Week 4 : Hecke actions

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The goal of this talk is to give a more detailed introduction to the spherical Hecke category in preparation for the three talks on geometric Satake.

Brief recollection of week 1 talk :

1. Throughout, we fixed a smooth projective curve X over a base field k with characteristic zero. For simplicity, one should assume it to be $\mathbb{C}$. We also fix a reductive group G over k.
2. Bun_G is the algebro-geometric version of the double coset of adeles in unramified Langlands of function fields over a finite field.
3. Hecke eigenforms, which are certain functions on the double coset space of the adeles, can be constructed from decategorifying perverse sheaves on Bun_G.
4. Over characteristic zero, perverse sheaves and D-modules are equivalent by the Riemann–Hilbert correspondence. Categorical geometric Langlands takes this as the starting point.
5. affine Grassmannians are the algebro-geometric versions of the homogeneous space $G((t))/G[\![t]\!]$ and hence lead to the geometric version of spherical Hecke algebras Sph_G.
6. Given a closed point x of the curve, there is an action of Sph_G on $D\mathbf{Mod}(\mathrm{Bun}_G)$ called the automorphic Hecke action.
7. Given a G^L local system σ on X, there is an action of $\mathrm{Rep}_k G^L$ on $D\mathbf{Mod}(\mathrm{Bun}_G)$ called the Galois Hecke action.
8. Geometric Satake gives $\mathrm{Sph}_G \simeq \mathrm{Rep}_k G^L$ as tensor categories.
9. The automorphic Hecke action across all points can be bundled to together, giving two actions of $\mathrm{Rep}_k G^L$ on $D\mathbf{Mod}(\mathrm{Bun}_G)$.
10. modulo details about associativity, a Hecke eigensheaf is a D-module on Bun_G "such that" the two actions of $\mathrm{Rep}_k G^L$ coincide. [1]

This talk focuses on giving more details on (5), (6), (9).

Setup some notation :

– Aff_k denotes the opposite of the category of k-algebras.

[1] We put "such that" in quotation marks because this is a structure, not a property.

– For an affine scheme $S = \operatorname{Spec} A$, $D_S^\wedge, D_S, D_S^\circ$ will denote $\operatorname{Spf} A[\![t]\!], \operatorname{Spec} A[\![t]\!], \operatorname{Spec} A((t))$. These are the relative formal disk, disk, punctured disk over S.

Contents

1 Arc groups, Loop groups and Affine Grassmannians

Definition – Arc and loop group

The *arc group of* G is defined as the functor on Aff_k

$$L^+G(S) := G(D_S)$$

and the *loop group of* G is defined as the functor

$$LG(S) := G(D_S^\circ)$$

Ideally, L^+G should be thought of as the mapping space $\underline{\mathrm{Hom}}(D, G)$ from the disk D to G, and similarly LG should be $\underline{\mathrm{Hom}}(D^\circ, G)$. The issue is $A \otimes_k k[\![t]\!] \not\simeq A[\![t]\!]$ so this is not literally true. This can be fixed by using the formal disk $D^\wedge := \operatorname{Spf} k[\![t]\!]$ instead since $\operatorname{Spec} A \times \operatorname{Spf} k[\![t]\!] = \operatorname{Spf} A[\![t]\!]$ and G is affine. However, this fix does not work for the loop group LG because "one cannot Spf the topological ring $A((t))$".

Restricting along $D_S^\circ \to D_S$ defines a morphism $L^+G \to LG$ of group functors. One can show this is a closed embedding.

> *Lemma. The morphism $L^+G \to LG$ is a closed embedding.*
>
> *Proof.* Since G is affine, there exists a closed embedding $i : G \to \mathbb{A}^N$ for some finite N. Let $s : S \to LG$ be a point of the loop group. Using i, we see that s is given by N-coordinates $(s_1, \dots, s_N) \in \mathbb{A}^N(D_S^\circ)$. The fiber product $L^+G \times_{LG} S$ is computed as the vanishing locus of the finitely many polar terms of $s_1, \dots, s_N$. ■

Here's the answer to the question of what kind of objects are the arc and loop group.

Proposition

The arc group L^+G is a scheme and the loop group LG is an ind-affine scheme.

Proof. We define intermediate spaces : the space of n-jets L^nG is defined as a functor by

$$L^nG := \underline{\mathrm{Hom}}(\operatorname{Spec} k[t]/(t^{n+1}), G)$$

Note that $n = 1$ recovers the tangent bundle of G. It is not hard to show that in the case of $G = \mathbb{A}^1$ that $L^n G$ is affine. Since G is affine finite type, it can be put into a cartesian square

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}^N \arrow[d] \\ 0 \arrow[r] & \mathbb{A}^r \end{tikzcd}$$

Since L^n is right adjoint to $\_ \times \operatorname{Spec} k[t]/(t^{n+1})$, is preserves limits so $L^n G$ is affine. Finally, $L^+ G \simeq \varprojlim_n L^n G$ is hence also affine.

Now for the loop group. It is *not* true that $LG = \underline{\operatorname{Hom}}(D^\circ, G)$, but the formula

$$(LG)(S) = G(D_S^\circ) \overset{\text{Yoneda}}{\simeq} \operatorname{Hom}(D_S^\circ, G)$$

still shows that L commutes with limits. By the same argument as above, we reduce to $G = \mathbb{A}^1$. One then computes

$$\mathbb{A}^1((t))(A) = \mathbb{A}^1(A((t))) = A((t)) \simeq \varinjlim_n A[\![t]\!] = (\varinjlim_n \mathbb{A}^1[\![t]\!])(A)$$

where the system we are taking colimits is

$$\mathbb{A}^1[\![t]\!] \xrightarrow{t} \mathbb{A}^1[\![t]\!] \xrightarrow{t} \cdots$$

Concretely, $\mathbb{A}^1[\![t]\!] \simeq \mathbb{A}^{\mathbb{N}}$ and the above transition maps shift the sequence of coefficients to the right by one. One sees from this that each copy $\mathbb{A}^1[\![t]\!]$ is the closed subscheme of the next copy of sequences where the first coefficient is zero. We have thus shown $\mathbb{A}^1((t))$ is a ind-affine scheme. $\square$

Definition – Affine Grassmannian for general G

Let G be an algebraic group scheme over k. For an affine S, define the groupoid $\operatorname{Gr}_G(S)$ as follows :

- Objects are G-bundles on D_S together with a trivialisation $s : \operatorname{Triv} \simeq P|_{D_S^\circ}$ on the punctured disk D_S°.
- A morphism $(P, s) \to (Q, t)$ is a morphism $\alpha : P \to Q$ of G-bundles on D_S respecting the trivialisations, i.e. $t = \alpha s$.

Given a map $T \to S$ of affines, there is a functor $\operatorname{Gr}_G(S) \to \operatorname{Gr}_G(T)$ given by pulling back along $D_T \to D_S$. The above forms a groupoid valued (pseudo-)functor on Aff_k, which we call the *affine Grassmannian of G*.

[a]

[a] Question to self : Why does Gaitsgory bother writing this using Tannaka formalism when G-bundles over D_S and D_S° are perfectly well defined?

Although Gr_G is a-priori valued in groupoids, one can show that it is in fact valued in discrete groupoids by the following lemma. Hence Gr_G is equivalently a set-valued functor sending affines S to the isomorphism class of G-bundles on D_S equipped with a trivialisation over D_S°.

Lemma. Let P be a G-bundle on D_S where S is affine and $s : \mathrm{Triv} \simeq P|_{D_S^\circ}$ a trivialisation on the punctured disk. Suppose $\alpha : P \to P$ is an (auto)morphism of G-bundles on D_S such that $\alpha s = s$ over D_S°. Then α must be the identity.

Proof. (Example of trivial line bundle) Suppose $G = \mathrm{GL}_1$ and replace GL_1-bundles with line bundles and consider the case of $P = \mathbb{A}^1_{D_S}$. The trivialisation s is equivalent to an element $\mathcal{O}(S)((t))^\times$. Similarly, α is equivalent to an element $\alpha \in \mathcal{O}(S)[\![t]\!]^\times$. Then we have $\alpha s = s$ as elements in $\mathcal{O}(S)((t))^\times$. Since s is a unit, $\alpha = 1$ in $\mathcal{O}(S)((t))^\times$ and hence in $\mathcal{O}(S)[\![t]\!]^\times$.

(General case) Using the same strategy as in the example, we see that we only need to prove two things :

1. P can be trivialised after $D_{\tilde{S}} \to D_S$ by some fpqc cover $\tilde{S} \to S$.
2. After (1), we can assume $P = G \times D_S$. We then need to show $G(D_S) \to G(D_S^\circ)$ is injective.

For (1), the special fiber P_0 over $\{0\}_S \simeq S$ is also a G-bundle, and hence trivialises upon passing to some fpqc cover $\tilde{S} \to S$. The trivialisation of P_0 over $\tilde{S} \simeq \{0\}_{\tilde{S}}$ extends to $D_{\tilde{S}}$ by smoothness of G.

For (2), using affinity of G we can realise G as a closed subscheme of $\mathbb{A}^N$ for some large N. WLOG $N = 1$. This reduces to showing $\mathbb{A}^1(D_S) \to \mathbb{A}^1(D_S^\circ)$ is injective. This is precisely the inclusion $\mathcal{O}(S)[\![t]\!] \subseteq \mathcal{O}(S)((t))$.

■

We now give the description of the affine Grassmannian as a homogeneous space of the loop group.

Definition

Define an action $\mathrm{Gr}_G \times LG \to \mathrm{Gr}_G$ as follows

$$((P, s), g) \mapsto (P, sg)$$

Proposition

The map $LG \to \mathrm{Gr}_G$ sending $g \mapsto (\mathrm{Triv}, g)$ is a L^+G-bundle.

Proof. We need to show the following :

1. L^+G acts trivially on Gr_G
2. the map $LG \to \mathrm{Gr}_G$ is L^+G equivariant
3. For every point $(P, s) : S \to \mathrm{Gr}_G$ of Gr_G with S affine, there exists an fpqc cover $\tilde{S} \to S$ such that the pullback of LG to $\tilde{S}$ is isomorphic to the trivial L^+G bundle.

$$\begin{tikzcd}
LG \arrow[d] & LG \times_{\mathrm{Gr}_G} S \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] & \tilde{S} \times L^+G \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\
\mathrm{Gr}_G & S \arrow[l, "(P,s)"'] & \tilde{S} \arrow[l, "\text{fpqc}"']
\end{tikzcd}$$

Goal (1) and (2) are clear. (3) follows from the fact we've already seen : there exists fpqc cover $\tilde{S} \to S$ such that $D_{\tilde{S}} \to D_S$ trivialises P.

$\square$

2 Spherical D-modules, Convolution

[Can skip until...] I find it useful to make the following auxiliary definitions, which are not explicitly present in Gaitsgory's notes.

> **Definition**
> The *stack of G-bundles on the disk* is defined as by the groupoid-valued functor
>
> $$(\operatorname{Bun}_G D)(S) := \operatorname{Hom}(D_S, BG)$$
>
> And the *stack of G-bundles on the punctured disk* is defined by the groupoid-valued functor
>
> $$(\operatorname{Bun}_G D^\circ)(S) := \operatorname{Hom}(D_S^\circ, BG)$$

Recall that the definition of $\operatorname{Bun}_G X$ is the internal hom $\underline{\operatorname{Hom}}(X, BG)$. This begs the question whether $\operatorname{Bun}_G D, \operatorname{Bun}_G D^\circ$ are also internal homs.

1. For $\operatorname{Bun}_G D$ this is true by a combination of the Tannakian formalism and the fact that for Noetherian A with ideal I, finitely generated projective modules over $\operatorname{Spec} A_I^\wedge$ and $\operatorname{Spf} A_I^\wedge$ are equivalent :

$$\begin{aligned}
\operatorname{Hom}(D_S, BG) &\simeq \operatorname{Fun}^{\otimes}_{\text{ex, cts}}(\operatorname{Rep} G, \operatorname{QCoh} D_S) \simeq \operatorname{Fun}^{\otimes}_{\text{ex}}(\operatorname{Rep}_{\text{f.d.}} G, \operatorname{Perf} D_S) \\
&\simeq \operatorname{Fun}^{\otimes}_{\text{ex}}(\operatorname{Rep}_{\text{f.d.}} G, \varprojlim_n \operatorname{Perf}(n \cdot 0_S)) \\
&\simeq \varprojlim_n \operatorname{Fun}^{\otimes}_{\text{ex}}(\operatorname{Rep}_{\text{f.d.}} G, \operatorname{Perf}(n \cdot 0_S)) \\
&\simeq \varprojlim_n \operatorname{Hom}(n \cdot 0_S, BG) \simeq \operatorname{Hom}(\varinjlim_n n \cdot 0_S, BG) \\
&\simeq \operatorname{Hom}(S \times D^\wedge, BG) \simeq \underline{\operatorname{Hom}}(D^\wedge, BG)(S)
\end{aligned}$$

2. For $\operatorname{Bun}_G D^\circ$ this is *not* true, again essentially because $\operatorname{Spec} A((t))$ is not computed as the fiber product $\operatorname{Spec} A \times \operatorname{Spec} k((t))$.

> **Definition**
> The local Hecke correspondence is defined as the fiber product
>
> $$\begin{tikzcd} \operatorname{Bun}_G D \arrow[d] & \operatorname{Hecke}^{\text{Loc}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\ \operatorname{Bun}_G D^\circ & \operatorname{Bun}_G D \arrow[l] \end{tikzcd}$$

[... here.] Unraveling the definition, a map $S \to \mathrm{Hecke}^{\mathrm{Loc}}$ from an affine is the same as the data (P_0, P_1, α) where P_i are G-bundles on D_S and $\alpha : P_0|_{D_S^\circ} \simeq P_1|_{D_S^\circ}$. This is related to the affine Grassmannian in the following way :

> **Proposition**
> Define $\mathrm{Hecke}^{\mathrm{Loc}}_{\square}$ by the cartesian square
>
> $$\begin{tikzcd} \mathrm{Hecke}^{\mathrm{Loc}} \arrow[d] & \mathrm{Hecke}^{\mathrm{Loc}}_{\square} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ \mathrm{Bun}_G D \times \mathrm{Bun}_G D & \mathrm{pt} \arrow[l, "(\mathrm{Triv},\mathrm{Triv})"'] \end{tikzcd}$$
>
> Then
>
> 1. $L^+G \backslash \mathrm{Hecke}^{\mathrm{Loc}}_{\square} / L^+G \simeq \mathrm{Hecke}^{\mathrm{Loc}}$
> 2. $\mathrm{Hecke}^{\mathrm{Loc}}_{\square} \simeq LG$ respecting the left and right actions of L^+G. Hence $\mathrm{Hecke}^{\mathrm{Loc}} \simeq L^+G \backslash \mathrm{Gr}_G$

Proof. (1) $\mathrm{Hecke}^{\mathrm{Loc}}_{\square}$ parameterises $(P_0, P_1, \alpha, s_0 : \mathrm{Triv} \to P_0, s_1 : \mathrm{Triv} \to P_1)$ where P_i are G-bundles on the disk. Quotienting by L^+G on both sides forgets the trivialisations s_i.

(2) We give mutual inverse functors and omit the check that it works.

$LG \to \mathrm{Hecke}^{\mathrm{Loc}}_{\square}$ takes $g \in LG(S)$ to $(\mathrm{Triv}, \mathrm{Triv}, g, \mathbb{1}, \mathbb{1})$.

$\mathrm{Hecke}^{\mathrm{Loc}}_{\square} \to LG$ takes an S-point $(P_0, P_1, \alpha, s_0, s_1)$ to $s_1^{-1} \alpha s_0 \in LG(S)$. $\square$

We now want to define the spherical Hecke category, which D-module version of the categorified spherical Hecke category. Ideally, we want the following definition :

$$\mathrm{Sph}_G := D\mathbf{Mod}(\mathrm{Hecke}^{\mathrm{Loc}})$$

The issue is that at this point, we only really know how to do D-modules on varieties. The idea is to use $\mathrm{Hecke}^{\mathrm{Loc}} \simeq L^+G \backslash \mathrm{Gr}_G$, to reduce to the case of varieties. There are two parts to this reduction :

1. Gr_G is ind-projective, meaning we can write it as the colimit of projective varieties along closed embeddings. This will be addressed in next week's talk on the geometry of the affine Grassmannian.
2. D-modules on quotient stacks. It will also be shown next week that one can present Gr_G as a filtered colimit of projective varieties X_i along closed embeddings such that each X_i is L^+G-invariant and the action in fact factors through an algebraic quotient of L^+G. In preparation for that, we give a short exposition of D-modules on quotient stacks of varieties by algebraic groups in the next section.

Provided that we know how to deal with D-modules on infinite type schemes like Gr_G, LG and their quotients nice enough actions by pro-algebraic groups like L^+G, we can now define a monoidal structure on Sph_G via convolution.

Definition – Convolution product
The multiplication on the loop group induces a diagram involving $\mathrm{Hecke}^{\mathrm{Loc}}$

$$\begin{tikzcd} L^+G\backslash LG/L^+G & L^+G\backslash LG \times^{L^+G} LG/L^+G \arrow[l, "p_0"'] \arrow[r, "m"] \arrow[d, "p_1"'] & L^+G\backslash LG/L^+G \\ & L^+G\backslash LG/L^+G & \end{tikzcd}$$

The convolution product is defined by the following formula

$$\mathrm{Sph}_G \times \mathrm{Sph}_G \to \mathrm{Sph}_G, (\mathcal{F}_0, \mathcal{F}_1) \mapsto \mathcal{F} * \mathcal{G} := m_!(p_0^! \mathcal{F}_0 \otimes p_1^! \mathcal{F}_1)$$

and defines a monoidal structure on Sph_G.

3 D-modules on quotient stacks

Let X be a variety over k and G an algebraic group acting on X. With this, one can form the truncated Cech nerve of $X \to X/G$.

$$X \times_{X/G} X \times_{X/G} X \begin{array}{c} \xrightarrow{p_{01}} \\ \xrightarrow{p_{12}} \\ \xrightarrow{p_{02}} \end{array} X \times_{X/G} X \begin{array}{c} \xrightarrow{p_0} \\ \xrightarrow{p_1} \end{array} X$$

where we have the projections

- $p_i : X \times_{X/G} X \to X, (x_0, x_1) \mapsto x_i$
- $p_{ij} : X \times_{X/G} X \times_{X/G} X \to X \times_{X/G} X, (x_0, x_1, x_2) \mapsto (x_i, x_j)$

One can show that this is precisely the usual action groupoid :

$$G \times G \times X \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} G \times X \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} X$$

For an affine S, an S-point of $X \times_{X/G} X$ is a triple (x_0, x_1, ϕ) where $x_i \in X(S)$ and

$$\begin{tikzcd} S \times G \arrow[r] \arrow[d] \arrow[dr, "\phi"] & X \\ S & S \times G \arrow[l] \arrow[u] \end{tikzcd}$$

where ϕ is G-equivariant. The data of ϕ is equivalent to $g \in G(S)$ such that $x_1 = g \cdot x_0$.

Definition – G-equivariance

The category $(\mathrm{QCoh}\,X)^G$ of quasi-coherent sheaves on X equipped with G-equivariance is defined as follows :

– An object of $(\mathrm{QCoh}\,X)^G$ is a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}\,X$ equipped with a $\phi : p_0^*\mathcal{F} \cong p_1^*\mathcal{F}$ in $\mathrm{QCoh}(X \times_{X/G} X)$ such that

1. $\phi = \mathbb{1}$ when restricted to the diagonal section $\Delta : X \to X \times_{X/G} X$
2. the following diagram commute in $\mathrm{QCoh}(X \times_{X/G} X \times_{X/G} X)$:

$$\begin{tikzcd}
 & p_{01}^*p_1^*\mathcal{F} \arrow[dr, "\sim"] & \\
p_{01}^*p_0^*\mathcal{F} \arrow[ur, "p_{01}^*(\phi)"] \arrow[d, "\sim"'] & & p_{12}^*p_0^*\mathcal{F} \arrow[d, "p_{12}^*(\phi)"] \\
p_{02}^*p_0^*\mathcal{F} \arrow[dr, "p_{02}^*(\phi)"'] & & p_{12}^*p_1^*\mathcal{F} \arrow[dl, "\sim"] \\
 & p_{02}^*p_1^*\mathcal{F} &
\end{tikzcd}$$

– a morphism $\eta : (\mathcal{F}, \phi) \to (\mathcal{G}, \psi)$ is a morphism $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{QCoh}\,X$ such that the following commutes :

$$\begin{tikzcd}
p_0^*\mathcal{F} \arrow[r, "\phi"] \arrow[d, "p_0^*(\eta)"'] & p_1^*\mathcal{F} \arrow[d, "p_1^*(\eta)"] \\
p_0^*\mathcal{G} \arrow[r, "\psi"'] & p_1^*\mathcal{G}
\end{tikzcd}$$

A weakly G-equivariant D-module on X is a D-module on X equipped with G-equivariance structure ϕ which is a morphism of $\mathcal{O}_G \boxtimes D_X$-modules, where we used $G \times X \simeq X \times_{X/G} X$.

A strongly G-equivariant D-module on X is the same except we require ϕ is be a morphism of $D_{G\times X}$-modules.

Using the equivalence of D-modules and crystals :

$$D\mathbf{Mod}(X) \simeq \mathrm{QCoh}\,X_{\mathrm{dR}}$$

the above definitions can be rephrased as follows :

1. The action of G on X induces an action of G on X_{dR}. Thus, we have an action groupoid for X_{dR} :

$$\begin{tikzcd}
G \times G \times X_{\mathrm{dR}} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \times X_{\mathrm{dR}} \arrow[r, shift left=1] \arrow[r, shift right=1] & X_{\mathrm{dR}}
\end{tikzcd}$$

Then a weakly G-equivariant D-module on X is equivalently a G-equivariant quasi-coherent sheaf on X_{dR}.

2. Taking de Rham spaces commutes with limits of presheaves. [1] So $(G \times X)_{\mathrm{dR}} = G_{\mathrm{dR}} \times X_{\mathrm{dR}}$. Then the action of G on X induces an action of G_{dR} on X_{dR}, giving us the action groupoid

$$\begin{tikzcd} G_{\mathrm{dR}} \times G_{\mathrm{dR}} \times X_{\mathrm{dR}} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G_{\mathrm{dR}} \times X_{\mathrm{dR}} \arrow[r, shift left] \arrow[r, shift right] & X_{\mathrm{dR}} \end{tikzcd}$$

A strongly G-equivariant D-module on X is thus equivalently a G_{dR}-equivariant quasi-coherent sheaf on X_{dR}.

We have the following.

> **Proposition**
> There is an equivalence of categories :
> $$(\mathrm{QCoh}\, X_{\mathrm{dR}})^{G_{\mathrm{dR}}} \simeq D\mathbf{Mod}(X/G)$$
> i.e. strongly G-equivariant D-modules on X are equivalent to D-modules on X/G.

Proof. (Sketch)

First, we define the functor $(\mathrm{QCoh}\, X_{\mathrm{dR}})^{G_{\mathrm{dR}}} \to D\mathbf{Mod}(X/G)$. Recall for a stack $\mathscr{X}$, $\mathcal{D}_{\mathscr{X}}$-modules on it are defined locally in the smooth topology $\mathscr{X}_{sm}$. The objects of $\mathscr{X}_{sm}$ are pairs (S, π_S) where S is a scheme and $\pi_S : S \to \mathscr{X}$ is a smooth 1-morphism. [BD, Section 1.1.2] The morphisms between pairs (S, π_S) and $(S', \pi_{S'})$ are pairs (φ, α) where $\varphi : S \to S'$ is a smooth morphism and $\alpha : \pi_S \xrightarrow{\sim} \pi_{S'}\varphi$ is a 2-morphism, i.e., it satisfies some kind of cocycle condition. Suppose we now have a strongly equivariant D-module M on X. Goal : for each $(S, \pi_S) \in (X/G)_{sm}$, we need to give a $\mathcal{D}_S$-module M_S, and, given a morphism $(\varphi, \alpha) : (S, \pi_S) \to (S', \pi_{S'})$ we need to give an isomorphism of $\mathcal{D}_S$-modules $\beta : \varphi^* M_{S'} \xrightarrow{\sim} M_S$ satisfying a cocycle condition. The data of (S, π_S) is equivalent to giving a principal G-bundle $p : P \to S$ together with a smooth G-equivariant morphism $\gamma : P \to X$.

$$\begin{tikzcd} P \arrow[r, "\gamma"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ S \arrow[r] & X/G \end{tikzcd} \tag{1}$$

The point now is that M pullback to a strongly equivariant D-module N on P. The G-invariant sections N^G of N then determine a $\mathcal{D}_S$-module M_S on S. This is because $\mathcal{D}_S$ embeds into $(\mathcal{D}_P)^G$: if $P = G \times S$ is trivial then $\mathcal{D}_S \hookrightarrow (\mathcal{D}_{G\times S})^G$, and one can use a fpqc-local triviality of G-bundles to reduce to this case. One can check that this procedure produces a well-defined D-module on X/G.

For a quasi-inverse functor, note that amongst $(X/G)_{\mathrm{sm}}$ are the following fiber squares

[1] In fact, it commutes with colimits of presheaves too.

$$
\begin{tikzcd}
X \times_{X/G} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
X \arrow[r] & X/G
\end{tikzcd}
\qquad \simeq \qquad
\begin{tikzcd}
G \times X \arrow[r, "\text{act}"] \arrow[d, "{g,x \mapsto x}"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
X \arrow[r] & X/G
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \times_{X/G} X \times_{X/G} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
X \times_{X/G} X \arrow[r] & X/G
\end{tikzcd}
\qquad \simeq \qquad
\begin{tikzcd}
G \times G \times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
G \times X \arrow[r] & X/G
\end{tikzcd}
$$

There are three of these second kind corresponding to the three projections p_{01}, p_{12}, p_{02} of the action groupoid of G on X. [1] Now given a D-module $\mathscr{N}$ on X/G, the cartesian square of the first kind gives a D-module N on X. There are morphisms between the cartesian square of the first kind and the cartesian squares of the second kind in $(X/G)_{\mathrm{sm}}$. These produce the transition map for a strong G-equivariance structure on N and the cocycle condition for $\mathscr{N}$ gives the cocycle condition for the the strong G-equivariance.

□

4 Relativising to the entire curve

Back to Hecke actions. The previous discussion of Sph_G does not involve the curve X. Given a closed point x of the curve X and a choice $k[\![t]\!] \simeq \mathcal{O}_x^\wedge$, we can replace the disk and punctured disk around 0 in $\mathbb{A}^1$ by the disk and punctured disk around x in X. (If need more material, can define $D_{x_S}, D_{x_S}^\circ$.) In other words, we have isomorphisms :

$$LG \simeq L_x G \quad L^+G \simeq L_x^+ G$$

where $L_xG(S) = G(D_{x_S}^\circ)$ and $L_x^+G(S) = G(D_{x_S})$. These isomorphisms induce

$$\mathrm{Sph}_G \simeq \mathrm{Sph}_{G,x}$$

where the latter is the category of D-modules on $L_x^+G\backslash(L_xG/L_x^+G)$. From the talk in week 1, we can make an action of $\mathrm{Sph}_{G,x}$ on $D\mathbf{Mod}(\mathrm{Bun}_G)$ by considering the L_x^+G-bundle $\mathrm{Bun}_{G,x}^{\mathrm{lvl}} \to \mathrm{Bun}_G$ and the regluing action of L_xG on $\mathrm{Bun}_{G,x}^{\mathrm{lvl}}$. I now justify my claim in week 1 that there is a way of doing all of this in families above X.

The trick is to consider the "bundle of uniformisers" :

$$\widehat{X}^+(S) := \{(x, \alpha) \,|\, x \in X(S), \alpha : D_{x_S} \simeq D_S \text{ preserving } S\}$$

Examples of points of this is given by [Stacks, Tag 05D5]. The forgetful map $\widehat{X}^+ \to X$ is a bundle for the pro-algebraic group $\mathrm{Aut}^+ D$ of automorphisms of D preserving 0. There is an action of $\mathrm{Aut}^+ D$ on LG, L^+G, Gr_G

[1] The action map $G \times X \to X$ is smooth because it is the composition of $G \times X \simeq G \times X, (g, x) \mapsto (g, gx)$ with the second projection.

then the relativisation precedure is twisting these using $\widehat{X}^+$.

$$\begin{aligned} L_X^+ G &:= \widehat{X}^+ \times^{\mathrm{Aut}^+ D} L^+ G \\ L_X G &:= \widehat{X}^+ \times^{\mathrm{Aut}^+ D} LG \\ \mathrm{Gr}_{G,X} &:= \widehat{X}^+ \times^{\mathrm{Aut}^+ D} \mathrm{Gr}_G \end{aligned}$$

The final object here is called the *Beilinson–Drinfeld Grassmannian* and it will play a crucial role in proving that the monoidal structure on Sph_G is symmetric, i.e. the categorification of the fact that convolution product on the spherical Hecke algebra is commutative.

References

[BD] A. Beilinson and V. Drinfeld. *Quantization of Hitchin's Integrable System and Hecke Eigensheaves*. URL: https://math.uchicago.edu/~drinfeld/langlands/QuantizationHitchin.pdf.

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