The adic Fargues–Fontaine curve

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Abstract

These are expanded notes for a talk on the adic Fargues–Fontaine curve.

Contents

1 Motivation : geometrization of ℓ-adic local Langlands

This section is largely informal. We begin with local class field theory.

> *Example.*
> *Let $\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p$ be the maximal abelian extension of $\mathbb{Q}_p$. The local Kronecker–Weber theorem says $\mathbb{Q}_p^{\mathrm{ab}} = \mathbb{Q}_p(\mu_{\mathbb{N}}) = \mathbb{Q}_p(\mu_{p^\infty})\breve{\mathbb{Q}}_p$ where $\breve{\mathbb{Q}}_p = \mathbb{Q}_p(\bigcup_{(N,p)=1} \mu_N)$ is the maximal unramified extension of $\mathbb{Q}_p$. Then we have :*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p^{\mathrm{ur}}) \arrow[r] & \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) \arrow[r] & \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \arrow[r] \arrow[u, phantom, "\simeq"] & W^{\mathrm{ab}} \arrow[r] \arrow[u, hook] & \mathrm{Frob}^{\mathbb{Z}} \arrow[r] \arrow[u, hook] & 1 \\
1 \arrow[r] & \mathbb{Z}_p^\times \arrow[r] \arrow[u, phantom, "\simeq"] & \mathbb{Q}_p^\times \arrow[r] \arrow[u, phantom, "\simeq"] & p^{\mathbb{Z}} \arrow[r] \arrow[u, phantom, "\simeq"] & 1
\end{tikzcd}$$

Local class field theory is the isomorphism of the middle row to the bottom row. The isomorphism $\mathbb{Q}_p^\times \simeq W^{\mathrm{ab}}$ is the local Artin reciprocity map.

The above can be seen as saying we have a correspondence

$$\text{1-dim reps of } \mathrm{GL}_1(\mathbb{Q}_p) \leftrightarrow \text{ 1-dim reps of } W$$

where $W \subseteq \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is the preimage of $\mathrm{Frob}^{\mathbb{Z}} \subseteq \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. More generally, ℓ-adic local Langlands for GL_n and $\mathbb{Q}_p$ says there is a "nice" correspondence

$$\ell\text{-adic reps of } \mathrm{GL}_n(\mathbb{Q}_p) \leftrightarrow \text{ group morphisms } W \to \mathrm{GL}_n(\mathbb{Z}_\ell)$$

More generally still, ℓ-adic local Langlands for a connected reductive group G over $\mathbb{Q}_p$ is about understanding the relation between

$$\ell\text{-adic reps of } G(\mathbb{Q}_p) \overset{?}{\leftrightsquigarrow} \text{continuous group sections } W \to \widehat{G}(\mathbb{Z}_\ell) \rtimes W$$

where

1. $\widehat{G}$ is the connected reductive group over $\mathbb{Z}_\ell$ with root datum dual to $G \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$.
2. The Weil group W acts on the root datum of $G \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ and hence on $\widehat{G}$ and its $\mathbb{Z}_\ell$-points. The semi-direct product $\widehat{G}(\mathbb{Z}_\ell) \rtimes W$ is formed using this action.

Objects on the right are called *L-parameters*. [FS24, Def. VIII.1.1.] We have the following questions :

- Q1 How does the dual group $\widehat{G}$ appear?
- Q2 What does W have to do with the representation theory of $G(E)$?

The heuristic of [FS24] is as follows :

- Step 0 In the setting of geometric Langlands of function fields of smooth projective curves X over either finite fields or $\mathbb{C}$, the above two questions are understood better.
  - A1 $\widehat{G}$ comes from the study of modifications of G-bundles locally around a point, i.e. geometric Satake.
  - A2 This comes from excursion operators of [Laf18].

Step 1 The map $\operatorname{Spec}\mathbb{C}_p \to (\operatorname{Spec}\breve{\mathbb{Q}}_p)/\varphi^{\mathbb{Z}}$ is a W-torsor so at least in the GL_n case, we should think about L-parameters as ℓ-adic rank n local systems on $(\operatorname{Spec}\breve{\mathbb{Q}}_p)/\varphi^{\mathbb{Z}}$. This raises the idea of "unramified geometric Langlands over $(\operatorname{Spec}\breve{\mathbb{Q}}_p)/\varphi^{\mathbb{Z}}$".

Step 2 For any hope of geometric Satake and excursion operators over $(\operatorname{Spec}\breve{\mathbb{Q}}_p)/\varphi^{\mathbb{Z}}$, we need a moduli of G-torsors over $(\operatorname{Spec}\breve{\mathbb{Q}}_p)/\varphi^{\mathbb{Z}}$. Observe that

$$(\operatorname{Spec}\breve{\mathbb{Q}}_p)/\varphi^{\mathbb{Z}} = (\operatorname{Spec} W(\overline{\mathbb{F}_p})[1/p])/\varphi^{\mathbb{Z}}$$

where $W(\_)$ is the functor of Witt vectors on perfect $\mathbb{F}_p$-algebras and φ comes from lifting the Frobenius automorphism of $\overline{\mathbb{F}_p}$. The idea of [FS24] is to think of the above as

$$\text{"}(\operatorname{Spec}\overline{\mathbb{F}_p} \times \operatorname{Spec}\mathbb{Q}_p)/(\varphi^{\mathbb{Z}} \times \mathbb{1})\text{"}$$

and replace $\overline{\mathbb{F}_p}$ with perfectoids S over $\mathbb{F}_p$. We thus want to consider :

$$X_S := \text{"}(S \times \operatorname{Spec}\mathbb{Q}_p)/(\varphi^{\mathbb{Z}} \times \mathbb{1})\text{"}$$

The formula for $\breve{\mathbb{Q}}_p$ suggests the following definition.

> **Definition**
> For $S = \operatorname{Spa}(C, \mathcal{O}_C)$ where C is a perfectoid field of characteristic p,
>
> $$X_C := (\operatorname{Spa} W(\mathcal{O}_C) \setminus V([\pi]p))/\varphi^{\mathbb{Z}}$$
>
> where $\pi \in \mathcal{O}_C$ is a pseudo-uniformizer of C. This is the *adic Fargues–Fontaine curve over C*. More generally, for $S = \operatorname{Spa}(A, A^+)$ an affinoid perfectoid in characteristic p,
>
> $$X_S := (\operatorname{Spa} W(A^+) \setminus V([\pi]p))/\varphi^{\mathbb{Z}}$$
>
> where $\pi \in A^+$ is a pseudo-uniformizer of A. This is the *relative Fargues–Fontaine curve X_S over S*.

The product $S \times \operatorname{Spec}\mathbb{Q}_p$ gives the empty scheme, and similarly for $S \times_{\operatorname{Spa}\mathbb{Z}} \operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. But one can show that the associated diamond of X_S can be identified with

$$X_S^{\diamond} = (S \times \operatorname{Spd}\mathbb{Q}_p)/(\varphi^{\mathbb{Z}} \times \mathbb{1})$$

Warning : Since $(W(A^+), W(A^+))$ is not discrete, nor finite type over a Noetherian ring, nor strongly Noetherian, nor sousperfectoid, it is a priori not even clear that we have an adic space before modding by the action of φ. We make this precise in the section on adic spaces as functors.

2 Holomorphic functions in the variable p

Before anything, we give some intuition for the the case over a point. First, a reminder of how the tilting equivalence works, with a few extra details.

Proposition – Tilting correspondence / Classification of untilts
Define the following category :

- objects are (F, I) where F is a perfectoid field of characteristic p and $I \subseteq W(F^\circ)$ which is generated by a *primitive element of degree 1*. These are elements $x \in W(F^\circ)$ with Teichmuller expansion

$$x = [\overline{x}_0] + [\overline{x}_1]p^1 + \cdots$$

where $\overline{x}_0 \in F^{\circ\circ}$ and $\overline{x}_1 \in (F^\circ)^\times$.[a]

- a morphism $(F, I) \to (F_1, I_1)$ is a continuous ring morphism $F \to F_1$ which sends I into I_1 under the corresponding ring morphism $W(F) \to W(F_1)$.

We have an equivalence :

$$\begin{array}{c}\text{category of perfectoid fields } K \\ \text{with continuous ring morphisms}\end{array} \xleftrightarrow{\sim} \text{category of } (F, I)$$

The two functors on objects are

1. Given a perfectoid field K, take $(K^\flat, \ker\theta \subseteq W(K^{\flat,\circ}))$. The norm on $K^\flat$ can be defined through

$$K^{\flat,\circ} \xrightarrow{-^\sharp} K^\circ \xrightarrow{|-|} \mathbb{R}_{\geq 0}$$

2. Given (F, I), form $F^{\sharp,\circ} := W(F^\circ)/I$. For $x \in F^{\sharp,\circ}$ there exists $\tilde{x} \in F^\circ$ unique up to $(F^\circ)^\times$ such that $[\tilde{x}] = x$ in $F^{\sharp,\circ}$. Defining $|x| := |\tilde{x}|$ defines a norm on $F^{\sharp,\circ}$ making it a rank 1 valuation ring and $F^\sharp := \operatorname{Frac} F^{\sharp,\circ}$ a perfectoid field.

[Ked15, Theorem 1.5.1]

[a] In [FF18, Def. 2.2.1], $\overline{x}_0$ is required to be non-zero. This has the effect of excluding the (unique up to isomorphism) characteristic p untilt. The definition here argees with [SW20, Def. 6.2.9].

Definition – Radius of untilts
Fix a perfectoid field F of characteristic p. An untilt of F is defined as the data (K, ι) where

- K is a perfectoid field
- $\iota : K^\flat \simeq F$ is a bi-continuous ring isomorphism.

By the theory of Witt vectors, ι corresponds to a morphism between Witt vectors :

$$\begin{tikzcd} W(K^{\flat,\circ}) \arrow[r, two heads, "\theta"] \arrow[d, "\sim"'] & K^\circ \arrow[d, "\sim"] \\ W(F^\circ) \arrow[r, two heads] & W(F^\circ)/(p - [\iota(p^\flat)]) \end{tikzcd}$$

which yields a bi-continuous ring isomorphism $K \simeq \operatorname{Frac} W(F^\circ)/(p - [\iota(p^\flat)])$. Here $p^\flat \in K^\flat$ is either

1. zero when K is characteristic p
2. a pseudo-uniformizer coming from a choice of perfectoid pseudo-uniformizer in K.

In either cases, the norm of $p^\flat$ is independent of choices, and so we have a well-defined number

$$r(K, \iota) := \left|\iota(p^\flat)\right|_F \in [0, 1)$$

This is called the *radius*. A morphism of untilts $(K_1, \iota_1) \to (K_2, \iota_2)$ is a continuous ring morphism $\alpha : K_1 \to K_2$ such that $\iota_2 = \alpha^\flat \iota_1$.

Isomorphic untilts gives rise to the same radius and up to isomorphism F is the only untilt of itself with radius zero. This gives rise to the following heuristic :

$\lvert\mathcal{Y}_F\rvert :=$ {Untilts of F} / $\simeq$	$\{\lvert z\rvert < 1\} \subseteq \mathbb{C}$
$p \in W(F^\circ)$	coordinate function z
$K \mapsto \lvert p\rvert_K$	$z \mapsto \lvert z\rvert$
$f \in W(F^\circ)$	$\sum_{n=0}^{\infty} c_n z^n$ s.t. $\lvert c_n\rvert$ bounded
???	ring of holomorphic functions on $\{\lvert z\rvert < 1\} \subseteq \mathbb{C}$

Intuitively, a holomorphic function on $\{|z| < 1\}$ is a compatible family of holomorphic functions on $\{a \leq |z| \leq b\}$ ranging over $[a, b] \subseteq [0, 1)$. Making rigorous the rings of "holomorphic functions in variable p on $a \leq |p| \leq b$" will give us an analytic space over $\mathbb{Z}_p$ playing the role of $\{|z| < 1\} \subseteq \mathbb{C}$. For the purposes of defining the Fargues–Fontaine curve, we will jump forward to describing the holomorphic functions on "$0 < |p| < 1$".

3 Defining adic spaces without using adic spectrum

I want to define adic spaces without using adic spectrum because :

1. As mentioned before, $W(R^+)$ is not sheafy so $\operatorname{Spa} W(R^+)$ already falls out of category (V). (For the definition, see [SW20, Def. 3.2.1].)
2. [SW20, Prop. 11.2.1] does not explicitly define $\mathcal{Y}_S := \operatorname{Spa} W(R^+) \setminus V([\pi])$ as an object of category (V). The underlying topological space is conceivable, however defining a structure presheaf and showing it is a sheaf looks like non-trivial work.
3. Verifying the sheaf condition on structure presheaves requires an understanding of the underlying topological space. I find adic spectra difficult to compute.

In the end, no matter what theory of adic spaces one chooses, it should always be the case that a filtered colimit of sheafy affinoids along rational localizations gives an adic space. The relative Fargues–Fontaine curve X_S will be such a colimit. We develop adic geometry completely analogously to algebraic geometry using functor of points in the following steps :

1. We define affinoids as dual to sheafy complete Huber pairs and introduce the Grothendieck topology given by rational covers.

2. Define adic spaces as living in the sheaf topos over affinoids in the same way as schemes can be defined for algebraic geometry.

This is more restrictive than the theory of pre-adic spaces in [SW20, Section 3.4]. We chose to do this because we think sheafifying representables is too opaque; every reasonable site structure should be subcanonical. A different fix, which requires more machinary, is to realise that every complete Huber pair is sheafy in a derived sense, e.g. in the Clausen–Scholze theory of analytic stacks.

Definition

Define $\widetilde{\mathrm{Aff}}$ as opposite to the category of complete Huber pairs with continuous ring morphisms preserving subrings of integral elements. For (A, A^+) complete Huber, we write $\widetilde{\mathrm{Spa}}\,(A, A^+)$ for the corresponding object in $\widetilde{\mathrm{Aff}}$.

Define $\mathrm{Aff} \subseteq \widetilde{\mathrm{Aff}}$ as the full subcategory dual to sheafy complete Huber pairs. We write $\mathrm{Spa}\,(A, A^+)$ similarly.

We identify $\widetilde{\mathrm{Aff}}, \mathrm{Aff}$ with their essential images in presheaves under the Yoneda embedding. We reserve the term *affinoid* for objects in Aff.

A reminder for the procedure to compute rational localizations. This works for $\widetilde{\mathrm{Aff}}$.

Proposition – Rational localization

Let (R, R^+) be a complete Huber pair. Let $s \in R$ and $T \subseteq R$ finite subset with $TR \subseteq R$ open. Consider the subfunctor $\{|t| \leq |s| \neq 0 \text{ s.t. } t \in T\} \subseteq \widetilde{\mathrm{Spa}}\,(R, R^+)$ defined by

$$\{|t| \leq |s| \neq 0 \text{ s.t. } t \in T\}\,(A, A^+) := \left\{\, \alpha \in \widetilde{\mathrm{Spa}}\,(R, R^+)(A, A^+) \text{ s.t. } s \in A^\times \text{ and } T/s \subseteq A^+ \,\right\}$$

Then this is representable by $(R, R^+) \to (R\langle T/s\rangle, R\langle T/s\rangle^+)$ defined as

1. Choose $I \subseteq R_0 \subseteq R$ with R_0 a ring of definition and I finitely generated ideal of definition.
2. Make $R[1/s]$ into a topological ring by using the ideal and ring of definition $IR_0[T/s] \subseteq R_0[T/s] \subseteq R[1/s]$.
3. Complete the Huber ring $R[1/s]$ which can be computed as

$$\widehat{R[1/s]} := \widehat{R[1/s]}_0 \otimes_{R_0[T/s]} R[1/s]$$
$$\widehat{R[1/s]}_0 := \left(\varprojlim_{n \geq 0} \frac{R_0[T/s]}{(IR_0[T/s])^n} \right)$$

   where $\widehat{R[1/s]}_0$ and $I\widehat{R[1/s]}_0$ serves as a ring and ideal of definition. It can be shown that $\widehat{R[1/s]}_0$ is precisely the topological closure of $R_0[T/s]$ inside $\widehat{R[1/s]}$.
4. Define $R[1/s]^+$ as the integral closure of $R^+[T/s]$ inside $R[1/s]$. Finally, define the complete Huber

pair $(R\langle T/s\rangle, R\langle T/s\rangle^+)$ as

$$\begin{aligned} R\langle T/s\rangle &:= \widehat{R[1/s]} \\ R\langle T/s\rangle^+ &:= \text{topological closure of } R[1/s]^+ \text{ in } R\langle T/s\rangle \end{aligned}$$

[SW20, Theorem 3.1.3]

Definition – Analytic topology on Aff

A standard rational cover of $\mathrm{Spa}\,(A, A^+) \in \mathrm{Aff}$ is a collection of morphisms $(U_t \to \mathrm{Spa}\,(A, A^+))_{t\in T}$ where $T \subseteq A$ is a finite subset with $(T) = A$ and

$$U_t := \{|t'| \leq |t| \neq 0 \text{ s.t. } t \neq t' \in T\}$$

[Mor19, Def. IV.2.3.1] One can show standard rational covers gives a site structure on Aff. We call the associated Grothendieck topology the *analytic topology* on Aff. It is subcanonical.

Definition – Adic spaces as functors on a subcanonical site

We say $j : U \to \mathrm{Spa}\,(R, R^+)$ is an *open immersion with affinoid target* when it is a monomorphism of presheaves and there exists a collection $(U_i)_{i\in I}$ of rational opens of $\mathrm{Spa}\,(R, R^+)$ such that

1. for all $i \in I$ we have $U_i \subseteq U$
2. for all $\mathrm{Spa}\,(A, A^+) \to U$, there exists a standard rational cover $(W_j)_{j\in J}$ of $\mathrm{Spa}\,(A, A^+)$ such that each $W_j \to \mathrm{Spa}\,(A, A^+) \to U$ factors through some $U_i \to U$.

We say $j : U \to X$ in $\mathrm{PSh}\,\mathrm{Aff}$ is an *open immersion* when the base change to all affinoids mapping into X gives open immersions with affinoid target.

An *adic space* is a sheaf X over Aff for the analytic topology such that there exists a collection of open immersions $(U_i \to X)_{i\in I}$ where

1. each U_i is affinoid
2. for all $\mathrm{Spa}\,(A, A^+) \to X$, there exists a standard rational cover $(W_j)_{j\in J}$ of $\mathrm{Spa}\,(A, A^+)$ such that each $W_j \to \mathrm{Spa}\,(A, A^+) \to X$ factors through some $U_i \to X$.

Such a collection is called an atlas (for the analytic topology).

Proposition

Let $X : I \to \mathrm{Aff}$ be a filtered system of affinoids with transition maps which are rational localizations. Let $X := \varinjlim_{i\in I} X_i$ be the colimit in the category of *presheaves* over Aff. Then X is in fact the colimit of the same diagram in $\mathrm{Sh}\,\mathrm{Aff}$ and is an adic space.

Proof. To show the presheaf colimit is already a sheaf, it follows from representables being quasi-compact. The key fact is that every $\mathrm{Spa}\,(A, A^+) \to X$ admits a factoring $\mathrm{Spa}\,(A, A^+) \to X_i \to X$. Then the fact that each $X_j \to X$ is an open immersion follows from the fact that rational localizations of affinoids are preserved under base change. The key fact also implies $(X_i)_{i\in I}$ form an atlas for X. $\square$

Here is an example to demonstrating you do not need the adic spectrum.

Example (*Affine line over* $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ *as an adic space*).
We work with $\mathrm{Aff}/\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. *Consider the functor*

$$\mathbb{A}^1 : \mathrm{Spa}(A, A^+) \mapsto A$$

We show that this is an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. *Consider* $p\mathbb{Z}_p \subseteq \mathbb{Z}_p \subseteq \mathbb{Q}_p[T]$. *We topologize* $\mathbb{Q}_p[t]$ *using* $(p^n\mathbb{Z}_p)_{n\geq 0}$. *The ring* $\mathbb{Z}_p$ *is integrally closed in* $\mathbb{Q}_p[T]$ *so we have a complete Huber pair* $(\mathbb{Q}_p[T], \mathbb{Z}_p)$.

> *Lemma.* $\widetilde{\mathrm{Spa}}(\mathbb{Q}_p[T], \mathbb{Z}_p)$ *on* $\mathrm{Aff}/\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ *is isomorphic to* $\mathbb{A}^1$.
>
> *Proof.* Let $a \in (A, A^+)$ with $(\mathbb{Q}_p, \mathbb{Z}_p) \to (A, A^+)$. To suffices to check the algebra morphism $\mathbb{Q}_p[T] \to A, T \mapsto a$ is continuous. Pick a ring and ideal of definition $I \subseteq A_0 \subseteq A$. By continuity of $\mathbb{Q}_p \to A$, there exists $n \geq 0$ with $p^n\mathbb{Z}_p \subseteq \mathbb{Q}_p$ landing in I. This implies $p^n\mathbb{Z}_p \subseteq \mathbb{Q}_p[T]$ also lands in I. ∎

Unfortunately, we don't know a way for checking if $(\mathbb{Q}_p[T], \mathbb{Z}_p)$ *is sheafy :* $\mathbb{Q}_p[T]$ *is not finite type over the ring of definition* $\mathbb{Z}_p$. *But consider*

$$\mathbb{D} := \{|T| \leq 1\} : \mathrm{Spa}(A, A^+) \mapsto A^+$$

Idea : we can write $\mathbb{A}^1$ as union of larger and larger disks. *Let* $\alpha : (\mathbb{Q}_p, \mathbb{Z}_p) \to (A, A^+)$ *of complete Huber pairs. Since* $A^+ \subseteq A^\circ$, *there exists a ring and ideal of definition* $I \subseteq A_0 \subseteq A$ *such that* $a \in A_0$.

1. $\mathbb{Z}_p \to A_0$ *: By continuity of* $\mathbb{Q}_p \to A$, *there exists* $n \geq 0$ *with* $\mathbb{Z}_p/p^n\mathbb{Z}_p \to A_0/I$. *Since* A_0 *is* I*-adically complete, this defines a continuous morphism* $\mathbb{Z}_p \to A_0$ *which is a priori different from the given map* $\mathbb{Z}_p \to A$. *However,* $\mathbb{Z} \to \mathbb{Z}_p \to A_0 \to A$ *agrees with the given* $\mathbb{Z} \to \mathbb{Z}_p \to A$. *Since* $\mathbb{Z} \subseteq \mathbb{Z}_p$ *is dense, we obtain that the two morphisms* $\mathbb{Z}_p \to A$ *agree and hence* $\mathbb{Z}_p \to A_0$.

2. *We extend* $\mathbb{Z}_p \to A_0$ *to* $\mathbb{Z}_p[T] \to A_0$ *by* $T \mapsto a$. *Since* $p^n \in \mathbb{Z}_p[T]$ *maps into* I *we have* $p^n\mathbb{Z}_p[T]$ *maps into* I. *By* I*-adic completeness of* A_0, *this extends uniquely to a continuous morphism*

$$\mathbb{Z}_p\langle T\rangle := \varprojlim_{n\geq 0} \frac{\mathbb{Z}_p[T]}{p^{n+1}\mathbb{Z}_p[T]} \to A_0$$

   Since $\mathbb{Z}_p \to A^+$ *we have* $\mathbb{Z}_p[T]$ *maps into* A^+. *By definition of* (A, A^+) *being complete,* A^+ *is closed in* A *under the* I*-adic topology. Given that* $\mathbb{Z}_p[T] \to \mathbb{Z}_p\langle T\rangle$ *has dense image, it follows that* $\mathbb{Z}_p\langle T\rangle$ *is mapped into* A^+.

3. *By assumption,* $p \in A^\times$ *and* $\mathbb{Q}_p\langle T\rangle := \mathbb{Z}_p\langle T\rangle \otimes_{\mathbb{Z}_p[T]} \mathbb{Q}_p[T] = \mathbb{Z}_p\langle T\rangle \otimes_{\mathbb{Z}_p[T]} (\mathbb{Z}_p[T][1/p]) = \mathbb{Z}_p\langle T\rangle[1/p]$ *so the continuous morphism* $\mathbb{Z}_p\langle T\rangle \to A^+$ *extends uniquely to* $\mathbb{Q}_p\langle T\rangle \to A$. *We have rediscovered the classical Tate algebra!*

We have $\mathbb{D} \simeq \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$. *The fact that this is sheafy is Tate's acyclicity theorem.* [SW20, *Theorem 3.1.8.(3)*] *If we instead wanted to do* $\{|T| \leq |1/p^n|\} := \{|p^nT| \leq 1\}$ *parameterizing* $a \in A$ *such that* $p^n a \in A^+$, *we would get* $\mathrm{Spa}(\mathbb{Q}_p\langle p^nT\rangle, \mathbb{Z}_p\langle p^nT\rangle)$. *Since* $A = A^+[1/p]$ *we have*

$$\mathbb{A}^1 \simeq \bigcup_{n\geq 0} \mathrm{Spa}(\mathbb{Q}_p\langle p^nT\rangle, \mathbb{Z}_p\langle p^nT\rangle)$$

where the latter is a filtered colimit of affinoids along rational localizations. Hence $\mathbb{A}^1$ is an adic space as it should be.

Just to be sure : $\mathbb{Z}_p\langle pT\rangle$ is computed by taking the inverse limit of

$$\begin{gathered}
\mathbb{Z}_p[pT]/p\mathbb{Z}_p[pT] \simeq \mathbb{F}_p \\
\mathbb{Z}_p[pT]/p^2\mathbb{Z}_p[pT] \simeq (\mathbb{Z}_p/p^2)[pT] \subseteq (\mathbb{Z}_p/p^2)[T] \\
\mathbb{Z}_p[pT]/p^3\mathbb{Z}_p[pT] \simeq (\mathbb{Z}_p/p^3)[pT] \subseteq (\mathbb{Z}_p/p^3)[T] \\
\vdots
\end{gathered}$$

we obtain an injective ring morphism $\mathbb{Z}_p\langle pT\rangle \to \mathbb{Z}_p[[T]]$ with image

$$\left\{\sum_{n=0}^{\infty} a_n p^n T^n \textit{ s.t. } \lim_{n\to\infty} |a_n| = 0\right\} = \left\{\sum_{n=0}^{\infty} b_n T^n \textit{ s.t. } \lim_{n\to\infty} \left|p^{-n} b_n\right| \to 0\right\}$$

The above computation for $\{|T| \leq 1\}$ generalises over any Tate affinoid.

Lemma. Let $\mathrm{Spa}\,(R, R^+) \in \mathrm{Aff}$ which is Tate. Define $\{|T| \leq 1\}$ as a presheaf over $\mathrm{Aff}/\mathrm{Spa}\,(R, R^+)$ sending $\mathrm{Spa}\,(A, A^+) \mapsto A^+$. Then there exists a complete Huber pair whose $\widetilde{\mathrm{Spa}}$ is isomorphic to $\{|T| \leq 1\}$.

Proof. By [SW20, Lem. 5.1.2] the morphism $\alpha : (R, R^+) \to (A, A^+)$ is adic i.e. there exists $I \subseteq R_0 \subseteq R$ ideal and ring of definition and $A_0 \subseteq A$ ring of definition with $\alpha R_0 \subseteq A_0$ and $(\alpha I)A_0 \subseteq A_0$ giving an ideal of definition. If A_0 is complete w.r.t. some ideal of definition, then it is complete w.r.t. any ideal of definition. In particular A_0 is αI-adically complete.

Now let $f \in A^+$. There exists a ring of definition $A_0' \subseteq A$ with $f \in A_0'$. There exists a ring of definition $A_0'' \subseteq A$ which contains both A_0, A_0'. Since α is adic, $(\alpha I)A_0'' \subseteq A_0''$ is an ideal of definition. So WLOG $f \in A_0$.

We extend $R_0 \to A_0$ to $R_0[T] \to A_0$ by $T \mapsto f$. We have that $IR_0[T]$ maps into αI. By αI-adic completeness of A_0 we have a unique extension of ring morphisms

$$\begin{tikzcd}
R_0[T] \arrow[r] \arrow[d] & A_0 \\
R_0[T]^{\wedge}_{IR_0[T]} \arrow[ur, dashed, "\exists!"'] & :=
\end{tikzcd}$$

We have an explicit description of the adic completion as an $R_0[T]$ algebra :

$$R_0[T]^{\wedge}_{IR_0[T]} \simeq R_0\langle T\rangle := \left\{\sum_{n=0}^{\infty} a_n T^n \in R_0[[T]] \text{ s.t. } \lim_{n\to\infty} a_n = 0\right\}$$

This is continuous when we equip $R_0\langle T\rangle$ with the $IR_0\langle T\rangle$-adic topology. Since R is Tate, there exists $\pi \in R^{\circ\circ} \cap R^\times$ with $R = R_0[1/\pi]$ and $\pi \in R_0$. The following ideal of a subring of a ring then defines a complete Huber ring :

$$IR_0\langle T\rangle \subseteq R_0\langle T\rangle \subseteq R_0\langle T\rangle[1/\pi]$$

We have the same explicit description :

$$R_0\langle T\rangle[1/\pi] \simeq R\langle T\rangle := \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] \text{ s.t. } \lim_{n\to\infty} a_n = 0 \right\}$$

The morphism $R_0\langle T\rangle \to A_0$ extends uniquely to a morphism $R\langle T\rangle \to A$ of complete Huber rings.

So far, the ideal and ring of definition $I \subseteq R_0 \subseteq R$ depended on α. However, for any ideal of ring of definition $\widetilde{I} \subseteq \widetilde{R_0} \subseteq R$ and pseudo-uniformizer $\widetilde{\pi}$ of R with $\widetilde{R_0}[1/\widetilde{\pi}] = R$ and additionally $R_0 \subseteq \widetilde{R_0}$, we have an isomorphism of complete Huber rings

$$R\langle T\rangle \simeq \widetilde{R_0}[T]^{\wedge}_{\widetilde{I_0}\widetilde{R_0}[T]}[1/\widetilde{\pi}]$$

Since the set of rings of definition is filtered, we obtain that the topology on $R\langle T\rangle$ is independent of the choice of $I \subseteq R_0 \subseteq R$.

Define $R\langle T\rangle^+$ as the topological closure in $R\langle T\rangle$ of the integral closure of $A^+[T]$ in $R[T]$. Then the complete Huber pair $(R\langle T\rangle, R\langle T\rangle^+)$ represents $\{|T| \leq 1\}$ over $\mathrm{Aff}/\mathrm{Spa}\,(R, R^+)$. ∎

Example (The complement of $\mathbb{G}_m$ in $\mathbb{A}^1$).
We continue to work over $\mathrm{Spa}\,(\mathbb{Q}_p, \mathbb{Z}_p)$ for simplicity. Let $\mathbb{G}_m := \{|t| \neq 0\} \subseteq \mathbb{A}^1$. Although $\mathbb{A}^1 = \widetilde{\mathrm{Spa}}\,(\mathbb{Q}_p[T], \mathbb{Z}_p)$ is not affinoid, we can still compute a complete Huber pair which represents this functor.

1. *The rings and ideal of definition for $(\mathbb{Q}_p[T], \mathbb{Z}_p)$ are $p\mathbb{Z}_p \subseteq \mathbb{Z}_p \subseteq \mathbb{Q}_p[T]$.*
2. *The topological ring with its ideal and ring of definition are $p\mathbb{Z}_p \subseteq \mathbb{Z}_p \subseteq \mathbb{Q}_p[T, 1/T]$.*
3. *Since $\mathbb{Z}_p$ is already p-adically complete, the completed Huber ring is still $\mathbb{Q}_p[T, 1/T]$.*
4. *Integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p[T, 1/T]$ is just $\mathbb{Z}_p$. This is closed so the new completed Huber pair is $(\mathbb{Q}_p[T, 1/T], \mathbb{Z}_p)$.*

$\mathbb{G}_m \simeq \widetilde{\mathrm{Spa}}\,(\mathbb{Q}_p[T, 1/T], \mathbb{Z}_p)$ is a filtered colimit of closed annuli centred at the origin along rational localizations and thus is an adic space.

Question : what is the complement of $\mathbb{G}_m$? In scheme theory, this is $\mathrm{Spf}\ \mathbb{Q}_p[[T]]$. To understand what's going on, we first try the complement of $\mathbb{G}_m \cap \mathbb{D}$. Let $f : \mathrm{Spa}\,(A, A^+) \to \mathbb{D}$ be a morphism of affinoids. This lies in the complement when for all $n \geq 0$, the preimage of $\{|p^n/T| \leq 1\}$ in $\mathrm{Spa}\,(A, A^+)$ is empty. We have

$$\begin{tikzcd}
\mathrm{Spa}(A, A^+) \arrow[r, "f"] & \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle) \\
\{|p^n/f| \leq 1\} \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \{|p^n/T| \leq 1\} \arrow[u]
\end{tikzcd}$$

Since A is sheafy, $\{|p^n/f| \leq 1\}$ is an affinoid. ***TODO the topological ring $A\langle X\rangle/(Xg-1)$ is Huber for any $g \in A$ and we have an isomorphic of complete Huber rings***

$$A\langle 1/g\rangle \simeq A\langle X\rangle/(Xg-1)$$

Claim : this is the zero ring iff $g \in A^{\circ\circ}$. Assuming it is zero, we have $h \in A\langle X\rangle$ with $(Xg-1)h = 0$. Writing $h = \sum_{n=0}^{\infty} h_n T^n$ and expanding, we find $h_n = g^n$. Since $\lim_{n\to\infty} h_n = 0$ we have $g \in A^{\circ\circ}$. Conversely if $g \in A^{\circ\circ}$ then any $h \in A\langle X\rangle/(Xg-1)$ can be written as $h = g^{-n}hg^n$. But since g^{-1} is power bounded in $A\langle 1/g\rangle$, $g^{-\mathbb{N}}h$ is bounded and so $h \in \bigcap_{n\geq 0} g^{-\mathbb{N}}hg^n = (0)$.

We use the above for $g = p^{-n}f$ and obtain that f lands in the complement of $\{|p^n/T| \leq 1\}$ iff $p^{-n}f \in A^{\circ\circ}$. So f lands in the complement of $\mathbb{G}_m \cap \mathbb{D}$ iff $f \in \bigcap_{n\geq 0} p^n A^{\circ\circ} = \bigcap_{n\geq 0} p^n A^{\circ\circ} A^{\circ}$. The same result happens for larger closed disks so the complement of $\mathbb{G}_m$ is

$$\mathrm{Spa}\,(A, A^+) \mapsto \bigcap_{n\geq 0} p^n A^{\circ}$$

This is the †-nilradical!

4 The Fargues–Fontaine curve is an adic space

We can now be precise about Y_S being an adic space. Let $\mathrm{Aff\,Perf}$ denote the category of affinoid perfectoid characteristic p.

Proposition

Let $S = \mathrm{Spa}\,(R, R^+) \in \mathrm{Aff\,Perf}$. Choose $\pi \in R^\times \cap R^{\circ\circ}$ so that $R = R^+[1/\pi]$. Define

$$Y_S := \{p[\pi] \neq 0\} \subseteq \widetilde{\mathrm{Spa}}\,(W(R^+), W(R^+))$$

which we restrict to a presheaf on Aff. Then Y_S is the filtered colimit of affinoids along rational localizations and hence an adic space. This is independent of the choice of π. Furthermore, if we base change to $\mathbb{Q}_p^\infty := \mathbb{Q}_p(p^{1/p^\infty})$, then

$$Y_S \times_{\mathrm{Spa}\,(\mathbb{Q}_p, \mathbb{Z}_p)} \mathrm{Spa}\,(\mathbb{Q}_p^\infty, \mathbb{Z}_p^\infty)$$

is a perfectoid space whose tilt is isomorphic to the perfectoid punctured unit disk over S

$$S \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\,(\mathbb{F}_p((t^{1/p^\infty})), \mathbb{F}_p[[t^{1/p^\infty}]]_t^\wedge)$$

We try to give extra detail in constructing the atlas so that the reader knows the precise amount of data that goes into constructing an adic space.

Proof. A morphism $\mathrm{Spa}\,(A, A^+) \to \{p[\pi] \neq 0\}$ is equivalent to the data of

1. a continuous ring morphism $\alpha W(R^+) \to A$ such that
2. $\alpha W(R^+) \subseteq A^+$
3. $\alpha([\pi]) \in A^\times$
4. $\alpha(p) \in A^\times$

But p is topologically nilpotent in $W(R^+)$ so for all $b > 0$ there exists $a > 0$ with $\alpha(p)^a/\alpha([\pi])^b \in A^+$. Similarly, for all $d > 0$ there exists $c > 0$ with $\alpha([\pi])^c/p^d \in A^+$. Thus we have

$$Y_S := \{p[\pi] \neq 0\} = \bigcup_{a,b,c,d>0} Y_{S,[p^{-a/b},p^{-d/c}]}$$
$$Y_{S,[p^{-a/b},p^{-d/c}]} := \left\{|p^a| \leq \left|[\pi]^b\right| \neq 0\right\} \cap \left\{|[\pi]^c| \leq \left|p^d\right| \neq 0\right\}$$

The union is a presheaf filtered colimit. We show that $Y_{S,[p^{-a/b},p^{-d/c}]}$ is an affinoid over $\mathrm{Spa}\,(\mathbb{Q}_p, \mathbb{Z}_p)$ and for inclusion of such intervals

$$I \subseteq J \quad \Rightarrow \quad Y_{S,I} \subseteq Y_{S,J} \text{ rational localization}$$

(Rational localization) Follows from the description of $Y_{S,I}$.

(Affinoid) We have

$$Y_{S,[p^{-a/b},p^{-d/c}]} = \left\{\left|p^{a+d}\right|, \left|[\pi]^c[\pi]^b\right| \leq \left|[\pi]^b p^d \neq 0\right|\right\}$$

Following the procedure to compute rational localization, we get :

1. The topology on $W(R^+)$ is the so-called weak topology, which comes from the bijection

$$(R^+)^{\mathbb{N}} \xrightarrow{\sim} W(R^+), (x_n) \mapsto \sum_{n=0}^{\infty} [x_n]p^n$$

   taking the product topology on the LHS with each R^+ the π-adic topology. This turns out to be precisely the $(p, [\pi])$-adic topology and π-adic completeness of R^+ implies $(p, [\pi])$-adic completeness of $W(R^+)$. A proof in the case of $R = F, R^+ = F^\circ$ with F a characteristic p perfect complete NA field, see [FF18, Prop. 1.4.11].

2. The underlying ring is
$$B_S^b := W(R^+)[1/[\pi]^b p^d] = W(R^+)[1/[\pi]^b, 1/p^d]$$

3. The ring of definition is $W(R^+)\left[\frac{p^{a+d}}{[\pi]^b p^d}, \frac{[\pi]^c[\pi^b]}{[\pi]^b p^d}\right] = W(R^+)\left[\frac{p^a}{[\pi]^b}, \frac{[\pi]^c}{p^d}\right]$.

4. The ideal of definition is $(p, [\pi])W(R^+)\left[\frac{p^a}{[\pi]^b}, \frac{[\pi]^c}{p^d}\right]$. But since we have $(p^a) \subseteq (p, [\pi])$ and $(p, [\pi])^{2c} \subseteq (p)$, an equivalent ideal of definition is $pW(R^+)\left[\frac{p^a}{[\pi]^b}, \frac{[\pi]^c}{p^d}\right]$.

5. Our completed ring of definition is thus the p-adic completion

$$B_{S,[p^{-a/b},p^{-d/c}],0} := W(R^+)\left[\frac{p^a}{[\pi]^b}, \frac{[\pi]^c}{p^d}\right]_p^\wedge$$

equipped with the p-adic topology.

6. Our completed Huber ring is

$$B_{S,[p^{-a/b},p^{-d/c}]} := W(R^+)\left[\frac{p^a}{[\pi]^b}, \frac{[\pi]^c}{p^d}\right]_p^\wedge \left[\frac{1}{[\pi]^b}, \frac{1}{p^d}\right] = W(R^+)\left[\frac{p^a}{[\pi]^b}, \frac{[\pi]^c}{p^d}\right]_p^\wedge \left[\frac{1}{p}\right]$$

because $1 = \frac{1}{p^a}\frac{p^a}{[\pi]^b}[\pi]^b$ implies inverting p already makes $[\pi]^b$ invertible.

7. Since the integral elements of $W(R^+)$ was just $W(R^+)$, to compute the new set of integral elements, one must first take integral closure of $W(R^+)\left[\frac{p^a}{[\pi]^b}, \frac{[\pi]^c}{p^d}\right]$ inside B_S^b, then take topological closure inside $B_{S,[p^{-a/b},p^{-d/c}],0}$. **This is probably just the same ring again but I struggle to compute integral closures.** Since p in topologically nilpotent in $W(R^+)$ and $p \in B^\times_{S,[p^{-N},p^{-1/M}]}$, our completed Huber pair receives a unique morphism from $(\mathbb{Q}_p, \mathbb{Z}_p)$.

For sheafiness, it suffices to show the above complete Huber pair is sousperfectoid. Consider the extension of NA fields $\mathbb{Q}_p \to \mathbb{Q}_p(p^{1/p^\infty})$. Since $\mathbb{Q}_p$ is discretely valued, any closed subspace of a $\mathbb{Q}_p$-Banach space has a complement. [Sch13, Prop. 10.5] This gives a splitting of $\mathbb{Q}_p \to \mathbb{Q}_p(p^{1/p^\infty})$ in topological $\mathbb{Q}_p$-vector spaces, which gives a splitting of the base change

$$B_{S,[p^{-a/b},p^{-d/c}]} \to \mathbb{Q}_p(p^{1/p^\infty})\widehat{\otimes}_{\mathbb{Q}_p} B_{S,[p^{-a/b},p^{-d/c}]} =: B$$

in topological $B_{S,[p^{-a/b},p^{-d/c}]}$-modules. We are now reduced to the final statement.

(Base change to $\mathbb{Q}_p^\infty$ is perfectoid) What's useful is that the topology on $B_{S,I}$ actually comes from a power-multiplicative norm.

> *Lemma. Let $|\_|_R$ be a power-multiplicative norm on R which induces its topology. For $0<\rho<1$, define*
>
> $$|\_|_\rho : B_S^b \to \mathbb{R}_{\geq 0}, x = \sum_{n>>-\infty}[x_n]p^n \mapsto \sup_{n\in\mathbb{Z}} |x_n|_R\, \rho^n$$
>
> *Then this is a power-multiplicative norm making B_S^b into a normed $\mathbb{Q}_p$-algebra. For $I = [\rho_1,\rho_2] \subseteq (0,1)$, completing w.r.t. $|\_|_{\rho_1}, |\_|_{\rho_2}$ is the same as completing w.r.t. $|\_|_I := \max(|\_|_{\rho_1}, |\_|_{\rho_2})$. [Far24, Section 7.1.1]*

Furthermore, it should be the case when $\rho_1 = |[\pi]|^{a/b}, \rho_2 = |[\pi]|^{d/c}$ then this completion matches what we previously computed[1]

$$\left\{x \in B_F^b \text{ s.t. } |x|_I \leq 1\right\} = W(R^+)[p^a/[\pi]^b, [\pi]^c/p^d]$$

[1]Neither [KL15, Section 5.1] nor [FS24, Prop. II.1.1] is clear so this is my guess.

Since $\mathbb{Q}_p^\infty$ and $B_{S,I}$ are both $\mathbb{Q}_p$-Banach algebras with power-multiplicative norms, we can compute their completed tensor product easily : take closed unit balls which are $\mathbb{Z}_p$-algebras, take p-adically completion, then invert p. I'm hoping the completed tensor product (the projective one) of two power-multiplicatively normed Banach algebras give another power-multiplicatively norm. This gives uniformity of the base change to $\mathbb{Q}_p^\infty$.

It remains to show the surjectivity of Frobenius on $B^\circ/(p)$. Whatever B° is, its reduction mod p will be

$$B^\circ/(p) \simeq \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge/(p) \otimes_{\mathbb{F}_p} W(R^+)[p^a/[\pi]^b, [\pi]^c/p^d]_p^\wedge/(p)$$

We get

$$B^\circ/(p) \simeq \mathbb{F}_p[t^{1/p^\infty}]/(t) \otimes_{\mathbb{F}_p} R^+/(\pi^c) \simeq (R^+)[t^{1/p^\infty}]/(\pi^c, t)$$

Since R is perfect and R^+ is integrally closed, R^+ is perfect as well. This implies $R^+/(\pi^c)$ has surjective Frobenius and hence $B^\circ/(p)$ has surjective Frobenius.

(The tilt of the base change) We write $Y_S \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^\infty$ for the base change. Any morphism from an affinoid into $Y_S \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^\infty$ must factor through some $Y_{S,I} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^\infty$. Since the tilt of a rational localization is the "same" rational localization of the tilt [SW20, Theorem 7.1.1], it follows that it suffices to show

$$(Y_{S,I} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^\infty)^\flat \simeq \mathbb{D}_{S,I}$$

where $\mathbb{D}_{S,I}$ is the adic closed annuli over S with inner and outter radii specified by I.

Notice that since $\mathbb{Q}_p$ is Tate, the morphisms $\mathbb{Q}_p \to \mathbb{Q}_p^\infty$ and $\mathbb{Q}_p \to B_{S,I}$ of complete Tate rings are adic. It follows from [SW20, Prop. 5.1.5.(2), Rmk. 5.1.6] that $\mathbb{Q}_p^\infty \widehat{\otimes}_{\mathbb{Q}_p} B_{S,I}$ is the pushout in the category of complete Tate rings. Thus for any $T = \mathrm{Spa}\,(A, A^+) \in \mathrm{Aff\,Perf}$,

1. a morphism of affinoids $T \to (Y_{S,I} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^\infty)^\flat$ is equivalent to
2. an untilt $T^\sharp$ together with a morphism of affinoids $T^\sharp \to Y_{S,I} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^\infty$, which is equivalent to
3. an untilt $T^\sharp$ together with a pair of morphisms of affinoids

$$T^\sharp \to \mathrm{Spa}\,(\mathbb{Q}_p^\infty, \mathbb{Q}_p^{\infty,\circ})$$
$$T^\sharp \to \mathrm{Spa}\,(B_{S,I}, B_{S,I}^+)$$

   which is equivalent to

4. by the tilting correspondence, a morphism of affinoids

$$T \to \mathrm{Spa}\,(\mathbb{F}_p((t^{1/p^\infty})), \mathbb{F}_p[[t^{1/p^\infty}]]_t^\wedge)$$

   and an untilt $T^\sharp$ together with a continuous ring morphism

$$W(R^+)\left[\frac{p^a}{[\pi]^b}, \frac{[\pi]^c}{p^d}\right]_p^\wedge \to \mathcal{O}(T^\sharp)^+$$

   Such a continuous ring morphism is equivalent to a continuous ring morphism $W(R^+) \to \mathcal{O}(T^\sharp)^+$ such that $p^a/[\pi]^b, [\pi]^c/p^d \in \mathcal{O}(T^\sharp)^+$ as well. So we have a further equivalence to

5. a morphism of affinoids
$$T \to \operatorname{Spa}(\mathbb{F}_p((t^{1/p^\infty})), \mathbb{F}_p[[t^{1/p^\infty}]]_t^\wedge)$$
and a continuous ring morphism
$$\alpha : R^+ \to A^+$$
such that $t^a/\alpha(\pi^b), \alpha(\pi^c)/t^d$ into A^+, where we have used $t = p^\flat$ and the following facts :
   - (Fact 1) For R^+ perfect and $A^\sharp$ perfectoid, we have
$$\begin{aligned} \{\text{cts } W(R^+) \to A^{\sharp,\circ}\} &\xrightarrow[\sim]{\text{mod } p} \{\text{cts } R^+ \to A^{\sharp,\circ}/(p)\} && \text{theory of Witt vectors} \\ &\xrightarrow[\sim]{} \{\text{cts } R^+ \to A^\circ\} && R^+ \text{ perfect, } A \text{ perfectoid} \end{aligned}$$
   - (Fact 2) [SW20, Lem. 6.2.5] Since $A^\sharp$ is perfectoid, we have an isomorphism of multiplicative monoids
$$\begin{array}{ccccc} \varprojlim_{x\mapsto x^p} A^{\sharp,+} & \subseteq & \varprojlim_{x\mapsto x^p} A^{\sharp,\circ} & \subseteq & \varprojlim_{x\mapsto x^p} A^{\sharp} \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ A^+ & \subseteq & A^\circ & \subseteq & A \end{array}$$
     The middle vertical map has an explicit inverse $x \mapsto (x^\sharp, (x^{1/p})^\sharp, \cdots)$.

   Finally, this is equivalent to

6. a morphism of affinoids $T \to \mathbb{D}_{S,I} := \{|t^a| \le |\pi^b| \ne 0\} \cap \{|\pi^c| \le |t|^d \ne 0\}$

$\square$

Proposition
Let $S = \operatorname{Spa}(A, A^+) \in \operatorname{Aff\,Perf}$. By the theory of Witt vectors, the Frobenius automorphism on S induces an automorphism φ of $\widetilde{\operatorname{Spa}}(W(R^+), W(R^+))$, and hence on Y_S. Define
$$X_S := Y_S/\varphi^{\mathbb{Z}}$$
where the quotient is as sheaves over Aff for the analytic topology. Then X_S is an adic space. In fact, one can obtain X_S as the quotient of $Y_{S,[p^{-p},p^{-1}]}$ by identifying the rational subspaces $\varphi : Y_{S,[p^{-p},p^{-p}]} \simeq Y_{S,[p^{-1},p^{-1}]}$. In particular, X_S is quasi-compact.

The picture one should have is of the Tate elliptic curve: closed annuli on the punctured open unit disk are identified by radial scaling, forming a torus which looks like the complex points of an elliptic curve.

Proof. Concretely, $\varphi : W(R^+) \to W(R^+)$ maps
$$x = \sum_{n=0}^{\infty} [x_n] p^n \mapsto \sum_{n=0}^{\infty} [x_n^p] p^n$$

So if we start with $\operatorname{Spa}(A, A^+) \to Y_{S,[p^{-N}, p^{-1/M}]}$, which means on $\operatorname{Spa}(A, A^+)$ we have

$$|p^a| \leq \left|[\pi]^b\right| \text{ and } |[\pi]^c| \leq \left|p^d\right|$$

Composing with $Y_S \to Y_S$ coming from φ on algebras, we get the following condition on $\operatorname{Spa}(A, A^+)$

$$|p^a| \leq \left|[\pi]^{bp}\right| \text{ and } |[\pi]^{cp}| \leq \left|p^d\right|$$

which is equivalent to $\operatorname{Spa}(A, A^+) \to Y_{S,[p^{-a/bp}, p^{-d/cp}]}$ It follows that φ induces an isomorphism

$$Y_{S,[p^{-a/b}, p^{-d/c}]} \xrightarrow[\sim]{\varphi} Y_{S,[p^{-a/bp}, p^{-d/cp}]}$$

i.e. "the closed annuli moves towards radius 1". For closed intervals $I \subseteq (0,1)$ of the form $I = [p^{-a/b}, p^{-d/c}]$ we have commutative diagrams

$$\begin{tikzcd}
Y_{S,I} \arrow[r, "\varphi", "\sim"'] \arrow[d, "\text{rat. loc.}"'] & Y_{S,\varphi I} \arrow[d, "\text{rat. loc.}"] \\
Y_{S,J} \arrow[r, "\sim", "\varphi"'] & Y_{S,\varphi J}
\end{tikzcd}$$

Identifying these affinoids under the action of φ thus gives another filtered system of affinoids along rational localizations. It follows that the presheaf colimit is the sheaf colimit and is an adic space. The universal property is clear.

(Quasi-compact) For compact intervals $I \subsetneq (0,1)$ such that $I \cap I^{1/p} = \varnothing$, the morphism $Y_{S,I} \to X_S$ is an open immersion. Since $(0,1) = \bigcup_{n\in\mathbb{Z}} [p^{-p/p^n}, p^{-1/p^n}]$, the result follows. $\square$

> **Proposition**
> Let $S = \operatorname{Spa}(R, R^+) \in \operatorname{Aff}\operatorname{Perf}$. Then we have
>
> $$Y_S^\diamond = S \times \operatorname{Spd}\mathbb{Q}_p$$
>
> $$X_S^\diamond = (S \times \operatorname{Spd}\mathbb{Q}_p)/(\varphi^{\mathbb{Z}} \times \mathbb{1})$$
>
> [FS24, Prop. II.1.17] This is the rational version of [SW20, Prop. 11.2.1].

Proof. The description of $X_S^\diamond$ follows from that of $Y_S^\diamond$. For $Y_S^\diamond$, note that we have already proved

$$\begin{tikzcd}
S \arrow[d] & S \times \operatorname{Spd}\mathbb{Q}_p \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & (Y_S \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^\infty)^\diamond \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\bullet & \operatorname{Spd}\mathbb{Q}_p \arrow[l] & \operatorname{Spd}\mathbb{Q}_p^\infty \arrow[l]
\end{tikzcd}$$

To compute $S \times \operatorname{Spd}\mathbb{Q}_p$, by descent in toposes it suffices to compute $S \times \operatorname{Spd}\mathbb{Q}_p^\infty$ with its $\mathbb{Z}_p^\times$ equivariant projection to $\operatorname{Spd}\mathbb{Q}_p^\infty$. The computation $(Y_S \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^\infty)^\diamond \simeq S \times D_S$ did exactly this. $\square$

Warning : the morphism $Y_S^\diamond \to S$ does *not* come from a morphism $Y_S \to S$ of adic spaces. (Indeed p is invertible on Y_S whilst p is zero on S.) Furthermore, $X_S^\diamond$ does not have a morphism to S even as pro-étale sheaves over $\mathrm{Aff\,Perf}$. Nonetheless, a morphism $T \to S$ in $\mathrm{Aff\,Perf}$ induces morphisms of adic spaces $Y_T \to Y_S$ such that if $S \to Y_S^\diamond$ corresponds to an untilt $S^\sharp \to Y_S$ then by [SW20, Theorem 6.2.11] the untilt $T^\sharp \to S^\sharp$ under the tilting equivalence fits in a cartesian square :

$$\begin{tikzcd} S^\sharp \arrow[d] & T^\sharp \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ Y_S & Y_T \arrow[l] \end{tikzcd}$$

We delay addressing the elephant in the room : *In what sense is X_S a relative curve over S?*

5 Vector bundles on the Fargues–Fontaine curve

We study vector bundles on X_S. In the motivation, we got the idea for X_S from the space $\mathrm{Spec}\, W(\overline{\mathbb{F}_p})[1/p]/\varphi^{\mathbb{Z}}$. Vector bundles on the latter are called *isocrystals over $\overline{\mathbb{F}_p}$*. These are very well understood. We write $\breve{\mathbb{Q}}_p$ for $\breve{\mathbb{Q}}_p$.

Proposition – Dieudonné classification of isocrystals

Define the category $\varphi\mathrm{Mod}(\breve{\mathbb{Q}}_p)$ of isocrystals over $\overline{\mathbb{F}_p}$ as the category of finite dimensional $\breve{\mathbb{Q}}_p$-vector spaces V together with a linear isomorphism

$$\varphi_V : \varphi^* V \simeq V$$

Then $\varphi\mathrm{Mod}(\breve{\mathbb{Q}}_p)$ is semisimple with simple objects, up to isomorphism, given by

$$D(\lambda) := (\breve{\mathbb{Q}}_p[T]/(T^r - p^d), T)$$

with $\lambda = d/r \in \mathbb{Q}$, $r > 0$, $(d, r) = 1$. The division algebra $\mathrm{End}\, D(\lambda)$ over $\mathbb{Q}_p$ is central with invariant $-[\lambda] \in \mathbb{Q}/\mathbb{Z} \simeq \mathrm{Br}(\mathbb{Q}_p)$.

A proof of the above uses the so-called HN formalism, which appears in the study of vector bundles on curves in algebraic geometry.

Proposition – Harder–Narasimhan formalism

Let C be an exact category[a] with two functions

$$\deg : \mathrm{Obj}\, C \to \mathbb{Z}$$

$$\mathrm{rk} : \mathrm{Obj}\, C \to \mathbb{N}$$

which are additive on short exact sequences. Assume the existence of an exact faithful functor

$$F : C \to A$$

to an abelian category A and an additive function $\mathrm{rk} : \mathrm{Obj}\, A \to \mathbb{N}$ extending $\mathrm{rk} : \mathrm{Obj}\, C \to \mathbb{N}$ along F. Furthermore, require three conditions :

1. F induces for each $\mathcal{E} \in C$ a bijection

$$\{\text{strict subobjects of } \mathcal{E}\} \simeq \{\text{subobjects of } F(\mathcal{E})\}$$

   where strict subobjects are subobjects fitting in a SES.

2. For $\mathcal{E} \in C$ $\mathrm{rk}\, \mathcal{E} = 0$ iff $\mathcal{E} = 0$.
3. Given $u : \mathcal{E} \to \mathcal{F}$ in C with $F(u)$ an isomorphism, then $\deg \mathcal{E} \leq \deg \mathcal{F}$ with equality iff u is an isomorphism.

Given the above situation, define the *slope* of $\mathcal{E} \in C$ to be

$$\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}\, \mathcal{E} \in \mathbb{Q} \cup \{\infty\}$$

An object $\mathcal{E} \in C$ is called *semistable* when for all non-zero strict subobjects $\mathcal{F} \subseteq \mathcal{E}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. Then for each $\mathcal{E} \in C$ there exists a filtration

$$0 = \mathcal{E}_0 \subseteq \cdots \subseteq \mathcal{E}_r = \mathcal{E}$$

unique with respect to the properties :

1. $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable
2. $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ strictly decreasing as i increases.

[FF18, Section 5.5]

[a] Here we mean in the sense of Quillen. This can be defined as full additive subcategories of abelian categories which are closed under isomorphisms and extensions.

Proof. Skipping over many details, existence is by induction on rank of $\mathcal{E}$. Uniqueness is by proving that given $\mathcal{E}, \mathcal{F}$ semistable with $\mu(\mathcal{E}) > \mu(\mathcal{F})$ then $C(\mathcal{E}, \mathcal{F}) = 0$. This can also be used to show the filtration is functorial in $\mathcal{E}$. □

The point is that this applies to

- $C = A = \varphi\mathrm{Mod}_{\breve{\mathbb{Q}}_p}$
- $\mathrm{rk}\, V := \dim_{\breve{\mathbb{Q}}_p} V$
- Using the observation

$$\pi_0 \{\text{rank 1 isocrystals}\} \simeq \breve{E}^\times/\varphi\text{-conjugacy} \simeq \mathbb{Z}$$

  define $\deg V := \deg \bigwedge^{\mathrm{rk}\, V} V$

To get the classification, one must calculate what these semistable objects are and show that the HN filtration splits. This suggests some ideas :

Q1 Is there a functor from isocrystals over $\overline{\mathbb{F}_p}$ to vector bundles on X_S.

Q2 Does the HN formalism apply to vector bundles on X_S? If so, how does the classification compare with isocrystals?

The answer to (Q1) is affirmative.

Definition – φ-modules to vector bundles on the curve

Let $S = \mathrm{Spa}\,(R, R^+) \in \mathrm{Aff\,Perf}$. The extension $\overline{\mathbb{F}_p} \to R$ gives $\breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}_p}) \to W(R^+)$ by functoriality. This induces a morphism

$$Y_S \to \mathrm{Spa}\,(\breve{\mathbb{Q}}_p, \breve{\mathbb{Q}}_p^\circ)$$

so we can pullback finite dimensional vector spaces over $\breve{\mathbb{Q}}_p$ to get vector bundles on Y_S. Define

$$\varphi\mathrm{Mod}(\breve{\mathbb{Q}}_p) \to \mathrm{VB}(X_S)$$

$$(V, \varphi_V) \mapsto \mathcal{E}(V, \phi_V)$$

where the latter is obtained by taking the trivial vector bundle over Y_S with fiber V, then descending it along $Y_S \to X_S$ by letting φ act via φ_V. We write $\mathcal{O}(d/r) := \mathcal{E}(D(-d/r))$ where $D(-d/r)$ is simple.

To study vector bundles, the following GAGA result says we can WLOG work in scheme theory.

Proposition – GAGA

Let $(X, \mathcal{O}_X)$ be a locally ringed spectral space. Suppose we have a line bundle $\mathcal{O}_X(1)$ such that

1. there exists $N \geq 0$ such that for all $n \geq N$ and all vector bundles $\mathcal{E}$ on X, $\mathcal{E}(n)$ is globally generated.
2. for all vector bundles $\mathcal{E}$ on X and for all $i > 0$ there exists n such that $H^i(X, \mathcal{E}(n)) = 0$.

Let $P := \bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(n))$. This is a graded ring so we can define $X^{\mathrm{alg}} := \mathrm{Proj}P$. Then there exists a morphism of locally ringed spaces $(X, \mathcal{O}_X) \to X^{\mathrm{alg}}$ such that

1. pullback induces an equivalences an equivalence of categories of vector bundles
2. for any vector bundle $\mathcal{E}$ on X with corresponding vector bundle $\mathcal{E}^{\mathrm{alg}}$ on X^{alg}, for all $i \geq 0$ we have an induced isomorphism

$$H^i(X^{\mathrm{alg}}, \mathcal{E}^{\mathrm{alg}}) \xrightarrow{\sim} H^i(X, \mathcal{E})$$

In this case, the tautological line bundle on X^{alg} pulls back to $\mathcal{O}_X(1)$. [FS24, Prop. II.2.7]

The relative Fargues–Fontaine curves satisfy this.

Proposition – Schematic relative Fargues–Fontaine curve

Let S continue to be in Aff Perf. Let $\mathcal{E}$ be a vector bundle on X_S. Then there exists $N \geq 0$ such that for all $n \geq N$ we have a surjection for some $m \geq 0$

$$\mathcal{O}_{X_S}^m \twoheadrightarrow \mathcal{E}(n)$$

and moreover $H^{i>0}(X_S, \mathcal{E}(n)) = 0$. We write X_S^{alg} for the scheme obtained using GAGA.

Proof. We can show that if $\mathcal{E}$ is a vector bundle over X_S, then $H^i(X_S, \mathcal{E}) = 0$ for $i \neq 0, 1$. For $\mathcal{O}$-modules on sheafy (A, A^+) associated to finite A-modules, there is no higher cohomology. [KL15, Theorem 2.5.20] Combining this with the fact that pushforward of $\mathcal{O}$-modules along open immersions of ringed spaces is exact, it suffices to find an open cover of X_S by two affinoids. For p odd, take $Y_{S,[p^{-p}, p^{-(p-1)/2p}]}$ and $Y_{S,[p^{-(p-1)/2p}, p^{-1}]}$. For $p = 2$, take $Y_{S,[1/4,3/8]}$ and $Y_{S,[3/8,1/2]}$.

See [FS24, Theorem II.2.6] for the "ampleness" of $\mathcal{O}_{X_S}(1)$. ☐

We can now address the question : in what sense is X_S a relative curve over S? Fargues and Fontaine define curves as follows :

Definition

A curve is a smooth, Noetherian, connected, separated scheme X of Krull dimension one together with a positive integer $\deg x$ for every closed point $x \in X$. [FF18, Def. 5.1.1]

Let X be a curve. Consider the exact sequence

$$1 \to \mathcal{O}(X)^\times \to \mathcal{M}(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div} X \to \mathrm{Pic} X \to 0$$

Define

$$\deg : \mathrm{Div} X \to \mathbb{Z}, \sum_{x \in |X|} m_x x \mapsto \sum_{x \in |X|} m_x \deg(x)$$

We say X is *complete* when $\deg$ factors through $\mathrm{Pic} X$, i.e. for any non-zero rational function f we have $\deg \mathrm{div} f = 0$.

Example.
Let k be a field and X a smooth projective curve over k in the usual sense. Then assigning every closed point x the value $\deg x := [\kappa(x) : k]$ defines a curve in the sense of [FF18].

We specialise to the case of a geometric point.

Proposition

Let $S = \mathrm{Spa}(F, F^\circ)$ where F is a complete algebraically closed NA field of characteristic p. Then the following is true :

1. The closed points of X_F^{alg} biject with characteristic zero untilts of F up to Frobenius.
2. X_F^{alg} is a complete curve in the sense of [FF18] when we set $\deg(x) = 1$ for all closed points.
3. For any closed point $x_{F^\sharp}$ of X_F^{alg}, the complement $X_F^{\mathrm{alg}} \setminus \{x_{F^\sharp}\}$ is the spectrum of a PID.

[FF18, Theorem 6.5.2]

For general $S \in \mathrm{Aff\,Perf}$, one should think of X_S as a family of curves $X_{\kappa(s)}$ where s ranges over points of S valued in completed algebraically closed NA fields (up to equivalence).

For S a geometric point, we have the following answer to (Q2) and (Q3) :

Proposition – Classification of vector bundles on the Fargues–Fontaine curve

Let $S = \mathrm{Spa}\,(F, F^\circ)$ where F is a complete algebraically closed NA field of characteristic p.

1. [FS24, Prop. II.2.12] The HN formalism applies to the category of vector bundles on X_F with the usual notion of degree and rank. Thus every vector bundle $\mathcal{E}$ has a unique exhaustive separating $\mathbb{Q}$-indexed filtration with factors which are semi-stable with single slope and the slope strictly decreases going up towards all of $\mathcal{E}$. This is called the HN filtration and it is respected by morphisms of vector bundles.
2. [FS24, Prop. II.2.13] For $F \to F'$ a continuous extension of complete algebraically closed NA fields of characteristic p, pullback along $X_{F'} \to X_F$ sends HN filtrations to HN filtrations.
3. [FS24, Prop. II.2.14] The HN filtration splits and if $\mathcal{E}$ is semistable with single slope λ, then $\mathcal{E} \simeq \mathcal{O}_{X_F}(\lambda)^m$ for some $m \geq 0$. Thus any vector bundle $\mathcal{E}$ on X_F is a direct sum of vector bundles of the form $\mathcal{O}_{X_F}(\lambda)$ for $\lambda \in \mathbb{Q}$.

In particular, the functor $(V, \varphi_V) \mapsto \mathcal{E}(V, \varphi_V)$ induces a bijection on the sets of isomorphism classes of isocrystals over $\overline{\mathbb{F}_p}$ and vector bundles on X_F.

We will need the following computation of global sections :

Proposition

Let $\lambda = d/r \in \mathbb{Q}$ with $(d, r) = 1$ and $r > 0$.

	$d < 0$	$d = 0$	$0 < d$
$H^0(X, \mathcal{O}(d/r))$	0	$\mathbb{Q}_p$	$B^{\varphi^r = p^d} \neq 0$
$H^1(X, \mathcal{O}(d/r))$	big	0	0

Proof. See [FF18, Prop. 8.2.3]. $H^0(X, \mathcal{O}) = \mathbb{Q}_p$ can be done by developing a theory of Newton polygons on the ring B^b. □

6 Application : Étale fundamental group of the Fargues–Fontaine curve

Fix F a complete algebraically closed NA field of characteristic p. We prove :

Proposition

Base change along $X_F \to X_F^{\mathrm{alg}} \to \mathrm{Spec}\,\mathbb{Q}_p$ induces an equivalence between finite étale morphisms over X_F and finite étale algebras over $\mathbb{Q}_p$. Consequently, we have

$$\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \simeq \pi_1^{\text{ét}}(X_F, x)$$

Proof. We follow [SW20, Theorem 13.5.7]. By GAGA, we can just work with the schematic $X := X_F^{\mathrm{alg}}$. Let $\widetilde{X} \to X$ be a finite étale cover and let $\mathcal{E}$ be the corresponding finite étale $\mathcal{O}_X$-algebra. It suffices to show $\mathcal{E}$ is a trivial vector bundle since $H^0(X, \mathcal{O}) = \mathbb{Q}_p$.

Since $\widetilde{X} \to X$ is finite, flat, locally finitely presented, $\mathcal{E}$ is locally free finite rank. By the classification of vector bundles on X, we can write $\mathcal{E} \simeq \mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_s)$. It suffices to show all $\lambda_i = 0$.

Since $\mathcal{E}$ is a vector bundle, one has a well-defined trace map $\mathrm{tr} : \mathcal{E} \to \mathcal{O}_X$. Étaleness implies the pairing $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \to \mathcal{O}_X$ is non-generate which exhibits $\mathcal{E}$ as its own $\mathcal{O}_X$-linear dual. [Stacks, Tag 0BVH] Since $\det(\mathcal{E}^\vee) \simeq (\det \mathcal{E})^{-1}$, we have $\deg \mathcal{E} = \deg \mathcal{E}^\vee = -\deg \mathcal{E}$. This implies $\sum_s \lambda_s = 0$.

Let $\lambda := \max(\lambda_s)$ so that $\lambda \geq 0$. It remains to show that $\lambda = 0$. Suppose for a contradiction that $\lambda > 0$. Then composition

$$\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda) \to \mathcal{E} \otimes \mathcal{E} \xrightarrow{\mathrm{mul}} \mathcal{E}$$

gives a global section $f \in \mathrm{Hom}(\mathcal{O}(\lambda)^{\otimes 2}, \mathcal{E}) \simeq \mathrm{Hom}(\mathcal{O}, \mathcal{E} \otimes \mathcal{O}(-2\lambda)) = H^0(X, \mathcal{E} \otimes \mathcal{O}(-2\lambda))$.

> *Lemma. For $(V, \varphi_V), (W, \varphi_W) \in \varphi\mathrm{Mod}(\breve{\mathbb{Q}}_p)$, define*
>
> $$(V, \varphi_V) \otimes (W, \varphi_W) := (V \otimes W, \varphi_V \otimes \varphi_W)$$
>
> *For $d_1/r_1, d_2/r_2 \in \mathbb{Q}$ with $(d_i, r_i) = 1$ and $r_i > 0$, we have*
>
> $$D\left(\frac{d_1}{r_1}\right) \otimes D\left(\frac{d_2}{r_2}\right) \simeq D\left(\frac{d_1}{r_1} + \frac{d_2}{r_2}\right)^{\oplus (r_1, r_2)}$$
>
> *Hence we have the following isomorphism of vector bundles on X_S :*
>
> $$\mathcal{O}_{X_S}\left(\frac{d_1}{r_1}\right) \otimes \mathcal{O}_{X_S}\left(\frac{d_2}{r_2}\right) \simeq \mathcal{O}_{X_S}\left(\frac{d_1}{r_1} + \frac{d_2}{r_2}\right)^{\oplus (r_1, r_2)}$$
>
> [*FF18, Prop. 5.6.23*]

By the lemma, all slopes in the decomposition of $\mathcal{E} \otimes \mathcal{O}(-2\lambda)$ are negative, so $H^0(X, \mathcal{E} \otimes \mathcal{O}(-2\lambda)) = 0$. This implies that for all $f \in H^0(X, \mathcal{O}(\lambda)) \subseteq H^0(X, \mathcal{E})$ we have $f^2 = 0$. Since $\mathcal{E}$ is étale, $H^0(X, \mathcal{E})$ is a reduced ring so $H^0(X, \mathcal{O}(\lambda)) = 0$, which is a contradiction because $\lambda > 0$. $\square$

7 Proof of classification of vector bundles

Interestingly, [FF18, Section 5.6] gives a classification of vector bundles on any "generalised Riemann sphere", which axiomitizes properties of $\mathbb{P}^1$ needed to classify its vector bundles.

TODO

8 Other facts about the curve

Other facts about the curve in the case of S being a geometric point with references from [FF18].

Proposition – Gauss norms
For $0 < \rho < 1$ a real number, define

$$|\_|_\rho : B_F^b \to \mathbb{R}_{\geq 0}, x = \sum_{n>>-\infty} [x_n]p^n \mapsto \sup_{n \in \mathbb{Z}} |x_n|_F \, \rho^n$$

Then we have :

1. [FF18, Prop. 1.4.3] $|x+y|_\rho \leq \max(|x|_\rho, |y|_\rho)$
2. [FF18, Remark 1.4.4] For $\lambda \in \mathbb{Q}_p$, $|x|_\rho = \rho^{-v_p(x)}$.
3. [FF18, Prop. 1.4.9] $|xy|_\rho = |x|_\rho |y|_\rho$.
4. [FF18, Prop. 1.4.11.(3)] The topology on $W(F^\circ)$ induced by $|\_|_\rho$ is the $(p, [\pi])$-adic topology. Consequently, $|\_|_\rho$ is a norm on B_F^b as a $\mathbb{Q}_p$-vector space.
5. For $0 < a \leq b \leq c < 1$ we have $|x|_b \leq \max(|x|_a, |x|_c)$.
6. [FF18, Example 1.6.3] For $I = [\rho_1, \rho_2] \subseteq (0,1)$ compact interval, define $B_{F,I}$ as the completion of B_F^b w.r.t. all the norms $|\_|_\rho$ with $\rho \in I$. By the previous point, this is equivalent to the completion w.r.t. just the norm $|\_|_I := \max(|\_|_{\rho_1}, |\_|_{\rho_2})$. If $\rho_1 = |a|, \rho_2 = |b|$ for $a, b \in F^\circ$ then this completion matches what we previously computed

$$\left\{x \in B_F^b \text{ s.t. } |x|_I \leq 1\right\} = W(F^\circ)\left[\frac{[a]}{p}, \frac{p}{[b]}\right]$$
$$B_{F,I} \simeq W(F^\circ)\left[\frac{[a]}{p}, \frac{p}{[b]}\right]_p^\wedge [1/p]$$

7. [FF18, Lem. 2.4.7, 2.4.8] Let $|Y_F|$ denote the set of characteristic zero untilts of F up to isomorphism. For a compact interval $I \subseteq (0,1)$, let

$$|Y_{F,I}| := \{y \in |Y_F| \text{ s.t. } r(y) \in I\}$$

For each $y \in |Y_{F,I}|$, writing $\theta_y : W(F^\circ) \to K_y$ for the surjection to the untilt, it extends uniquely to a continuous surjection $\theta_y : B_{F,I} \to K_y$.

$$\begin{tikzcd}
W(F^\circ) \arrow[r] \arrow[d, two heads, "\theta_y"'] & B_F^b \arrow[r] \arrow[dl, two heads] & B_{F,I} \arrow[dll, two heads, "\widetilde{\theta_y}"] \\
K_y & &
\end{tikzcd}$$

In fact, $(\ker \theta_y) B_{F,I} = \ker \widetilde{\theta}_y$.

8. [FF18, Theorem 2.5.1] For compact intervals $I \subseteq (0,1)$, $B_{F,I}$ is a PID and the previous point induces a bijection

$$|Y_{F,I}| \xrightarrow{\sim} \operatorname{MaxSpec} B_{F,I}$$

Recall we defined X by gluing $\mathrm{Spa}\,(B_{[a,b]}, B_{[a,b]^+})$. In fact, one can glue $\mathrm{Spec}\, B_{[a,b]}$ in the same way to obtain X^{alg}. It follows that the set of closed points can be obtained as

$$|X| \simeq |Y|/\varphi^{\mathbb{Z}}$$

For $y \in |Y|$, $r(\varphi(y)) = r(y)^{1/p}$. For intervals $[a,b] \subseteq (0,1)$ such that $[a,b] \cap [a^{1/p}, b^{1/p}] = \varnothing$, one can show that the morphism

$$\mathrm{Spa}\,(B_{[a,b]}, B_{[a,b]^+}) \to X$$

is an open immersion. **Warning** : the corresponding morphism of schemes

$$\mathrm{Spec}\, B_{[a,b]} \to X^{\mathrm{alg}}$$

is *not* an open immersion, because it is not the complement of finitely many points! Nonetheless, we take the following definition :

> **Proposition**
>
> Let $x \in |X|$ and let $y \in |Y|$ be a lift of x. This corresponds to an untilt (C, ι) of F over characteristic zero. Let $(\xi) = \ker\theta \subseteq W(F^\circ) \to C^\circ$. Then for any compact interval $I \subseteq (0,1)$ containing $r(y)$, the morphism of schemes
>
> $$\mathrm{Spec}\, B^b \to \mathrm{Spec}\, B_{[a,b]} \to X^{\mathrm{alg}}$$
>
> induces isomorphisms of rings
>
> $$\left(B^b\right)^\wedge_\xi \xrightarrow{\sim} \left(B_{[a,b]}\right)^\wedge_{\mathfrak{m}_y} \xrightarrow{\sim} \mathcal{O}^\wedge_{X,x}$$
>
> We identify these rings and call it $B^+_{\mathrm{dR}}(C)$. [FF18, Def. 2.7.1] This is a complete DVR.

Proof. The first isomorphism is quite formal. The second requires more work [FF18, Theorem 6.5.2.(5)]. Complete DVR part follows from previous proposition. □

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