Definition – $\mathbb{Z}$-Functor

Define $\mathbb{M} := \mathbb{Z}\mathbf{Alg}^{op}$. The *category of $\mathbb{Z}$-functors* $\mathbb{M}\mathbf{Set}$ is define to be the category of presheaves of sets on $\mathbb{M}$. For $X \in \mathbb{M}\mathbf{Set}$ and $A^{op} \in \mathbb{M}$, elements of $X(A^{op})$ are called *morphisms from A^{op} to X*.

Define $\mathrm{Sp} : \mathbb{M} \to \mathbb{M}\mathbf{Set}$ to be the fully faithful Yoneda embedding $A^{op} \mapsto \mathbb{M}(-, A^{op})$. For $A \in \mathbb{Z}\mathbf{Alg}$, $\mathrm{Sp}\, A$ is called the *spectrum of A*. The *category of affine schemes* is defined to be the essential image of Sp. We will denote it with $\mathbf{Aff}$.

Proposition – Yoneda

The following are true :

- ("Morphisms from A^{op} to X are Morphisms from $\mathrm{Sp}\, A$ to X") For $X \in \mathbb{M}\mathbf{Set}$ and $A^{op} \in \mathbb{M}$,

$$\mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X) \xrightarrow{\sim} X(A^{op})$$

  by $\alpha \mapsto \alpha_{A^{op}}(\mathbb{1}_{A^{op}})$. Furthermore, this is functorial in A^{op} and X.

- (Density of Representables / "The data of X is precisely how test spaces map into it") For $X \in \mathbb{M}\mathbf{Set}$, X is the colimit of the diagram $\mathrm{Sp} \downarrow X \to \mathbb{M}\mathbf{Set}$, i.e. morphisms out of X are determined by restricting along morphisms from affine schemes to X.

Proof. Straightforward. □

Proposition – Affine Line

Let $n \in \mathbb{N}$. Define *affine n-space* to be $\mathbb{A}^n \in \mathbb{M}\mathbf{Set}$ sending $A^{op} \mapsto A^n$. Then

- $\mathbb{A}^n$ is representable by $\mathbb{Z}[T_1, \ldots, T_n]^{op}$. Hence $\mathbb{A}^n \in \mathbf{Aff}$.

- for $n = 1$, $\mathbb{A}^1$ is a ring object in $\mathbb{M}\mathbf{Set}$. Hence for $X \in \mathbb{M}\mathbf{Set}$, $\mathcal{O}(X) := \mathbb{M}\mathbf{Set}(X, \mathbb{A}^1) \in \mathbb{Z}\mathbf{Alg}$. This is called the *ring of global functions on X* and gives a functor $\mathcal{O}(\star) : \mathbb{M}\mathbf{Set} \to \mathbb{Z}\mathbf{Alg}^{op}$. We call elements of $\mathcal{O}(X)$ *functions on X*.

  Concretely on morphisms $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$, the corresponding ring morphism $\varphi^\flat : \mathcal{O}(X) \to \mathcal{O}(Y)$ is given by $f \mapsto f \circ \varphi$, i.e. pulling back along φ.

- (Spec, Global Function Adjunction)

$$\mathbb{M}\mathbf{Set}(-, \mathrm{Sp}\, \star) \cong \mathbb{Z}\mathbf{Alg}(\star, \mathcal{O}(-))$$

- (Canonical Choice of Spectrum for Affine Schemes) for $X \in \mathbb{M}$, X is affine if and only if $\mathbb{1}^{\perp}_{\mathcal{O}(X)} : X \to \mathrm{Sp}\, \mathcal{O}(X)$ is an isomorphism of $\mathbb{Z}$-functors, where $\mathbb{1}^{\perp}_{\mathcal{O}(X)}$ is the adjunct of $\mathbb{1}_{\mathcal{O}(X)}$ under the previous adjunction.

Proof. (*Spec, Global Function Adjunction*) Follows from this chain of bijections functorial in A and X given by the density of representables :

$$\begin{aligned}\mathbb{M}\mathbf{Set}(X, \operatorname{Sp} A) &\cong \varprojlim_{(B,\beta)\in \operatorname{Sp}\downarrow X} \mathbb{M}\mathbf{Set}(\operatorname{Sp} B, \operatorname{Sp} A) \cong \varprojlim_{(B,\beta)\in \operatorname{Sp}\downarrow X} \mathbb{Z}\mathbf{Alg}(A, B) \cong \mathbb{Z}\mathbf{Alg}\left(A, \varprojlim_{(B,\beta)\in \operatorname{Sp}\downarrow X} B\right) \\ &\cong \mathbb{Z}\mathbf{Alg}\left(A, \varprojlim_{(B,\beta)\in \operatorname{Sp}\downarrow X} \mathbb{M}\mathbf{Set}(\operatorname{Sp} B, \mathbb{A}^1)\right) \cong \mathbb{Z}\mathbf{Alg}(A, \mathbb{M}\mathbf{Set}(X, \mathbb{A}^1)) = \mathbb{Z}\mathbf{Alg}(A, \mathcal{O}(X))\end{aligned}$$

(*Affine Schemes*) The reverse implication is clear. Let $X \xrightarrow{\sim} \operatorname{Sp} A$ be an isomorphism in $\mathbb{M}\mathbf{Set}$. The adjunction $\mathbb{M}\mathbf{Set}(X, \operatorname{Sp} A) \cong \mathbb{Z}\mathbf{Alg}(A, \mathcal{O}(X)) \cong \mathbb{M}\mathbf{Set}(\operatorname{Sp} \mathcal{O}(X), \operatorname{Sp} A)$ gives the commutative diagram :

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d] & \operatorname{Sp} A \\ \operatorname{Sp} \mathcal{O}(X) \arrow[ur, dashed] & \end{tikzcd}$$

where the dashed has the composition of the other two morphisms as an inverse, and hence an isomorphism.

□

Remark – Intuition of Affine n-Space. For a smooth manifold X, a smooth map $\varphi : X \to \mathbb{R}^n$ is equivalent to the data of n global smooth functions $f_1, \ldots, f_n$ on X, i.e.

$$C^\infty\mathbf{Mfd}(X, \mathbb{R}^n) \cong C^\infty(X)^n$$

"$\mathbb{R}^n$ is the classifying space of n-tuples of global smooth functions." In the functorial POV of algebraic geometry, we take this as our definition of affine n-space.

> **Proposition – Categorical Properties of $\mathbb{Z}$-Functors**
> The following are true :
>
> – (Completeness and Cocompleteness) $\mathbb{M}\mathbf{Set}$ has all (small) limits and colimits, all computed pointwise.
>
>   In particular, a morphism $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$ is respectively a mono/epi/isomorphism if and only if for all $A \in \mathbb{M}^{op}$, $\varphi_A : Y(A) \to X(A)$ is injective/surjective/bijective.
>
> – (Base Change) For $K \in \mathbf{CRing}$, define the *category of K-functors* to be the over-category $\mathbb{M}\mathbf{Set}/\operatorname{Sp} K$. In particular, we call $\mathbf{Aff}/\operatorname{Sp} K$ the *category of affine K-schemes*.
>
>   Then we have for $\varphi \in \mathbb{M}\mathbf{Set}(\operatorname{Sp} L, \operatorname{Sp} K)$, we have the following adjunction
>
> $$(\mathbb{M}\mathbf{Set}/\operatorname{Sp} L)\,(-, \operatorname{Sp} L \times_{\operatorname{Sp} K} (\star)) \cong (\mathbb{M}\mathbf{Set}/\operatorname{Sp} K)\,(-, \star)$$
>
>   Furthermore, this restricts to an adjunction between $\mathbf{Aff}/\operatorname{Sp} L$ and $\mathbf{Aff}/\operatorname{Sp} K$, i.e. the pullback of affine schemes is affine.

Proof. (*Base Change*) The first adjunction is categorical. For the restriction to affine schemes over K and L, note that for any K-algebra A,

$$\operatorname{Sp} L \times_{\operatorname{Sp} K} \operatorname{Sp} A \text{ ``$=$'' } \operatorname{Sp}(L \otimes_K A)$$

$\square$