Remark. A scheme will be a "space modelled on $\mathbb{M}$ with an open cover by affine schemes". This section defines the notion of "open subfunctors" of a $\mathbb{Z}$-functor and an "open cover". Since $\mathbb{Z}$-functors are meant to be modelled on $\mathbb{M}$, we first define everything for affine schemes.

Somehow, no one could eradicate the special role of fields as "points".

Definition – Points and Covers

[a] Define the *category of test points*, Pts, to be the full subcategory of $\mathbb{M}$ consisting of K^{op} where K is a field. For $X \in \mathbb{M}\mathbf{Set}$, a *point of* X is defined to be a morphism $x \in \mathbb{M}\mathbf{Set}(\operatorname{Sp} K, X)$ where $K^{op} \in \mathrm{Pts}$. For $\varphi \in \mathbb{M}\mathbf{Set}(X, Y)$, we will use $\varphi(x)$ to denote $\varphi \circ x$. For $f \in \mathcal{O}(X)$, we will use ev_x to denote the pullback $x^\flat : \mathcal{O}(X) \to \mathcal{O}(K)$.

For $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$, we will use $Y^{\mathrm{Pts}}, X^{\mathrm{Pts}}$ to denote the restriction of Y, X to Pts^{op} and $\varphi^{\mathrm{Pts}} : Y^{\mathrm{Pts}} \to X^{\mathrm{Pts}}$ the restricted morphism. Then we say φ is *surjective on points* when φ^{Pts} is an epimorphism (equivalently, component-wise surjective).

For a subset $\mathcal{U} \subseteq \mathbb{M}\mathbf{Set} \downarrow X$ for any $X \in \mathbb{M}\mathbf{Set}$, we say $\mathcal{U}$ *covers* X when $\coprod \mathcal{U} \to X$ is surjective on points. Equivalently, for all points $x : \operatorname{Sp} K \to X$ of X, there exists $U \in \mathcal{U}$ with a factoring

$$\begin{tikzcd} U \arrow[r] & X \\ & \operatorname{Sp} K \arrow[u, "x"'] \arrow[ul] \end{tikzcd}$$

[a] Pts and "surjective on points" is non-standard definition.

Counter Example (*Surjective on Points implies Surjective*).
Consider $\operatorname{Sp} \mathbb{F}_2 \to \operatorname{Sp} \mathbb{F}_2[dT]$ *where* $dT^2 := 0$. $(\operatorname{Sp} \mathbb{F}_2)^{\mathrm{Pts}} \cong (\operatorname{Sp} \mathbb{F}_2[dT])^{\mathrm{Pts}}$ *but* $(\operatorname{Sp} \mathbb{F}_2[dT])(\mathbb{F}_2[dT])$ *bijects with* $\mathbb{F}_2$ *whilst* $(\operatorname{Sp} \mathbb{F}_2)(\mathbb{F}_2[dT])$ *is singleton.*

Proposition – Base Change of Cover

Let $\mathcal{U}$ be a cover of $X \in \mathbb{M}\mathbf{Set}$. Then for all $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$, the set $\varphi^{-1}\mathcal{U}$ of pullbacks of morphisms in $\mathcal{U}$ forms a cover of Y.

Proof. Follows from fiber product in $\mathbb{M}\mathbf{Set}$ being computed component-wise. □

Proposition – Multiplicative Group Scheme

Consider the functor $\mathbb{G}^\times \in \mathbb{M}\mathbf{Set}$ defined by $A \in \mathbb{M}^{op} \mapsto A^\times$. Then

- $\mathbb{G}^\times$ is representable by the ring $\mathbb{Z}[T, T^{-1}]$ and hence affine.
- $\mathbb{G}^\times$ is a group object in $\mathbb{M}\mathbf{Set}$. In fact, for any $X \in \mathbb{M}\mathbf{Set}$, $\mathbb{M}\mathbf{Set}(X, \mathbb{G}^\times) = \mathcal{O}(X)^\times$.

Proof. UP of $\mathbb{Z}[T, T^{-1}]$ implies it represents $\mathbb{G}^\times$. The second property can be straightforwardly deduced either from the Spec-global functions adjunction or elementarily. □

Proposition – Basic Opens of a $\mathbb{Z}$-Functor

Let $X \in \mathbb{M}\mathbf{Set}$ and $f \in \mathcal{O}(X)$. The *support of* f, X_f, is defined as the subfunctor of X sending $A \in \mathbb{M}^{op}$ to the set of $\alpha \in \mathbb{M}\mathbf{Set}(\operatorname{Sp} A, X)$ such that $\varphi^\flat(f) \in A^\times$.

Then X_f is the pullback of $\mathbb{G}^\times$ along $f : X \to \mathbb{A}^1$.

$$\begin{tikzcd} X_f \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{G}^\times \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

Subfunctors of X of the form X_f are called *basic opens*.

Proof. Easy. □

Remark – Intuition of Multiplicative Group Scheme and Basic Opens. For a smooth manifold X,

$$C^\infty\mathbf{Mfd}(X, \mathbb{R}^\times) \cong C^\infty(X)^\times$$

"$\mathbb{R}^\times$ is classifying space for invertible global functions on X." One can thus think of $\mathbb{G}^\times$ as "$\mathbb{A}^1 \setminus \{0\}$". A basic open X_f is then just the preimage of "$\mathbb{A}^1 \setminus \{0\}$" under $f : X \to \mathbb{A}^1$.

Proposition – Opens of Affine Schemes

Let $X \in \mathbf{Aff}$. For $I \in \operatorname{Ideal} \mathcal{O}(X)$, define $D(I) \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$ by

$$A \in \mathbb{M}^{op} \mapsto \left\{\varphi \in \mathbf{Aff}(\operatorname{Sp} A, X) \mid A\varphi^\flat I = A\right\}$$

In particular, for $I = (f)$, we have $D(I) = X_f$. Sometimes, we use $D(f)$ to denote X_f.

Then

- (Basic Opens are Affine) for $f \in \mathcal{O}(X)$, $D(f)$ is representable by $\mathcal{O}(X)[f^{-1}]$.
- (Ideals to Opens) For $I, J \in \operatorname{Ideal} \mathcal{O}(X)$, $I \subseteq J$ implies $D(I) \subseteq D(J)$.

  This defines $D : \operatorname{Ideal} \mathcal{O}(X) \to \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$. The category $\operatorname{Open} X$ of *opens of* X is defined as the essential image of D. We call $U \in \operatorname{Open} X$ an *open of* X.
- (Intuitive Definition of Opens) for $I \in \operatorname{Ideal} \mathcal{O}(X)$, $\{D(f)\}_{f \in I}$ covers $D(I)$.
- (Partition of Unity) For $I \in \operatorname{Ideal} \mathcal{O}(X)$, $D(I)$ covers X if and only if there exists finite $I_0 \subseteq I$ such that $AI_0 = A$. Such $I_0 \subseteq A$ are called *partitions of unity*.
- (Base Change / "Preimage of Opens are Open") Let $\varphi \in \mathbf{Aff}(Y, X)$, $I \in \operatorname{Ideal} \mathcal{O}(X)$. Let the following be a pullback diagram :

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] & X \\
\varphi^{-1}D(I) \arrow[u] \arrow[r] & D(I) \arrow[u]
\end{tikzcd}$$

Then $\varphi^{-1}D(I) = D(\varphi^\flat I)$.

– (Fiber Product) Let $X, X_1 \in \mathbf{Aff}/Y$ where $Y \in \mathbf{Aff}$. Let I, I_1 be ideals of $\mathcal{O}(X)$, $\mathcal{O}(X_1)$ respectively. Then we have the pullback diagram :

$$\begin{tikzcd}
D(I) \arrow[r] & Y \\
D(I \otimes_{\mathcal{O}(Y)} I_1) \arrow[u] \arrow[r] & D(I_1) \arrow[u]
\end{tikzcd}$$

In particular, $X = X_1 = Y$ implies the intersection of two opens of X is an open of X. The special case of $D(I) = D(f)$ and $D(I_1) = D(f_1)$ yields $D(f) \cap D(f_1) = D(ff_1)$.

Proof.

(*Basic Opens are Affine*) UP of $\mathcal{O}(X)[f^{-1}]$ as an $\mathcal{O}(X)$ algebra.

(*Intuitive Def of Opens*) Let $x : \operatorname{Sp} K \to X$ be a point of X. Then $Kx^\flat I = K$ if and only if there exists $f \in I$ with $f(x) \in K^\times$.

(*Partition of Unity*) Having finite $I_0 \subseteq I$ with $AI_0 = A$ is equivalent to $AI = A$. Clearly, $AI = A$ implies $D(I)$ covers $\operatorname{Sp} A$. Conversely, suppose $AI \subsetneq A$. $D(I)$ not covering $\operatorname{Sp} A$ is the same as it "missing a point of $\operatorname{Sp} A$", that is to say we are looking for a point $x : \operatorname{Sp} K \to \operatorname{Sp} A$ of $\operatorname{Sp} A$ that doesn't admit a lift across $D(I) \to \operatorname{Sp} A$. This is the same as $I \subseteq \ker \operatorname{ev}_x$. Well, $AI \subsetneq A$ implies by Zorn's lemma the existence of a map $\operatorname{ev}_x : A \to K$ where K is a field with the desired property. □

Counter Example ($\bigcup_{f \in I} D(f) = D(I)$).
Consider the ring $\mathbb{F}_2 \times \mathbb{F}_2$ *and elements* $(1,0), (0,1)$. *The ideal* I *generated by these is the whole ring. But* $D((1,0))(\mathbb{F}_2 \times \mathbb{F}_2) \cup D((0,1))(\mathbb{F}_2 \times \mathbb{F}_2) \subsetneq D(I)(\mathbb{F}_2 \times \mathbb{F}_2)$ *since the ring endomorphism* $(a,b) \mapsto (b,a)$ *doesn't map any of* $(1,0), (0,1)$ *to units. Thus* $D((1,0)) \cup D((0,1)) \subsetneq D(I)$.

Definition – Open Subfunctor

Let $X \in \mathbb{M}\mathbf{Set}$. For $U \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$, U is called *open* when for all $\varphi : \operatorname{Sp} A \to X$, the pullback $\varphi^{-1}U$ of U along φ is an open of $\operatorname{Sp} A$.

$$\begin{array}{ccc} \mathrm{Sp}\, A & \xrightarrow{\varphi} & X \\ \uparrow & & \uparrow \\ \varphi^{-1}U & \longrightarrow & U \end{array}$$

We will use $\operatorname{Open} X$ to denote the full subcategory of opens of X in $\mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$. When X is affine, this agrees with our specialized notion for affine schemes.

Proposition – Basic Facts about Open Subfunctors

The following are true :

- ("Extensionality") Let $U, V \in \operatorname{Open} X$. Then $U = V$ if and only if $U^{\mathrm{Pts}} = V^{\mathrm{Pts}}$.
- (Composition) Let $V \in \operatorname{Open} U$, $U \in \operatorname{Open} X$, $X \in \mathbb{M}\mathbf{Set}$. Then $V \in \operatorname{Open} X$.
- (Base Change/"Preimage of Opens are Opens") Let $X \in \mathbb{M}\mathbf{Set}$, $U \in \operatorname{Open} X$ and $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$. Then the pullback $\varphi^{-1}U$ of U along φ is an open of Y.

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow & & \uparrow \\ \varphi^{-1}U & \longrightarrow & U \end{array}$$

- (Fiber Product) Let U, U_1 be opens of $X, X_1 \in \mathbb{M}\mathbf{Set}$ respectively. Then for any $X \to S, X_1 \to S$, the fiber product $U \times_S U_1$ is an open of $X \times_S X_1$. In particular, for $X = X_1 = S$, this proves the intersection of two opens of X is again an open of X.

Proof. (Extensionality) Reduce to affine global case and use partition of unity. $\square$