steps : (Skip all diagram and say "replace x with X". Except for final formula.)

1. The arc and loop groups relativises to group presheaves over X

$$\begin{tikzcd}
G(\mathcal{O}_X^\wedge) \arrow[rr] \arrow[dr] & & G(K_X^\wedge) \arrow[dl] \\
& X &
\end{tikzcd}$$

The local Hecke correspondences also relativises

$$\mathrm{Hecke}_X^{\mathrm{Loc}} := G(\mathcal{O}_X^\wedge)\backslash G(K_X^\wedge)/G(\mathcal{O}_X^\wedge) \to X$$

2. The family $\mathrm{Bun}_{G,x}^{\mathrm{lvl}} \to \mathrm{Bun}_G$ relativises to

$$\mathrm{Bun}_G^{\mathrm{lvl}} \to X \times \mathrm{Bun}_G$$

and the action so does the action of the loop group

$$G(K_X^\wedge) \times_X \mathrm{Bun}_G^{\mathrm{lvl}} \to \mathrm{Bun}_G^{\mathrm{lvl}}$$

3. The convolution diagram relativises :

$$\begin{tikzcd}
G(\mathcal{O}_X^\wedge)\backslash G(K_X^\wedge)/G(\mathcal{O}_X^\wedge) & G(\mathcal{O}_X^\wedge)\backslash G(K_X^\wedge) \times^{G(\mathcal{O}_X^\wedge)} \mathrm{Bun}_{G,X}^{\mathrm{lvl}} \arrow[l, "p_0"'] \arrow[r, "\mathrm{ReGlue}"'] \arrow[d, "p_1"'] & G(\mathcal{O}_X^\wedge)\backslash \mathrm{Bun}_{G,X}^{\mathrm{lvl}} \simeq X \times \mathrm{Bun}_G \\
& G(\mathcal{O}_X^\wedge)\backslash \mathrm{Bun}_{G,X}^{\mathrm{lvl}} \arrow[d] & \\
& \mathrm{Bun}_G &
\end{tikzcd}$$

This defines the *global Hecke action* : [1]

$$\begin{aligned} \mathrm{Sph}_{G,X} \times D\mathbf{Mod}(\mathrm{Bun}_G) &\to D\mathbf{Mod}(X \times \mathrm{Bun}_G) \\ \mathcal{S}, \mathcal{F} &\mapsto H(\mathcal{S}, \mathcal{F}) := (p_1)_* \left(p^! S \otimes p_0^! \mathcal{F}\right) \end{aligned}$$

4 Hecke action from Galois side and Hecke eigensheaves

The previous section was all on the automorphic side. Finally, we make contact with the Galois side.

Proposition – Geometric Satake
Consider the spherical Hecke category

$$\mathrm{Sph}_G := D\mathbf{Mod}(\mathrm{Hecke}^{\mathrm{Loc}})$$

where $\mathrm{Hecke}^{\mathrm{Loc}} := G[\![t]\!]\backslash G((t))/G[\![t]\!]$. This is the version of $\mathrm{Sph}_{G,x}$ for $X = \mathbb{A}^1, x = 0$. [a] Then the convolution product on Sph_G makes Sph_G into a rigid tensor category. [b] There exists a (faithful exact) tensor functor $F : \mathrm{Sph}_G \to \mathbf{Vec}_k$ which induces an equivalence of tensor categories

$$\mathrm{Sph}_G \simeq \mathrm{Rep}_k G^L, \mathcal{S}_V \leftarrow\!\shortmid V$$

via the Tannaka reconstruction theorem. [c] [d] [e]

[a] We don't use the projectivity of X for geometric Satake. Indeed, since geometric Satake is *local* it should be independent of

[1] If anyone asks : give geometric interpretation of middle term as Hecke correspondence.

the choice of curve X.

[b]Possible question someone might ask : is the convolution product "commutative" like in the arithmetic case for GL_n? The categorification of "commutative" is symmetric monoidal. The answer is yes, but requires hard geometry. See [Zhu16, Section 5.2].

[c]If anyone asks : roughly speaking, this recovers the classical Satake isomorphism by taking Grothendieck rings on both sides. RHS has basis by dominant coweights $X_\bullet(T)^+$. See [Zhu16, Section 5.6].

[d]If anyone asks : the reason we need to use fpqc quotients instead of just prestack quotients is that the geometry of the affine Grassmannian $\mathrm{Gr}_G := G((t))/G[\![t]\!]$ plays an important role in proving geometric Satake. If we didn't sheafify, the space would simply be wrong.

[e]If anyone asks : Tannaka reconstruction can be seen as a generalisation of Pontrjagin duality to affine algebraic groups. For non-commutative groups, instead of characters, i.e. 1-dimensional reps, one needs all simple reps, which is why the tensor category of finite reps serves as the "dual" to the group.

The fibers of $\mathrm{Hecke}_X^{\mathrm{Loc}} \to X$ are the local Hecke correspondences $\mathrm{Hecke}_x^{\mathrm{Loc}}$ at each point $x \in X(k)$,

$$\begin{tikzcd}
\mathrm{Hecke}_X^{\mathrm{Loc}} \arrow[d] & \mathrm{Hecke}_x^{\mathrm{Loc}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \arrow[r, phantom, "\overset{\text{choice of } \mathbb{C}[\![t]\!] \simeq \mathcal{O}_x^\wedge}{\simeq}"] & \mathrm{Hecke}^{\mathrm{Loc}} \\
X & \mathrm{pt} \arrow[l, "x"] &
\end{tikzcd}$$

and given a choice of uniformizer at x, we have $\mathrm{Sph}_G \simeq \mathrm{Sph}_{G,x}$ as tensor categories. We take on faith that this idea allows one to take $\mathcal{S} \in \mathrm{Sph}_G$ and extend it across $\mathrm{Hecke}^{\mathrm{Loc}}$ in the X-direction, [1] giving a functor

$$\begin{aligned} \mathrm{Sph}_G &\to \mathrm{Sph}_{G,X} \\ \mathcal{S} &\mapsto \tilde{\mathcal{S}} \end{aligned}$$

This gives us *two* actions of $\mathrm{Sph}_G \simeq \mathrm{Rep}_k G^L$ on $D\mathbf{Mod}(\mathrm{Bun}_G)$.

1. On the automorphic side : we let Sph_G act through all points at once via $\mathrm{Sph}_{G,X}$

$$\begin{aligned} \mathrm{Sph}_G \times D\mathbf{Mod}(\mathrm{Bun}_G) &\to D\mathbf{Mod}(X \times \mathrm{Bun}_G) \\ \mathcal{S}, \mathcal{F} &\mapsto H(\tilde{\mathcal{S}}, \mathcal{F}) \end{aligned}$$

2. On the Galois side : given a G^L local system σ on X, we can pullback representations of G^L along σ :

$$X_{\mathrm{dR}} \xrightarrow[\sigma]{} BG^L \qquad \rightsquigarrow \qquad D\mathbf{Mod}(X) \xleftarrow{\sigma^*} \mathrm{Rep}_k G^L$$

This sends a representation V to the associated bundle V_σ which has a connection via σ. So we have the action :

$$\begin{aligned} \mathrm{Rep}_k G^L \times D\mathbf{Mod}(\mathrm{Bun}_G) &\to D\mathbf{Mod}(X \times \mathrm{Bun}_G) \\ V, \mathcal{F} &\mapsto V_\sigma \boxtimes \mathcal{F} \end{aligned}$$

[1]An account of this is given in [Gomez, Section 4.2]. However, it is somewhat unsatisfactory because the choice of a point $x \in X(k)$ is used whereas the global Hecke action doesn't mention any point in particular. A fix should be $\mathrm{Hecke}_X^{\mathrm{Loc}} \simeq X^\wedge \times^{\mathrm{Aut}\, D^\wedge} \mathrm{Hecke}^{\mathrm{Loc}}$ where $X^\wedge$ is the moduli space of formal parameters of X and $\mathrm{Aut}\, D^\wedge$ is the pro-algebraic group of automorphisms of the formal disk preserving 0.

Asking for these two actions to coincide is (roughly) the definition of a Hecke eigensheaf.

Definition

Let $\sigma \in \mathrm{LocSys}_{G^L} X(k)$ be a G^L local system on X. Then a *Hecke eigensheaf with eigenvalue* σ is a $\mathcal{F} \in D\mathbf{Mod}(\mathrm{Bun}_G)$ equipped with isomorphisms for each $V \in \mathrm{Rep}_k G^L$,[a]

$$H(\tilde{\mathcal{S}_V}, \mathcal{F}) \simeq V_\sigma \boxtimes \mathcal{F}$$

[a] Technically, there are more conditions to make the action "agree". See [Gomez, Section 4.3].

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