Week 5 : Geometric Satake Part I - Affine Grassmannian

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Summer 2023

Appendices contain details added after the talk.

Contents

Let G be a reductive group over $k = \mathbb{C}$. Recall the rough statement of geometric Satake.

> **Proposition**
> There is an equivalence of tensor abelian categories
> $$\mathrm{Sph}_G \simeq \mathrm{Rep}_{\mathbb{C}} G^L$$

Here, the spherical Hecke category A.K.A. Satake category is the abelian category of L^+G-equivariant (regular holonomic) D-modules[1] on Gr_G.

$$\mathrm{Sph}_G := (D\mathbf{Mod}_{\mathrm{rh}}\,\mathrm{Gr}_G)^{L^+G} \simeq (\mathrm{Perv}\,\mathrm{Gr}_G)^{L^+G}$$

Today we focus on answering these two questions :

1. What is the algebro-geometric structure on Gr_G?

[1] or alternatively, under the Riemann–Hilbert correspondence, perverse sheaves

2. How exactly do we work with equivariant D-modules on Gr_G?

1 The structure of the affine Grassmannian

Recall from the talk in week 4 that the affine Grassmannian is (equivalent to) the following set-valued functor

$$\mathrm{Gr}_G(A) = \left\{ (P, s) \,\middle|\, \begin{matrix} G\text{-bundle } P \text{ on } D_A \\ s : \mathrm{Triv}|_{D_A^\circ} \simeq P|_{D_A^\circ} \end{matrix} \right\} / \simeq$$

where

- $D_A = \mathrm{Spec}\, A[\![t]\!]$
- $D_A^\circ = \mathrm{Spec}\, A(\!(t)\!)$
- $\mathrm{Triv}|_S = S \times G$ trivial G-bundle [1]

Alternatively, we have

$$\mathrm{Gr}_G \simeq LG/L^+G$$

where the quotient is as fpqc sheaves.

We will see now that the affine Grassmannian is an *ind-scheme*.

Definition
A functor $X \in \mathrm{PSh\,Aff}$ is called an ind-scheme when it is a colimit of schemes along closed embeddings.

Proposition
Gr_G is an ind-scheme. Moreover, we can write $\mathrm{Gr}_G \simeq \varinjlim_{N \in \mathbb{N}} X_N$ where

1. X_N is projective schemes over k
2. transition maps are closed embeddings
3. each X_N is L^+G stable
4. The action of L^+G on X_N factors through some $L^M G$ for sufficiently large M.

Proof. ($G = \mathrm{GL}_n$ case) In this case we have equivalence between groupoids of

- GL_n-bundles on D_A
- rank n vector bundles on D_A
- rank n projective modules over $A[\![t]\!]$

This allows us to make the following simplicification of $\mathrm{Gr}_{\mathrm{GL}_n}$: it is isomorphic to the functor

$$A \mapsto \text{ set of } A[\![t]\!]\text{-lattices in } A(\!(t)\!)^n$$

See Appendix : Affine Grassmannian for GL_n.

[1] Recall G-bundles means fpqc G-torsor.

Let Λ_0 be the standard lattice $A[\![t]\!]^n$ in $A((t))^n$. Then for any $A[\![t]\!]$-lattice Λ one can find $N \leq 0$ such that

$$t^N \Lambda_0 \subseteq \Lambda \subseteq t^{-N} \Lambda_0$$

This condition defines a subfunctor $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$ for each N and we have

$$\mathrm{Gr}_{\mathrm{GL}_n} \simeq \varinjlim_N \mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$$

We choose $X_N = \mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$ for the result.

Proof of (1) :

- Idea : each $\Lambda \in \mathrm{Gr}^{(N)}_{\mathrm{GL}_n}(A)$ gives a subspace

$$0 \to \Lambda/t^N\Lambda_0 \to t^{-N}\Lambda_0/t^N\Lambda_0 \simeq A^{2nN}$$

  So Λ should give an A-point in the usual Grassmannian $\mathrm{Gr}(2nN)$.
- Issue : $\mathrm{Gr}(2nN)(A) =$ set of direct summands of A^{2nN}.
- Fix : we show that $\Lambda/t^N\Lambda_0$ has a complement in $t^{-N}\Lambda_0/t^N\Lambda_0$. It suffices to show

$$\frac{t^{-N}\Lambda_0/t^N\Lambda_0}{\Lambda/t^N\Lambda_0} \simeq \frac{t^{-N}\Lambda_0}{\Lambda}$$

  is projective over A. This fits in an SES :

$$0 \to \frac{t^{-N}\Lambda_0}{\Lambda} \to \frac{A((t))^n}{\Lambda} \to \frac{A((t))^n}{t^{-N}\Lambda_0} \to 0$$

  It suffices to show the right two modules are projective A-modules. The right-most module is free over A. For the middle module, we have

$$\frac{A((t))^n}{\Lambda} = \frac{\bigcup_{0\leq k} t^{-k}\Lambda}{\Lambda} \simeq \bigoplus_{0\leq k} \frac{t^{-(k+1)}\Lambda}{t^{-k}} \simeq \bigoplus_{0\leq k} \frac{\Lambda}{t\Lambda}$$

  where the first isomorphism uses the fact that $\Lambda/t\Lambda$ is projective over A.

Proof of (2) : The above defines a map $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n} \to \mathrm{Gr}(2nN)$ which is injective, which we need to show is a closed embedding. This follows from the observation that subspaces of $t^{-N}\Lambda_0/t^N\Lambda_0$ of the form $\Lambda/t^N\Lambda_0$ are precisely those which are stable under the action of t. If we use t again to denote the corresponding A-linear endomorphism on A^{2nN} under $A^{2nN} \simeq t^{-N}\Lambda_0/t^N\Lambda_0 \simeq A^{2nN}$, then we have

$$\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}(A) \simeq \{M \in \mathrm{Gr}(2nN)(A) \,|\, tM \subseteq M\}$$

The condition $tM \subseteq M$ is equivalent to $(M + tM)/M = 0$ which defines a closed subscheme of $\mathrm{Spec}\, A$. This shows $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n} \to \mathrm{Gr}(2nN)$ is a closed embedding, and hence a projective scheme over k.

Proof of (3) : Elements $g \in L^+\mathrm{GL}_n(A) = \mathrm{GL}_n(A[\![t]\!])$ stabilise the standard lattice $\Lambda_0 = A[\![t]\!]^n$. It follows that for any $\Lambda \in \mathrm{Gr}^{(N)}_{\mathrm{GL}_n}(A)$, $g\Lambda$ is still in $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}(A)$.

Proof of (4) : Recall that $L^M G(A) = G(A[t]/(t^{M+1}))$. The kernel of $L^+ \mathrm{GL}_n(A) \to L^M \mathrm{GL}_n(A)$ consists of elements of the form $1 + t^{M+1}A$ with $A \in M_{n\times n}(A[\![t]\!])$. So the action of L^+G on X_N factors through $L^M G$ if and only if all $1 + t^{M+1}A$ act trivially on any points of X_N. We claim that $M + 1 = 2N$ works. It suffices to show $(1 + t^{2N}A)\Lambda \subseteq \Lambda$ for any $\Lambda \in X_N$. This follows from

$$(1 + t^{2N}A)\Lambda \subseteq \Lambda + t^{2N}At^{-N}\Lambda_0 \subseteq \Lambda + t^N\Lambda_0 \subseteq \Lambda$$

(General reductive G) One can show that Gr_G is ind-projective using the following argument : There exists $i : G \to \mathrm{GL}_n$ a closed embedding with affine homogeneous space GL_n /G of finite type. [1] To show Gr_G is ind-projective, it suffices to show the induced functor $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a closed embedding. See Appendix : Ind-projectivity of Gr_G for general G.

□

2 Schubert varieties and cells

This section is roughly based on [**Zhu-17**].

To define and study L^+G-equivariant D-modules on Gr_G, we need to understand the L^+G-orbits (of k-valued points). But we know that for a finite type scheme X,

$$D\mathbf{Mod}(X) \xrightarrow{\sim} D\mathbf{Mod}(X_{\mathrm{red}})$$

so as far as D-modules are concerned, we can work with $(\mathrm{Gr}_G)_{\mathrm{red}} = \varinjlim_n (X_n)_{\mathrm{red}}$ instead. This is now an ind-variety[2] and hence we can work at the level of k-valued points. We have[3]

$$(\mathrm{Gr}_G)_{\mathrm{red}}(k) = \mathrm{Gr}_G(k) \simeq LG(k)/L^+G(k) = G(k((t)))/G(k[\![t]\!])$$

The $L^+G(k)$-orbits of $\mathrm{Gr}_G(k)$ are given by the *Cartan decomposition*. Begin by choosing a maximal torus[4] and Borel $T \subseteq B \subseteq G$ and let $W := N(T)/T$ be the Weyl group. Recall the coweight lattice of T

$$\mathbf{X}_\bullet(T) := \underline{\mathrm{Hom}}(\mathbb{G}_m, T)$$

For any F/k algebraically closed,

$$\begin{aligned} \mathbf{X}_\bullet(T) &\to LG(F) \\ \lambda &\mapsto t^\lambda := \lambda(t) \end{aligned}$$

[1] All algebraic groups admit a closed embedding $G \to \mathrm{GL}_n$ for some n. [Mil17, Theorem 4.8] The fact that the quotient GL_n /G must be affine finite type is Matsushima's criterion.

[2] where variety means reduced, separated, finite type scheme over k

[3] We showed in week 5 that $\mathrm{Gr}_G = LG/L^+G$ where the quotient is as fpqc sheaves, so it does not immediately follow that $\mathrm{Gr}_G(k) = LG(k)/L^+G(k)$. However, this is nonetheless true because G-bundles on D_k can be trivialised without passing through $D_S \to D_k$ via an fpqc cover $S \to \mathrm{Spec}\, k$.

[4] automatically split because k is algebraically closed

Proposition – Cartan decomposition

For F/k algebraically closed, we have the isomorphism[a]

$$\mathbf{X}_\bullet(T)^+ \simeq \mathbf{X}_\bullet(T)/W \simeq L^+G(F)\backslash LG(F)/L^+G(F)$$

[a] Note that this isomorphism is functorial in F.

Definition – Schubert cell, variety

For $\lambda \in \mathbf{X}_\bullet(T)^+$, the associated Schubert cell is defined as

$$\mathrm{Gr}_G^\lambda := \ L^+G(k)\text{-orbit of } t^\lambda$$

and the associated Schubert variety is defined the Zariski closure $\overline{\mathrm{Gr}_G^\lambda}$.

Proposition – Properties of Schubert cells and varieties

The following are true :

1. $\mathrm{Gr}_G^\lambda \to \mathrm{Gr}_G$ is locally closed variety
2. $\overline{\mathrm{Gr}_G^\lambda}$ is a projective variety
3. For all $x \in \mathrm{Gr}_G(k)$, $\mathrm{Stab}_{L^+G}(x)$ is connected
4. $\dim \mathrm{Gr}_G^\lambda = (2\rho, \lambda)$
5. $\overline{\mathrm{Gr}_G^\lambda} = \bigcup_{\mu\leq\lambda} \mathrm{Gr}_G^\mu$
6. The connected components of Gr_G are

$$\bigcup_{\mu\in\omega} \mathrm{Gr}_G^\mu \text{ where } \omega \in \mathbf{X}_\bullet(T)/\mathbb{Z}\Phi^\vee$$

Proof. Recall that we could write $\mathrm{Gr}_G = \varinjlim_n X_n$ where the transition maps are closed embeddings and each X_n is a L^+G-stable projective variety with L^+G acting through a finite type quotient. For $\lambda \in \mathbf{X}_\bullet(T)^+$, $t^\lambda \in \mathrm{Gr}_G(k)$ must lie in some X_n and hence $\mathrm{Gr}_G^\lambda \subseteq X_n$ in that X_n. (1) and (2) now follow from [Mil17, Prop. 9.4].

(6) follows from (5). The proof of (3), (4), (5) was omitted in the talk but can be found at [**Zhu-16**].

□

3 Equivariant D-modules

We recall some facts about D-modules on a smooth variety X over $k = \mathbb{C}$.

1. We have the Riemann–Hilbert correspondence [HTT07, Theorem 7.2.2]

$$\begin{array}{ccccccc} & & D^b_{rh}(D_X) & \simeq & D^b_c(X^{\mathrm{an}}) & & \\ & & \uparrow & & \uparrow & & \\ D\mathbf{Mod}_{rh}(X) & \simeq & D^b_{rh}(D_X)^\heartsuit & \simeq & D^b_c(X^{\mathrm{an}})^\heartsuit & =: & \mathrm{Perv}(X^{\mathrm{an}}) \end{array}$$

The LHS is the derived category of complexes of D-modules with bounded cohomologies which are all regular holonomic. The RHS is the derived category of complexes of $\mathbb{C}$-valued sheaves with respect to the analytic topology on X, with bounded cohomologies which are all constructible. The t-structure on the LHS is the standard one and it identifies with the perverse t-structure on the right. For this study group, we are using the D-modules side, but Jonas is secretly using perverse sheaves so to be safe, we will assume all D-modules are regular holonomic. [1]

2. There are six derived functors $f^!, f_!, f_*, f^*, \otimes, \underline{\mathrm{Hom}}$ on the $D^b(D_X)$. In general, none of these preserve the heart $D\mathbf{Mod}(\_)$. We list some special cases.
   - If i is a closed embedding, then $i_* = i_!$ preserves $D\mathbf{Mod}(\_)$.
   - If j is an open embedding, then $j^* = j^!$ preserves $D\mathbf{Mod}(\_)$.
   - If f is smooth of relative dimension d, then $f^\dagger := f^*[d] = f^![-d]$ preserves $D\mathbf{Mod}(\_)$

Recall from the previous talk that given an algebraic group H acting on X we can consider the category $(D\mathbf{Mod}X)^H$ of (strongly) H-equivariant D-modules on X. Given a morphism $H' \to H$ of algebraic groups, we have a restriction functor

$$\mathrm{Res}^H_{H'} : (D\mathbf{Mod}X)^H \to (D\mathbf{Mod}X)^{H'}$$

Concretely, $H' \to H$ induces a map of the action groupoid of H' on X to that of H, allowing restriction of H-equivariance to H'-equivariance. For the example of $H' = 1 \to H$, we obtain the "forgetful functor"

$$(D\mathbf{Mod}X)^H \to D\mathbf{Mod}X$$

This is fully faithful when H is connected. [**Zhu-16**]

We have some further properties :

1. $i_* = i_!, j^* = j^!, f^\dagger$ still make sense for (strongly) equivariant (regular holonomic) D-modules.
2. Given $N \trianglelefteq H$ normal closed subgroup acting trivially on X, then

$$(D\mathbf{Mod}X)^{H/N} \xrightarrow{\sim} (D\mathbf{Mod}X)^H$$

3. Given $N \trianglelefteq H$ normal closed subgroup such that N acts freely on X and the fpqc quotient X/N is in fact an algebraic space, then

$$(D\mathbf{Mod}X/N)^{H/N} \xrightarrow{\sim} (D\mathbf{Mod}X)^H$$

From the above, we can show that H-equivariant D-modules on homogeneous spaces of H are very simple.

[1]Holonomicity and regularity can be see as finiteness conditions [HTT07, Theorem 3.3.1, Definition 6.1].

Proposition

Let X be a homogeneous space for H such that the stabilisers for $x \in X(k)$ are connected. [a] Then

$$(D\mathbf{Mod}X)^H \simeq \mathbf{Vec}_k^{\text{f.d.}}$$

[a] You only need one to be connected for all to be connected.

Proof. Pick $x \in X(k)$. Then $X \simeq H/H_x$. Then

$$(D\mathbf{Mod}X)^H \overset{(3)}{\simeq} (D\mathbf{Mod}H)^{H \times H_x} \overset{(3)}{\simeq} (D\mathbf{Mod}H/H)^{H_x} \simeq (D\mathbf{Mod}\mathrm{pt})^{H_x} \overset{(2)}{\simeq} D\mathbf{Mod}\mathrm{pt} \simeq \mathbf{Vec}_k^{\text{f.d.}}$$

□

This can be applied inductively using *recollement* [Ach21, Exercises A.7.4 to A.7.7] to obtain the following :

Proposition

Let H act on X with finitely many orbits and suppose the stabiliser of all $x \in X(k)$ are connected. Then

$$\begin{aligned} H\text{-orbits in } X \;\longleftrightarrow\; & \text{Simples in } (D\mathbf{Mod}X)^H \\ Hx \mapsto & \, \mathrm{IC}_{Hx} \end{aligned}$$

where IC_{Hx} is such that

- $\operatorname{supp} \mathrm{IC}_{Hx} = \overline{Hx}$
- $\mathrm{IC}_{Hx}|_{Hx} \simeq \mathcal{O}_{Hx}$ where restriction means $*$-pullback.

The definition of IC_{Hx} can be found in [Ach21, Section 3.3].

4 D-modules on ind-varieties

The situation is X is a ind-variety with an action from a pro-algebraic group H. We require a presentation $X \simeq \varinjlim_{i \in I} X_i$ where X_i is H-stable and H acts through a finite type quotient H_i.

Our desired situation is $X = \mathrm{Gr}_{\mathrm{GL}_n}$, $H = L^+ \mathrm{GL}_n$, $X_i = \mathrm{Gr}_G^{(i)}$ and $H_i = L^N \mathrm{GL}_n$ for large enough N.

We now define

$$(D\mathbf{Mod}X_i)^H := (D\mathbf{Mod}X_i)^{H_i}$$

Property (2) from the previous section ensures this is independent of H_i up to equivalence of categories. We can now define

$$(D\mathbf{Mod}X)^H := \varinjlim_i (D\mathbf{Mod}X_i)^H$$

where

- the objects are $(i, \mathcal{F}_i)$ where $\mathcal{F}_i \in (D\mathbf{Mod}X_i)^H$

– morphisms $(i, \mathcal{F}_i) \to (j, \mathcal{F}_j)$ are morphisms $\alpha_* \mathcal{F}_i \to \beta_* \mathcal{F}_j$ where

$$X_i \xrightarrow{\alpha} X_k \xleftarrow{\beta} X_j$$

This is well-defined because $*$-pushfoward is fully faithful along closed embeddings.

This category is independent of the presentation of X up to equivalence. Finally, we now have a definition of the spherical Hecke category. [1]

> **Definition**
> $\mathrm{Sph}_G := (D\mathbf{Mod}\,\mathrm{Gr}_G)^{L^+G}$

5 Structure of Sph_G

Given a closed subvariety X of Gr_G which is L^+G-stable, it must be a union

$$X = \bigcup_{\lambda \in S} \mathrm{Gr}_G^\lambda$$

for some finite $S \subseteq \mathbf{X}_\bullet(T)^+$ because X is finite dimensional and Gr_G^λ have arbitrarily large dimension for larger λ. It follows that L^+G acts on X through a finite type quotient so $(D\mathbf{Mod}X)^{L^+G}$ makes sense. Then we have a bijection

$$S \leftrightarrow \left\{ \text{simples in } (D\mathbf{Mod}X)^{L^+G} \right\}$$

Since any $\mathcal{F} \in (D\mathbf{Mod}\,\mathrm{Gr}_G)^{L^+G}$ is supported on some closed subvariety X of Gr_G, we deduce

> **Proposition**
> There is a bijection
>
> $$\mathbf{X}_\bullet(T)^+ \longleftrightarrow \left\{ \text{simples in } (D\mathbf{Mod}\,\mathrm{Gr}_G)^{L^+G} \right\}$$
> $$\lambda \mapsto \mathrm{IC}_\lambda := \mathrm{IC}_{\mathrm{Gr}_G^\lambda}$$

Let us also address convolution with a bit more detail. Goal : construct a functor

$$\_ * \_ : \mathrm{Sph}_G \times \mathrm{Sph}_G \to \mathrm{Sph}_G$$

This should come from the diagram :

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} LG \times \mathrm{Gr}_G \xrightarrow{q} LG \times^{L^+G} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G$$

[1]Thought to myself : the collection of Schubert varieties indexed by $\mathbf{X}_\bullet(T)^+$ should be a more canonical presentation of Gr_G than the one provided by $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$. The issue is that this is really a presentation of $(\mathrm{Gr}_G)_{\mathrm{red}}$ not Gr_G. But after speaking with Jonas, this seems to be a non-issue because $D\mathbf{Mod}(X) \simeq D\mathbf{Mod}(X_{\mathrm{red}})$.

The morphism q is a L^+G-torsor so it should induce an equivalence

$$q^* : D\mathbf{Mod}(LG \times^{L^+G} \mathrm{Gr}_G)^{L^+G} \xrightarrow{\sim} D\mathbf{Mod}(LG \times \mathrm{Gr}_G)^{L^+G \times L^+G}$$

Given two arc group equivariant D-modules $\mathcal{F}, \mathcal{G}$ on Gr_G, $\mathcal{F} \boxtimes \mathcal{G}$ is $L^+G \times L^+G$ equivariant. Since p is $L^+G \times L^+G$ equivariant, we have $p^*(\mathcal{F} \boxtimes \mathcal{G})$ is $L^+G \times L^+G$ and hence it descends along q to give an arc group equivariant D-module $(q^*)^{-1}p^*(\mathcal{F} \boxtimes \mathcal{G})$ on $LG \times^{L^+G} \mathrm{Gr}_G$. Finally, we can !-pushforward along m. All in all, the formula is

$$\mathcal{F} * \mathcal{G} := m_!(q^*)^{-1}p^*(\mathcal{F} \boxtimes \mathcal{G})$$

Here are some problems of the above argument :

1. $LG \times \mathrm{Gr}_G$ is *not* an ind-variety so we technically don't know how to do D-modules on it!
2. does $m_!$ preserve L^+G-equivariance? (The answer is yes!)

We present a fix for (1). [Ach21, Section 9.2] Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sph}_G$. Then we can find Y, Z closed projective subvarieties of Gr_G such that

- Y, Z are stable under L^+G
- L^+G acts on Y through a finite type quotient H.

Let $K = \ker(L^+G \to H)$. Then

$$\begin{tikzcd}
 & LG/K \arrow[d, "H\text{-torsor}"'] & \tilde{Y} \arrow[l] \arrow[d, "H\text{-torsor}"] \arrow[dl, phantom, "\llcorner"] \\
\mathrm{Gr}_G \simeq & LG/L^+G & Y \arrow[l]
\end{tikzcd}$$

It follows that $\tilde{Y}$ is also a variety. Then we can define $\mathcal{F} * \mathcal{G} := m_!(q^*)^{-1}p^*(\mathcal{F} \boxtimes \mathcal{G})$ by the diagram

$$Y \times Z \xleftarrow{p} \tilde{Y} \times Z \xrightarrow{q} \tilde{Y} \times^H Z \xrightarrow{m} \mathrm{Gr}_G$$

This makes sense because all the spaces are now ind-varieties.

One can show that $\mathcal{F} * \mathcal{G}$ is semisimple. Even better, we have

> **Proposition**
> Sph_G is semi-simple.

6 Appendix : Affine Grassmannian for GL_n

> *Lemma. The affine Grassmannian for GL_n is isomorphic to the set-valued functor*
>
> $$A \mapsto \text{ set of } A[\![t]\!]\text{-lattices in } A((t))^n$$

where an $A[\![t]\!]$-lattice in $A((t))^n$ is defined as a $A[\![t]\!]$ submodule Λ of $A((t))^n$ which is

1. *finitely generated projective over $A[\![t]\!]$*
2. *the inclusion $\Lambda \subseteq A((t))^n$ induces $\Lambda \otimes_{A[\![t]\!]} A((t)) \simeq A((t))^n$*

Proof. Let $\mathrm{Gr}'_{\mathrm{GL}_n}$ denote our target functor. Let Λ be an A-family of lattices in $A((t))^n$. Then Λ is a rank n vector bundle on D_A. The fact that the inclusion $\Lambda \to A((t))^n$ induces $\Lambda \otimes_{A[\![t]\!]} A((t)) \simeq A((t))^n$ supplies a trivialisation of $\Lambda|_{D_S^\circ}$. This defines $\mathrm{Gr}_{\mathrm{GL}_n} \to \mathrm{Gr}'_{\mathrm{GL}_n}$.

Conversely, suppose we are given a rank n vector bundle M over D_A and a trivialisation s of M over D_A°. Goal : give an $A[\![t]\!]$-lattice M in $A((t))^n$. It suffices to show

$$M \to M \otimes_{A[\![t]\!]} A((t)), m \mapsto m \otimes 1$$

is an injective morphism of $A[\![t]\!]$ modules since the image of M under $s : M \otimes_{A[\![t]\!]} A((t)) \simeq A((t))^n$ gives an $A[\![t]\!]$-lattice in $A((t))^n$. We have a SES of $A[\![t]\!]$ modules.

$$0 \to A[\![t]\!] \to A((t)) \to Q \to 0$$

We reach our goal by applying $M \otimes_{A[\![t]\!]} \_$ and projectivity of M. This defines $\mathrm{Gr}'_{\mathrm{GL}_n} \to \mathrm{Gr}_{\mathrm{GL}_n}$.

We omit checking these are mutual inverses. ∎

7 Appendix : Ind-projectivity of Gr_G for general G

Goal : given an affine S and a point $(P, s) : S \to \mathrm{Gr}_{\mathrm{GL}_n}$, we need to show the right vertical morphism

$$\begin{tikzcd} \mathrm{Gr}_G \arrow[d] & ? \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ \mathrm{Gr}_{\mathrm{GL}_n} & S \arrow[l, "(P,s)"'] \end{tikzcd}$$

is a closed embedding. By looking at the points of the fiber product in question, one sees that one needs to understand more about reductions of bundles, leading one to the following lemma :

Lemma. Let $G_1 \to G_2$ be a closed subgroup inclusion of algebraic groups and and X a functor acted on by G_2. Then we have a cartesian square :

$$\begin{tikzcd} X/G_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{pt}/G_1 \arrow[d] \\ X/G_2 \arrow[r] & \mathrm{pt}/G_2 \end{tikzcd}$$

Proof. We only sketch comparison functors. [a] A point of X/G_1 is a diagram

$$\begin{tikzcd} P \arrow[r, "G_1\text{-equiv}"] \arrow[d, "G_1\text{-bundle}"'] & X \\ S \end{tikzcd}$$

The map $X/G_1 \to \mathrm{pt}/G_1$ forgets the map $P \to X$. The map $X/G_1 \to X/G_2$ takes the above such diagrams to

$$\begin{tikzcd} P \times^{G_1} G_2 \arrow[r, "G_2\text{-equiv}"] \arrow[d, "G_2\text{-bundle}"'] & X \\ S \end{tikzcd}$$

The above defines a morphism $X/G_1 \to (X/G_2) \times_{\mathrm{pt}/G_2} (\mathrm{pt}/G_1)$.

We construct an inverse. A point of the fiber product is the data of an affine S, a G_1-bundle P_1 on S, a G_2-bundle P_2 on S, a G_2-equivariant map $P_2 \to X$ and an isomorphism of G_2-bundles $P_1 \times^{G_1} G_2 \simeq P_2$. Given this, we make the diagram

$$\begin{tikzcd} P_1 \arrow[r, "G_1\text{-equiv}"] \arrow[d, "G_1\text{-bundle}"'] & P_1 \times^{G_1} G_2 \arrow[r, "\sim"] & P_2 \arrow[r, "G_2\text{-equiv}"] & X \\ S \end{tikzcd}$$

■

[a] Note to self: actually I don't know where I used the fact that $G_1 \to G_2$ is a closed embedding.

Applying to our situation $G_1 = G, G_2 = \mathrm{GL}_n, X = P$ shows that the groupoid of G-reductions of P is equivalent to the groupoid of sections of $P/G \to D_S$. [1]

Now we can describe the points of the fiber product in question. First, see s as a section of $P|_{D_S^\circ}$. Then given any affine $t : T \to S$ over S, then s gives rise to a section s_t of the pullback P_t to D_T over D_T°.

$$\begin{tikzcd} (P_t)^\circ \arrow[r] \arrow[d] & P_t \arrow[d] \\ (P_t)^\circ/G \arrow[r] \arrow[d] & P_t/G \arrow[d] \\ D_T^\circ \arrow[r] \arrow[uu, bend left, "s_t"] & D_T \end{tikzcd}$$

[1] The degenerate case of $G_1 = 1, G_2 = G$ says that a reduction of a G-bundle to a 1-bundle is the same thing as a global section.

Since s_t trivialises P_t°, it also provides a G-reduction of P_t°. The question is : what are all the extensions of this G-reduction to all of D_T? Using the above lemma, this is equivalent to asking for the groupoid of sections of P_t/G extending s_t from D_T° to D_T. Note that the data of extending s_t means this groupoid is discrete. Another way of seeing P_t/G is as $P \times^{\mathrm{GL}_n} (\mathrm{GL}_n/G)$. By our assumption of GL_n/G being affine and finite type, it follows that P_t/G is relatively affine and finite type over D_T. We can thus give a closed embedding

$$\begin{tikzcd} P_t/G \arrow[r, "\text{c.emb}"] \arrow[d] & \mathbb{A}^N_{D_T} \arrow[dl] \\ D_T & \end{tikzcd}$$

for some large N. The section s_t of P_t/G over D_T° can be seen as the data

$$s_t = (s_t^1, \ldots, s_t^N) \in (\mathcal{O}(T)((z)))^N$$

One sees now that the locus of points of D_T admitting extensions of s_t is given by the vanishing of the polar terms in the Laurent expansions of the coordinates of s_t. Furthermore, any extension is unique. We have thus shown that the fiber product in question is a close subscheme of S.

8 Appendix : Schematic definition of Schubert cells and varieties

The following is my understanding on how to define the Schubert cells and varieties schematically without working at the level of k-valued points. As discussed with Jonas, since D-modules do not see the difference between schemes and their reduction, this is purely a matter of taste.

> **Definition**
>
> Let F/k be algebraically closed. Let P_0, P_1 be G-bundles on D_F and $\beta : P_0|_{D_F^\circ} \to P_1|_{D_F^\circ}$. Given trivialisations s_i of P_i on D_F, one obtains an automorphism $s_1^{-1}\beta s_0$ of $\mathrm{Triv}|_{D_F^\circ}$, equivalently an element of $LG(F)$. Quotienting by the choice of trivialisations, we obtain a well-defined element
>
> $$\mathrm{Inv}(\beta) \in L^+G(F)\backslash LG(F)/L^+G(F) \simeq \mathbf{X}_\bullet(T)^+$$
>
> This is called the *relative position of* β.
>
> For F not necessarily algebraically closed, one can choose an algebraic closure $\overline{F}/F$ and define $\mathrm{Inv}(\beta)$ by first base changing along $D_{\overline{F}} \to D_F$ when do it over $D_{\overline{F}}$. The resulting element of $\mathbf{X}_\bullet(T)^+$ is independent of the choice of $\overline{F}$ by the Cartan decomposition.
>
> Now let $\operatorname{Spec} R$ be an affine scheme and $(P, s) \in \mathrm{Gr}_G(R)$. For a topological point $x \in \operatorname{Spec} R$, define
>
> $$\mathrm{Inv}_x\, s := \text{ relative difference of } P_{D_{\kappa(x)}} \xrightarrow{s} \mathrm{Triv}_{D_{\kappa(x)}} \text{ over } D^\circ_{\kappa(s)}$$
>
> For each $\mu \in \mathbf{X}_\bullet(T)^+$, we now define the Schubert variety associated to μ as a subfunctor of Gr_G by the

formula

$$\mathrm{Gr}_G^{\leq\mu}(R) := \{(P,s) : \forall\, x \in \operatorname{Spec} R, \operatorname{Inv}_x s \leq \mu\}$$

We also define the Schubert cell of μ as

$$\mathrm{Gr}_G^{\mu} := \mathrm{Gr}_G^{\leq\mu} \setminus \bigcup_{\lambda<\mu} \mathrm{Gr}_G^{\leq\lambda}$$

At this point, one should be able to either show that Gr_G^{λ} and $\mathrm{Gr}_G^{\leq\lambda}$ have reductions giving back the definition in the talk, and maybe even show that these are reduced already. But I have not had the time to think about this nor have I found a reference.

References

[Ach21] P. Achar. *Perverse Sheaves and Applications to Representation Theory*. Mathematical Surveys and Monographs. American Mathematical Society, 2021. URL: https://books.google.co.uk/books?id=8-J9zgEACAAJ.

[HTT07] R. Hotta, K. Takeuchi, and T. Tanisaki. *D-modules, perverse sheaves, and representation theory*. Vol. 236. Springer Science & Business Media, 2007.

[Mil17] J. S. Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017.