Notes on Primary Decomposition : "Geometrically"

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Definition – Support of a Module
Let $M \in A\mathbf{Mod}$.

$$\operatorname{Supp} M := \{p \in \operatorname{Spec} A \mid M_p \neq 0\}$$

In particular, for $m \in M$, $\operatorname{Supp} m := \operatorname{Supp} Am$.

Proposition – Basic Properties of Support
Let $M \in A\mathbf{Mod}$. Then

- (Triviality) $\operatorname{Supp} M \neq \varnothing$ if and only if $M \neq 0$.
- (Closure Supports and Annihilator) $\overline{\operatorname{Supp} M} = V(\operatorname{Ann} M)$.

  In particular, for any section m, $\operatorname{Supp} m = V(\operatorname{Ann} m)$.
- (SES) Let $0 \to N \to M \to P \to 0$ be a short exact sequence of A-modules.
  Then $\operatorname{Supp} M = \operatorname{Supp} N \cup \operatorname{Supp} P$.
- (Support is Union of Support of Non-Zero Sections) $\operatorname{Supp} M = \bigcup_{0 \neq m} \operatorname{Supp} m$.

  In particular if M finite, then this is a finite union of closed subsets and hence $\operatorname{Supp} M = V(\operatorname{Ann} M)$.

Proof. (*Triviality*) being zero is stalk-local.

(*Ann*) For $p \in \operatorname{Spec} A$,

$$\begin{aligned} p \notin \overline{\operatorname{Supp} M} &\Leftrightarrow \exists\, f \in A, p \in D(f) \text{ and } D(f) \cap \operatorname{Supp} M = \varnothing. \\ &\Leftrightarrow \exists\, f \in A, p \in D(f) \text{ and } M_f = 0. \\ &\Leftrightarrow \operatorname{Ann} M \not\subseteq I(p) \Leftrightarrow p \notin V(\operatorname{Ann} M) \end{aligned}$$

Second equivalence : $D(f) = \operatorname{Spec} A_f$ so $D(f) \cap \operatorname{Supp} M = \varnothing$ if and only if $M_f = 0$ since being zero is stalk-local. Third equivalence : $\Leftarrow$ is clear. Conversely, given $p \in D(f)$ and $M_f = 0$, M_f implies there exists $N > 0$, $f^N \in \operatorname{Ann} M$. But $D(f) = D(f^N)$, so $f^N \in \operatorname{Ann} M \setminus I(p)$.

For a section $m \in M$, $(Am)_p = 0$ if and only if $(Am)_f = 0$ for some f with $p \in D(f)$, hence $\operatorname{Supp} m = \overline{\operatorname{Supp} m}$.

(SES) Let $p \in \operatorname{Spec} A$. Taking stalks is exact so we have exactness of $0 \to N_p \to M_p \to Q_p \to 0$. The result follows since $M_p = 0$ if and only if $N_p = 0 = Q_p$.

(*Union*) $\bigcup_{0 \neq m} \operatorname{Supp} m \subseteq \operatorname{Supp} M$ by (SES). The other inclusion is clear.

Now assume $M = \sum_{i=1}^{n} Am_i$. Then for any $p \in \operatorname{Spec} A$, $M_p = 0$ if and only if $(Am_i)_p = 0$ for all i.

□

Proposition – Primary Submodules
Let $M \in A\mathbf{Mod}$, $Q \lneq M$. For $f \in A$,

- TFAE :
  1. $f \in \bigcup_{0 \neq m} \operatorname{Ann} m$
  2. $f \in \bigcup_{0 \neq m} \sqrt{\operatorname{Ann} m}$
  3. there exists $0 \neq m \in M$, $\operatorname{Supp} m \subseteq V(f)$.

  f *is a zero-divisor in* M := any (and thus all) of the above.
- TFAE :
  1. $f \in \sqrt{\operatorname{Ann} M}$.
  2. $V(\operatorname{Ann} M) \subseteq V(f)$.
  3. $\operatorname{Supp} M \subseteq V(f)$.

  f *is nilpotent in* M := any (and thus all) of the above.

Hence TFAE :

1. All zero-divisors of M are nilpotent.
2. For all $0 \neq m \in M$, $\operatorname{Supp} m = \operatorname{Supp} M$.

Q *primary* := M/Q satisfies any (and thus all) of the above.

Proof. (*zero-divisor*) $2 \Leftrightarrow 3$ is ok. $1 \Rightarrow 2$ also ok. Suffices to prove $2 \Rightarrow 1$. Let $0 \neq m$, $f^N m = 0$. Let N be minimal. Then $m \neq 0$ gives $0 < N$ and $ff^{N-1}m = 0$ with $f^{N-1}m \neq 0$, so $f \in \operatorname{Ann}(f^{N-1}m)$.

(*nilpotent*) ($1 \Leftrightarrow 2$) by ideal-subset adjunction. ($2 \Leftrightarrow 3$) by $\overline{\operatorname{Supp} M} = V(\operatorname{Ann} M)$.

(*primary*) ($1 \Rightarrow 2$) Let $0 \neq m \in M$. Then

$$\operatorname{Supp} m = V(\operatorname{Ann} m) = \bigcap_{f \in \operatorname{Ann} m} V(f) \supseteq \operatorname{Supp} M \supseteq \operatorname{Supp} m$$

since all $f \in \operatorname{Ann} m$ are zero-divisors.

($2 \Rightarrow 1$) clear.

□

Proposition – Existence of Primary Decomposition in Noetherian Setting
For $N \lneq M \in A\mathbf{Mod}$. define N *irreducible* := for all $N_0, N_1 \leq M$, $N = N_0 \cap N_1$ implies $N = N_0$ or $N = N_1$. Let M be Noetherian. Then

1. For all $Q \lneq M$, Q irreducible implies Q primary.
2. For all $N \lneq M$ there exists finite $\nu \subseteq \mathbf{Sub}A\mathbf{Mod}(M)$ such that $N = \bigcap_{Q \in \nu} Q$ and all $Q \in \nu$ are irreducible, and hence primary.

For $N \lneq M$, a finite set $\nu \subseteq \mathbf{Sub}A\mathbf{Mod}(M)$ such that $N = \bigcap_{Q \in \nu} Q$ and all $Q \in \nu$ primary is called a *primary decomposition of* N.

Proof. (1) Let $Q \lneq M$ be irreducible. Let $f \in A$ be a zero-divisor in M/Q, i.e. let $0 \neq m \in M/Q$ with $fm = 0$. We want $N > 0$ such that $f^N M/Q = 0$. Since $0 \neq m$ and Q is irreducible, it suffices to give an $N > 0$ such that $Am \cap f^N M/Q = 0$. Well, M Noetherian implies M/Q Noetherian, so the chain

$$\ker f \subseteq \ker f^2 \subseteq \cdots$$

is constant after some $N > 0$. Then $Am \cap f^N M/Q \subseteq f^N \ker f^{N+1} = f^N \ker f^N = 0$ as desired.

(2) Decomposition into irreducibles exists by Noetherian induction on the poset of proper submodules of M.

$\square$

Proposition – Supports of Primary Submodules have Generic Points
Let $Q \lneq M \in A\mathbf{Mod}$. Then

1. Q primary $\Rightarrow \sqrt{\operatorname{Ann} M/Q}$ prime.

   Hence, there exists a unique $p_Q \in \operatorname{Spec} A$ such that $\operatorname{Supp} M/Q = \overline{\{p_Q\}}$.
2. $\sqrt{\operatorname{Ann} M/Q}$ maximal $\Rightarrow Q$ primary.

Proof. (1) Let $fg \in \sqrt{\operatorname{Ann} M/Q} = \sqrt{\operatorname{Ann} m}$ where m is non-zero section. WLOG $f \notin \sqrt{\operatorname{Ann} m}$. Then there exists $N > 0$ where $g \in \sqrt{\operatorname{Ann} f^N m} = \sqrt{\operatorname{Ann} M/Q}$, so $\sqrt{\operatorname{Ann} M/Q}$ is prime.

For "hence", $\operatorname{Supp} M/Q = \operatorname{Supp} m$ for some non-zero section m of M/Q. So $\operatorname{Supp} M/Q = \overline{\operatorname{Supp} M/Q} = \overline{\{p_Q\}}$ where p_Q is the point corresponding to $\sqrt{\operatorname{Ann} M/Q}$. Uniqueness of p_Q follows from $\operatorname{Spec} A$ being sober.

(2) Let $p \in \operatorname{Spec} A$ with $I(p) = \sqrt{\operatorname{Ann} M/Q}$. Then

$$\overline{\operatorname{Supp} M/Q} = V(\operatorname{Ann} M/Q) = V(I(p)) = \overline{\{p\}} = \{p\}$$

So $\operatorname{Supp} M/Q = \{p\}$. Since any non-zero section m of M/Q has $\varnothing \neq \operatorname{Supp} m$, the result follows.

$\square$

Example (Primary Components not Invariant).
Let K be a field. Consider the ideal $I := (X^2, XY)$ in $A := K[X,Y]$. $(X^2, XY) = (X) \cap (X,Y)^2$ but also $(X^2, XY) = (X) \cap (X^2, Y)$.

Note that for $p, q \in \operatorname{Spec} A$, $(A/I(p))_q = 0$ if and only if $q \notin \overline{\{p\}}$. So $\operatorname{Supp} A/I(p) = \overline{\{p\}}$. But $(A/I(p))_p = \kappa(p)$, so for all $0 \neq f \in A/I(p)$, $p \in \operatorname{Supp} f$ and hence $\operatorname{Supp} f = \operatorname{Supp} A/I(p)$. Thus, $I(p)$ is a primary submodule of A.

Then the above two intersections are distinct primary decompositions of the same ideal. Furthermore, this is a counter example to converse of previous proposition part (1) : $\sqrt{\operatorname{Ann} A/I} = \sqrt{I} = \sqrt{(X)} \cap \sqrt{(X,Y)^2} = (X) \cap (X,Y) = (X)$ is prime, but I is not primary.

The uniqueness in primary decomposition is hidden elsewhere.

Definition – Associated Points (according to Atiyah-MacDonald)
Let $M \in A\mathbf{Mod}$, $p \in \operatorname{Spec} A$. For any $m \in M$, the following are equivalent :

- (Generic point of support of sections) $\overline{\{p\}} = \operatorname{Supp} m$.
- (Algebraic side) $I(p) = \sqrt{\operatorname{Ann} m}$.

Then we say p *is AM-associated to* M when there exists $m \in M$ satisfying any of the above. We use $\operatorname{Ass}_{\mathrm{AM}} M$ to denote the set of points AM-associated to M.

Proposition – 1st Uniqueness of Primary Decomposition
Let $N \lneq M \in A\mathbf{Mod}$ and $N = \bigcap_{Q \in \nu} Q$ a primary decomposition of N. For $Q \in \nu$, let $p_Q \in \operatorname{Spec} A$ correspond to $\sqrt{\operatorname{Ann} M/Q}$.

Call ν *minimal* when

- for all $Q, Q_1 \in \nu$, $Q \neq Q_1$ implies $p_Q \neq p_{Q_1}$.
- for all $Q \in \nu$, $\bigcap_{Q_1 \in \nu \setminus \{Q\}} Q_1 \nsubseteq Q$.

Then

1. the primary decomposition ν contains a minimal one.
2. For ν minimal, $\{p_Q \mid Q \in \nu\} = \operatorname{Ass}_{\mathrm{AM}} M/N$. Hence, the left set is independent of the minimal primary decomposition ν.

Proof. (1) it suffices to prove that for any two primary Q_0, Q_1 with $\operatorname{Supp} M/Q_0 = \overline{\{p\}} = \operatorname{Supp} M/Q_1$. we have $Q_0 \cap Q_1$ primary as well, with $\operatorname{Supp} M/Q_0 \cap Q_1 = \overline{\{p\}}$. Note that we already have :

$$\operatorname{Supp} M/Q_0 \cap Q_1 = \operatorname{Supp} M/Q_0 \cup \operatorname{Supp} M/Q_1 = \overline{\{p\}}$$

For $m \in M/Q_0 \cap Q_1$, let m_0 and m_1 be the reduction of m mod Q_0 and Q_1 respectively. Then $\operatorname{Ann} m = \operatorname{Ann} m_0 \cap \operatorname{Ann} m_1$ with $m \neq 0$ gives

$$\operatorname{Supp} m = \operatorname{Supp} m_0 \cup \operatorname{Supp} m_1 = \overline{\{p\}} = \operatorname{Supp} M/Q_0 \cap Q_1$$

(2) For $0 \neq m \in M/N$, let m_Q be the reduction of m mod Q. Then $\mathrm{Ann}(m) = \bigcap_{Q \in \nu} \mathrm{Ann}(m_Q)$ implies

$$\operatorname{Supp} m = \bigcup_{Q \in \nu} \operatorname{Supp} m_Q = \bigcup_{m_Q \neq 0} \operatorname{Supp} M/Q = \bigcup_{m_Q \neq 0} \overline{\{p_Q\}}$$

So if $\operatorname{Supp} m = \overline{\{p\}}$, then there exists $Q \in \nu$ where $\overline{\{p\}} = \overline{\{p_Q\}}$, and hence $p = p_Q$. Conversely, for $Q \in \nu$, minimality of ν gives $0 \neq m^Q \in M/N$ with $m^Q_Q \neq 0$ and $m^Q_{Q_1} = 0$ for $Q_1 \neq Q$. So $\operatorname{Supp} m^Q = \overline{\{p_Q\}}$.

$\square$