Notes Category Theory

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Contents

In this paper, we do not pay attention to foundational set-theoretic issues like the "set" of all sets should be a class.

1 Categories

Definition – Categories

A *category* $\mathcal{C}$ is defined by the following data :

1. A set of *objects*, $\mathrm{Obj}(\mathcal{C})$.
2. For every $U, V \in \mathrm{Obj}(\mathcal{C})$, a set of $\mathcal{C}$-*morphisms* from U to V, denoted $\mathcal{C}(U, V)$. We denote $f : U \xrightarrow{\mathcal{C}} V$ for $f \in \mathcal{C}(U, V)$.
3. For every $U, V, W \in \mathrm{Obj}(\mathcal{C})$, $f : U \xrightarrow{\mathcal{C}} V$ and $g : V \xrightarrow{\mathcal{C}} W$, a $\mathcal{C}$-morphism called the *composition of* f *with* g, denoted $g \circ f : U \xrightarrow{\mathcal{C}} W$.
4. Associativity of $\circ$.

5. For every $U \in \mathrm{Obj}(\mathcal{C})$, an *identity morphism* $\mathbb{1}_U : U \xrightarrow{\mathcal{C}} U$.
6. For all $U, V, W \in \mathrm{Obj}(\mathcal{C})$, $f : U \xrightarrow{\mathcal{C}} V$ and $g : W \xrightarrow{\mathcal{C}} U$, we have $f \circ \mathbb{1}_U = f$ and $\mathbb{1}_U \circ g = g$.

Definition – Isomorphisms
Let $\mathcal{C}$ be a category, $U, V \in \mathrm{Obj}(\mathcal{C})$, $f : U \xrightarrow{\mathcal{C}} V$. Then f is called an *isomorphism* when there exists $g : V \xrightarrow{\mathcal{C}} U$ such that $g \circ f = \mathbb{1}_U$ and $f \circ g = \mathbb{1}_V$. In this case, we denote $f : U \xrightarrow[\sim]{\mathcal{C}} V$. When there exists an isomorphism from U to V, we say they are *isomorphic* and write $U \cong V$.

Definition – Subcategories
Let $\mathcal{C}, \mathcal{D}$ be categories. Then $\mathcal{D}$ is called a *subcategory of* $\mathcal{C}$ when $\mathrm{Obj}(\mathcal{D}) \subseteq \mathrm{Obj}(\mathcal{C})$ and for all $U, V \in \mathrm{Obj}(\mathcal{D})$, $\mathcal{D}(U, V) \subseteq \mathcal{C}(U, V)$.

Example (*Standard Categories*).

1. $\mathbf{Set}$ *denotes the* category of sets, *where* $\mathrm{Obj}(\mathbf{Set})$ *contains sets and for* $U, V \in \mathrm{Obj}(\mathbf{Set})$, $\mathbf{Set}(U, V)$ *is the set of maps from* U *to* V.
2. $\mathbf{Top}$ *denotes the* category of topological spaces, *where* $\mathrm{Obj}(\mathbf{Top})$ *contains topological spaces and for* $U, V \in \mathrm{Obj}(\mathbf{Top})$, $\mathbf{Top}(U, V)$ *is the set of continuous maps from* U *to* V. $\mathbf{Top}$ *is a subcategory of* $\mathbf{Set}$.
3. *The* category of groups $\mathbf{Grp}$ *has* $\mathrm{Obj}(\mathbf{Grp})$ *containing groups and* $\mathbf{Grp}(U, V)$ *containing group homomorphisms from* U *to* V. $\mathbf{Grp}$ *is a subcategory of* $\mathbf{Set}$.
4. *The* category of abelian groups $\mathbf{Ab}$ *has* $\mathrm{Obj}(\mathbf{Ab})$ *containing abelian groups and* $\mathbf{Ab}(U, V)$ *containing group homomorphisms from* U *to* V. $\mathbf{Ab}$ *is a subcategory of* $\mathbf{Grp}$.
5. *The* category of rings $\mathbf{Ring}$ *has* $\mathrm{Obj}(\mathbf{Ring})$ *containing rings and* $\mathbf{Ring}(U, V)$ *containing ring homomorphisms from* U *to* V. $\mathbf{Ring}$ *is a subcategory of* $\mathbf{Set}$.
6. *The* category of commutative rings $\mathbf{CRing}$ *has* $\mathrm{Obj}(\mathbf{CRing})$ *containing commutative rings and* $\mathbf{CRing}(U, V)$ *containing ring homomorphisms from* U *to* V. $\mathbf{CRing}$ *is a subcategory of* $\mathbf{Ring}$.
7. *Let* R *be a commutative ring. Then the* category of R-modules $\mathbf{Mod}(R)$ *has* $\mathrm{Obj}(\mathbf{Mod}(R))$ *containing* R*-modules and* $\mathbf{Mod}(R)(U, V)$ *contains* R*-linear maps from* U *to* V. *This is a subcategory of* $\mathbf{Ab}$.
8. *Let* R *be a commutative ring. Then the* category of R-algebras $\mathbf{Alg}(R)$ *has* $\mathrm{Obj}(\mathbf{Alg}(R))$ *containing pairs* (S, σ) *where* $\sigma : R \xrightarrow{\mathbf{CRing}} S$. $\mathbf{Alg}(R)((U, u), (V, v))$ *contains* $f : U \xrightarrow{\mathbf{CRing}} V$ *such that* $f \circ u = v$.

Example (*Preordered Sets as Categories*).
Let I *be a set,* $\leq$ *a relation on* I. *Then* $(I, \leq)$ *is called a* preordered set *when* $\leq$ *satisfies all of the following* :

1. (*Reflexivity*) *For all* $i \in I$, $i \leq i$.
2. (*Transitivity*) *For all* $i, j, k \in I$, $i \leq j$ *and* $j \leq k$ *implies* $i \leq k$.

If $(I, \leq)$ *is a preordered set where* $\leq$ *is clear, we abbreviate to* I.

Let I *be a preordered set. Then we can turn* I *into a category as follows :*

1. $\mathrm{Obj}(I)$ *is* I.
2. *For* $i, j \in \mathrm{Obj}(I)$, $I(i,j)$ *is singleton when* $i \leq j$ *and empty otherwise.*

Things get meta. We can form the category of preordered sets $\mathbf{Ord}$ *where* $\mathrm{Obj}(\mathbf{Ord})$ *contains preoredered sets and* $\mathbf{Ord}(I, J)$ *contains* $f : I \xrightarrow{\mathbf{Set}} J$ *such that for all* $i, j \in I$, $i \leq j$ *implies* $f(i) \leq f(j)$.

Example (*Category of Partially Ordered Sets*).
Let $I \in \mathrm{Obj}(\mathbf{Ord})$. *Then* I *is called a* partially ordered set *when* $\leq$ *is* antisymmetric, *i.e. for all* $i, j \in I$, $i \leq j$ *and* $j \leq i$ *implies* $i = j$. *We thus have the* category of partially ordered sets $\mathbf{PoSet}$ *where* $\mathrm{Obj}(\mathbf{PoSet})$ *contains partially ordered sets and* $\mathbf{PoSet}(I, J) = \mathbf{Ord}(I, J)$. *We see that* $\mathbf{PoSet}$ *is a subcategory of* $\mathbf{Ord}$.

Example (*Partially Ordered Sets*).

1. *Let* X *be a set. Then its powerset* $2^X \in \mathrm{Obj}(\mathbf{PoSet})$.
2. *Let* X *be a topological space. Then the set of its open sets* $\mathrm{Open}\, X$ *is a partially ordered set.*
3. *Let* G *be a group. Then the set of its subgroups is in* $\mathrm{Obj}(\mathbf{PoSet})$.
4. *Let* R *be a commutative ring and* M *be an* R*-module. Then the set of* R*-submodules of* M *is in* $\mathrm{Obj}(\mathbf{PoSet})$.
5. *Let* R *be a commutative ring and* (S, σ) *an* R*-algebra. Then the set of all* R*-subalgebras of* S *is in* $\mathrm{Obj}(\mathbf{PoSet})$.
6. *Consider the relation on* $\mathbb{N}$ *that is* $a \mid b$. *This is a partial order on* $\mathbb{N}$.

Example (*A Group as a Category*).
A group G *is equivalent to a category* G *where there is only one object* $\bullet$ *and all morphisms are isomorphisms.*

A direct generalization is a groupoid *: a category where every morphism is an isomorphism.*

2 Functors

Definition – Functors

Let $\mathcal{C}, \mathcal{D}$ be categories. Then a *functor* F *from* $\mathcal{C}$ *to* $\mathcal{D}$ is defined by the following data :

1. A map of objects $\mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, which we will denote by the same name F.
2. A map of morphisms for all $U, V \in \mathrm{Obj}(\mathcal{C})$, $\mathcal{C}(U, V) \to \mathcal{D}(F(U), F(V))$, which we will also

denote by the same name F.

3. (Compositions are Preserved) For all $f : U \xrightarrow{\mathcal{C}} V$ and $g : V \xrightarrow{\mathcal{C}} W$, $F(g \circ f) = F(g) \circ F(f)$.
4. (Identity Morphisms are Preserved) For all $U \in \mathrm{Obj}\,(\mathcal{C})$, $F(\mathbb{1}_U) = \mathbb{1}_{F(U)}$.

Definition – Category of Categories
We define the *category of categories* $\mathbf{Cat}$,

1. $\mathrm{Obj}\,(\mathbf{Cat})$ consists of categories.
2. For $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}\,(\mathbf{Cat})$, $\mathbf{Cat}\,(\mathcal{C}, \mathcal{D})$ consists of functors from $\mathcal{C}$ to $\mathcal{D}$.
3. For $\mathcal{C} \in \mathrm{Obj}\,(\mathbf{Cat})$, $\mathbb{1}_{\mathcal{C}}$ is the obvious thing.

Definition – Faithful, Full, Fully Faithful
Let $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then F is called

1. *faithful* when for all $U, V \in \mathrm{Obj}\,(\mathcal{C})$, $F : \mathcal{C}\,(U, V) \to \mathcal{D}\,(F(U), F(V))$ is injective.
2. *full* when for all $U, V \in \mathrm{Obj}\,(\mathcal{C})$, $F : \mathcal{C}\,(U, V) \to \mathcal{D}\,(F(U), F(V))$ is surjective.
3. *fully faithful* when for all $U, V \in \mathrm{Obj}\,(\mathcal{C})$, $F : \mathcal{C}\,(U, V) \to \mathcal{D}\,(F(U), F(V))$ is bijective.

Proposition – Fully Faithful Functors are Injective
Let $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ be fully faithful, $U, V \in \mathrm{Obj}\,(\mathcal{C})$ such that $F(U) \cong F(V)$. Then $U \cong V$.

Proof. Let $f_1 \in \mathcal{D}(F(U), F(V))$ and $f_2 \in \mathcal{D}(F(V), F(U))$ such that $\mathbb{1}_{F(U)} = f_2 \circ f_1$ and $\mathbb{1}_{F(V)} = f_1 \circ f_2$. Then f_1, f_2 corresponds respectively to $g_1, g_2 \in \mathcal{C}(U, V), \mathcal{C}(V, U)$ through F. We thus have

$$F(g_2 \circ g_1) = F(g_2) \circ F(g_1) = f_2 \circ f_1 = \mathbb{1}_{F(U)} = F(\mathbb{1}_U)$$

which by F fully faithful gives $g_2 \circ g_1 = \mathbb{1}_U$. Similarly, $g_1 \circ g_2 = \mathbb{1}_V$. $\square$

Definition – Natural Transformations
Let $F, G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then a *natural transformation* η *from* F *to* G is defined by the following data :

1. For all $U \in \mathrm{Obj}\,(\mathcal{C})$, $\eta_U : F(U) \xrightarrow{\mathcal{D}} G(U)$.
2. (Naturality) For all $U, V \in \mathrm{Obj}\,(\mathcal{C})$ and $f : U \xrightarrow{\mathcal{C}} V$, we have the following commutative diagram.

$$
\begin{array}{ccc}
F(U) & \xrightarrow{\eta_U} & G(U) \\
{\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\
F(V) & \xrightarrow[\eta_V]{} & G(V)
\end{array}
$$

Definition – Category of Functors
Let $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}\,(\mathbf{Cat})$. Then the *category of functors from $\mathcal{C}$ to $\mathcal{D}$*, denoted $\mathcal{D}^{\mathcal{C}}$, is defined by

1. $\mathrm{Obj}\left(\mathcal{D}^{\mathcal{C}}\right) := \mathbf{Cat}\,(\mathcal{C}, \mathcal{D})\,.$
2. For all $F, G \in \mathrm{Obj}\left(\mathcal{D}^{\mathcal{C}}\right)$, $\mathcal{D}^{\mathcal{C}}\,(F, G)\,:=$ the set of natural transformations from F to G.
3. For all $F \in \mathrm{Obj}\left(\mathcal{D}^{\mathcal{C}}\right)$, $\mathbb{1}_F$ is the obvious thing.

Definition – Equivalence of Categories
Let $\mathcal{C}, \mathcal{D}$ be categories, $F \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$. Then F is called an *equivalence of categories* when there exists $G \in \mathbf{Cat}(\mathcal{D}, \mathcal{C})$ such that $G \circ F \cong \mathbb{1}_{\mathcal{C}}$ and $F \circ G \cong \mathbb{1}_{\mathcal{D}}$.

Definition – Essentially Surjective
Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. The *essential image of F* is defined as the set of $X \in \mathcal{D}$ such that there exists $U \in \mathcal{C}$ where $F(U) \cong X$. F is called *essentially surjective* when its essential image is the whole of $\mathcal{D}$.

Proposition – Characterisation of Equivalence of Categories
Let $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then F is an equivalence of categories if and only if F is fully faithful and essentially surjective.

Proof. ($\Rightarrow$) Let $G : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $\varepsilon : \mathbb{1}_{\mathcal{C}} \xrightarrow[\sim]{} G \circ F$, $\eta : F \circ G \xrightarrow[\sim]{} \mathbb{1}_{\mathcal{D}}$. It is clear that F is essentially surjective. For faithful, let $f, g \in \mathcal{C}(U, V)$ such that $F(f) = F(g)$. Then by naturality of ε, we have the following commutative diagram :

$$
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\downarrow{\scriptstyle \varepsilon_U} & & \downarrow{\scriptstyle \varepsilon_V} \\
GF(U) & \xrightarrow{GF(f)} & GF(V)
\end{array}
$$

Then $f = g$ follows from $\varepsilon_U, \varepsilon_V$ being isomorphisms.

For fullness, let $f \in \mathcal{D}(F(U), F(V))$. The guess is that by mapping f back to $\mathcal{C}$, we get the morphism that maps to f. That is, we claim that $F(\varepsilon_V^{-1} \circ G(f) \circ \varepsilon_U) = f$. Since the arguments of the above paragraph also applies to G, we have G is faithful and hence it suffices to show that $GF(\varepsilon_V^{-1} \circ G(f) \circ \varepsilon_U) = G(f)$. We first show that perhaps unsurprisingly, $GF(\varepsilon_V^{-1}) = \varepsilon_{GF(V)}^{-1}$. By functoriality of GF, it suffices to show that $GF(\varepsilon_V) = \varepsilon_{GF(V)}$. This follows from ε_V being an isomorphism and the following commutative diagram due to the naturality of ε :

$$\begin{tikzcd} V \arrow[r, "\varepsilon_V"] \arrow[d, "\varepsilon_V"] & GF(V) \arrow[d, "GF(\varepsilon_V)"] \\ GF(V) \arrow[r, "\varepsilon_{GF(V)}"] & GFGF(V) \end{tikzcd}$$

It now remains to show $GF(G(f) \circ \varepsilon_U) = \varepsilon_{GF(V)} \circ G(f)$. This follows from naturality of ε and ε_U being an isomorphism :

$$\begin{tikzcd} U \arrow[r, "G(f)\circ\varepsilon_U"] \arrow[d, "\varepsilon_U"] & GF(V) \arrow[d, "\varepsilon_{GF(V)}"] \\ GF(U) \arrow[r, "GF(G(f)\circ\varepsilon_U)"'] & GFGF(V) \end{tikzcd}$$

($\Leftarrow$) Using the axiom of choice, for each $X \in \mathrm{Obj}\,(\mathcal{D})$, let $G(X) \in \mathrm{Obj}\,(\mathcal{C})$ and $\eta_X \in \mathcal{D}(FG(X), X)$ such that η_X is an isomorphism. For $f \in \mathcal{D}(X, Y)$, by full faithfulness of F let $G(f) \in \mathcal{C}(G(X), G(Y))$ be the unique morphism such that $FG(f) = \eta_Y^{-1} \circ f\eta_X$. It then follows from uniqueness of the above morphisms and functoriality of F that G is a functor. Note that by construction, the collection of η_X gives a natural isomorphism $\eta : F \circ G \to \mathbb{1}_{\mathcal{D}}$.

It remains to give a natural isomorphism $\varepsilon : \mathbb{1}_{\mathcal{C}} \to G \circ F$. For $U \in \mathrm{Obj}\,(\mathcal{C})$, we are looking for a morphism $U \to GF(U)$. Feeling optimistic, we use full and faithfulness of F to define $\varepsilon_U \in \mathcal{C}(U, GF(U))$ as the unique morphism such that $F(\varepsilon_U) = \eta_{F(U)}^{-1}$. From $\eta_{F(U)}$ being an isomorphism and full faithfulness of F, it follows that ε_U is also an isomorphism. Finally, to check naturality of ε, let $f \in \mathcal{C}(U, V)$. We need $\varepsilon_V \circ f = GF(f) \circ \varepsilon_U$. But since F is faithful, it suffices that F applied to these morphisms are equal. Well, indeed we have it

$$F(\varepsilon_V \circ f) = \eta_{F(V)}^{-1} \circ F(f) = \eta_{F(V)}^{-1} \circ F(f) \circ \eta_U \circ \eta_U^{-1} = \eta_{F(V)}^{-1} \circ F(f) \circ \eta_U \circ F(\varepsilon_U) = F(GF(f) \circ \varepsilon_U)$$

$\square$

> *Example* (*Functors and Natural Transformations*).
> *The following is but a small sample of the vast sea of functors that appear in mathematics. There is no need to "memorize" these. You will spot them when they appear.*
>
> *The first list gives constructions of "structures" between subcategories of sets. The theme is that these are all free functors adjoint to some kind of forgetful functor. Details of this are explained in the section on adjunctions.*
>
> - *Forgetful functor Given any subcategory $\mathcal{C}$ of $\mathcal{D}$, there is an "obvious" functor from $\mathcal{C}$ to $\mathcal{D}$ that maps $\mathrm{Obj}\,(\mathcal{C}) \to \mathrm{Obj}\,(\mathcal{D})$ by doing nothing and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$ by doing nothing. Functors of this form are often called the* forgetful functor.

Here is a graph showing subcategories of set and their "inclusions".

$$\begin{tikzcd}
\mathbf{Set} & \mathbf{Grp} \arrow[l] & \mathbf{Ab} \arrow[l] & \mathbf{Mod}(R) \arrow[l] \\
\mathbf{Top} \arrow[u] & \mathbf{Ring} \arrow[u] \arrow[ur] & \mathbf{CRing} \arrow[l] \arrow[u] & \mathbf{Alg}(R) \arrow[l] \arrow[u]
\end{tikzcd}$$

in the diagram, R is a commutative ring with unity. The maps from $\mathbf{Ring}, \mathbf{CRing}$ into $\mathbf{Ab}$ take (commutative) rings to their underlying abelian groups.

- (*Free Group*) *For each set S, the free group over S is an object $\langle S \rangle \in \mathbf{Grp}$ the comes with a morphism of sets $\uparrow \colon S \to \langle S \rangle$ such that for any group G and $\phi \in \mathbf{Set}(S, G)$, there is a unique morphism of groups $\langle \phi \rangle \in \mathbf{Grp}(\langle S \rangle, G)$ such that $\langle \phi \rangle \circ \uparrow = \phi$. This makes $G \mapsto \langle G \rangle$ into a functor from $\mathbf{Set}$ to $\mathbf{Grp}$.*

- (*Free Module over a Ring*) *Let A be a commutative ring. For each set S, the free A-module over S is an object $A^{\oplus S} \in \mathbf{Mod}(A)$ that comes with a morphism of sets $\uparrow \colon S \to A^{\oplus S}$ such that for any A-module M and $f \in \mathbf{Set}(S, M)$, there is a unique A-linear map $A^{\oplus f} : \oplus_{s \in S} A \to M$ such that $A^{\oplus f} \circ \uparrow = f$. This makes $S \mapsto A^{\oplus S}$ into a functor from $\mathbf{Set}$ to $\mathbf{Mod}(A)$.*

  In particular, for a fixed $M \in \mathbf{Mod}(A)$ and $S \subseteq M$, S is called respectively linearly independent, spanning, a basis if and only if $A^{\oplus f}$ is injective, surjective, an isomorphism.

  Note that the above covers $\mathbf{Ab}$, since $\mathbf{Ab}$ is nothing more than $\mathbf{Mod}(\mathbb{Z})$.

- (*Free Algebra over a Ring*) *Let K be a commutative ring. For each set S, the free K-algebra over S is an object $K[S] \in \mathbf{Alg}(K)$ that comes with a morphism of sets $\uparrow \colon S \to K[S]$ such that for any K-algebra A and $a \in \mathbf{Set}(S, A)$, there exists a unique K-algebra morphism $ev_a : K[S] \to A$ such that $ev_a \circ \uparrow = a$. This makes $S \mapsto K[S]$ into a functor from $\mathbf{Set}$ to $\mathbf{Alg}(K)$.*

  These free algebras are not unfamiliar. For instance, the polynomial ring in $K[T]$ over K is precisely $K[\{*\}]$ where $\{*\}$ is the singleton set. For any K-algebra A, a set morphism $a : \{*\} \to A$ is nothing more than an element in A. So as suggested by the notation, ev_a is precisely evaluation of polynomials $f \mapsto f(a)$ where we have identified the set morphism a with the unique element in its image. Generalizing, for an arbitrary set S, $K[S]$ is precisely the K-algebra of polynomials with variables indexed by S. In particular, for a K-algebra A and $S \subseteq A$, S is called respectively algebraically independent over A, generating when $ev_S : K[S] \to A$ is injective, surjective.

- (*Tensor Product, Extension and Contraction of Scalars*) *Let B be an A-algebra where A is a commutative ring. Every B-module N already has an A-module structure. This gives a forgetful functor from $\mathbf{Mod}(B)$ to $\mathbf{Mod}(A)$.*

  "Conversely", for any A-module M, $B \otimes_A M$ has an obvious B-module structure. Then for any A-linear map $f \in \mathbf{Mod}(A)(M, N)$, $\mathbb{1}_B \otimes_A f \in \mathbf{Mod}(B)(B \otimes_A M, B \otimes_A N)$. This makes $B \otimes_A (-)$ into a functor from $\mathbf{Mod}(A)$ to $\mathbf{Mod}(B)$. In analogy with the prior examples, extension of scalars can be seen as "taking the free B-module over an A-module".

- (*Localization of Modules*) *Let A be a commutative ring and $S \subseteq A$ multiplicative. Define the category $\mathbf{Mod}(A_S)$ as the full subcategory of $\mathbf{Mod}(A)$ with objects consisting of $M \in \mathrm{Obj}\,(\mathbf{Mod}(A))$ such that for all $f \in S$, scalar multiplication by f on M is an isomorphism, i.e. f is an "invertible" scalar for M. There is an obvious forgetful functor from $\mathbf{Mod}(A_S)$ to $\mathbf{Mod}(A)$.*

  "Conversely", for an A-module M, the localization of M with respect to S is is an object M_S of $\mathbf{Mod}(A_S)$ that comes with an A-linear map $\uparrow\colon M \to M_S$ such that for all $N \in \mathbf{Mod}(A_S)$ and $f \in \mathbf{Mod}(A)(M, N)$, there is a unique $f_S \in \mathbf{Mod}(A_S)(M_S, N)$ where $f_S \circ \uparrow = f$. This gives a functor $\mathbf{Mod}(A) \to \mathbf{Mod}(A_S), M \mapsto M_S$ and morphisms are mapped to induced morphisms. In particular, the localization A_S of A itself has an obvious ring structure. This realizes $\mathbf{Mod}(A_S)$ as the category of modules over A_S.

- (*Group Algebra*) *The following is similar to the free algebra construction. Let K be a commutative ring. Then for $A \in \mathbf{Alg}(K)$, $A^\times \in \mathbf{Ab}$. Any $f \in \mathbf{Alg}(K)(A, B)$, let $f^\times$ denote the restriction of f onto $A^\times$. Then $f^\times$ is automatically a morphism of abelian groups (keeping in mind the group operation is multiplication). This gives a "forgetful functor" from $\mathbf{Alg}(K)$ to $\mathbf{Ab}$.*

  "Conversely", for any abelian group G, the group K-algebra over G is a K-algebra $K[G]$ that comes with a morphism of abelian groups $\uparrow\colon G \to K[G]^\times$ such that for any other K-algebra A and $\phi \in \mathbf{Ab}(G, A^\times)$, there exists a unique $K[\phi] \in \mathbf{Alg}(K)(K[G], A)$ such that $K[\phi] \circ \uparrow = \phi$. This property makes $G \mapsto K[G]$ into a functor from $\mathbf{Ab}$ to $\mathbf{Alg}(K)$. In analogy to the prior examples, this may be seen as taking the "free K-algebra on G". In particular, $K[\mathbb{Z}]$ is precisely the localization $K[T, T^{-1}]$.

- (*Symmetric Algebra*) *The following is similar to the group algebra construction. Let A be a commutative ring. Then for any A-module M, the symmetric algebra $Symm\, M$ is an object in $\mathbf{Alg}(A)$ that comes with an A-linear map $\uparrow\colon M \to Symm\, M$ such that for any A-algebra B and $\phi \in \mathbf{Mod}(A)(M, B)$, there exists a unique A-algebra morphism $Symm\, \phi : Symm\, M \to B$ such that $Symm\, \phi \circ \uparrow = \phi$. In analogy with the prior examples, this may be seen as taking the "free A-algebra over M".*

- (*Discrete Topology*) *For any set X, $(X, 2^X)$ where 2^X is the powerset of X is a topological space. Then for any topological space Y and $f \in \mathbf{Set}(X, Y)$, f is automatically continuous with respect to the discrete topology 2^X. This gives rise to a functor $\mathbf{Set} \to \mathbf{Top}$. In analogy with the prior examples, this seen as taking the "free topological space on X".*

The next list is themed "moving structures on objects across morphisms".

- *Image, Preimage of subsets*
- *Image Preimage of subgroups*
- *Image Preimage of Subrings*
- *Image Preimage of Submodules*
- *Image, Preimage of filters*

This list contains more exotic "algebraic constructions".

- *Fundamental groups*
- *Singular Complex*

- *Classical Galois Correspondence*
- *Vanishing, Ideal*
- *Spec of a commutative ring*

The final list consists of miscellaneous "algebraic constructions" :

- (*Vector Spaces with a Basis*) *Let K be a field. Define a category $\mathcal{C}$ as follows :*
  - *objects are pairs (V, B) where V is a K-vector space and B is a basis of V.*
  - *For $(V, B_V), (W, B_W)$ objects in $\mathcal{C}$, define $\mathcal{C}((V, B_V), (W, B_W))$ as the set of K-linear maps from V to W such that maps B_V into B_W.*
  - *For every K-vector space with a basis (V, B_V), $\mathbb{1}_{(V,B_V)}$ is defined to be the identity map of V.*
  - *Composition of the underlying K-linear maps of morphisms yields another morphism in this category.*

  Define the functor $F : \mathcal{C} \to \mathcal{C}$ that "takes components" as follows :

  - *For (V, B_V) in $\mathcal{C}$, let $F((V, B_V)) := (K^{\oplus B_V}, E)$ where $K^{\oplus B_V}$ is the free K-vector space on B_V and E is the standard basis.*
  - *For a morphism $f \in \mathcal{C}((V, B_V), (W, B_W))$, since $fB_V \subseteq B_W$, this determines a map from the standard basis of $K^{\oplus B_V}$ to the standard basis of $K^{\oplus B_W}$, thus extending to a unique K-linear map $F(f) : K^{\oplus B_V} \to K^{\oplus B_W}$.*
  - *Identity morphisms are clearly respected.*
  - *Composition of morphisms are clearly respected.*

  There is a natural isomorphism between "taking components" and the identity functor : For each (V, B_V) in $\mathcal{C}$, consider the K-linear map $[-]_{B_V} : V \to K^{\oplus B_V}$ that that takes vectors to their components with respect to B_V. This is well-defined and an isomorphism by B_V being a basis. (*In fact, this can serve as a* definition *of B_V being a basis.*) *Then we have naturality :*

$$\begin{tikzcd}
(V, B_V) \arrow[r, "f"] \arrow[d, "{[-]_{B_V}}"', "\sim"] & (W, B_W) \arrow[d, "\sim"', "{[-]_{B_W}}"] \\
(K^{\oplus B_V}, E) \arrow[r, "F(f)"] & (K^{\oplus B_W}, E)
\end{tikzcd}$$

  In particular for a fixed K-vector space V and two finite bases B, B_1, any total ordering on B, B_1 gives rise to a unique $f \in \mathcal{C}((V, B), (V, B_1))$. Then the (*iso*)*morphism $F(f)$ is what is usually known as* change of basis.

- *Dual Module*

- ($\textit{Power Set as an } \mathbb{F}_2\textit{-Algebra}$) *For any set* X*, we can see the power set* 2^X *as* $\mathbb{F}_2^X$ *the set of set morphisms from* X *to the field with two elements* $\mathbb{F}_2$*. Then* $\mathbb{F}_2^X$ *naturally has a structure of an* $\mathbb{F}_2$*-algebra. Explicitly, for two subsets* $f, g \in \mathbb{F}_2^X$*,* $fg = f \cap g$ *and* $f + g = (f \cup g) \setminus (f \cap g)$*. The additive identity is* $\varnothing$ *and the multiplicative identity is* X*. One can see that preimage functor* $X \mapsto 2^X = \mathbb{F}_2^X$ *upgrades to a contravariant functor from* $\mathbf{Set}$ *to* $\mathbf{Alg}(\mathbb{F}_2)$*.*
- *Tangent space of pointed differentiable manifold*

3 Universal Morphisms

Definition – Comma Category
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $X \in \operatorname{Obj}(\mathcal{D})$. Then the *comma category* $X \downarrow G$ is defined as follows.

1. $\operatorname{Obj}(X \downarrow G)$ consists of pairs (U, u) where $U \in \operatorname{Obj}(\mathcal{C})$ and $u : X \xrightarrow{\mathcal{D}} G(U)$.
2. For $(U, u), (V, v) \in \operatorname{Obj}(X \downarrow G)$, $X \downarrow G\left((U, u), (V, v)\right)$ consists of $f : U \xrightarrow{\mathcal{C}} V$ such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "v"'] & G(U) \arrow[d, "G(f)"] \\ & G(V) \end{tikzcd}$$

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$ and $U \in \operatorname{Obj}(\mathcal{C})$. Then the *comma category* $F \downarrow U$ is defined as follows.

1. $\operatorname{Obj}(F \downarrow U)$ consists of pairs (X, x) where $X \in \operatorname{Obj}(\mathcal{D})$ and $x : F(X) \xrightarrow{\mathcal{C}} U$.
2. For $(X, x), (Y, y) \in \operatorname{Obj}(F \downarrow U)$, $X \downarrow F\left((X, x), (Y, y)\right)$ consists of $g : X \xrightarrow{\mathcal{D}} Y$ such that

$$\begin{tikzcd} F(X) \arrow[d, "F(g)"'] \arrow[dr, "x"] & \\ F(Y) \arrow[r, "y"'] & U \end{tikzcd}$$

Remark. Here is a special case of the comma category worth noting.

Definition – Over Category
Let $\mathcal{C}$ be a category and $U \in \operatorname{Obj}(\mathcal{C})$. Then the *over category* $\mathcal{C} \downarrow U$ is defined as $\mathbb{1}_{\mathcal{C}} \downarrow U$.

Dually, the *under category* $U \downarrow \mathcal{C}$ is defined as $U \downarrow \mathbb{1}_{\mathcal{C}}$.

Example (*Over and Under Categories*).

1. *Let* $R \in \operatorname{Obj}(\mathbf{Ring})$*. Then* $\mathbf{Alg}(R) = R \downarrow \mathbf{Ring}$*.*
2. *Let* $X \in \operatorname{Obj}(\mathbf{Top})$*. Then we have the category of covering spaces of* X *which is the subcategory of* $\mathbf{Top} \downarrow X$ *where objects are* $(\tilde{X}, p)$ *with* p *a covering map.*

Definition – Universal Morphism
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $X \in \mathrm{Obj}\,(\mathcal{D})$. Then a *universal morphism from X to G* is the following data.

1. An object $(F(X), \eta_X)$ of the comma category $X \downarrow G$.
2. (Universal Property) For all $(V, v) \in \mathrm{Obj}\,(X \downarrow G)$, there exists a unique morphism $(F(X), \eta_X) \xrightarrow{X \downarrow G} (V, v)$.

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$ and $U \in \mathrm{Obj}\,(\mathcal{C})$. Then a *universal morphism from F to U* is the following data.

1. An object $(G(U), \varepsilon_U)$ of the comma category $F \downarrow U$.
2. (Universal Property)For all $(Y, y) \in \mathrm{Obj}\,(F \downarrow U)$, there exists a unique morphism $(Y, y) \xrightarrow{F \downarrow U} (G(U), \varepsilon_U)$.

Proposition – Unique up to Unique Isomorphism
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $X \in \mathrm{Obj}\,(\mathcal{D})$, $(U, u), (V, v) \in X \downarrow G$ both universal morphisms from X to G. Then there exists a unique $f : (U, u) \xrightarrow{X \downarrow G} (V, v)$ and $g : (V, v) \xrightarrow{X \downarrow G} (U, u)$ such that $g \circ f = \mathbb{1}_{(U,u)}$ and $f \circ g = \mathbb{1}_{(V,v)}$. Thus, if a universal morphism exists, we say it is *unique up to unique isomorphism*.

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $U \in \mathrm{Obj}\,(\mathcal{C})$, $(X, x), (Y, y) \in F \downarrow U$ both universal morphisms from F to U. Then there exists a unique $f : (X, x) \xrightarrow{F \downarrow U} (Y, y)$ and $g : (Y, y) \xrightarrow{F \downarrow u} (X, x)$ such that $g \circ f = \mathbb{1}_{(X,x)}$ and $f \circ g = \mathbb{1}_{(Y,y)}$.

Proof. (Shorter proof that does not go through Yoneda).

By the universal property of (U, u), There exists a unique $f : (U, u) \xrightarrow{X \downarrow G} (V, v)$. Similarly, there exists a unique $g : (Y, y) \xrightarrow{F \downarrow u} (X, x)$. But then $g \circ f : (U, u) \xrightarrow{X \downarrow G} (U, u)$. By applying the universal property of (U, u) with itself, we see that $\mathbb{1}_{(U,u)}$ is the unique $(U, u) \xrightarrow{X \downarrow G} (U, u)$. In particular, we have $g \circ f = \mathbb{1}_{(U,u)}$. Similarly, we have $f \circ g = \mathbb{1}_{(V,v)}$. Since f and g are the *only* morphisms between (U, u), (V, v), they are *the* unique isomorphism between (U, u) and (V, v). $\square$

Remark – "Canonically Isomorphic". It is common in category theory and maths at large to *equate* two objects that satisfy the same universal property, since they are not only isomorphic, but also isomorphic in a unique way. Some also call these *canonically isomorphic*.

Proposition – Isomorphic to Universal implies Universal
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $X \in \mathrm{Obj}\,(\mathcal{D})$, $(U, u), (V, v) \in X \downarrow G$ where $(U, u) \cong_{X \downarrow G} (V, v)$ and (U, u) is a universal morphism from X to G. Then (V, v) is a universal morphism from X to G.

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $U \in \mathrm{Obj}\,(\mathcal{C})$, $(X, x), (Y, y) \in F \downarrow U$ where $(X, x) \cong_{F \downarrow U} (Y, y)$ and (X, x) is a universal morphism from F to U. Then (Y, y) is a universal morphism from F to U.

Proof. Let $f : (U,u) \xrightarrow[\sim]{X\downarrow G} (V,v)$. Let $(W,w) \in \operatorname{Obj}(X \downarrow G)$. Then f induces a bijection between $X \downarrow G((U,u),(W,w))$ and $X \downarrow G((V,v),(W,w))$. Since the former is singleton, so is the latter.

The dual has a similar argument. □

4 Yoneda's Lemma

> **Definition – Dual Categories**
> Let $\mathcal{C} \in \operatorname{Obj}(\mathbf{Cat})$. Then the *dual category of* $\mathcal{C}$, denoted $\mathcal{C}^{op}$, is defined by :
>
> 1. $\operatorname{Obj}(\mathcal{C}^{op}) := \operatorname{Obj}(\mathcal{C})$.
> 2. For all $U, V \in \operatorname{Obj}(\mathcal{C}^{op})$, $\mathcal{C}^{op}(U,V) := \mathcal{C}(V,U)$.

> **Definition – Contravariant Functors**
> Let $\mathcal{C}, \mathcal{D} \in \operatorname{Obj}(\mathbf{Cat})$. Then a *contravariant functor from* $\mathcal{C}$ *to* $\mathcal{D}$ is just a functor $\mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Functors $\mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ are henceforth called *covariant functors from* $\mathcal{C}$ *to* $\mathcal{D}$.

> **Definition – Morphism Functor**
> Let $\mathcal{C}$ be a category and $U \in \operatorname{Obj}(\mathcal{C})$. Then $h_U : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$ is defined as :
>
> 1. For all $V \in \operatorname{Obj}(\mathcal{C}^{op})$, $h_U(V) := \mathcal{C}(V,U)$.
> 2. For all $V, W \in \operatorname{Obj}(\mathcal{C}^{op})$ and $f : V \xrightarrow{\mathcal{C}^{op}} W$, $h_U(f) : h_U(V) \to h_U(W), g \mapsto g \circ f$.
>
> Similarly, $h^U : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$ is defined as :
>
> 1. For all $V \in \operatorname{Obj}(\mathcal{C})$, $h^U(V) := \mathcal{C}(U,V)$.
> 2. For all $V, W \in \operatorname{Obj}(\mathcal{C})$ and $f : V \xrightarrow{\mathcal{C}} W$, $h^U(f) : h^U(V) \to h^U(W), g \mapsto f \circ g$.

> **Proposition – Morphism Functor is Functorial**
> Let $\mathcal{C}$ be a category. Then $h_\star : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}^{op}}$. Similarly, $h^\star : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}}$.

Remark – Functor of Points. Because of its relevance in algebraic geometry, h_U is called the *functor of points of* U.

> **Proposition – Yoneda's Lemma**
> Let $\mathcal{C}$ be a category. Then $h_\star : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}^{op}}$ is fully faithful. Since fully faithful functors are injective, $h_\star$ is called the *Yoneda embedding*.
>
> More generally, for any $U \in \operatorname{Obj}(\mathcal{C})$ and $F \in \operatorname{Obj}\left(\mathbf{Set}^{\mathcal{C}^{op}}\right)$, $\mathbf{Set}^{\mathcal{C}^{op}}(h_U, F)$ bijects with $F(U)$ via $s \mapsto s_U(\mathbb{1}_U)$ and this bijection is natural in both U and F.
>
> Dually, $h^\star : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}}$ is fully faithful and more generally, for any $U \in \operatorname{Obj}(\mathcal{C}^{op})$ and $F \in$

Obj $\left(\mathbf{Set}^{\mathcal{C}}\right)$, $\mathbf{Set}^{\mathcal{C}}\left(h^U, F\right)$ naturally bijects with $F(U)$ via $s \mapsto s_U(\mathbb{1}_U)$.

Proof. We first prove the general statement. Let $U \in \text{Obj}\,(\mathcal{C})\,, F \in \text{Obj}\left(\mathbf{Set}^{\mathcal{C}^{op}}\right)$. Given an element $s \in F(U)$, we are tasked with constructing a natural transformation $h_U \to F$. For $V \in \mathcal{C}$ we want to map elements $f \in h_U(V)$ to some element of $F(V)$. Well, f is a morphism from V to U, so $F(f)$ is a morphism from $F(U)$ to $F(V)$, and we are given an element $s \in F(U)$. So define $\alpha_V^s : h_U(V) \to F(V) := f \mapsto F(f)(s)$. For the collection of α_V^s to form a natural transformation, we need naturality. So given $f \in \mathcal{C}(V, W)$, we need the following diagram to commute :

$$\begin{tikzcd}
h_U(W) \arrow[r, "\alpha_W^s"] \arrow[d, "h_U(f)"] & F(W) \arrow[d, "F(f)"] \\
h_U(V) \arrow[r, "\alpha_V^s"] & F(V)
\end{tikzcd}$$

For $g \in h_U(W)$, then we have as desired

$$\alpha_V^s \circ h_U(f)(g) = \alpha_V^s(g \circ f) = F(g \circ f)(x) = F(f) \circ F(g)(x) = F(f) \circ \alpha_W^s(g)$$

So $\alpha^s : h_U \to F$ is a natural transformation.

Note that we can recover s from α^s by $\alpha_U^s(\mathbb{1}_U) = s$. This motivates us to define the inverse map by $\alpha \in \mathbf{Set}^{\mathcal{C}^{op}}(h_U, F) \mapsto \alpha_U(\mathbb{1}_U)$. To show these two maps are indeed inverses, first consider the following diagram where $\alpha : h_U \to F$ is a natural tranformation, $W \in \text{Obj}\,(\mathcal{C})$ and $f \in h_U(W)$:

$$\begin{tikzcd}
h_U(U) \arrow[rrr, "a_U"] \arrow[ddd, "h_U(f)"'] & & & F(U) \arrow[ddd, "F(f)"] \\
& \mathbb{1}_U \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_U(\mathbb{1}_U) \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \alpha_W(f) = F(f)(\alpha_U(\mathbb{1}_U)) & \\
h_U(W) \arrow[rrr, "\alpha_W"'] & & & F(W)
\end{tikzcd}$$

The above diagram commutes by naturality of α. What it shows is that α_W is completely determined by $\alpha_U(\mathbb{1}_U)$, and hence α is completely determined by $\alpha_U(\mathbb{1}_U)$. This proves one side of the inverse situation. The other side is clear. Thus we have a bijection between $\mathbf{Set}^{\mathcal{C}^{op}}(h_U, F) \cong F(U)$.

At this point, we can already get $h_\star$ fully faithful by applying the above bijection to $F = h_\star$ itself and noting the bijection turns $f \in h_V(U)$ into h_f.

For naturality in the first component, let $f : U \xrightarrow{\mathcal{C}} V$. Then we have the following commutative diagram.

$$
\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{op}}(h_U, F) \arrow[rrr] & & & F(U) \\
& \alpha \circ h^f \arrow[r, mapsto] & (\alpha \circ h^f)_U(\mathbb{1}_U) = \alpha_V(f) = F(f) \circ \alpha_V(\mathbb{1}_V) & \\
& \alpha \arrow[u, mapsto] \arrow[r, mapsto] & \alpha_V(\mathbb{1}_V) \arrow[u, mapsto] & \\
\mathbf{Set}^{\mathcal{C}^{op}}(h_V, F) \arrow[rrr] \arrow[uuu, "(\star \circ h^f)"] & & & F(V) \arrow[uuu, "F(f)"']
\end{tikzcd}
$$

For naturality in the second component, let $\phi : F \xrightarrow{\mathbf{Set}^{\mathcal{C}^{op}}} G$. Then we have the following commutative diagram.

$$
\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{op}}(h_U, F) \arrow[rrr] \arrow[ddd, "(\phi \circ \star)"'] & & & F(U) \arrow[ddd, "\phi_U"] \\
& \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_U(\mathbb{1}_U) \arrow[d, mapsto] & \\
& \phi \circ \alpha \arrow[r, mapsto] & (\phi \circ \alpha)_U(\mathbb{1}_U) = \phi_U \circ \alpha_U(\mathbb{1}_U) & \\
\mathbf{Set}^{\mathcal{C}^{op}}(h_U, G) \arrow[rrr] & & & G(U)
\end{tikzcd}
$$

We thus have the desired result. □

Definition – Representable Functors

Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$ be a covariant functor. Then a *representation of* G is a $(U, u) \in h^\star \downarrow G$ where $u : h^\star \xrightarrow[\sim]{\mathbf{Set}^{\mathcal{C}}} G$.

Dually, let $F : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$ be a contravariant functor. Then a *representation of* F is a $(U, u) \in h_\star \downarrow F$ where $u : h_\star \xrightarrow[\sim]{\mathbf{Set}^{\mathcal{C}^{op}}} F$.

A functor (covariant or contravariant) that has a representation is called *representable*.

Remark. If a functor has a representation, Yoneda's lemma implies it is canonical. This is the next result.

Before this, we first relate universal morphisms to representable functors. This is important as it leads to the notion of *adjunction*.

Proposition – Universal iff Represents

Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $X \in \mathrm{Obj}\,(\mathcal{D})$, $(L(X), \eta_X) \in \mathrm{Obj}\,(X \downarrow R)$. Then the following are equivalent :

1. $(L(X), \eta_X)$ is a universal morphism from X to R.
2. $L(X)$ represents the covariant functor $\mathcal{D}(X, R(\star))$ and $\mathbb{1}_{L(X)}$ corresponds to η_X.

Dually, let $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $U \in \mathrm{Obj}(\mathcal{C})$, $(R(U), \varepsilon_U) \in \mathrm{Obj}(L \downarrow U)$. Then the following are equivalent :

1. $(R(U), \varepsilon_U)$ is a universal morphism from L to U.
2. $R(U)$ represents the contravariant functor $\mathcal{C}(L(\star), U)$ and $\mathbb{1}_{R(U)}$ corresponds to ε_U.

Proof. (Universal implies Represents) Let $(L(X), \eta_X)$ be a universal morphism from X to R. Define the following natural transformation,

$$\begin{aligned} &h^{L(X)} \xrightarrow{\mathbf{Set}^{\mathcal{C}}} \mathcal{D}(X, R(\star)) := \\ &W \in \mathrm{Obj}(\mathcal{C}) \mapsto \left[f \in h^{L(X)}(W) \mapsto R(f) \circ \eta_X \in \mathcal{D}(X, R(W)) \right] \end{aligned}$$

Then for every $W \in \mathrm{Obj}(\mathcal{C})$, this is an isomorphism between $h^{L(X)}(W)$ and $\mathcal{D}(X, R(W))$, and hence a natural isomorphism. Indeed, $\mathbb{1}_{L(X)}$ corresponds to η_X under this natural isomorphism.

(Represents implies Universal) Let $\alpha : h^{L(X)} \xrightarrow{\mathbf{Set}^{\mathcal{C}}} \mathcal{D}(X, R(\star))$ be a natural isomorphism where at $L(X)$, $\alpha_{L(X)}(\mathbb{1}_{L(X)}) = \eta_X$. Let $(V, v) \in \mathrm{Obj}(X \downarrow R)$. For any $f : L(X) \xrightarrow{\mathcal{C}} V$, consider the following commutative diagram.

$$\begin{tikzcd}
h^{L(X)}(L(X)) \arrow[rrr, "\alpha_{L(X)}"] \arrow[ddd, "h^{L(X)}(f)"'] & & & \mathcal{D}(X, RL(X)) \arrow[ddd, "\mathcal{D}(X,R(f))"] \\
& \mathbb{1}_{L(X)} \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_{L(X)}(\mathbb{1}_{L(X)}) = \eta_X \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \alpha_V(f) = R(f) \circ \eta_X & \\
h^{L(X)}(V) \arrow[rrr, "\alpha_V"'] & & & \mathcal{D}(X, R(V))
\end{tikzcd}$$

Thus $f : (L(X), \eta_X) \xrightarrow{X \downarrow R} (V, v)$ if and only if $\alpha_V(f) = v$. Then $\alpha_V^{-1}(v)$ is the unique morphism $(L(X), \eta_X) \xrightarrow{X \downarrow R} (V, v)$. Since there exists a unique $(L(X), \eta_X) \xrightarrow{X \downarrow R} (V, v)$, $(L(X), \eta_X)$ is universal.

The dual equivalence has an analogous proof. □

Proposition – Canonical Representation
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $(U, u) \in h^{\star} \downarrow G$. Then the following are equivalent :

1. (U, u) is a representation of G.

2. (U, u) is a universal morphism from $h^\star$ to G.

In particular, representations of G are canonically isomorphic.

Dually, let $F : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $(V, v) \in h_\star \downarrow F$. Then the following are equivalent :

1. (V, v) is a representation of F.
2. (V, v) is a universal morphism from $h_\star$ to F.

In particular, representations of F are canonically isomorphic.

Proof. (Representation implies Universal) Let $(W, w) \in \mathrm{Obj}\,(h^\star \downarrow G)$. Then $u^{-1} \circ w : h^W \xrightarrow{\mathbf{Set}^{\mathcal{C}}} h^U$. By Yoneda's lemma, there exists a unique $u(W, w) : U \xrightarrow{\mathcal{C}} W$ such that $u^{-1} \circ w = h^{u(W,w)}$. Hence $u(W, w)$ is the unique morphism $(W, w) \xrightarrow{h^\star \downarrow G} U, u$.

(Universal implies Representation) By universal iff represents and Yoneda's lemma, we have the following diagram.

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}}\,(h^\star, G) \arrow[rr, "V\in\mathrm{Obj}(\mathcal{C})\mapsto\left[s\in\mathbf{Set}^{\mathcal{C}}\left(h^V,G\right)\mapsto s_V(\mathbb{1}_V)\right]", "\sim"'] & & G \\
h^U \arrow[u, "V\in\mathrm{Obj}(\mathcal{C})\mapsto\left[f\in h^U(V)\mapsto u\circ h^f\right]", "\sim"'] \arrow[urr, "u"] & &
\end{tikzcd}$$

The claim is that the above commutes, and hence u is an isomorphism. Let $V \in \mathrm{Obj}\,(\mathcal{C})$ and $f \in h^U(V)$. Then

$$(h^f \circ u)_V(\mathbb{1}_V) = u_V \circ \left(h^f\right)_V(\mathbb{1}_V) = u_V(f)$$

So the above diagram commutes.

For the dual, the argument is similar. $\square$

5 Adjoint Functors

Definition – Adjoint Functors

Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then R is a *right adjoint* when there exists $L : \mathrm{Obj}\,(\mathcal{D}) \to \mathrm{Obj}\,(\mathcal{C})$ and $\eta \in \Pi X \in \mathrm{Obj}\,(\mathcal{D}), \mathcal{D}\,(X, RL(X))$ such that for all $X \in \mathrm{Obj}\,(\mathcal{D})$, $(L(X), \eta(X))$ is a universal morphism from X to R. In this case, L is called the *left adjoint of* R.

Dually, let $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then L is a *left adjoint* when there exists $R : \mathrm{Obj}\,(\mathcal{C}) \to \mathrm{Obj}\,(\mathcal{C})$

and $\varepsilon \in \Pi U \in \mathrm{Obj}(\mathcal{C}), \mathcal{C}(LR(U), U)$ such that for all $U \in \mathrm{Obj}(\mathcal{C})$, $(R(U), \varepsilon(U))$ is a universal morphism from L to U. In this case, R is called the *right adjoint of* L.

Definition – Product Category
Let $\mathcal{C}, \mathcal{D}$ be categories. Then the *product category of* $\mathcal{C}, \mathcal{D}$ is denoted $\mathcal{C} \times \mathcal{D}$ and is defined as follows.

1. $\mathrm{Obj}(\mathcal{C} \times \mathcal{D}) := \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$.
2. For $(U, X), (V, Y) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D}), \mathcal{C} \times \mathcal{D}((U, X), (V, Y)) := \mathcal{C}(U, V) \times \mathcal{D}(X, Y)$.

Proposition – Natural Transformations on Product Category
Let $F, G : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, $\alpha \in \Pi(U, X) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D}), \mathcal{E}(F(U), G(X))$. Then the following are equivalent.

1. $\alpha : F \to G$.
2. For all $(U, X) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $\alpha(U, -) : F(U, -) \to G(U, -)$ and $\alpha(-, X) : F(-, X) \to G(-, X)$.

Definition – Adjunction
Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. We have the two functors $\mathcal{C}(L(\star), -), \mathcal{D}(\star, R(-)) : \mathcal{D}^{op} \times \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$. Then (L, R) is an *adjunction* when $\mathcal{C}(L(\star), -), \mathcal{D}(\star, R(-))$ are naturally isomorphic.

In this case, R is called the *right adjoint of* L and L is called the *left adjoint of* R. The isomorphism is called the *adjunction isomorphism*. For all $f : L(X) \xrightarrow{\mathcal{C}} U$, the image of f under the adjunction isomorphism is called the *adjunct of* f, denoted $f^{\perp}$. Similarly for $g : X \xrightarrow{\mathcal{D}} R(U)$, we have the *adjunct of* g, denoted $g^{\perp}$.

Proposition – Universal Morphism Characterisation of Adjunction
Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then the following are equivalent :

1. R is a right adjoint.
2. There exists $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$ such that (L, R) is an adjunction.

Dually, let $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then the following are equivalent :

1. L is a left adjoint.
2. There exists $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ such that (L, R) is an adjunction.

Proof. $(\Rightarrow)$ Let R be a right adjoint, $L : \mathrm{Obj}(\mathcal{D}) \to \mathrm{Obj}(\mathcal{C})$, $\eta \in \Pi X \in \mathrm{Obj}(\mathcal{D}), \mathcal{D}(X, RL(X))$, for all $X \in \mathrm{Obj}(\mathcal{D})$, $(L(X), \eta(X))$ universal morphism from X to R.

The universal properties at every $X \in \mathrm{Obj}(\mathcal{D})$ implies L is functorial. By universal iff represents, for all $X \in \mathrm{Obj}(\mathcal{D})$, we have $\mathcal{C}(L(X), -) \cong \mathcal{D}(X, R(-))$ as functors $\mathcal{C} \to \mathbf{Set}$. Let $f : X \xrightarrow{\mathcal{D}} Y$ and $U \in \mathrm{Obj}(\mathcal{C})$. Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{C}(L(X), U) \arrow[rr, "R(-)\circ\eta(X)"] & & \mathcal{D}(X, R(U)) \\
g \circ L(f) \arrow[r, mapsto] & R(g \circ L(f)) \circ \eta(X) = R(g) \circ \eta(Y) \circ f & \\
g \arrow[u, mapsto] \arrow[r, mapsto] & R(g) \circ \eta(Y) \arrow[u, mapsto] & \\
\mathcal{C}(L(Y), U) \arrow[uuu, "h^{L(f)}"] \arrow[rr, "R(-)\circ\eta(Y)"'] & & \mathcal{D}(Y, R(U)) \arrow[uuu, "h^f"']
\end{tikzcd}$$

Thus the isomorphism $\mathcal{C}(L(X), -) \cong \mathcal{D}(X, R(-))$ is functorial in X, and hence an isomorphism between $\mathcal{C}(L(\star), -) \cong \mathcal{D}(\star, R(-))$.

($\Leftarrow$) Let $L : \mathrm{Obj}(\mathcal{D}) \xrightarrow{\mathbf{Cat}} \mathcal{C}$ such that (L, R) is an adjunction. Then for each $X \in \mathrm{Obj}(\mathcal{D})$, $\mathcal{C}(L(X), -) \cong \mathcal{D}(X, R(-))$. Let $\eta(X)$ be the adjunct of $\mathbb{1}_{L(X)}$. By universal iff represents, $(L(X), \eta(X))$ is a universal morphism from X to R.

The dual has a similar argument. $\square$

Proposition – Uniqueness of Adjoints
Let $R, R_1 : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $L, L_1 : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then

1. If (L, R) and (L, R_1) are both adjunctions, then $R \cong R_1$ as functors.
2. If (L, R) and (L_1, R) are both adjunctions, then $L \cong L_1$ as functors.

Proof. (1) Let $(L, R), (L, R_1)$ both be adjunctions. Let $f : U \xrightarrow{\mathcal{C}} V$. We have an isomorphism between the functors $\mathcal{D}(-, R(U))$ and $\mathcal{D}(-, R_1(U))$ for all $U \in \mathrm{Obj}(\mathcal{C})$. By Yoneda's lemma, these isomorphisms are equal to h_{α_U} for some unique morphism $\alpha_U : R(U) \xrightarrow{\mathcal{D}} R_1(U)$. So we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{D}(-, R(U)) \arrow[r, "h_{\alpha_U}", "\sim"'] \arrow[d, "h_{R(f)}"'] & \mathcal{D}(-, R_1(U)) \arrow[d, "h_{R_1(f)}"] \\
\mathcal{D}(-, R(V)) \arrow[r, "\sim", "h_{\alpha_V}"'] & \mathcal{D}(-, R_1(V))
\end{tikzcd}$$

Again by Yoneda, we have $R_1(f) \circ \alpha_U = \alpha_V \circ R(f)$. The fact that h_{α_U} is an isomorphism implies α_U is an isomorphism. Thus α is a natural isomorphism between R, R_1.

(2) Analogous. $\square$

Remark. There is another characterisation of adjunctions.

Proposition – Unit/Counit Characterisation of Adjunction
Let $R : \mathcal{C} \xrightarrow{\textbf{Cat}} \mathcal{D}$ and $L : \mathcal{D} \xrightarrow{\textbf{Cat}} \mathcal{C}$. Then the following are equivalent :

1. (Morphism Isomorphism) (R, L) is an adjunction.
2. (Unit-Counit) There exists $\eta : \mathbb{1}_{\mathcal{D}} \to RL$ and $\varepsilon : LR \to \mathbb{1}_{\mathcal{C}}$ such that
   (a) $\mathbb{1}_L = \varepsilon L \circ L\eta$, that is to say for all $X \in \mathrm{Obj}\,(D)$, we have the following commutative diagram.

$$\begin{tikzcd} L(X) \arrow[r, "L(\eta(X))"] \arrow[dr, "\mathbb{1}_{L(X)}"'] & LRL(X) \arrow[d, "\varepsilon(L(X))"] \\ & L(X) \end{tikzcd}$$

   (b) $\mathbb{1}_R = R\varepsilon \circ \eta R$, i.e. for all $U \in \mathrm{Obj}\,(\mathcal{C})$, we have the following commutative diagram.

$$\begin{tikzcd} R(U) \arrow[r, "\eta(R(U))"] \arrow[dr, "\mathbb{1}_{R(U)}"'] & RLR(U) \arrow[d, "R(\varepsilon(U))"] \\ & R(U) \end{tikzcd}$$

   The above two equations are often called *triangle-identities*.

Proof. $(1 \Rightarrow 2)$ For all $X \in \mathrm{Obj}\,(\mathcal{D})$, the adjunction isomorphism gives an isomorphism of functors

$$\mathcal{C}\,(L(X), -) \cong \mathcal{D}\,(X, R(-))$$

Define $\eta(X) := \mathbb{1}_{L(X)}^{\perp}$. Then by universal iff represents, $(L(X), \eta(X))$ is a universal morphism from X to R. We claim that $\eta : \mathbb{1}_{\mathcal{D}} \to RL$.

Let $f : X \xrightarrow{\mathcal{D}} Y$. Then by the universal property of $(L(X), \eta(X))$, we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "\eta(X)"] \arrow[d, "f"'] & RL(X) \arrow[d, "RL(f)"] \\ Y \arrow[r, "\eta(Y)"'] & RL(Y) \end{tikzcd}$$

i.e. η is a natural transformation as desired. We similarly define $\varepsilon(U) := \mathbb{1}_{R(U)}^{\perp}$ for $U \in \mathrm{Obj}\,(\mathcal{C})$ and see that $\varepsilon : LR \to \mathbb{1}_{\mathcal{C}}$.

To prove (a), let $X \in \mathrm{Obj}\,(\mathcal{D})$. Then

$$\mathbb{1}_{L(X)} = \left(\mathbb{1}_{L(X)}^{\perp}\right)^{\perp} = (\eta(X))^{\perp} = \varepsilon(L(X)) \circ L(\eta(X))$$

where the last equality follows from the universal property of $(RL(X), \varepsilon(L(X)))$. Similarly for (b), we have for $U \in \mathrm{Obj}\,(\mathcal{C})$,

$$\mathbb{1}_{R(U)} = \left(\mathbb{1}_{R(U)}^{\perp}\right)^{\perp} = (\varepsilon(U))^{\perp} = R(\varepsilon(U)) \circ \eta(R(U))$$

where the last equality is by the universal property of $(LR(U), \eta(R(U)))$.

$(2 \Rightarrow 1)$ Let $(X, U) \in \mathrm{Obj}\,(\mathcal{D}^{op} \times \mathcal{C})$. Since $(L(X), \eta(X))$ is supposed to be a universal morphism from X to R, we define the adjunction map to be

$$\begin{aligned} \mathcal{C}\,(L(X), U) &\xleftrightarrow{\perp} \mathcal{D}\,(X, R(U)) \\ f &\longmapsto R(f) \circ \eta(X) \\ \varepsilon(U) \circ L(g) &\longleftarrow\!\shortmid\, g \end{aligned}$$

Then for $f : L(X) \xrightarrow{\mathcal{C}} U$,

$$\begin{aligned} \left(f^{\perp}\right)^{\perp} &= \varepsilon(U) \circ L(f^{\perp}) = \varepsilon(U) \circ L\left(R(f) \circ \eta(X)\right) \\ &= \varepsilon(U) \circ LR(f) \circ L(\eta(X)) = f \circ \varepsilon(L(X)) \circ L(\eta(X)) = f \end{aligned}$$

Similarly, $\left(g^{\perp}\right)^{\perp} = g$. So $\perp$ is an isomorphism at all (X, U).

It remains to show naturality. It suffices to show that the isomorphism is natural in both components. Let $f : X \xrightarrow{\mathcal{D}^{op}} Y$. Then we have the following diagram.

$$\begin{tikzcd} \mathcal{C}\,(L(X), U) \arrow[r, "\perp"] \arrow[d, "h^{L(f)}"'] & \mathcal{D}\,(X, R(U)) \arrow[d, "h^{f}"] \\ \mathcal{C}\,(L(Y), U) \arrow[r, "\perp"'] & \mathcal{D}\,(Y, R(U)) \end{tikzcd}$$

It follows from $\eta : \mathbb{1}_{\mathcal{D}} \to RL$ that the above commutes. Similarly, naturality of ε implies naturality in the second component. Hence $\perp$ is a natural isomorphism as desired. □

Remark. The following is a special case of adjunction that is worth noting.

> **Definition – Galois Connection**
> Let I, J be partially ordered sets. Then I, J can be seen as categories. A *monotone Galois connection between* I, J is an adjunction between I, J. A *antitone Galois connection between* I, J is an adjunction

between I^{op}, J.

Remark. The unit/counit characterisation of adjunctions shows that if (R, L) is a Galois connection (mono or anti) between partially ordered sets I, J, then R and L are bijective on their images.

Definition – Free Functors

6 Limits and Colimits

Definition – (Co)Diagrams

Let $\mathcal{I}, \mathcal{C}$ be categories. Then an *$\mathcal{I}$-diagram in $\mathcal{C}$* is a covariant functor from $\mathcal{I}$ to $\mathcal{C}$. Dually, an *$\mathcal{I}$-codiagram* is a contravariant functor from $\mathcal{I}$ to $\mathcal{C}$, i.e. an $\mathcal{I}^{op}$-diagram.

Remark. Often, it is easier to take $\mathcal{I}$ to be a subcategory of $\mathcal{C}$.

Definition – Constant (Co)Diagrams

Let $\mathcal{I}, \mathcal{C}$ be categories and $U \in \mathrm{Obj}(\mathcal{C})$. Then define the *constant diagram* $\Delta(U)$ as follows.

1. For all $i \in \mathcal{I}$, $\Delta(U)(i) := U$.
2. For all $\phi : i \xrightarrow{\mathcal{I}} j$, $\Delta(U)(\phi) := \mathbb{1}_U$.

Dually, we have the *constant codiagram* $\Delta^{op}(U)$ defined as :

1. For all $i \in \mathrm{Obj}(\mathcal{I})$, $\Delta^{op}(U)(i) := U$.
2. For all $\phi : i \xrightarrow{\mathcal{I}^{op}} j$, $\Delta(U)(\phi) := \mathbb{1}_U$.

Proposition – Functoriality of Constant (Co)Diagrams

Let $\mathcal{I}, \mathcal{C}$ be categories. Then $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$. Dually, $\Delta^{op} : \mathcal{C} \to \mathcal{C}^{\mathcal{I}^{op}}$.

Definition – (Co)Limits of (Co)Diagrams

Let $\mathcal{I}, \mathcal{C}$ be categories, X a $\mathcal{I}$-diagram in $\mathcal{C}$, and Y a $\mathcal{I}$-codiagram in $\mathcal{C}$.

Then a *limit of X* is a universal morphism from Δ to X. If a limit of X exists, it is canonical and referred to as *the* limit, denoted $(\varprojlim X, \pi_X)$.

Dually, a *colimit of Y* is a universal morphism from Y to Δ^{op}. If a colimit of Y exists, it is canonical and referred to as *the* colimit, denoted with $(\varinjlim Y, \iota_Y)$.

Remark. Sometimes limits are also called *projective limits*, and colimits are called *injective limits*.

Definition – (Co)Completeness

Let $\mathcal{C}$ be a category. Then it is called *complete* when for all "small" categories $\mathcal{I}$ and diagrams $X : \mathcal{I} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, there exists the limit of X.

Dually, it is called *cocomplete* when for all "small" categories $\mathcal{I}$ and codiagrams $Y : \mathcal{I}^{op} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, there exists the colimit of Y.

Remark. We now cover important examples of limits and colimits.

Definition – Discrete Category

For $I \in \mathrm{Obj}(\mathbf{Set})$, I can be turned into a category by having elements as objects and the only morphisms being identity morphisms. Categories obtained in this way are called *discrete categories*.

Remark. Note that for a discrete category $\mathcal{I}$, $\mathcal{I}$ and $\mathcal{I}^{op}$ are isomorphic in an obvious way. Consequently, it is best to think of $\mathcal{I}$-diagrams and $\mathcal{I}$-codiagrams as "the same".

Definition – (Co)Products

Let $\mathcal{C}$ be a category and $\mathcal{I}$ a discrete category.

Let X be an $\mathcal{I}$-diagram in $\mathcal{C}$. Then the limit of X is called the *product of* $X(i)$.

Dually, let Y be an $\mathcal{I}$-codiagram in $\mathcal{C}$. Then the colimit of Y is called the *coproduct of* $Y(i)$.

In the special case of $I = \varnothing$, the product is called the *final object of* $\mathcal{C}$. Dually, the coproduct is called the *initial object of* $\mathcal{C}$.

Example (*Final Objects*).

Example (*Initial Objects*).

Example (*Products*).

Example (*Coproducts*).

Definition – (Co)Equalizers

Let $\mathcal{C}$ be a category. Let I be an arbitrary set and $\mathcal{I}$ be the following category.

$$\begin{tikzcd} 0 \arrow[loop above, "\mathbb{1}_0"] \arrow[r, "i"] & 1 \arrow[loop above, "\mathbb{1}_1"] \end{tikzcd}$$

where there is a morphism $i : 0 \xrightarrow{\mathcal{I}} 1$ for all $i \in I$.

Let X be an $\mathcal{I}$-diagram in $\mathcal{C}$. Then the limit of X is called the *equalizer of* $X(i)$*'s*. Dually, let Y be an $\mathcal{I}$-codiagram in $\mathcal{C}$. Then the colimit of Y is called the *coequalizer of* $Y(i)$*'s*.

Example (*Equalizers*).

Example (*Coequalizers*).

Definition – Pullbacks and Pushouts

Let $\mathcal{C}$ be a category, $U \in \mathrm{Obj}(\mathcal{C})$. Then a *pullback over* U is a product in the category $\mathcal{C} \downarrow U$. Dually, a *pushout under* U is a coproduct in the category $U \downarrow \mathcal{C}$.

Let I be an arbitrary set and $\mathcal{I}$ the following category.

$$\begin{tikzcd} i \arrow[loop above, "\mathbb{1}_0"] \arrow[r, "\phi(i)"] & * \arrow[loop above, "\mathbb{1}_1"] \end{tikzcd}$$

1. $\mathrm{Obj}(\mathcal{I}) = I \sqcup \{*\}$.
2. For all $x \in \mathrm{Obj}(\mathcal{I})$, $\mathcal{I}(x, x) = \{\mathbb{1}_x\}$.
3. For all $i \in I$, $\mathcal{I}(i, *) = \{\phi(i)\}$.

Then a pullback over U is equivalently the limit of an $\mathcal{I}$-diagram X with $X(*) = U$. Dually, a pushout under U is equivalently the colimit of an $\mathcal{I}$-codiagram Y with $Y(*) = U$.

7 Completeness

Proposition – Characterisation of Completeness, Cocompleteness

Proposition – Set Complete

Proposition – Top Complete

Proposition – Grp Complete

Proposition – Ring Complete

Proposition – Mod Complete

Proposition – Set-theoretic Characterisation of Limits and Colimits
Let X be an $\mathcal{I}$-shaped diagram in a category $\mathcal{C}$ and $(U, u) \in \operatorname{Obj}(\Delta \downarrow X)$. We have an $\mathcal{I}$-shaped diagram in $\mathbf{Set}^{\mathcal{C}^{op}}$ that is $h_X := h_\star \circ X$. We also have $h_{\Delta(U)} = \Delta(h_U)$ and the natural transformation $h_u : \Delta(h_U) \to h_X$. So $(h_U, h_u) \in \operatorname{Obj}(\Delta \downarrow h_X)$. Then the following are equivalent.

1. (U, u) is a limit of X.
2. (h_U, h_u) is a limit of h_X.

Corollary – Right Adjoints commute with Limits, Left Adjoints commute with Colimits

Definition – Filtered Sets and Filtered Colimits

Proposition – Filtered Colimits commute with Finite Limits

8 Abelian Categories

Definition – Zero Objects

Definition – Kernels and Cokernels