A Study in 'A Study in Derived Algebraic Geometry' : Ind-Coherent Sheaves , Duality and Crystals

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Contents

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Declaration : The following is my own work except otherwise stated.

1 Introduction

1.1 Why crystals?

The goal is to provide a setting where representation theory can be interpreted in algebraic geometry. Here,

- "representation theory" refers to the study of Lie algebras and their representations
- "algebraic geometry" means being able to pushforward and pullback sheaves on spaces where computations ultimately come down to commutative algebra.

One key example of something from representation theory we would like to interpret algebro-geometrically is Beilinson-Bernstein localisation. This result played a central role in solving the Kazhdan-Lusztig conjectures and is one of the founding results of modern representation theory. Roughly speaking, this says that given a semi-simple Lie algebra $\mathfrak{g}$, there is an equivalence

$$L : \mathfrak{g}\mathrm{Mod}_0 \xrightarrow{\sim} \mathrm{Diff}_X\mathrm{Mod}$$

where

- $\mathfrak{g}\mathrm{Mod}_0$ is the category of left modules over the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ where the center of $U(\mathfrak{g})$ acts trivially.
- $\mathrm{Diff}_X\mathrm{Mod}$ is the category of left D-modules on X with X being the flag variety of $\mathfrak{g}$.

We refer the reader to [Keller] for an introduction to Beilinson-Bernstein localisation. A more detailed treatment with relations to Kazhdan-Lusztig conjectures can be found in [H.T.T., Ch 11, Ch 12].

Let us describe what the Beilinson-Bernstein localisation functor looks like in this new algebro-geometric framework. Let k be a field of characteristic zero and $\bullet := \operatorname{Spec} k$. Firstly, we would have a correspondence between Lie algebras over k and "formal groups" over k. [GR2, Ch 7, 3.1.4]

$$\mathrm{Lie} : \mathrm{Grp}(\mathrm{FormMod}/\bullet) \simeq \mathrm{LieAlg}(\mathrm{QCoh}\, k) : \exp$$

- $\mathrm{FormMod}/\bullet$ refers to *formal moduli problems over* k [Lurie-DAGX, Ch 2][GR2, Ch 5, 1.1.] and group objects in there are what GR calls *formal groups over* k. $\mathrm{Grp}(\mathrm{FormMod}/\bullet)$ denotes the category of these.

– For the full correspondence, we need to pass from Lie algebras to *differential graded Lie algebras*. So $\mathrm{QCoh}\, k$ refers to the derived category of chain complexes of k-vector spaces and $\mathrm{Lie}(\mathrm{QCoh}\, k)$ refers to Lie algebra objects in there. [Lurie-DAGX, p. 2.1.5]

Furthermore, the action of a $\mathfrak{g} \in \mathrm{LieAlg}(\mathrm{QCoh}\, k)$ on a scheme X via vector fields should be equivalent to an action of the formal group $\exp \mathfrak{g}$ on X.

Secondly, given a scheme X equipped with an action from a formal group G, we should be able to *form the quotient X by G*, which should be a morphism $X \to B_X G$. In general, we should be able to *quotient by formal groupoids $\mathcal{G}$*, giving under object $X \to B_X \mathcal{G}$ under X such that $\mathcal{G}$ is recovered as $X \times_{B_X \mathcal{G}} X$. [GR2, Ch 5 , 2.3.2]

$$B_X : \mathrm{FormGrpd}(X) \simeq \mathrm{FormMod}_{X/} : \text{ fiber product}$$

– $\mathrm{FormGrpd}(X)$ refers to the category of formal groupoids over X. This includes $G \times X$ coming from the action of a formal group G on X. [GR2, Ch 5, 2.2.1]

– $\mathrm{FormMod}_{X/}$ denotes the category of *formal moduli problems under X*. [GR2, Ch 5, 1.3]

As a corollary in the situation of $X = \bullet$, we obtain an equivalence :

$$B : \mathrm{Grp}(\mathrm{FormMod}/\bullet) \simeq \mathrm{Pt}(\mathrm{FormMod}/\bullet) : \Omega$$

where B takes a formal group G to its *classifying space BG* and Ω takes $s : \bullet \to Y$ to $\bullet \times_Y \bullet$. Furthermore, in the special case of $G = \exp \mathfrak{g} \in \mathrm{Grp}(\mathrm{FormMod}/\bullet)$ the formal group corresponding to a $\mathfrak{g} \in \mathrm{LieAlg}(\mathrm{QCoh}\, k)$, we should have an equivalence :

$$\begin{tikzcd}
\mathrm{QCoh}(BG) \arrow[rr, "\sim"] \arrow[dr, shift left, "s^!"] & & \mathfrak{g}\mathrm{Mod}(\mathrm{QCoh}\, k) \arrow[dl, shift left, "\mathrm{forget}"] \\
& \mathrm{QCoh}\, k \arrow[ul, shift left, "s_*"] \arrow[ur, shift left, "\mathrm{ind}"] \arrow[ul, phantom, "\top"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

which realises pullback $s^!$ along s as the forgetful functor $\mathfrak{g}\mathrm{Mod} \to \mathrm{QCoh}\, k$ and pushforward s_* inducing representations (from the trivial Lie algebra).

Thirdly, within the category of formal moduli problem under a scheme X, there is a space called the *de Rham space* $p_{\mathrm{dR}} : X \to X_{\mathrm{dR}}$. The formal groupoid $X \times_{X_{\mathrm{dR}}} X$ over X corresponding to the de Rham space is called the *infinitesimal groupoid*. Intuitively, p_{dR} is the quotient of X by all points that are "infinitesimally close", and thus sheaves on X_{dR} are supposed to be sheaves on X equipped with "infinitesimal equivariance". This idea leads to an equivalence under the case of smooth X :

$$\begin{tikzcd}
\mathrm{QCoh}(X_{\mathrm{dR}}) \arrow[rr, "\sim"] \arrow[dr, shift left, "p_{\mathrm{dR}}^!"] & & \mathrm{Diff}_X\mathrm{Mod} \arrow[dl, shift left, "\mathrm{oblv}"] \\
& \mathrm{QCoh}\, X \arrow[ul, shift left, "(p_{\mathrm{dR}})_*"] \arrow[ur, shift left, "\mathrm{ind}"] \arrow[ul, phantom, "\top"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

which realises sheaves on X_{dR} as D-modules on X, induction ind of D-modules as pushforward $(p_{\mathrm{dR}})_*$ and the forgetful functor oblv of D-modules as pullback $p^!_{\mathrm{dR}}$. The category of sheaves on X_{dR} is called the category of *(left) crystals on X*.

To put everything together, let $\mathfrak{g} \in \mathrm{LieAlg}(\mathrm{QCoh}\, k)$ and X a scheme be equipped with an action from the formal group $\exp \mathfrak{g}$ corresponding to $\mathfrak{g}$. Then we obtain a *correspondence of spaces* :

$$\begin{tikzcd} \exp \mathfrak{g} \arrow[d] & (\exp \mathfrak{g}) \times X \arrow[l] \arrow[d, "\text{action}"'] \arrow[dr] & \\ \bullet & X \arrow[l] & X \times_{X_{\mathrm{dR}}} X \arrow[l] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} B(\exp \mathfrak{g}) & B_X(\exp \mathfrak{g}) \arrow[l, "q"'] \arrow[d, "p"] \\ & X_{\mathrm{dR}} \end{tikzcd}$$

Intuitively, q comes from factoring $X \to \bullet \to B(\exp \mathfrak{g})$ through $X \to B_X(\exp \mathfrak{g})$ and p comes from the morphism of formal groupoids $(\exp \mathfrak{g}) \times X \to X \times_{X_{\mathrm{dR}}} X$ over X. Finally, in the case of $\mathfrak{g}$ being a Lie algebra concentrated in degree zero that is semi-simple, and X its flag variety, we recover the Beilinson-Bernstein localisation functor as pullback then pushforward across the above correspondence.

$$L \simeq p_* q^! : \mathfrak{g}\mathrm{Mod} \simeq \mathrm{QCoh}(B \exp \mathfrak{g}) \to \mathrm{QCoh}(X_{\mathrm{dR}}) \simeq \mathrm{Diff}_X \mathrm{Mod}$$

The end goal of this paper is to understand just one part of this story : the equivalence between (left) crystals and (left) D-modules on a smooth scheme X.

1.2 Why stable infinity categories?

Let us be clear that all of the categories of sheaves and categories of modules mentioned in the previous section are *stable infinity categories*.

We know from classical algebraic geometry that when considering derived categories of sheaves, we want to consider them as at least triangulated categories. However, triangulated categories suffer some drawbacks :

1. cones are not functorial
2. limits of triangulated categories are hard to deal with

By passing to stable infinity categories, the situation is improved :

– Cones are colimits and thus can be made functorial, and easier to work with abstractly.

– Stable infinity categories are closed under limits. This is what will allow us to define sheaves on arbitrary prestacks rather simply and be able to argue with them abstractly.

– Analogous to the theory of compactly generated triangulated categories and colimit-preserving functors, there is a theory of compactly generated stable infinity categories and colimit-preserving functors which makes computations possible. This is used, for example, in giving a working theory of integral transforms

– due to the existence of a tensor product, duality statements can be phrased nicely in terms of derived categories of sheaves being *self-dual*. Examples include self duality of $\mathrm{QCoh}\, X$ for quasi-compact schemes X with affine diagonal and Serre duality.

1.3 Why derived algebraic geometry?

There is another direction in which infinity categories are getting involved : we require the use of *derived algebraic geometry*, meaning that the basic building blocks of schemes are not commutative rings but *commutative differential graded algebras*.

A simple reason is that base change theorems break without flatness assumptions. Consider the following example. Let $A := k[t]/(t)^2$ where k is a field and let $\mathrm{Spec}\, k \to \mathrm{Spec}\, A$ be the closed embedding of $t = 0$. Then the *classical* fiber product gives the following.

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r, "\mathbb{1}"] \arrow[d, "\mathbb{1}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Spec}\, k \arrow[d, "i"] \\ \mathrm{Spec}\, k \arrow[r, "i"'] & \mathrm{Spec}\, A \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathrm{QCoh}\, k \arrow[d, "\mathbb{1}"'] \arrow[dr, Rightarrow, "\not\simeq" description] & \mathrm{QCoh}\, k \arrow[l, "\mathbb{1}"'] \arrow[d, "Ri_*"] \\ \mathrm{QCoh}\, k & \mathrm{QCoh}\, A \arrow[l, "Li^*"] \end{tikzcd}$$

Indeed the diagram on the right hand side does not commute up to isomorphism since we have :

$$\begin{tikzcd}[column sep=small] & & & & & & \vdots \arrow[d, "t\otimes 1"'] & & \vdots \arrow[d, "0"'] & & \\ & & & & & & A \otimes_A k \arrow[d, "t\otimes 1"'] & & k \arrow[d, "0"'] & & \\ Li^*(Ri_* k) & \simeq & Li^*(k) & \simeq & k \otimes_A^L k & \simeq & A \otimes_A k & \simeq & k & \not\simeq & k \end{tikzcd}$$

Going derived fixes this because we have the isomorphism

$$\_ \otimes_A^L k \simeq \_ \otimes_k^L (k \otimes_A^L k)$$

Another reason is that the theory of formal moduli problems mentioned in the first section also involves derived affine schemes, rather than just affine schemes. However, we will not get this in this paper.

1.4 Why ind-coherent sheaves?

Let us also be clear that it will not be enough to consider quasi-coherent sheaves alone. There are a few reasons for this :

– In the first section, we saw that many of adjunctions of representation theory under the interpreted using sheaves requires the pullback of sheaves to be the *right adjoint* to pushforward, rather than the left adjoint as it usual is for quasi-coherent sheaves on schemes.

– As mentioned, for computations with derived categories of sheaves as stable infinity categories, it is desirable to stay in the world of compactly generated stable infinity categories and colimit-preserving functors.

The example of the closed embedding $i : \operatorname{Spec} k \to \operatorname{Spec} A = \operatorname{Spec} k[t]/(t)^2$ again demonstrates both of the above points. Let us explain.

We will show later that $\operatorname{QCoh} k$ and $\operatorname{QCoh} A$ are both compactly generated, meaning they are the ind-completions of their full subcategory of compact objects. Now let us impose the above two points on the derived pushforward $i_* : \operatorname{QCoh} k \to \operatorname{QCoh} A$, i.e. let's assume that i_* has a right adjoint $i^!$ that preserves small colimits. Then

$$(\operatorname{QCoh} A)(i_* V, M) \simeq (\operatorname{QCoh} k)(V, i^! M)$$

for any $V \in \operatorname{QCoh} k, M \in \operatorname{QCoh} A$. It follows that i_* preserves compact objects if and only if $i^!$ preserves small coproducts. By the assumption, we thus obtain that $i_* k$ must be compact as an object in $\operatorname{QCoh} A$. However, it is a classical result that for A Noetherian, a module M over A is compact in $\operatorname{QCoh} A$ if and only if it is finitely generated and admits a finite projective resolution. This in particular implies that $\operatorname{Tor}^n_A(i_* k, i_* k) = 0$ for large enough n. But this is a contradiction since for all $n \geq 0$,

$$\operatorname{Tor}^n_A(i_* k, i_* k) \simeq H^n(k \overset{L}{\otimes}_A k) \simeq k \not\simeq 0$$

The issue above can be seen as the failure of i_* to preserve compact objects. The idea behind *ind-coherent sheaves* is this : since i is proper, i_* sends $\operatorname{Coh} k$ to $\operatorname{Coh} A$ where $\operatorname{Coh} \_$ denotes the full subcategory $\operatorname{QCoh} \_$ of $\mathcal{F}$ with finitely many non-zero cohomologies, all of which are coherent sheaves. So we replace

$$\begin{array}{ccccc} \operatorname{QCoh} k \simeq \operatorname{Ind} \operatorname{Perf} k & \rightsquigarrow & \operatorname{IndCoh} k & := & \operatorname{Ind}(\operatorname{Coh} k) \\ \big\downarrow{\scriptstyle i_*} & & \big\downarrow{\scriptstyle i_*} & & \\ \operatorname{QCoh} A \simeq \operatorname{Ind} \operatorname{Perf} A & & \operatorname{IndCoh} A & := & \operatorname{Ind}(\operatorname{Coh} A) \end{array}$$

where Ind denotes *ind-completion* and i_* comes from the universal property of ind-completions and by definition preserves small colimits, and hence has a right adjoint $i^!$ by adjoint functor theorem. (We will cover this later.) Furthermore, this right adjoint also preserves small coproducts because it can be shown that the full subcategory of compact objects of $\operatorname{IndCoh} k, \operatorname{IndCoh} A$ are $\operatorname{Coh} k, \operatorname{Coh} A$, which are preserved by i_* by construction.

Thus, a large part of the theory of crystals is in developing the theory of ind-coherent sheaves together with pushforward, pullback, tensor, proper base change, etc for a large enough class of spaces including de Rham spaces, formal groups, quotients by formal groupoids. Crystals on a scheme X are then defined as

$$\operatorname{Crys} X := \operatorname{IndCoh} X_{\mathrm{dR}}$$

1.5 How crystals compare with usual D-modules?

The classical theory of D-modules has its advantage in how explicit it is and hence how indispensible it is for calculations in examples. However, it is known that for singular varieties Z, even in the affine case the ring of differential operators can be rather complicated. [BGG] gives an example where Noetherianness of the ring of differential operators fails. One approach is via Kashiwara's theorem : find a closed embedding $Z \to X$ where X is smooth then define D-modules on Z to be D-modules on X which are supported on Z. One then has to do work to show that this is independent of the closed embedding $Z \to X$. One advantage of crystals is that they are intrinsic to the space, and Kashiwara's theorem can be deduced [Crys, Ch 2, 2.5.6] rather than taken as a definition.

The main advantage of crystals over D-modules is in the perspective it brings : one can now think of D-modules as quasi-coherent / ind-coherent sheaves, and hence operations to do with D-modules as usual operations to do with quasi-coherent / ind-coherent sheaves. This includes interpretation as Verdier duality for D-modules as self-duality of $\mathrm{Crys}\, X := \mathrm{IndCoh}\, X_{\mathrm{dR}}$, which places it on the same footing as Serre duality. Of course, it is also made with higher stacks in mind, which makes it indispensible for the Geometric Langlands programme.

1.6 Guide to Reading

The structure of the paper is as follows :

- In section 2 we first explain how to work with "dg categories" in the sense of Gaitsgory–Rozenblyum, gathering all results needed to do reduces computations to the level of triangulated categories. Then we describe the types of spaces IndCoh will be made from, namely, prestacks of locally almost finite type.
- In section 3, we describe how ind-coherent sheaves work for derived schemes. We first take a look at the definition, pushforward, pullback for open immersions, pullback for proper morphisms, and proper base change. Then we describe the theory of integral transforms for both quasi-coherent and ind-coherent sheaves, which is central to Gaitsgory–Rozenblyum's proof of the equivalence of (left) crystals and left D-modules for smooth classical schemes. On the way, we describe self-dualities of $\mathrm{QCoh}\, X$ and $\mathrm{IndCoh}\, X$ on derived schemes of almost finite type and how they relate to the classical procedure of taking duals of perfect complexes and Serre-duals of coherent sheaves. Finally, we describe a comparison Υ from QCoh to IndCoh which exist not only for derived schemes of almost finite type but all prestacks of locally almost finite type.
- In section 4, we first introduce de Rham spaces, left and right crystals. Then induction of crystals is obtained from Lurie's very general theory of descent and the fact that IndCoh satisfies base change for *ind-proper morphisms*. Finally, we prove the equivalence of left crystals and left D-modules.

Remark. The proofs of all of the results in this paper are well-known to the experts. However, I have tried to make proofs more explicit wherever possible by adding more details, or slightly altering the argument so that things are hopefully more digestable. I always indicate when I do this.

2 Setting

2.1 How to work with "DG Categories"

Gaitsgory–Rozenblyum has a highly abstract but conceptual definition of dg-categories, which can be summarised in the following diagram :

$$\begin{tikzcd}
\mathrm{Set} \arrow[r] \arrow[d] & \text{1-Cat} \arrow[d, shift right, "N"'] \arrow[d, phantom, "\vdash"] & \mathrm{Cat}_{\mathrm{dg}}k \arrow[d, "N_{\mathrm{dg}}"] & & & \\
\mathrm{Spc} \arrow[r] & (\infty,1)\text{-Cat} \arrow[u, shift right, "h"'] & (\infty,1)\text{-Cat}^{\mathrm{ex}} \arrow[l] & \mathrm{Vec}^c\text{-Mod} \arrow[l] & =: & \mathrm{DGCat} \\
& & (\infty,1)\text{-Cat}^{\mathrm{ex}}_{\mathrm{cts}} \arrow[u] & \text{Vec-Mod} \arrow[l] \arrow[u] & =: & \mathrm{DGCat}_{\mathrm{cts}}
\end{tikzcd}$$

This diagram should be used as a map for the next few sections.

2.2 All you need to know about infinity categories

We will use "infinity category" to refer only to $(\infty,1)$-categories. The word "category" will exclusively refer to 1-categories.

$(\infty,1)$-Cat denotes the infinity category of small infinity categories. $(\infty,1)$-Cat has all small limits [Kerodon, Cor 7.4.1.11] and small colimits [Kerodon, Cor 7.4.3.13] and is cartesian closed. We use $\mathrm{Fun}(C,D)$ to denote the infinity category of functors from C to D.

There is an adjunction $h \dashv N : (\infty,1)\text{-Cat} \rightleftarrows \text{1-Cat}$. N is called the *nerve functor* and it is fully faithful, allowing us to see 1-categories as ∞-categories. In particular, we use Δ^n to denote the *n-simplex*, the ∞-category obtained from the linear order $[n] = \{0 \leq \cdots \leq n\}$. Given an infinity category C, objects of C are the same as functors $\Delta^0 \to C$ and morphisms in C are the same as functors $\Delta^1 \to C$.

There is a full subcategory Spc of $(\infty,1)$-Cat consisting of ∞-categories X where all morphisms are isomorphisms. These are called *∞-groupoids* but also *spaces* by homotopy theorists. The infinity category Spc plays the role of Set in 1-category theory in the sense that given an infinity category C, the fiber of $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$ over (X,Y), denoted $C(X,Y)$, is in fact a space. The nerve functor N lands Set inside Spc, where given a set S and two points $x, y \in S$, we have $NS(x,y) \simeq \varnothing$ the empty space. In other words, sets are "discrete spaces".

For any infinity category C, we define $\mathrm{PSh}\, C := \mathrm{Fun}(C^{op}, \mathrm{Spc})$ and refer to its objects as *presheaves in C*. Infinity categories of presheaves are significant since we will be working with derived algebraic geometry functorially. We will use the following universal property of $\mathrm{PSh}\, C$ many times.

Proposition – Universal Property of Presheaf ∞-Categories

Let S be a small ∞-category.

- There is a fully faithful functor $S \to \mathrm{PSh}\, S$ which takes each object x in S to the functor $S(\_, x)$:

$C^{op} \to \mathrm{Spc}$ taking points y to $S(y,x)$. This is called the *Yoneda embedding*.

- [Lurie-HTT, Prop 5.1.2.3] $\mathrm{PSh}\, S$ has small colimits and small limits. In fact, they are computed pointwise.
- [Lurie-HTT, Prop 5.1.5.6, 5.2.6.5] For C be an ∞-category with small colimits, let $\mathrm{Fun}^L(\mathrm{PSh}\, S, C)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{PSh}\, S, C)$ consisting of functors preserving small colimits. Then restricting along the Yoneda embedding $S \to \mathrm{PSh}\, S$ gives an equivalence of ∞-categories :

$$\mathrm{Fun}^L(\mathrm{PSh}\, S, C) \xrightarrow{\sim} \mathrm{Fun}(S, C)$$

An inverse functor is given by left Kan extension. In particular, for $u_! \in \mathrm{Fun}^L(\mathrm{PSh}\, S, C)$ corresponding to $u \in \mathrm{Fun}(S, C)$, we have for every $X \in \mathrm{PSh}\, S$ that $u_!$ exhibits $u_!(X)$ as the colimit of the diagram $S_{/X} \to S \to C$. Furthermore, if we assume C is locally small, then we have an adjunction

$$u_! \dashv u^* : \mathrm{PSh}\, S \rightleftarrows C$$

where u^* is given by the composition

$$C \xrightarrow{\text{Yoneda}} \mathrm{PSh}\, C = \mathrm{Fun}(C, \mathrm{Spc}^{\mathrm{op}}) \xrightarrow{\text{restrict along } u} \mathrm{Fun}(S, \mathrm{Spc}^{\mathrm{op}}) = \mathrm{PSh}\, S$$

An immediate consequence of the above is the following, which is key for defining de Rham spaces.

Lemma (*Left and Right Kan Extensions of Presheaves*). *Let $u : S \to T$ be a functor between small ∞-categories. Then we have a triple of adjoints :*

$$\begin{tikzcd} \mathrm{PSh}\, S \arrow[r, shift left=4, "u_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "u_*"'] & \mathrm{PSh}\, T \arrow[l, "u^*" description] \end{tikzcd}$$

where

- *$u_!$ is the left Kan extension of $S \to T \to \mathrm{PSh}\, T$*
- *u^* is the composition $\mathrm{PSh}\, T \to \mathrm{PSh}\,\mathrm{PSh}\, T \to \mathrm{PSh}\, S$*
- *u_* is the composition $\mathrm{PSh}\, S \to \mathrm{PSh}\,\mathrm{PSh}\, S \to \mathrm{PSh}\, T$*

In particular, for $X \in \mathrm{PSh}\, S$, $u_!(X)$ is the left Kan extension of X along $S \to T$ and $u_*(X)$ is the right Kan extension of X along $S \to T$.

Proof. For $u_! \dashv u^*$, we apply the universal property of presheaf ∞-categories to $S \to T \to \mathrm{PSh}\, T$. Now for $u^* \dashv u_*$, note that since colimits in $\mathrm{PSh}\, T$ are computed pointwise, we have that u^* preserves small colimits. This means we can apply the universal property of presheaf ∞-categories again, but this time to the composition $T \to \mathrm{PSh}\, T \to \mathrm{PSh}\, S$. This gives $u^* \dashv u_*$. ■

2.3 All you need know about stable infinity categories

$(\infty,1)\text{-Cat}^{\text{ex}}$ denotes subcategory of $(\infty,1)\text{-Cat}$ consisting of *stable infinity categories* and *exact functors*. It contains all small limits and the "inclusion" $(\infty,1)\text{-Cat}^{\text{ex}} \to (\infty,1)\text{-Cat}$ preserves small limits [Lurie-HA, Prop 1.1.4.4].

Stable infinity categories are basically triangulated categories where exact triangles are determined by an infinity-categorical universal property. Here is the definition.

> **Definition**
>
> Let C be an infinity category. We say C has a *zero object* when it has an object that is both initial and final. [Lurie-HA, Def 1.1.1.1]
>
> Now assume C have a zero object. Then a *triangle* in C is defined as a diagram in C of the form :
>
> $$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$
>
> A triangle is called a *fiber sequence* when it is a cartesian and a *cofiber sequence* when it is cocartesian. [Lurie-HA, Def 1.1.1.4] In the first case, we say $Y \to Z$ *admits a fiber* and refer to X as the fiber, and in the other case we say $X \to Y$ *admits a cofiber* and refer to Z as the cofiber.
>
> C is called *stable* when the following are true :
>
> – every morphism has both a fiber and a cofiber.
>
> – A triangle is fiber sequence iff it is a cofiber sequence. Such triangles are called *exact triangles*.
>
> [Lurie-HA, Prop 1.1.1.9]
>
> An exact functor $F : C \to D$ between stable infinity categories is one which satisfy any of the following equivalent conditions : [Lurie-HA, Prop 1.1.4.1]
>
> – F preserves exact triangles
>
> – F preserves finite limits
>
> – F preserves finite colimits.
>
> For stable ∞-categories C, D the full subcategory $\text{Fun}^{\text{ex}}(C, D)$ of $\text{Fun}(C, D)$ consisting of exact functors is also stable.[a]
>
> [a] [GR1, Ch 1, 5.1.4]claims this. [Lurie-HA, Prop 1.1.3.1] shows that $\text{Fun}(K, C)$ is stable for any K and stable C. The result follows given that finite (co)limit-preserving functors are closed under finite (co)limits.

To help build intuition of "stable infinity categories as fixed triangulated categories", we record here the important parts of the procedure of extracting a triangulated category from a stable infinity category.

Proposition

Let C be a stable infinity category. Then the following defines a triangulated structure on the 1-category hC :

- Define the *suspension functor* $\Sigma : C \to C$ by pushout against zeros :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

  Since the above square is a cofiber sequence, it is also a fiber sequence. This shows that *looping* $\Omega : C \to C$, given by pullback against zeros, gives an inverse for Σ and hence shows that Σ is an equivalence. Taking homotopy categories, we obtain an equivalence $[1] : hC \xrightarrow{\sim} hC$, which we use as the shift functor for the triangulated structure.

- We call a diagram
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
  in hC an exact triangle (in the triangulated categorical sense) when it comes from a diagram of the following form in C :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

  i.e. two exact triangles (in the stable infinity categorical sense).

- For X, Y objects of C, we have

$$\begin{aligned} C(X,Y) &\simeq C(\Sigma\Omega X, Y) \simeq \Omega C(\Omega X, Y) \\ &\simeq C(\Sigma^2\Omega^2 X, Y) \simeq \Omega^2 C(\Omega^2 X, Y) \end{aligned}$$

  Upon taking π_0 , we obtain

$$hC(X,Y) := \pi_0 C(X,Y) \simeq \pi_1 C(\Omega X, Y) \simeq \pi_2(\Omega^2 X, Y)$$

  where the last isomorphism is a group morphism. For π_2 of any "space" [a] the obvious group structure given by is abelian, this gives $hC(X,Y)$ an abelian group structure, making hC into an additive category.

[Lurie-HA, Thm 1.1.2.14]

For X, Y objects in C, we define the abelian group $\mathrm{Ext}^n_C(X,Y) := hC(X, Y[n])$. [Lurie-HA, Notation 1.1.2.17]

[a]In the quasi-category model of infinity categories, $C(X,Y)$ is a Kan complex, which one can take homotopy groups of.

A t-structure on a stable ∞-category is simply a t-structure on its homotopy category. [Lurie-HA, Def 1.2.1.4]

Definition

Let $F : C \to D$ be a functor in $(\infty, 1)\text{-Cat}^{\mathrm{ex}}$ where C and D are equipped with t-structures. Then F is called *right t-exact* when $FC_{0\leq} \subseteq D_{0\leq}$. It is called *left t-exact* when $FC_{\leq 0} \subseteq D_{\leq 0}$. We say F is *t-exact* when it is both left and right t-exact.

[Lurie-HA, Def 1.3.3.1]

2.4 All you need to know about compactly generated stable infinity categories

We are now ready for compactly generated ∞-categories. We will only make use of the case of *stable* compactly generated ∞-categories since many definitions then admit alternative characterisations which can be checked at the level of triangulated categories.

We first note that the theory of colimits simplifies in the stable case.

Proposition – For a stable ∞-category C, TFAE:

  - admitting small colimits
  - admitting small filtered colimits
  - admitting small coproducts

- For a functor $F : C \to D$ between stable ∞-categories which admit small colimits, TFAE:
  - preserving small colimits
  - preserving small filtered colimits
  - preserving small coproducts

  Any functor satisfying the above, GR calls *continuous*.

[Lurie-HA, Prop 1.4.4.1]

We now explain compact generation. The starting point is the theory of *inductive completions*[1]. Here are the main results concerning ind-completions in the stable case.

Proposition – Ind-completions of Stable ∞-Categories

Let C be a small ∞-category and κ a regular cardinal.[a] Then the Yoneda embedding $C \to \mathrm{PSh}\, C$ factors through a full subcategory $\mathrm{Ind}_\kappa(C)$ with the following properties :

- [Lurie-HTT, Prop 5.3.5.3] $\mathrm{Ind}_\kappa(C)$ has all small κ-filtered colimits and the inclusion $\mathrm{Ind}_\kappa(C) \subseteq$

[1]This is a bit of a misnomer because intuitively we are adding filtered *colimits*, not limits.

$\mathrm{PSh}\, C$ preserves them

– [Lurie-HTT, Prop 5.3.5.4] An object X in $\mathrm{PSh}\, C$ is in $\mathrm{Ind}_\kappa(C)$ iff it is a κ-filtered colimit of representables iff $X : C^{op} \to \mathrm{Spc}$ preserves κ-small limits.

– [Lurie-HA, Prop 1.1.3.6] If C is stable then so is $\mathrm{Ind}_\kappa(C)$.

– [Lurie-HTT, Prop 5.3.5.10] For any ∞-category D admitting small κ-filtered colimits, we have the following equivalence of functor ∞-categories :

$$\mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D)$$

where

  – the left category denotes the full subcategory of $\mathrm{Fun}(\mathrm{Ind}_\kappa(C), D)$ consisting of functors preserving κ-filtered colimits.[b]

  – the forward functor is given by restricting along the Yoneda embedding $C \to \mathrm{Ind}_\kappa(C)$

  – the inverse functor is given by left Kan extension.

Assuming C, D are stable and κ is the cardinality of $\mathbb{N}$, the above equivalence restricts to an equivalence between the following two full subcategories :

$$\mathrm{Fun}^{\mathrm{ex}}_{\mathrm{cts}}(\mathrm{Ind}_\kappa(C), D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(C, D)$$

where the left is the ∞-category of exact continuous functors from $\mathrm{Ind}_\kappa(C)$ to D.

When κ is the cardinality of $\mathbb{N}$, we write Ind instead of Ind_κ.

[a] A regular cardinal κ is a cardinality that is "sufficiently large" in the sense that the 1-category $\mathrm{Set}_{<\kappa}$ of sets with cardinality strictly less than κ has all colimits of size strictly less than κ. The cardinality of $\mathbb{N}$ is an example, since a finite colimit of finite sets is still finite.

[b] At [Lurie-HTT, Prop 5.3.4.5], these are called κ-continuous functors. Taking the minimal case of $\kappa = |\mathbb{N}|$, it seems only reasonable to refer to functors preserving filtered colimits as *continuous* functors. This is a potential explanation of GR's choice of terminology for continuous functors.

For a regular cardinal κ and an ∞-category C, we say C is κ-compactly generated when it has all small colimits and there exists a small ∞-category C^0 with an equivalence $\mathrm{Ind}_\kappa(C^0) \xrightarrow{\sim} C$.[1] For the case of $\kappa =$ cardinality of $\mathbb{N}$, we simply say *compactly generated*. A *presentable* ∞-category is one that is κ-compactly generated for some κ. We use $(\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{cts}}$ to denote the subcategory of $(\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}}$ whose objects are presented stable ∞-categories and morphisms are exact functors preserving small coproducts.

One appeal of presentable stable ∞-categories is that we have the adjoint functor theorem at our disposal.

Proposition – Adjoint Functor Theorem for Presentable Stable ∞-Categories

Let $F : C \to D$ be a functor between presentable ∞-categories.

[1] This is unraveled from [Lurie-HTT, Def 5.5.7.1] [Lurie-HTT, Def 5.5.0.18] [Lurie-HTT, Def 5.4.2.1]. In particular, the second condition is usually called κ-accessibility, but we have no need for such terminology.

> – F is a left adjoint iff it preserves small colimits.
>
> – F is a right adjoint iff it preserves small limits and there exists a regular cardinal κ such that C has small κ-filtered colimits and F preserves them.
>
> [Lurie-HTT, Prop 5.5.2.9]

We also have a way of detecting when a right adjoint is continuous.

> **Proposition – Continuity of Right Adjoint**
> Let $F \dashv G : C \rightleftarrows D$ be an adjunction between compactly generated stable ∞-categories. Then G is continuous if and only if F preserve compact objects.

Proof. Direct application of [Lurie-HTT, Prop 5.5.7.2]. We saw the idea of the proof in 1.4. □

For functors out of $\mathrm{Ind}_\kappa(C)$, fully faithfulness and equivalence can be detected at the level of C.

> **Proposition – Functors out of κ-Compactly Generated Categories**
> Let C^0 be a small ∞-category, κ a regular cardinal, $C = \mathrm{Ind}_\kappa(C^0)$ and D an ∞-category admitting κ-filtered colimits. Let D^κ be the full subcategory of D consisting of κ-compact objects. Let $F : C \to D$ be a functor preserving κ-filtered colimits and $F_0 : C^0 \to D$ its restriction along the Yoneda embedding $C^0 \to C$.
>
> – If F_0 is fully faithful and its essential image lands in D^κ, then F is fully faithful.
>
> – F is an equivalence iff the following are true :
>
>   – F_0 fully faithful
>
>   – F_0 factors through D^κ
>
>   – all objects of D are κ-filtered colimits of diagrams in D with objects in the image of F_0.
>
>   In particular, for any full subcategory $\tilde{C} \subseteq C^\kappa$ which generates C under κ-filtered colimits, we have $\mathrm{Ind}_\kappa(\tilde{C}) \xrightarrow{\sim} C$.
>
> [Lurie-HTT, Prop 5.3.5.11]

Furthermore, the Yoneda embedding $C^0 \to C$ factors through C^κ. The fully faithful functor $C^0 \to C^\kappa$ is not in general an equivalence, however it does exhibit C^κ as the *idempotent completion* of C^0. (See in next proposition.) So if C^0 is idempotent complete, then we recover the κ-compact objects of C as precisely (the essential image of) C^0.

There is a complication with defining idempotents in an ∞-category C. All we need to know is that they are diagrams $\mathrm{Idem} \to C$ from a certain "shape"[1] and C is called idempotent complete when all such diagrams admit a colimit.

[1] meaning simplicial set, if we choose to use the quasi-categories model for ∞-categories.

The following propositions contain all we need about idempotent completions. One should not worry about having to check idempotent completeness ∞-categorically because we will only be in the stable setting, in which we have a characterisation at the level of triangulated categories.

Proposition – Idempotent Completions

Let C be an ∞-category.

- If C has small colimits, then C is idempotent complete.
- ([Lurie-HTT, Def 5.1.4.1, Prop 5.1.4.9, Lem 5.1.4.7]) A functor $F : C \to D$ in $(\infty, 1)\text{-Cat}$ is said to *exhibit D as the idempotent completion of C* when F is fully faithful, D is idempotent complete and every object of D is a retract of some $F(x)$ with $x \in C$.

  Suppose $F : C \to D$ is such a functor. Then for any idempotent complete ∞-category E, we have an equivalence :
  $$\mathrm{Fun}(D, E) \xrightarrow{\sim} \mathrm{Fun}(C, E)$$
  given by restriction along F. An inverse functor is given by left Kan extension along F.
- [Lurie-HTT, Prop 5.4.2.4] Let C be small and κ a regular cardinal. Then $C \to (\mathrm{Ind}\, C)^\kappa$ exhibits the latter ∞-category as the idempotent completion of C. In particular, the latter is equivalent to a small ∞-category.
- [Lurie-HA, Prop 1.2.4.6] Let C be a stable ∞-category. Then C is idempotent complete iff hC is as a 1-category, i.e. for every morphism $e : B \to B$ such that $e^2 = e$, there exists a retract of $s : A \rightleftarrows B : r$ that exhibits $e = rs$.

  If the above is the case and C is small, then for any regular cardinal κ we have $C \xrightarrow{\sim} (\mathrm{Ind}_\kappa(C))^\kappa$.

But how do I actually get my hands on a presentable stable ∞-category? Worry not! All the computable examples are in the compactly generated case. As it turns out, this can also be checked at the level of the triangulated categories.

Proposition – Compact Generation of Stable Infinity Categories

Let C be a stable ∞-category. We say an object X *generates* C when for all objects Y in C, $hC(X, Y) = 0$ implies $Y \simeq 0$. Additionally, we say a set I of objects in C *compactly generates* C when I consists of compact objects and $X := \bigoplus_{X_i \in I} X_i$ generates C.

Then the following are true :

1. Suppose we have that :
   - C has small coproducts
   - hC is locally small
   - There exists a compact object X which generates C.

Define the following sequence of full subcategories of C :

$$\begin{aligned} C(0) &:= \text{full subcategory of } C \text{ spanned by } \{X[n]\}_{n\in\mathbb{Z}} \\ C(k+1) &:= \text{full subcategory of } C \text{ spanned by cofibers of morphisms in } C(k) \\ C(\omega) &:= \bigcup_{n\in\mathbb{N}} C(n) \end{aligned}$$

Then

(a) $C(\omega)$ is the smallest stable full subcategory of C containing X and is equivalent to a small ∞-category.

(b) $\operatorname{Ind} C(\omega) \xrightarrow{\sim} C$. In particular, C is compactly generated.

2. [Lurie-HA, Prop 1.4.4.1] For an object X in C, X is compact if and only if for every morphism $f : X \to \coprod_{i\in I} Y_i$ in C, there exists a finite subset $I_0 \subseteq I$ such that in hC, f factors through $\coprod_{i\in I_0} Y_i \to \coprod_{i\in I} Y_i$.

3. Suppose $F \in \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{cts}}(C, D)$ where both C, D satisfies the conditions in (1) with compact generators $X \in C, F(X) \in D$ respectively. Suppose the induced morphism $C(X, X) \to D(F(X), F(X))$ is an equivalence. Then F is an equivalence.[a]

[a] The idea that "for an equivalence it suffices that compact generators and their Ext's match" is old, but I could not find a formalisation of this idea in the stable ∞-categorical set up. The proof I wrote here is likely to be faulty so I would be grateful if someone can check the proof.

Proof. The following proof of (1) is adapted from the proof of [Lurie-HA, Prop 1.4.4.2] which is about general presentable ∞-categories, rather than the compactly generated special case.

We first show $C(\omega)$ is equivalent to a small ∞-category. This part is the same as in [Lurie-HA, Prop 1.4.4.2]. It is a set-theoretic issue that's not very interesting, so we will use the following result without proof.

> *Lemma. Let C be an ∞-category and κ a regular cardinal which is uncountable. Then there exists a κ-small ∞-category $D \xrightarrow{\sim} C$ if and only if hC is κ-small and C is locally κ-small, i.e. for every morphism $f : X \to Y$ in C, $\pi_0 C(X, Y)$ and $\pi_{i>0}(C(X, Y), f)$ are κ-small.*
>
> *Proof.* See [Lurie-HTT, Prop 5.4.1.2] ∎

Since $hC(\omega)$ is small, in order to show $C(\omega)$ is equivalent to a small ∞-category, it remains to show that C is locally small. We have $\pi_0(X, Y) = hC(X, Y)$ is small by assumption. For the higher homotopy groups, note that $C(X, Y) \simeq \Omega C(\Omega X, Y)$ therefore the we can WLOG assume $f = 0$ for computation of the higher homotopy groups. Then $\pi_{i>0}(C(X, Y), 0) \simeq hC(X[i], Y)$ which is small again.

Now we may WLOG assume $C(\omega)$ is small. By construction, $C(\omega)$ is closed under translations and cofibers. It follows from the stability of C that $C(\omega)$ is also closed under fibers and hence a stable full subcategory. We thus have (a) by construction.

To prove (b), by the universal property of ind-completions, the inclusion $C(\omega) \to C$ factors into

$$C(\omega) \xrightarrow{\text{Yoneda}} \operatorname{Ind} C(\omega) \xrightarrow{j} C$$

where j is the left Kan extension of $C(\omega) \to C$. The inclusion $C(\omega) \subseteq C$ is fully faithful and every object in $C(\omega)$ is compact, therefore j is fully faithful.

It remains to show the essential image of j is all of C. We will achieve this by explicitly computing an inverse. We saw above that C is locally small. So by the universal property of presheaf ∞-categories applied to the inclusion $i : C(\omega) \to C$, we have an adjunction

$$\begin{tikzcd}
\operatorname{Ind} C(\omega) \arrow[d, "\subseteq"'] \arrow[dr, "j"] & \\
\operatorname{PSh} C(\omega) \arrow[r, shift left, "i_!"] \arrow[r, phantom, "\perp"] & C \arrow[l, shift left, "i^*"]
\end{tikzcd}$$

where $i_!$ is the left Kan extension of i along $C(\omega) \to \operatorname{PSh} C(\omega)$. The above diagram commutes up to isomorphism because both j and $i_!$ restrict to give $i : C(\omega) \to C$. Now, the fact that i preserves finite colimits implies that $i^* C \subseteq \operatorname{Ind} C(\omega)$. Thus, we have an adjunction $j \dashv i^* : \operatorname{Ind} C(\omega) \rightleftarrows C$. It suffices to show that for all Y in C, $j(i^*Y) \xrightarrow{\sim} Y$. Let $K \to j(i^*Y) \to Y$ be a fiber sequence. It suffices to show $K \simeq 0$. Since X is a generator of C, this is equivalent to showing

$$0 \simeq hC(X, K) \simeq \pi_0(\operatorname{Ind} C(\omega))(X, i^*K)$$

where the latter isomorphism comes from the adjunction $j \dashv i^*$. This follows by applying the right adjoint i^* to the fiber sequence :

$$\begin{tikzcd}
K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & j(i^*Y) \arrow[d] \\
0 \arrow[r] & Y
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
i^*K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & i^*(j(i^*Y)) \arrow[d, "\sim"] \\
0 \arrow[r] & i^*Y
\end{tikzcd}$$

The equivalence $i^*ji^*Y \to i^*Y$ comes from the adjunction $j \dashv i^*$. Therefore $i^*K \simeq 0$ and hence we have $\pi_0(\operatorname{Ind} C(\omega))(X, i^*K) \simeq 0$ as desired.

(2) We refer the reader to the reference.

(3) By compact generation of C and D, F is an equivalence iff F is fully faithful on the full subcategory C^c of compact objects of C and $FC^c \subseteq D^c$. By the theory of idempotent completions, $C(\omega) \to C^c$ exhibits the latter as the idempotent completion of the former, the smallest stable full subcategory of C containing X. We have the same situation with $D(\omega) \to D^c$. It thus suffices to show that $F : C(\omega) \to D$ factors through $D(\omega)$ and gives an equivalence $F : C(\omega) \xrightarrow{\sim} D(\omega)$. We check this inductively. For the base case, we have $F : C(0) \xrightarrow{\sim} D(0)$ because the homs in $C(0), D(0)$ are given by suspensions and loopings of respectively $C(X, X), D(F(X), F(X))$, which we assumed are equivalent under F. For $k+1$, exactness of F and the inductive hypothesis that $F : C(k) \simeq D(k)$, we obtain that the essential image of $C(k+1)$ under F is $D(k+1)$. Since homs between cofibers of morphisms in $C(k)$ are determined by fibers and cofibers of homs in $C(k)$, and F is exact, we obtain that $F : C(k+1) \to D(k+1)$ is fully faithful. Therefore, F induces an equivalence $C(k+1) \simeq D(k+1)$.

□

Finally, there is one hard fact we shall use without motivation.

> **Proposition**
> Let $C \in (\infty, 1)\text{-Cat}^{\text{ex}}_{\text{cts}}$. Then C has all small limits. [Lurie-HTT, Prop 5.5.2.4]

2.5 "DG Categories" Finally

An important example of a compactly generated stable ∞-category is Vec.

> **Proposition**
> Let k be a field. Then there exists an ∞-category Vec called the *right derived ∞-category of k-vector spaces* with the following properties :
>
> – [Lurie-HA, Prop 1.3.2.18] Vec is stable.
>
> – (Universal Property as Localisation [Lurie-HA, Prop 1.3.4.4]) There is a functor $l : \mathrm{Ch}^{-}(k) \to \mathrm{Vec}$ with the property that for all ∞-categories E, restricting along l yields a fully faithful functor $\mathrm{Fun}(\mathrm{Vec}, E) \to \mathrm{Fun}(\mathrm{Ch}^{-}(k), E)$ with essential image consisting of functors $\mathrm{Ch}^{-}(k) \to E$ which invert quasi-isomorphisms.
>
> – [Lurie-HA, Prop 1.3.2.9] $h\mathrm{Vec}$ gives the usual 1-category right derived category of k-vector spaces.
>
> Consequently, for $X, Y \in \mathrm{Ch}^{-}(k)$, we have
>
> $$\pi_n \mathrm{Vec}(X, Y) = \pi_0 \Omega^n \mathrm{Vec}(X, Y) \simeq \pi_0 \mathrm{Vec}(X, Y[n]) =: \mathrm{Ext}^n(X, Y)$$
>
> – Derived tensor $\otimes$ makes Vec into a symmetric monoidal infinity category.
>
> – Vec is compactly generated. See later.

Practically speaking, computations tensor product are done by using the projective model structure on the category of complexes of k-vectors spaces.

$(\mathrm{Vec}, \otimes)$ can be seen as an commutative algebra object in the symmetric monoidal infinity category $(\infty, 1)\text{-Cat}^{\text{ex}}_{\text{cts}}$.

> **Definition**
> $\mathrm{DGCat}_{\text{cts}}$ denotes the infinity category of left modules over Vec inside $(\infty, 1)\text{-Cat}^{\text{ex}}_{\text{cts}}$.
>
> [GR1, Ch 1, 3.4, 10.3.3,]

It is beyond the scope of this paper to give a precise definition of this. We will however describe some properties sufficient for our purposes.

Let $C \in \mathrm{DGCat}_{\text{cts}}$. Then there will be a functor $\mathrm{Vec} \otimes C \to C$ in $(\infty, 1)\text{-Cat}^{\text{ex}}$. Since the unit for the symmetric monoidal structure of Vec is k, the functor $k \otimes \_ \in \mathrm{Fun}^{\text{ex}}_{\text{cts}}(C, C)$ will be isomorphic to the the identity functor of C. Fixing $x \in C$, we obtain a functor $\_ \otimes x \in (\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{cts}}(\mathrm{Vec}, C)$. By the adjoint functor theorem, we have an adjunction

$$\_ \otimes x \dashv \mathrm{Hom}_C(x, \_) : \mathrm{Vec} \rightleftarrows C$$

in $(\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}}$. This way for every pair of objects $x, y \in C$, we have a complex of k-vector spaces $\mathrm{Hom}_C(x,y) \in \mathrm{Vec}$.

There is also a notion of *tensor product of dg-categories over* Vec with the expected universal property.

Proposition

Let $C, D \in \mathrm{DGCat}_{\mathrm{cts}}$. Then there exists $C \otimes_{\mathrm{Vec}} D \in \mathrm{DGCat}_{\mathrm{cts}}$ together with a functor $\boxtimes : C \times D \to C \otimes_{\mathrm{Vec}} D$ that is Vec-linear and continuous in each component and for any $E \in \mathrm{DGCat}_{\mathrm{cts}}$, we have an equivalence

$$\mathrm{DGCat}_{\mathrm{cts}}(C \otimes_{\mathrm{Vec}} D, E) \xrightarrow{\sim} \mathrm{DGCat}_{\mathrm{Bi-Cts}}(C \times D, E)$$

given by restriction along $\boxtimes$ and the latter is the full subcategory of $\mathrm{DGCat}(C \times D, E)$ which is Vec-linear and continuous in each component. [GR1, Ch 1, 10.4]

2.6 Derived Schemes

Now that we have explained the kind of category of sheaves, we describe the spaces the sheaves will be on. The big picture to follow is the following :

$$\begin{tikzcd}
\mathrm{Aff}^{\mathrm{red}} \arrow[r, "\subseteq"] \arrow[d] & \mathrm{Aff} \arrow[r, "\subseteq"] \arrow[d] & \mathrm{dAff} \arrow[d] & \mathrm{PSh}\,\mathrm{dAff}^{\leq n} \arrow[r, shift right] \arrow[d, shift right, "\mathrm{Lan}"'] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] & \mathrm{PSh}\,\mathrm{dAff}^{\leq n}_{\mathrm{ft}} \arrow[l, shift right, "\mathrm{Lan}"'] \arrow[d, shift right, "\mathrm{Lan}"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{PStk}_{\mathrm{red}} \arrow[r, shift left, "\mathrm{Lan}"] \arrow[r, phantom, "\perp"] & \mathrm{PStk}_{\mathrm{cl}} \arrow[l, shift left, "\mathrm{red}"] \arrow[r, shift left, "\mathrm{Lan}"] \arrow[r, phantom, "\perp"] & \mathrm{PStk} \arrow[l, shift left, "\mathrm{cl}"] \arrow[r, shift right] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "L"'] \arrow[d, phantom, "\dashv"] & \mathrm{PSh}\,\mathrm{dAff}^{<\infty} \arrow[l, shift right] \arrow[u, shift right, "\leq n"'] \arrow[r, shift left] \arrow[r, phantom, "\top"] \arrow[d, shift left, "\mathrm{Ran}"] \arrow[d, phantom, "\dashv"] & \mathrm{PSh}\,\mathrm{dAff}^{<\infty}_{\mathrm{ft}} \arrow[l, shift left, "\mathrm{Lan}"] \arrow[u, shift right] \\
 & & \mathrm{Stk}_{\acute{e}t} \arrow[u, shift right, "\subseteq"'] & \mathrm{PStk}_{\mathrm{laft}} \arrow[ul, "\subseteq"] \arrow[u, shift left] \arrow[ur, "\sim"] &
\end{tikzcd}$$

$$\mathrm{dSch}_{\mathrm{aft}} := \mathrm{dSch}_{\mathrm{qc}} \cap \mathrm{Stk}_{\acute{e}t} \cap \mathrm{PStk}_{\mathrm{laft}}$$

Let us explain.

- dAff refers to the opposite of the infinity category of commutative algebra objects in Vec.

  Practically speaking, $\mathrm{dAff}^{\mathrm{op}}$ can be realised as the localisation of commutative dg-algebras concentrated in non-positive degree with respect to a suitable model structure. (Alternatively, localisation of simplicial commutative rings w.r.t. suitable model structure. However, you would then need to show is equivalent to commutative algebra objects in Vec.) We use cohomological grading so homological degree refer to negative cohomological degree.

  Aff is the full subcategory of dAff of classical affine schemes, and $\mathrm{Aff}^{\mathrm{red}}$ the reduced classical affine schemes, which we will simply refer to as *reduced* for brevity.

We use Spec to denote the Yoneda embedding of any of $\mathrm{Aff}^{\mathrm{red}}, \mathrm{Aff}, \mathrm{dAff}$ into their presheaf ∞-categories.

– The left Kan extension functors $\mathrm{PStk}_{\mathrm{cl}} \to \mathrm{PStk}$ are obtained from the universal property of presheaf ∞-categories to inclusion of full subcategories $\mathrm{Aff} \subseteq \mathrm{dAff}$.

  Note that since small colimits in presheaf ∞-categories are computed pointwise, res preserves small colimits. We obtain that $\mathrm{res}\,\mathrm{Lan} \xrightarrow{\sim} \mathbb{1}$ since this is the case on derived affines. Thus, Lan is fully faithful. In other words, we can safely think of classical prestacks as special cases of prestacks.

  The same reasoning applies for $\mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}_{\mathrm{cl}}$ and the composite $\mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}$. So we can think of reduced prestacks as special cases of classical prestacks, and also of prestacks.

  We henceforth view $\mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}_{\mathrm{cl}} \to \mathrm{PStk}$ as inclusions of full subcategories.

– For $X \in \mathrm{PStk}$, we say X has an underlying scheme when the underlying classical prestack is a scheme.

– [GR1, Ch 2, 1.2.3, 1.2.7] $\mathrm{dAff}^{<\infty} := \bigcup_{0\leq n} \mathrm{dAff}^{\leq n}$ where $\mathrm{dAff}^{\leq n}$ is the full subcategory of dAff consisting of $\operatorname{Spec} A$ such that $\pi_{i>n} A \simeq H^{i<-n} A \simeq 0$. Derived affine schemes in $\mathrm{dAff}^{<\infty}$ are called *eventually coconnective*.[1]

  Since $\mathrm{dAff}^{<\infty} \subseteq \mathrm{dAff}$ is a full subcategory, we obtain a triple of adjoints $\mathrm{Lan} \dashv \mathrm{res} \dashv \mathrm{Ran} : \mathrm{PStk} \rightleftarrows \mathrm{PSh}\,\mathrm{dAff}^{<\infty}$. Then essential image of Ran is of importance due to the following characterisation.

  > **Proposition**
  >
  > Let $X \in \mathrm{PStk}$. Then X is in the essential image of $\mathrm{Ran} : \mathrm{PSh}\,\mathrm{dAff}^{<\infty} \to \mathrm{PStk}$ if and only if for all $\operatorname{Spec} A \in \mathrm{dAff}$, we have $X(A) \xrightarrow{\sim} \varprojlim_{0\leq n} X(\tau^{\leq n} A)$. We call such prestacks *convergent*. [GR1, Ch2, 1.4.7]

– $\mathrm{dAff}_{\mathrm{aft}}$ is defined as the full subcategory of dAff consisting of derived affines of *almost finite type*. This means $\operatorname{Spec} A$ where $H^0 A$ is finite type over the base field k and $H^{i>0} A$ are finite generated over $H^0 A$.[GR1, Ch 2, 1.7.1] We also define

$$\mathrm{dAff}^{\leq n}_{\mathrm{ft}} := \mathrm{dAff}^{\leq n} \cap \mathrm{dAff}_{\mathrm{aft}}$$

  With the same argument as in previous points, we obtain an adjunction $\mathrm{Lan} \dashv \mathrm{res} : \mathrm{PSh}\,\mathrm{dAff}^{\leq n}_{\mathrm{ft}} \rightleftarrows \mathrm{PSh}\,\mathrm{dAff}^{\leq n}$ where Lan is fully faithful so we can see the former as a full subcategory of the latter. The objects of this subategory has the following charactersation :

  > **Proposition**
  >
  > Let $X \in \mathrm{PSh}\,\mathrm{dAff}^{\leq n}$. Then X lies in $\mathrm{PSh}\,\mathrm{dAff}^{\leq n}_{\mathrm{ft}}$ if and only if as a functor $(\mathrm{dAff}^{\leq n})^{\mathrm{op}} \to \mathrm{Spc}$, it preserves small filtered colimits. [GR1, Ch 2, 1.6.4]

  Finally, the building blocks of derived algebraic geometry which will allow us to do computations

$$\mathrm{dAff}^{<\infty}_{\mathrm{ft}} := \mathrm{dAff}^{<\infty} \cap \mathrm{dAff}_{\mathrm{aft}} = \bigcup_{0\leq n} \mathrm{dAff}^{\leq n}_{\mathrm{ft}}$$

  We have a characterisation of the prestacks determined by how $S \in \mathrm{dAff}^{<\infty}_{\mathrm{ft}}$ map into them. These are the *laft prestacks*.

[1] The "coconnectivity" refers to *cohomological grading*. So eventually coconnective means the same thing as *"homologically eventually connective"*.

Proposition – Characterisation of Prestacks of Locally Almost Finite Type
Restriction along $\mathrm{dAff}_{\mathrm{ft}}^{<\infty} \subseteq \mathrm{dAff}$ gives an equivalence

$$\begin{tikzcd}
\mathrm{PStk}_{\mathrm{laft}} \arrow[rr, shift left, "\mathrm{res}"] & & \mathrm{PSh}\, \mathrm{dAff}_{\mathrm{ft}}^{<\infty} \arrow[ll, shift left, "\sim"] \arrow[dl, "\mathrm{Lan}"] \\
& \mathrm{PSh}\, \mathrm{dAff}^{<\infty} \arrow[ul, "\mathrm{Ran}"] &
\end{tikzcd}$$

where

- $\mathrm{PStk}_{\mathrm{laft}}$ is the full subcategory of PStk consisting of X such that
  1. X is convergent.
  2. X takes filtered limits in $\mathrm{dAff}^{\leq n}$ to filtered colimits in Spc, equivalently $X^{\leq n}$ lies in $\mathrm{PSh}\, \mathrm{dAff}_{\mathrm{ft}}^{\leq n}$.
- an inverse is given by first *left* Kan extending along $\mathrm{dAff}_{\mathrm{ft}}^{<\infty} \subseteq \mathrm{dAff}^{<\infty}$ then *right* Kan extending along $\mathrm{dAff}^{<\infty} \subseteq \mathrm{dAff}$.

[GR1, Ch 2, 1.7.6]

This is the class of prestacks for which IndCoh will be developed.

We also have that $\mathrm{PStk}_{\mathrm{laft}} \subseteq \mathrm{PStk}$ preserves finite limits, which will be useful for us later. [GR1, Ch 2 , 1.7.10]

One can also show that $\mathrm{dAff}_{\mathrm{aft}} = \mathrm{dAff} \cap \mathrm{PStk}_{\mathrm{laft}}$. [GR1, Ch 2 , 1.7.2]

- In the adjunction $L \dashv \text{forget} : \mathrm{PStk} \rightleftarrows \mathrm{Stk}_{\text{ét}}$, we have $\mathrm{Stk}_{\text{ét}}$ denoting the full subcategory of PStk consisting of prestacks satisfying étale descent. L is the sheafification functor. We refer the reader to [GR1, Ch 2, 2.3] for details of the definition.
- dSch denotes the full subcategory of PStk consisting of *derived schemes*. See [GR1, Ch 2, 3.1.1] for the definition. A thing to note is that the derived schemes in [GR1] have diagonals that are closed embeddings and affine by definition.

  $\mathrm{dSch}_{\mathrm{qc}}$ denotes the full subcategory of dSch consisting of derived schemes which admit a finite Zariski cover by derived affines.

  Finally, $\mathrm{dSch}_{\mathrm{aft}}$ denotes the ∞-category of *derived schemes of almost finite type*. [GR1, Ch 2, 3.5] The theory of IndCoh will be built from these spaces.

3 IndCoherent Sheaves on Derived Schemes

3.1 Quasi-Coherent Sheaves

The theory of ind-coherent sheaves does not stand alone from the theory of quasi-coherent sheaves. So before starting with ind-coherent sheaves, we briefly gather some basic facts about quasi-coherent sheaves. Specifically, how they are set up in derived algebraic geometry.

> **Proposition – Contravariant** QCoh
> There is a functor $\mathrm{QCoh}^* : \mathrm{dAff} \to (\infty, 1)\text{-}\mathrm{Cat}^{\mathrm{op}}$ that assigns to each $A \in \mathrm{dAff}$ the *derived category of A-modules*, denoted $A\text{-Mod}$.
>
> For $\mathrm{Spec}\, A \in \mathrm{dAff}$, $A\text{-Mod}$ is ∞-category of left modules over A in Vec. It can be realised as localisation of left modules over A as a commutative dg-algebra w.r.t. some model structure. $A\text{-Mod} = \mathrm{QCoh}\, A$ is a cocomplete and complete stable infinity category.
>
> For $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, $f^* : \mathrm{QCoh}\, A \to \mathrm{QCoh}\, B$ can be realised as derived tensoring $A \overset{L}{\otimes}_k \_$.

Remark – Concrete description of $A\text{-Mod}$*.* For A discrete (i.e. a commutative ring), there is the following description of $A\text{-Mod}$ under the quasi-category model of ∞-categories summarised in a single diagram :

$$\begin{tikzcd}
N(\mathrm{Ch}\, A) \arrow[d, "\subseteq"'] \arrow[dr, "W^{-1}"] & & & \\
N_{dg}(\mathrm{Ch}\, A) \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & N_{dg}((\mathrm{Ch}\, A)_f) \arrow[l, shift left, "\supseteq"] \arrow[d, "h"] & =: & A\text{-Mod} \\
& D(A) & &
\end{tikzcd}$$

Explanation :

- $\mathrm{Ch}\, A$ is the honest-to-god dg-category of chain complexes of honest-to-god A-modules and $D(A)$ is the category of complexes of injectives
- $\mathrm{Ch}\, A$ has a model structure such that cofibrations are degree-wise injections and weak equivalences are quasi-isomorphisms. [Lurie-HA, Prop 1.3.5.3] Although the class of fibrations are defined abstractly as those satisfying right lifting with respect to acyclic cofibrations, it turns out that any fibrant complex must be degree-wise injective, and partially conversely, any bounded above complex of injectives is fibrant. [Lurie-HA, Prop 1.3.5.6] $(\mathrm{Ch}\, A)_f$ denotes the full subcategory of fibrant complexes.
- N denotes the nerve functor which converts 1-categories to simplicial sets, which have the property of being ∞-categories. N_{dg} denotes the dg-nerve functor which achieves the same thing for honest-to-god dg-catgeory categories. (See [Kerodon, Subsection 2.5.3] for a construction.)

  We have that the homotopy category of $A\text{-Mod}$ gives the usual derived category of A-modules, as in classical algebraic geometry.

– L is a left adjoint to the inclusion $N_{dg}((\mathrm{Ch}\, A)_f) \subseteq N_{dg}(\mathrm{Ch}\, A)$. Intuitively, for every complex $M_\bullet$, there exists a acyclic cofibration $M_\bullet \to I_\bullet$ to fibrant $I_\bullet$ and this is initial in the category of arrows from $M_\bullet$ into $N_{dg}((\mathrm{Ch}\, A)_f)$. [Lurie-HA, Prop 1.3.5.12] This means for each $M_\bullet$, such a morphism $M_\bullet \to I_\bullet$ is unique up to equivalence and assembles to the desired functor L. Practically speaking, $L(M_\bullet) \simeq I_\bullet$.

– The composition $N(\mathrm{Ch}\, A) \to N_{dg}((\mathrm{Ch}\, A)_f)$ exhibits the latter as the ∞-categorical localisation of the former at quasi-isomorphisms. [Lurie-HA, Prop 1.3.5.15] This matches the standard treatment in classical algebraic geometry : the localisation functor from $\mathrm{Ch}\, A$ to $D(A)$ takes a complex and resolves it by injecting it quasi-isomorphically into a complex of injectives.

Remark. For $\mathrm{Spec}\, A \in \mathrm{dAff}$, $\mathrm{QCoh}\, A$ has obvious t-structure.

Proposition – Pullback is Symmetric Monoidal

The functor $\mathrm{QCoh}^* : \mathrm{dAff} \to (\infty,1)\text{-}\mathrm{Cat}^{\mathrm{op}}$ admits a lifts to a functor $\mathrm{QCoh}^* : \mathrm{dAff} \to (\mathrm{cMon}\,((\infty,1)\text{-}\mathrm{Cat}))^{\mathrm{op}}$, where the latter is the opposite of the ∞-category of commutative monoid objects in $(\infty,1)$-Cat, A.K.A. symmetric monoidal ∞-categories with symmetric monoidal functors.

[Lurie-HA, Prop 4.5.3.1]

Remark. We take for granted that $\mathrm{QCoh}^* : \mathrm{dAff} \to (\infty,1)\text{-}\mathrm{Cat}^{\mathrm{op}}$ as mentioned so far can be lifted to $\mathrm{QCoh}^* : \mathrm{dAff} \to \mathrm{DGCat}^{\mathrm{op}}_{\mathrm{cts}}$. For individual $\mathrm{QCoh}\, A$, this is not hard to believe since the hom complexes of $\mathrm{QCoh}\, A$ are left A-modules, and hence in particular, complexes of k-vector spaces.

With that, we are ready for quasi-coherent sheaves on all prestacks.

Definition – Quasi-coherent Sheaves on Prestacks

We define $\mathrm{QCoh}^* : \mathrm{PStk} \to \mathrm{DGCat}^{\mathrm{op}}_{\mathrm{cts}}$ as the left Kan extension of $\mathrm{QCoh}^* : \mathrm{dAff} \to \mathrm{DGCat}^{\mathrm{op}}_{\mathrm{cts}}$.

Remark. For $X \in \mathrm{PStk}$, $\mathrm{QCoh}\, X$ has obvious t-structure that is checked pointwise. We refer the reader to [GR1, Ch 3, 1.5].

For $X \in \mathrm{dSch}$, the heart of $\mathrm{QCoh}\, X$ gives the abelian 1-category of quasi-coherent sheaves.

There is also a covariant QCoh.

Proposition

There exists a functor $\mathrm{QCoh}_* : \mathrm{PStk} \to \mathrm{DGCat}$ which at the level of objects does the same as QCoh^* but at the level of morphisms $f : X \to Y$, its image f_* is obtained as the right adjoint to f^* via the adjoint functor theorem. [GR1, Ch 3, 2.1.1]

Remark. It is important to note that f_* is *not* in general continuous.

3.2 Ind-Coherent Sheaves on a single Derived Scheme

Definition – IndCoh

Let $X \in \mathrm{dSch}_{\mathrm{aft}}$. For $\mathcal{F} \in \mathrm{QCoh}\, X$, we say $\mathcal{F}$ is *coherent* when it has finitely many non-zero cohomologies and for all $i \in \mathbb{Z}$, $H^i\mathcal{F} \in (\mathrm{QCoh}\, X)^\heartsuit$ is coherent in the classical sense. We use $\mathrm{Coh}\, X$ to denote the full subcategory of $\mathrm{QCoh}\, X$ consisting of coherent sheaves. [GR1, Intro, 1.1.]

The ∞-category $\mathrm{IndCoh}\, X$ is defined as the ind-completion $\mathrm{Ind}\,(\mathrm{Coh}\, X)$. [GR1, Ch 4, 1.1.1]

Lemma (*t-structure on* IndCoh). *By the universal property of ind-completions, we have*

$$\begin{tikzcd} \mathrm{Coh}\, X \arrow[r] \arrow[d] & \mathrm{QCoh}\, X \\ \mathrm{IndCoh}\, X \arrow[ur, dashed, "\Psi_X"'] & \end{tikzcd}$$

where Ψ_X *left Kan extends the fully faithful inclusion* $\mathrm{Coh}\, X \to \mathrm{QCoh}\, X$.

Then there is an unique t-structure on $\mathrm{IndCoh}\, X$ *such that*

- $(\mathrm{IndCoh}\, X)^{\leq 0} = \Psi_X^{-1}(\mathrm{QCoh}\, X)^{\leq 0}$ *full subcat* $\mathrm{IndCoh}\, X$
- Ψ_X *is t-exact.*
- $\mathrm{IndCoh}(X)^{0\leq}$ *is closed under filtered colimits*
- *for any* $n \in \mathbb{Z}$, Ψ_X *gives an equivalence* $\mathrm{IndCoh}(X)^{n\leq} \xrightarrow{\sim} \mathrm{QCoh}(X)^{n\leq}$.

In particular, $\mathrm{Coh}\, X \subseteq (\mathrm{IndCoh}\, X)^{+}$.

Proof. See [GR1, Ch 4, 1.2.2] ■

Proposition
Let $X \in \mathrm{dSch}_{\mathrm{aft}}$. Then $\mathrm{Coh}\, X$ is idempotent complete. Hence the Yoneda embedding $\mathrm{Coh}\, X \xrightarrow{\sim} (\mathrm{IndCoh}\, X)^c$ is an equivalence.

[GR1, Ch 4 , 1.2.8]

Proof. This can be checked at the level of the triangulated category $h\mathrm{Coh}\, X$ since $\mathrm{Coh}\, X$ is a stable ∞-category. □

3.3 Pushforward and Pullback across Various Morphisms

We describe what IndCoh does to morphisms in $\mathrm{dSch}_{\mathrm{aft}}$. Pushforward for IndCoh is easy to describe. It follows from the definition of the t-structure of IndCoh.

Proposition – Pushforward for IndCoh **on** $\mathrm{dSch}_{\mathrm{aft}}$
Let $f : X \to Y$ be in $\mathrm{dSch}_{\mathrm{aft}}$. Then there exists a unique $f_*^{\mathrm{IndCoh}} \in \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{IndCoh}\, X, \mathrm{IndCoh}\, Y)$ that is compatible with Ψ meaning the following commutes :

$$\begin{tikzcd} \mathrm{IndCoh}\, X \arrow[r, "\Psi_X"] \arrow[d, "f_*^{\mathrm{IndCoh}}"'] & \mathrm{QCoh}\, X \arrow[d, "f_*^{\mathrm{QCoh}}"] \\ \mathrm{IndCoh}\, Y \arrow[r, "\Psi_Y"] & \mathrm{QCoh}\, Y \end{tikzcd}$$

Proof. This is a sketch of [GR1, Ch 4 , 2.1.2].

This crucially uses the fact that Ψ gives an equivalence $(\mathrm{IndCoh}\, X)^+ \xrightarrow{\sim} (\mathrm{QCoh}\, X)^+$. Using the t-structure of $\mathrm{IndCoh}\, X$ and the universal property of ind-completions in order to give f_*^{IndCoh} it suffices to show that $f_*^{\mathrm{QCoh}}\Psi_X$ maps $(\mathrm{IndCoh}\, X)^+$ into $(\mathrm{QCoh}\, Y)^+$. This comes down to the fact that f_*^{QCoh} is a right derived functor and hence sends $(\mathrm{QCoh}\, X)^+$ to $(\mathrm{QCoh}\, Y)^+$. $\square$

In [GR1, Ch 4, 2.2], functoriality of pushforward is shown.

Proposition
There exists a functor
$$\mathrm{IndCoh}_* : \mathrm{dSch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cts}}$$
equipped with a natural transformation
$$\Psi : \mathrm{IndCoh}_* \to \mathrm{QCoh}_*$$
such that on objects and morphisms it yields
$$\mathrm{IndCoh}^*(X) = \mathrm{IndCoh}(X) \quad f : X \to Y \rightsquigarrow f_* : \mathrm{IndCoh}\, X \to \mathrm{IndCoh}\, Y$$
[GR1, Ch 4, 2.2.3]

We now describe pullback for IndCoh. This is more delicate because it needs to achieve two things :

- for proper $f : X \to Y$ in $\mathrm{dSch}_{\mathrm{aft}}$, we want to have an adjunction
$$f_* \dashv f^! : \mathrm{IndCoh}\, X \rightleftarrows \mathrm{IndCoh}\, Y$$
inside $\mathrm{DGCat}_{\mathrm{cts}}$.
- for an open embedding $j : U \to X$ in $\mathrm{dSch}_{\mathrm{aft}}$, we want to have an adjunction
$$j^! \dashv j_* : \mathrm{IndCoh}\, U \rightleftarrows \mathrm{IndCoh}\, X$$

Let us first show how this pullback functors exist in the individual cases.

Proposition – Pullback for IndCoh along Proper Morphisms
Let $f : X \to Y$ be a proper morphism in $\mathrm{dSch}_{\mathrm{aft}}$. Then $f_* : \mathrm{IndCoh}\, X \to \mathrm{IndCoh}\, Y$ maps $\mathrm{Coh}\, X$ into $\mathrm{Coh}\, Y$. Hence we have an adjunction
$$f_* \dashv f^! : \mathrm{IndCoh}\, X \rightleftarrows \mathrm{IndCoh}\, Y$$

where $f^! \in \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{IndCoh}\, Y, \mathrm{IndCoh}\, X)$.

Proof. This is a sketch of [GR1, Ch 4, 5.1.4]. By construction of f_* for IndCoh, it suffice to show the result for $f_* : \mathrm{QCoh}\, X \to \mathrm{QCoh}\, Y$. By a standard argument with t-structures, it suffices to show $f_*(\mathrm{Coh}\, X)^\heartsuit \subseteq \mathrm{Coh}\, Y$. We try to the case of X, Y both classical. First, notice that the inclusion $i : X^{\mathrm{cl}} \to X$ induces an equivalence $(\mathrm{Coh}\, X)^\heartsuit \simeq (\mathrm{Coh}\, X)^\heartsuit$. Combining this with the fact that the inclusion $Y^{\mathrm{cl}} \to Y$ maps $\mathrm{Coh}\, Y^{\mathrm{cl}} \to \mathrm{Coh}\, Y$, it suffices to prove the result for $X^{\mathrm{cl}} \to Y^{\mathrm{cl}}$. We are now in the classical case [Stacks, Prop 30.19.1]. This proves $f^*\mathrm{Coh}\, X \subseteq \mathrm{Coh}\, Y$.

The fact that we have an adjunction $f_* \dashv f^!$ is simply an application of the adjoint functor theorem since f_* is by definition in $\mathrm{DGCat}_{\mathrm{cts}}$. The content of this proposition is that since $\mathrm{Coh}\, X \sim (\mathrm{IndCoh}\, X)^c$ and f_* preserves compact objects, it follows that $f^!$ is continuous. ☐

Proposition – Proper Base Change for IndCoh

Suppose we have the following cartesian square in $\mathrm{dSch}_{\mathrm{aft}}$:

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

where g and v are proper. Then the dashed morphism below obtained from $g_*u_* \simeq f_*v_*$ is an equivalence :

$$\begin{tikzcd} u_*v^! \arrow[r, dashed] \arrow[d] & g^! f_* \\ g^! g_* u_* v^! \arrow[r, "\sim"'] & g^! f_* v_* u^! \arrow[u] \end{tikzcd}$$

Proof. This is [GR1, Ch 4, 5.2.2]. The proof follows basically from the compatibility of the continuous right adjoint $f^!$ for IndCoh and the *not necessarily continuous* right adjoint $f^!$ for QCoh under the equivalence Ψ : $(\mathrm{IndCoh}\,\_)^+ \simeq (\mathrm{IndCoh}\,,\_)^+$, the adjunction $f_* \dashv f^! : \mathrm{QCoh}\, X \rightleftarrows \mathrm{QCoh}\, Y$ from classical algebraic geometry, and finally base change for QCoh along quasi-compact schematic morphisms. [GR1, Ch 3, 2.2.2] ☐

Proposition – Pullback for IndCoh across Open Embeddings

Let $j : U \to X$ be an open embedding. Then we have an adjunction

$$j^! \dashv j_* : \mathrm{IndCoh}\, U \rightleftarrows \mathrm{IndCoh}\, X$$

where j_* is fully faithful.

Proof. This is a shortened version of [GR1, Ch 4, 3] applied to the special case of open embeddings. First note that $j^* : \mathrm{QCoh}\, X \to \mathrm{QCoh}\, U$ maps $\mathrm{Coh}\, X$ into $(\mathrm{QCoh}\, U)^+ \simeq (\mathrm{IndCoh}\, U)^+ \subseteq \mathrm{IndCoh}\, U$. So by the universal property of ind-completions, we obtain $j^* \in \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{IndCoh}\, X, \mathrm{IndCoh}\, U)$. We claim that $j^! := j^*$ works.

To show $j^! \dashv j_*$ on IndCoh, since both are continuous, it suffices to check on the full subcategory of compact objects. This is $\mathrm{Coh}\, X$, respectively $\mathrm{Coh}\, U$. The adjunction now follows from $j^* \dashv j_* : \mathrm{QCoh}\, X \rightleftarrows \mathrm{QCoh}\, U$.

The fully faithfulness of j_*, equivalently the fact that $j^* j_* \to \mathbb{1}$ is an equivalence is checked in the same way : on compact objects. ☐

Now, the idea for $f^!$ for a general $f : X \to Y$ in $\mathrm{dSch}_{\mathrm{aft}}$ is to use Nagata's compactification theorem [Stacks, Thm 38.33.8] to factor $f = pj$ where p is proper and j is an open embedding and define

$$f^! := p^! j^!$$

In [GR1, Ch 5 , 2.1], it is then shown that the ∞-category $\mathrm{Factor}(f)$ of factorings of f into an open embedding followed by a proper morphism is contractible and that this gives functoriality for !-pullback. Unfortunately, the proof is much too complicated so we will omit the proof.

> **Proposition – Functoriality of !-Pullback**
> There exists a functor $\mathrm{IndCoh}^! : \mathrm{dSch}_{\mathrm{aft}} \to (\mathrm{DGCat}_{\mathrm{cts}})^{\mathrm{op}}$ such that for proper f in $\mathrm{dSch}_{\mathrm{aft}}$, the image of f recovers $f^!$ and for open embeddings j in $\mathrm{dSch}_{\mathrm{aft}}$, the image of j recovers $j^!$. [GR1, Ch 5 , 3.1.4]

Finally, we obtain ind-coherent sheaves on all laft prestacks.

> **Definition – Ind-coherent sheaves on laft prestacks**
> We define $\mathrm{IndCoh}^! : \mathrm{PStk}_{\mathrm{laft}} \to (\mathrm{DGCat}_{\mathrm{cts}})^{\mathrm{op}}$ as the left Kan extension of $\mathrm{IndCoh}^! : \mathrm{dAff}_{\mathrm{ft}}^{<\infty} \to (\mathrm{DGCat}_{\mathrm{cts}})^{\mathrm{op}}$ via the universal property of presheaf ∞-categories . [GR1, Ch 5 , 3.4.1]

3.4 Integral Transforms and Duality for Quasi-Coherent Sheaves

Let $X, Y \in \mathrm{PStk}$. The theory of integral transforms for quasi-coherent sheaves says that under suitable conditions on X, Y, there is an equivalence in $\mathrm{DGCat}_{\mathrm{cts}}$:

$$\mathrm{QCoh}(Y \times X) \simeq \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{QCoh}\, X, \mathrm{QCoh}\, Y)$$

which sends a sheaf $\mathcal{Q}$ on $Y \times X$ to the functor $(p_Y)_*(p_X^*(\_) \otimes \mathcal{Q})$. For intuition, there is a nice analogy with linear algebra :

k base field	Vec
1-category of k-modules $k\mathrm{Mod}$	$\mathrm{DGCat}_{\mathrm{cts}} = \mathrm{VecMod}$
Set	PStk
$k^{-} = \mathrm{Set}(\_, k) : \mathrm{Set} \to (k\mathrm{Mod})^{\mathrm{op}}$	$\mathrm{QCoh}^* : \mathrm{PStk} \to (\mathrm{DGCat}_{\mathrm{cts}})^{\mathrm{op}}$
s, t finite sets	X, Y aft derived schemes
$t \leftarrow t \times s \to s$	$X \leftarrow X \times Y \to Y$
$k^s \to k^{t \times s}, [v_j]_{j \in s} \mapsto [v_j]_{i,j \in t \times s}$	p_X^*
$k^{t \times s} \to k^t, [A_{i,j}]_{i,j \in t \times s} \mapsto \left[\sum_{j \in s} A_{i,j}\right]_{i \in t}$	$(p_Y)_*$
$k^s \simeq k^{\oplus s}$	$\mathrm{QCoh}\, X \simeq \mathrm{Ind}\left((\mathrm{QCoh}\, X)^c\right)$
pointwise multiplcation on $k^{t \times s}$	$\otimes$ on $\mathrm{QCoh}(Y \times X)$
$k^{t \times s} \simeq k^s \otimes k^t$	$\mathrm{QCoh}(X \times Y) \simeq \mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, Y$
$\langle \_, \_ \rangle : k^s \otimes k^s \to k$	$\langle \_, \_ \rangle : \mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, X \to \mathrm{Vec}$
$\mathbb{1} \otimes \langle \_, \_ \rangle : k^t \otimes k^s \otimes k^s \to k^t$	$\mathbb{1} \otimes \langle \_, \_ \rangle : \mathrm{QCoh}\, Y \otimes \mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, X \to \mathrm{QCoh}\, Y$
$k^s \otimes k^t \simeq \mathrm{Hom}_k(k^s, k^t)$	$\mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, Y \simeq \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{QCoh}\, X, \mathrm{QCoh}\, Y)$
$k^{t \times s} \simeq \mathrm{Hom}_k(k^s, k^t)$	$\mathrm{QCoh}(Y \times X) \simeq \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{QCoh}\, X, \mathrm{QCoh}\, Y)$
$A \mapsto \left(v \mapsto \left[\sum_{j \in s} A_{i,j} v_j\right]_{i \in t}\right)$	$\mathcal{Q} \mapsto (p_2)_*(p_1^*(\_) \otimes \mathcal{Q})$

Following the analogy, the ansatz for the integral transform equivalence is the following composition :

$$\mathrm{QCoh}(Y \times X) \xleftarrow[(1)]{\sim} \mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, Y \xrightarrow[(2)]{\sim} \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{QCoh}\, X, \mathrm{QCoh}\, Y)$$

where (2) should come from $\mathrm{QCoh}\, X$ being "self-dual" in a suitable sense and (1) comes from a computation on "basis elements".

Let us first address the topic of duality. There are many definitions of dualisability in a symmetric monoidal ∞-category. We will take the following to be equivalent for granted.

> **Proposition – Dualisable Objects**
>
> Let C be a symmetric monoidal ∞-category. Let $e : x^\vee \otimes x \to 1_C$ where 1_C is the unit. Then the following are equivalent :
>
> 1. (duality datum) there exists $c : 1_C \to x \otimes x^\vee$ such that we have the following commuting triangles in hC :

$$\begin{tikzcd} x \arrow[r, "c\otimes\mathbb{1}"] \arrow[dr, "\mathbb{1}"'] & x\otimes x^\vee \otimes x \arrow[d, "\mathbb{1}\otimes e"] \\ & x \end{tikzcd} \qquad \begin{tikzcd} x^\vee \arrow[d, "\mathbb{1}\otimes c"'] \arrow[dr, "\mathbb{1}"] & \\ x^\vee\otimes x\otimes x^\vee \arrow[r, "e\otimes\mathbb{1}"'] & x^\vee \end{tikzcd}$$

2. (Adjunction) for all $a \in C$, the morphism $\mathbb{1} \otimes e : a \otimes x^\vee \otimes x \to a$ induces an equivalence

$$\begin{tikzcd} C(b, a\otimes x^\vee) \arrow[r, "\_\otimes x"] \arrow[dr, "\sim"'] & C(b\otimes x, a\otimes x^\vee \otimes x) \arrow[d, "(\mathbb{1}\otimes e)\_"] \\ & C(b\otimes x, a) \end{tikzcd}$$

3. (Internal Hom) Under the extra condition that C has internal hom $\underline{\mathrm{Hom}}$, for any object $a \in C$, the morphism $e \otimes \mathbb{1} : a \otimes x^\vee \otimes x \to a$ induces an equivalence

$$a \otimes x^\vee \to \underline{\mathrm{Hom}}(x, a)$$

[Lurie-HA, Prop 4.6.1.6] An object x is called *dualisable* when there exists another object $x^\vee$ together with a morphisms $e : x^\vee \otimes x \to 1_C$ satisfying any (and thus all) of the above.

Remark. In the definition of dualisable objects, we assumed C is symmetric monoidal. This implies given a dualisable pair $e : x^\vee \otimes x \to 1_C$, $e_{\text{transposed}} : x \otimes x^\vee \to 1_C$ is also a dualisable pair. In particular, $(x^\vee)^\vee \simeq x$.

Definition
Let $C \in \mathrm{DGCat}_{\mathrm{cts}}$. Then C is called *dualisable* when it is dualisable as an object in $\mathrm{DGCat}_{\mathrm{cts}} = \mathrm{VecMod}$.

Remark. Let's unpack the above a bit. To say $C \in \mathrm{DGCat}_{\mathrm{cts}}$ is dualisable means there exists $C^\vee \in \mathrm{DGCat}_{\mathrm{cts}}$ and $e \in \mathrm{DGCat}_{\mathrm{cts}}(C^\vee \otimes C, \mathrm{Vec})$ such that for all $D \in \mathrm{DGCat}_{\mathrm{cts}}$, we have an equivalence

$$\begin{tikzcd} \mathrm{DGCat}_{\mathrm{cts}}(E, D\otimes C^\vee) \arrow[r, "\_\otimes C"] \arrow[dr, "\sim"'] & \mathrm{DGCat}_{\mathrm{cts}}(E\otimes C, D\otimes C^\vee \otimes C) \arrow[d, "(\mathbb{1}\otimes e)\_"] \\ & \mathrm{DGCat}_{\mathrm{cts}}(E\otimes C, D) \end{tikzcd}$$

In particular, applying $D = \mathrm{Vec}$, gives $C^\vee \simeq \mathrm{DGCat}_{\mathrm{cts}}(C, \mathrm{Vec})$. On the other hand, applying $E = \mathrm{Vec}$ and a transposition yields $C^\vee \otimes D \simeq \mathrm{DGCat}_{\mathrm{cts}}(C, D)$.

The following shows that compact generation of C is sufficient for being dualisable and in fact, the dual $C^\vee$ has a explicit description.

Proposition – Duality for Compactly Generated DG ∞-Categories
Let $C \in \mathrm{DGCat}_{\mathrm{cts}}$ be compactly generated. Consider the following :

$$\begin{tikzcd}
(C^c)^{\mathrm{op}} \times C^c \arrow[r] \arrow[dr, "\mathrm{Hom}_C"'] & \mathrm{Ind}((C^c)^{\mathrm{op}}) \times C \arrow[r] \arrow[d, dashed] & \mathrm{Ind}((C^c)^{\mathrm{op}}) \otimes C \arrow[dl, dashed, "e"] \\
& \mathrm{Vec} &
\end{tikzcd}$$

The enriched hom bi-functor Hom_C on compact objects is Vec-linear and exact in both variables hence the middle dashed morphism by left Kan extension in each variable/ This is Vec-linear and continuous in both variables, hence the right dashed morphism in $\mathrm{DGCat}_{\mathrm{cts}}$ by the universal property of tensors. Then e exhibits $\mathrm{Ind}((C^c)^{\mathrm{op}})$ as the dual of C. [GR1, Ch 1, 7.3.2]

Proof. For any $E \in \mathrm{DGCat}_{\mathrm{cts}}$, we have the following equivalences

$$\begin{aligned}
\mathrm{DGCat}_{\mathrm{cts}}(\mathrm{DGCat}_{\mathrm{cts}}(C,D),E) &\simeq \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{DGCat}(C^c,D),E) \\
&\simeq \mathrm{DGCat}_{\mathrm{cts}}(E^{\mathrm{op}},\mathrm{DGCat}(C^c,D)^{\mathrm{op}}) \\
&\simeq \mathrm{DGCat}_{\mathrm{cts}}(E^{\mathrm{op}},\mathrm{DGCat}((C^c)^{\mathrm{op}},D^{\mathrm{op}})) \\
&\simeq \mathrm{DGCat}((C^c)^{\mathrm{op}},\mathrm{DGCat}_{\mathrm{cts}}(E^{\mathrm{op}},D^{\mathrm{op}})) \\
&\simeq \mathrm{DGCat}((C^c)^{\mathrm{op}},\mathrm{DGCat}_{\mathrm{cts}}(D,E)) \\
&\simeq \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{Ind}((C^c)^{\mathrm{op}}),\mathrm{DGCat}_{\mathrm{cts}}(D,E)) \\
&\simeq \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{Ind}((C^c)^{\mathrm{op}}) \otimes D,E)
\end{aligned}$$

where we have used the following equivalence without proof :

$$\mathrm{DGCat}_{\mathrm{cts}}(\_,\star) \xrightarrow{\sim} \mathrm{DGCat}_{\mathrm{cts}}(\star^{\mathrm{op}},\_^{\mathrm{op}})$$

where the forward direction is obtained by taking right adjoints via the adjoint functor theorem and the right side is well-defined because presentable ∞-categories have small limits . ☐

Proposition – $\mathrm{QCoh}\,A$ Compactly Generated (Derived Affine Case)
Let $\mathrm{Spec}\,A \in \mathrm{dAff}$ and $C := \mathrm{QCoh}\,A$. Then $C \simeq \mathrm{Ind}\,C(\omega)$ where $C(\omega)$ is the smallest stable full subcategory of C containing A. In particular, C is compactly generated.

Furthermore, for $M \in C$, the following are equivalent :

- M is a retract of objects in $C(\omega)$.
- M is compact
- M is dualisable

Proof. Let $C := \mathrm{QCoh}\,A$. One can check that A is a compact object of $\mathrm{QCoh}\,A$ by using the 1-categorical characterisation of compact objects and the fact that homotopy coproducts in $h\mathrm{QCoh}\,A$ coincide with taking degree-wise coproducts. Similarly, one can check that A is a generator at the level of the 1-categorical derived category $h\mathrm{QCoh}\, A$. Hence $C \simeq \mathrm{Ind}\, C(\omega)$. We also deduce that $C(\omega) \to C^c$ exhibits C^c as the idempotent completion of $C(\omega)$. Therefore compact objects coincide with retracts of objects in $C(\omega)$.

We now show an $M \in C$ is compact if and only if it is dualisable. This is a slightly different proof to [BZFN, Lem 3.4].

Suppose M is compact. Our ansatz for $M^\vee$ is $\underline{\mathrm{Hom}}(M, 1)$, where $\underline{\mathrm{Hom}} := \underline{\mathrm{Hom}}_S$ is the internal hom of $\mathrm{QCoh}\, S$. For $e : M \otimes \underline{\mathrm{Hom}}(M, 1) \to 1$, we choose it to correspond to the identity morphism $\underline{\mathrm{Hom}}(M, 1) \to \underline{\mathrm{Hom}}(M, 1)$ under the adjunction $M \otimes \_ \dashv \underline{\mathrm{Hom}}(M, \_)$. It remains to show that the natural transformation $\_ \otimes \underline{\mathrm{Hom}}(M, 1) \to \underline{\mathrm{Hom}}(M, \_)$ coming from $\mathbb{1} \otimes e : \_ \otimes M \otimes \underline{\mathrm{Hom}}(M, 1) \to \_$ is an equivalence. By assumption, $\underline{\mathrm{Hom}}(M, \_)$ preserves small coproducts. Hence, both $\underline{\mathrm{Hom}}(M, \_)$ and $\_ \otimes \underline{\mathrm{Hom}}(M, 1)$ preserves small colimits. Since $\mathrm{QCoh}\, S =: C \simeq \mathrm{Ind}\, C(\omega)$, it suffices to show that $\_ \otimes \underline{\mathrm{Hom}}(M, 1) \to \underline{\mathrm{Hom}}(M, \_)$ is an equivalence on $C(\omega)$. Since $C(\omega)$ is obtained from $1 = \mathcal{O}(S)$ by iteratively adding it cofibers and both functors preserve small colimits, it suffices to show that $1 \otimes \underline{\mathrm{Hom}}(M, 1) \to \underline{\mathrm{Hom}}(M, 1)$ is an equivalence. This is clear.

Now assume M is dualisable. Then $\underline{\mathrm{Hom}}(M, \_) \simeq \_ \otimes M^\vee$ and hence preserves small coproducts. Therefore M is compact.

$\square$

Proposition – $\mathrm{QCoh}\, X$ Compactly Generated (Derived Scheme Case)
Let $X \in \mathrm{dSch}_{\mathrm{qc}}$. Then $\mathrm{QCoh}\, X \simeq \mathrm{Ind}\,\mathrm{Perf}\, X$ where $\mathcal{F}$ is in the full subcategory $\mathrm{Perf}\, X$ if and only if any of the following equivalent conditions are true :

- $\mathcal{F}$ is dualisable
- $\mathcal{F}$ is compact
- for all $x : S \to X$ where S derived affine, $\mathcal{F}_x \in \mathrm{QCoh}\, S$ is compact.

Proof. We omit the proof of $\mathrm{QCoh}\, X \simeq \mathrm{Ind}\,\mathrm{Perf}\, X$ for brevity. See [BZFN, Prop 3.19] for a proof. For $\mathcal{F} \in \mathrm{QCoh}\, X$, the fact that $\mathcal{F}$ is dualisable if and only if it is fiberwise dualisable is proved in [BZFN, Prop 3.6]. The equivalence of dualisability and compactness is shown in [BZFN, Prop 3.9]. $\square$

We now show that $\mathrm{QCoh}\, X$ has a *self-duality* when X is a quasi-compact derived scheme.

Proposition – Self Duality of $\mathrm{QCoh}\, X$
Let $X \in \mathrm{dSch}_{\mathrm{qc}}$. Define

$$\langle \_, \_ \rangle : \mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, X \xrightarrow{\otimes} \mathrm{QCoh}\, X \xrightarrow{(p_X)_*} \mathrm{Vec}$$

Then $\langle \_, \_ \rangle$ exhibits $\mathrm{QCoh}\, X$ as its own dual. Let $\mathbb{D} : \mathrm{QCoh}\, X \simeq (\mathrm{QCoh}\, X)^\vee$ correspond to $\langle \_, \_ \rangle$. We refer to $\mathbb{D}$ as the *naive dualisation functor*.

Proof. This proved in [GR1, Ch 1 , 9.2] for general stable rigid monoidal ∞-categories. What follows is an application of the general proof to this example.

We know that the dual of $\mathrm{QCoh}\, X \simeq \mathrm{Ind}\,\mathrm{Perf}\, X$ is $\mathrm{Ind}((\mathrm{Perf}\, X)^{\mathrm{op}})$. We thus compute the naive dualisation functor as :

$$\begin{aligned} \mathrm{QCoh}\, X \to \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{QCoh}\, X, \mathrm{Vec}) &\xleftarrow{\sim} (\mathrm{QCoh}\, X)^\vee \\ \mathcal{F} \mapsto \langle \_, \mathcal{F}\rangle \simeq \mathrm{Hom}_X(\mathbb{D}\mathcal{F}, \_) &\mapsfrom \mathbb{D}\mathcal{F} \end{aligned}$$

It suffices to show $\mathbb{D}$ is an equivalence. Since $\mathrm{QCoh}\, X$ is the ind-completion of $\mathrm{Perf}\, X$, it suffices to show that $\mathbb{D}$ maps $\mathrm{Perf}\, X$ fully faithfully into $(\mathrm{Perf}\, X)^{\mathrm{op}}$ inside $(\mathrm{QCoh}\, X)^\vee$.

Let $\mathcal{F} \in \mathrm{Perf}\, X$. Then

$$\mathrm{Hom}_X(\mathbb{D}\mathcal{F}, \_) \simeq \langle \_, \mathcal{F}\rangle \simeq \mathrm{Hom}_X(\mathcal{O}_X, \_ \otimes \mathcal{F}) \simeq \mathrm{Hom}_X(\mathcal{F}^\vee, \_) \simeq \mathrm{Hom}_X(\underline{\mathrm{Hom}}_X(\mathcal{F}, \mathcal{O}_X), \_)$$

Therefore on $\mathrm{Perf}\, X$, the dualisation functor $\mathbb{D}$ lands in the image under Yoneda of $(\mathrm{Perf}\, X)^{\mathrm{op}}$. Identifying $(\mathrm{Perf}\, X)^{\mathrm{op}}$ as a full subcategory of $(\mathrm{QCoh}\, X)^\vee$, we see that $\mathbb{D}(\_) \simeq \_^\vee \simeq \underline{\mathrm{Hom}}_X(\_, \mathcal{O}_X)$ which is an equivalence $\mathrm{Perf}\, X \simeq (\mathrm{Perf}\, X)^{\mathrm{op}}$ because this is simply taking dual objects.

□

This sorts out the equivalence (2) for $X \in \mathrm{dSch}_{\mathrm{qc}}$. We now address the equivalence (1).

Proposition – Integral Transform for Quasi-Coherent Sheaves

Let $X, Y \in \mathrm{PStk}$. Suppose $\mathrm{QCoh}\, X$ is dualisable in $\mathrm{DGCat}_{\mathrm{cts}}$. Then we have an equivalence

$$\begin{tikzcd} \boxtimes \quad : & \mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, Y \arrow[r, "\sim"] & \mathrm{QCoh}(Y \times X) \\ & \mathrm{QCoh}\, X \times \mathrm{QCoh}\, Y \arrow[u] \arrow[ur, "p_X^*(\_)\otimes p_Y^*(\_)"'] & \end{tikzcd}$$

[GR1, Ch 3, 3.1.7]

Proof. What follows is the proof of [GR1, Ch 3 , 3.1.7] completed with details from [BZFN].

By the universal property of the tensor product in $\mathrm{DGCat}_{\mathrm{cts}}$, it suffices to give an abstract equivalence $\mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, Y \simeq \mathrm{QCoh}(Y \times X)$ that restricts to $p_X^* \otimes p_Y^*$ on $\mathrm{QCoh}\, X \times \mathrm{QCoh}\, Y$.

Let us show that it suffices to show for any derived affines S, T, we have that $\boxtimes : \mathrm{QCoh}\, S \otimes \mathrm{QCoh}\, T \to$

$\mathrm{QCoh}(T \times S)$ is an equivalence. Assume this. Then we have the following chain of equivalences :

$$
\begin{aligned}
&\mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, Y \\
&\xrightarrow{\sim} \mathrm{QCoh}\, X \otimes \left(\varprojlim_{T \in \mathrm{dAff}/Y} \mathrm{QCoh}\, T \right) && \text{by } \mathrm{QCoh}^* \text{ left Kan extension} \\
&\xrightarrow{\sim} \varprojlim_{T \in \mathrm{dAff}/Y} \mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, T && \text{by } \mathrm{QCoh}\, X \text{ dualisable} \\
&\xrightarrow{\sim} \varprojlim_{T \in \mathrm{dAff}/Y} \left(\varprojlim_{S \in \mathrm{dAff}/X} \mathrm{QCoh}\, S \right) \otimes \mathrm{QCoh}\, T && \text{by } \mathrm{QCoh}^* \text{ left Kan extension} \\
&\xrightarrow{\sim} \varprojlim_{T \in \mathrm{dAff}/Y} \varprojlim_{S \in \mathrm{dAff}/X} \mathrm{QCoh}\, S \otimes \mathrm{QCoh}\, T && \text{by } \mathrm{QCoh}\, T \text{ dualisable} \\
&\xrightarrow{\sim} \varprojlim_{(T,S) \in \mathrm{dAff}/Y \times \mathrm{dAff}/X} \mathrm{QCoh}(S \times T) && \text{by assumption and limits commute with limits} \\
&\xleftarrow{\sim} \varprojlim_{R \in \mathrm{dAff}/Y \times X} \mathrm{QCoh}\, R && \text{by } \mathrm{dAff}/Y \times \mathrm{dAff}/X \to \mathrm{dAff}/Y \times X \text{ cofinal} \\
&\xleftarrow{\sim} \mathrm{QCoh}(Y \times X) && \text{by } \mathrm{QCoh}^* \text{ left Kan extension}
\end{aligned}
$$

The fact that restricting along $\otimes : \mathrm{QCoh}\, X \times \mathrm{QCoh}\, Y \to \mathrm{QCoh}\, X \otimes \mathrm{QCoh}\, Y$ produces $p_X^*(\_) \otimes p_Y^*(\_)$ comes from its definition, which amounts to the following commuting diagram :

$$
\begin{tikzcd}
\mathrm{QCoh}\, X \times \mathrm{QCoh}\, Y \arrow[r, "p_X^*(\_) \otimes p_Y^*(\_)"] \arrow[d, "\sim"'] & \mathrm{QCoh}(Y \times X) \arrow[d, "\sim"] \\
\varprojlim_{(T,S) \in \mathrm{dAff}/Y \times \mathrm{dAff}/X} \mathrm{QCoh}\, S \times \mathrm{QCoh}\, T \arrow[r, "\varprojlim p_S^*(\_) \otimes p_T^*(\_)"'] & \varprojlim_{T \times S \in \mathrm{dAff}/Y \times X} \mathrm{QCoh}\,(T \times S)
\end{tikzcd}
$$

We thus reduce to the special case of derived affines. We need to compute with the tensor product of dg-∞-categories.

> *Lemma* (*GR1 Ch1 7.4.2*). *Let $C, D \in \mathrm{DGCat}_{\mathrm{cts}}$ with compactly generating sets of objects $C_0 \subseteq C$ and $D_0 \subseteq D$. Then the set $\{c_0 \otimes d_0 \,|\, c_0 \in C_0, d_0 \in D_0\}$ compactly generates $C \otimes D$. Furthermore, for $c_0, d_0 \in C_0, D_0$ and $c, d \in C, D$, we have an equivalence*
>
> $$\mathrm{Hom}_C(c_0, c) \otimes \mathrm{Hom}_D(d_0, d) \simeq \mathrm{Hom}_{C \otimes D}(c_0 \otimes d_0, c \otimes d)$$
>
> [*GR1, Ch 1 , 7.4.2*]
>
> *Proof.* $C \simeq \mathrm{Ind}\, C_0$ implies $C^\vee := \mathrm{Ind}(C_0^{\mathrm{op}})$ is a dual for C. Then we have
>
> $$C \otimes D \simeq \mathrm{DGCat}_{\mathrm{cts}}(C^\vee, D) \simeq \mathrm{DGCat}(C_0^{\mathrm{op}}, \mathrm{DGCat}(D_0^{\mathrm{op}}, \mathrm{Vec})) \simeq \mathrm{Fun}_{\mathrm{Vec}}^{\mathrm{Bi-Ex}}(C_0^{\mathrm{op}} \times D_0^{\mathrm{op}}, \mathrm{Vec})$$
>
> where the latter is the dg-∞-category of functors $C_0^{\mathrm{op}} \times D_0^{\mathrm{op}}$ which are Vec-linear and exact in each

variable. Under these equivalences, the functor $\boxtimes : C \times D \to C \otimes D$ corresponds to

$$\begin{aligned} C \times D &\to \mathrm{DGCat}(C_0^{\mathrm{op}}, \mathrm{Vec}) \times \mathrm{DGCat}(D_0^{\mathrm{op}}, \mathrm{Vec}) \quad \text{enriched Yoneda} \\ &\xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Vec}}^{\mathrm{Bi-Ex}}(C_0^{\mathrm{op}} \times D_0^{\mathrm{op}}, \mathrm{Vec} \times \mathrm{Vec}) \\ &\to \mathrm{Fun}_{\mathrm{Vec}}^{\mathrm{Bi-Ex}}(C_0^{\mathrm{op}} \times D_0^{\mathrm{op}}, \mathrm{Vec}) \quad \text{compose with } \otimes \end{aligned}$$

Let h the above composition. Let Hom denote the enriched hom of $\mathrm{Fun}_{\mathrm{Vec}}^{\mathrm{Bi-Ex}}(C_0^{\mathrm{op}} \times D_0^{\mathrm{op}}, \mathrm{Vec})$. Now applying the tensor-hom adjunction in Vec with two applications of enriched Yoneda's lemma shows that for any $F \in \mathrm{Fun}_{\mathrm{Vec}}^{\mathrm{Bi-Ex}}(C_0^{\mathrm{op}} \times D_0^{\mathrm{op}}, \mathrm{Vec})$, and $(c_0, d_0) \in C_0 \times D_0$, we have

$$\mathrm{Hom}(h(c_0, d_0), F) \xrightarrow{\sim} F(c_0, d_0) \text{ in } \mathrm{Vec}$$

Since h corresponds to $\boxtimes$, this shows that $\{c_0 \otimes d_0 \mid c_0 \in C_0, d_0 \in D_0\}$ compactly generates $C \otimes D$.

Finally, for formula, $c \otimes d$ and $c_0 \otimes d_0$ transported across the equivalences yield $h(c, d)$ and $h(c_0, d_0)$. So we have

$$\mathrm{Hom}_{C \otimes D}(c_0 \otimes d_0, c \otimes d) \simeq \mathrm{Hom}(h(c_0, d_0), h(c, d)) \simeq \mathrm{Hom}_C(c_0, c) \otimes \mathrm{Hom}_D(d_0, d)$$

as desired.

∎

Finally, we are ready.

Lemma (Integral Transforms for Quasi-Coherent Sheaves on Derived Affines). For $S, T \in \mathrm{dAff}$, we have the equivalence $\boxtimes : \mathrm{QCoh}\, S \otimes \mathrm{QCoh}\, T \xrightarrow{\sim} \mathrm{QCoh}(S \times T)$.

Proof. We follow the approach of [BZFN, Prop 4.6]. By the theory of compactly generated stable ∞-categories , it suffices to show that $\mathrm{QCoh}\, S \otimes \mathrm{QCoh}\, T$ has a small compactly generating full subcategory that gets sent fully faithfully to a compactly generating full subcategory of $\mathrm{QCoh}(S \otimes T)$. Of course, we choose to show that

1. $\{M \boxtimes N \in \mathrm{QCoh}(S \times T) \mid M \in (\mathrm{QCoh}\, S)^c, N \in (\mathrm{QCoh}\, T)^c\}$ compactly generates $\mathrm{QCoh}(S \times T)$
2. for all $M, M_1 \in (\mathrm{QCoh}\, S)^c$ and $N, N_1 \in (\mathrm{QCoh}\, T)^c$, we have

$$\mathrm{Hom}_S(M, M_1) \otimes \mathrm{Hom}_T(N, N_1) \xrightarrow{\sim} \mathrm{Hom}_{S \times T}(M \boxtimes N, M_1 \boxtimes N_1)$$

For (1), we use that fact that for $M \in \mathrm{QCoh}\, S$ where S is derived affine, M is compact if and only if it is dualisable. Therefore, if $M, N \in \mathrm{QCoh}\, S, \mathrm{QCoh}\, T$ are compact, then they are dualisable. Since for any morphism f of derived affines, f^* is symmetric monoidal, we obtain that $p_S^*(M), p_T^*(N)$ are both dualisable in $\mathrm{QCoh}(S \times T)$. It is not hard to show that the tensor product of two dualisable objects is again dualisable, so we obtain $M \boxtimes N$ is dualisable, and hence compact in $\mathrm{QCoh}(S \times T)$.

Now we show that $\{M \boxtimes N \mid M, N \text{ compact } \in \mathrm{QCoh}\, S, \mathrm{QCoh}\, T\}$ generates $\mathrm{QCoh}(S \times T)$. What follows is essentially a series of definitions. Let $P \in \mathrm{QCoh}(S \times Y)$ and assume that

$$\pi_0 \mathrm{Hom}_{S\times T}((\mathrm{QCoh}\, S)^c \boxtimes (\mathrm{QCoh}\, T)^c, P) \simeq 0$$

Then for any $N \in (\mathrm{QCoh}\, T)^c$, we have

$$0 \simeq \pi_0 \mathrm{Hom}_{S\times T}((\mathrm{QCoh}\, S)^c \boxtimes N, P) \simeq \pi_0 \mathrm{Hom}_S((\mathrm{QCoh}\, S)^c, (p_S)_* \underline{\mathrm{Hom}}_{S\times T}(p_T^* N, P))$$

Since $(\mathrm{QCoh}\, S)^c$ compactly generates $\mathrm{QCoh}\, S$, we obtain $0 \simeq (p_S)_* \underline{\mathrm{Hom}}_{S\times T}(p_T^* N, P)$ for any $N \in (\mathrm{QCoh}\, T)^c$.

We are now in the world of algebra. For $N \in (\mathrm{QCoh}\, T)^c$, $0 \simeq (p_S)_* \underline{\mathrm{Hom}}_{S\times T}(p_T^* N, P)$ says that the $\mathcal{O}(S \times T)$-module $\underline{\mathrm{Hom}}_{S\times T}(p_T^* N, P)$ is zero after applying the forgetful functor $\mathcal{O}(S\times T)\mathrm{Mod} \to \mathcal{O}(S)\mathrm{Mod}$. Thus

$$\begin{aligned} &0 \simeq \underline{\mathrm{Hom}}_{S\times T}(p_T^* N, P) \\ &\Rightarrow 0 \simeq \mathrm{Hom}_{S\times T}(\mathcal{O}(S\times T), \underline{\mathrm{Hom}}_{S\times T}(p_T^* N, P)) \simeq \mathrm{Hom}_{S\times T}(p_T^* N, P) \simeq \mathrm{Hom}_T(N, (p_T)_* P) \end{aligned}$$

This holds for all $N \in (\mathrm{QCoh}\, T)^c$. So by compact generation of $\mathrm{QCoh}\, T$, we see that $0 \simeq (p_T)_* P$. This just says $0 \simeq P$ as an $\mathcal{O}(T)$-module. This clearly implies $0 \simeq P$ as an $\mathcal{O}(S \times T)$-module, finishing the proof of (1).

(2)

$$\begin{aligned}
& \mathrm{Hom}_{S\times T}(M \boxtimes N, M_1 \boxtimes N_1) && \\
\simeq\ & \Gamma(S\times T, (p_S^*(M) \otimes p_T^*(N))^\vee \otimes p_S^*(M_1) \otimes p_T^*(N_1)) && \text{tensor of dualisables is dualisable} \\
\simeq\ & \Gamma(S\times T, p_S^*(M^\vee) \otimes p_T^*(N^\vee) \otimes p_S^*(M_1) \otimes p_T^*(N_1)) && p^* \text{ preserve dualisables} \\
\simeq\ & \Gamma\left(S\times T, p_S^*(M^\vee \otimes M_1) \otimes p_T^*(N^\vee \otimes N_1)\right) && p^* \text{ symmetric monoid} \\
\simeq\ & \Gamma\left(S\times T, p_S^*(\underline{\mathrm{Hom}}_S(M, M_1) \otimes p_T^*(\underline{\mathrm{Hom}}_T(N, N_1)))\right) && \text{internal hom and dualisable} \\
\simeq\ & \Gamma\left(T, (p_T)_* \left(p_S^*(\underline{\mathrm{Hom}}_S(M, M_1) \otimes p_T^*(\underline{\mathrm{Hom}}_T(N, N_1)))\right)\right) && p_{T*} \text{ definition} \\
\simeq\ & \Gamma\left(T, (p_T)_* p_S^*(\underline{\mathrm{Hom}}_S(M, M_1)) \otimes \underline{\mathrm{Hom}}_T(N, N_1)\right) && \text{projection formula for } S\times T \to T \\
\simeq\ & \mathrm{Hom}_S(M, M_1) \otimes \mathrm{Hom}_T(N, N_1) && ???
\end{aligned}$$

The special case of projection formula for $S \times T \to T$ can be deduced as follows. We wish to show that for all $M \in \mathrm{QCoh}(S \times T)$, we have an equivalence :

$$((p_T)_* M) \otimes \_ \xrightarrow{\sim} (p_T)_* \left(M \otimes p_T^* \_\right)$$

as endo-functors of $\mathrm{QCoh}\, T$. Let $C := \mathrm{QCoh}\, T$. We have seen that $C \simeq \mathrm{Ind}\, C(\omega)$ where $C(\omega)$ is the smallest stable full subcategory of C containing $\mathcal{O}(T)$. It suffices that the above two functors are continuous and that they agree on $C(\omega)$. Since $\otimes$ is continuous in each variable, $((p_T)_* M) \otimes \_$ is continuous. Continuity of $(p_T)_* \left(M \otimes p_T^* \_\right)$ follows from continuity of $p_T^*, M \otimes \_, (p_T)_*$ where the last one uses our specific situation $S \times T \to T$ of derived affines. Now, to show the two functors agree on $C(\omega)$, recall that $C(\omega)$ is obtained from $\mathcal{O}(T)$ by first taking all shifts, then iteratively adding in cofibers. Since the two functors preserve shifts and cofibers, it suffices that they agree on $\mathcal{O}(T)$. This is now clear. ■

□

3.5 Integral Transforms and Duality for Ind-Coherent Sheaves

The integral transform theorem for IndCoh follows the same strategy as that of QCoh.

$$\mathrm{IndCoh}(Y \times X) \xleftarrow[(1)]{\sim} \mathrm{IndCoh}\, X \otimes \mathrm{IndCoh}\, Y \xrightarrow[(2)]{\sim} \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{IndCoh}\, X, \mathrm{IndCoh}\, Y)$$

As in the argument for QCoh, (2) comes from a self-duality of $\mathrm{IndCoh}\, X$. But let us address (1) first.

Proposition – Integral Transform for Ind-Coherent Sheaves
Let $X, Y \in \mathrm{PStk}_{\mathrm{laft}}$. Assume $\mathrm{IndCoh}\, X$ is dualisable. Let us also assume that the base field k is perfect. Then

$$\boxtimes : \mathrm{IndCoh}\, X \otimes \mathrm{IndCoh}\, Y \xrightarrow{\sim} \mathrm{IndCoh}(X \times Y)$$

Proof. As in the first part of the proof of integral transforms for quasi-coherent sheaves, we reduce to the case of $X, Y \in \mathrm{dAff}_{\mathrm{ft}}^{<\infty}$. The result is in fact true for all $X, Y \in \mathrm{dSch}_{\mathrm{aft}}$.

Lemma. Let $X, Y \in \mathrm{dSch}_{\mathrm{aft}}$. Then

$$\boxtimes : \mathrm{IndCoh}\, X \otimes \mathrm{IndCoh}\, Y \xrightarrow{\sim} \mathrm{IndCoh}(X \times Y)$$

is fully faithful. If the base field k is perfect, then $\boxtimes$ is an equivalence.

Proof. We refer the reader to [GR1, Ch 4 , 6.3.4]. ■

□

It remains to describe self-duality for $\mathrm{IndCoh}\, X$ when $X \in \mathrm{dSch}_{\mathrm{aft}}$. Note that by definition, $\mathrm{IndCoh}\, X$ is compactly generated and hence dualisable. What's interesting is that the self-duality recovers the classical Serre duality for coherent sheaves. First, we need to give $\mathrm{IndCoh}\, X$ a symmetric monoidal structure.

Proposition
Let $X, Y \in \mathrm{dSch}_{\mathrm{aft}}$. Then there exists a unique functor

$$\boxtimes \in \mathrm{DGCat}_{\mathrm{cts}}\left(\mathrm{IndCoh}\, X \otimes \mathrm{IndCoh}\, Y, \mathrm{IndCoh}(Y \times X)\right)$$

that preserves compact objects and makes the following diagram commute :

$$\begin{tikzcd}
\operatorname{IndCoh} X \otimes \operatorname{IndCoh} Y \arrow[r, "\boxtimes"] \arrow[d, "\Psi_X \otimes \Psi_Y"'] & \operatorname{IndCoh}(Y \times X) \arrow[d, "\Psi_{Y \times X}"] \\
\operatorname{QCoh} X \otimes \operatorname{QCoh} Y \arrow[r, "\boxtimes"'] & \operatorname{QCoh}(Y \times X)
\end{tikzcd}$$

Proof. Technical. See [GR1, Ch 4 , 6.3.2] for details. □

In [GR1, Ch 5 , 4.1], functoriality of ⊠ is shown.

Proposition
There is a symmetric monoidal structure on $\operatorname{IndCoh}^! : \operatorname{dSch}_{\text{aft}}^{\text{op}} \to \operatorname{DGCat}_{\text{cts}}$ such that for $X, Y \in \operatorname{dSch}_{\text{aft}}$, the morphism

$$\operatorname{IndCoh} X \otimes \operatorname{IndCoh} Y \to \operatorname{IndCoh}(Y \times X)$$

from the symmetric monoidal structure of $\operatorname{IndCoh}^!$ recovers ⊠.

Proof. Omitted. See [GR1, Ch 5 , 4.1]. □

Remark. Assuming the above result, we can give each $\operatorname{IndCoh} X$ a symmetric monoidal structure by the following trick from Gaitsgory–Rozenblyum. A symmetric monoidal dg-∞-category is by their definition a commutative algebra object in $\operatorname{DGCat}_{\text{cts}}$. The idea is that the source of commutative algebra structure on $\operatorname{IndCoh} X$ actually comes a commutative algebra structure on X *itself*. This means the following.

Lemma (*Symmetric Monoidal Structure on* PStk). *There is a symmetric monoidal structure* $\otimes$ *on* $\operatorname{PStk}^{\text{op}}$ *that gives for each* $X, Y \in \operatorname{PStk}^{\text{op}}$,

$$X \otimes Y \simeq X \times Y \in \operatorname{PStk}^{\text{op}}$$

Furthermore, we have a functor $\operatorname{PStk}^{\text{op}} \to \operatorname{cAlg}(\operatorname{PStk}^{\text{op}})$ *which for every* $X \in \operatorname{PStk}^{\text{op}}$, *gives it a commutative algebra structure such that the multiplication*

$$X \times X \to X$$

is given by (*the opposite of*) *the diagonal* Δ^{op}.

Proof. The proof is not so important for our purposes. We refer the reader to [Lurie-HA, Prop 2.4.1.5] where the result is proved for any ∞-category admitting all finite products. ■

Definition
The symmetric monoidal functor $\operatorname{IndCoh}^! : \operatorname{dSch}_{\text{aft}}^{\text{op}} \to \operatorname{DGCat}_{\text{cts}}$ sends commutative algebra objects to commutative algebra objects. So we have the following :

$$\begin{tikzcd}
\mathrm{dSch}_{\mathrm{aft}}^{\mathrm{op}} \arrow[r, "\mathrm{IndCoh}^!"] \arrow[d, shift right, "\text{use diagonal}"'] & \mathrm{DGCat}_{\mathrm{cts}} \\
\mathrm{cAlg}(\mathrm{dSch}_{\mathrm{aft}}^{\mathrm{op}}) \arrow[r, "\mathrm{IndCoh}^!"'] \arrow[u, shift right, "\text{“forget”}"'] & \mathrm{cAlg}(\mathrm{DGCat}_{\mathrm{cts}}) \arrow[u, "\text{“forget”}"']
\end{tikzcd}$$

Hence we obtain a factoring of $\mathrm{IndCoh}^!$ through the forgetful functor $\mathrm{cAlg}(\mathrm{DGCat}_{\mathrm{cts}}) \to \mathrm{DGCat}_{\mathrm{cts}}$.

For $X \in \mathrm{dSch}_{\mathrm{aft}}$, we use $\overset{!}{\otimes}$ to denote the symmetric monoidal operation on $\mathrm{IndCoh}\, X$.

[GR1, Ch 5 , 4.1.3]

Remark. Tracing through the above functor $\mathrm{IndCoh}^! : \mathrm{dSch}_{\mathrm{aft}}^{\mathrm{op}} \to \mathrm{cAlg}(\mathrm{dSch}_{\mathrm{aft}}^{\mathrm{op}}) \to \mathrm{cAlg}(\mathrm{DGCat}_{\mathrm{cts}})$, we see that for $X \in \mathrm{dSch}_{\mathrm{aft}}$, the symmetric monoidal operation for $\mathrm{IndCoh}\, X$ is the following :

$$\begin{tikzcd}
X \times X \arrow[d] & \rightsquigarrow & \mathrm{IndCoh}\, X \otimes \mathrm{IndCoh}\, X \arrow[r, "\boxtimes"] \arrow[dr, "\overset{!}{\otimes}"'] & \mathrm{IndCoh}(X \times X) \arrow[d, "\Delta_X^!"] \\
X & & & \mathrm{IndCoh}\, X
\end{tikzcd}$$

Proposition – “Serre Duality”
Let $X \in \mathrm{dSch}_{\mathrm{aft}}$ and the base field k be perfect. Define

$$\langle \_, \_ \rangle : \mathrm{IndCoh}\, X \otimes \mathrm{IndCoh}\, X \xrightarrow{\overset{!}{\otimes}} \mathrm{IndCoh}\, X \xrightarrow{(p_X)_*} \mathrm{Vec}$$

Then $\langle \_, \_ \rangle$ exhibits $\mathrm{IndCoh}\, X$ as its own dual. Let $\mathbb{D}_{\mathrm{Serre}} : \mathrm{IndCoh}\, X \simeq (\mathrm{IndCoh}\, X)^\vee$ correspond to $\langle \_, \_ \rangle$. We refer to $\mathbb{D}_{\mathrm{Serre}}$ as the *Serre duality functor*.

Proof. Let us first say that the way in which [GR1] obtains $\mathbb{D}_{\mathrm{Serre}}$ is highly abstract. This is to ensure the duality has higher categorical functoriality. However, at [GR1, Ch 5 , Rmk 4.2.8], it is mentioned that one does not need all the machinery of *correspondences* to deduce Serre duality for each individual $X \in \mathrm{dSch}_{\mathrm{aft}}$ and a method is briefly explained. What follows is an expansion of the remark.

Let $e := \langle \_, \_ \rangle$. Following , we will explicitly give a morphism $c : \mathrm{Vec} \to \mathrm{IndCoh}\, X \otimes \mathrm{IndCoh}\, X$ in $\mathrm{DGCat}_{\mathrm{cts}}$ making (c, e) into duality datum. The idea of [GR1] is that the duality datum (e, c) should come from a duality datum on the underlying space X with itself. This is the philosophy of *correspondences*. The key behind X being self-dual is essentially the following :

$$\begin{tikzcd}
Y \times X & R \arrow[l] \arrow[d] & \leftrightsquigarrow & Y & \\
& Z & & R \arrow[u] \arrow[r] & X \times Z
\end{tikzcd}$$

Applying the above to the identity correspondence $X \leftarrow X \rightarrow X$ yields two correspondences :

$$\begin{tikzcd} X \times X & X \arrow[l] \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} k & \\ X \arrow[u] \arrow[r] & X \times X \end{tikzcd}$$

Using the fact $\boxtimes$ is an equivalence for IndCoh, the left correspondence gives e and the right correspondence leads us to define $c := (\Delta_X)_* p_X^!$. We need to show the following triangles commute in $\mathrm{DGCat}_{\mathrm{cts}}$:

$$\begin{tikzcd} \operatorname{IndCoh} X \arrow[r, "c \otimes \mathbb{1}"] \arrow[dr, "\mathbb{1}"'] & (\operatorname{IndCoh} X)^{\otimes 3} \arrow[d, "\mathbb{1} \otimes e"] \\ & \operatorname{IndCoh} X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{IndCoh} X \arrow[d, "\mathbb{1} \otimes c"'] \arrow[dr, "\mathbb{1}"] & \\ (\operatorname{IndCoh} X)^{\otimes 3} \arrow[r, "e \otimes \mathbb{1}"'] & \operatorname{IndCoh} X \end{tikzcd}$$

Again, the source of the above commuting triangles should be at the level of spaces. Indeed, the functors in the above diagram come from the following correspondences :

$$\begin{tikzcd} X & X \times X \arrow[l, "(p_X, \mathbb{1})"'] \arrow[d, "(\Delta_X, \mathbb{1})"'] & X \arrow[l, "\Delta_X"] \arrow[d, "\Delta_X"] \arrow[dl, phantom, "\llcorner", very near start] \\ & X \times X \times X & X \times X \arrow[l, "(\mathbb{1}, \Delta_X)"] \arrow[d, "(\mathbb{1}, p_X)"] \\ & & X \end{tikzcd} \qquad \begin{tikzcd} X & & \\ X \times X \arrow[u, "(\mathbb{1}, p_X)"] \arrow[r, "(\mathbb{1}, \Delta_X)"] & X \times X \times X & \\ X \arrow[u, "\Delta_X"] \arrow[r, "\Delta_X"] \arrow[ur, phantom, "\urcorner", very near start] & X \times X \arrow[u, "(\Delta_X, \mathbb{1})"'] \arrow[r, "(p_X, \mathbb{1})"'] & X \end{tikzcd}$$

Thus, we are reduced to showing IndCoh base change for the cartesian square :

$$\begin{tikzcd} X \times X \arrow[d, "(\Delta_X, \mathbb{1})"'] & X \arrow[l, "\Delta_X"'] \arrow[d, "\Delta_X"] \arrow[dl, phantom, "\llcorner", very near start] \\ X \times X \times X & X \times X \arrow[l, "(\mathbb{1}, \Delta_X)"] \end{tikzcd}$$

where all morphisms are closed embeddings (since in [GR1]'s definition of dSch the diagonal morphism is assumed to be a closed embedding). This follows from proper base change for IndCoh.

$\square$

3.6 Relation between Quasi-Coherent Sheaves and Ind-Coherent Sheaves

There is also a functor turning quasi-coherent sheaves into ind-coherent sheaves for all laft prestacks. This is crucial for the proving the equivalence of left and right crystals.

Definition – Action of $\mathrm{QCoh}\,X$ **on** $\mathrm{IndCoh}\,X$

Let $X \in \mathrm{dSch}_{\mathrm{aft}}$. We have seen that $\mathrm{QCoh}\,X = \mathrm{Ind}((\mathrm{QCoh}\,X)^c)$.

Then the tensor product $\mathrm{QCoh}\,X \otimes \mathrm{QCoh}\,X \to \mathrm{QCoh}\,X$ which is a morphism in $\mathrm{DGCat}_{\mathrm{cts}}$ restricts to a tensor product $\mathrm{Perf}\,X \otimes \mathrm{Coh}\,X \to \mathrm{Coh}\,X$ which is a morphism in DGCat. By taking ind-objects, we obtain the following morphism in DGCat,

$$\mathrm{QCoh}\,X \otimes \mathrm{IndCoh}\,X \to \mathrm{IndCoh}\,X$$

Restricting along $\mathrm{Vec} \to \mathrm{IndCoh}\,X$ corresponding to the unit ω_X of $\otimes^!$ gives

$$\Upsilon_X : \mathrm{QCoh}\,X \to \mathrm{IndCoh}\,X$$

which on objects gives $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_X$.

It is shown in [GR1, Ch 6 , 3.3] that Υ can be made into a natural transformation $\mathrm{QCoh}^* \to \mathrm{IndCoh}^!$ as functors $\mathrm{dSch}_{\mathrm{aft}} \to (\mathrm{DGCat}_{\mathrm{cts}})^{op}$. To extend to all laft prestacks, we first restrict to $\mathrm{dAff}_{\mathrm{ft}}^{<\infty} \subseteq \mathrm{dSch}_{\mathrm{aft}}$.

$$\begin{tikzcd}
\mathrm{QCoh}^* \arrow[r, "\Upsilon"] & \mathrm{IndCoh}^! & \text{in} & \mathrm{Fun}(\mathrm{dAff}_{\mathrm{ft}}^{<\infty}, \mathrm{DGCat}_{\mathrm{cts}}^{\mathrm{op}}) \arrow[d, squiggly, "\mathrm{Lan}"'] \\
\widetilde{\mathrm{QCoh}}^* \arrow[r, "\Upsilon"] & \mathrm{IndCoh}^! & \text{in} & \mathrm{Fun}(\mathrm{PStk}_{\mathrm{laft}}, \mathrm{DGCat}_{\mathrm{cts}}^{\mathrm{op}})
\end{tikzcd}$$

For each individual $X \in \mathrm{PStk}_{\mathrm{laft}}$, this means

$$\widetilde{\mathrm{QCoh}}^*(X) \simeq \varprojlim_{S \in \mathrm{dAff}_{\mathrm{ft}}^{<\infty}/X} \mathrm{QCoh}^*(X)$$
$$\mathrm{IndCoh}^!(X) \simeq \varprojlim_{S \in \mathrm{dAff}_{\mathrm{ft}}^{<\infty}/X} \mathrm{IndCoh}^!(X)$$

Finally, we obtain $\Upsilon : \mathrm{QCoh}^* \to \mathrm{IndCoh}^!$ on all of $\mathrm{PStk}_{\mathrm{laft}}$ as the following composition :

$$\begin{tikzcd}
\mathrm{QCoh}^* \arrow[r] \arrow[dr, "\Upsilon"'] & \widetilde{\mathrm{QCoh}}^* \arrow[d, "\Upsilon"] \\
& \mathrm{IndCoh}^!
\end{tikzcd}$$

where the top morphism has components $\mathrm{QCoh}^*(X) \to \widetilde{\mathrm{QCoh}}^*(X)$ induced by the inclusion $\mathrm{dAff}^{<\infty}_{\mathrm{ft}}/X \to \mathrm{dAff}/X$. [GR1, Ch 6 , 3.3.4]

Consider the following situation in linear algebra : Suppose you have two finite dimensional vector spaces V, W equipped $\langle \_, \_ \rangle : V \otimes V \to k$ and $\langle \_, \_ \rangle : W \otimes W \to k$ inducing $V \simeq V^\vee, W \simeq W^\vee$. Then given any linear map $A : V \to W$, there is a unique $A^\vee : W \to V$ such that

$$\langle A\_, \star \rangle = \langle \_, A^\vee \star \rangle$$

The above phenomenon is true for any pair of self-dual objects V, W in an ∞-category of modules over a commutative algebra object k in a symmetric monoidal ∞-category Vec. In particular, we have an analogous result for $\mathbb{D}_{\mathrm{naive}}$ and $\mathbb{D}_{\mathrm{Serre}}$.

Proposition

Let $X \in \mathrm{dSch}_{\mathrm{aft}}$ and the base field k be perfect. Then under the self-dualities $\mathbb{D}_{\mathrm{naive}} : \mathrm{QCoh}\, X \simeq (\mathrm{QCoh}\, X)^\vee$ and $\mathbb{D}_{\mathrm{Serre}} : \mathrm{IndCoh}\, X \simeq (\mathrm{IndCoh}\, X)^\vee$, we have

$$\Upsilon_X \simeq \Psi_X^\vee$$

More concretely, we have the following equivalence of functors $\mathrm{IndCoh}\, X \otimes \mathrm{QCoh}\, X \to \mathrm{Vec}$,

$$\langle \Psi_X \_, \star \rangle \simeq \langle \_, \Upsilon_X \star \rangle$$

[INDCOH, Ch 9, 9.3]

Proof. This follows from the computation :

$$\begin{aligned}\langle \Psi_X \_, \star \rangle &\simeq \Gamma(X, \Psi_X \_ \otimes \star) \simeq \Gamma(X, \Psi_X(\star \otimes \_)) \\ &\simeq \Gamma(X, \Psi_X(\star \otimes (\_ \overset{!}{\otimes} \omega_X))) \simeq \Gamma(X, \Psi_X(\_ \overset{!}{\otimes} (\star \otimes \omega_X))) \simeq \langle \_, \Upsilon_X \star \rangle\end{aligned}$$

□

4 Crystals and D-modules

4.1 Left and Right Crystals

Definition – de Rham space

By the theory of left and right Kan extensions, we have the following adjunctions :

$$\mathrm{Aff}^{\mathrm{red}} \xrightarrow{\subseteq} \mathrm{dAff} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathrm{PStk}_{\mathrm{red}} \arrow[r, shift left=2ex, "\mathrm{Lan}"] \arrow[r, shift right=2ex, "\mathrm{Ran}"'] & \mathrm{PStk} \arrow[l, "\mathrm{res}"'] \end{tikzcd}$$

The functor which takes de Rham space is defined to be $\mathrm{dR} := \mathrm{Ran}\,\mathrm{res}$.

> The unit of the adjunction $\mathrm{res} \dashv \mathrm{Ran}$ is a natural transformation $\mathbb{1}_{\mathrm{PStk}} \to \mathrm{dR}$. For $X \in \mathrm{PStk}$, we write $p_{\mathrm{dR}} : X \to X_{\mathrm{dR}}$ for the component of the natural transformation.
>
> [Crys, Section 1.1]

Remark. Let $X \in \mathrm{PStk}$ and $\mathrm{Spec}\, A \in \mathrm{dAff}$. Then points of X_{dR} has the following description :

$$\mathrm{PStk}(\mathrm{Spec}\, A, X_{\mathrm{dR}}) \simeq \mathrm{PStk}_{\mathrm{red}}(\mathrm{Spec}\, A_{\mathrm{red}}, X_{\mathrm{red}}) \simeq \mathrm{PStk}(\mathrm{Spec}\, A_{\mathrm{red}}, X)$$

The first equivalence is by definition of Ran and the second is by the fully faithful embedding $\mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}$. In particular, the morphism $p_{\mathrm{dR}} : X \to X_{\mathrm{dR}}$ gives at the level of points :

$$\mathrm{PStk}(\mathrm{Spec}\, A, X) \to \mathrm{PStk}(\mathrm{Spec}\, A_{\mathrm{red}}, X)$$

Remark – Relation to Grothendieck's infinitesimal site. Let $X \in \mathrm{PStk}$. For simplicity, we assume the underlying classical prestack of X is zero-truncated, meaning for every $\mathrm{Spec}\, A \in \mathrm{Aff}$ we have $X(A)$ is a set. (E.g. if X has an underlying scheme.) We will describe the underlying classical prestack of X_{dR}, i.e. the over category $\mathrm{Aff}/X_{\mathrm{dR}}$.

First note that since X^{cl} is zero-truncated, so is $X^{\mathrm{cl}}_{\mathrm{dR}}$. This means for $\mathrm{Spec}\, A \in \mathrm{Aff}$, $X_{\mathrm{dR}}(A)$ is a set so we do not have to worry about any homotopical phenomena. Then given two classical points $x : \mathrm{Spec}\, A \to X_{\mathrm{dR}}$ and $y : \mathrm{Spec}\, B \to X_{\mathrm{dR}}$, a morphism $f : y \to x$ over X_{dR} is precisely a morphism $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ such that $xf = y$ at the reduced level.

$$\begin{tikzcd}
\mathrm{Spec}\, B \arrow[d, "f"'] \arrow[dr, "y"] & \\
\mathrm{Spec}\, A \arrow[r, "x"'] & X_{\mathrm{dR}}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathrm{Spec}\, B \arrow[d, "f"'] & \mathrm{Spec}\, B^{\mathrm{red}} \arrow[l] \arrow[d, "f^{\mathrm{red}}"'] \arrow[dr, "y^{\mathrm{red}}"] & \\
\mathrm{Spec}\, A & \mathrm{Spec}\, A^{\mathrm{red}} \arrow[l] \arrow[r, "x^{\mathrm{red}}"'] & X
\end{tikzcd}$$

Now assuming X has a underlying scheme, Grothendieck's infinitesimal site of X is precisely the full subcategory of $\mathrm{Aff}/X_{\mathrm{dR}}$ with objects $x : \mathrm{Spec}\, A \to X_{\mathrm{dR}}$ such that $\mathrm{Spec}\, A^{\mathrm{red}} \to X$ is an open immersion. [Stacks, Def 60.9.1]

Remark – What about $\mathrm{Lan}\,\mathrm{res}$. From the discussion on prestacks, the left Kan extension functor $\mathrm{Lan} : \mathrm{PStk}_{\mathrm{red}} \to \mathrm{PStk}$ simply takes a reduced prestack and views it as a general prestack. Thus $\mathrm{Lan}\,\mathrm{res}$ is the functor that takes the underlying reduced prestack of a general prestack.

Remark. We actually have more than just an inclusion $\mathrm{Aff}^{\mathrm{red}} \to \mathrm{dAff}$ but an adjunction :

$$\begin{tikzcd}
\mathrm{Aff}^{\mathrm{red}} \arrow[r, shift left=2, "\subseteq"] \arrow[r, phantom, "\perp"] & \mathrm{dAff} \arrow[l, shift left=2, "\mathrm{Spec}\,(\pi_0 A)_{\mathrm{red}} \leftarrow\!\shortmid \mathrm{Spec}\, A"]
\end{tikzcd}$$

From this, we can also deduce that dR preserves both small colimits and small limits which is useful.

Proposition

The functor dR taking de Rham spaces preserves small colimits and small limits.

Proof. We proceed by a slightly different proof to [Crys, Lem 1.1.4]. Basically, this is an exercise in the theory of presheaf ∞-categories. To show $\mathrm{dR} = \mathrm{Ran}\,\mathrm{res}$ preserves small colimits and small limits, it suffices that Ran is a left adjoint. By applying the theory of left and right Kan extensions of presheaves to the adjunction $\mathrm{Aff}^{\mathrm{red}} \rightleftarrows \mathrm{dAff}$, we obtain *two* adjoint triples :

$$\begin{tikzcd} \mathrm{Aff}^{\mathrm{red}} \arrow[r, shift left=1.5, "\subseteq"] \arrow[r, phantom, "\perp"] & \mathrm{dAff} \arrow[l, shift left=1.5, "\mathrm{red}"] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{array}{c} \begin{tikzcd} \mathrm{PStk}_{\mathrm{red}} \arrow[r, shift left=3, "\mathrm{Lan}"] \arrow[r, shift right=3, "\mathrm{Ran}"'] & \mathrm{PStk} \arrow[l, "\mathrm{res}" description] \end{tikzcd} \\[2em] \begin{tikzcd} \mathrm{PStk}_{\mathrm{red}} \arrow[r, "\mathrm{res}" description] & \mathrm{PStk} \arrow[l, shift right=3, "\mathrm{Lan}"'] \arrow[l, shift left=3, "\mathrm{Ran}"] \end{tikzcd} \end{array}$$

The point is that we have a commuting square :

$$\begin{tikzcd} \mathrm{Aff}^{\mathrm{red}} \arrow[d] & \mathrm{dAff} \arrow[l, "\mathrm{red}"'] \arrow[d] \\ \mathrm{PStk}_{\mathrm{red}} & \mathrm{PStk} \arrow[l, "\mathrm{res}"] \end{tikzcd}$$

So res in the top adjunction is isomorphic to Lan in the bottom adjunction, hence by uniqueness of adjoint functors we obtain a quadruple of adjoints :

$$\begin{tikzcd} \mathrm{PStk}_{\mathrm{red}} \arrow[r, shift left=4.5, "\mathrm{Lan}"] \arrow[r, shift right=1.5, "\mathrm{Ran}" description] & \mathrm{PStk} \arrow[l, shift right=-1.5, "\mathrm{res}" description] \arrow[l, shift left=4.5] \end{tikzcd}$$

This proves Ran is a left adjoint and hence preserves small colimits.

□

Definition – Left and Right Crystals

Let $X \in \mathrm{PStk}_{\mathrm{laft}}$. Then we define the ∞*-category of left crystals on* X to be

$$\mathrm{Crys}^L(X) := \mathrm{QCoh}\, X_{\mathrm{dR}}$$

[Crys, Def 2.1.1] The ∞*-category of right crystals on* X is defined to be

$$\mathrm{Crys}^R(X) := \mathrm{IndCoh}\, X_{\mathrm{dR}}$$

[Crys, Def 2.3.2]

Technically speaking, for the definition of right crystals to make sense, we need to show that X_{dR} is also locally almost finite type. Indeed, we have the following.

Proposition – $\mathrm{PStk}_{\mathrm{laft}}$ is closed under dR
Let $X \in \mathrm{PStk}_{\mathrm{laft}}$. Then $X_{\mathrm{dR}} \in \mathrm{PStk}_{\mathrm{laft}}$. [Crys, prop 1.3.3]

Proof. We first show the convergence of X_{dR}. This follows from the following diagram which commutes up to isomorphism :

$$\begin{tikzcd} X(A_{\mathrm{red}}) \arrow[r, "\sim"] \arrow[d, "="'] & \varprojlim_n X(A_{\mathrm{red}}) \arrow[d, "\sim"] \\ X_{\mathrm{dR}}(A) \arrow[r] & \varprojlim_n X_{\mathrm{dR}}(\tau^{\leq n} A) \end{tikzcd}$$

It remains to show X_{dR} takes filtered limits in $\mathrm{dAff}^{\leq n}$ to filtered colimits in Spc. By assumption, X does this so it suffices to show that $\mathrm{red} : \mathrm{dAff}^{\leq n} \to \mathrm{Aff}^{\mathrm{red}}$ preserves filtered limits. This is a composition $\mathrm{dAff}^{\leq n} \to \mathrm{Aff} \to \mathrm{Aff}^{\mathrm{red}}$ where the first part takes H^0 and the second part takes reduction. Both functors preserve filtered limits. $\square$

Remark. A specific situation worth noting is when the morphism $X \to X_{\mathrm{dR}}$ exhibits X_{dR} as the geometric realisation of its Cech nerve $\check{C}(X/X_{\mathrm{dR}})$, i.e. when $X \to X_{\mathrm{dR}}$ is an effective epimorphism.[1]

Proposition
Let $X \in \mathrm{PStk}_{\mathrm{laft}}$. Then the following are equivalent :

1. For all $S \in \mathrm{dAff}$, the map of sets $\pi_0 X(S) \to \pi_0 X(S_{\mathrm{red}})$ is surjective.
2. $X \to X_{\mathrm{dR}}$ is an effective epimorphism in PStk.

We say X is *classically formally smooth* when any (and thus all) of the above are satisfied. In this case, we have

[1] For a definition of Cech nerves, see [Lurie-HTT, Prop 6.1.2.11].

$$\mathrm{Crys}^L X \xrightarrow{\sim} \varprojlim \mathrm{QCoh}(\check{C}(X/X_{\mathrm{dR}}))$$

$$\mathrm{Crys}^R X \xrightarrow{\sim} \varprojlim \mathrm{IndCoh}(\check{C}(X/X_{\mathrm{dR}}))$$

i.e. "crystals are the same as sheaves equipped with equivariance with respect to the infinitesimal groupoid".[a]

[a] Crys 3.1.3 actually proves this for general dg ind-scheme X of locally almost finite type without assumptions of classically formally smooth. However, we focused on this because it is all that's needed for the equivalence crystals and D-modules.

Proof. Let us remark that [Crys, Lem 1.2.4] only proves (1) implies (2), but we think the two are actually equivalent. For the equivalence of (1) and (2), we use that colimits in $\mathrm{PStk} = \mathrm{PSh}\,\mathrm{dAff}$ are computed pointwise, the lemma is equivalent to claiming that $X(S) \to X(S_{\mathrm{red}})$ is effectively epic if and only if it is surjective on π_0. This is a non-trivial fact about ∞-groupoids which we will assume.

> *Lemma. Let $q : X \to Y$ be a morphism of ∞-groupoids. Then q is an effective epimorphism if and only if it is surjective on π_0.*
>
> *Proof.* We defer the interested reader to [Lurie-HTT, Prop 7.2.1.15].
>
> ■

Now assume X is classically formally smooth. Since $\mathrm{QCoh}^* : \mathrm{PStk} \to (\mathrm{DGCat}_{\mathrm{cts}})^{\mathrm{op}}$ is the left Kan extension of its restriction to dAff, it preserves small colimits and in particular geometric realisations. We thus obtain the equivalence $\mathrm{Crys}^L X \xrightarrow{\sim} \varprojlim \mathrm{QCoh}(\check{C}(X/X_{\mathrm{dR}}))$.

We want to use the same argument for right crystals, however the subtlety is that $\mathrm{IndCoh}^!$ is *not* defined on all of PStk but rather just $\mathrm{PStk}_{\mathrm{laft}}$.[1] It suffices to show that the Cech nerve of $X \to X_{\mathrm{dR}}$ lies in $\mathrm{PStk}_{\mathrm{laft}}$ and that $X \to X_{\mathrm{dR}}$ is an effective epimorphism *in* $\mathrm{PStk}_{\mathrm{laft}}$. For the first part, note that the morphism $X \to X_{\mathrm{dR}}$ is in $\mathrm{PStk}_{\mathrm{laft}}$ because dR preserves laft-ness. Since $\mathrm{PStk}_{\mathrm{laft}}$ is closed under finite limits, we obtain that the Cech nerve of $X \to X_{\mathrm{dR}}$ in $\mathrm{PStk}_{\mathrm{laft}}$ agrees with the one in PStk. For the second part, we use $\mathrm{res} : \mathrm{PStk}_{\mathrm{laft}} \xrightarrow{\sim} \mathrm{PSh}\,\mathrm{dAff}_{\mathrm{ft}}^{<\infty}$. Since colimits are computed pointwise, by the lemma above it suffices to show that $X(A) \to X_{\mathrm{dR}}(A)$ is a surjection on π_0 for all $\mathrm{Spec}\, A \in \mathrm{dAff}_{\mathrm{ft}}^{<\infty}$. This is true by assumption.

□

Here is a sufficient condition to be classically formally smooth, whose proof we won't go into.

Proposition – Comparison of classically formally smooth and smoothness

[1] This is [Crys, Lem 2.3.11]. However, the proof in the reference simply claims that the argument is the same as for left crystals. We provide some unclear details.

Let X be a laft smooth classical scheme of finite type over k. Then X is classically formally smooth when considered as a prestack.

Proof. Omitted. It is shown in [DGINDSCH, Prop 8.4.2] that smooth classical schemes of finite type over k are *formally smooth* when considered as prestacks.[1] With the additional assumption of laft-ness, this implies classically formally smooth. $\square$

4.2 Equivalence of Left and Right Crystals

Proposition – Equivalence of Left and Right Crystals

Let $X \in \mathrm{PStk}_{\mathrm{laft}}$. Then we have an equivalence

$$\Upsilon_{X_{\mathrm{dR}}} : \mathrm{Crys}^L X \xrightarrow{\sim} \mathrm{Crys}^R X$$

Proof. We filled in some minor details from [Crys, Prop 2.4.4].

(Step 0 - Reduce to derived affine) Since $\mathrm{Crys}^L, \mathrm{Crys}^R$ are both left Kan extensions from $\mathrm{dAff}_{\mathrm{ft}}^{<\infty}$, we can WLOG assume $X \in \mathrm{dAff}_{\mathrm{aft}}$.

(Step 1 - Reduce to classically formally smooth) Writing $X = \operatorname{Spec} A$, we have that $H^0 A$ is a finite type algebra over k. Choose a surjection $k[\mathbb{A}^n] \to H^0 A \simeq A_0/\mathrm{Im} d^1$ of algebras over k and choose a lift $k[\mathbb{A}^n] \to A_0$ where A_0 is the degree zero part of A, which is also an algebra over k. Then this defines a morphism of commutative dg algebras $k[\mathbb{A}^n] \to A$ Viewing things geometrically, we have found a closed embedding $i : X \to Z$ for some smooth classical affine scheme Z.

Now consider the *formal completion of Z along X*, which is defined by the following fiber product :

$$\begin{tikzcd}
Z_{\widehat{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\
X_{\mathrm{dR}} \arrow[r] & Z_{\mathrm{dR}}
\end{tikzcd}$$

By the universal property of fiber products, we have a morphism $X \to Z_{\widehat{X}}$. We claim that this induces

$$X_{\mathrm{dR}} \xrightarrow{\sim} (Z_{\widehat{X}})_{\mathrm{dR}}$$

Indeed, since taking the underlying reduced prestack is a right adjoint, fiber products are preserved and we get

$$\begin{tikzcd}
Z_{\widehat{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\
X_{\mathrm{dR}} \arrow[r] & Z_{\mathrm{dR}}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
(Z_{\widehat{X}})^{\mathrm{red}} \arrow[r] \arrow[d, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z^{\mathrm{red}} \arrow[d, "\sim"] \\
(X_{\mathrm{dR}})^{\mathrm{red}} \arrow[r] & (Z_{\mathrm{dR}})^{\mathrm{red}}
\end{tikzcd}$$

[1]Gaitsgory–Rozenblyum's usage of the term "formally smooth" does *not* immediately match the usual usage. The interested reader may find the definition in the reference provided.

Since $Z_{\widehat{X}} \to X_{\mathrm{dR}}$ induces an equivalence on reduced points, we get $(Z_{\widehat{X}})_{\mathrm{dR}} \simeq X_{\mathrm{dR}}$. Therefore we can replace X by $Z_{\widehat{X}}$.

What do we gain from this? Well, note that Z is formally classically smooth, i.e. $Z \to Z_{\mathrm{dR}}$ is an effective epimorphism. By [Lurie-HTT, Prop 6.2.3.15], the pullback of an effective epimorphism in a presheaf ∞-category is an effective epimorphism, so $Z_{\widehat{X}} \to X_{\mathrm{dR}}$ is also an effective epimorphism. But we saw earlier that this morphism equivalent to $Z_{\widehat{X}} \to (Z_{\widehat{X}})_{\mathrm{dR}}$. Therefore $Z_{\widehat{X}}$ is classically formally smooth. In other words, we can WLOG assume X is classically formally smooth.

(Step 2 - Reduce to $\check{C}^i(X/X_{\mathrm{dR}})$) Since $X \to X_{\mathrm{dR}}$ is an effective epimorphism, we have the commutative square :

$$\begin{tikzcd}
\mathrm{Crys}^L X \arrow[rr, "\Upsilon_{X_{\mathrm{dR}}}"] \arrow[d, "\sim"'] & & \mathrm{Crys}^R X \arrow[d, "\sim"] \\
\varprojlim \mathrm{QCoh}\,\check{C}(X/X_{\mathrm{dR}}) \arrow[rr, "\varprojlim \Upsilon_i"'] & & \varprojlim \mathrm{IndCoh}\,\check{C}(X/X_{\mathrm{dR}})
\end{tikzcd}$$

where $\Upsilon_i : \mathrm{QCoh}\,\check{C}^i(X/X_{\mathrm{dR}}) \to \mathrm{IndCoh}\,\check{C}^i(X/X_{\mathrm{dR}})$ comes from the i-simplices of the Cech nerve. So for $\Upsilon_{X_{\mathrm{dR}}}$ to be an equivalence, it suffices for each Υ_i to be an equivalence.

(Step 3 - Reduction to smooth classical affines) Recall from step 1 that we have $X \simeq X_{\mathrm{dR}} \times_{Z_{\mathrm{dR}}} Z$. Then by the pasting lemma for (∞-categorical) pullbacks

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z^i \arrow[d] \\
X_{\mathrm{dR}} \arrow[r, "\Delta"'] & X^i_{\mathrm{dR}} \arrow[r] & Z^i_{\mathrm{dR}}
\end{tikzcd}$$

we see that $\check{C}^i(X/X_{\mathrm{dR}}) \simeq Z^i_{\widehat{X}}$ the formal completion of Z^i along X closed embedded in the diagonal. Let $U^i \subseteq Z^i$ be the open complement of X.

We now have the following descriptions of QCoh and IndCoh on formal completions along closed embeddings.

> *Lemma (Sheaves on Formal Completions along Closed Embeddings). Let $i : X \to Z$ be a closed embedding in $\mathrm{dAff}_{\mathrm{aft}}$. Let $j : U \to Z$ be the open complement of X. Define the following full dg-subcategories :*
>
> $$\begin{tikzcd}
(\mathrm{QCoh}\,Z)_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{QCoh}\,Z \arrow[d, "j^*"] \\
0 \arrow[r] & \mathrm{QCoh}\,U
\end{tikzcd} \qquad \begin{tikzcd}
(\mathrm{IndCoh}\,Z)_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{IndCoh}\,Z \arrow[d, "j^!"] \\
0 \arrow[r] & \mathrm{IndCoh}\,U
\end{tikzcd}$$

Then

– [DGINDSCH, *Prop 7.1.3*] *There is an equivalence* $\mathrm{QCoh}(Z_{\widehat{X}}) \simeq (\mathrm{QCoh}\, Z)_X$.

– [DGINDSCH, *Prop 7.4.5*] *There is an equivalence* $\mathrm{IndCoh}(Z_{\widehat{X}}) \simeq (\mathrm{IndCoh}\, Z)_X$.

Proof. Omitted. ∎

Thus, we obtain the following "double short exact sequence" in $\mathrm{DGCat}_{\mathrm{cts}}$:

$$\begin{tikzcd} \mathrm{QCoh}\, Z_{\widehat{X}} \arrow[d, "\Upsilon_{Z_{\widehat{X}}}"] & \simeq & \mathrm{Ker}\, j^* \arrow[r] \arrow[d] & \mathrm{QCoh}\, Z^i \arrow[r, "j^*"] \arrow[d, "\Upsilon_{Z^i}"] & \mathrm{QCoh}\, U_i \arrow[d, "\Upsilon_{U_i}"] \\ \mathrm{IndCoh}\, Z_{\widehat{X}} & \simeq & \mathrm{Ker}\, j^! \arrow[r] & \mathrm{IndCoh}\, Z^i \arrow[r, "j^!"] & \mathrm{IndCoh}\, U_i \end{tikzcd}$$

Thus is suffices to show Υ_S is an equivalence for smooth classical affines S.

This is true because we saw that under the self-dualities $\mathbb{D}_{\mathrm{naive}} : \mathrm{QCoh}\, S \simeq (\mathrm{QCoh}\, S)^\vee$ and $\mathbb{D}_{\mathrm{Serre}} : \mathrm{IndCoh}\, S \simeq (\mathrm{IndCoh}\, S)^\vee$, we have $\Upsilon_S^\vee \simeq \Psi_S$. This is an equivalence because it is well-known that for smooth classical affines S, $\mathrm{Perf}\, S = \mathrm{Coh}\, S$. Thus, we are done.

□

4.3 Induction of Left and Right Crystals

The main tool is the theory of descent as formulated in [Lurie-HA, Prop 4.7.5.2].

Proposition – Descent

Let $C^\bullet : \Delta \to (\infty, 1)\text{-Cat}$. Suppose $C^\bullet$ has the following property :

– for all $\alpha : [m] \to [n]$ in Δ, then the following commutative square is *left-adjointable* :

$$\begin{tikzcd} C^m \arrow[r, "d_0"] \arrow[d] & C^{m+1} \arrow[d] \\ C^n \arrow[r, "d_0"'] & C^{n+1} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} C^m \arrow[d] & C^{m+1} \arrow[l, "\delta_0"'] \arrow[d] \\ C^n & C^{n+1} \arrow[l, "\delta_0"] \end{tikzcd}$$

meaning the coface morphisms $d^0 : C^k \to C^{k+1}$ admit left adjoints $F(k) : C^{k+1} \to C^k$ and the commutative square on the left hand side induces the commutative square on the right by passing to left adjoints.

Let C be the underlying ∞-category of the limit $\varprojlim C^\bullet$. Then

– the functor $G : C \to C^0$ admits a left adjoint F.

– the following commutative square is left-adjointable :

$$\begin{tikzcd} C \arrow[r, "G"] \arrow[d, "G"'] & C^0 \arrow[d, "d^1"] \\ C^0 \arrow[r, "d^0"'] & C^1 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} C \arrow[d, "G"'] & C^0 \arrow[l, "F"'] \arrow[d, "d^1"] \\ C^0 & C^1 \arrow[l, "F(0)"] \end{tikzcd}$$

– the adjunction $F \dashv G : C^0 \rightleftarrows C$ is monadic.

[Lurie-HA, Prop 4.7.5.2]

Under the simplifying assumption of $X \in \mathrm{PStk}_{\mathrm{laft}}$ being classically formally smooth so that

$$\mathrm{Crys}^R X \xrightarrow{\sim} \varprojlim \mathrm{IndCoh}\left(\check{C}(X/X_{\mathrm{dR}})\right)$$

we see that to obtain the induction functor for right crystals, we want the projections $p_1, p_2 : X \times_{X_{\mathrm{dR}}} X \rightrightarrows X$ to give $p_1^!, p_2^!$ which admit left adjoints and satisfies base change. This is indeed possible and is what [GR2, Ch 3, 2.1] achieves.

Proposition – Base Change for IndCoh on Ind-Schematic, Ind-Proper Morphisms
Suppose we have the following cartesian square in $\mathrm{PStk}_{\mathrm{laft}}$:

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

where g is *ind-schematic* and *ind-proper*. Then $g^!, v^!$ respectively admit left adjoints g_*, v_* such that we have an equivalence

$$v_* u^! \xrightarrow{\sim} f^! g_*$$

arising by adjunction from the equivalence $f_* v_* \simeq g_* u_*$.

Proof. Omitted since this would require developing the theory of ind-schemes for derived algebraic geometry. See [GR2, Ch 3, 2.1] for the proof. □

Due to the constraint in space, we also choose to omit the verification that the projections $p_1, p_2 : X \times_{X_{\mathrm{dR}}} X \rightrightarrows X$ are ind-schematic and ind-proper. We defer the reader to the definitions at [GR2, Ch 2, 1.6.5] and [GR2, Ch 2, 1.6.11]. We thus obtain induction for right crystals.

Proposition – Induction for Right Crystals (Crys 3.3.2)

Let $X \in \mathrm{PStk}_{\mathrm{laft}}$ such that

$$\mathrm{Crys}^R X \xrightarrow{\sim} \varprojlim \mathrm{IndCoh}\left(\check{C}(X/X_{\mathrm{dR}})\right)$$

(such as X classically formally smooth). Consider $p : X \to X_{\mathrm{dR}}$ and $p_s, p_t : X \times_{X_{\mathrm{dR}}} X \rightrightarrows X$. Then

1. the functor $\mathrm{oblv}^R = p^! : \mathrm{Crys}^R X \to \mathrm{IndCoh}\, X$ has a left adjoint ind^R.
2. we have an equivalence of functors

$$\mathrm{oblv}^R \mathrm{ind}^R \simeq (p_t)_* \, p_s^!$$

3. the adjunction $\mathrm{ind}^R \dashv \mathrm{oblv}^R$ is monadic. In particular, if $\mathrm{IndCoh}\, X$ is compactly generated, so is $\mathrm{Crys}^R X$.[Crys, Cor 3.3.3]

[Crys, Prop 3.3.2]

Proof. We've already discussed how (1) and (2) follow from Lurie's theory of descent.

For (3), note that since $\mathrm{ind}^R \dashv \mathrm{oblv}^R$ is monadic, oblv^R must be conservative. This implies that the image of the set of objects of $(\mathrm{IndCoh}\, X)^c$ compactly generates $\mathrm{Crys}^R X$. ☐

The induction functor for left crystals $\mathrm{ind}^L : \mathrm{QCoh}\, X \to \mathrm{Crys}^L X$ is obtained by transferring across the equivalence $\Upsilon_{X_{\mathrm{dR}}} : \mathrm{Crys}^L X \simeq \mathrm{Crys}^R X$.

4.4 Equivalence of Left Crystals and D-modules

Proposition – Main Result

Let X be a smooth proper classical scheme of locally almost finite type. Then there is an equivalence $\mathrm{Crys}^L(X) \simeq \mathrm{Diff}_X \mathrm{Mod}$ such that the following commutes up to natural isomorphism :

$$\begin{tikzcd} \mathrm{Crys}^L(X) \arrow[r, "\sim"] \arrow[d, "\mathrm{oblv}^L"'] & \mathrm{Diff}_X \mathrm{Mod} \arrow[d] \\ \mathrm{QCoh}\, X \arrow[r, "\mathbb{1}"'] & \mathrm{QCoh}\, X \end{tikzcd}$$

The right vertical morphism is the forgetful functor. [Crys, Section 5.5.5]

Proof. 1. $\mathrm{Crys}^L X \rightleftarrows \mathrm{QCoh}\, X$ and $\mathrm{Diff}_X \mathrm{Mod} \rightleftarrows \mathrm{QCoh}\, X$ are both monadic.

$\mathrm{Diff}_X \mathrm{Mod}$ is by definition the ∞-category of modules in $\mathrm{QCoh}\, X$ over the monad corresponding to $\mathrm{Diff}_X \in \mathrm{QCoh}(X \times X)$, so $\mathrm{Diff}_X \mathrm{Mod} \rightleftarrows \mathrm{QCoh}\, X$ is monadic.

For $\mathrm{Crys}^L X$, by [Lurie-HA, Prop 4.7.0.3], it suffices to show that

- oblv^L is conservative
- $\mathrm{Crys}^L X$ has geometric realisations and oblv^L preserves them.

The second point is taken care of by the fact that $\mathrm{QCoh}^* : \mathrm{PStk} \to (\mathrm{DGCat}_{\mathrm{cts}})^{\mathrm{op}}$. For the first point, since X is a laft smooth classical scheme, it is classically formally smooth. This implies the equivalence between left crystals and infinitesimally equivariant quasi-coherent sheaves, as well as the equivalence between right crsytals and infinitesimally equivariant ind-coherent sheaves :

$$\mathrm{Crys}^L X \xrightarrow{\sim} \varprojlim \mathrm{QCoh}(\check{C}(X/X_{\mathrm{dR}}))$$

$$\mathrm{Crys}^R X \xrightarrow{\sim} \varprojlim \mathrm{IndCoh}(\check{C}(X/X_{\mathrm{dR}}))$$

Under the equivalence of left and right crystals and the equivalence $\Upsilon_X : \mathrm{QCoh}\, X \simeq \mathrm{IndCoh}\, X$ due to smoothness of X, we obtain $\mathrm{oblv}^L \simeq \Upsilon_X^{-1} \mathrm{oblv}^R \Upsilon_{X_{\mathrm{dR}}}$. Since oblv^R is conservative, we obtain the same for oblv^L.

So the two ∞-categories $\mathrm{Crys}^L X$ and $\mathrm{Diff}_X\mathrm{Mod}$ are determined by their monads.

2. Under the integral transform equivalence $\mathrm{QCoh}(X \times X) \simeq \mathrm{DGCat}_{\mathrm{cts}}(\mathrm{QCoh}\, X, \mathrm{QCoh}\, X)$, algebra objects on the left correspond to algebra objects on the right, i.e. monads. Therefore the monads $\mathrm{oblv}^L \mathrm{ind}^L$ and $\mathrm{Diff}_X \otimes \_$ correspond to quasi-coherent sheaves on $X \times X$.
3. Of course, $\mathrm{Diff}_X \otimes \_$ corresponds to Diff_X on $X \times X$.
4. Let $\mathscr{D}_X^L \in \mathrm{QCoh}(X \times X)$ correspond to $\mathrm{oblv}^L \mathrm{ind}^L$. [Crys, Prop 5.3.6] shows

   $$\mathscr{D}_X^L \simeq (\omega_X \boxtimes \mathcal{O}_X) \otimes \mathrm{Fiber}(\mathcal{O}_{X\times X} \to j_* j^* \mathcal{O}_{X\times X})$$

   lies $\mathrm{QCoh}(X \times X)^\heartsuit$, with $j : X \times X \setminus \Delta_X \to X \times X$.
5. In [Crys, Section 5.4.1], we have for any $\mathcal{F}, \mathcal{G} \in (\mathrm{QCoh}\, X)^\heartsuit$ and $\mathcal{Q} \in \mathrm{QCoh}(X \times X)^\heartsuit$ with set-theoretically supported on the diagonal,

   $$(p_2)_*(p_1^* \mathcal{F} \otimes \mathcal{Q}) \to \mathcal{G} \quad \rightsquigarrow \quad \mathcal{Q} \to \mathrm{Diff}_X(\mathcal{F}, \mathcal{G})$$

   Applying to $\mathcal{F} = \mathcal{G} = \mathcal{O}_X, \mathcal{Q} = \mathscr{D}_X^L$, we obtain a morphism $\mathscr{D}_X^L \to \mathrm{Diff}_X$. This is a morphism of algebra objects on $\mathrm{QCoh}(X \times X)$.
6. As in [Crys, Section 5.4.3], $\mathscr{D}_X^L \xrightarrow{\sim} \mathrm{Diff}_X$ is a classical computation by using the fact that when X is smooth proper Noetherian classical over k, ω_X has an explicit description as shifted top forms. [Stacks, Lem 48.15.7]

□

5 Bibliography

References

[BZFN] D. Ben-Zvi, J. Francis, and D. Nadler. "Integral transforms and Drinfeld centers in derived algebraic geometry". In: *Journal of the American Mathematical Society* 23.4 (2010), pp. 909–966.

[BGG] I. Bernstein, I. M. Gel'fand, and S. Gel'fand. "Differential operators on a cubic cone". In: *Russian Mathematical Surveys* 27.1 (1972), pp. 169–174.

[INDCOH] D. Gaitsgory. *Ind-coherent sheaves*. 2011. URL: https://arxiv.org/abs/1105.4857.

[Crys] D. Gaitsgory and N. Rozenblyum. "Crystals and D-modules". In: *Pure and Applied Mathematics Quarterly* 10 (Nov. 2011).

[DGINDSCH] D. Gaitsgory and N. Rozenblyum. "DG indschemes". In: *Perspectives in representation theory* 610 (2014), pp. 139–251.

[GR2] D. Gaitsgory and N. Rozenblyum. *A Study in Derived Algebraic Geometry: Volume II: Deformations, Lie Theory and Formal Geometry*. Vol. 221. 2017.

[GR1] D. Gaitsgory and N. Rozenblyum. *A study in derived algebraic geometry: Volume I: correspondences and duality*. Vol. 221. American Mathematical Society, 2019.

[H.T.T.] R. Hotta, K. Takeuchi, and T. Tanisaki. *D-modules, perverse sheaves, and representation theory*. Vol. 236. Springer Science & Business Media, 2007.

[Keller] J. Keller. *Beilinson–Bernstein Localization*. http://math.uchicago.edu/~may/REU2018/REUPapers/Keller.pdf. Accessed: 2022. 2018.

[Lurie-HTT] J. Lurie. "Higher Topos Theory". In: Sept. 2006.

[Lurie-DAGX] J. Lurie. *Derived Algebraic Geometry X: Formal Moduli Problems*. https://people.math.harvard.edu/~lurie/papers/DAG-X.pdf. Accessed: 2022. Nov. 2011.

[Lurie-HA] J. Lurie. "Higher algebra". In: (2017).

[Kerodon] J. Lurie. *Kerodon*. https://kerodon.net/. Accessed: 2022. 2022.

[Stacks] *Stacks Project*. https://stacks.math.columbia.edu/. Accessed: 2022. 2022.