Definition – Zariski-Local Properties

Let $P : \mathbf{Sch} \to \mathbf{Prop}$ be a predicate on schemes. Then P is called *Zariski-local* when for all $X \in \mathbf{Sch}$ and Zariski covers $\mathcal{X}$ of X, $P(X)$ is true if and only if for all $X_i \in \mathcal{X}$, $P(X_i)$ is true.

Definition – Affine-Local Properties

Let $P : \mathbf{Aff} \to \mathbf{Prop}$ be a predicate on affine schemes. Then P is called *affine-local* when for all $X \in \mathbf{Aff}$ and basic Zariski covers $\mathcal{X}$ of X, $P(X)$ is true if and only if for all $X_i \in \mathcal{U}$, $P(X_i)$ is true.

Proposition – Affine-Locality

Let $P : \mathbf{Aff} \to \mathbf{Prop}$ be affine-local. Define the predicate locally $P : \mathbf{Sch} \to \mathbf{Prop}$ by setting X is locally P when there exists an affine Zariski-cover $\mathcal{X}$ of X such that all $X_i \in \mathcal{X}$ satisfy P.

Then TFAE :

1. X is locally P
2. All opens U of X are locally P.
3. All affine opens U of X satisfy P.
4. There exists an affine Zariski cover $\mathcal{X}$ of X where all $X_i \in \mathcal{X}$ satisfy P.
5. There exists a Zariski cover $\mathcal{X}$ of X where all $X_i \in \mathcal{X}$ are locally P.

In particular, "locally P" is a Zariski-local property of schemes.

Proof.

$(1 \Rightarrow 2)$ Let $U \in \operatorname{Open} X$. Let $\mathcal{X}$ be an affine Zariski cover of X where all $X_i \in \mathcal{X}$ satisfy P. For each X_i, $X_i \cap U$ is an open of X_i and hence admits a Zariski covering $\mathcal{U}_i$ by basic opens of X_i. Since $P(X_i)$ is true, for every $U_{i,j} \in \mathcal{U}_i$, $P(U_{i,j})$ is true as well. Then note that $U_{i,j}$ are affine since X_i is and also open in U so the composite $\mathcal{U} := \bigcup_{X_i \in \mathcal{X}} \mathcal{U}_i$ gives an Zariski cover of $\mathcal{U}$ consisting of affines satisfying P.

$(2 \Rightarrow 3)$ Let $U \in \operatorname{Open} X$ be affine. Then U has a basic Zariski cover $\mathcal{U}$. Applying $(1 \Rightarrow 2)$ to U, we see that all opens in $\mathcal{U}$ satisfy P, and hence U satisfies P by P being affine-local.

$(3 \Rightarrow 4)$ By X being a scheme. $(4 \Rightarrow 5)$ OK. $(5 \Rightarrow 1)$ Composites of open covers.

□

Proposition – Examples of Affine-Local Properties

The following predicates on $\mathbf{Aff}$ are affine-local :

1. For $X \in \mathbf{Aff}$, say X is *Noetherian* when $\mathcal{O}(X)$ is Noetherian.
2. For $X \in \mathbf{Aff}$, say X is *reduced* when $\mathcal{O}(X)$ has no nilpotent elements.

Definition – Globally

Let P be a affine-local property of affine schemes and $X \in \mathbf{Sch}$. We say X is *globally* P[a] when X is locally P and X is quasi-compact.

[a]This is non-standard terminology. A lot of affine-local properties are extended to schemes by quasi-compact + locally P. In this case, it is standard terminology to say simply say "X is P". However, this clashes with properties of schemes *not* coming from affine-local properties. The addition of the adverb "globally" is an attempt to highlight the fact that the property P is affine-local.