0.1 How to work with "DG Categories"

Gaitsgory-Rozenblyum has a highly abstract but clean definition of dg-categories, which can be summarised in the following diagram :

$$\begin{tikzcd}
\mathrm{Set} \arrow[r] \arrow[d] & 1\text{-}\mathrm{Cat} \arrow[d, shift right, "N"'] \arrow[d, phantom, "\vdash"] & \mathrm{Cat}_{\mathrm{dg}} k \arrow[d, "N_{\mathrm{dg}}"] & & & \\
\mathrm{Spc} \arrow[r] & (\infty,1)\text{-}\mathrm{Cat} \arrow[u, shift right, "h"'] & (\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}} \arrow[l] & \mathrm{Vec}^c\text{-}\mathrm{Mod} \arrow[l] & =: & \mathrm{DGCat} \\
& & (\infty,1)\text{-}\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{cts}} \arrow[u] & \mathrm{Vec}\text{-}\mathrm{Mod} \arrow[l] \arrow[u] & =: & \mathrm{DGCat}_{\mathrm{cts}}
\end{tikzcd}$$

Some explanations are due :

- We will use "infinity category" to refer only to $(\infty, 1)$-categories. The word "category" will exclusively refer to 1-categories.

  $(\infty, 1)$-Cat denotes the infinity category of small infinity categories. $(\infty, 1)$-Cat has all small limits (Kerodon 7.4.1.11) and small colimits (Kerodon 7.4.3.13) and is cartesian closed. We use $\mathrm{Fun}(C, D)$ to denote the infinity category of functors from C to D.

  There is an adjunction $h \dashv N : (\infty, 1)\text{-}\mathrm{Cat} \rightleftarrows 1\text{-}\mathrm{Cat}$. N is called the *nerve functor* and it is fully faithful, allowing us to see 1-categories as ∞-categories. In particular, we use Δ^n to denote the *n-simplex*, the ∞-category obtained from the linear order $[n] = \{0 \leq \cdots \leq n\}$. Given an infinity category C, objects of C are the same as functors $\Delta^0 \to C$ and morphisms in C are the same as functors $\Delta^1 \to C$.

  There is a full subcategory Spc of $(\infty, 1)$-Cat consisting of ∞-categories X where all morphisms are isomorphisms. These are called *∞-groupoids* but also *spaces* by homotopy theorists. The infinity category Spc plays the role of Set in 1-category theory in the sense that given an infinity category C, the fiber of $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$ over (X, Y), denoted $C(X, Y)$, is in fact a space. The nerve functor N lands Set inside Spc, where given a set S and two points $x, y \in S$, we have $NS(x, y) \simeq \varnothing$ the empty space. In other words, sets are "discrete spaces".

  For any infinity category C, we define $\mathrm{PSh}\, C := \mathrm{Fun}(C^{op}, \mathrm{Spc})$ and refer to its objects as *presheaves in* C. Infinity categories of presheaves are significant since we will be working with derived algebraic geometry functorially. We will use the following universal property of $\mathrm{PSh}\, C$ many times.

  > **Proposition – Universal Property of Presheaf Categories (Lurie HTT 5.1.5.6)**
  > Let S be a small ∞-category.
  >
  > - There is a fully faithful functor $S \to \mathrm{PSh}\, S$ which takes each object x in S to the functor $S(\_, x) : C^{op} \to \mathrm{Spc}$ taking points y to $S(y, x)$. This is called the *Yoneda embedding*.
  > - $\mathrm{PSh}\, S$ has small colimits.
  > - For C an ∞-category with small colimits, let $\mathrm{Fun}^L(\mathrm{PSh}\, S, C)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{PSh}\, S, C)$ consisting of functors preserving small colimits. Then restricting along

> the Yoneda embedding $S \to \mathrm{PSh}\, S$ gives an equivalence of ∞-categories :
>
> $$\mathrm{Fun}^L(\mathrm{PSh}\, S, C) \xrightarrow{\sim} \mathrm{Fun}(S, C)$$
>
> An inverse functor is given by left Kan extension. In particular, for $F \in \mathrm{Fun}^L(\mathrm{PSh}\, S, C)$ corresponding to $F_0 \in \mathrm{Fun}(S, C)$, we have for every $X \in \mathrm{PSh}\, S$ that F exhibits $F(X)$ as the colimit of the diagram $S_{/X} \to S \to C$.

– $(\infty, 1)\text{-}\mathrm{Cat}^{\mathrm{ex}}$ denotes subcategory of $1\text{-}\mathrm{Cat}$ consisting of *stable infinity categories* and *exact functors*. It contains all small limits and the "inclusion" $1\text{-}\mathrm{Cat}^{ex} \to 1\text{-}\mathrm{Cat}$ preserves small limits (Lurie HA 1.1.4.4).

  Stable infinity categories are basically triangulated categories where exact triangles are determined by an infinity-categorical universal property. Here is the definition.

> **Definition**
>
> Let C be an infinity category. We say C has a *zero object* when it has an object that is both initial and final. (Lurie HA 1.1.1.1.)
>
> Now assume C have a zero object. Then a *triangle* in C is defined as a diagram in C of the form :
>
> $$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$
>
> A triangle is called a *fiber sequence* when it is a cartesian and a *cofiber sequence* when it is cocartesian. (Lurie HA 1.1.1.4.) In the first case, we say $Y \to Z$ *admits a kernel* and refer to X as the kernel, and in the other case we say $X \to Y$ *admits a cokernel* and refer to Z as the cokernel. [a]
>
> C is called *stable* when the following are true :
>
> – every morphism has both a kernel and a cokernel.
>
> – A triangle is fiber sequence iff it is a cofiber sequence. Such triangles are called *exact triangles*.
>
> (Lurie HA 1.1.1.9.)
>
> An exact functor $F : C \to D$ between stable infinity categories is one which satisfy any of the following equivalent conditions : (Lurie HA 1.1.4.1)
>
> – F preserves exact triangles
>
> – F preserves finite limits
>
> – F preserves finite colimits.

For stable ∞-categories C, D the full subcategory $\mathrm{Fun}^{\mathrm{ex}}(C, D)$ of $\mathrm{Fun}(C, D)$ consisting of exact functors is also stable.[b]

[a] In Lurie HA, kernels are called fibers and cokernels are called cofibers.

[b] GRI Chapter 1 5.1.4 claims this. Lurie HA 1.1.3.1 shows that $\mathrm{Fun}(K, C)$ is stable for any K and stable C. The result follows given that finite (co)limit-preserving functors are closed under finite (co)limits.

To help build intuition of "stable infinity categories as fixed triangulated categories", we record here the important parts of the procedure of extracting a triangulated category from a stable infinity category.

Proposition – Lurie 1.1.2.14

Let C be a stable infinity category. Then the following defines a triangulated structure on the 1-category hC :

- Define the *suspension functor* $\Sigma : C \to C$ by pushout against zeros :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

  Since the above square is a cofiber sequence, it is also a fiber sequence. This shows that *looping* $\Omega : C \to C$, given by pullback against zeros, gives an inverse for Σ and hence shows that Σ is an equivalence. Taking homotopy categories, we obtain an equivalence $[1] : hC \xrightarrow{\sim} hC$, which we use as the shift functor for the triangulated structure.

- We call a diagram
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
  in hC an exact triangle (in the triangulated categorical sense) when it comes from a diagram of the following form in C :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X[1] \end{tikzcd}$$

  i.e. two exact triangles (in the stable infinity categorical sense).

- For X, Y objects of C, we have

$$\begin{aligned} C(X, Y) &\simeq C(\Sigma\Omega X, Y) \simeq \Omega C(\Omega X, Y) \\ &\simeq C(\Sigma^2\Omega^2 X, Y) \simeq \Omega^2 C(\Omega^2 X, Y) \end{aligned}$$

> Upon taking π_0 , we obtain
>
> $$hC(X,Y) := \pi_0 C(X,Y) \simeq \pi_1 C(\Omega X, Y) \simeq \pi_2(\Omega^2 X, Y)$$
>
> where the last isomorphism is a group morphism. For π_2 of any "space" [a] the obvious group structure given by is abelian, this gives $hC(X,Y)$ an abelian group structure, making hC into an additive category.
>
> For X, Y objects in C, we define the abelian group $\mathrm{Ext}^n_C(X,Y) := hC(X, Y[n])$. (Lurie HA 1.1.2.17)
>
> [a] In the quasi-category model of infinity categories, $C(X,Y)$ is a Kan complex, which one can take homotopy groups of.

(IP : t-structures, truncation as reflective localisation, $(D^-(A))^\heartsuit \simeq A$ Lurie 1.3.2.19)

– We are now ready for compactly generated ∞-categories. We will only make use of the case of *stable* compactly generated ∞-categories since many definitions then admit alternative characterisations which can be checked at the level of triangulated categories.

We first note that the theory of colimits simplifies in the stable case.

> **Proposition – Lurie HA 1.4.4.1** – For a stable ∞-category C, TFAE:
>
> * admitting small colimits
> * admitting small filtered colimits
> * admitting small coproducts
>
> – For a functor $F : C \to D$ between stable ∞-categories which admit small colimits, TFAE:
>
>   * preserving small colimits
>   * preserving small filtered colimits
>   * preserving small coproducts
>
>   Any functor satisfying the above, GR calls *continuous*.

We now explain compact generation. The starting point is the theory of *inductive cocompletions*[1]. Here are the main results concerning ind-completions in the stable case.

> **Proposition – Ind-completions of Stable ∞-Categories**
> Let C be a small ∞-category and κ a regular cardinal.[a] Then the Yoneda embedding $C \to \mathrm{PSh}\, C$ factors through a full subcategory $\mathrm{Ind}_\kappa(C)$ with the following properties :
>
> – (Lurie HTT 5.3.5.3) $\mathrm{Ind}_\kappa(C)$ has all small κ-filtered colimits and the inclusion $\mathrm{Ind}_\kappa(C) \subseteq \mathrm{PSh}\, C$ preserves them

[1] In the literature, this is called ind-completion. This is a bit of a misnomer because intuitively we are adding filtered *colimits*, not limits.

– (Lurie HTT 5.3.5.4) An object X in $\mathrm{PSh}\, C$ is in $\mathrm{Ind}_\kappa(C)$ iff it is a κ-filtered colimit of representables iff $X : C^{op} \to \mathrm{Spc}$ preserves κ-small limits.

– (Lurie HA 1.1.3.6) If C is stable then so is $\mathrm{Ind}_\kappa(C)$.

– (Lurie HTT 5.3.5.10) For any ∞-category D admitting small κ-filtered colimits, we have the following equivalence of functor ∞-categories :

$$\mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D)$$

where

* the left category denotes the full subcategory of $\mathrm{Fun}(\mathrm{Ind}_\kappa(C), D)$ consisting of functors preserving κ-filtered colimits.[b]
* the forward functor is given by restricting along the Yoneda embedding $C \to \mathrm{Ind}_\kappa(C)$
* the inverse functor is given by left Kan extension.

Assuming C, D are stable and κ is the cardinality of $\mathbb{N}$, the above equivalence restricts to an equivalence between the following two full subcategories :

$$\mathrm{Fun}^{\mathrm{ex}}_{\mathrm{cts}}(\mathrm{Ind}_\kappa(C), D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(C, D)$$

where the left is the ∞-category of exact continuous functors from $\mathrm{Ind}_\kappa(C)$ to D.

When κ is the cardinality of $\mathbb{N}$, we write Ind instead of Ind_κ.

[a] A regular cardinal κ is a cardinality that is "sufficiently large" in the sense that the 1-category $\mathrm{Set}_{<\kappa}$ of sets with cardinality strictly less than κ has all colimits of size strictly less than κ. The cardinality of $\mathbb{N}$ is an example, since a finite colimit of finite sets is still finite.

[b] At Lurie HTT 5.3.4.5, these are called κ-continuous functors. Taking the minimal case of $\kappa = |\mathbb{N}|$, it seems only reasonable to refer to functors preserving filtered colimits as *continuous* functors. This is a potential explanation of GR's choice of terminology for continuous functors.

For a regular cardinal κ and an ∞-category C, we say C is κ-compactly generated when it has all small colimits and there exists a small ∞-category C^0 with an equivalence $\mathrm{Ind}_\kappa(C^0) \xrightarrow{\sim} C$.[1] For the case of $\kappa =$ cardinality of $\mathbb{N}$, we simply say *compactly generated*. A *presentable* ∞-category is one that is κ-compactly generated for some κ. We use $(\infty, 1)\text{-}\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{cts}}$ to denote the subcategory of $(\infty, 1)\text{-}\mathrm{Cat}^{\mathrm{ex}}$ whose objects are presented stable ∞-categories and morphisms are exact functors preserving small coproducts.

One should not worry about having to check presentability of a stable ∞-category since all the computable examples are in the compactly generated case. As it turns out, this can be checked at the level of the triangulated categories.

Proposition – Criterion for Compact Generation of Stable Infinity Categories

Let C be a stable ∞-category. We say an object X *generates* C when for all objects Y in C,

[1] This is unraveled from Lurie HTT 5.5.7.1, 5.5.0.18, and 5.4.2.1. In particular, the second condition is usually called κ-accessibility, but we have no need for such terminology.

$hC(X, Y) = 0$ implies $Y \simeq 0$. Then the following are true :

- C is compactly generated iff the following are true :
  * C has small coproducts
  * hC is locally small
  * There exists a regular cardinal κ and a κ-compact object X which generates C.
- For an object X in C, X is compact if and only if for every morphism $f : X \to \coprod_{i \in I} Y_i$ in C, there exists a finite subset $I_0 \subseteq I$ such that in hC, f factors through $\coprod_{i \in I_0} Y_i \to \coprod_{i \in I} Y_i$.

One appeal of presentable stable ∞-categories is that we have the adjoint functor theorem at our disposal.

Proposition – Adjoint Functor Theorem for Presentable ∞-Categories (Lurie HTT 5.5.2.9)
Let $F : C \to D$ be a functor between presentable ∞-categories.

- F is a left adjoint iff it preserves small colimits.
- Assuming C, D are κ-compactly generated, F is a right adjoint iff it preserves small limits and κ-filtered colimits.

For functors out of $\mathrm{Ind}_\kappa(C)$, fully faithfulness and equivalence can be detected at the level of C.

Proposition – Functors out of Compactly Generated Categories (Lurie 5.3.5.11)
Let C^0 be a small ∞-category, κ a regular cardinal, $C = \mathrm{Ind}_\kappa(C^0)$ and D an ∞-category admitting κ-filtered colimits. Let D^κ be the full subcategory of D consisting of κ-compact objects. Let $F : C \to D$ be a functor preserving κ-filtered colimits and $F_0 : C^0 \to D$ its restriction along the Yoneda embedding $C^0 \to C$.

- If F_0 is fully faithful and its essential image lands in D^κ, then F is fully faithful.
- F is an equivalence iff the following are true :
  * F_0 fully faithful
  * F_0 factors through D^κ
  * all objects of D are κ-filtered colimits of diagrams in D with objects in the image of F_0.

  In particular, for any full subcategory $\tilde{C} \subseteq C^\kappa$ which generates C under κ-filtered colimits, we have $\mathrm{Ind}_\kappa(\tilde{C}) \xrightarrow{\sim} C$.

Furthermore, the Yoneda embedding $C^0 \to C$ factors through C^κ. The fully faithful functor $C^0 \to C^\kappa$ is not in general an equivalence, however it does exhibit C^κ as the *idempotent completion* of C^0. (Lurie 5.4.2.4.) So if C^0 is idempotent complete, then we recover the κ-compact objects of C as precisely (the essential image of) C^0. One doesn't need to know the general ∞-categorical definition for idempotent completion since we will only be in the stable setting, in which we have the following characterisation.

> **Proposition – Idempotent Completeness in Stable Case (Lurie HA 1.2.4.6)**
> Let C be a stable ∞-category. Then C is idempotent complete iff hC is as a 1-category, i.e. for every morphism $e : B \to B$ such that $e^2 = e$, there exists a retract of $s : A \rightleftarrows B : r$ that exhibits $e = rs$.
>
> If the above is the case, then for any regular cardinal κ we have $C \xrightarrow{\sim} (\mathrm{Ind}_\kappa(C))^\kappa$.

– An important example of a compactly generated stable ∞-category is Vec.

> **Proposition**
> Let k be a field. Then there exists an ∞-category Vec called the *right derived ∞-category of k-vector spaces* with the following properties :
>
> – (Lurie HA 1.3.2.18) Vec is stable.
>
> – (Lurie HA 1.3.4.4 - Universal Property as Localisation) There is a functor $l : \mathrm{Ch}^-(k) \to \mathrm{Vec}$ with the property that for all ∞-categories E, restricting along l yields a fully faithful functor $\mathrm{Fun}(\mathrm{Vec}, E) \to \mathrm{Fun}(\mathrm{Ch}^-(k), E)$ with essential image consisting of functors $\mathrm{Ch}^-(k) \to E$ which invert quasi-isomorphisms.
>
> – (Lurie HA 1.3.2.9) $h\mathrm{Vec}$ gives the usual 1-category right derived category of k-vector spaces.
>
> Consequently, for $X, Y \in \mathrm{Ch}^-(k)$, we have
>
> $$\pi_n \mathrm{Vec}(X, Y) = \pi_0 \Omega^n \mathrm{Vec}(X, Y) \simeq \pi_0 \mathrm{Vec}(X, Y[n]) =: \mathrm{Ext}^n(X, Y)$$

– $1\text{-}\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{cts}}$ has symmetric monoidal structure $\otimes$ via the *Lurie tensor product*.

We won't really need to know anything about the tensor product other than its universal property.

(IP : Explain how practically speaking, it suffices to know the univeral property because we will work with compactly generated dg-categories, meaning computation will come down to compact generators and their homs.)

(IP : Give impression of symmetric monoidal infinity categories via Lawvere theory perspective.)

Unanswered Q : is this only used because the theory of commutative dg-algebras compromises in positive characteristic?

– $(\mathrm{Vec}, \otimes)$ is the stable symmetric monoidal (right bounded) derived infinity category of complexes of k-vector spaces. Its homotopy category $h(\mathrm{Vec})$ is 1-categorical localisation of the category of complexes of k-vectors spaces at quasi-isomorphisms, and has the usual t-structure. The heart of Vec is the usual abelian category of k-vector spaces. We use cohomological degree, where negative cohomological degree refers to homological degree.

Practically speaking, computations tensor product are done by using the projective model structure on the category of complexes of k-vectors spaces.

– $(\mathrm{Vec}, \otimes)$ can be seen as an commutative algebra object in the symmetric monoidal infinity category $1\text{-}\mathrm{Cat}^{st,cocompl}_{cts}$. Then dgCat_{cts} denotes the infinity category of modules over Vec inside $1\text{-}\mathrm{Cat}^{st,cocompl}_{cts}$.

– derived rings stuff - Lurie HTT 5.5.9.3

– modules over derived rings as symmetric monoidal ∞-cat - Lurie HA 7.1.2.13. Subtlety about different model structures is Lurie HA 7.1.2.9.

0.2 Quasi-Coherent Sheaves

Definition – Covariant QCoh

There is a functor $\mathrm{QCoh}_* : \mathrm{dAff} \to \mathrm{dgCat}_{cts}$ that assigns to each $A \in \mathrm{dAff}$ the *derived category of A-modules*, denoted A-Mod.

For A discrete (i.e. a commutative ring), there is the following description of A-Mod under the quasi-category model of ∞-categories summarised in a single diagram :

$$\begin{tikzcd}
N(\mathrm{Ch}\,A) \arrow[d, "\subseteq"'] \arrow[dr, "W^{-1}"] & & & \\
N_{dg}(\mathrm{Ch}\,A) \arrow[r, shift left=2, "L"] \arrow[r, phantom, "\perp"] & N_{dg}((\mathrm{Ch}\,A)_f) \arrow[l, shift left=2, "\supseteq"] \arrow[d, "h"] & =: & A\text{-}\mathrm{Mod} \\
& D(A) & &
\end{tikzcd}$$

Details :

– $\mathrm{Ch}\,A$ is the honest-to-god dg-category of chain complexes of honest-to-god A-modules and $D(A)$ is the category of complexes of injectives

– $\mathrm{Ch}\,A$ has a model structure such that cofibrations are degree-wise injections and weak equivalences are quasi-isomorphisms (Lurie HA 1.3.5.3). Although the class of fibrations are defined abstractly as those satisfying right lifting with respect to acyclic cofibrations, it turns out that any fibrant complex must be degree-wise injective, and partially conversely, any bounded above complex of injectives is fibrant (Lurie HA 1.3.5.6). $(\mathrm{Ch}\,A)_f$ denotes the full subcategory of fibrant complexes.

– N denotes the nerve functor which converts 1-categories to simplicial sets, which have the property of being ∞-categories. N_{dg} denotes the dg-nerve functor which achieves the same thing for honest-to-god dg-catgeory categories. (See Kerodon 2.5.3 for a construction.)

– h is the truncation of an infinity category to a 1-category by taking its homotopy catgeory. It is the left adjoint to N. (See Kerodon 1.2.5 for a construction.)

We have that the homotopy category of A-Mod gives the usual derived category of A-modules, as in classical algebraic geometry.

– L is a left adjoint to the inclusion $N_{dg}((\mathrm{Ch}\,A)_f) \subseteq N_{dg}(\mathrm{Ch}\,A)$. Intuitively, for every complex

$M_\bullet$, there exists a acyclic cofibration $M_\bullet \to I_\bullet$ to fibrant $I_\bullet$ and this is initial in the category of arrows from $M_\bullet$ into $N_{dg}((\mathrm{Ch}\, A)_f)$ (Lurie HA 1.3.5.12). This means for each $M_\bullet$, such a morphism $M_\bullet \to I_\bullet$ is unique up to equivalence and assembles to the desired functor L. Practically speaking, $L(M_\bullet) \simeq I_\bullet$.

– The composition $N(\mathrm{Ch}\, A) \to N_{dg}((\mathrm{Ch}\, A)_f)$ exhibits the latter as the ∞-categorical localisation of the former at quasi-isomorphisms (Lurie HA 1.3.5.15). This matches the standard treatment in classical algebraic geometry : the localisation functor from $\mathrm{Ch}\, A$ to $D(A)$ takes a complex and resolves it by injecting it quasi-isomorphically into a complex of injectives.

I'm sick of preliminaries for now.