

# Week 5 : Geometric Satake Part I - Affine Grassmannian

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Appendices contain details added after the talk.

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Let  $G$  be a reductive group over  $k = \mathbb{C}$ . Recall the rough statement of geometric Satake.

### Proposition

There is an equivalence of tensor abelian categories

$$\mathrm{Sph}_G \simeq \mathrm{Rep}_{\mathbb{C}} G^L$$

Here, the spherical Hecke category A.K.A. Satake category is the abelian category of  $L^+G$ -equivariant (regular holonomic)  $D$ -modules<sup>1</sup> on  $Gr_G$ .

$$\mathrm{Sph}_G := (D\mathbf{Mod}_{\mathrm{th}} Gr_G)^{L^+G} \simeq (\mathrm{Perv} Gr_G)^{L^+G}$$

Today we focus on answering these two questions :

1. What is the algebro-geometric structure on  $Gr_G$ ?
2. How exactly do we work with equivariant  $D$ -modules on  $Gr_G$ ?

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<sup>1</sup>or alternatively, under the Riemann–Hilbert correspondence, perverse sheaves

# 1 The structure of the affine Grassmannian

Recall from the talk in week 4 that the affine Grassmannian is (equivalent to) the following set-valued functor

$$\mathrm{Gr}_G(A) = \left\{ (P, s) \mid \begin{array}{l} G\text{-bundle } P \text{ on } D_A \\ s : \mathrm{Triv}|_{D_A^\circ} \simeq P|_{D_A^\circ} \end{array} \right\} / \simeq$$

where

- $D_A = \mathrm{Spec} A[[t]]$
- $D_A^\circ = \mathrm{Spec} A((t))$
- $\mathrm{Triv}|_S = S \times G$  trivial  $G$ -bundle <sup>1</sup>

Alternatively, we have

$$\mathrm{Gr}_G \simeq LG/L^+G$$

where the quotient is as fpqc sheaves.

We will see now that the affine Grassmannian is an *ind-scheme*.

## Definition

A functor  $X \in \mathrm{PSh} \mathrm{Aff}$  is called an ind-scheme when it is a colimit of schemes along closed embeddings.

## Proposition

$\mathrm{Gr}_G$  is an ind-scheme. Moreover, we can write  $\mathrm{Gr}_G \simeq \varinjlim_{N \in \mathbb{N}} X_N$  where

1.  $X_N$  is projective schemes over  $k$
2. transition maps are closed embeddings
3. each  $X_N$  is  $L^+G$  stable
4. The action of  $L^+G$  on  $X_N$  factors through some  $L^M G$  for sufficiently large  $M$ .

*Proof.* ( $G = \mathrm{GL}_n$  case) In this case we have equivalence between groupoids of

- $\mathrm{GL}_n$ -bundles on  $D_A$
- rank  $n$  vector bundles on  $D_A$
- rank  $n$  projective modules over  $A[[t]]$

This allows us to make the following simplification of  $\mathrm{Gr}_{\mathrm{GL}_n}$  : it is isomorphic to the functor

$$A \mapsto \text{set of } A[[t]]\text{-lattices in } A((t))^n$$

See [Appendix : Affine Grassmannian for  \$\mathrm{GL}\_n\$](#) .

Let  $\Lambda_0$  be the standard lattice  $A[[t]]^n$  in  $A((t))^n$ . Then for any  $A[[t]]$ -lattice  $\Lambda$  one can find  $N \leq 0$  such that

$$t^N \Lambda_0 \subseteq \Lambda \subseteq t^{-N} \Lambda_0$$

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<sup>1</sup>Recall  $G$ -bundles means fpqc  $G$ -torsor.

This condition defines a subfunctor  $\text{Gr}_{\text{GL}_n}^{(N)}$  for each  $N$  and we have

$$\text{Gr}_{\text{GL}_n} \simeq \varinjlim_N \text{Gr}_{\text{GL}_n}^{(N)}$$

We choose  $X_N = \text{Gr}_{\text{GL}_n}^{(N)}$  for the result.

Proof of (1) :

- Idea : each  $\Lambda \in \text{Gr}_{\text{GL}_n}^{(N)}(A)$  gives a subspace

$$0 \rightarrow \Lambda/t^N \Lambda_0 \rightarrow t^{-N} \Lambda_0/t^N \Lambda_0 \simeq A^{2nN}$$

So  $\Lambda$  should give an  $A$ -point in the usual Grassmannian  $\text{Gr}(2nN)$ .

- Issue :  $\text{Gr}(2nN)(A) = \text{set of direct summands of } A^{2nN}$ .
- Fix : we show that  $\Lambda/t^N \Lambda_0$  has a complement in  $t^{-N} \Lambda_0/t^N \Lambda_0$ . It suffices to show

$$\frac{t^{-N} \Lambda_0/t^N \Lambda_0}{\Lambda/t^N \Lambda_0} \simeq \frac{t^{-N} \Lambda_0}{\Lambda}$$

is projective over  $A$ . This fits in an SES :

$$0 \rightarrow \frac{t^{-N} \Lambda_0}{\Lambda} \rightarrow \frac{A((t))^n}{\Lambda} \rightarrow \frac{A((t))^n}{t^{-N} \Lambda_0} \rightarrow 0$$

It suffices to show the right two modules are projective  $A$ -modules. The right-most module is free over  $A$ . For the middle module, we have

$$\frac{A((t))^n}{\Lambda} = \frac{\bigcup_{0 \leq k} t^{-k} \Lambda}{\Lambda} \simeq \bigoplus_{0 \leq k} \frac{t^{-(k+1)} \Lambda}{t^{-k}} \simeq \bigoplus_{0 \leq k} \frac{\Lambda}{t \Lambda}$$

where the first isomorphism uses the fact that  $\Lambda/t \Lambda$  is projective over  $A$ .

Proof of (2) : The above defines a map  $\text{Gr}_{\text{GL}_n}^{(N)} \rightarrow \text{Gr}(2nN)$  which is injective, which we need to show is a closed embedding. This follows from the observation that subspaces of  $t^{-N} \Lambda_0/t^N \Lambda_0$  of the form  $\Lambda/t^N \Lambda_0$  are precisely those which are stable under the action of  $t$ . If we use  $t$  again to denote the corresponding  $A$ -linear endomorphism on  $A^{2nN}$  under  $A^{2nN} \simeq t^{-N} \Lambda_0/t^N \Lambda_0 \simeq A^{2nN}$ , then we have

$$\text{Gr}_{\text{GL}_n}^{(N)}(A) \simeq \{M \in \text{Gr}(2nN)(A) \mid tM \subseteq M\}$$

The condition  $tM \subseteq M$  is equivalent to  $(M + tM)/M = 0$  which defines a closed subscheme of  $\text{Spec } A$ . This shows  $\text{Gr}_{\text{GL}_n}^{(N)} \rightarrow \text{Gr}(2nN)$  is a closed embedding, and hence a projective scheme over  $k$ .

Proof of (3) : Elements  $g \in L^+ \text{GL}_n(A) = \text{GL}_n(A[[t]])$  stabilise the standard lattice  $\Lambda_0 = A[[t]]^n$ . It follows that for any  $\Lambda \in \text{Gr}_{\text{GL}_n}^{(N)}(A)$ ,  $g\Lambda$  is still in  $\text{Gr}_{\text{GL}_n}^{(N)}(A)$ .

Proof of (4) : Recall that  $L^M G(A) = G(A[t]/(t^{M+1}))$ . The kernel of  $L^+ \text{GL}_n(A) \rightarrow L^M \text{GL}_n(A)$  consists of elements of the form  $1 + t^{M+1} A$  with  $A \in M_{n \times n}(A[[t]])$ . So the action of  $L^+ G$  on  $X_N$  factors through  $L^M G$

if and only if all  $1 + t^{M+1}A$  act trivially on any points of  $X_N$ . We claim that  $M + 1 = 2N$  works. It suffices to show  $(1 + t^{2N}A)\Lambda \subseteq \Lambda$  for any  $\Lambda \in X_N$ . This follows from

$$(1 + t^{2N}A)\Lambda \subseteq \Lambda + t^{2N}At^{-N}\Lambda_0 \subseteq \Lambda + t^N\Lambda_0 \subseteq \Lambda$$

(General reductive  $G$ ) One can show that  $\mathrm{Gr}_G$  is ind-projective using the following argument : There exists  $i : G \rightarrow \mathrm{GL}_n$  a closed embedding with affine homogeneous space  $\mathrm{GL}_n/G$  of finite type.<sup>1</sup> To show  $\mathrm{Gr}_G$  is ind-projective, it suffices to show the induced functor  $\mathrm{Gr}_G \rightarrow \mathrm{Gr}_{\mathrm{GL}_n}$  is a closed embedding. See [Appendix : Ind-projectivity of  \$\mathrm{Gr}\_G\$  for general  \$G\$](#) .

□

## 2 Schubert varieties and cells

This section is roughly based on [Zhu-17].

To define and study  $L^+G$ -equivariant  $D$ -modules on  $\mathrm{Gr}_G$ , we need to understand the  $L^+G$ -orbits (of  $k$ -valued points). But we know that for a finite type scheme  $X$ ,

$$D\mathrm{Mod}(X) \xrightarrow{\sim} D\mathrm{Mod}(X_{\mathrm{red}})$$

so as far as  $D$ -modules are concerned, we can work with  $(\mathrm{Gr}_G)_{\mathrm{red}} = \varinjlim_n (X_n)_{\mathrm{red}}$  instead. This is now an ind-variety<sup>2</sup> and hence we can work at the level of  $k$ -valued points. We have<sup>3</sup>

$$(\mathrm{Gr}_G)_{\mathrm{red}}(k) = \mathrm{Gr}_G(k) \simeq LG(k)/L^+G(k) = G(k((t)))/G(k[[t]])$$

The  $L^+G(k)$ -orbits of  $\mathrm{Gr}_G(k)$  are given by the *Cartan decomposition*. Begin by choosing a maximal torus<sup>4</sup> and Borel  $T \subseteq B \subseteq G$  and let  $W := N(T)/T$  be the Weyl group. Recall the coweight lattice of  $T$

$$\mathbf{X}_\bullet(T) := \underline{\mathrm{Hom}}(\mathbb{G}_m, T)$$

For any  $F/k$  algebraically closed,

$$\begin{aligned} \mathbf{X}_\bullet(T) &\rightarrow LG(F) \\ \lambda &\mapsto t^\lambda := \lambda(t) \end{aligned}$$

### Proposition – Cartan decomposition

For  $F/k$  algebraically closed, we have the isomorphism<sup>a</sup>

$$\mathbf{X}_\bullet(T)^+ \simeq \mathbf{X}_\bullet(T)/W \simeq L^+G(F) \backslash LG(F) / L^+G(F)$$

<sup>1</sup>All algebraic groups admit a closed embedding  $G \rightarrow \mathrm{GL}_n$  for some  $n$ . [M17, Theorem 4.8] The fact that the quotient  $\mathrm{GL}_n/G$  must be affine finite type is Matsushima's criterion.

<sup>2</sup>where variety means reduced, separated, finite type scheme over  $k$

<sup>3</sup>We showed in week 5 that  $\mathrm{Gr}_G = LG/L^+G$  where the quotient is as fpqc sheaves, so it does not immediately follow that  $\mathrm{Gr}_G(k) = LG(k)/L^+G(k)$ . However, this is nonetheless true because  $G$ -bundles on  $D_k$  can be trivialised without passing through  $D_S \rightarrow D_k$  via an fpqc cover  $S \rightarrow \mathrm{Spec} k$ .

<sup>4</sup>automatically split because  $k$  is algebraically closed

<sup>a</sup>Note that this isomorphism is functorial in  $F$ .

### Definition – Schubert cell, variety

For  $\lambda \in \mathbf{X}_\bullet(T)^+$ , the associated Schubert cell is defined as

$$\mathrm{Gr}_G^\lambda := L^+G(k)\text{-orbit of } t^\lambda$$

and the associated Schubert variety is defined the Zariski closure  $\overline{\mathrm{Gr}_G^\lambda}$ .

### Proposition – Properties of Schubert cells and varieties

The following are true :

1.  $\mathrm{Gr}_G^\lambda \rightarrow \mathrm{Gr}_G$  is locally closed variety
2.  $\overline{\mathrm{Gr}_G^\lambda}$  is a projective variety
3. For all  $x \in \mathrm{Gr}_G(k)$ ,  $\mathrm{Stab}_{L^+G}(x)$  is connected
4.  $\dim \mathrm{Gr}_G^\lambda = (2\rho, \lambda)$
5.  $\overline{\mathrm{Gr}_G^\lambda} = \bigcup_{\mu \leq \lambda} \mathrm{Gr}_G^\mu$
6. The connected components of  $\mathrm{Gr}_G$  are

$$\bigcup_{\mu \in \omega} \mathrm{Gr}_G^\mu \text{ where } \omega \in \mathbf{X}_\bullet(T)/\mathbb{Z}\Phi^\vee$$

*Proof.* Recall that we could write  $\mathrm{Gr}_G = \varinjlim_n X_n$  where the transition maps are closed embeddings and each  $X_n$  is a  $L^+G$ -stable projective variety with  $L^+G$  acting through a finite type quotient. For  $\lambda \in \mathbf{X}_\bullet(T)^+$ ,  $t^\lambda \in \mathrm{Gr}_G(k)$  must lie in some  $X_n$  and hence  $\mathrm{Gr}_G^\lambda \subseteq X_n$  in that  $X_n$ . (1) and (2) now follow from [M17, Prop. 9.4].

(6) follows from (5). The proof of (3), (4), (5) was omitted in the talk but can be found at [Zhu16, Prop. 2.1.5].

□

## 3 Equivariant $D$ -modules

We recall some facts about  $D$ -modules on a smooth variety  $X$  over  $k = \mathbb{C}$ .

1. We have the Riemann–Hilbert correspondence [H.T.T., Theorem 7.2.2]

$$\begin{array}{ccccc}
& D_{rh}^b(D_X) & \simeq & D_c^b(X^{\text{an}}) & \\
& \uparrow & & \uparrow & \\
D\mathbf{Mod}_{rh}(X) & \simeq & D_{rh}^b(D_X)^\heartsuit & \simeq & D_c^b(X^{\text{an}})^\heartsuit =: \text{Perv}(X^{\text{an}})
\end{array}$$

The LHS is the derived category of complexes of  $D$ -modules with bounded cohomologies which are all regular holonomic. The RHS is the derived category of complexes of  $\mathbb{C}$ -valued sheaves with respect to the analytic topology on  $X$ , with bounded cohomologies which are all constructible. The  $t$ -structure on the LHS is the standard one and it identifies with the perverse  $t$ -structure on the right. For this study group, we are using the  $D$ -modules side, but Jonas is secretly using perverse sheaves so to be safe, we will assume all  $D$ -modules are regular holonomic. <sup>1</sup>

2. There are six derived functors  $f^!$ ,  $f_!$ ,  $f_*$ ,  $f^*$ ,  $\otimes$ ,  $\underline{\text{Hom}}$  on the  $D^b(D_X)$ . In general, none of these preserve the heart  $D\mathbf{Mod}(\_)$ . We list some special cases.

- If  $i$  is a closed embedding, then  $i_* = i_!$  preserves  $D\mathbf{Mod}(\_)$ .
- If  $j$  is an open embedding, then  $j^* = j^!$  preserves  $D\mathbf{Mod}(\_)$ .
- If  $f$  is smooth of relative dimension  $d$ , then  $f^\dagger := f^*[d] = f^![-d]$  preserves  $D\mathbf{Mod}(\_)$

Recall from the previous talk that given an algebraic group  $H$  acting on  $X$  we can consider the category  $(D\mathbf{Mod}X)^H$  of (strongly)  $H$ -equivariant  $D$ -modules on  $X$ . Given a morphism  $H' \rightarrow H$  of algebraic groups, we have a restriction functor

$$\text{Res}_{H'}^H : (D\mathbf{Mod}X)^H \rightarrow (D\mathbf{Mod}X)^{H'}$$

Concretely,  $H' \rightarrow H$  induces a map of the action groupoid of  $H'$  on  $X$  to that of  $H$ , allowing restriction of  $H$ -equivariance to  $H'$ -equivariance. For the example of  $H' = 1 \rightarrow H$ , we obtain the “forgetful functor”

$$(D\mathbf{Mod}X)^H \rightarrow D\mathbf{Mod}X$$

This is fully faithful when  $H$  is connected. [Zhu16, Lem. A.1.2]

We have some further properties :

1.  $i_* = i_!$ ,  $j^* = j^!$ ,  $f^\dagger$  still make sense for (strongly) equivariant (regular holonomic)  $D$ -modules.
2. Given  $N \trianglelefteq H$  normal closed subgroup acting trivially on  $X$ , then

$$(D\mathbf{Mod}X)^{H/N} \xrightarrow{\sim} (D\mathbf{Mod}X)^H$$

3. Given  $N \trianglelefteq H$  normal closed subgroup such that  $N$  acts freely on  $X$  and the fpqc quotient  $X/N$  is in fact an algebraic space, then

$$(D\mathbf{Mod}X/N)^{H/N} \xrightarrow{\sim} (D\mathbf{Mod}X)^H$$

From the above, we can show that  $H$ -equivariant  $D$ -modules on homogeneous spaces of  $H$  are very simple.

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<sup>1</sup>Holonomicity and regularity can be seen as finiteness conditions [H.T.T., Theorem 3.3.1, Definition 6.1].

### Proposition

Let  $X$  be a homogeneous space for  $H$  such that the stabilisers for  $x \in X(k)$  are connected. <sup>a</sup> Then

$$(D\mathbf{Mod}X)^H \simeq \mathbf{Vec}_k^{\text{f.d.}}$$

<sup>a</sup>You only need one to be connected for all to be connected.

*Proof.* Pick  $x \in X(k)$ . Then  $X \simeq H/H_x$ . Then

$$(D\mathbf{Mod}X)^H \stackrel{(3)}{\simeq} (D\mathbf{Mod}H)^{H \times H_x} \stackrel{(3)}{\simeq} (D\mathbf{Mod}H/H)^{H_x} \simeq (D\mathbf{Modpt})^{H_x} \stackrel{(2)}{\simeq} D\mathbf{Modpt} \simeq \mathbf{Vec}_k^{\text{f.d.}}$$

□

This can be applied inductively using *recollement* [A21, Exercises A.7.4 to A.7.7] to obtain the following :

### Proposition

Let  $H$  act on  $X$  with finitely many orbits and suppose the stabiliser of all  $x \in X(k)$  are connected. Then

$$\begin{aligned} H\text{-orbits in } X &\longleftrightarrow \text{Simples in } (D\mathbf{Mod}X)^H \\ Hx &\mapsto \text{IC}_{Hx} \end{aligned}$$

where  $\text{IC}_{Hx}$  is such that

- $\text{supp IC}_{Hx} = \overline{Hx}$
- $\text{IC}_{Hx}|_{Hx} \simeq \mathcal{O}_{Hx}$  where restriction means  $*$ -pullback.

The definition of  $\text{IC}_{Hx}$  can be found in [A21, Section 3.3].

## 4 $D$ -modules on ind-varieties

The situation is  $X$  is a ind-variety with an action from a pro-algebraic group  $H$ . We require a presentation  $X \simeq \varinjlim_{i \in I} X_i$  where  $X_i$  is  $H$ -stable and  $H$  acts through a finite type quotient  $H_i$ .

Our desired situation is  $X = \text{Gr}_{\text{GL}_n}$ ,  $H = L^+ \text{GL}_n$ ,  $X_i = \text{Gr}_G^{(i)}$  and  $H_i = L^N \text{GL}_n$  for large enough  $N$ .

We now define

$$(D\mathbf{Mod}X_i)^H := (D\mathbf{Mod}X_i)^{H_i}$$

Property (2) from the previous section ensures this is independent of  $H_i$  up to equivalence of categories. We can now define

$$(D\mathbf{Mod}X)^H := \varinjlim_i (D\mathbf{Mod}X_i)^H$$

where

- the objects are  $(i, \mathcal{F}_i)$  where  $\mathcal{F}_i \in (D\mathbf{Mod}X_i)^H$

- morphisms  $(i, \mathcal{F}_i) \rightarrow (j, \mathcal{F}_j)$  are morphisms  $\alpha_* \mathcal{F}_i \rightarrow \beta_* \mathcal{F}_j$  where

$$X_i \xrightarrow{\alpha} X_k \xleftarrow{\beta} X_j$$

This is well-defined because  $*$ -pushforward is fully faithful along closed embeddings.

This category is independent of the presentation of  $X$  up to equivalence. Finally, we now have a definition of the spherical Hecke category.<sup>1</sup>

#### Definition

$$\mathrm{Sph}_G := (D\mathrm{Mod} \, \mathrm{Gr}_G)^{L^+G}$$

Given a closed subvariety  $X$  of  $\mathrm{Gr}_G$  which is  $L^+G$ -stable, it must be a union

$$X = \bigcup_{\lambda \in S} \mathrm{Gr}_G^\lambda$$

for some finite  $S \subseteq \mathbf{X}_\bullet(T)^+$  because  $X$  is finite dimensional and  $\mathrm{Gr}_G^\lambda$  have arbitrarily large dimension for larger  $\lambda$ . It follows that  $L^+G$  acts on  $X$  through a finite type quotient so  $(D\mathrm{Mod} \, X)^{L^+G}$  makes sense. Then we have a bijection

$$S \leftrightarrow \left\{ \text{simples in } (D\mathrm{Mod} \, X)^{L^+G} \right\}$$

Since any  $\mathcal{F} \in (D\mathrm{Mod} \, \mathrm{Gr}_G)^{L^+G}$  is supported on some closed subvariety  $X$  of  $\mathrm{Gr}_G$ , we deduce

#### Proposition

There is a bijection

$$\begin{aligned} \mathbf{X}_\bullet(T)^+ &\longleftrightarrow \left\{ \text{simples in } (D\mathrm{Mod} \, \mathrm{Gr}_G)^{L^+G} \right\} \\ \lambda &\mapsto \mathrm{IC}_\lambda := \mathrm{IC}_{\mathrm{Gr}_G^\lambda} \end{aligned}$$

TODO :

- the technical issues with defining convolution
- the fix
- the sketch of proof that  $\mathrm{Sph}_G$  is semi-simple

## 5 Appendix : Affine Grassmannian for $\mathrm{GL}_n$

*Lemma. The affine Grassmannian for  $\mathrm{GL}_n$  is isomorphic to the set-valued functor*

$$A \mapsto \text{set of } A[[t]]\text{-lattices in } A((t))^n$$

<sup>1</sup>Thought to myself : the collection of Schubert varieties indexed by  $\mathbf{X}_\bullet(T)^+$  should be a more canonical presentation of  $\mathrm{Gr}_G$  than the one provided by  $\mathrm{Gr}_G \rightarrow \mathrm{Gr}_{\mathrm{GL}_n}$ . The issue is that this is really a presentation of  $(\mathrm{Gr}_G)_{\mathrm{red}}$  not  $\mathrm{Gr}_G$ . But after speaking with Jonas, this seems to be a non-issue because  $D\mathrm{Mod}(X) \simeq D\mathrm{Mod}(X_{\mathrm{red}})$ .



where an  $A[[t]]$ -lattice in  $A((t))^n$  is defined as a  $A[[t]]$  submodule  $\Lambda$  of  $A((t))^n$  which is

1. *finitely generated projective over  $A[[t]]$*
2. *the inclusion  $\Lambda \subseteq A((t))^n$  induces  $\Lambda \otimes_{A[[t]]} A((t)) \simeq A((t))^n$*

*Proof.* Let  $\text{Gr}'_{\text{GL}_n}$  denote our target functor. Let  $\Lambda$  be an  $A$ -family of lattices in  $A((t))^n$ . Then  $\Lambda$  is a rank  $n$  vector bundle on  $D_A$ . The fact that the inclusion  $\Lambda \rightarrow A((t))^n$  induces  $\Lambda \otimes_{A[[t]]} A((t)) \simeq A((t))^n$  supplies a trivialisation of  $\Lambda|_{D_A^\circ}$ . This defines  $\text{Gr}_{\text{GL}_n} \rightarrow \text{Gr}'_{\text{GL}_n}$ .

Conversely, suppose we are given a rank  $n$  vector bundle  $M$  over  $D_A$  and a trivialisation  $s$  of  $M$  over  $D_A^\circ$ . Goal : give an  $A[[t]]$ -lattice  $M$  in  $A((t))^n$ . It suffices to show

$$M \rightarrow M \otimes_{A[[t]]} A((t)), m \mapsto m \otimes 1$$

is an injective morphism of  $A[[t]]$  modules since the image of  $M$  under  $s : M \otimes_{A[[t]]} A((t)) \simeq A((t))^n$  gives an  $A[[t]]$ -lattice in  $A((t))^n$ . We have a SES of  $A[[t]]$  modules.

$$0 \rightarrow A[[t]] \rightarrow A((t)) \rightarrow Q \rightarrow 0$$

We reach our goal by applying  $M \otimes_{A[[t]]} -$  and projectivity of  $M$ . This defines  $\text{Gr}'_{\text{GL}_n} \rightarrow \text{Gr}_{\text{GL}_n}$ .

We omit checking these are mutual inverses. ■

## 6 Appendix : Ind-projectivity of $\text{Gr}_G$ for general $G$

Goal : given an affine  $S$  and a point  $(P, s) : S \rightarrow \text{Gr}_{\text{GL}_n}$ , we need to show the right vertical morphism

$$\begin{array}{ccc} \text{Gr}_G & \xleftarrow{\quad} & ? \\ \downarrow & \lrcorner & \downarrow \\ \text{Gr}_{\text{GL}_n} & \xleftarrow{(P,s)} & S \end{array}$$

is a closed embedding. By looking at the points of the fiber product in question, one sees that one needs to understand more about reductions of bundles, leading one to the following lemma :

*Lemma.* Let  $G_1 \rightarrow G_2$  be a closed subgroup inclusion of algebraic groups and  $X$  a functor acted on by  $G_2$ . Then we have a cartesian square :

$$\begin{array}{ccc} X/G_1 & \longrightarrow & \text{pt}/G_1 \\ \downarrow & \lrcorner & \downarrow \\ X/G_2 & \longrightarrow & \text{pt}/G_2 \end{array}$$

*Proof.* We only sketch comparison functors. <sup>a</sup> A point of  $X/G_1$  is a diagram

$$\begin{array}{ccc} P & \xrightarrow{G_1\text{-equiv}} & X \\ G_1\text{-bundle} \downarrow & & \\ S & & \end{array}$$

The map  $X/G_1 \rightarrow \text{pt}/G_1$  forgets the map  $P \rightarrow X$ . The map  $X/G_1 \rightarrow X/G_2$  takes the above such diagrams to

$$\begin{array}{ccc} P \times^{G_1} G_2 & \xrightarrow{G_2\text{-equiv}} & X \\ G_2\text{-bundle} \downarrow & & \\ S & & \end{array}$$

The above defines a morphism  $X/G_1 \rightarrow (X/G_2) \times_{\text{pt}/G_2} (\text{pt}/G_1)$ .

We construct an inverse. A point of the fiber product is the data of an affine  $S$ , a  $G_1$ -bundle  $P_1$  on  $S$ , a  $G_2$ -bundle  $P_2$  on  $S$ , a  $G_2$ -equivariant map  $P_2 \rightarrow X$  and an isomorphism of  $G_2$ -bundles  $P_1 \times^{G_1} G_2 \simeq P_2$ . Given this, we make the diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{G_1\text{-equiv}} & P_1 \times^{G_1} G_2 & \xrightarrow{\sim} & P_2 & \xrightarrow{G_2\text{-equiv}} & X \\ G_1\text{-bundle} \downarrow & & & & & & \\ S & & & & & & \end{array}$$

■

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<sup>a</sup>Note to self: actually I don't know where I used the fact that  $G_1 \rightarrow G_2$  is a closed embedding.

Applying to our situation  $G_1 = G, G_2 = \text{GL}_n, X = P$  shows that the groupoid of  $G$ -reductions of  $P$  is equivalent to the groupoid of sections of  $P/G \rightarrow D_S$ .<sup>1</sup>

Now we can describe the points of the fiber product in question. First, see  $s$  as a section of  $P|_{D_S^\circ}$ . Then given any affine  $t : T \rightarrow S$  over  $S$ , then  $s$  gives rise to a section  $s_t$  of the pullback  $P_t$  to  $D_T$  over  $D_T^\circ$ .

$$\begin{array}{ccc} (P_t)^\circ & \longrightarrow & P_t \\ \downarrow & & \downarrow \\ s_t(P_t)^\circ/G & \longrightarrow & P_t/G \\ \downarrow & & \downarrow \\ D_T^\circ & \longrightarrow & D_T \end{array}$$

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<sup>1</sup>The degenerate case of  $G_1 = 1, G_2 = G$  says that a reduction of a  $G$ -bundle to a 1-bundle is the same thing as a global section.

Since  $s_t$  trivialises  $P_t^\circ$ , it also provides a  $G$ -reduction of  $P_t^\circ$ . The question is : what are all the extensions of this  $G$ -reduction to all of  $D_T$ ? Using the above lemma, this is equivalent to asking for the groupoid of sections of  $P_t/G$  extending  $s_t$  from  $D_T^\circ$  to  $D_T$ . Note that the data of extending  $s_t$  means this groupoid is discrete. Another way of seeing  $P_t/G$  is as  $P \times^{\mathrm{GL}_n} (\mathrm{GL}_n/G)$ . By our assumption of  $\mathrm{GL}_n/G$  being affine and finite type, it follows that  $P_t/G$  is relatively affine and finite type over  $D_T$ . We can thus give a closed embedding

$$\begin{array}{ccc} P_t/G & \xrightarrow{\text{c.emb}} & \mathbb{A}_{D_T}^N \\ \downarrow & \swarrow & \\ D_T & & \end{array}$$

for some large  $N$ . The section  $s_t$  of  $P_t/G$  over  $D_T^\circ$  can be seen as the data

$$s_t = (s_t^1, \dots, s_t^N) \in (\mathcal{O}(T)((z)))^N$$

One sees now that the locus of points of  $D_T$  admitting extensions of  $s_t$  is given by the vanishing of the polar terms in the Laurent expansions of the coordinates of  $s_t$ . Furthermore, any extension is unique. We have thus shown that the fiber product in question is a close subscheme of  $S$ .

## 7 Appendix : Schematic definition of Schubert cells and varieties

The following is my understanding on how to define the Schubert cells and varieties schematically without working at the level of  $k$ -valued points. As discussed with Jonas, since  $D$ -modules do not see the difference between schemes and their reduction, this is purely aesthetical.

### Definition

Let  $F/k$  be algebraically closed. Let  $P_0, P_1$  be  $G$ -bundles on  $D_F$  and  $\beta : P_0|_{D_F^\circ} \rightarrow P_1|_{D_F^\circ}$ . Given trivialisations  $s_i$  of  $P_i$  on  $D_F$ , one obtains an automorphism  $s_1^{-1}\beta s_0$  of  $\mathrm{Triv}|_{D_F^\circ}$ , equivalently an element of  $LG(F)$ . Quotienting by the choice of trivialisations, we obtain a well-defined element

$$\mathrm{Inv}(\beta) \in L^+G(F) \backslash LG(F) / L^+G(F) \simeq \mathbf{X}_\bullet(T)^+$$

This is called the *relative position* of  $\beta$ .

For  $F$  not necessarily algebraically closed, one can choose an algebraic closure  $\overline{F}/F$  and define  $\mathrm{Inv}(\beta)$  by first base changing along  $D_{\overline{F}} \rightarrow D_F$  when do it over  $D_{\overline{F}}$ . The resulting element of  $\mathbf{X}_\bullet(T)^+$  is independent of the choice of  $\overline{F}$  by the Cartan decomposition.

Now let  $\mathrm{Spec} R$  be an affine scheme and  $(P, s) \in \mathrm{Gr}_G(R)$ . For a topological point  $x \in \mathrm{Spec} R$ , define

$$\mathrm{Inv}_x s := \text{relative difference of } P_{D_{\kappa(x)}} \xrightarrow{s} \mathrm{Triv}_{D_{\kappa(x)}} \text{ over } D_{\kappa(x)}^\circ$$

For each  $\mu \in \mathbf{X}_\bullet(T)^+$ , we now define the Schubert variety associated to  $\mu$  as a subfunctor of  $\mathrm{Gr}_G$  by the

formula

$$\mathrm{Gr}_G^{\leq \mu}(R) := \{(P, s) : \forall x \in \mathrm{Spec} R, \mathrm{Inv}_x s \leq \mu\}$$

We also define the Schubert cell of  $\mu$  as

$$\mathrm{Gr}_G^\mu := \mathrm{Gr}_G^{\leq \mu} \setminus \bigcup_{\lambda < \mu} \mathrm{Gr}_G^{\leq \lambda}$$

At this point, one should be able to either show that  $\mathrm{Gr}_G^\lambda$  and  $\mathrm{Gr}_G^{\leq \lambda}$  have reductions giving back the definition in the talk, and maybe even show that these are reduced already. But I have not had the time to think about this nor have I found a reference.

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