Proposition – Big Zariski Site on $\mathbb{M}\mathrm{Set}$

For $X \in \mathbb{M}\mathbf{Set}$ and $\mathcal{U} \subseteq \mathbb{M}\mathbf{Set}/X$ a collection of morphisms into X, define $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ when "$\mathcal{U}$ is isomorphic to an open cover", meaning there exists $\{U_i\}_{i\in\mathcal{U}} \subseteq \mathrm{Open}\, X$ such that $\{U_i\}_{i\in\mathcal{U}}$ is a cover of X and for all $i \in \mathcal{U}$, $(i : s(i) \to X) \cong (U_i \to X)$ in $\mathbb{M}\mathbf{Set}/X$. Then the above defines a Grothendieck pretopology of $\mathbb{M}\mathbf{Set}$. Specifically :

- (Isomorphisms are Covers) For $X \in \mathbb{M}\mathbf{Set}$ and $\varphi \in \mathbb{M}\mathbf{Set}(U, X)$, φ iso implies $\{\varphi\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.
- (Pullback of Covers) For all $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$ and $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$,
  $\varphi^{-1}\mathcal{U} := \{Y \times_X s(i) \to Y\}_{i\in\mathcal{U}} \in \mathrm{Cov}_{\mathrm{Zar}}(Y)$.
- (Composite of Covers) Let $X \in \mathbb{M}\mathbf{Set}$, $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ and for each $i \in \mathcal{U}$, let $\mathcal{U}_i \in \mathrm{Cov}_{\mathrm{Zar}}(s(i))$. Then $\{s(j_i) \to s(i) \to X \mid i \in \mathcal{U}, j_i \in \mathcal{U}_i\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.

We will use $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$ to denote the site $\mathbb{M}\mathbf{Set}$ endowed with the topology generated by the above pretopology. We will call $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$ the *big Zariski site*. $\mathcal{X} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ are called *Zariski covers of X*.

Proof. Only slightly non-trivial part is pullback of covers. Use opens and covers preserved under base change. □

Remark – Intuition of Sheaves on $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$. For $X \in \mathbb{M}\mathbf{Set}$, if X is to be a "space" then for any other $Y \in \mathbb{M}\mathbf{Set}$ and open cover $\mathcal{U}$ of Y, the data of a morphism $Y \to X$ should be the same as a collection of morphisms $(U \to X)_{U\in\mathcal{U}}$ that agree on pairwise intersection. This is precisely what it means for $\mathbb{M}\mathbf{Set}(-, X)$ to be a sheaf on the site $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Remark. The following is a smaller site $\mathbf{Aff}_{\mathrm{Zar}}$ on affine schemes, where open covers consists only of basic opens. Since basic opens generate opens for affine schemes and affine schemes generate $\mathbb{M}\mathbf{Set}$ with compatible notion of opens, sheaves on $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$ will be the same as sheaves of $\mathbf{Aff}_{\mathrm{Zar}}$. This gives an easier check for when $X \in \mathbb{M}\mathbf{Set}$ is a sheaf on $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Proposition – Small Zariski Site on Aff

For $X \in \mathbf{Aff}$ and $\mathcal{U} \subseteq \mathbf{Aff}/X$, $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ when "$\mathcal{U}$ is isomorphic to a cover of X by basic opens", meaning there exists a cover $\{X_{f_\iota}\}_{\iota\in\mathcal{U}}$ where for all $\iota \in \mathcal{U}$, $(s(\iota) \to X) \cong (D(f_\iota) \to X)$ in $\mathbf{Aff}/X$. Then the above defines a Grothendieck pretopology on $\mathbf{Aff}$, specifically :

- (Isomorphisms are Covers) For all $X \in \mathbf{Aff}$ and $\iota \in \mathbf{Aff}(U, X)$, ι isomorphism implies $\{\iota\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.
- (Pullback of Covers) For all $\varphi \in \mathbf{Aff}(Y, X)$ and $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$,
  $\varphi^{-1}\mathcal{U} := \{Y \times_X s(\iota) \to Y \mid \iota \in \mathcal{U}\} \in \mathrm{Cov}_{\mathrm{Zar}}(Y)$.
- (Composite of Covers) Let $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ and for each $i \in \mathcal{U}$, let $\mathcal{U}_i \in \mathrm{Cov}_{\mathrm{Zar}}(s(i))$. Then $\{s(j_i) \to s(i) \to X \mid i \in \mathcal{U}, j_i \in \mathcal{U}_i\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.

We will use $\mathbf{Aff}_{\mathrm{Zar}}$ to denote the site $\mathbf{Aff}$ with the topology given by the above pretopology. We will call $\mathbf{Aff}_{\mathrm{Zar}}$ the *small Zariski site*. $\mathcal{X} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ will be called *basic Zariski covers of X*. [a]

[a] This is non-standard terminology, but helps avoid confusion between the topology on $\mathbf{Aff}$ just defined and the induced topology from $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Proof. UP of tensor products and localization. □

> **Proposition – Sheaves on Big and Small Zariski Site are the Same**
> Let $X \in \mathbb{M}\mathbf{Set}$. Then $\mathbb{M}\mathbf{Set}(-, X) \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$ if and only if $\mathbb{M}\mathbf{Set}(-, X) \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$.

Proof. Forward implication follows since the covers in $\mathbf{Aff}_{\mathrm{Zar}}$ are covers in $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Now let $\mathbb{M}\mathbf{Set}(-, X) \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$. Let $U \in \mathbb{M}\mathbf{Set}$ and $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(U)$. Then for $(A, \alpha) \in \mathrm{Sp} \downarrow U$, the pullback $\alpha^{-1}\mathcal{U}$ of $\mathcal{U}$ is a cover of $\mathrm{Sp}\, A$ in the big Zariski site. The chain of isomorphisms to be justified is :

$$\begin{aligned}\mathbb{M}\mathbf{Set}(U, X) &\overset{(1)}{\cong} \varprojlim_{(A,\alpha)\in\mathrm{Sp}\downarrow U} \mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X) \overset{(2)}{\cong} \varprojlim_{(A,\alpha)\in\mathrm{Sp}\downarrow U}\ \varprojlim_{V,W\in\mathcal{U}} \mathbb{M}\mathbf{Set}(\alpha^{-1}V \cap \alpha^{-1}W, X) \\ &\overset{(3)}{\cong} \varprojlim_{V,W\in\mathcal{U}}\ \varprojlim_{(A,\alpha)\in\mathrm{Sp}\downarrow U} \mathbb{M}\mathbf{Set}(\alpha^{-1}V \cap \alpha^{-1}W, X) \overset{(4)}{\cong} \varprojlim_{V,W\in\mathcal{U}} \mathbb{M}\mathbf{Set}(V \cap W, X)\end{aligned}$$

(1) Density of representables. (3) Limits commute with limits.

(4) We know $\alpha^{-1}(V \cap W) = \alpha^{-1}V \cap \alpha^{-1}W$, so it suffices to prove the following.

> *Lemma. For $U \in \mathbb{M}\mathbf{Set}$ and $Z \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(U)$, we have $Z = \varprojlim_{(A,\alpha)\in\mathrm{Sp}\downarrow U} \alpha^{-1}Z$*

Proof. The forgetful functor $\mathrm{Sp} \downarrow Z \to \mathrm{Sp} \downarrow U$ is a "section" of the pullback functor $\mathrm{Sp} \downarrow U \to \mathrm{Sp} \downarrow Z$, meaning for $(A, \alpha) \in \mathrm{Sp} \downarrow Z$, the following is a pullback diagram :

$$\begin{tikzcd} Z \arrow[r] & U \\ \mathrm{Sp}\, A \arrow[u, "\alpha"] \arrow[r, "\mathbb{1}"] & \mathrm{Sp}\, A \arrow[u] \end{tikzcd}$$

This implies pulling the diagram $\mathrm{Sp} \downarrow U$ back to $\mathrm{Sp} \downarrow Z$ only introduces duplicate objects with identity morphisms in between them. Hence $\varprojlim_{(A,\alpha)\in\mathrm{Sp}\downarrow U} \alpha^{-1}Z = \varprojlim_{(A_1,\alpha_1)\in\mathrm{Sp}\downarrow Z} \mathrm{Sp}\, A = Z$ by the density of representables. ■

(2) We need to show that $\mathbb{M}\mathbf{Set}(-, X)$ is a sheaf for $\mathbf{Aff}$ with covers from the big Zariski site $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$. The key is that basic opens cover opens for affine schemes.

Let $A \in \mathbb{M}^{op}$ and $\mathcal{U}$ be a $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$-cover of $\mathrm{Sp}\, A$. For each $i \in \mathcal{U}$, let $I_i \in \mathrm{Ideal}\, A$ with $i = D(I_i)$. Let $I := \bigsqcup_{i\in\mathcal{U}} I_i$. Then since $\{D(f)\}_{f\in I_i}$ is a $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$-cover of i for every $i \in \mathcal{U}$, $\{D(f)\}_{f\in I}$ is also a $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$-cover of $\mathrm{Sp}\, A$. We then have the commutative diagram :

$$\begin{tikzcd} \mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X) \arrow[r] \arrow[d, "\mathbb{1}"] & \varprojlim_{i,j\in\mathcal{U}} \mathbb{M}\mathbf{Set}(i \cap j, X) \arrow[d, "\sim"] \\ \mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X) \arrow[r, "\sim"] & \varprojlim_{f,g\in I} \mathbb{M}\mathbf{Set}(D(f) \cap D(g), X) \end{tikzcd}$$

where the horizontal isomorphism to due to $\mathbb{M}\mathbf{Set}(-, X)$ being a sheaf on $\mathbf{Aff}_{\mathrm{Zar}}$.

It remains to justify the vertical isomorphism. To do this, we apply the same argument as we're trying to do now, but on $i \cap j$. It's easy to see that $\{D(f) \cap D(g)\}_{f \in I_i, g \in I_j}$ covers $i \cap j$, so we get

$$\begin{aligned}\mathbb{M}\mathbf{Set}(i \cap j, X) &\cong \varprojlim_{(A_1,\alpha_1) \in \mathrm{Sp}\downarrow(i \cap j)} \mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A_1, X) \cong \varprojlim_{(A_1,\alpha_1) \in \mathrm{Sp}\downarrow(i \cap j)} \varprojlim_{f \in I_i, g \in I_j} \mathbb{M}\mathbf{Set}(D(\alpha_1^\flat(f)) \cap D(\alpha_1^\flat(g)), X) \\ &\cong \varprojlim_{f \in I_i, g \in I_j} \varprojlim_{(A_1,\alpha_1) \in \mathrm{Sp}\downarrow U} \mathbb{M}\mathbf{Set}(\alpha_1^{-1}\left(D(f) \cap D(g)\right), X) \overset{(4)}{\cong} \varprojlim_{f \in I_i, g \in I_j} \mathbb{M}\mathbf{Set}(D(f) \cap D(g), X)\end{aligned}$$

where (4) is as before.

$\square$