M4R : Crystals

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Contents

"IP" stands for "indefinitely postponed", which means the details are at the bottom of the priority list, and I will only fill them in if I have time at the end.

1 Introduction

1.1 Why?

To give a setting where representation theory can be interpreted algebro-geometrically. More concretely : everything should be seen as pushing or pulling quasi-coherent sheaves across spaces. A key example of something we would like to reinterpret is the Beilinson-Bernstein localisation functor.

(TODO : talk about what BB looks like in this framework, how this project will only cover the basic part about crystals.)

2 Setting

2.1 How to work with "DG Categories"

Gaitsgory-Rozenblyum has a highly abstract but clean definition of dg-categories, which can be summarised in the following diagram :

$$\begin{tikzcd}
\mathrm{Set} \arrow[r] \arrow[d] & \text{1-}\mathrm{Cat} \arrow[d, shift right, "N"'] \arrow[d, phantom, "\vdash"] & \mathrm{Cat}_{\mathrm{dg}} k \arrow[d, "N_{\mathrm{dg}}"] & & & \\
\mathrm{Spc} \arrow[r] & (\infty,1)\text{-}\mathrm{Cat} \arrow[u, shift right, "h"'] & (\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}} \arrow[l] & \mathrm{Vec}^c\text{-}\mathrm{Mod} \arrow[l] & =: & \mathrm{DGCat} \\
 & & (\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{cts}} \arrow[u] & \mathrm{Vec}\text{-}\mathrm{Mod} \arrow[l] \arrow[u] & =: & \mathrm{DGCat}_{\mathrm{cts}}
\end{tikzcd}$$

Some explanations are due :

– We will use "infinity category" to refer only to $(\infty, 1)$-categories. The word "category" will exclusively refer to 1-categories.

  $(\infty, 1)\text{-Cat}$ denotes the infinity category of small infinity categories. $(\infty, 1)\text{-Cat}$ has all small limits (Kerodon 7.4.1.11) and small colimits (Kerodon 7.4.3.13) and is cartesian closed. We use $\mathrm{Fun}(C, D)$ to denote the infinity category of functors from C to D.

  There is an adjunction $h \dashv N : (\infty, 1)\text{-Cat} \rightleftarrows 1\text{-Cat}$. N is called the *nerve functor* and it is fully faithful, allowing us to see 1-categories as ∞-categories. In particular, we use Δ^n to denote the *n-simplex*, the ∞-category obtained from the linear order $[n] = \{0 \leq \cdots \leq n\}$. Given an infinity category C, objects of C are the same as functors $\Delta^0 \to C$ and morphisms in C are the same as functors $\Delta^1 \to C$.

  There is a full subcategory Spc of $(\infty, 1)\text{-Cat}$ consisting of ∞-categories X where all morphisms are isomorphisms. These are called *∞-groupoids* but also *spaces* by homotopy theorists. The infinity category Spc plays the role of Set in 1-category theory in the sense that given an infinity category C, the fiber of $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$ over (X, Y), denoted $C(X, Y)$, is in fact a space. The nerve functor N lands Set inside Spc, where given a set S and two points $x, y \in S$, we have $NS(x, y) \simeq \varnothing$ the empty space. In other words, sets are "discrete spaces".

  For any infinity category C, we define $\mathrm{PSh}\, C := \mathrm{Fun}(C^{op}, \mathrm{Spc})$ and refer to its objects as *presheaves in C*. Infinity categories of presheaves are significant since we will be working with derived algebraic geometry functorially. We will use the following universal property of $\mathrm{PSh}\, C$ many times.

Proposition – Universal Property of Presheaf ∞-Categories
Let S be a small ∞-category.

- There is a fully faithful functor $S \to \mathrm{PSh}\, S$ which takes each object x in S to the functor $S(\_, x) : C^{op} \to \mathrm{Spc}$ taking points y to $S(y, x)$. This is called the *Yoneda embedding*.
- (Lurie HTT 5.1.2.3) $\mathrm{PSh}\, S$ has small colimits and small limits. In fact, they are computed pointwise.
- (Lurie HTT 5.1.5.6 and 5.2.6.5) For C be an ∞-category with small colimits, let $\mathrm{Fun}^L(\mathrm{PSh}\, S, C)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{PSh}\, S, C)$ consisting of functors preserving small colimits. Then restricting along the Yoneda embedding $S \to \mathrm{PSh}\, S$ gives an equivalence of ∞-categories :
$$\mathrm{Fun}^L(\mathrm{PSh}\, S, C) \xrightarrow{\sim} \mathrm{Fun}(S, C)$$
An inverse functor is given by left Kan extension. In particular, for $u_! \in \mathrm{Fun}^L(\mathrm{PSh}\, S, C)$ corresponding to $u \in \mathrm{Fun}(S, C)$, we have for every $X \in \mathrm{PSh}\, S$ that $u_!$ exhibits $u_!(X)$ as the colimit of the diagram $S_{/X} \to S \to C$. Furthermore, if we assume C is locally small, then we have an adjunction
$$u_! \dashv u^* : \mathrm{PSh}\, S \rightleftarrows C$$
where u^* is given by the composition
$$C \xrightarrow{\text{Yoneda}} \mathrm{PSh}\, C = \mathrm{Fun}(C, \mathrm{Spc}^{\mathrm{op}}) \xrightarrow{\text{restrict along } u} \mathrm{Fun}(S, \mathrm{Spc}^{\mathrm{op}}) = \mathrm{PSh}\, S$$

An immediate consequence of the above is the following, which is key for defining de Rham spaces.

Lemma (*Left and Right Kan Extensions of Presheaves*). *Let $u : S \to T$ be a functor between small ∞-categories. Then we have a triple of adjoints :*

$$\begin{tikzcd} \mathrm{PSh}\, S \arrow[r, shift left=2, "u_!"] \arrow[r, phantom, shift left=1, "\perp"] \arrow[r, shift right=2, "u_*"'] \arrow[r, phantom, shift right=1, "\perp"] & \mathrm{PSh}\, T \arrow[l, "u^*" description] \end{tikzcd}$$

where

- *$u_!$ is the left Kan extension of $S \to T \to \mathrm{PSh}\, T$*
- *u^* is the composition $\mathrm{PSh}\, T \to \mathrm{PSh}\, \mathrm{PSh}\, T \to \mathrm{PSh}\, S$*
- *u_* is the composition $\mathrm{PSh}\, S \to \mathrm{PSh}\, \mathrm{PSh}\, S \to \mathrm{PSh}\, T$*

In particular, for $X \in \mathrm{PSh}\, S$, $u_!(X)$ is the left Kan extension of X along $S \to T$ and $u_*(X)$ is the right Kan extension of X along $S \to T$.

Proof. For $u_! \dashv u^*$, we apply the universal property of presheaf ∞-categories to $S \to T \to \mathrm{PSh}\, T$. Now for $u^* \dashv u_*$, note that since colimits in $\mathrm{PSh}\, T$ are computed pointwise, we have that u^* preserves small colimits. This means we can apply the universal property of presheaf

∞-categories again, but this time to the composition $T \to \mathrm{PSh}\, T \to \mathrm{PSh}\, S$. This gives $u^* \dashv u_*$. ■

– $(\infty, 1)\text{-Cat}^{\mathrm{ex}}$ denotes subcategory of 1-Cat consisting of *stable infinity categories* and *exact functors*. It contains all small limits and the "inclusion" $1\text{-Cat}^{ex} \to 1\text{-Cat}$ preserves small limits (Lurie HA 1.1.4.4).

Stable infinity categories are basically triangulated categories where exact triangles are determined by an infinity-categorical universal property. Here is the definition.

Definition

Let C be an infinity category. We say C has a *zero object* when it has an object that is both initial and final. (Lurie HA 1.1.1.1.)

Now assume C have a zero object. Then a *triangle* in C is defined as a diagram in C of the form :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

A triangle is called a *fiber sequence* when it is a cartesian and a *cofiber sequence* when it is cocartesian. (Lurie HA 1.1.1.4.) In the first case, we say $Y \to Z$ *admits a kernel* and refer to X as the kernel, and in the other case we say $X \to Y$ *admits a cokernel* and refer to Z as the cokernel. [a]

C is called *stable* when the following are true :

- every morphism has both a kernel and a cokernel.
- A triangle is fiber sequence iff it is a cofiber sequence. Such triangles are called *exact triangles*.

(Lurie HA 1.1.1.9.)

An exact functor $F : C \to D$ between stable infinity categories is one which satisfy any of the following equivalent conditions : (Lurie HA 1.1.4.1)

- F preserves exact triangles
- F preserves finite limits
- F preserves finite colimits.

For stable ∞-categories C, D the full subcategory $\mathrm{Fun}^{\mathrm{ex}}(C, D)$ of $\mathrm{Fun}(C, D)$ consisting of exact functors is also stable.[b]

[a] In Lurie HA, kernels are called fibers and cokernels are called cofibers.

[b] GRI Chapter 1 5.1.4 claims this. Lurie HA 1.1.3.1 shows that $\mathrm{Fun}(K, C)$ is stable for any K and stable C. The result follows given that finite (co)limit-preserving functors are closed under finite (co)limits.

To help build intuition of "stable infinity categories as fixed triangulated categories", we record here the important parts of the procedure of extracting a triangulated category from a stable infinity category.

Proposition – Lurie 1.1.2.14

Let C be a stable infinity category. Then the following defines a triangulated structure on the 1-category hC :

- Define the *suspension functor* $\Sigma : C \to C$ by pushout against zeros :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

  Since the above square is a cofiber sequence, it is also a fiber sequence. This shows that *looping* $\Omega : C \to C$, given by pullback against zeros, gives an inverse for Σ and hence shows that Σ is an equivalence. Taking homotopy categories, we obtain an equivalence $[1] : hC \xrightarrow{\sim} hC$, which we use as the shift functor for the triangulated structure.

- We call a diagram
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
  in hC an exact triangle (in the triangulated categorical sense) when it comes from a diagram of the following form in C :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

  i.e. two exact triangles (in the stable infinity categorical sense).

- For X, Y objects of C, we have

$$\begin{aligned} C(X,Y) &\simeq C(\Sigma\Omega X, Y) \simeq \Omega C(\Omega X, Y) \\ &\simeq C(\Sigma^2\Omega^2 X, Y) \simeq \Omega^2 C(\Omega^2 X, Y) \end{aligned}$$

  Upon taking π_0 , we obtain

$$hC(X,Y) := \pi_0 C(X,Y) \simeq \pi_1 C(\Omega X, Y) \simeq \pi_2(\Omega^2 X, Y)$$

  where the last isomorphism is a group morphism. For π_2 of any "space" [a] the obvious group structure given by is abelian, this gives $hC(X,Y)$ an abelian group structure, making hC into an additive category.

> For X, Y objects in C, we define the abelian group $\mathrm{Ext}^n_C(X,Y) := hC(X, Y[n])$. (Lurie HA 1.1.2.17)
>
> [a] In the quasi-category model of infinity categories, $C(X,Y)$ is a Kan complex, which one can take homotopy groups of.

(IP : t-structures, truncation as reflective localisation, $(D^-(A))^\heartsuit \simeq A$ Lurie 1.3.2.19)

> **Definition**
> Let $F : C \to D$ be a functor in 1-$\mathrm{Cat}^{\mathrm{ex}}$ where C and D are equipped with t-structures. Then F is called *right t-exact* when $FC_{0\leq} \subseteq D_{0\leq}$. It is called *left t-exact* when $FC_{\leq 0} \subseteq D_{\leq 0}$. We say F is *t-exact* when it is both left and right t-exact.

– We are now ready for compactly generated ∞-categories. We will only make use of the case of *stable* compactly generated ∞-categories since many definitions then admit alternative characterisations which can be checked at the level of triangulated categories.

We first note that the theory of colimits simplifies in the stable case.

> **Proposition – Lurie HA 1.4.4.1** – For a stable ∞-category C, TFAE:
>
> * admitting small colimits
> * admitting small filtered colimits
> * admitting small coproducts
>
> – For a functor $F : C \to D$ between stable ∞-categories which admit small colimits, TFAE:
>
>   * preserving small colimits
>   * preserving small filtered colimits
>   * preserving small coproducts
>
>   Any functor satisfying the above, GR calls *continuous*.

We now explain compact generation. The starting point is the theory of *inductive completions*[1]. Here are the main results concerning ind-completions in the stable case.

> **Proposition – Ind-completions of Stable ∞-Categories**
> Let C be a small ∞-category and κ a regular cardinal.[a] Then the Yoneda embedding $C \to \mathrm{PSh}\, C$ factors through a full subcategory $\mathrm{Ind}_\kappa(C)$ with the following properties :
>
> – (Lurie HTT 5.3.5.3) $\mathrm{Ind}_\kappa(C)$ has all small κ-filtered colimits and the inclusion $\mathrm{Ind}_\kappa(C) \subseteq \mathrm{PSh}\, C$ preserves them
>
> – (Lurie HTT 5.3.5.4) An object X in $\mathrm{PSh}\, C$ is in $\mathrm{Ind}_\kappa(C)$ iff it is a κ-filtered colimit of representables iff $X : C^{op} \to \mathrm{Spc}$ preserves κ-small limits.

[1] This is a bit of a misnomer because intuitively we are adding filtered *colimits*, not limits.

- (Lurie HA 1.1.3.6) If C is stable then so is $\mathrm{Ind}_\kappa(C)$.
- (Lurie HTT 5.3.5.10) For any ∞-category D admitting small κ-filtered colimits, we have the following equivalence of functor ∞-categories :

$$\mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D)$$

  where

  * the left category denotes the full subcategory of $\mathrm{Fun}(\mathrm{Ind}_\kappa(C), D)$ consisting of functors preserving κ-filtered colimits.[b]
  * the forward functor is given by restricting along the Yoneda embedding $C \to \mathrm{Ind}_\kappa(C)$
  * the inverse functor is given by left Kan extension.

  Assuming C, D are stable and κ is the cardinality of $\mathbb{N}$, the above equivalence restricts to an equivalence between the following two full subcategories :

$$\mathrm{Fun}^{\mathrm{ex}}_{\mathrm{cts}}(\mathrm{Ind}_\kappa(C), D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(C, D)$$

  where the left is the ∞-category of exact continuous functors from $\mathrm{Ind}_\kappa(C)$ to D.

When κ is the cardinality of $\mathbb{N}$, we write Ind instead of Ind_κ.

[a] A regular cardinal κ is a cardinality that is "sufficiently large" in the sense that the 1-category $\mathrm{Set}_{<\kappa}$ of sets with cardinality strictly less than κ has all colimits of size strictly less than κ. The cardinality of $\mathbb{N}$ is an example, since a finite colimit of finite sets is still finite.

[b] At Lurie HTT 5.3.4.5, these are called κ-continuous functors. Taking the minimal case of $\kappa = |\mathbb{N}|$, it seems only reasonable to refer to functors preserving filtered colimits as *continuous* functors. This is a potential explanation of GR's choice of terminology for continuous functors.

For a regular cardinal κ and an ∞-category C, we say C is κ-compactly generated when it has all small colimits and there exists a small ∞-category C^0 with an equivalence $\mathrm{Ind}_\kappa(C^0) \xrightarrow{\sim} C$.[1] For the case of $\kappa =$ cardinality of $\mathbb{N}$, we simply say *compactly generated*. A *presentable* ∞-category is one that is κ-compactly generated for some κ. We use $(\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{cts}}$ to denote the subcategory of $(\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}}$ whose objects are presented stable ∞-categories and morphisms are exact functors preserving small coproducts.

One appeal of presentable stable ∞-categories is that we have the adjoint functor theorem at our disposal.

Proposition – Adjoint Functor Theorem for Presentable ∞-Categories (Lurie HTT 5.5.2.9)

Let $F : C \to D$ be a functor between presentable ∞-categories.

- F is a left adjoint iff it preserves small colimits.
- Assuming C, D are κ-compactly generated, F is a right adjoint iff it preserves small limits and κ-filtered colimits.

[1] This is unraveled from Lurie HTT 5.5.7.1, 5.5.0.18, and 5.4.2.1. In particular, the second condition is usually called κ-accessibility, but we have no need for such terminology.

For functors out of $\mathrm{Ind}_\kappa(C)$, fully faithfulness and equivalence can be detected at the level of C.

> **Proposition – Functors out of Compactly Generated Categories (Lurie 5.3.5.11)**
> Let C^0 be a small ∞-category, κ a regular cardinal, $C = \mathrm{Ind}_\kappa(C^0)$ and D an ∞-category admitting κ-filtered colimits. Let D^κ be the full subcategory of D consisting of κ-compact objects. Let $F : C \to D$ be a functor preserving κ-filtered colimits and $F_0 : C^0 \to D$ its restriction along the Yoneda embedding $C^0 \to C$.
>
> - If F_0 is fully faithful and its essential image lands in D^κ, then F is fully faithful.
> - F is an equivalence iff the following are true :
>   - F_0 fully faithful
>   - F_0 factors through D^κ
>   - all objects of D are κ-filtered colimits of diagrams in D with objects in the image of F_0.
>
>   In particular, for any full subcategory $\tilde{C} \subseteq C^\kappa$ which generates C under κ-filtered colimits, we have $\mathrm{Ind}_\kappa(\tilde{C}) \xrightarrow{\sim} C$.

Furthermore, the Yoneda embedding $C^0 \to C$ factors through C^κ. The fully faithful functor $C^0 \to C^\kappa$ is not in general an equivalence, however it does exhibit C^κ as the *idempotent completion* of C^0. (Lurie 5.4.2.4.) So if C^0 is idempotent complete, then we recover the κ-compact objects of C as precisely (the essential image of) C^0. One doesn't need to know the general ∞-categorical definition for idempotent completion since we will only be in the stable setting, in which we have the following characterisation.

> **Proposition – Idempotent Completeness in Stable Case (Lurie HA 1.2.4.6)**
> Let C be a stable ∞-category. Then C is idempotent complete iff hC is as a 1-category, i.e. for every morphism $e : B \to B$ such that $e^2 = e$, there exists a retract of $s : A \rightleftarrows B : r$ that exhibits $e = rs$.
>
> If the above is the case, then for any regular cardinal κ we have $C \xrightarrow{\sim} (\mathrm{Ind}_\kappa(C))^\kappa$.

One should not worry about having to check presentability of a stable ∞-category since all the computable examples are in the compactly generated case. As it turns out, this can be checked at the level of the triangulated categories.

> **Proposition – Compact Generation of Stable Infinity Categories**
> Let C be a stable ∞-category. We say an object X *generates* C when for all objects Y in C, $hC(X, Y) = 0$ implies $Y \simeq 0$. Then the following are true :
>
> 1. Suppose we have that :
>   - C has small coproducts
>   - hC is locally small
>   - There exists a compact object X which generates C.

Define the following sequence of full subcategories of C :

$$\begin{aligned} C(0) &:= \text{full subcategory of } C \text{ spanned by } \{X[n]\}_{n\in\mathbb{Z}} \\ C(k+1) &:= \text{full subcategory of } C \text{ spanned by finite colimits of objects in } C(k) \\ C(\omega) &:= \bigcup_{n\in\mathbb{N}} C(n) \end{aligned}$$

Then $C(\omega)$ equivalent to a small ∞-category and $\operatorname{Ind} C(\omega) \xrightarrow{\simeq} C$. In particular, C is compactly generated.

2. (Lurie HA 1.4.4.1) For an object X in C, X is compact if and only if for every morphism $f : X \to \coprod_{i\in I} Y_i$ in C, there exists a finite subset $I_0 \subseteq I$ such that in hC, f factors through $\coprod_{i\in I_0} Y_i \to \coprod_{i\in I} Y_i$.

Proof. We omit the proof of (2). The following proof of (1) is adapted from the proof of Lurie HA 1.4.4.2 which is about general presentable ∞-categories, rather than the compactly generated special case.

Showing $C(\omega)$ is equivalent to a small ∞-category is a set-theoretic issue that's not very interesting, so we will use the following result without proof.

> *Lemma. Let C be an ∞-category and κ a regular cardinal which is uncountable. Then there exists a κ-small ∞-category $D \xrightarrow{sim} C$ if and only if hC is κ-small and C is locally κ-small, i.e. for every morphism $f : X \to Y$ in C, $\pi_0 C(X, Y)$ and $\pi_{i>0}(C(X, Y), f)$ are κ-small.*
>
> *Proof.* See Lurie HTT 5.4.1.2. ■

This part of the proof is as in Lurie HA 1.4.4.2. Since $hC(\omega)$ is small, in order to show $C(\omega)$ is equivalent to a small ∞-category, it remains to show that C is locally small. We have $\pi_0(X, Y) = hC(X, Y)$ is small by assumption. For the higher homotopy groups, note that $C(X, Y) \simeq \Omega C(\Omega X, Y)$ therefore the we can WLOG assume $f = 0$ for computation of the higher homotopy groups. Then $\pi_{i>0}(C(X, Y), 0) \simeq hC(X[i], Y)$ which is small again.

Now we may WLOG assume $C(\omega)$ is small. By construction, $C(\omega)$ is closed under translations and cofibers. It follows from the stability of C that $C(\omega)$ is also closed under fibers and hence a stable full subcategory. Then by the universal property of ind-completions, the inclusion $C(\omega) \to C$ factors into

$$C(\omega) \xrightarrow{\text{Yoneda}} \operatorname{Ind} C(\omega) \xrightarrow{j} C$$

where j is the left Kan extension of $C(\omega) \to C$. The inclusion $C(\omega) \subseteq C$ is fully faithful and every object in $C(\omega)$ is compact, therefore j is fully faithful.

It remains to show the essential image of j is all of C. We will achieve this by explicitly computing an inverse. We saw above that C is locally small. So by the universal property of presheaf ∞-categories applied to the inclusion $i : C(\omega) \to C$, we have an adjunction

$$\begin{tikzcd}
\operatorname{Ind} C(\omega) \arrow[d, "\subseteq"'] \arrow[drr, "j"] & & \\
\operatorname{PSh} C(\omega) \arrow[rr, shift left, "i_!"] \arrow[rr, phantom, "\perp"] & & C \arrow[ll, shift left, "i^*"]
\end{tikzcd}$$

where $i_!$ is the left Kan extension of i along $C(\omega) \to \operatorname{PSh} C(\omega)$. The above diagram commutes up to isomorphism because both j and $i_!$ restrict to give $i : C(\omega) \to C$. Now, the fact that i preserves finite colimits implies that $i^*C \subseteq \operatorname{Ind} C(\omega)$. Thus, we have an adjunction $j \dashv i^* : \operatorname{Ind} C(\omega) \rightleftarrows C$. It suffices to show that for all Y in C, $j(i^*Y) \xrightarrow{\sim} Y$. Let $K \to j(i^*Y) \to Y$ be a fiber sequence. It suffices to show $K \simeq 0$. Since X is a generator of C, this is equivalent to showing

$$0 \simeq hC(X, K) \simeq \pi_0(\operatorname{Ind} C(\omega))(X, i^*K)$$

where the latter isomorphism comes from the adjunction $j \dashv i^*$. This follows by applying the right adjoint i^* to the fiber sequence :

$$\begin{tikzcd}
K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & j(i^*Y) \arrow[d] \\
0 \arrow[r] & Y
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
i^*K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & i^*(j(i^*Y)) \arrow[d, "\sim"] \\
0 \arrow[r] & i^*Y
\end{tikzcd}$$

The equivalence $i^*ji^*Y \to i^*Y$ comes from the adjunction $j \dashv i^*$. Therefore $i^*K \simeq 0$ and hence $\pi_0(\operatorname{Ind} C(\omega))(X, i^*K) \simeq 0$ as desired.

$\square$

- An important example of a compactly generated stable ∞-category is Vec.

> **Proposition**
>
> Let k be a field. Then there exists an ∞-category Vec called the *right derived ∞-category of k-vector spaces* with the following properties :
>
> - (Lurie HA 1.3.2.18) Vec is stable.
> - (Lurie HA 1.3.4.4 - Universal Property as Localisation) There is a functor $l : \operatorname{Ch}^-(k) \to \operatorname{Vec}$ with the property that for all ∞-categories E, restricting along l yields a fully faithful functor $\operatorname{Fun}(\operatorname{Vec}, E) \to \operatorname{Fun}(\operatorname{Ch}^-(k), E)$ with essential image consisting of functors $\operatorname{Ch}^-(k) \to E$ which invert quasi-isomorphisms.
> - (Lurie HA 1.3.2.9) $h\operatorname{Vec}$ gives the usual 1-category right derived category of k-vector spaces.

Consequently, for $X, Y \in \mathrm{Ch}^-(k)$, we have

$$\pi_n \mathrm{Vec}(X,Y) = \pi_0 \Omega^n \mathrm{Vec}(X,Y) \simeq \pi_0 \mathrm{Vec}(X, Y[n]) =: \mathrm{Ext}^n(X,Y)$$

– $1\text{-}\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{cts}}$ has symmetric monoidal structure $\otimes$ via the *Lurie tensor product*.

We won't really need to know anything about the tensor product other than its universal property.

(IP : Explain how practically speaking, it suffices to know the univeral property because we will work with compactly generated dg-categories, meaning computation will come down to compact generators and their homs.)

(IP : Give impression of symmetric monoidal infinity categories via Lawvere theory perspective.)

Unanswered Q : is this only used because the theory of commutative dg-algebras compromises in positive characteristic? A : No. Characteristic zero is also used later in equivalence of Lie algebras and formal moduli problems. But I have no time to look into this.

– $(\mathrm{Vec}, \otimes)$ is the stable symmetric monoidal (right bounded) derived infinity category of complexes of k-vector spaces. Its homotopy category $h(\mathrm{Vec})$ is 1-categorical localisation of the category of complexes of k-vectors spaces at quasi-isomorphisms, and has the usual t-structure. The heart of Vec is the usual abelian category of k-vector spaces. We use cohomological degree, where negative cohomological degree refers to homological degree.

Practically speaking, computations tensor product are done by using the projective model structure on the category of complexes of k-vectors spaces.

– $(\mathrm{Vec}, \otimes)$ can be seen as an commutative algebra object in the symmetric monoidal infinity category $1\text{-}\mathrm{Cat}^{st,cocompl}_{cts}$. Then DGCat_{cts} denotes the infinity category of left modules over Vec inside $(\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{cts}}$.

– IP : derived rings stuff - Lurie HTT 5.5.9.3

– IP : modules over derived rings as symmetric monoidal ∞-cat - Lurie HA 7.1.2.13. Subtlety about different model structures is Lurie HA 7.1.2.9.

2.2 Derived Schemes

Remark – Why use derived-ness?. A simple reason is that base change theorems break without flatness assumptions. Consider the following example. Let $A := k[t]/(t)^2$ where k is a field and let $\mathrm{Spec}\, k \to \mathrm{Spec}\, A$ be the closed embedding of $t = 0$. Then the *classical* fiber product gives the following.

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r, "\mathbb{1}"] \arrow[d, "\mathbb{1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}\, k \arrow[d, "i"] \\ \mathrm{Spec}\, k \arrow[r, "i"'] & \mathrm{Spec}\, A \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathrm{QCoh}\, k \arrow[d, "\mathbb{1}"'] \arrow[dr, phantom, "\not\simeq"] & \mathrm{QCoh}\, k \arrow[l, "\mathbb{1}"'] \arrow[d, "Ri_*"] \\ \mathrm{QCoh}\, k & \mathrm{QCoh}\, A \arrow[l, "Li^*"] \end{tikzcd}$$

Indeed the diagram on the right hand side does not commute up to isomorphism since we have :

$$
\begin{tikzcd}
 & & & \vdots \arrow[d, "t\otimes 1"'] & \vdots \arrow[d, "0"'] & \\
 & & & A \otimes_A k \arrow[d, "t\otimes 1"'] & k \arrow[d, "0"'] & \\
Li^*(Ri_* k) \arrow[r, phantom, "\simeq"] & Li^*(k) \arrow[r, phantom, "\simeq"] & k \otimes^L_A k \arrow[r, phantom, "\simeq"] & A \otimes_A k \arrow[r, phantom, "\simeq"] & k \arrow[r, phantom, "\not\simeq"] & k
\end{tikzcd}
$$

Going derived fixes this because we have the isomorphism

$$\_ \otimes^L_A k \simeq \_ \otimes^L_k (k \otimes^L_A k)$$

IP : explain the setup of prestacks, i.e. the following diagram.

GR1, Ch 2, 1.7.6.

$$
\begin{tikzcd}
\mathrm{Aff}^{\mathrm{red}} \arrow[r, "\subseteq"] \arrow[d] & \mathrm{Aff} \arrow[r, "\subseteq"] \arrow[d] & \mathrm{dAff} \arrow[d] & \mathrm{PSh}\,\mathrm{dAff}^{\leq n} \arrow[r, shift right] \arrow[d, shift right, "\mathrm{Lan}"'] \arrow[d, phantom, "\dashv"] \arrow[r, phantom, "\perp"] & \mathrm{PSh}\,\mathrm{dAff}^{\leq n}_{\mathrm{ft}} \arrow[l, shift right, "\mathrm{Lan}"'] \arrow[d, shift right, "\mathrm{Lan}"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{PStk}_{\mathrm{red}} \arrow[r, shift left, "\mathrm{Lan}"] \arrow[r, phantom, "\perp"] & \mathrm{PStk}_{\mathrm{cl}} \arrow[l, shift left, "\mathrm{red}"] \arrow[r, shift left, "\mathrm{Lan}"] \arrow[r, phantom, "\perp"] & \mathrm{PStk} \arrow[l, shift left, "\mathrm{cl}"] \arrow[r, shift right] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "L"'] \arrow[d, phantom, "\dashv"] & \mathrm{PSh}\,\mathrm{dAff}^{<\infty} \arrow[l, shift right] \arrow[u, shift right, "\leq n"'] \arrow[r, shift left] \arrow[r, phantom, "\top"] \arrow[d, shift left, "\mathrm{Ran}"] \arrow[d, phantom, "\dashv"] & \mathrm{PSh}\,\mathrm{dAff}^{<\infty}_{\mathrm{ft}} \arrow[l, shift left, "\mathrm{Lan}"] \arrow[u, shift right] \\
 & & \mathrm{Stk} \arrow[u, shift right, "\subseteq"'] & \mathrm{PStk}_{\mathrm{laft}} \arrow[ul, "\subseteq"] \arrow[u, shift left] \arrow[ur, "\sim"'] &
\end{tikzcd}
$$

$$\mathrm{dSch}_{\mathrm{aft}} := \mathrm{dSch}_{\mathrm{qc}} \cap \mathrm{Stk}_{\text{ét}} \cap \mathrm{PStk}_{\mathrm{laft}}$$

- The left Kan extension functors $\mathrm{PStk}_{\mathrm{cl}} \to \mathrm{PStk}$ are obtained from the universal property of presheaf ∞-categories to inclusion of full subcategories $\mathrm{Aff} \subseteq \mathrm{dAff}$.

  Note that since small colimits in presheaf ∞-categories are computed pointwise, res preserves small colimits. We obtain that $\mathrm{res}\,\mathrm{Lan} \xrightarrow{\sim} \mathbb{1}$ since this is the case on derived affines. Thus, Lan is fully faithful. In other words, we can safely think of classical prestacks as special cases of prestacks.

  The same reasoning applies for $\mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}_{\mathrm{cl}}$ and the composite $\mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}$. So we can think of reduced prestacks as special cases of classical prestacks, and also of prestacks.

  We henceforth view $\mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}_{\mathrm{cl}} \to \mathrm{PStk}$ as inclusions of full subcategories.

- For $X \in \mathrm{PStk}$, we say X has an underlying scheme when the underlying classical prestack is a scheme.

– (GR1 Ch2 1.2.3, 1.2.7) $\mathrm{dAff}^{<\infty} := \bigcup_{0\leq n} \mathrm{dAff}^{\leq n}$ where $\mathrm{dAff}^{\leq n}$ is the full subcategory of dAff consisting of $\mathrm{Spec}\, A$ such that $\pi_{i>n} A \simeq H^{i<-n} A \simeq 0$. Derived affine schemes in $\mathrm{dAff}^{<\infty}$ are called *eventually coconnective*.[1]

Since $\mathrm{dAff}^{<\infty} \subseteq \mathrm{dAff}$ is a full subcategory, we obtain a triple of adjoints $\mathrm{Lan} \dashv \mathrm{res} \dashv \mathrm{Ran} : \mathrm{PStk} \rightleftarrows \mathrm{PSh}\, \mathrm{dAff}^{<\infty}$. Then essential image of Ran is of importance due to the following characterisation.

> **Proposition – GR1 Ch2 1.4.7**
> Let $X \in \mathrm{PStk}$. Then X is in the essential image of $\mathrm{Ran} : \mathrm{PSh}\, \mathrm{dAff}^{<\infty} \to \mathrm{PStk}$ if and only if for all $\mathrm{Spec}\, A \in \mathrm{dAff}$, we have $X(A) \xrightarrow{\sim} \varprojlim_{0\leq n} X(\tau^{\leq n} A)$. We call such prestacks *convergent*.

– $\mathrm{dAff}_{\mathrm{aft}}$ is defined as the full subcategory of dAff consisting of derived affines of *almost finite type*. This means $\mathrm{Spec}\, A$ where $H^0 A$ is finite type over the base field k and $H^{i>0} A$ are finite generated over $H^0 A$.

$\mathrm{dAff}_{\mathrm{ft}}^{\leq n} := \mathrm{dAff}^{\leq n} \cap \mathrm{dAff}_{\mathrm{aft}}$

With the same argument as in previous points, we obtain an adjunction $\mathrm{Lan} \dashv \mathrm{res} : \mathrm{PSh}\, \mathrm{dAff}_{\mathrm{ft}}^{\leq n} \rightleftarrows \mathrm{PSh}\, \mathrm{dAff}^{\leq n}$ where Lan is fully faithful so we can see the former as a full subcategory of the latter. The objects of this subategory has the following charactersation :

> **Proposition**
> Let $X \in \mathrm{PSh}\, \mathrm{dAff}^{\leq n}$. Then X lies in $\mathrm{PSh}\, \mathrm{dAff}_{\mathrm{ft}}^{\leq n}$ if and only if as a functor $(\mathrm{dAff}^{\leq n})^{\mathrm{op}} \to \mathrm{Spc}$, it preserves small filtered colimits.

$\mathrm{dAff}_{\mathrm{ft}}^{<\infty} := \mathrm{dAff}^{<\infty} \cap \mathrm{dAff}_{\mathrm{aft}} = \bigcup_{0\leq n} \mathrm{dAff}_{\mathrm{ft}}^{\leq n}$.

> **Proposition – Characterisation of Prestacks of Locally Almost Finite Type**
> Restriction along $\mathrm{dAff}_{\mathrm{ft}}^{<\infty} \subseteq \mathrm{dAff}$ gives an equivalence
>
> $$\begin{tikzcd} \mathrm{PStk}_{\mathrm{laft}} \arrow[rr, shift left, "\mathrm{res}"] & & \mathrm{PSh}\, \mathrm{dAff}_{\mathrm{ft}}^{<\infty} \arrow[ll, shift left, "\sim"] \arrow[dl, "\mathrm{Lan}"] \\ & \mathrm{PSh}\, \mathrm{dAff}^{<\infty} \arrow[ul, "\mathrm{Ran}"] & \end{tikzcd}$$
>
> where
>
> – $\mathrm{PStk}_{\mathrm{laft}}$ is the full subcategory of PStk consisting of X such that
>   1. X is convergent.
>   2. X takes filtered limits in $\mathrm{dAff}^{\leq n}$ to filtered colimits in Spc, equivalently $X^{\leq n}$ lies in $\mathrm{PSh}\, \mathrm{dAff}_{\mathrm{ft}}^{\leq n}$.

[1]The "coconnectivity" refers to *cohomological grading*. So eventually coconnective means the same thing as *"homologically eventually connective"*.

- an inverse is given by first *left* Kan extending along $\mathrm{dAff}_{\mathrm{ft}}^{<\infty} \subseteq \mathrm{dAff}^{<\infty}$ then *right* Kan extending along $\mathrm{dAff}^{<\infty} \subseteq \mathrm{dAff}$.

$\mathrm{PStk}_{\mathrm{laft}} \subseteq \mathrm{PStk}$ preserves finite limits. (GR1 Ch2 1.7.10)

$\mathrm{dAff}_{\mathrm{aft}} = \mathrm{dAff} \cap \mathrm{PStk}_{\mathrm{laft}}$ (GR1 Ch2 1.7.2)

3 IndCoherent Sheaves on Derived Schemes

3.1 Quasi-Coherent Sheaves

Definition – Covariant QCoh

There is a functor $\mathrm{QCoh}^* : \mathrm{dAff} \to \mathrm{DGCat}_{cts}^{\mathrm{op}}$ that assigns to each $A \in \mathrm{dAff}$ the *derived category of A-modules*, denoted A-Mod.

For A discrete (i.e. a commutative ring), there is the following description of A-Mod under the quasi-category model of ∞-categories summarised in a single diagram :

$$\begin{tikzcd}
N(\mathrm{Ch}\,A) \arrow[d, "\subseteq"'] \arrow[dr, "W^{-1}"] & & & \\
N_{dg}(\mathrm{Ch}\,A) \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & N_{dg}((\mathrm{Ch}\,A)_f) \arrow[l, shift left, "\supseteq"] \arrow[d, "h"] & =: & A\text{-}\mathrm{Mod} \\
 & D(A) & &
\end{tikzcd}$$

Details :

- $\mathrm{Ch}\,A$ is the honest-to-god dg-category of chain complexes of honest-to-god A-modules and $D(A)$ is the category of complexes of injectives
- $\mathrm{Ch}\,A$ has a model structure such that cofibrations are degree-wise injections and weak equivalences are quasi-isomorphisms (Lurie HA 1.3.5.3). Although the class of fibrations are defined abstractly as those satisfying right lifting with respect to acyclic cofibrations, it turns out that any fibrant complex must be degree-wise injective, and partially conversely, any bounded above complex of injectives is fibrant (Lurie HA 1.3.5.6). $(\mathrm{Ch}\,A)_f$ denotes the full subcategory of fibrant complexes.
- N denotes the nerve functor which converts 1-categories to simplicial sets, which have the property of being ∞-categories. N_{dg} denotes the dg-nerve functor which achieves the same thing for honest-to-god dg-catgeory categories. (See Kerodon 2.5.3 for a construction.)
- h is the truncation of an infinity category to a 1-category by taking its homotopy catgeory. It is

the left adjoint to N. (See Kerodon 1.2.5 for a construction.)

We have that the homotopy category of A-Mod gives the usual derived category of A-modules, as in classical algebraic geometry.

- L is a left adjoint to the inclusion $N_{dg}((\mathrm{Ch}\,A)_f) \subseteq N_{dg}(\mathrm{Ch}\,A)$. Intuitively, for every complex $M_\bullet$, there exists a acyclic cofibration $M_\bullet \to I_\bullet$ to fibrant $I_\bullet$ and this is initial in the category of arrows from $M_\bullet$ into $N_{dg}((\mathrm{Ch}\,A)_f)$ (Lurie HA 1.3.5.12). This means for each $M_\bullet$, such a morphism $M_\bullet \to I_\bullet$ is unique up to equivalence and assembles to the desired functor L. Practically speaking, $L(M_\bullet) \simeq I_\bullet$.
- The composition $N(\mathrm{Ch}\,A) \to N_{dg}((\mathrm{Ch}\,A)_f)$ exhibits the latter as the ∞-categorical localisation of the former at quasi-isomorphisms (Lurie HA 1.3.5.15). This matches the standard treatment in classical algebraic geometry : the localisation functor from $\mathrm{Ch}\,A$ to $D(A)$ takes a complex and resolves it by injecting it quasi-isomorphically into a complex of injectives.

- $\mathrm{dAff}^{\mathrm{op}}$ is localisation of commutative dg-algebras w.r.t. suitable model structure. (Alternatively, localisation of simplicial commutative rings w.r.t. suitable model structure. However, then need to show is equivalent to commutative algebra objects in Vec.)
- for $\mathrm{Spec}\,A \in \mathrm{dAff}$, A-Mod is ∞-category of left modules over A in Vec. Can be realised as localisation of left modules over A as a commutative dg-algebra w.r.t. some model structure.
- $\mathrm{QCoh}^* : \mathrm{dAff} \to \mathrm{DGCat}_{\mathrm{cts}}^{\mathrm{op}}$ works and for each $f : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$, $f^* : \mathrm{QCoh}\,A \to \mathrm{QCoh}\,B$ can be realised as $A \otimes_k^L \_$.
- for $\mathrm{Spec}\,A \in \mathrm{dAff}$, $\mathrm{QCoh}\,A$ has obvious t-structure.

Definition – Quasi-coherent Sheaves on Prestacks
We define $\mathrm{QCoh}^* : \mathrm{PStk} \to \mathrm{DGCat}_{\mathrm{cts}}^{\mathrm{op}}$ as the left Kan extension of $\mathrm{QCoh}^* : \mathrm{dAff} \to \mathrm{DGCat}_{\mathrm{cts}}^{\mathrm{op}}$.

TODO : t-structure, perfect sheaves, coherent sheaves.

3.2 Why Ind-Coherent Sheaves?

- Want to compute. Work with compactly generated $C \in \mathrm{DGCat}_{cts}$ so that computations come down to compact generators and their homs.

  $\mathrm{QCoh}\,X$ compactly generated. But compact objects of $\mathrm{QCoh}\,X$ aren't preserved under pushforward of proper morphisms. However, $\mathrm{Coh}\,X$ is. So use $\mathrm{IndCoh}\,X$ to make $\mathrm{Coh}\,X$ into the compact objects.
- Want to interpret rep theory in AG terms. In particular, induction of reps is *left* adjoint to pullback. But pushforward for QCoh is *right* adjoint to pullback. IndCoh fixes this.
- Integral transform for *inf-schemes*. (Includes de Rham spaces, quotients by formal groupoids)
- Serre duality as self-duality of $\mathrm{IndCoh}\,X$, also for *inf-schemes*. Explains classical Serre duality and duality for D-modules as same phenomenon.

Also, six functor formalism should be viewed from perspective of correspondences.

3.3 Correspondences and Six Functor Formalisms

One of the main points of Gaitsgory-Rozenblyum is that the assignment $X \rightsquigarrow \mathrm{IndCoh}\, X$ should be viewed not as two functors $\mathrm{IndCoh}_* : \mathrm{dSch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cts}}$ and $\mathrm{IndCoh}^! : \mathrm{dSch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cts}}$ but rather as a single functor between $(\infty, 2)$-*categories* :

$$\mathrm{IndCoh} : \mathrm{Corr}(\mathrm{dSch}_{\mathrm{aft}})^{proper}_{all,all} \to \mathrm{DGCat}^{(\infty,2)}_{\mathrm{cts}}$$

The precise construction of IndCoh in this form is beyond the scope of this paper. However, let us describe what this gives us and leave the interested reader to find details in GR1, Ch 5.

IP : describe how correspondences encode integral transforms, proper base change.

3.4 Ind-Coherent Sheaves on a single Derived Scheme

3.5 Relation between Quasi-Coherent Sheaves and Ind-Coherent Sheaves

Definition – Ind-coherent sheaves on laft prestacks (GR1 Ch5 3.4.1)

We define $\mathrm{IndCoh}^! : \mathrm{PStk}_{\mathrm{laft}} \to (\mathrm{DGCat}_{\mathrm{cts}})^{\mathrm{op}}$ as the left Kan extension of $\mathrm{IndCoh}^! : \mathrm{dAff}^{<\infty}_{\mathrm{ft}} \to (\mathrm{DGCat}_{\mathrm{cts}})^{\mathrm{op}}$ via the universal property of presheaf ∞-categories .

(Something about $\mathrm{IndCoh}^! \to \mathrm{QCoh}^*$)

There is also a functor turning quasi-coherent sheaves into ind-coherent sheaves.

Definition – Action of $\mathrm{QCoh}\, X$ on $\mathrm{IndCoh}\, X$

Let $X \in \mathrm{dSch}_{\mathrm{aft}}$. It is shown in BZFN-3.19 that $\mathrm{QCoh}\, X = \mathrm{Ind}((\mathrm{QCoh}\, X)^c)$, where $(\mathrm{QCoh}\, X)^c$ is the full subcategory of compact objects, which in this case coincides with $\mathrm{Perf}\, X$, the full subcategory of perfect complexes.

Then the tensor product $\mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, X \to \mathrm{QCoh}\, X$ which is a morphism in $\mathrm{DGCat}_{\mathrm{cts}}$ restricts to a tensor product $\mathrm{Perf}\, X \otimes \mathrm{Coh}\, X \to \mathrm{Coh}\, X$ which is a morphism in DGCat. By taking ind-objects, we obtain the following morphism in DGCat,

$$\mathrm{QCoh}\, X \otimes \mathrm{IndCoh}\, X \to \mathrm{IndCoh}\, X$$

Restricting along $\mathrm{Vec} \to \mathrm{IndCoh}\, X$ corresponding to the unit ω_X of $\otimes^!$ gives

$$\Upsilon_X : \mathrm{QCoh}\, X \to \mathrm{IndCoh}\, X$$

which on objects gives $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_X$.

It is shown in GR1, Ch 6, 3.3 that Υ can be made into a natural transformation $\mathrm{QCoh}^* \to \mathrm{IndCoh}^!$ as functors $\mathrm{dSch}_{\mathrm{aft}} \to (\mathrm{DGCat}_{\mathrm{cts}})^{op}$. To extend to all laft prestacks, we first restrict to $\mathrm{dAff}^{<\infty}_{\mathrm{ft}} \subseteq \mathrm{dSch}_{\mathrm{aft}}$.

$$\begin{tikzcd} \mathrm{QCoh}^* \arrow[r, "\Upsilon"] & \mathrm{IndCoh}^! & \text{in} & \mathrm{Fun}(\mathrm{dAff}_{\mathrm{ft}}^{<\infty}, \mathrm{DGCat}_{\mathrm{cts}}^{\mathrm{op}}) \arrow[d, rightsquigarrow, "\mathrm{Lan}"'] \\ \widetilde{\mathrm{QCoh}}^* \arrow[r, "\Upsilon"] & \mathrm{IndCoh}^! & \text{in} & \mathrm{Fun}(\mathrm{PStk}_{\mathrm{laft}}, \mathrm{DGCat}_{\mathrm{cts}}^{\mathrm{op}}) \end{tikzcd}$$

For each individual $X \in \mathrm{PStk}_{\mathrm{laft}}$, this means

$$\widetilde{\mathrm{QCoh}}^*(X) \simeq \varprojlim_{S \in \mathrm{dAff}_{\mathrm{ft}}^{<\infty}/X} \mathrm{QCoh}^*(X)$$
$$\mathrm{IndCoh}^!(X) \simeq \varprojlim_{S \in \mathrm{dAff}_{\mathrm{ft}}^{<\infty}/X} \mathrm{IndCoh}^!(X)$$

Finally, we obtain $\Upsilon : \mathrm{QCoh}^* \to \mathrm{IndCoh}^!$ on all of $\mathrm{PStk}_{\mathrm{laft}}$ as the following composition :

$$\begin{tikzcd} \mathrm{QCoh}^* \arrow[r] \arrow[dr, "\Upsilon"'] & \widetilde{\mathrm{QCoh}}^* \arrow[d, "\Upsilon"] \\ & \mathrm{IndCoh}^! \end{tikzcd}$$

where the top morphism has components $\mathrm{QCoh}^*(X) \to \widetilde{\mathrm{QCoh}}^*(X)$ induced by the inclusion $\mathrm{dAff}_{\mathrm{ft}}^{<\infty}/X \to \mathrm{dAff}/X$.

3.6 Pushward and Pullback across Various Morphisms

TODO

- $j^! = j^*$ for open immersion
- $p^!$ for proper morphisms
- how $p^!$ relates to sheaves supported on closed subscheme when p is closed immersion
- proper base change for IndCoh.
- A note on how GR makes $p^!$ for general morphisms by proving the space of factorisations of a morphism f into $f = pj$ with j open immersion, p proper is contractible.

3.7 Integral Transforms

TODO :

- integral transform for QCoh only needs integral transform for derived affines + dualizability.

- integal transform for derived affines
- integral transform for IndCoh.

3.8 Serre Duality

TODO :

- describe how GR's Serre duality concretely recovers classical Serre duality
- IP : proof of $\omega_X = \Omega_X^d[d]$ in classical smooth case.

4 Crystals and D-modules

4.1 A note on inf-schemes

Say something about making IndCoh base change for larger class of spaces and morphisms that includes $X \to X_{\mathrm{dR}}$.

4.2 Left and Right Crystals

> **Definition – de Rham space**
>
> By the theory of left and right Kan extensions, we have the following adjunctions :
>
> $$\mathrm{Aff}^{\mathrm{red}} \xrightarrow{\subseteq} \mathrm{dAff} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathrm{PStk}_{\mathrm{red}} \arrow[r, bend left, "\mathrm{Lan}"] \arrow[r, bend right, "\mathrm{Ran}"'] & \mathrm{PStk} \arrow[l, "\mathrm{res}"'] \end{tikzcd}$$
>
> The functor which takes de Rham space is defined to be $\mathrm{dR} := \mathrm{Ran}\,\mathrm{res}$.
>
> The unit of the adjunction $\mathrm{res} \dashv \mathrm{Ran}$ is a natural transformation $\mathbb{1}_{\mathrm{PStk}} \to \mathrm{dR}$. For $X \in \mathrm{PStk}$, we write $p_{\mathrm{dR}} : X \to X_{\mathrm{dR}}$ for the component of the natural transformation.

Remark. Let $X \in \mathrm{PStk}$ and $\mathrm{Spec}\, A \in \mathrm{dAff}$. Then points of X_{dR} has the following description :

$$\mathrm{PStk}(\mathrm{Spec}\, A, X_{\mathrm{dR}}) \simeq \mathrm{PStk}_{\mathrm{red}}(\mathrm{Spec}\, A_{\mathrm{red}}, X_{\mathrm{red}}) \simeq \mathrm{PStk}(\mathrm{Spec}\, A_{\mathrm{red}}, X)$$

The first equivalence is by definition of Ran and the second is by the fully faithful embedding $\mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}$. In particular, the morphism $p_{\mathrm{dR}} : X \to X_{\mathrm{dR}}$ gives at the level of points :

$$\mathrm{PStk}(\mathrm{Spec}\, A, X) \to \mathrm{PStk}(\mathrm{Spec}\, A_{\mathrm{red}}, X)$$

Remark – Relation to Grothendieck's infinitesimal site. Let $X \in \mathrm{PStk}$. For simplicity, we assume the underlying classical prestack of X is zero-truncated, meaning for every $\mathrm{Spec}\, A \in \mathrm{Aff}$ we have $X(A)$ is a set. (E.g. if X has an underlying scheme.) We will describe the underlying classical prestack of X_{dR}, i.e. the over category $\mathrm{Aff}/X_{\mathrm{dR}}$.

First note that since X^{cl} is zero-truncated, so is $X^{\mathrm{cl}}_{\mathrm{dR}}$. This means for $\mathrm{Spec}\, A \in \mathrm{Aff}$, $X_{\mathrm{dR}}(A)$ is a set so we do not have to worry about any homotopical phenomena. Then given two classical points $x : \mathrm{Spec}\, A \to X_{\mathrm{dR}}$

and $y : \operatorname{Spec} B \to X_{\mathrm{dR}}$, a morphism $f : y \to x$ over X_{dR} is precisely a morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ such that $xf = y$ at the reduced level.

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d, "f"'] \arrow[dr, "y"] & \\ \operatorname{Spec} A \arrow[r, "x"'] & X_{\mathrm{dR}} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \operatorname{Spec} B \arrow[d, "f"'] & \operatorname{Spec} B^{\mathrm{red}} \arrow[l, hook'] \arrow[d, "f^{\mathrm{red}}"'] \arrow[dr, "y^{\mathrm{red}}"] & \\ \operatorname{Spec} A & \operatorname{Spec} A^{\mathrm{red}} \arrow[l, hook'] \arrow[r, "x^{\mathrm{red}}"'] & X \end{tikzcd}$$

Now assuming X has a underlying scheme, Grothendieck's infinitesimal site of X is precisely the full subcategory of $\mathrm{Aff}/X_{\mathrm{dR}}$ with objects $x : \operatorname{Spec} A \to X_{\mathrm{dR}}$ such that $\operatorname{Spec} A^{\mathrm{red}} \to X$ is an open immersion.

Remark – What about $\operatorname{Lan}\operatorname{res}$. From the discussion on prestacks, the left Kan extension functor $\operatorname{Lan} : \mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}$ simply takes a reduced prestack and views it as a general prestack. Thus $\operatorname{Lan}\operatorname{res}$ is the functor that takes the underlying reduced prestack of a general prestack.

Remark. We actually have more than just an inclusion $\mathrm{Aff}^{\mathrm{red}} \to \mathrm{dAff}$ but an adjunction :

$$\begin{tikzcd}[column sep=huge] \mathrm{Aff}^{\mathrm{red}} \arrow[r, shift left=2, "\subseteq"] \arrow[r, phantom, "\perp"] & \mathrm{dAff} \arrow[l, shift left=2, "\operatorname{Spec}(\pi_0 A)_{\mathrm{red}} \leftarrow\!\shortmid \operatorname{Spec} A"] \end{tikzcd}$$

From this, we can also deduce that dR preserves both small colimits and small limits which is useful.

> **Proposition**
> The functor dR taking de Rham spaces preserves small colimits and small limits.

Proof. We proceed by a slightly different proof to GR Crys 1.1.4. To show $\mathrm{dR} = \operatorname{Ran}\operatorname{res}$ preserves small colimits and small limits, it suffices that Ran is a left adjoint. By applying the theory of left and right Kan extensions of presheaves to the adjunction $\mathrm{Aff}^{\mathrm{red}} \rightleftarrows \mathrm{dAff}$, we obtain *two* adjoint triples :

$$\begin{tikzcd} \mathrm{Aff}^{\mathrm{red}} \arrow[r, shift left=2, "\subseteq"] \arrow[r, phantom, "\perp"] & \mathrm{dAff} \arrow[l, shift left=2, "\mathrm{red}"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{array}{c} \begin{tikzcd}[column sep=huge] \mathrm{PStk}_{\mathrm{red}} \arrow[r, shift left=4, "\operatorname{Lan}"] \arrow[r, shift right=4, "\operatorname{Ran}"'] & \mathrm{PStk} \arrow[l, "\mathrm{res}"' description] \end{tikzcd} \\ \\ \begin{tikzcd}[column sep=huge] \mathrm{PStk}_{\mathrm{red}} \arrow[r, "\mathrm{res}" description] & \mathrm{PStk} \arrow[l, shift right=4, "\operatorname{Lan}"'] \arrow[l, shift left=4, "\operatorname{Ran}"] \end{tikzcd} \end{array}$$

The point is that we have a commuting square :

$$\begin{tikzcd}
\mathrm{Aff}^{\mathrm{red}} \arrow[d] & \mathrm{dAff} \arrow[l, "\mathrm{red}"'] \arrow[d] \\
\mathrm{PStk}_{\mathrm{red}} & \mathrm{PStk} \arrow[l, "\mathrm{res}"]
\end{tikzcd}$$

So res in the top adjunction is isomorphic to Lan in the bottom adjunction, hence by uniqueness of adjoint functors we obtain a quadruple of adjoints :

$$\begin{tikzcd}
\mathrm{PStk}_{\mathrm{red}} \arrow[r, shift left=6, "\mathrm{Lan}"] \arrow[r, shift right=2, "\mathrm{Ran}" description] & \mathrm{PStk} \arrow[l, shift right=-2, "\mathrm{res}" description] \arrow[l, shift left=6]
\end{tikzcd}$$

This proves Ran is a left adjoint and hence preserves small colimits.

□

> **Definition – Left and Right Crystals**
> Let $X \in \mathrm{PStk}_{\mathrm{laft}}$. Then we define the *$\infty$-category of left crystals on X* to be
>
> $$\mathrm{Crys}^L(X) := \mathrm{QCoh}\, X_{\mathrm{dR}}$$
>
> The *∞-category of right crystals on X* is defined to be
>
> $$\mathrm{Crys}^R(X) := \mathrm{IndCoh}\, X_{\mathrm{dR}}$$

Technically speaking, for the definition of right crystals to make sense, we need to show that X_{dR} is also locally almost finite type. Indeed, we have the following.

> **Proposition – $\mathrm{PStk}_{\mathrm{laft}}$ is closed under dR (GR Crys 1.3.3)**
> Let $X \in \mathrm{PStk}_{\mathrm{laft}}$. Then $X_{\mathrm{dR}} \in \mathrm{PStk}_{\mathrm{laft}}$.

Proof. We first show the convergence of X_{dR}. This follows from the following diagram which commutes up to isomorphism :

$$\begin{tikzcd}
X(A_{\mathrm{red}}) \arrow[r, "\sim"] \arrow[d, "="'] & \varprojlim_n X(A_{\mathrm{red}}) \arrow[d, "\sim"] \\
X_{\mathrm{dR}}(A) \arrow[r] & \varprojlim_n X_{\mathrm{dR}}(\tau^{\leq n} A)
\end{tikzcd}$$

It remains to show X_{dR} takes filtered limits in $\mathrm{dAff}^{\leq n}$ to filtered colimits in Spc. By assumption, X does this so it suffices to show that $\mathrm{red} : \mathrm{dAff}^{\leq n} \to \mathrm{Aff}^{\mathrm{red}}$ preserves filtered limits. This is a composition $\mathrm{dAff}^{\leq n} \to \mathrm{Aff} \to \mathrm{Aff}^{\mathrm{red}}$ where the first part takes H^0 and the second part takes reduction. Both functors preserve filtered limits. $\square$

Remark. A specific situation worth noting is when the morphism $X \to X_{\mathrm{dR}}$ exhibits X_{dR} as the geometric realisation of its Cech nerve $\check{C}(X/X_{\mathrm{dR}})$, i.e. when $X \to X_{\mathrm{dR}}$ is an effective epimorphism.[1]

Proposition

Let $X \in \mathrm{PStk}_{\mathrm{laft}}$. Then the following are equivalent :

1. For all $S \in \mathrm{dAff}$, the map of sets $\pi_0 X(S) \to \pi_0 X(S_{\mathrm{red}})$ is surjective.
2. $X \to X_{\mathrm{dR}}$ is an effective epimorphism in PStk.

We say X is *classically formally smooth* when any (and thus all) of the above are satisfied. In this case, we have

$$\mathrm{Crys}^R X \xrightarrow{\sim} \varprojlim \mathrm{IndCoh}(\check{C}(X/X_{\mathrm{dR}}))$$

$$\mathrm{Crys}^L X \xrightarrow{\sim} \varprojlim \mathrm{QCoh}(\check{C}(X/X_{\mathrm{dR}}))$$

i.e. "crystals are the same as sheaves equipped with equivariance with respect to the infinitesimal groupoid".

Proof. For the equivalence of (1) and (2). Colimits in $\mathrm{PStk} = \mathrm{PSh}\,\mathrm{dAff}$ are computed pointwise, the lemma is equivalent to claiming that $X(S) \to X(S_{\mathrm{red}})$ is effectively epic if and only if it is surjective on π_0. This is a non-trivial fact about ∞-groupoids which we will assume.

[1]For a definition of Cech nerves, see Lurie HTT 6.1.2.11.

Lemma. *Let $q : X \to Y$ be a morphism of ∞-groupoids. Then q is an effective epimorphism if and only if it is surjective on π_0.*

Proof. We defer the interested reader to Lurie HTT 6.1.2.14. ■

Now assume X is classically formally smooth. Since $\mathrm{QCoh}^* : \mathrm{PStk} \to (\mathrm{DGCat}_{\mathrm{cts}})^{\mathrm{op}}$ is the left Kan extension of its restriction to dAff, it preserves small colimits and in particular geometric realisations. We thus obtain the equivalence $\mathrm{Crys}^L X \xrightarrow{\sim} \varprojlim \mathrm{QCoh}(\check{C}(X/X_{\mathrm{dR}}))$.

We want to use the same argument for right crystals, however the subtlety is that $\mathrm{IndCoh}^!$ is *not* defined on all of PStk but rather just $\mathrm{PStk}_{\mathrm{laft}}$.[1] It suffices to show that the Cech nerve of $X \to X_{\mathrm{dR}}$ lies in $\mathrm{PStk}_{\mathrm{laft}}$ and that $X \to X_{\mathrm{dR}}$ is an effective epimorphism *in* $\mathrm{PStk}_{\mathrm{laft}}$. For the first part, note that the morphism $X \to X_{\mathrm{dR}}$ is in $\mathrm{PStk}_{\mathrm{laft}}$ because dR preserves laft-ness. Since $\mathrm{PStk}_{\mathrm{laft}}$ is closed under finite limits, we obtain that the Cech nerve of $X \to X_{\mathrm{dR}}$ in $\mathrm{PStk}_{\mathrm{laft}}$ agrees with the one in PStk. For the second part, we use $\mathrm{res} : \mathrm{PStk}_{\mathrm{laft}} \xrightarrow{\sim} \mathrm{PSh}\,\mathrm{dAff}_{\mathrm{ft}}^{<\infty}$. Since colimits are computed pointwise, by the lemma above it suffices to show that $X(A) \to X_{\mathrm{dR}}(A)$ is a surjection on π_0 for all $\mathrm{Spec}\, A \in \mathrm{dAff}_{\mathrm{ft}}^{<\infty}$. This is true by assumption. □

Here is a sufficient condition to be classically formally smooth, whose proof we won't go into.

Proposition – Comparison of classically formally smooth and smoothness

Let X be a laft smooth classical scheme of finite type over k. Then X is classically formally smooth when considered as a prestack.

Proof. Omitted. It is shown in GR DGINDSCH 8.4.2 that smooth classical schemes of finite type over k are *formally smooth* when considered as prestacks.[2] With the additional assumption of laft-ness, this implies classically formally smooth. □

4.3 Induction of Left and Right Crystals

The main tool is the theory of descent as formulated in Lurie HA 4.7.5.2.

Proposition – Descent

Let $C^\bullet : \Delta \to (\infty, 1)\text{-Cat}$. Suppose $C^\bullet$ has the following property :

- for all $\alpha : [m] \to [n]$ in Δ, then the following commutative square is *left-adjointable* :

Let C be the underlying ∞-category of the limit $\varprojlim C^\bullet$. (TODO)

TODO : Describe induction of right crystals, how to get induction for left crystals from induction of right crystals.

[1] This is GR Crys 2.3.11. However, the proof in the reference simply claims that the argument is the same as for left crystals. We provide some unclear details.

[2] GR's usage of the term "formally smooth" does *not* immediately match the usual usage. The interested reader may find the definition in the reference provided.

4.4 Equivalence of Left Crystals and D-modules

Proposition

Let X be a smooth classical scheme of locally almost finite type. Then there is an equivalence $\mathrm{Crys}^L(X) \simeq \mathrm{Diff}_X\mathrm{Mod}$ such that the following commutes up to natural isomorphism :

$$\begin{tikzcd} \mathrm{Crys}^L(X) \arrow[r, "\sim"] \arrow[d, "\mathrm{oblv}^L"'] & \mathrm{Diff}_X\mathrm{Mod} \arrow[d] \\ \mathrm{QCoh}\, X \arrow[r, "\mathbb{1}"'] & \mathrm{QCoh}\, X \end{tikzcd}$$

The right vertical morphism is the forgetful functor.

Proof. Outline :

1. $\mathrm{Crys}^L \rightleftarrows \mathrm{QCoh}\, X$ and $\mathrm{Diff}_X\mathrm{Mod} \rightleftarrows \mathrm{QCoh}\, X$ are both monadic. So they are determined by their monads.
2. Under the integral transform equivalence $\mathrm{QCoh}(X \times X) \simeq \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{QCoh}\, X, \mathrm{QCoh}\, X)$, algebra objects on the left correspond to algebra objects on the right, i.e. monads. Therefore the monads $\mathrm{oblv}^L \mathrm{ind}^L$ and $\mathrm{Diff}_X \otimes \_$ correspond to quasi-coherent sheaves on $X \times X$.
3. Of course, $\mathrm{Diff}_X \otimes \_$ corresponds to Diff_X on $X \times X$.
4. Let $\mathscr{D}_X^L \in \mathrm{QCoh}(X \times X)$ correspond to $\mathrm{oblv}^L \mathrm{ind}^L$. GR Crystals 5.3.6 shows $\mathscr{D}_X^L \simeq (\omega_X \boxtimes {}_X) \otimes \mathrm{Fiber}({}_{X\times X} \to j_* j^* {}_{X \times X})$ lies $\mathrm{QCoh}(X \times X)^\heartsuit$, with $j : X \times X \setminus \Delta_X \to X \times X$.
5. GR Crystals 5.4.1 claims for any $\mathcal{F}, \mathcal{G} \in (\mathrm{QCoh}\, X)^\heartsuit$ and $\mathcal{Q} \in \mathrm{QCoh}(X \times X)^\heartsuit$ with set-theoretically supported on the diagonal,
$$(p_2)_*(p_1^*\mathcal{F} \otimes \mathcal{Q}) \to \mathcal{G} \quad \rightsquigarrow \quad \mathcal{Q} \to \mathrm{Diff}_X(\mathcal{F}, \mathcal{G})$$
Applying to $\mathcal{F} = \mathcal{G} = \mathcal{O}_X, \mathcal{Q} = \mathscr{D}_X^L$, we obtain a morphism $\mathscr{D}_X^L \to \mathrm{Diff}_X$. This is a morphism of algebra objects on $\mathrm{QCoh}(X \times X)$.
6. GR Crystals 5.4.3 claims $\mathscr{D}_X^L \xrightarrow{\sim} \mathrm{Diff}_X$ is a classical computation.

(1) $\mathrm{Diff}_X\mathrm{Mod}$ is by definition the ∞-category of modules in $\mathrm{QCoh}\, X$ over the monad corresponding to $\mathrm{Diff}_X \in \mathrm{QCoh}(X \times X)$, so $\mathrm{Diff}_X\mathrm{Mod} \rightleftarrows \mathrm{QCoh}\, X$ is monadic.

For $\mathrm{Crys}^L X$, by Lurie HA 4.7.0.3, it suffices to show that

- oblv^L is conservative
- $\mathrm{Crys}^L X$ has geometric realisations and oblv^L preserves them.

Since X is a laft smooth classical scheme, it is classically formally smooth. This implies the equivalence between left crystals and infinitesimally equivariant quasi-coherent sheaves : $\mathrm{Crys}^L X \xrightarrow{\sim} \varprojlim \mathrm{QCoh}(\check{C}(X/X_{\mathrm{dR}}))$.

Now take $\mathcal{F} \in \mathrm{Crys}^L X$ and assume $\mathrm{oblv}^L \mathcal{F} \simeq 0$. The "forgetful functor" oblv^L is the morphism from $\mathrm{Crys}^L X$ to the zero-th part of the Cech nerve. So $\mathrm{oblv}^L \mathcal{F} \simeq 0$ implies $\mathcal{F}$ is zero in $\varprojlim \mathrm{QCoh}(\check{C}(X/X_{\mathrm{dR}}))$ and hence in $\mathrm{Crys}^L X$ as desired.

(2)

GR2 Ch2 1.6.11 - definition of ind-proper morphism.

(3)

(4)

(5)

(6)

$\square$