# Notes on Category Theory

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Current view of category theory : a single language / point-of-view in which you can do most of mathematics. (No one can save the analysts.) It is a nice tool for organization of results. When approaching new content, I now constantly ask myself these three questions :

- Is this a universal property?
- Is this an adjunction?
- Is this just Yoneda?

In this document, we do not pay attention to foundational set-theoretic issues like the "set" of all sets being be a proper class.

## 1 Categories

**Definition – Categories**

A *category* $\mathcal{C}$ is defined by the following data :

1. A set of *objects*, $\mathrm{Obj}(\mathcal{C})$.
2. For every $U, V \in \mathrm{Obj}(\mathcal{C})$, a set of $\mathcal{C}$-*morphisms* from $U$ to $V$, denoted $\mathcal{C}(U, V)$. We denote $f : U \xrightarrow{\mathcal{C}} V$ for $f \in \mathcal{C}(U, V)$.
3. For every $U, V, W \in \mathrm{Obj}(\mathcal{C})$, $f : U \xrightarrow{\mathcal{C}} V$ and $g : V \xrightarrow{\mathcal{C}} W$, a $\mathcal{C}$-morphism called the *composition of $f$ with $g$*, denoted $g \circ f : U \xrightarrow{\mathcal{C}} W$.
4. Associativity of $\circ$.
5. For every $U \in \mathrm{Obj}(\mathcal{C})$, an *identity morphism* $\mathbb{1}_U : U \xrightarrow{\mathcal{C}} U$.
6. For all $U, V, W \in \mathrm{Obj}(\mathcal{C})$, $f : U \xrightarrow{\mathcal{C}} V$ and $g : W \xrightarrow{\mathcal{C}} U$, we have $f \circ \mathbb{1}_U = f$ and $\mathbb{1}_U \circ g = g$.

*Remark.* Morphisms in a category do *not* have to be functions. See the example of preordered sets as categories at end of this section.

**Definition – Isomorphisms**

Let $\mathcal{C}$ be a category, $U, V \in \mathrm{Obj}(\mathcal{C})$, $f : U \xrightarrow{\mathcal{C}} V$. Then $f$ is called an *isomorphism* when there exists $g : V \xrightarrow{\mathcal{C}} U$ such that $g \circ f = \mathbb{1}_U$ and $f \circ g = \mathbb{1}_V$. In this case, we denote $f : U \xrightarrow[\sim]{\mathcal{C}} V$. When there exists an isomorphism from $U$ to $V$, we say they are *isomorphic* and write $U \cong V$.

**Definition – Subcategories**

Let $\mathcal{C}, \mathcal{D}$ be categories. Then $\mathcal{D}$ is called a *subcategory of $\mathcal{C}$* when $\mathrm{Obj}(\mathcal{D}) \subseteq \mathrm{Obj}(\mathcal{C})$ and for all $U, V \in \mathrm{Obj}(\mathcal{D})$, $\mathcal{D}(U, V) \subseteq \mathcal{C}(U, V)$.

*Example (Standard Categories).*

1. $\mathbf{Set}$ *denotes the* category of sets, *where* $\mathrm{Obj}(\mathbf{Set})$ *contains sets and for* $U, V \in \mathrm{Obj}(\mathbf{Set})$, $\mathbf{Set}(U, V)$ *is the set of maps from $U$ to $V$.*
2. $\mathbf{Top}$ *denotes the* category of topological spaces, *where* $\mathrm{Obj}(\mathbf{Top})$ *contains topological spaces and for* $U, V \in \mathrm{Obj}(\mathbf{Top})$, $\mathbf{Top}(U, V)$ *is the set of continuous maps from $U$ to $V$.* $\mathbf{Top}$ *is a subcategory of* $\mathbf{Set}$.
3. *The* category of groups $\mathbf{Grp}$ *has* $\mathrm{Obj}(\mathbf{Grp})$ *containing groups and* $\mathbf{Grp}(U, V)$ *containing group homomorphisms from $U$ to $V$.* $\mathbf{Grp}$ *is a subcategory of* $\mathbf{Set}$.
4. *The* category of abelian groups $\mathbf{Ab}$ *has* $\mathrm{Obj}(\mathbf{Ab})$ *containing abelian groups and* $\mathbf{Ab}(U, V)$ *containing group homomorphisms from $U$ to $V$.* $\mathbf{Ab}$ *is a subcategory of* $\mathbf{Grp}$.
5. *The* category of rings $\mathbf{Ring}$ *has* $\mathrm{Obj}(\mathbf{Ring})$ *containing rings and* $\mathbf{Ring}(U, V)$ *containing ring homomorphisms from $U$ to $V$.* $\mathbf{Ring}$ *is a subcategory of* $\mathbf{Set}$.
6. *The* category of commutative rings $\mathbf{CRing}$ *has* $\mathrm{Obj}(\mathbf{CRing})$ *containing commutative rings and* $\mathbf{CRing}(U, V)$ *containing ring homomorphisms from $U$ to $V$.* $\mathbf{CRing}$ *is a subcategory of* $\mathbf{Ring}$.
7. *Let $R$ be a ring. Then the* category of left $R$-modules $R\mathbf{Mod}$ *has* $\mathrm{Obj}(R\mathbf{Mod})$ *containing left*

*$R$-modules and $R\mathbf{Mod}(U,V)$ contains $R$-linear maps from $U$ to $V$. This is a subcategory of $\mathbf{Ab}$.*

8. *Let $R$ be a commutative ring. Then the* category of $R$-algebras $R\mathbf{Alg}$ *has* $\mathrm{Obj}\,(R\mathbf{Alg})$ *containing pairs $(S,\sigma)$ where $\sigma : R \xrightarrow{\mathbf{Ring}} S$. $R\mathbf{Alg}((U,u),(V,v))$ contains $f : U \xrightarrow{\mathbf{Ring}} V$ such that $f \circ u = v$.*

*Example (Preordered Sets as Categories).*
*Let $I$ be a set, $\leq$ a relation on $I$. Then $(I,\leq)$ is called a* preordered set *when $\leq$ satisfies all of the following :*

1. *(Reflexivity) For all $i \in I$, $i \leq i$.*
2. *(Transitivity) For all $i,j,k \in I$, $i \leq j$ and $j \leq k$ implies $i \leq k$.*

*If $(I,\leq)$ is a preordered set where $\leq$ is clear, we abbreviate to $I$.*

*Let $I$ be a preordered set. Then we can turn $I$ into a category as follows :*

1. $\mathrm{Obj}\,(I)$ *is $I$.*
2. *For $i,j \in \mathrm{Obj}\,(I)$, $I(i,j)$ is singleton when $i \leq j$ and empty otherwise.*

*Things get meta. We can form the* category of preordered sets $\mathbf{Ord}$ *where* $\mathrm{Obj}\,(\mathbf{Ord})$ *contains preoredered sets and $\mathbf{Ord}(I,J)$ contains $f : I \xrightarrow{\mathbf{Set}} J$ such that for all $i,j \in I$, $i \leq j$ implies $f(i) \leq f(j)$.*

*Example (Category of Partially Ordered Sets).*
*Let $I \in \mathrm{Obj}\,(\mathbf{Ord})$. Then $I$ is called a* partially ordered set *when $\leq$ is* antisymmetric, *i.e. for all $i,j \in I$, $i \leq j$ and $j \leq i$ implies $i = j$. We thus have the* category of partially ordered sets $\mathbf{PoSet}$ *where* $\mathrm{Obj}\,(\mathbf{PoSet})$ *contains partially ordered sets and $\mathbf{PoSet}(I,J) = \mathbf{Ord}(I,J)$. We see that $\mathbf{PoSet}$ is a subcategory of $\mathbf{Ord}$.*

*Example (Partially Ordered Sets).*

1. *Let $X$ be a set. Then its powerset $(2^X, \subseteq) \in \mathrm{Obj}\,(\mathbf{PoSet})$.*
2. *Let $X$ be a topological space. Then the set of its opens $(\tau_X, \subseteq) \in \mathrm{Obj}\,(\mathbf{PoSet})$.*
3. *Let $G$ be a group. Then the set of its subgroups $(\mathbf{SubGrp}(G), \subseteq) \in \mathrm{Obj}\,(\mathbf{PoSet})$.*
4. *Let $R$ be a ring and $M$ be a left $R$-module. Then the set of left $R$-submodules of $M$, $R\mathbf{SubMod}(M)$, is in $\mathrm{Obj}\,(\mathbf{PoSet})$.*
5. *Let $R$ be a commutative ring and $(S,\sigma)$ an $R$-algebra. Then the set of all $R$-subalgebras of $S$, $R\mathbf{SubAlg}(S)$, is in $\mathrm{Obj}\,(\mathbf{PoSet})$.*
6. *Let $X$ be a set and $\mathrm{Fil}\,X$ the set of all filters on $X$. Then $(\mathrm{Fil}\,X, \subseteq) \in \mathrm{Obj}\,(\mathbf{PoSet})$.*
7. *Consider the relation on $\mathbb{N}$ that is $a \mid b$. This is a partial order on $\mathbb{N}$.*

*Example* (*A Group as a Category*).
*The data of a group $G$ is equivalent to a category $G$ where there is only one object $\bullet$ and all morphisms are isomorphisms.*

*A direct generalization is a* groupoid *: a category where every morphism is an isomorphism.*

*Example* (*Vector Spaces together with a Basis*).
*Let $K$ be a field. Define a category $\mathcal{C}$ as follows :*

- *objects are pairs $(V, B)$ where $V$ is a $K$-vector space and $B$ is a basis of $V$.*
- *For $(V, B_V), (W, B_W)$ objects in $\mathcal{C}$, define $\mathcal{C}((V, B_V), (W, B_W))$ as the set of $K$-linear maps from $V$ to $W$ such that maps $B_V$ into $B_W$.*
- *For every $K$-vector space with a basis $(V, B_V)$, $\mathbb{1}_{(V,B_V)}$ is defined to be the identity map of $V$.*
- *Composition of the underlying $K$-linear maps of morphisms yields another morphism in this category.*

*This category has nice connections to change of basis.*

# 2 Functors

**Definition – Functors**

Let $\mathcal{C}, \mathcal{D}$ be categories. Then a *functor $F$ from $\mathcal{C}$ to $\mathcal{D}$* is defined by the following data :

1. A map of objects $\mathrm{Obj}\,(\mathcal{C}) \to \mathrm{Obj}\,(\mathcal{D})$, which we will denote by the same name $F$.
2. A map of morphisms for all $U, V \in \mathrm{Obj}\,(\mathcal{C})$, $\mathcal{C}\,(U, V) \to \mathcal{D}\,(F(U), F(V))$, which we will also denote by the same name $F$.
3. (Compositions are Preserved) For all $f : U \xrightarrow{\mathcal{C}} V$ and $g : V \xrightarrow{\mathcal{C}} W$, $F(g \circ f) = F(g) \circ F(f)$.
4. (Identity Morphisms are Preserved) For all $U \in \mathrm{Obj}\,(\mathcal{C})$, $F(\mathbb{1}_U) = \mathbb{1}_{F(U)}$.

**Definition – Category of Categories**

We define the *category of categories* $\mathbf{Cat}$,

1. $\mathrm{Obj}\,(\mathbf{Cat})$ consists of categories.[a]
2. For $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}\,(\mathbf{Cat})$, $\mathbf{Cat}\,(\mathcal{C}, \mathcal{D})$ consists of functors from $\mathcal{C}$ to $\mathcal{D}$.
3. For $\mathcal{C} \in \mathrm{Obj}\,(\mathbf{Cat})$, $\mathbb{1}_{\mathcal{C}}$ is the obvious thing.

[a] This is technically not a set since it contains the category $\mathbf{Set}$, but we do not lose sleep over such issues.

**Definition – Faithful, Full, Fully Faithful**

Let $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then $F$ is called

1. *faithful* when for all $U, V \in \mathrm{Obj}\,(\mathcal{C})$, $F : \mathcal{C}\,(U, V) \to \mathcal{D}\,(F(U), F(V))$ is injective.

2. *full* when for all $U, V \in \mathrm{Obj}\,(\mathcal{C})$, $F : \mathcal{C}\,(U,V) \to \mathcal{D}\,(F(U),F(V))$ is surjective.
3. *fully faithful* when for all $U, V \in \mathrm{Obj}\,(\mathcal{C})$, $F : \mathcal{C}\,(U,V) \to \mathcal{D}\,(F(U),F(V))$ is bijective.

**Proposition – Fully Faithful Functors are Injective**
Let $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ be fully faithful, $U, V \in \mathrm{Obj}\,(\mathcal{C})$ such that $F(U) \cong F(V)$. Then $U \cong V$.

*Proof.* Let $f_1 \in \mathcal{D}(F(U),F(V))$ and $f_2 \in \mathcal{D}(F(V),F(U))$ such that $\mathbb{1}_{F(U)} = f_2 \circ f_1$ and $\mathbb{1}_{F(V)} = f_1 \circ f_2$. Then $f_1, f_2$ corresponds respectively to $g_1, g_2 \in \mathcal{C}(U,V), \mathcal{C}(V,U)$ through $F$. We thus have

$$F(g_2 \circ g_1) = F(g_2) \circ F(g_1) = f_2 \circ f_1 = \mathbb{1}_{F(U)} = F(\mathbb{1}_U)$$

which by $F$ fully faithful gives $g_2 \circ g_1 = \mathbb{1}_U$. Similarly, $g_1 \circ g_2 = \mathbb{1}_V$. □

**Definition – Natural Transformations**
Let $F, G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then a *natural transformation $\eta$ from $F$ to $G$* is defined by the following data :

1. For all $U \in \mathrm{Obj}\,(\mathcal{C})$, $\eta_U : F(U) \xrightarrow{\mathcal{D}} G(U)$.
2. (Naturality) For all $U, V \in \mathrm{Obj}\,(\mathcal{C})$ and $f : U \xrightarrow{\mathcal{C}} V$, we have the following commutative diagram.

$$\begin{tikzcd} F(U) \arrow[r, "\eta_U"] \arrow[d, "F(f)"'] & G(U) \arrow[d, "G(f)"] \\ F(V) \arrow[r, "\eta_V"'] & G(V) \end{tikzcd}$$

**Definition – Category of Functors**
Let $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}\,(\mathbf{Cat})$. Then the *category of functors from $\mathcal{C}$ to $\mathcal{D}$*, denoted $\mathcal{D}^{\mathcal{C}}$, is defined by

1. $\mathrm{Obj}\left(\mathcal{D}^{\mathcal{C}}\right) := \mathbf{Cat}\,(\mathcal{C},\mathcal{D})$.
2. For all $F, G \in \mathrm{Obj}\left(\mathcal{D}^{\mathcal{C}}\right)$, $\mathcal{D}^{\mathcal{C}}\,(F,G)$ := the set of natural transformations from $F$ to $G$.
3. For all $F \in \mathrm{Obj}\left(\mathcal{D}^{\mathcal{C}}\right)$, $\mathbb{1}_F$ is the obvious thing.
4. The obvious way to define composition of natural tranformations is "component-wise".

**Definition – Equivalence of Categories**
Let $\mathcal{C}, \mathcal{D}$ be categories, $F \in \mathbf{Cat}(\mathcal{C},\mathcal{D})$. Then $F$ is called an *equivalence of categories* when there exists $G \in \mathbf{Cat}(\mathcal{D},\mathcal{C})$ such that $G \circ F \cong \mathbb{1}_{\mathcal{C}}$ and $F \circ G \cong \mathbb{1}_{\mathcal{D}}$.

**Definition – Essentially Surjective**
Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. The *essential image of* $F$ is defined as the set of $X \in \mathcal{D}$ such that there exists $U \in \mathcal{C}$ where $F(U) \cong X$. $F$ is called *essentially surjective* when its essential image is the whole of $\mathcal{D}$.

**Proposition – Characterisation of Equivalence of Categories**
Let $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then $F$ is an equivalence of categories if and only if $F$ is fully faithful and essentially surjective.

*Proof.* ($\Rightarrow$) Let $G : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $\varepsilon : \mathbb{1}_{\mathcal{C}} \xrightarrow[\sim]{} G \circ F$, $\eta : F \circ G \xrightarrow[\sim]{} \mathbb{1}_{\mathcal{D}}$. It is clear that $F$ is essentially surjective. For faithful, let $f, g \in \mathcal{C}(U, V)$ such that $F(f) = F(g)$. Then by naturality of $\varepsilon$, we have the following commutative diagram :

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varepsilon_U"] & V \arrow[d, "\varepsilon_V"] \\ GF(U) \arrow[r, "GF(f)"] & GF(V) \end{tikzcd}$$

Then $f = g$ follows from $\varepsilon_U, \varepsilon_V$ being isomorphisms.

For fullness, let $f \in \mathcal{D}(F(U), F(V))$. The guess is that by mapping $f$ back to $\mathcal{C}$, we get the morphism that maps to $f$. That is, we claim that $F(\varepsilon_V^{-1} \circ G(f) \circ \varepsilon_U) = f$. Since the arguments of the above paragraph also applies to $G$, we have $G$ is faithful and hence it suffices to show that $GF(\varepsilon_V^{-1} \circ G(f) \circ \varepsilon_U) = G(f)$. We first show that perhaps unsurprisingly, $GF(\varepsilon_V^{-1}) = \varepsilon_{GF(V)}^{-1}$. By functoriality of $GF$, it suffices to show that $GF(\varepsilon_V) = \varepsilon_{GF(V)}$. This follows from $\varepsilon_V$ being an isomorphism and the following commutative diagram due to the naturality of $\varepsilon$ :

$$\begin{tikzcd} V \arrow[r, "\varepsilon_V"] \arrow[d, "\varepsilon_V"] & GF(V) \arrow[d, "GF(\varepsilon_V)"] \\ GF(V) \arrow[r, "\varepsilon_{GF(V)}"] & GFGF(V) \end{tikzcd}$$

It now remains to show $GF(G(f) \circ \varepsilon_U) = \varepsilon_{GF(V)} \circ G(f)$. This follows from naturality of $\varepsilon$ and $\varepsilon_U$ being an isomorphism :

$$\begin{tikzcd} U \arrow[r, "G(f) \circ \varepsilon_U"] \arrow[d, "\varepsilon_U"] & GF(V) \arrow[d, "\varepsilon_{GF(V)}"] \\ GF(U) \arrow[r, "GF(G(f) \circ \varepsilon_U)"'] & GFGF(V) \end{tikzcd}$$

($\Leftarrow$) Using the axiom of choice, for each $X \in \operatorname{Obj}(\mathcal{D})$, let $G(X) \in \operatorname{Obj}(\mathcal{C})$ and $\eta_X \in \mathcal{D}(FG(X), X)$ such that $\eta_X$ is an isomorphism. For $f \in \mathcal{D}(X, Y)$, by full faithfulness of $F$ let $G(f) \in \mathcal{C}(G(X), G(Y))$ be the unique morphism such that $FG(f) = \eta_Y^{-1} \circ f \eta_X$. It then follows from uniqueness of the above morphisms and functoriality of $F$ that $G$ is a functor. Note that by construction, the collection of $\eta_X$ gives a natural isomorphism $\eta : F \circ G \to \mathbb{1}_{\mathcal{D}}$.

It remains to give a natural isomorphism $\varepsilon : \mathbb{1}_{\mathcal{C}} \to G \circ F$. For $U \in \operatorname{Obj}(\mathcal{C})$, we are looking for a morphism $U \to GF(U)$. Feeling optimistic, we use full and faithfulness of $F$ to define $\varepsilon_U \in \mathcal{C}(U, GF(U))$ as the unique morphism such that $F(\varepsilon_U) = \eta_{F(U)}^{-1}$. From $\eta_{F(U)}$ being an isomorphism and full faithfulness of $F$, it follows that $\varepsilon_U$ is also an isomorphism. Finally, to check naturality of $\varepsilon$, let $f \in \mathcal{C}(U, V)$. We need $\varepsilon_V \circ f = GF(f) \circ \varepsilon_U$. But since $F$ is faithful, it suffices that $F$ applied to these morphisms are equal. Well, indeed we have it

$$F(\varepsilon_V \circ f) = \eta_{F(V)}^{-1} \circ F(f) = \eta_{F(V)}^{-1} \circ F(f) \circ \eta_U \circ \eta_U^{-1} = \eta_{F(V)}^{-1} \circ F(f) \circ \eta_U \circ F(\varepsilon_U) = F(GF(f) \circ \varepsilon_U)$$

$\square$

*Example* (*Functors and Natural Transformations*).
*The following is but a small sample of the vast sea of functors that appear in mathematics.*

*The first list concerns "freely creating structure from a subcategory to an ambient category". It begins with the concept of a* forgetful functor. *Details of this are explained in the section on adjunctions.*

– (*Forgetful functor*) *Given any subcategory $\mathcal{C}$ of $\mathcal{D}$, there is an "obvious" functor from $\mathcal{C}$ to $\mathcal{D}$ that maps $\operatorname{Obj}(\mathcal{C}) \to \operatorname{Obj}(\mathcal{D})$ by doing nothing and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$ by doing nothing. Functors of this form are called* forgetful functors*.*

  *Here is a graph showing subcategories of set and their "inclusions".*

$$\begin{tikzcd}
\mathbf{Set} & \mathbf{Grp} \arrow[l] & \mathbf{Ab} \arrow[l] & \mathbf{Ring} \arrow[l] & \mathbf{CRing} \arrow[l] \\
\mathbf{Top} \arrow[u] & & R\mathbf{Mod} \arrow[u] & \mathbf{Alg}(R) \arrow[l] \arrow[u] & \mathbf{CAlg}(R) \arrow[l] \arrow[u]
\end{tikzcd}$$

  *For the category $R\mathbf{Mod}$ of left modules over $R$, $R$ need only be a ring. For $\mathbf{Alg}(R)$, $R$ needs to be commutative with unity. The maps from $\mathbf{Ring}$, $R\mathbf{Mod}$ into $\mathbf{Ab}$ takes the underlying abelian groups.*

  *In particular, we call a subcategory $\mathcal{C}$ of $\mathcal{D}$, we say $\mathcal{C}$ is a* full subcategory of *$\mathcal{D}$ when the forgetful functor is full. Faithfulness is given since morphism sets of $\mathcal{C}$ are by definition subsets of morphism sets of $\mathcal{D}$.*

– (*Free Group*) *For each set $S$, the* free group over *$S$ is an object $\langle S \rangle \in \mathbf{Grp}$ the comes with a morphism of sets $\uparrow : S \to \langle S \rangle$ such that for any group $G$ and $\phi \in \mathbf{Set}(S, G)$, there is a unique morphism of groups $\langle \phi \rangle \in \mathbf{Grp}(\langle S \rangle, G)$ such that $\langle \phi \rangle \circ \uparrow = \phi$. This makes $G \mapsto \langle G \rangle$ into a functor from* $\mathbf{Set}$ *to* $\mathbf{Grp}$*.*

  *In particular, for $S \subseteq G$ and $f$ the usual inclusion, $S$ is called* generating *when $\langle \iota \rangle$ is surjective, where $\iota \in \mathbf{Set}(S, G)$ is the usual inclusion.*

– (*Free Module over a Ring*) *Let $R$ be a ring. For each set $S$, the* free left $R$-module over $S$ *is an object $R^{\oplus S} \in R\mathbf{Mod}$ that comes with a morphism of sets $\uparrow : S \to R^{\oplus S}$ such that for any $R$-module $M$ and*

*$f \in \mathbf{Set}(S, M)$, there is a unique $R$-linear map $R^{\oplus f} : \oplus_{s \in S} R \to M$ such that $R^{\oplus f} \circ \uparrow = f$. This makes $S \mapsto R^{\oplus S}$ into a functor from $\mathbf{Set}$ to $R\mathbf{Mod}$.*

*In particular, for $S \subseteq M$ and $f$ the usual inclusion, $S$ is called respectively linearly independent, spanning, a basis when $R^{\oplus f}$ is injective, surjective, an isomorphism.*

*Note that the above covers $\mathbf{Ab}$, since $\mathbf{Ab}$ is nothing more than $\mathbb{Z}\mathbf{Mod}$.*

– *(Free Commutative Algebra over a Ring) Let $K$ be a commutative ring. For each set $S$, the free commutative $K$-algebra over $S$ is an object $K[S] \in \mathbf{CAlg}(K)$ that comes with a morphism of sets $\uparrow : S \to K[S]$ such that for any $K$-algebra $A$ and $a \in \mathbf{Set}(S, A)$, there exists a unique $K$-algebra morphism $ev_a : K[S] \to A$ such that $ev_a \circ \uparrow = a$. This makes $S \mapsto K[S]$ into a functor from $\mathbf{Set}$ to $\mathbf{CAlg}(K)$.*

  *These free commutative algebras are not unfamiliar. For instance, the polynomial ring in $K[T]$ over $K$ is precisely $K[\{*\}]$ where $\{*\}$ is the singleton set. For any commutative $K$-algebra $A$, a set morphism $a : \{*\} \to A$ is nothing more than an element in $A$. So as suggested by the notation, $ev_a$ is precisely evaluation of polynomials $f \mapsto f(a)$ where we have identified the set morphism $a$ with the unique element in its image. Generalizing, for an arbitrary set $S$, $K[S]$ is precisely the commutative $K$-algebra of polynomials with variables indexed by $S$. In particular, for a commutative $K$-algebra $A$ and $S \subseteq A$, $S$ is called respectively algebraically independent over $A$, generating when $ev_S : K[S] \to A$ is injective, surjective.*

– *(Tensor Product, Extension and Contraction of Scalars) Let $B$ be a commutative $A$-algebra where $A$ is a commutative ring. Every $B$-module $N$ has an obvious $A$-module structure. This gives a forgetful functor from $B\mathbf{Mod}$ to $A\mathbf{Mod}$.*

  *For any $M \in \mathrm{Obj}\,(A\mathbf{Mod})$, $M \otimes_A B$ the extension of scalars of $M$ has the property that it comes with an obvious morphism of $A$-modules $\uparrow : M \to M \otimes_A B$ such that for any $N \in \mathrm{Obj}\,(B\mathbf{Mod})$ and $f \in A\mathbf{Mod}(M, N)$, there exists a unique $A$-module morphism $f \otimes_A \mathbb{1}_B : M \otimes_A B \to N$ such that $(f \otimes_A \mathbb{1}_B) \circ \uparrow = f$.*

  *In analogy with the prior examples, extension of scalars can be seen as "taking the free $B$-module over an $A$-module".*

– *(Localization of Modules) Let $A$ be a commutative ring and $S \subseteq A$ multiplicative. Define the category $S^{-1}A\mathbf{Mod}$ as the full subcategory of $S^{-1}A\mathbf{Mod}$ with objects consisting of $M \in \mathrm{Obj}\,(A\mathbf{Mod})$ such that for all $f \in S$, scalar multiplication by $f$ on $M$ is an isomorphism, i.e. $f$ is an "invertible" scalar for $M$. We then have a forgetful functor from $S^{-1}A\mathbf{Mod}$ to $A\mathbf{Mod}$.*

  *Now, for an $A$-module $M$, the localization of $M$ with respect to $S$ is is an object $S^{-1}M$ of $S^{-1}A\mathbf{Mod}$ that comes with an $A$-linear map $\uparrow : M \to S^{-1}M$ such that for all $N \in S^{-1}A\mathbf{Mod}$ and $f \in A\mathbf{Mod}(M, N)$, there is a unique $S^{-1}(f) \in S^{-1}A\mathbf{Mod}(M_S, N)$ where $S^{-1}(f) \circ \uparrow = f$. This gives a functor $A\mathbf{Mod} \to S^{-1}A\mathbf{Mod}, M \mapsto S^{-1}M$ and morphisms are mapped to induced morphisms. In particular, the localization $S^{-1}A$ of $A$ itself has an obvious ring structure. One can then realize $S^{-1}A\mathbf{Mod}$ as the category of modules over $S^{-1}A$.*

  *In analogy with previous examples, $S^{-1}M$ can be seen as taking $M$ and "freely inverting the scalars in $S$".*

- ($Symmetric Algebra$) *The following is similar to the free commutative algebra construction. Let* $A$ *be a commutative ring. Then for any* $A$*-module* $M$*, the symmetric algebra* $Symm\, M$ *is an object in* $A\mathbf{CAlg}$ *that comes with an* $A$*-linear map* $\uparrow : M \to Symm\, M$ *such that for any commutative* $A$*-algebra* $B$ *and* $\phi \in A\mathbf{Mod}(M, B)$*, there exists a unique* $A$*-algebra morphism* $Symm\, \phi : Symm\, M \to B$ *such that* $Symm\, \phi \circ \uparrow = \phi$*. In analogy with the prior examples, this may be seen as taking the "free commutative* $A$*-algebra over* $M$*".*

- (*Discrete Topology*) *For any set* $X$*,* $(X, 2^X)$ *where* $2^X$ *is the powerset of* $X$ *is a topological space. Then for any topological space* $Y$ *and* $f \in \mathbf{Set}(X, Y)$*,* $f$ *is automatically continuous with respect to the discrete topology* $2^X$*. This gives rise to a functor* $\mathbf{Set} \to \mathbf{Top}$*. In analogy with the prior examples, this seen as taking the "free topological space on* $X$*".*

- (*Discrete Category*) *Given any set* $S$*, you can give it the structure of a category by declaring* $\mathrm{Obj}\,(S) := S$ *and the only morphisms to be identities. Then for any category* $\mathcal{C}$ *and function* $f : S \to \mathrm{Obj}\,(()\,\mathcal{C})$*, there is a unique functor* $f_{\mathbf{Cat}} : S \to \mathcal{C}$ *that agrees with* $f$ *on the objects. This can be seen as the "free category over* $S$*".*

*The next list is themed "moving structures on objects across morphisms".*

- (*Image, Preimage of "Subobjects"*) *Let* $f : U \to V$ *be a set morphism. Then taking the image of subsets of* $U$ *under* $f$ *gives a functor* $f : 2^U \to 2^V$*. In the other direction, taking preimages of subsets of* $V$ *under* $f$ *gives a functor* $f^{-1} : 2^V \to 2^U$*.*

  *If* $f : U \to V$ *is a morphism of groups, the above functors restrict to*

$$f_{\mathbf{Grp}} : \mathbf{SubGrp}(U) \to \mathbf{SubGrp}(V)$$
$$f^{-1}_{\mathbf{Grp}} : \mathbf{SubGrp}(V) \to \mathbf{SubGrp}(U)$$

  *Similar results for when* $f$ *is a morphism of* $R\mathbf{Mod}$*,* $R\mathbf{Alg}$*.*

- (*Image, Preimage of Filters*) *Let* $f : X \to Y$ *be a morphism of sets. Then the functor* $f : 2^X \to 2^Y$ *gives rise to the functor* $f^{-1} : 2^{2^Y} \to 2^{2^X}$*, which restricts to a functor* $f^{-1}_{\mathrm{Fil}} : \mathrm{Fil}\, Y \to \mathrm{Fil}\, X$*. This is the* inverse image functor for filters.

  *Unfortunately, given a* $F \in \mathrm{Fil}(X)$*, the set of subsets* $\{fU \mid U \in F\}$ *does not in general form a filter of* $Y$*. However, the set of subsets* $f_{\mathrm{Fil}} F := \left\{W \in 2^Y \mid \exists\, V \in F, fV \subseteq W\right\}$ *does form a filter on* $Y$*. This is equivalently the largest filter on* $Y$ *such that its inverse image filter is in* $F$*. It is not hard to check that* $f_{\mathrm{Fil}} : \mathrm{Fil}\, X \to \mathrm{Fil}\, Y$ *is functorial with respect to the category structure on* $\mathrm{Fil}\, X, \mathrm{Fil}\, Y$ *given by their* $\mathbf{PoSet}$ *structure.*

- (*"Functions" into a Fixed Object*) *For any topological space* $X$*,* $\mathbf{Top}(X, \mathbb{R})$ *with pointwise addition and multiplication is an object in* $\mathbb{R}\mathbf{CAlg}$*. Call* $\mathbf{Top}(X, \mathbb{R})$ *the* global functions on $X$*. Then for any topological morphism* $\phi : X \to Y$ *and we naturally get a set morphism* $(- \circ \phi) : \mathbf{Top}(X, \mathbb{R}) \to \mathbf{Top}(Y, \mathbb{R})$ *which turns out to be a morphism of* $\mathbb{R}$*-algebras. This yields a functor, not from* $\mathbf{Top}$ *to* $\mathbb{R}\mathbf{CAlg}$*, but rather "*$\mathbf{Top}^{op}$*" to* $\mathbb{R}\mathbf{CAlg}$ *where "*$\mathbf{Top}^{op}$*" looks like* $\mathbf{Top}$ *but all morphisms are reversed. See later for details.*

  *Many situations analogous to the above can be found. For instance, replacing* $\mathbf{Top}$ *with* $\mathbf{Set}$ *and* $\mathbb{R}$ *with* $\{0, 1\}$ *yields the*

*This list contains more exotic "algebraic constructions".*

- *Fundamental groups*
- *Singular Complex*
- *Classical Galois Correspondence*
- *Vanishing, Ideal*
- *Spec of a commutative ring*

*The final list consists of miscellaneous "algebraic constructions" :*

- (*Group Algebra*) *The following is similar to the free algebra construction. Let $K$ be a commutative ring. Then for $A \in \mathbf{Alg}(K)$, $A^\times \in \mathbf{Ab}$. Any $f \in \mathbf{Alg}(K)(A, B)$, let $f^\times$ denote the restriction of $f$ onto $A^\times$. Then $f^\times$ is automatically a morphism of abelian groups* (*keeping in mind the group operation is multiplication*).

  *"Conversely", for any abelian group $G$, the group $K$-algebra over $G$ is a $K$-algebra $K[G]$ that comes with a morphism of abelian groups $\uparrow\colon G \to K[G]^\times$ such that for any other $K$-algebra $A$ and $\phi \in \mathbf{Ab}(G, A^\times)$, there exists a unique $K[\phi] \in \mathbf{Alg}(K)(K[G], A)$ such that $K[\phi] \circ \uparrow = \phi$. This property makes $G \mapsto K[G]$ into a functor from $\mathbf{Ab}$ to $\mathbf{Alg}(K)$. In particular, $K[\mathbb{Z}]$ is precisely the localization $K[T, T^{-1}]$.*

  *Note that we can actually remove the commutativity hypothesis on groups and algebras, obtaining functors between $\mathbf{Grp}$ and the category of non-commutative algebras of $K$.*

- (*Vector Spaces with a Basis*) *Let $K$ be a field and $\mathcal{C}$ the category of $K$-vector spaces together with a basis. Define the functor $F : \mathcal{C} \to \mathcal{C}$ that "takes components" as follows :*

  - *For $(V, B_V)$ in $\mathcal{C}$, let $F((V, B_V)) := (K^{\oplus B_V}, E)$ where $K^{\oplus B_V}$ is the free $K$-vector space on $B_V$ and $E$ is the standard basis.*
  - *For a morphism $f \in \mathcal{C}((V, B_V), (W, B_W))$, since $fB_V \subseteq B_W$, this determines a map from the standard basis of $K^{\oplus B_V}$ to the standard basis of $K^{\oplus B_W}$, thus extending to a unique $K$-linear map $F(f) : K^{\oplus B_V} \to K^{\oplus B_W}$.*
  - *Identity morphisms are clearly respected.*
  - *Composition of morphisms are clearly respected.*

  *There is a natural isomorphism between "taking components" and the identity functor : For each $(V, B_V)$ in $\mathcal{C}$, consider the $K$-linear map $[-]_{B_V} : V \to K^{\oplus B_V}$ that that takes vectors to their components with respect to $B_V$. This is well-defined and an isomorphism by $B_V$ being a basis.* (*In fact, this can serve as a* definition *of $B_V$ being a basis.*) *Then we have naturality :*

$$\begin{array}{ccc}(V,B_V) & \xrightarrow{\quad f \quad} & (W,B_W) \\ {\scriptstyle [-]_{B_V}}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle [-]_{B_W}} \\ (K^{\oplus B_V},E) & \xrightarrow{F(f)} & (K^{\oplus B_W},E)\end{array}$$

*In particular for a fixed $K$-vector space $V$ and two finite bases $B, B_1$, any total ordering on $B, B_1$ gives rise to a unique $f \in \mathcal{C}((V,B),(V,B_1))$. Then the (iso)morphism $F(f)$ is what is usually known as* change of basis.

- *Dual Module*
- *(Power Set as an $\mathbb{F}_2$-Algebra) For any set $X$, we can see the power set $2^X$ as $\mathbb{F}_2^X$ the set of set morphisms from $X$ to the field with two elements $\mathbb{F}_2$. Then $\mathbb{F}_2^X$ naturally has a structure of an $\mathbb{F}_2$-algebra. Explicitly, for two subsets $f,g \in \mathbb{F}_2^X$, $fg = f \cap g$ and $f+g = (f \cup g) \setminus (f \cap g)$. The additive identity is $\varnothing$ and the multiplicative identity is $X$. One can see that preimage functor $X \mapsto 2^X = \mathbb{F}_2^X$ upgrades to a contravariant functor from $\mathbf{Set}$ to $\mathbf{Alg}(\mathbb{F}_2)$.*
- *Tangent space of pointed differentiable manifold*

# 3 Universal Morphisms

**Definition – Comma Category**

Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $X \in \mathrm{Obj}\,(\mathcal{D})$. Then the *comma category* $X \downarrow G$ is defined as follows.

1. $\mathrm{Obj}\,(X \downarrow G)$ consists of pairs $(U,u)$ where $U \in \mathrm{Obj}\,(\mathcal{C})$ and $u : X \xrightarrow{\mathcal{D}} G(U)$.
2. For $(U,u),(V,v) \in \mathrm{Obj}\,(X \downarrow G)$, $X \downarrow G\,((U,u),(V,v))$ consists of $f : U \xrightarrow{\mathcal{C}} V$ such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "v"'] & G(U) \arrow[d, "G(f)"] \\ & G(V) \end{tikzcd}$$

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$ and $U \in \mathrm{Obj}\,(\mathcal{C})$. Then the *comma category* $F \downarrow U$ is defined as follows.

1. $\mathrm{Obj}\,(F \downarrow U)$ consists of pairs $(X,x)$ where $X \in \mathrm{Obj}\,(\mathcal{D})$ and $x : F(X) \xrightarrow{\mathcal{C}} U$.
2. For $(X,x),(Y,y) \in \mathrm{Obj}\,(F \downarrow U)$, $X \downarrow F\,((X,x),(Y,y))$ consists of $g : X \xrightarrow{\mathcal{D}} Y$ such that

$$\begin{tikzcd} F(X) \arrow[d, "F(g)"'] \arrow[dr, "x"] & \\ F(Y) \arrow[r, "y"'] & U \end{tikzcd}$$

*Remark.* Here is a special case of the comma category worth noting.

**Definition – Over Category**
Let $\mathcal{C}$ be a category and $U \in \mathrm{Obj}(\mathcal{C})$. Then the *over category* $\mathcal{C} \downarrow U$ is defined as $\mathbb{1}_{\mathcal{C}} \downarrow U$.

Dually, the *under category* $U \downarrow \mathcal{C}$ is defined as $U \downarrow \mathbb{1}_{\mathcal{C}}$.

*Example (Over and Under Categories).*

1. *Let* $R \in \mathrm{Obj}(\mathbf{Ring})$. *Then* $\mathbf{Alg}(R) = R \downarrow \mathbf{Ring}$.
2. *Let* $X \in \mathrm{Obj}(\mathbf{Top})$. *Then we have the category of covering spaces of* $X$ *which is the subcategory of* $\mathbf{Top} \downarrow X$ *where objects are* $(\tilde{X}, p)$ *with* $p$ *a covering map.*

**Definition – Universal Morphism**
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $X \in \mathrm{Obj}(\mathcal{D})$. Then a *universal morphism from* $X$ *to* $G$ is the following data.

1. An object $(F(X), \eta_X)$ of the comma category $X \downarrow G$.
2. (Universal Property) For all $(V, v) \in \mathrm{Obj}(X \downarrow G)$, there exists a unique morphism $(F(X), \eta_X) \xrightarrow{X\downarrow G} (V, v)$.

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$ and $U \in \mathrm{Obj}(\mathcal{C})$. Then a *universal morphism from* $F$ *to* $U$ is the following data.

1. An object $(G(U), \varepsilon_U)$ of the comma category $F \downarrow U$.
2. (Universal Property)For all $(Y, y) \in \mathrm{Obj}(F \downarrow U)$, there exists a unique morphism $(Y, y) \xrightarrow{F\downarrow U} (G(U), \varepsilon_U)$.

**Proposition – Unique up to Unique Isomorphism**
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $X \in \mathrm{Obj}(\mathcal{D})$, $(U, u), (V, v) \in X \downarrow G$ both universal morphisms from $X$ to $G$. Then there exist unique $f : (U, u) \xrightarrow{X\downarrow G} (V, v)$ and $g : (V, v) \xrightarrow{X\downarrow G} (U, u)$ such that $g \circ f = \mathbb{1}_{(U,u)}$ and $f \circ g = \mathbb{1}_{(V,v)}$. Thus, if a universal morphism exists, we say it is *unique up to unique isomorphism*.

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $U \in \mathrm{Obj}(\mathcal{C})$, $(X, x), (Y, y) \in F \downarrow U$ both universal morphisms from $F$ to $U$. Then there exists a unique $f : (X, x) \xrightarrow{F\downarrow U} (Y, y)$ and $g : (Y, y) \xrightarrow{F\downarrow u} (X, x)$ such that $g \circ f = \mathbb{1}_{(X,x)}$ and $f \circ g = \mathbb{1}_{(Y,y)}$.

*Proof.* (Shorter proof that does not go through Yoneda).

By the universal property of $(U, u)$, There exists a unique $f : (U, u) \xrightarrow{X\downarrow G} (V, v)$. Similarly, there exists a unique $g : (Y, y) \xrightarrow{F\downarrow u} (X, x)$. But then $g \circ f : (U, u) \xrightarrow{X\downarrow G} (U, u)$. By applying the universal property of

$(U,u)$ with itself, we see that $\mathbb{1}_{(U,u)}$ is the unique $(U,u) \xrightarrow{X\downarrow G} (U,u)$. In particular, we have $g \circ f = \mathbb{1}_{(U,u)}$. Similarly, we have $f \circ g = \mathbb{1}_{(V,v)}$. Since $f$ and $g$ are the *only* morphisms between $(U,u)$, $(V,v)$, they are *the* unique isomorphism between $(U,u)$ and $(V,v)$. ☐

*Remark – "Canonically Isomorphic"*. It is common in category theory and maths at large to *equate* two objects that satisfy the same universal property, since they are not only isomorphic, but also isomorphic in a unique way. Some also call these *canonically isomorphic*.

> **Proposition – Isomorphic to Universal implies Universal**
> Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $X \in \mathrm{Obj}(\mathcal{D})$, $(U,u),(V,v) \in X \downarrow G$ where $(U,u) \cong_{X\downarrow G} (V,v)$ and $(U,u)$ is a universal morphism from $X$ to $G$. Then $(V,v)$ is a universal morphism from $X$ to $G$.
>
> Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $U \in \mathrm{Obj}(\mathcal{C})$, $(X,x),(Y,y) \in F \downarrow U$ where $(X,x) \cong_{F\downarrow U} (Y,y)$ and $(X,x)$ is a universal morphism from $F$ to $U$. Then $(Y,y)$ is a universal morphism from $F$ to $U$.

*Proof.* Let $f : (U,u) \xrightarrow[\sim]{X\downarrow G} (V,v)$. Let $(W,w) \in \mathrm{Obj}(X \downarrow G)$. Then $f$ induces a bijection between $X \downarrow G((U,u),(W,w))$ and $X \downarrow G((V,v),(W,w))$. Since the former is singleton, so is the latter.

The dual has a similar argument. ☐

# 4 Yoneda's Lemma

> **Definition – Dual Categories**
> Let $\mathcal{C} \in \mathrm{Obj}(\mathbf{Cat})$. Then the *dual category of* $\mathcal{C}$, denoted $\mathcal{C}^{op}$, is defined by :
>
> 1. $\mathrm{Obj}(\mathcal{C}^{op}) := \mathrm{Obj}(\mathcal{C})$.
> 2. For all $U,V \in \mathrm{Obj}(\mathcal{C}^{op})$, $\mathcal{C}^{op}(U,V) := \mathcal{C}(V,U)$.

> **Definition – Contravariant Functors**
> Let $\mathcal{C},\mathcal{D} \in \mathrm{Obj}(\mathbf{Cat})$. Then a *contravariant functor from* $\mathcal{C}$ *to* $\mathcal{D}$ is just a functor $\mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Functors $\mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ are henceforth called *covariant functors from* $\mathcal{C}$ *to* $\mathcal{D}$.

> **Definition – Morphism Functor**
> Let $\mathcal{C}$ be a category and $U \in \mathrm{Obj}(\mathcal{C})$. Then $h_U : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$ is defined as :
>
> 1. For all $V \in \mathrm{Obj}(\mathcal{C}^{op})$, $h_U(V) := \mathcal{C}(V,U)$.
> 2. For all $V,W \in \mathrm{Obj}(\mathcal{C}^{op})$ and $f : V \xrightarrow{\mathcal{C}^{op}} W$, $h_U(f) : h_U(V) \to h_U(W), g \mapsto g \circ f$.
>
> Similarly, $h^U : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$ is defined as :
>
> 1. For all $V \in \mathrm{Obj}(\mathcal{C})$, $h^U(V) := \mathcal{C}(U,V)$.

2. For all $V, W \in \mathrm{Obj}\,(\mathcal{C})$ and $f : V \xrightarrow{\mathcal{C}} W$, $h^U(f) : h^U(V) \to h^U(W), g \mapsto f \circ g$.

**Proposition – Morphism Functor is Functorial**
Let $\mathcal{C}$ be a category. Then $h_\star : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}^{op}}$. Similarly, $h^\star : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}}$.

*Remark – Functor of Points.* Because of its relevance in algebraic geometry, $h_U$ is called the *functor of points of* $U$.

**Proposition – Yoneda's Lemma**
Let $\mathcal{C}$ be a category. Then $h_\star : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}^{op}}$ is fully faithful. Since fully faithful functors are injective, $h_\star$ is called the *Yoneda embedding*.

More generally, for any $U \in \mathrm{Obj}\,(\mathcal{C})$ and $F \in \mathrm{Obj}\left(\mathbf{Set}^{\mathcal{C}^{op}}\right)$, $\mathbf{Set}^{\mathcal{C}^{op}}(h_U, F)$ bijects with $F(U)$ via $s \mapsto s_U(\mathbb{1}_U)$ and this bijection is natural in both $U$ and $F$.

Dually, $h^\star : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}}$ is fully faithful and more generally, for any $U \in \mathrm{Obj}\,(\mathcal{C}^{op})$ and $F \in \mathrm{Obj}\left(\mathbf{Set}^{\mathcal{C}}\right)$, $\mathbf{Set}^{\mathcal{C}}\left(h^U, F\right)$ naturally bijects with $F(U)$ via $s \mapsto s_U(\mathbb{1}_U)$.

*Proof.* We first prove the general statement. Let $U \in \mathrm{Obj}\,(\mathcal{C})\,, F \in \mathrm{Obj}\left(\mathbf{Set}^{\mathcal{C}^{op}}\right)$. Given an element $s \in F(U)$, we are tasked with constructing a natural transformation $h_U \to F$. For $V \in \mathcal{C}$ we want to map elements $f \in h_U(V)$ to some element of $F(V)$. Well, $f$ is a morphism from $V$ to $U$, so $F(f)$ is a morphism from $F(U)$ to $F(V)$, and we are given an element $s \in F(U)$. So define $\alpha_V^s : h_U(V) \to F(V) := f \mapsto F(f)(s)$. For the collection of $\alpha_V^s$ to form a natural transformation, we need naturality. So given $f \in \mathcal{C}(V, W)$, we need the following diagram to commute :

$$\begin{tikzcd}
h_U(W) \arrow[r, "\alpha_W^s"] \arrow[d, "h_U(f)"] & F(W) \arrow[d, "F(f)"] \\
h_U(V) \arrow[r, "\alpha_V^s"] & F(V)
\end{tikzcd}$$

For $g \in h_U(W)$, then we have as desired

$$\alpha_V^s \circ h_U(f)(g) = \alpha_V^s(g \circ f) = F(g \circ f)(x) = F(f) \circ F(g)(x) = F(f) \circ \alpha_W^s(g)$$

So $\alpha^s : h_U \to F$ is a natural transformation.

Note that we can recover $s$ from $\alpha^s$ by $\alpha_U^s(\mathbb{1}_U) = s$. This motivates us to define the inverse map by $\alpha \in \mathbf{Set}^{\mathcal{C}^{op}}(h_U, F) \mapsto \alpha_U(\mathbb{1}_U)$. To show these two maps are indeed inverses, first consider the following diagram where $\alpha : h_U \to F$ is a natural tranformation, $W \in \mathrm{Obj}\,(\mathcal{C})$ and $f \in h_U(W)$ :

$$
\begin{tikzcd}
h_U(U) \arrow[rrr, "a_U"] \arrow[ddd, "h_U(f)"'] & & & F(U) \arrow[ddd, "F(f)"] \\
& \mathbb{1}_U \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_U(\mathbb{1}_U) \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \alpha_W(f) = F(f)(\alpha_U(\mathbb{1}_U)) & \\
h_U(W) \arrow[rrr, "\alpha_W"'] & & & F(W)
\end{tikzcd}
$$

The above diagram commutes by naturality of $\alpha$. What it shows is that $\alpha_W$ is completely determined by $\alpha_U(\mathbb{1}_U)$, and hence $\alpha$ is completely determined by $\alpha_U(\mathbb{1}_U)$. This proves one side of the inverse situation. The other side is clear. Thus we have a bijection between $\mathbf{Set}^{\mathcal{C}^{op}}(h_U, F) \cong F(U)$.

At this point, we can already get $h_\star$ fully faithful by applying the above bijection to $F = h_\star$ itself and noting the bijection turns $f \in h_V(U)$ into $h_f$.

For naturality in the first component, let $f : U \xrightarrow{\mathcal{C}} V$. Then we have the following commutative diagram.

$$
\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{op}}(h_U, F) \arrow[rrr] & & & F(U) \\
& \alpha \circ h^f \arrow[r, mapsto] & (\alpha \circ h^f)_U(\mathbb{1}_U) = \alpha_V(f) = F(f) \circ \alpha_V(\mathbb{1}_V) & \\
& \alpha \arrow[r, mapsto] \arrow[u, mapsto] & \alpha_V(\mathbb{1}_V) \arrow[u, mapsto] & \\
\mathbf{Set}^{\mathcal{C}^{op}}(h_V, F) \arrow[rrr] \arrow[uuu, "(\star \circ h^f)"] & & & F(V) \arrow[uuu, "F(f)"']
\end{tikzcd}
$$

For naturality in the second component, let $\phi : F \xrightarrow{\mathbf{Set}^{\mathcal{C}^{op}}} G$. Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{op}}\left(h_U, F\right) \arrow[rr] \arrow[dd, "(\phi\circ\star)"'] & & F(U) \arrow[dd, "\phi_U"] \\
& \begin{array}{ccc} \alpha & \longmapsto & \alpha_U(\mathbb{1}_U) \\ \downarrow & & \downarrow \\ \phi\circ\alpha & \longmapsto & (\phi\circ\alpha)_U(\mathbb{1}_U) = \phi_U\circ\alpha_U(\mathbb{1}_U) \end{array} & \\
\mathbf{Set}^{\mathcal{C}^{op}}\left(h_U, G\right) \arrow[rr] & & G(U)
\end{tikzcd}$$

We thus have the desired result. $\square$

**Definition – Representable Functors**
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$ be a covariant functor. Then a *representation of* $G$ is a $(U, u) \in h^\star \downarrow G$ where $u : h^\star \xrightarrow[\sim]{\mathbf{Set}^{\mathcal{C}}} G$.

Dually, let $F : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$ be a contravariant functor. Then a *representation of* $F$ is a $(U, u) \in h_\star \downarrow F$ where $u : h_\star \xrightarrow[\sim]{\mathbf{Set}^{\mathcal{C}^{op}}} F$.

A functor (covariant or contravariant) that has a representation is called *representable*.

*Remark.* If a functor has a representation, Yoneda's lemma implies it is canonical. This is the next result.

Before this, we first relate universal morphisms to representable functors. This is important as it leads to the notion of *adjunction*.

**Proposition – Universal iff Represents**
Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $X \in \mathrm{Obj}\left(\mathcal{D}\right)$, $(L(X), \eta_X) \in \mathrm{Obj}\left(X \downarrow R\right)$. Then the following are equivalent :

1. $(L(X), \eta_X)$ is a universal morphism from $X$ to $R$.
2. $L(X)$ represents the covariant functor $\mathcal{D}\left(X, R(\star)\right)$ and $\mathbb{1}_{L(X)}$ corresponds to $\eta_X$.

Dually, let $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $U \in \mathrm{Obj}\left(\mathcal{C}\right)$, $(R(U), \varepsilon_U) \in \mathrm{Obj}\left(L \downarrow U\right)$. Then the following are equivalent :

1. $(R(U), \varepsilon_U)$ is a universal morphism from $L$ to $U$.
2. $R(U)$ represents the contravariant functor $\mathcal{C}\left(L(\star), U\right)$ and $\mathbb{1}_{R(U)}$ corresponds to $\varepsilon_U$.

*Proof.* (Universal implies Represents) Let $(L(X), \eta_X)$ be a universal morphism from $X$ to $R$. Define the following natural transformation,

$$\begin{aligned} &h^{L(X)} \xrightarrow{\mathbf{Set}^{\mathcal{C}}} \mathcal{D}\left(X, R(\star)\right) := \\ &W \in \mathrm{Obj}\left(\mathcal{C}\right) \mapsto \left[f \in h^{L(X)}(W) \mapsto R(f)\circ\eta_X \in \mathcal{D}\left(X, R(W)\right)\right] \end{aligned}$$

Then for every $W \in \text{Obj}(\mathcal{C})$, this is an isomorphism between $h^{L(X)}(W)$ and $\mathcal{D}(X, R(W))$, and hence a natural isomorphism. Indeed, $\mathbb{1}_{L(X)}$ corresponds to $\eta_X$ under this natural isomorphism.

(Represents implies Universal) Let $\alpha : h^{L(X)} \xrightarrow{\mathbf{Set}^{\mathcal{C}}} \mathcal{D}(X, R(\star))$ be a natural isomorphism where at $L(X)$, $\alpha_{L(X)}(\mathbb{1}_{L(X)}) = \eta_X$. Let $(V, v) \in \text{Obj}(X \downarrow R)$. For any $f : L(X) \xrightarrow{\mathcal{C}} V$, consider the following commutative diagram.

$$\begin{tikzcd}
h^{L(X)}(L(X)) \arrow[rrr, "\alpha_{L(X)}"] \arrow[ddd, "h^{L(X)}(f)"'] & & & \mathcal{D}(X, RL(X)) \arrow[ddd, "\mathcal{D}(X, R(f))"] \\
& \mathbb{1}_{L(X)} \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_{L(X)}(\mathbb{1}_{L(X)}) = \eta_X \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \alpha_V(f) = R(f) \circ \eta_X & \\
h^{L(X)}(V) \arrow[rrr, "\alpha_V"'] & & & \mathcal{D}(X, R(V))
\end{tikzcd}$$

Thus $f : (L(X), \eta_X) \xrightarrow{X \downarrow R} (V, v)$ if and only if $\alpha_V(f) = v$. Then $\alpha_V^{-1}(v)$ is the unique morphism $(L(X), \eta_X) \xrightarrow{X \downarrow R} (V, v)$. Since there exists a unique $(L(X), \eta_X) \xrightarrow{X \downarrow R} (V, v)$, $(L(X), \eta_X)$ is universal.

The dual equivalence has an analogous proof. □

> **Proposition – Canonical Representation**
> Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $(U, u) \in h^\star \downarrow G$. Then the following are equivalent :
>
> 1. $(U, u)$ is a representation of $G$.
> 2. $(U, u)$ is a universal morphism from $h^\star$ to $G$.
>
> In particular, representations of $G$ are canonically isomorphic.
>
> Dually, let $F : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $(V, v) \in h_\star \downarrow F$. Then the following are equivalent :
>
> 1. $(V, v)$ is a representation of $F$.
> 2. $(V, v)$ is a universal morphism from $h_\star$ to $F$.
>
> In particular, representations of $F$ are canonically isomorphic.

*Proof.* (Representation implies Universal) Let $(W, w) \in \text{Obj}(h^\star \downarrow G)$. Then $u^{-1} \circ w : h^W \xrightarrow{\mathbf{Set}^{\mathcal{C}}} h^U$. By Yoneda's lemma, there exists a unique $u(W, w) : U \xrightarrow{\mathcal{C}} W$ such that $u^{-1} \circ w = h^{u(W,w)}$. Hence $u(W, w)$ is the unique morphism $(W, w) \xrightarrow{h^\star \downarrow G} U, u$.

(Universal implies Representation) By universal iff represents and Yoneda's lemma, we have the following diagram.

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}}\left(h^{\star}, G\right) \arrow[r, "V\in\mathrm{Obj}(\mathcal{C})\mapsto\left[s\in\mathbf{Set}^{\mathcal{C}}\left(h^{V},G\right)\mapsto s_V(\mathbb{1}_V)\right]", "\sim"'] & G \\
h^U \arrow[u, "V\in\mathrm{Obj}(\mathcal{C})\mapsto\left[f\in h^U(V)\mapsto u\circ h^f\right]", "\sim"'] \arrow[ur, "u"] &
\end{tikzcd}$$

The claim is that the above commutes, and hence $u$ is an isomorphism. Let $V \in \mathrm{Obj}(\mathcal{C})$ and $f \in h^U(V)$. Then

$$(h^f \circ u)_V(\mathbb{1}_V) = u_V \circ \left(h^f\right)_V (\mathbb{1}_V) = u_V(f)$$

So the above diagram commutes.

For the dual, the argument is similar. □

# 5 Adjoint Functors

**Definition – Adjoint Functors**

Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then $R$ is a *right adjoint* when there exists $L : \mathrm{Obj}(\mathcal{D}) \to \mathrm{Obj}(\mathcal{C})$ and $(\eta_X) \in \prod_{X\in\mathrm{Obj}(\mathcal{D})} \mathcal{D}(X, RL(X))$ such that for all $X \in \mathrm{Obj}(\mathcal{D})$, $(L(X), \eta(X))$ is a universal morphism from $X$ to $R$. In this case, $L$ is called the *left adjoint of* $R$.

Dually, let $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then $L$ is a *left adjoint* when there exists $R : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ and $(\varepsilon_U) \in \prod_{U\in\mathrm{Obj}(\mathcal{C})} \mathcal{C}(LR(U), U)$ such that for all $U \in \mathrm{Obj}(\mathcal{C})$, $(R(U), \varepsilon(U))$ is a universal morphism from $L$ to $U$. In this case, $R$ is called the *right adjoint of* $L$.

**Definition – Product Category**

Let $\mathcal{C}, \mathcal{D}$ be categories. Then the *product category of* $\mathcal{C}, \mathcal{D}$ is denoted $\mathcal{C} \times \mathcal{D}$ and is defined as follows.

1. $\mathrm{Obj}(\mathcal{C} \times \mathcal{D}) := \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$.
2. For $(U, X), (V, Y) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $\mathcal{C} \times \mathcal{D}((U, X), (V, Y)) := \mathcal{C}(U, V) \times \mathcal{D}(X, Y)$.

**Proposition – Natural Transformations on Product Category**

Let $F, G : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, $(\alpha_{U,X}) \in \prod_{(U,X)\in\mathrm{Obj}(\mathcal{C}\times\mathcal{D})} \mathcal{E}(F(U, X), G(U, X))$. Then the following are equivalent.

1. $\alpha : F \to G$.
2. For all $(U, X) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $\alpha_{U,-} : F(U, -) \to G(U, -)$ and $\alpha_{-,X} : F(-, X) \to G(-, X)$.

*Proof.* Straight forward. □

**Definition – Adjunction**
Let $R : \mathcal{C} \xrightarrow{\textbf{Cat}} \mathcal{D}$ and $L : \mathcal{D} \xrightarrow{\textbf{Cat}} \mathcal{C}$. We have the two functors $\mathcal{C}\left(L(\star), -\right), \mathcal{D}\left(\star, R(-)\right) : \mathcal{D}^{op} \times \mathcal{C} \xrightarrow{\textbf{Cat}} \textbf{Set}$. Then $(L, R)$ is an *adjunction* when $\mathcal{C}\left(L(\star), -\right), \mathcal{D}\left(\star, R(-)\right)$ are naturally isomorphic.

In this case, $R$ is called the *right adjoint of* $L$ and $L$ is called the *left adjoint of* $R$. The isomorphism is called the *adjunction isomorphism*. For all $f : L(X) \xrightarrow{\mathcal{C}} U$, the image of $f$ under the adjunction isomorphism is called the *adjunct of* $f$, denoted $f^{\perp}$. Similarly for $g : X \xrightarrow{\mathcal{D}} R(U)$, we have the *adjunct of* $g$, denoted $g^{\perp}$.

**Proposition – Universal Morphism Characterisation of Adjunction**
Let $R : \mathcal{C} \xrightarrow{\textbf{Cat}} \mathcal{D}$. Then the following are equivalent :

1. $R$ is a right adjoint.
2. There exists $L : \mathcal{D} \xrightarrow{\textbf{Cat}} \mathcal{C}$ such that $(L, R)$ is an adjunction.

Dually, let $L : \mathcal{D} \xrightarrow{\textbf{Cat}} \mathcal{C}$. Then the following are equivalent :

1. $L$ is a left adjoint.
2. There exists $R : \mathcal{C} \xrightarrow{\textbf{Cat}} \mathcal{D}$ such that $(L, R)$ is an adjunction.

*Proof.* $(\Rightarrow)$ Let $R$ be a right adjoint, $L : \mathrm{Obj}\left(\mathcal{D}\right) \to \mathrm{Obj}\left(\mathcal{C}\right)$, $\eta \in \Pi X \in \mathrm{Obj}\left(\mathcal{D}\right), \mathcal{D}\left(X, RL(X)\right)$, for all $X \in \mathrm{Obj}\left(\mathcal{D}\right)$, $(L(X), \eta(X))$ universal morphism from $X$ to $R$.

The universal properties at every $X \in \mathrm{Obj}\left(\mathcal{D}\right)$ implies $L$ is functorial. By universal iff represents, for all $X \in \mathrm{Obj}\left(\mathcal{D}\right)$, we have $\mathcal{C}\left(L(X), -\right) \cong \mathcal{D}\left(X, R(-)\right)$ as functors $\mathcal{C} \to \textbf{Set}$. Let $f : X \xrightarrow{\mathcal{D}} Y$ and $U \in \mathrm{Obj}\left(\mathcal{C}\right)$. Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{C}\left(L(X), U\right) \arrow[rrr, "R(-)\circ\eta(X)"] & & & \mathcal{D}\left(X, R(U)\right) \\
 & g \circ L(f) \arrow[r, mapsto] & R(g \circ L(f)) \circ \eta(X) = R(g) \circ \eta(Y) \circ f & \\
 & g \arrow[u, mapsto] \arrow[r, mapsto] & R(g) \circ \eta(Y) \arrow[u, mapsto] & \\
\mathcal{C}\left(L(Y), U\right) \arrow[uuu, "(h^{L(f)})_U"] \arrow[rrr, "R(-)\circ\eta(Y)"'] & & & \mathcal{D}\left(Y, R(U)\right) \arrow[uuu, "(h^{f})_U"']
\end{tikzcd}$$

Thus the isomorphism $\mathcal{C}\left(L(X), -\right) \cong \mathcal{D}\left(X, R(-)\right)$ is functorial in $X$, and hence an isomorphism between $\mathcal{C}\left(L(\star), -\right) \cong \mathcal{D}\left(\star, R(-)\right)$.

($\Leftarrow$) Let $L : \mathrm{Obj}\,(\mathcal{D}) \xrightarrow{\mathbf{Cat}} \mathcal{C}$ such that $(L, R)$ is an adjunction. Then for each $X \in \mathrm{Obj}\,(\mathcal{D})$, $\mathcal{C}\,(L(X), -) \cong \mathcal{D}\,(X, R(-))$. Let $\eta(X)$ be the adjunct of $\mathbb{1}_{L(X)}$. By universal iff represents, $(L(X), \eta(X))$ is a universal morphism from $X$ to $R$.

The dual has a similar argument. $\square$

> **Proposition – Uniqueness of Adjoints**
> Let $R, R_1 : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $L, L_1 : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then
>
> 1. If $(L, R)$ and $(L, R_1)$ are both adjunctions, then $R \cong R_1$ as functors.
> 2. If $(L, R)$ and $(L_1, R)$ are both adjunctions, then $L \cong L_1$ as functors.

*Proof.* (1) Let $(L, R), (L, R_1)$ both be adjunctions. Let $f : U \xrightarrow{\mathcal{C}} V$. We have an isomorphism between the functors $\mathcal{D}\,(-, R(U))$ and $\mathcal{D}\,(-, R_1(U))$ for all $U \in \mathrm{Obj}\,(\mathcal{C})$. By Yoneda's lemma, these isomorphisms are equal to $h_{\alpha_U}$ for some unique morphism $\alpha_U : R(U) \xrightarrow{\mathcal{D}} R_1(U)$. So we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{D}\,(-, R(U)) \arrow[r, "h_{\alpha_U}", "\sim"'] \arrow[d, "h_{R(f)}"'] & \mathcal{D}\,(-, R_1(U)) \arrow[d, "h_{R_1(f)}"] \\
\mathcal{D}\,(-, R(V)) \arrow[r, "\sim", "h_{\alpha_V}"'] & \mathcal{D}\,(-, R_1(V))
\end{tikzcd}$$

Again by Yoneda, we have $R_1(f) \circ \alpha_U = \alpha_V \circ R(f)$. The fact that $h_{\alpha_U}$ is an isomorphism implies $\alpha_U$ is an isomorphism. Thus $\alpha$ is a natural isomorphism between $R, R_1$.

(2) Analogous. $\square$

*Remark.* There is another characterisation of adjunctions. This may be skipped on first reading since in practice, the special case of Galois connection happens more often and the proofs become much easier.

> **Proposition – Unit/Counit Characterisation of Adjunction**
> Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then the following are equivalent :
>
> 1. (Morphism Isomorphism) $(R, L)$ is an adjunction.
> 2. (Unit-Counit) There exists $\eta : \mathbb{1}_{\mathcal{D}} \to RL$ and $\varepsilon : LR \to \mathbb{1}_{\mathcal{C}}$ such that
>
>    (a) $\mathbb{1}_L = \varepsilon L \circ L\eta$, that is to say for all $X \in \mathrm{Obj}\,(D)$, we have the following commutative diagram.

$$\begin{tikzcd} L(X) \arrow[r, "L(\eta(X))"] \arrow[dr, "\mathbb{1}_{L(X)}"'] & LRL(X) \arrow[d, "\varepsilon(L(X))"] \\ & L(X) \end{tikzcd}$$

(b) $\mathbb{1}_R = R\varepsilon \circ \eta R$, i.e. for all $U \in \mathrm{Obj}\,(\mathcal{C})$, we have the following commutative diagram.

$$\begin{tikzcd} R(U) \arrow[r, "\eta(R(U))"] \arrow[dr, "\mathbb{1}_{R(U)}"'] & RLR(U) \arrow[d, "R(\varepsilon(U))"] \\ & R(U) \end{tikzcd}$$

The above two equations are often called *triangle-identities*.

*Proof.* $(1 \Rightarrow 2)$ For all $X \in \mathrm{Obj}\,(\mathcal{D})$, the adjunction isomorphism gives an isomorphism of functors

$$\mathcal{C}\,(L(X), -) \cong \mathcal{D}\,(X, R(-))$$

Define $\eta(X) := \mathbb{1}^{\perp}_{L(X)}$. Then by universal iff represents, $(L(X), \eta(X))$ is a universal morphism from $X$ to $R$. We claim that $\eta : \mathbb{1}_{\mathcal{D}} \to RL$.

Let $f : X \xrightarrow{\mathcal{D}} Y$. Then by the universal property of $(L(X), \eta(X))$, we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "\eta(X)"] \arrow[d, "f"'] & RL(X) \arrow[d, "RL(f)"] \\ Y \arrow[r, "\eta(Y)"'] & RL(Y) \end{tikzcd}$$

i.e. $\eta$ is a natural transformation as desired. We similarly define $\varepsilon(U) := \mathbb{1}^{\perp}_{R(U)}$ for $U \in \mathrm{Obj}\,(\mathcal{C})$ and see that $\varepsilon : LR \to \mathbb{1}_{\mathcal{C}}$.

To prove $(a)$, let $X \in \mathrm{Obj}\,(\mathcal{D})$. Then

$$\mathbb{1}_{L(X)} = \left(\mathbb{1}^{\perp}_{L(X)}\right)^{\perp} = (\eta(X))^{\perp} = \varepsilon(L(X)) \circ L(\eta(X))$$

where the last equality follows from the universal property of $(RL(X), \varepsilon(L(X)))$. Similarly for $(b)$, we have for $U \in \mathrm{Obj}\,(\mathcal{C})$,

$$\mathbb{1}_{R(U)} = \left(\mathbb{1}^{\perp}_{R(U)}\right)^{\perp} = (\varepsilon(U))^{\perp} = R(\varepsilon(U)) \circ \eta(R(U))$$

where the last equality is by the universal property of $(LR(U), \eta(R(U)))$.

$(2 \Rightarrow 1)$ Let $(X, U) \in \mathrm{Obj}\,(\mathcal{D}^{op} \times \mathcal{C})$. Since $(L(X), \eta(X))$ is supposed to be a universal morphism from $X$ to $R$, we define the adjunction map to be

$$\begin{aligned} \mathcal{C}\,(L(X), U) &\overset{\perp}{\longleftrightarrow} \mathcal{D}\,(X, R(U)) \\ f &\longmapsto R(f) \circ \eta(X) \\ \varepsilon(U) \circ L(g) &\longleftarrow g \end{aligned}$$

Then for $f : L(X) \xrightarrow{\mathcal{C}} U$,

$$\begin{aligned} \left(f^{\perp}\right)^{\perp} &= \varepsilon(U) \circ L(f^{\perp}) = \varepsilon(U) \circ L\left(R(f) \circ \eta(X)\right) \\ &= \varepsilon(U) \circ LR(f) \circ L(\eta(X)) = f \circ \varepsilon(L(X)) \circ L(\eta(X)) = f \end{aligned}$$

Similarly, $\left(g^{\perp}\right)^{\perp} = g$. So $\perp$ is an isomorphism at all $(X, U)$.

It remains to show naturality. It suffices to show that the isomorphism is natural in both components. Let $f : X \xrightarrow{\mathcal{D}^{op}} Y$. Then we have the following diagram.

$$\begin{tikzcd} \mathcal{C}\,(L(X), U) \arrow[r, "\perp"] \arrow[d, "h^{L(f)}"'] & \mathcal{D}\,(X, R(U)) \arrow[d, "h^f"] \\ \mathcal{C}\,(L(Y), U) \arrow[r, "\perp"'] & \mathcal{D}\,(Y, R(U)) \end{tikzcd}$$

It follows from $\eta : \mathbb{1}_{\mathcal{D}} \to RL$ that the above commutes. Similarly, naturality of $\varepsilon$ implies naturality in the second component. Hence $\perp$ is a natural isomorphism as desired. □

*Remark.* The following is a special case of adjunction that is worth noting.

> **Definition – Galois Connection**
>
> Let $I, J$ be partially ordered sets. Then $I, J$ can be seen as categories. A *monotone Galois connection between* $I, J$ is an adjunction between $I, J$. A *antitone Galois connection between* $I, J$ is an adjunction between $I^{op}, J$.

*Remark.* The unit/counit characterisation of adjunctions shows that if $(R, L)$ is a Galois connection (mono or anti) between partially ordered sets $I, J$, then $R$ and $L$ are bijective on their images.

> **Definition – Free Functors**
>
> Let $\mathcal{C}$ be a subcategory of $\mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$ be a functor. Then $L$ is called a *free functor from* $\mathcal{D}$ *to* $\mathcal{C}$ when it is left adjoint to the forgetful functor from $\mathcal{C}$ to $\mathcal{D}$. Note that if it exists, $L$ is then unique up to natural isomorphism, so it is customary to called it *the* free functor.

> *Example (Adjunctions).*
> *The adjunctions hinted at in section 2.*
>
> *First, a list of free, forgetful adjunctions.*
>
> – *(Free Group) What we previously said about taking free groups is precisely the a fact that it's a uni-*

*versal morphism from $S$ to the forgetful functor. The forgetful functor is a right adjoint. Its left adjoint is taking free groups.*

$$\mathbf{Grp}(\langle - \rangle, \star) \cong \mathbf{Set}(-, \star)$$

– (*Free Left Module over a Ring*) *Similarly, for a ring $R$, taking free left modules over a set is a left adjoint to forgetful $R\mathbf{Mod} \to \mathbf{Set}$.*

$$R\mathbf{Mod}(R^{\oplus(-)}, \star) \cong \mathbf{Set}(-, \star)$$

– (*Free Commutative Algebra over a Ring*) *For a commutative ring $K$,*

$$\mathbf{CAlg}(K)(K[-], \star) \cong \mathbf{Set}(-, \star)$$

– (*Extension and Contraction of Scalars*) *Let $B$ be a commutative algebra over $A$ ($A$ is commutative by definition). Then extension of scalars is the free functor left adjoint to the forgetful $B\mathbf{Mod} \to A\mathbf{Mod}$.*

$$B\mathbf{Mod}((-) \otimes_A B, \star) \cong A\mathbf{Mod}(-, \star)$$

– (*Localization of Modules*) *Let $A$ be a commutative ring and $S \subseteq A$ a multiplicative set. Then localization with respect to $S$ is the free functor left adjoint to the forgetful $S^{-1}A\mathbf{Mod} \to A\mathbf{Mod}$.*

$$S^{-1}A\mathbf{Mod}(S^{-1}(-), \star) \cong A\mathbf{Mod}(-, \star)$$

*Note that by uniqueness of adjoints, this can be seen as a special case of extension and contraction of scalars via the commutative $A$-algebra $S^{-1}A$.*

– (*Symmetric Algebra*) *Let $A$ be a commutative ring. Then taking symmetric algebas of modules is the free functor left adjoint to the forgetful $\mathbf{CAlg}(A) \to A\mathbf{Mod}$.*

$$\mathbf{CAlg}(A)(Symm(-), \star) \cong A\mathbf{Mod}(-, \star)$$

– (*Discrete Topology*) *Giving sets the discrete topology is the free functor left adjoint to the forgetful $\mathbf{Top} \to \mathbf{Set}$.*

$$\mathbf{Top}((-, 2^-), \star) \cong \mathbf{Set}(-, \star)$$

– (*Free Category on a Pre-Ordered Set*) *As previously discussed, every pre-ordered set can be made into a category by considering the relation $\leq$ as morphisms. This is the free functor left adjoint to the forgetful $\mathbf{Cat} \to \mathbf{Ord}$, where we see any category $\mathcal{C}$ as a pre-ordered set by declaring for $U, V \in \operatorname{Obj}(\mathcal{C})$, $U \leq V := \mathcal{C}(U, V) \neq \varnothing$.*

*A list of adjunctions, themed "moving structures on objects across morphisms".*

– (*Image, Preimage of "Substructures"*) *Let $f : U \to V$ be a morphism of sets. Then*

$$2^U(-, f^{-1}(\star)) \cong 2^V(f(-), \star)$$

*If $f$ is instead a morphism of groups, then this upgrades to*

$$\mathbf{SubGrp}(U)(-, f^{-1}(\star)) \cong \mathbf{SubGrp}(V)(f(-), \star)$$

*If $f$ is instead a morphism of rings,*

$$\mathbf{SubRing}(U)(-, f^{-1}(\star)) \cong \mathbf{SubRing}(V)(f(-), \star)$$

*If $f$ is instead a morphism of left $R$-modules for some ring $R$,*

$$\mathbf{Sub}R\mathbf{Mod}(U)(-, f^{-1}(\star)) \cong \mathbf{Sub}R\mathbf{Mod}(V)(f(-), \star)$$

*If $f$ is instead a morphism of $K$-algebras for some commutative ring $K$,*

$$\mathbf{Sub}K\mathbf{Alg}(U)(-, f^{-1}(\star)) \cong \mathbf{Sub}K\mathbf{Alg}(V)(f(-), \star)$$

– (*Image, Preimage of Filters*) *Let $f : U \to V$ be a morphism of sets. Then*

$$\mathrm{Fil}(U)(-, f^{-1}(\star)) \cong \mathrm{Fil}(V)(f_{\mathrm{Fil}}(-), \star)$$

*where for a filter $F$ on $U$, $f_{\mathrm{Fil}}(F) := \{W \subseteq V \mid f^{-1}W \in F\}$.*

*The following is a list of instances of "exponential objects".*

– *Let $Y$ be a set. Taking the cartesian product with $Y$, $(-) \times Y : \mathbf{Set} \to \mathbf{Set}$, is functorial. Then we have the adjunction between $(-) \times Y$ and $h^Y$ :*

$$\mathbf{Set}((-) \times Y, \star) \cong \mathbf{Set}(-, \mathbf{Set}(Y, \star))$$

– *Let $N$ be an $A$-module where $A$ is a commutative ring. Then $(-) \otimes_A N : A\mathbf{Mod} \to A\mathbf{Mod}$ is functorial. On the other hand, the coYoneda embedding $h^N$ of $N$ is naturally a functor $A\mathbf{Mod} \to A\mathbf{Mod}$. Then we have the adjunction between $(-) \otimes_A N$ and $h^N$ :*

$$A\mathbf{Mod}((-) \otimes_A N, \star) \cong A\mathbf{Mod}(-, h^N(\star))$$

– *Let $\mathcal{D}$ be a category. Then taking the product of categories with $\mathcal{D}$ yields a functor $(-) \times \mathcal{D} : \mathbf{Cat} \to \mathbf{Cat}$. On the other hand, the coYoneda embedding of $\mathcal{D}$ is naturally a functor $h^{\mathcal{D}} : \mathbf{Cat} \to \mathbf{Cat}$. Then we have the adjunction :*

$$\mathbf{Cat}((-) \times \mathcal{D}, \star) \cong \mathbf{Cat}(-, h^{\mathcal{D}}(\star))$$

(*TODO : Add to this list*)

# 6 Limits and Colimits

*Remark.* The idea that limits and colimits formalize is that of making new objects out of given objects within a category in the "most efficient way possible".

**Definition – Diagrams**
Let $\mathcal{I}, \mathcal{C}$ be categories. Then an *$\mathcal{I}$-shaped diagram in $\mathcal{C}$* is a covariant functor from $\mathcal{I}$ to $\mathcal{C}$.

*Remark.* Often, it is easier to take $\mathcal{I}$ to be a subcategory of $\mathcal{C}$.

**Definition – Constant (Co)Diagrams**
Let $\mathcal{I}, \mathcal{C}$ be categories and $U \in \operatorname{Obj}(\mathcal{C})$. Then define the *constant diagram* $\Delta(U)$ as follows.

1. For all $i \in \mathcal{I}$, $\Delta(U)(i) := U$.
2. For all $\phi : i \xrightarrow{\mathcal{I}} j$, $\Delta(U)(\phi) := \mathbb{1}_U$.

Then $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$.

**Definition – (Co)Limits of Diagrams**
Let $\mathcal{I}, \mathcal{C}$ be categories, $X$ a $\mathcal{I}$-shaped diagram in $\mathcal{C}$. Let $(U, u) \in \operatorname{Obj}(\Delta \downarrow X)$. Then the following are equivalent :

- $(U, u)$ is a universal morphism from $\Delta$ to $X$.
- $U$ represents the functor $\mathcal{C}^{\mathcal{I}}(\Delta(-), X) : \mathcal{C}^{op} \to \mathbf{Set}$, i.e. $\mathcal{C}^{\mathcal{I}}(\Delta(-), X) \cong \mathcal{C}(-, U)$ where $\mathbb{1}_U$ corresponds to $u$.

When any (and thus all) of these are true, $(U, u)$ is called a *limit of* $X$. By abuse of language, we sometimes simply say $U$ is a limit of $X$. If a limit of $X$ exists, it is canonical. So we will say *the* limit to refer to *any* limit, and denote it using $(\varprojlim X, \pi_X)$. When a limit exists, we will say *the* limit exists.

Dually, let $(V, v) \in \operatorname{Obj}(X \downarrow \Delta)$. Then the following are equivalent :

- $(V, v)$ is a universal morphism from $X$ to $\Delta$.
- $V$ represents the functor $\mathcal{C}^{\mathcal{I}}(X, \Delta(-)) : \mathcal{C} \to \mathbf{Set}$, i.e. $\mathcal{C}(V, -) \cong \mathcal{C}^{\mathcal{I}}(X, \Delta(-))$ where $\mathbb{1}_V$ corresponds to $v$.

When any (and thus all) of these are true, $(V, v)$ is called a *colimit of* $X$. By abuse of language, we sometimes simply say $V$ is a colimit of $X$. If a colimit of $X$ exists, it is canonical. So we will say *the* colimit to refer to *any* colimit, and denote it with $(\varinjlim X, \iota_X)$. When a colimit exists, we will say *the* colimit exists.

*Remark – Some Terminology.* Sometimes limits are also called *projective limits*, and colimits are called *injective limits*. Also, objects in $\Delta \downarrow X$ are often called *cones*. Dually, objects in $X \downarrow \Delta$ are called *cocones*. In my experience, being able to "see the (co)cones" has been great for keeping a clear mind in later proofs.

*Remark.* We now cover important instances of (co)limits. These are special in the sense that all (co)limits can be built from them.

**Definition – Discrete Category**
For $I \in \operatorname{Obj}(\mathbf{Set})$, $I$ can be turned into a category by having elements as objects and the only

morphisms being identity morphisms. Categories obtained in this way are called *discrete categories*.

*Remark.* Taking discrete categories is the free functor adjoint to the forgetful functor from **Cat** to **Set** by taking the set of objects. [1]

**Definition – (Co)Products**

Let $\mathcal{C}$ be a category and $\mathcal{I}$ a discrete category. Let $X$ be an $\mathcal{I}$-diagram in $\mathcal{C}$. If a limit of $X$ exists, it is called the *product of* $X(i)$. Dually, if a colimit of $X$ exists, it is called the *coproduct of* $X(i)$.

In the special case of $I = \varnothing$, the product is called the *final object of* $\mathcal{C}$. Dually, the empty coproduct is called the *initial object of* $\mathcal{C}$.

*Example* (*Final Objects*).
(*TODO*)

*Example* (*Initial Objects*).
(*TODO*)

*Example* (*Products*).
(*TODO*)

*Example* (*Coproducts*).
(*TODO*)

**Definition – (Co)Equalizers**

Let $\mathcal{C}$ be a category. Let $I$ be an arbitrary set and $\mathcal{I}$ be the following category.

$$\begin{tikzcd} 0 \arrow[loop above, "\mathbb{1}_0"] \arrow[r, "i"] & 1 \arrow[loop above, "\mathbb{1}_1"] \end{tikzcd}$$

where there is a morphism $i : 0 \xrightarrow{\mathcal{I}} 1$ for all $i \in I$.

Let $X$ be an $\mathcal{I}$-diagram in $\mathcal{C}$. If a limit of $X$ exists, it is called the *equalizer of* $X(i)$*'s*. Dually, if a colimit of $X$ exists, it is called the *coequalizer of* $X(i)$*'s*.

*Example* (*Equalizers*).
(*TODO*)

[1] Russell cries, but we shall ignore him.

*Example* (*Coequalizers*).
(*TODO*)

**Definition – Pullbacks and Pushouts**

Let $\mathcal{C}$ be a category, $U \in \mathrm{Obj}\,(\mathcal{C})$. Then a *pullback over* $U$ is a product in the category $\mathcal{C} \downarrow U$. Dually, a *pushout under* $U$ is a coproduct in the category $U \downarrow \mathcal{C}$.

Let $I$ be an arbitrary set and $\mathcal{I}$ the following category.

$$\begin{tikzcd} i \arrow[loop above, "\mathbb{1}_0"] \arrow[r, "\phi(i)"] & * \arrow[loop above, "\mathbb{1}_1"] \end{tikzcd}$$

1. $\mathrm{Obj}\,(\mathcal{I}) = I \sqcup \{*\}$.
2. For all $x \in \mathrm{Obj}\,(\mathcal{I})$, $\mathcal{I}\,(x, x) = \{\mathbb{1}_x\}$.
3. For all $i \in I$, $\mathcal{I}\,(i, *) = \{\phi(i)\}$.

Then a pullback over $U$ is equivalently the limit of an $\mathcal{I}$-shaped diagram $X$ with $X(*) = U$. Dually, a pushout under $U$ is equivalently the colimit of an $\mathcal{I}^{op}$-shaped diagram $Y$ with $Y(*) = U$. [a]

[a] We could have define colimits only for $\mathcal{I}^{op}$-shaped diagrams so that coproduct, coequalizer, pushout are all related to their respective duals in the same way. However, other than this, doing things this way has little practical nor theoretical significance, hence my choice of defining colimits over $\mathcal{I}$-shaped diagrams.

# 7 Completeness

**Definition – Finite Categories**

Let $\mathcal{C}$ be a category. Then it is called *finite* when $\mathrm{Obj}\,(\mathcal{C})$ is finite and for any $U, V \in \mathrm{Obj}\,(\mathcal{C})$, $\mathcal{C}(U, V)$ is finite.

**Definition – (Finite) (Co)Completeness**

Let $\mathcal{C}$ be a category. Then it is called (*finitely*) *complete* when for all (finite) categories $\mathcal{I}$ and diagrams $X : \mathcal{I} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, there exists the limit of $X$ in $\mathcal{C}$.

Dually, it is called (*finitely*) *cocomplete* when for all (finite) categories $\mathcal{I}$ and diagrams $X : \mathcal{I} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, there exists the colimit of $X$ in $\mathcal{C}$.

*Remark.* If a category $\mathcal{C}$ has nice constructions, then for any category $\mathcal{I}$, $\mathcal{C}^{\mathcal{I}}$ will often inherit these constructions. This is the following result.

**Proposition – Characterisation of (Co)Limits of Functors**

Let $\mathcal{C}, \mathcal{D}$ be categories. Let $X : \mathcal{I} \to \mathcal{D}^{\mathcal{C}}$ be an $\mathcal{I}$-shaped diagram of functors from $\mathcal{C}$ to $\mathcal{D}$.

Let $(F, \phi) \in \mathrm{Obj}\,(\Delta \downarrow X)$. For each $U \in \mathrm{Obj}\,(\mathcal{C})$, we obtain a functor $X_U : \mathcal{I} \to \mathcal{D}, i \in \mathrm{Obj}\,(\mathcal{I}) \mapsto X(i, U)$ by restricting the second argument to $U$ (mapping morphisms is obvious). In a similar way, we obtain $(F(U), \phi_U) \in \mathrm{Obj}\,(\Delta \downarrow X_U)$. Then the following are equivalent :

- For all $U \in \mathrm{Obj}\,(\mathcal{C})$, $(F(U), \phi_U)$ is a limit of $X_U$.
- $(F, \phi)$ is a limit of $X$ and for every $U \in \mathrm{Obj}\,(\mathcal{C})$, the limit of $X_U$ exists in $\mathcal{D}$.

Paraphrasing : if $\mathcal{D}$ has limits of $\mathcal{I}$-shaped diagrams, then so does $\mathcal{D}^{\mathcal{C}}$.

Dually, let $(F, \phi) \in \mathrm{Obj}\,(X \downarrow \Delta)$. For each $U \in \mathrm{Obj}\,(\mathcal{C})$, we obtain a functor $X^U : \mathcal{I} \to \mathcal{D}, i \in \mathrm{Obj}\,(\mathcal{I}) \mapsto X(i, U)$ (mapping morphisms is obvious). In a similar way, we obtain $(F(U), \phi^U) \in \mathrm{Obj}\,(X^U \downarrow \Delta)$. Then the following are equivalent :

- For all $U \in \mathrm{Obj}\,(\mathcal{C})$, $(F(U), \phi^U)$ is a colimit of $X^U$.
- $(F, \phi)$ is a colimit of $X$ and for every $U \in \mathrm{Obj}\,(\mathcal{C})$, the colimit of $X^U$ exists in $\mathcal{D}$.

Paraphrasing : if $\mathcal{D}$ has colimits of $\mathcal{I}$-shaped diagrams, then so does $\mathcal{D}^{\mathcal{C}}$.

*Proof.* ($\Rightarrow$) Suppose for all $U \in \mathrm{Obj}\,(\mathcal{C})$, $(F(U), \phi_U)$ is a limit of $X_U$. We show that $(F, \phi)$ has the desired UP. Let $(G, \gamma) \in \mathrm{Obj}\,(\Delta \downarrow X)$. For each $U \in \mathrm{Obj}\,(\mathcal{C})$, define $(G(U), \gamma_U) \in \mathrm{Obj}\,(\Delta \downarrow X_U)$ by restricting the second argument to $U$. Then by the UP of limits, we have a unique morphism $\varprojlim \gamma_U \in \Delta \downarrow X_U((G(U), \gamma_U), (F(U), \phi_U))$.

$$\begin{tikzcd} \Delta(G(U)) \arrow[r, dashed, "\Delta(\varprojlim \gamma_U)"] \arrow[dr, "\gamma_U"'] & \Delta(F(U)) \arrow[d, "\phi_U"] \\ & X_U \end{tikzcd} \qquad \begin{tikzcd} \Delta(G) \arrow[r, dashed, "\Delta(\varprojlim \gamma)"] \arrow[dr, "\gamma"'] & \Delta(F) \arrow[d, "\phi"] \\ & X \end{tikzcd}$$

Then again by the UP of limits, the left triangle is functorial in $U$, so the collection of $(\varprojlim \gamma_U)_{U \in \mathrm{Obj}(\mathcal{C})}$ forms a $\varprojlim \gamma \Delta \downarrow X((G, \gamma), (F, \phi))$. The uniqueness of $\varprojlim \gamma$ comes from the componentwise uniqueness of $(\varprojlim \gamma_U)_{U \in \mathrm{Obj}(\mathcal{C})}$.

($\Leftarrow$) Let $(F, \phi)$ be a limit of $X$ and suppose for every $U \in \mathrm{Obj}\,(\mathcal{C})$, the limit of $X_U$ exists. Then we can use the forward implication to explicitly construct a limit $(\overline{F}, \overline{\phi})$ of $X$ by defining the components $(\overline{F}(U), \overline{\phi}_U)$ to be limits of $X_U$. Functoriality of $\overline{F}$ comes from the UP of limits in each component. Since limits are canonical, we have a unique isomorphism between $(F, \phi)$ and $(\overline{F}, \overline{\phi})$. Restricting to each component yields an isomorphism of cones $(F(U), \phi_U) \cong (\overline{F}(U), \overline{\phi}_U)$ for every $U \in \mathrm{Obj}\,(\mathcal{C})$. Since $(\overline{F}(U), \overline{\phi}_U)$ is by construction a limit of $X_U$, and being isomorphic to the limit implies being a limit. We conclude that $(F(U), \phi_U)$ is a limit of $X_U$ for every $U$.

The dual argument is similar. □

*Remark.* The following result gives an easier check for (co)completeness.

**Proposition – Characterisation of Completeness, Cocompleteness**

Let $\mathcal{C}$ be a category. Then the following are equivalent :

- $\mathcal{C}$ is (finitely) complete.
- $\mathcal{C}$ has (finite) products and (finite) equalizers.

Dually, the following are equivalent :

- $\mathcal{C}$ is (finitely) cocomplete.
- $\mathcal{C}$ has (finite) coproducts and (finite) coequalizers.

*Proof.* For fun, we prove the dual argument instead for once. It suffices to prove the reverse implication. Let $X : \mathcal{I} \to \mathcal{C}$ be a diagram. Define the *arrow category of* $\mathcal{I}$ as follows : take the free category on the partially ordered set $\mathbb{2} := \{0, 1\}$ and $\mathrm{Arr}(\mathcal{I}) := \mathcal{I}^{\mathbb{2}}$. Define *source* and *target* functions $s, t : \mathrm{Obj}\,(\mathrm{Arr}(\mathcal{I})) \to \mathrm{Obj}\,(\mathcal{I}), (\phi : i \to j) \mapsto i, j$ respectively. Let $(\coprod_{\phi \in \mathrm{Obj}(\mathrm{Arr}\,\mathcal{I})} X_{s(\phi)}, \iota)$ be the coproduct of $X_{s(\phi)}$ across all arrows $\phi$ in $\mathcal{I}$ and $(\coprod_{i \in \mathrm{Obj}(\mathcal{I})} X_i, \uparrow)$ the coproduct of $X(i)$ across all $i$ in $\mathcal{I}$. Now consider the following diagram :

$$\coprod_{\phi \in \mathrm{Obj}(\mathrm{Arr}\,\mathcal{I})} X_{s(\phi)} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{i \in \mathrm{Obj}(\mathcal{I})} X_i$$

where $\alpha$ is defined using the UP of the coproduct via for all $\phi \in \mathrm{Obj}\,(\mathrm{Arr}\,\mathcal{I})$, $\alpha \circ \iota_\phi :=\uparrow_{s(\phi)}$; $\beta$ is also defined using the UP of the coproduct, but via for all $\phi \in \mathrm{Obj}\,(\mathrm{Arr}\,\mathcal{I})$, $\beta \circ \iota_\phi :=\uparrow_{t(a)} \circ X(\phi)$. It is then straightforward to check that the coequalizer of $\alpha, \beta$ with obvious morphisms from $X$ into it is a colimit of $X$. Note that the above proof still works for finite categories $\mathcal{I}$.

The non-dual argument is similar. $\square$

*Remark.* We now proceed to show the (co)completeness of some standard categories. As this is unimportant to the theory, the reader may skip to the next section.

For fun, we will not use the above characterisation, though the constructions are of similar nature.

**Proposition – Set (Co)Complete**

Let $I$ be a category and $X$ an $I$-shaped diagram in $\mathbf{Set}$.

Consider the set $\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)$ where 1 is a singleton set. There is an obvious $\pi_X : \Delta(\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)) \to X$. Then $(\mathbf{Set}^{\mathcal{I}}(\Delta(1), X), \pi_X)$ is a limit of $X$ in $\mathbf{Set}$.

Dually, we construct the colimit of $X$ as follows. For every $(i, x_i) \in \bigsqcup_{i \in I} X_i$, define $[(i, x_i)]$ to be the subset of $(j, x_j) \in \bigsqcup_{i \in I} X_i$ such that there exists a "zig zag of elements and morphisms in $X$ joining them" (unenlightening to write down fully) :

$$\begin{tikzcd}
(i, x_i) \arrow[dr] & & (k_1, x_1) \arrow[dl] \arrow[dr] & & (k_n, x_n) \arrow[dl] \arrow[dr] & & (j, x_j) \arrow[dl] \\
& (l_0, y_0) & & \cdots & & (l_n, y_n) &
\end{tikzcd}$$

Then $Q := \{[(i, x_i)] \mid (i, x_i) \in \bigsqcup_{i \in I} X(i)\}$ forms a partition of $\bigsqcup_{i \in I} X(i)$, corresponding to the minimal equivalence relation generated by $(i, x_i) \sim (j, x_j) := \exists \phi \in I(i, j), X(\phi)(x_i) = x_j$. There is an obvious morphism $\iota_X : X \to \Delta(Q)$. Then $(Q, \iota_X)$ is a colimit of $X$.

*Proof.* Straightforward. $\square$

**Proposition – Top Complete**

Let $I$ be a category and $X$ an $I$-shaped diagram in $\mathbf{Top}$.

Let $(\varprojlim_{\mathbf{Set}} X, \pi_X)$ be the limit of $X_{\mathbf{Set}}$, obtained by forgeting from $\mathbf{Top}$ to $\mathbf{Set}$. Let $\varprojlim X$ be endowed with the topology generated by preimages of opens in components. Then for all $(Y, y) \in \mathrm{Obj}\,(\Delta \downarrow X_{\mathbf{Set}})$ where $Y \in \mathrm{Obj}\,(\mathbf{Top})$ and $\overline{y} \in \Delta \downarrow X((Y, y), (\varprojlim_{\mathbf{Set}} X, \pi_X))$, $\overline{y} \in \mathbf{Top}(Y, \varprojlim X)$ if and only if $y \in \mathbf{Top}^{\mathcal{I}}(\Delta(Y), X)$. Hence, $\pi_X \in \mathbf{Top}^{\mathcal{I}}(\Delta(\varprojlim X), X)$ and $(\varprojlim X, \pi_X)$ is a limit of $X$ in $\mathbf{Top}$.

Dually, similar trick to turn the colimit in $\mathbf{Set}$ to colimit in $\mathbf{Top}$ : take the topology generated by sets whose preimages are open in every component of the diagram $X$.

*Proof.* Straightforward. □

**Proposition – Grp Complete**

Let $I$ be a category and $X$ an $I$-shaped diagram in $\mathbf{Grp}$.

The construction $\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)$ from the limit of $X$ as a diagram in $\mathbf{Set}$ has a group structure by multiplying "objectwise" in the diagram $X$. This makes $\pi_X \in \mathbf{Grp}^{\mathcal{I}}(\Delta(\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)), X)$, and hence $\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)$ a limit of $X$ in $\mathbf{Grp}$.

Dually, we imitate the colimit construction in $\mathbf{Set}$ and consider the free product quotiented by the subgroup generated by "eventually being equal in $X$" :

$$\left(\bigsqcup_{i \in \mathcal{I}} X_i\right) / \langle (i, x_i)(j, x_j)^{-1} \mid \exists\, \phi \in \mathcal{I}(i, j), X(\phi)(x_i) = x_j \rangle$$

Then the above is a colimit of $X$.

*Proof.* Straightforward. □

*Remark.* We will show (co)completeness of modules over a ring first, then show (co)completeness of rings, since for the case of colimits, we will make use of the fact that $\mathbf{Ring}$ forgets into $\mathbf{Ab} = \mathbb{Z}\mathbf{Mod}$.

**Proposition – $R$Mod Complete**

Let $R \in \mathbf{Ring}$, $\mathcal{I}$ a category, $X$ an $\mathcal{I}$-diagram in $R\mathbf{Mod}$.

The construction $\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)$ from the limit of $X$ as a diagram in $\mathbf{Set}$ has an $R\mathbf{Mod}$ structure by adding and scalar multiplying "objectwise" in the diagram $X$. This gives the morphism $\pi_X \in R\mathbf{Mod}^{\mathcal{I}}(\Delta(\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)), X)$, and hence $\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)$ a limit of $X$ in $R\mathbf{Mod}$.

Dually, we imitate the colimit construction in $\mathbf{Set}$ and consider the direct sum quotiented by the left $R$-submodule generated by "eventually being equal in $X$" :

$$\left(\bigoplus_{i \in \mathcal{I}} X_i\right) / R\,\{(i, x_i) - (j, x_j) \mid \exists\, \phi \in \mathcal{I}(i, j), X(\phi)(x_i) = x_j\}$$

Then the above is a colimit for $X$ in $R\mathbf{Mod}$.

*Proof.* Straightforward. $\square$

**Proposition – Ring Complete**

Let $\mathcal{I}$ be a category and $X$ an $\mathcal{I}$-shaped diagram in $\mathbf{Ring}$.

The construction $\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)$ from the limit of $X$ as a diagram in $\mathbf{Set}$ has a ring structure by adding and multiplying "objectwise" in the diagram $X$. Then $\pi_X \in \mathbf{Ring}^{\mathcal{I}}(\Delta(\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)), X)$, and hence $\mathbf{Set}^{\mathcal{I}}(\Delta(1), X)$ a limit of $X$ in $\mathbf{Ring}$.

Dually, things get nasty.

*Proof.* (Nasty things go here) (TODO) $\square$

*Remark.* One may wonder : (1) Why are all the limits for these categories the same as in $\mathbf{Set}$? (2) Why do the colimits differ to $\mathbf{Set}$ for everything except $\mathbf{Top}$?

The answer is the related to adjunctions, and is the focus of the next section. In a nutshell : all of the following categories are subcategories of $\mathbf{Set}$ which admit free functors. This implies that the forgetful functor preserves limits. This answers the first question. This is also why colimits differ to the ones in $\mathbf{Set}$ *except* for the case of $\mathbf{Top}$, since the forgetful functor also admits a *right adjoint*, namely taking the indiscrete topology. This then implies the forgetful functor also preserves colimits. This answers the second question.

# 8 Continuity

**Definition – (Co)Continuous Functors**

Let $F : \mathcal{C} \to \mathcal{D}$. Then $F$ is called *continuous* when for all categories $\mathcal{I}$ and $\mathcal{I}$-shaped diagrams $X$ with a limit $(U, u)$, $(F(U), F(u))$ is a limit of the $\mathcal{I}$-shaped diagram $F(X) := F \circ X$. More succinctly by abuse of notation, $F(\varprojlim X) = \varprojlim F(X)$.

Dually, $F$ is called *cocontinuous* when or all categories $\mathcal{I}$ and $\mathcal{I}$-shaped diagrams $X$ with a colimit $(U, u)$, $(F(U, F(u)))$ is a colimit of the $\mathcal{I}$-shaped diagram $F \circ X$. More succinctly by abuse of notation, $F(\varinjlim X) = \varinjlim F(X)$.

*Remark.* The focus of this section is to prove that *right adjoints are continuous* and *left adjoints are cocontinuous*. This is achieved by the special case of the (co)Yoneda embedding.

What this says is that via adjunctions, one can transfer constructions in one category to another.

**Proposition – Naturality of Taking Constant Diagrams**

Let $\mathcal{I}$ be a category. Then the collection of $(\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}})_{\mathcal{C} \in \mathrm{Obj}(\mathbf{Cat})}$ is a natural transformation from $\mathbb{1}_{\mathbf{Cat}}$ to $\mathbf{Cat}(\mathcal{I}, -) = h^{\mathcal{I}}$.

*Proof.* This is simply the commutativity of the following square given a functor $F : \mathcal{C} \to \mathcal{D}$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\Delta"] & \mathcal{D} \arrow[d, "\Delta"] \\ \mathcal{C}^{\mathcal{I}} \arrow[r, "F\circ -"] & \mathcal{D}^{\mathcal{I}} \end{tikzcd}$$

$\square$

**Proposition – (Co)Continuity of the (Co)Yoneda Embedding**

Let $X$ be an $\mathcal{I}$-shaped diagram in a category $\mathcal{C}$. We can embed $X$ into $\mathbf{Set}^{\mathcal{C}^{op}}$ by the Yoneda embedding to get the $\mathcal{I}$-shaped diagram $h_X := \mathcal{C}(-, X)$. There is an obvious morphism $\pi_X \in \left(\mathbf{Set}^{\mathcal{C}^{op}}\right)^{\mathcal{I}} \left(\Delta(\mathcal{C}^{\mathcal{I}}(\Delta(-), X)), h_X\right)$.

Then $(\mathcal{C}^{\mathcal{I}}(\Delta(-), X), \pi_X)$ is a limit of $h_X$ and for every $U \in \mathrm{Obj}\,(\mathcal{C}^{op})$, the limit of $h_X(U)$ exists. [a] Hence for any $(U, u) \in \mathrm{Obj}\,(\Delta \downarrow X)$,

$$(U, u) \text{ is a limit of } X \Leftrightarrow (h_U, h_u) \text{ is a limit of } h_X$$

More succinctly with abuse of notation, $h_{\varprojlim X} = \varprojlim h_X$ if either limit exists (then they both exist).

Dually, we can embed $X$ into $\mathbf{Set}^{\mathcal{C}}$ by the Yoneda embedding to get the $\mathcal{I}^{op}$-shaped diagram $h^X := \mathcal{C}(X, -)$. There is again an obvious morphism $\pi_X \in \left(\mathbf{Set}^{\mathcal{C}}\right)^{\mathcal{I}} \left(\Delta(\mathcal{C}^{\mathcal{I}}(X, \Delta(-))), h^X\right)$.

Then $(\mathcal{C}^{\mathcal{I}}(X, \Delta(-)), \pi_X)$ is a limit of $h^X$ and for every $U \in \mathrm{Obj}\,(\mathcal{C})$, the limit of $h^X(U)$ exists. Hence, for any $(U, u) \in \mathrm{Obj}\,(X \downarrow \Delta)$,

$$(U, u) \text{ is a colimit of } X \Leftrightarrow (h^U, h^u) \text{ is a limit of } h^X$$

More succinctly with abuse of notation, $h^{\varinjlim X} = \varprojlim h^X$ if either the colimit or the limit exists (then they both exist).

[a] Note that technically, we don't need to carry around this second piece of data since completeness of $\mathbf{Set}$ implies completeness of $\mathbf{Set}^{\mathcal{I}}$. I have decided to keep it so that should the reader have decided to skip the completeness results of the previous section, the proofs in this section still carry through without completeness assumptions.

*Proof.* By the characterisation of limits of functors, it suffices to show that for every $V \in \mathrm{Obj}\,(\mathcal{C}^{op})$, we have that $(\mathcal{C}^{\mathcal{I}}(\Delta(V), X), (\pi_X)_V)$ is a limit of $h_X(V)$, where $(\pi_X)_V \in \mathbf{Set}^{\mathcal{I}}(\Delta(\mathcal{C}^{\mathcal{I}}(\Delta(V), X)), h_X(V))$ is obtained by restricting $\pi_X$ in the second argument to $V$. The essence is that "a cone over $X$ is nothing more than a compatible system of morphisms from the tip to $X$".

Let $(S, \sigma) \in \mathrm{Obj}\,(\Delta \downarrow \mathcal{C}(V, X))$. We seek a unique morphism $(S, \sigma) \to (\mathcal{C}^{\mathcal{I}}(\Delta(V), X), (\pi_X)_V)$. By function extensionality, a collection of set morphisms $(\{s\} \to \mathcal{C}^{\mathcal{I}}(\Delta(V), X))_{s \in S}$ glues uniquely to a function $S \to \mathcal{C}^{\mathcal{I}}(\Delta(V), X)$. It thus suffice to do the case of $S = \{*\}$, a singleton.

$$\begin{tikzcd} \Delta(\{*\}) \arrow[r, dashed, "\Delta(\varprojlim \sigma)"] \arrow[dr, "\sigma"'] & \Delta(\mathcal{C}^{\mathcal{I}}(\Delta(V), X)) \arrow[d, "(\pi_X)_V"] \\ & \mathcal{C}(V, X) \end{tikzcd}$$

Then $\sigma : \Delta(\{*\}) \to \mathcal{C}(V, X)$ is equivalent to the data of a morphism $\Delta(V) \to X$, i.e. an element of $\mathcal{C}^{\mathcal{I}}(\Delta(V), X)$. This gives a unique $\varprojlim \sigma : \{*\} \to \mathcal{C}^{\mathcal{I}}(\Delta(V), X)$ such that $(\pi_X)_V \circ \Delta(\varprojlim \sigma) = \sigma$ as desired. Hence, $(\mathcal{C}^{\mathcal{I}}(\Delta(-), X), \pi_X)$ is a limit of $h_X$.

Now, $(U, u)$ is a limit of $X$ if and only if $\mathcal{C}(-, U) \cong \mathcal{C}^{\mathcal{I}}(\Delta(-), X)$ where $\mathbb{1}_U$ corresponds to $u$, if and only if $(h_U, h_u)$ is isomorphic to $(\mathcal{C}^{\mathcal{I}}(\Delta(-), X), \pi_X)$ in $\Delta \downarrow h_X$, if and only if $(h_U, h_u)$ is a limit of $h_X$.

The dual argument is similar, the only minor discrepancy being the conversion of colimits into limits. □

*Remark.* Note that what we not just shown continuity of the Yoneda embedding, but also that "limits in its image pull back to limits".

> **Proposition – Right Adjoints Continuous, Left Adjoints Cocontinuous**
> Let $R : \mathcal{C} \to \mathcal{D}$, $L : \mathcal{D} \to \mathcal{C}$ be functors such that $(L, R)$ forms an adjunctions. Let $\mathcal{I}$ be a category. Then
>
> – Given an $\mathcal{I}$-shaped diagram $X$ in $\mathcal{C}$ with a limit $(U, u)$, $(R(U), R(u))$ is a limit of the diagram $R(X) := R \circ X$. More succinctly with abuse of notation, $R(\varprojlim X) = \varprojlim R(X)$.
>
> – Given an $\mathcal{I}$-shaped diagram $X$ in $\mathcal{D}$ with colimit $(U, u)$, $(L(U), L(u))$ is a colimit of the diagram $L(X) := L \circ X$. More succinctly with abuse of notation, $L(\varinjlim X) = \varinjlim L(X)$.

*Proof.* Consider the following not necessarily commutative diagram :

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h_\star"] \arrow[d, "R"'] & \mathbf{Set}^{\mathcal{C}^{op}} \arrow[d, "(-\circ L^{op})"] \\ \mathcal{D} \arrow[r, "h_\star"'] & \mathbf{Set}^{\mathcal{D}^{op}} \end{tikzcd}$$

Starting on the top left, putting $u : \Delta(U) \to X$ through to the bottom right via the two different paths yields the following square (using naturality of constant diagrams along the way):

$$\begin{tikzcd} \Delta(h_{R(U)}) \arrow[r, "\sim"] \arrow[d, "h_{R(u)}"] & \Delta(h_U \circ L^{op}) \arrow[d, "h_u \circ L^{op}"] \\ h_{R(X)} \arrow[r, "\sim"] & h_X \circ L^{op} \end{tikzcd}$$

The horizontal isomorphisms come from the adjunction and the square commutes by naturality of the adjunction isomorphism. For $(R(U), R(u))$ to be a limit of $R(X)$, by continuity of the Yoneda embedding it suffices that $(h_{R(U)}, h_{R(u)})$ is a limit of $h_{R(X)}$. Then by the horizontal isomorphisms, it suffice that $(h_U \circ L^{op}, h_u \circ L^{op})$ is a limit of $h_X \circ L^{op}$.

By the characterisation of limits of functors, it suffices for every $V \in \text{Obj}(\mathcal{D}^{op})$ that the corresponding component $(h_U(L^{op}(V)), (h_u \circ L^{op})_V)$ is a limit of $h_X(L^{op}(V))$. Well, since $(U, u)$ is a limit of $X$, by continuity of the Yoneda embedding, we have $(h_U, h_u)$ is a limit of $h_X$ and for every $V \in \text{Obj}(\mathcal{C}^{op})$, the limit of $h_X(V)$ exists. Then again by the characterisation of limits of functors, we have in fact that the components of $(h_U, h_u)$ are limits of the components of $h_X$. In particular, for any $V \in \text{Obj}(\mathcal{D}^{op})$, $(h_U(L^{op}(V)), (h_u)_{L^{op}(V)})$ is a limit of $h_X(L^{op}(V))$ as desired.

The dual argument is similar. $\square$

*Remark.* The usual proof of the above given in texts is the following line of "equalities" :

$$\mathcal{D}(-, R(\varprojlim X)) = \mathcal{C}(L(-), \varprojlim X) = \varprojlim \mathcal{C}(L(-), X) = \varprojlim \mathcal{D}(-, R(X))$$

from which one concludes $R(\varprojlim X) = \varprojlim R(X)$ by continuity of the Yoneda embedding. The second and third "equality" can be obtained if one equates all limits of the same diagram.

However, the first equality is just a flat out lie. It corresponds to the top horizontal isomorphism in the second square of the proof given here. The usual proof can then be remedied by attaching an extra "equality" at the end "$= \mathcal{D}(-, \varprojlim R(X))$", for then by the Yoneda embedding being fully faithful, one obtains an isomorphism $R(\varprojlim X) \cong \varprojlim R(X)$. This does indeed prove that "$R(\varprojlim X)$" is a limit, however, we have lost the information about the morphism to the diagram $R(X)$. There is still work to be done to show that the morphism from $\Delta(R(\varprojlim X))$ to $R(X)$ is really the image of the original morphism $\pi_X : \Delta(\varprojlim X) \to X$ under $R$. This work is precisely the commutativity of the second square in the proof given here.

**Corollary – (Co)Limits commute with (Co)Limits**
(TODO)

**Definition – Filtered Categories, Diagrams, Colimits**
(TODO)

**Proposition – Characterisation of Filtered Diagrams in terms of Set**
(TODO)