Definition – Schemes

Let $X \in \mathbb{M}\mathbf{Set}$. Then X is called a *scheme* when we have :

- ("Is a Space") $X \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$, equivalently $\mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$.
- (Open cover by Affine schemes) there exists $\mathcal{U} \subseteq \mathrm{Open}\, X$ such that $\mathcal{U}$ covers X and every $U \in \mathcal{U}$ is affine.

We use $\mathbf{Sch}$ to denote the full subcategory of schemes in $\mathbb{M}\mathbf{Set}$.

Remark – Intuition of Definition of Schemes. In the same way that smooth manifolds are spaces modeled on $\mathbb{R}^n$ that is locally $\mathbb{R}^n$, schemes are spaces modeled on $\mathbb{M}$ that is "locally $\mathbb{M}$". In particular, objects of $\mathbb{M}$ ought to be schemes.

Proposition – Affine Schemes are Schemes

Let $X \in \mathbf{Aff}$. Then $X \in \mathbf{Sch}$.

Proof. X is an affine open cover of itself, so it suffices to check the sheaf condition. Since $\mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}) = \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$, it suffices to check that for $(A, \alpha) \in \mathrm{Sp} \downarrow X, \mathcal{U}$ a $\mathbf{Aff}_{\mathrm{Zar}}$-cover of $\mathrm{Sp}\, A$, we have

$$\mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X) \xrightarrow{\sim} \varprojlim_{U,V \in \mathcal{U}} \mathbb{M}\mathbf{Set}(U \cap V, X)$$

Since $X = \mathrm{Sp}\, \mathcal{O}(X)$ and $\mathrm{Sp} : \mathbb{M}^{op} \to \mathbb{M}\mathbf{Set}$ is fully faithful, this is equivalent to

$$\mathbb{Z}\mathbf{Alg}(\mathcal{O}(X), \mathcal{O}(\mathrm{Sp}\, A)) \xrightarrow{\sim} \mathbb{Z}\mathbf{Alg}(\mathcal{O}(X), \varprojlim_{U,V \in \mathcal{U}} \mathcal{O}(U \cap V))$$

It thus suffices to show that

$$A = \mathcal{O}(\mathrm{Sp}\, A) \xrightarrow{\sim} \varprojlim_{U,V \in \mathcal{U}} \mathcal{O}(U \cap V)$$

where the isomorphism is from the restriction maps $\mathcal{O}(\mathrm{Sp}\, A) \to \mathcal{O}(U \cap V)$.

Let $(f_V \in \mathcal{O}(V))_{V \in \mathcal{U}}$ that agrees on pairwise intersections. Suppose for a moment, for any finite subcover $\mathcal{U}_0 \subseteq \mathcal{U}$, we have a unique $f_{\mathcal{U}_0} \in A$ that agree with f_V on $V \in \mathcal{U}_0$. WLOG $\mathcal{U} = \{D(f)\}_{f \in I}$ for some $I \subseteq A$. Then $\mathcal{U}$ covers $\mathrm{Sp}\, A$ implies $D(I)$ covers $\mathrm{Sp}\, A$, which implies $AI = A$, which gives a *finite* subset $I_0 \subseteq I$ where $AI_0 = A$. Hence, we do have a finite subcover $\mathcal{U}_0$ and such $f_{\mathcal{U}_0}$. Furthermore, for any $V \in \mathcal{U}$, $f_{\mathcal{U}_0} = f_{\mathcal{U}_0 \cup \{V\}}$ by uniqueness of $f_{\mathcal{U}_0}$ on $\mathcal{U}_0$ so

$$\downarrow^V f_{\mathcal{U}_0} = \downarrow^V f_{\mathcal{U}_0 \cup \{V\}} = f_V$$

So $f_{\mathcal{U}_0}$ actually agrees with f_V on all $V \in \mathcal{U}$. Furthermore, it is unique, again by uniqueness on $\mathcal{U}_0$. Thus, it suffices to do the case of $\mathcal{U}$ *finite*.

The naive idea is this : if each $f_V = g_V / h_V$ with $V = D(h_V)$, then "agreeing on intersections" *should* mean $g_V h_W = g_W h_V \in \mathcal{O}(U) = A$. We can then use a partition of unity $1 = \sum_{V \in \mathcal{U}} \lambda_V h_V$ to patch :

$$f_V = \frac{g_V}{h_V} = \frac{\sum_{W \in \mathcal{U}} \lambda_W g_V h_W}{h_V} = \frac{\sum_{W \in \mathcal{U}} \lambda_W g_W h_V}{h_V} = \frac{\sum_{W \in \mathcal{U}} \lambda_W g_W}{1}$$

So $f := \sum_{W \in \mathcal{U}} \lambda_W g_W \in A = \mathcal{O}_{\mathrm{Spec}\, A}(U)$ is the guy we want. This is even unique since if we have another such f_1, then $f/1 = f_1/1 \in \mathcal{O}(V) \cong A_{h_V}$ implies the existence of $N_V \in \mathbb{N}$ such that $(f - f_1) h_V^{N_V} = 0$. By *finiteness of* $\mathcal{U}$, we can pick a single $N \in \mathbb{N}$ with $(f - f_1)h_V^N = 0$ for all $V \in \mathcal{U}$. Then using another partition of unity $1 = \sum_{V \in \mathcal{U}} \mu_V h_V^N$, we have

$$f - f_1 = \sum_{V \in \mathcal{U}} \mu_V (f - f_1) h_V^N = 0$$

It thus suffices to write for all $V \in \mathcal{U}$, $f_V = g_V / h_V$ such that for all $W \in \mathcal{U}$, $g_V h_W = g_W h_V$.

Well, for each $V \in \mathcal{U}$, let $h_V \in A$ with $V = D(h_V)$. Then $f_V = g_V / h_V^{n_V}$. Since $D(h_V) = D(h_V^{n_V})$, WLOG $f_V = g_V / h_V$ with $V = D(h_V)$ Now, since f_V and f_W agree on $V \cap W = D(h_V h_W)$, we have $g_V h_W / h_V h_W =\downarrow^{V \cap W} g_V / h_V =\downarrow^{V \cap W} g_W / h_W = g_W h_V / h_V h_W$ and so the existence of $n(V, W) \in \mathbb{N}$ such that

$$(g_V h_W - g_W h_V)(h_V h_W)^{n(V,W)} = 0$$

Smashing it again with *finiteness of* $\mathcal{U}$, we can choose a single $N \in \mathbb{N}$ such that

$$(g_V h_W - g_W h_V)(h_V h_W)^N = 0$$

for *all* $V, W \in \mathcal{U}$. Then, since $g_V / h_V = g_V h_V^N / h_V^{N+1}$ and $D(h_V) = D(h_V^{N+1})$, we can WLOG $f_V = g_V / h_V$ with $V = D(h_V)$ and

$$g_V h_W - g_W h_V = 0$$

as desired, finishing the proof.

$\square$

> **Proposition – Opens Subschemes**
> Let $X \in \mathbf{Sch}$, $U \in \mathrm{Open}\, X$. Then $U \in \mathbf{Sch}$. We call U an *open subscheme of* X.

Proof. (*Sheaf*)

> *Lemma* (*Opens of Sheaves are Sheaves*). *Let* $U \in \mathrm{Open}\, X$ *where* $X \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$. *Then* $U \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$.

Proof. Given $Y \in \mathbb{M}\mathbf{Set}$, a compatible system $(\varphi_i)_{Y_i \in \mathcal{Y}}$ of morphisms from an open cover $\mathcal{Y}$ of Y to U glues uniquely to a morphism $\varphi : Y \to X$. Factoring φ through U is equivalent to $\varphi^{-1}U = Y$, which is true from $\varphi^{-1}U$ "containing" the cover $\mathcal{Y}$, and so is single open covering Y, and hence is equal to Y by extensionality of opens. $\blacksquare$

(*Affine Open Cover*) Let $\mathcal{U} \subseteq \mathrm{Open}\, X$, $\mathcal{U}$ consists of affine opens and covers X. Since opens and covers are preserved under base change, $\{U \cap V\}_{V \in \mathcal{U}}$ is an open cover of U. For each $V \in \mathcal{U}$, $U \cap V$ is also an open of V. By affineness of V, $U \cap V$ has a cover by basic opens V_f of V. The V_f are open in $U \cap V$ by base change and hence open in U by composition. This gives an affine open cover of $U \cap V$, and hence an affine open cover of U by taking the composite of these covers.

$\square$

Proposition – Fiber Product of Schemes

Let $X, Y, S \in \mathbf{Sch}$ and $\varphi \in \mathbf{Sch}(X, S)$, $\psi \in \mathbf{Sch}(Y, S)$. Then the fiber product $X \times_S Y$ in $\mathbb{M}\mathbf{Set}$ is a scheme and is the fiber product of X, Y over S in $\mathbf{Sch}$.