# Notes Category Theory

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## Contents



In this paper, we do not pay attention to foundational set-theoretic issues like the "set" of all sets should be a class.

## 1 Categories

**Definition – Categories**

A *category* $\mathcal{C}$ is defined by the following data :

1. A set of *objects*, $\mathrm{Obj}\,(\mathcal{C})$.
2. For every $U, V \in \mathrm{Obj}\,(\mathcal{C})$, a set of $\mathcal{C}$-*morphisms* from $U$ to $V$, denoted $\mathcal{C}\,(U, V)$. We denote $f : U \xrightarrow{\mathcal{C}} V$ for $f \in \mathcal{C}\,(U, V)$.
3. For every $U, V, W \in \mathrm{Obj}\,(\mathcal{C})$, $f : U \xrightarrow{\mathcal{C}} V$ and $g : V \xrightarrow{\mathcal{C}} W$, a $\mathcal{C}$-morphism called the *composition of $f$ with $g$*, denoted $g \circ f : U \xrightarrow{\mathcal{C}} W$.
4. Associativity of $\circ$.

5. For every $U \in \operatorname{Obj}(\mathcal{C})$, an *identity morphism* $\mathbb{1}_U : U \xrightarrow{\mathcal{C}} U$.
6. For all $U, V, W \in \operatorname{Obj}(\mathcal{C})$, $f : U \xrightarrow{\mathcal{C}} V$ and $g : W \xrightarrow{\mathcal{C}} U$, we have $f \circ \mathbb{1}_U = f$ and $\mathbb{1}_U \circ g = g$.

**Definition – Isomorphisms**
Let $\mathcal{C}$ be a category, $U, V \in \operatorname{Obj}(\mathcal{C})$, $f : U \xrightarrow{\mathcal{C}} V$. Then $f$ is called an *isomorphism* when there exists $g : V \xrightarrow{\mathcal{C}} U$ such that $g \circ f = \mathbb{1}_U$ and $f \circ g = \mathbb{1}_V$. In this case, we denote $f : U \underset{\sim}{\xrightarrow{\mathcal{C}}} V$. When there exists an isomorphism from $U$ to $V$, we say they are *isomorphic* and write $U \cong V$.

**Definition – Subcategories**
Let $\mathcal{C}, \mathcal{D}$ be categories. Then $\mathcal{D}$ is called a *subcategory of* $\mathcal{C}$ when $\operatorname{Obj}(\mathcal{D}) \subseteq \operatorname{Obj}(\mathcal{C})$ and for all $U, V \in \operatorname{Obj}(\mathcal{D})$, $\mathcal{D}(U, V) \subseteq \mathcal{C}(U, V)$.

*Example (Standard Categories).*

1. $\mathbf{Set}$ *denotes the* category of sets, *where* $\operatorname{Obj}(\mathbf{Set})$ *contains sets and for* $U, V \in \operatorname{Obj}(\mathbf{Set})$, $\mathbf{Set}(U, V)$ *is the set of maps from* $U$ *to* $V$.
2. $\mathbf{Top}$ *denotes the* category of topological spaces, *where* $\operatorname{Obj}(\mathbf{Top})$ *contains topological spaces and for* $U, V \in \operatorname{Obj}(\mathbf{Top})$, $\mathbf{Top}(U, V)$ *is the set of continuous maps from* $U$ *to* $V$. $\mathbf{Top}$ *is a subcategory of* $\mathbf{Set}$.
3. *The* category of groups $\mathbf{Grp}$ *has* $\operatorname{Obj}(\mathbf{Grp})$ *containing groups and* $\mathbf{Grp}(U, V)$ *containing group homomorphisms from* $U$ *to* $V$. $\mathbf{Grp}$ *is a subcategory of* $\mathbf{Set}$.
4. *The* category of abelian groups $\mathbf{Ab}$ *has* $\operatorname{Obj}(\mathbf{Ab})$ *containing abelian groups and* $\mathbf{Ab}(U, V)$ *containing group homomorphisms from* $U$ *to* $V$. $\mathbf{Ab}$ *is a subcategory of* $\mathbf{Grp}$.
5. *The* category of rings $\mathbf{Ring}$ *has* $\operatorname{Obj}(\mathbf{Ring})$ *containing rings and* $\mathbf{Ring}(U, V)$ *containing ring homomorphisms from* $U$ *to* $V$. $\mathbf{Ring}$ *is a subcategory of* $\mathbf{Set}$.
6. *The* category of commutative rings $\mathbf{CRing}$ *has* $\operatorname{Obj}(\mathbf{CRing})$ *containing commutative rings and* $\mathbf{CRing}(U, V)$ *containing ring homomorphisms from* $U$ *to* $V$. $\mathbf{CRing}$ *is a subcategory of* $\mathbf{Ring}$.
7. *Let* $R$ *be a commutative ring. Then the* category of $R$-modules $R$-$\mathbf{Mod}$ *has* $\operatorname{Obj}(R\text{-}\mathbf{Mod})$ *containing* $\mathbb{R}$*-modules and* $R$-$\mathbf{Mod}(U, V)$ *contains* $R$*-linear maps from* $U$ *to* $V$*. This is a subcategory of* $\mathbf{Ab}$.
8. *Let* $R$ *be a commutative ring. Then the* category of $R$-algebras $R$-$\mathbf{Alg}$ *has* $\operatorname{Obj}(R\text{-}\mathbf{Alg})$ *containing pairs* $(S, \sigma)$ *where* $\sigma : R \xrightarrow{\mathbf{CRing}} S$. $R$-$\mathbf{Alg}((U, u), (V, v))$ *contains* $f : U \xrightarrow{\mathbf{CRing}} V$ *such that* $f \circ u = v$.

*Example (Preordered Sets as Categories).*
*Let* $I$ *be a set,* $\leq$ *a relation on* $I$*. Then* $(I, \leq)$ *is called a* preordered set *when* $\leq$ *satisfies all of the following :*

1. (*Reflexivity*) *For all* $i \in I$, $i \leq i$.
2. (*Transitivity*) *For all* $i, j, k \in I$, $i \leq j$ *and* $j \leq k$ *implies* $i \leq k$.

*If* $(I, \leq)$ *is a preordered set where* $\leq$ *is clear, we abbreviate to* $I$.

*Let* $I$ *be a preordered set. Then we can turn* $I$ *into a category as follows :*

1. $\mathrm{Obj}\,(I)$ *is* $I$.
2. *For* $i, j \in \mathrm{Obj}\,(I)$, $I(i, j)$ *is singleton when* $i \leq j$ *and empty otherwise.*

*Things get meta. We can form the* category of preordered sets $\mathbf{Ord}$ *where* $\mathrm{Obj}\,(\mathbf{Ord})$ *contains preoredered sets and* $\mathbf{Ord}(I, J)$ *contains* $f : I \xrightarrow{\mathbf{Set}} J$ *such that for all* $i, j \in I$, $i \leq j$ *implies* $f(i) \leq f(j)$.

*Example* (*Category of Partially Ordered Sets*).
*Let* $I \in \mathrm{Obj}\,(\mathbf{Ord})$. *Then* $I$ *is called a* partially ordered set *when* $\leq$ *is* antisymmetric, *i.e. for all* $i, j \in I$, $i \leq j$ *and* $j \leq i$ *implies* $i = j$. *We thus have the* category of partially ordered sets $\mathbf{PoSet}$ *where* $\mathrm{Obj}\,(\mathbf{PoSet})$ *contains partially ordered sets and* $\mathbf{PoSet}(I, J) = \mathbf{Ord}(I, J)$. *We see that* $\mathbf{PoSet}$ *is a subcategory of* $\mathbf{Ord}$.

*Example* (*Partially Ordered Sets*).

1. *Let* $X$ *be a set. Then its powerset* $2^X \in \mathrm{Obj}\,(\mathbf{PoSet})$.
2. *Let* $X$ *be a topological space. Then the set of its open sets* $\mathrm{Open}\,X$ *is a partially ordered set.*
3. *Let* $G$ *be a group. Then the set of its subgroups is in* $\mathrm{Obj}\,(\mathbf{PoSet})$.
4. *Let* $R$ *be a commutative ring and* $M$ *be an* $R$*-module. Then the set of* $R$*-submodules of* $M$ *is in* $\mathrm{Obj}\,(\mathbf{PoSet})$.
5. *Let* $R$ *be a commutative ring and* $(S, \sigma)$ *an* $R$*-algebra. Then the set of all* $R$*-subalgebras of* $S$ *is in* $\mathrm{Obj}\,(\mathbf{PoSet})$.
6. *Consider the relation on* $\mathbb{N}$ *that is* $a \mid b$. *This is a partial order on* $\mathbb{N}$.

*Example* (*A Group as a Category*).
*A group* $G$ *is equivalent to a category* $G$ *where there is only one object* $\bullet$ *and all morphisms are isomorphisms.*

*A direct generalization is a* groupoid *: a category where every morphism is an isomorphism.*

# 2 Functors

**Definition – Functors**

Let $\mathcal{C}, \mathcal{D}$ be categories. Then a *functor* $F$ *from* $\mathcal{C}$ *to* $\mathcal{D}$ is defined by the following data :

1. A map of objects $\mathrm{Obj}\,(\mathcal{C}) \to \mathrm{Obj}\,(\mathcal{D})$, which we will denote by the same name $F$.
2. A map of morphisms for all $U, V \in \mathrm{Obj}\,(\mathcal{C})$, $\mathcal{C}\,(U, V) \to \mathcal{D}\,(F(U), F(V))$, which we will also

denote by the same name $F$.

3. (Compositions are Preserved) For all $f : U \xrightarrow{\mathcal{C}} V$ and $g : V \xrightarrow{\mathcal{C}} W$, $F(g \circ f) = F(g) \circ F(f)$.
4. (Identity Morphisms are Preserved) For all $U \in \mathrm{Obj}(\mathcal{C})$, $F(\mathbb{1}_U) = \mathbb{1}_{F(U)}$.

**Definition – Category of Categories**
We define the *category of categories* $\mathbf{Cat}$,

1. $\mathrm{Obj}(\mathbf{Cat})$ consists of categories.
2. For $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathbf{Cat})$, $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ consists of functors from $\mathcal{C}$ to $\mathcal{D}$.
3. For $\mathcal{C} \in \mathrm{Obj}(\mathbf{Cat})$, $\mathbb{1}_{\mathcal{C}}$ is the obvious thing.

**Definition – Faithful, Full, Fully Faithful**
Let $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then $F$ is called

1. *faithful* when for all $U, V \in \mathrm{Obj}(\mathcal{C})$, $F : \mathcal{C}(U, V) \to \mathcal{D}(F(U), F(V))$ is injective.
2. *full* when for all $U, V \in \mathrm{Obj}(\mathcal{C})$, $F : \mathcal{C}(U, V) \to \mathcal{D}(F(U), F(V))$ is surjective.
3. *fully faithful* when for all $U, V \in \mathrm{Obj}(\mathcal{C})$, $F : \mathcal{C}(U, V) \to \mathcal{D}(F(U), F(V))$ is bijective.

**Proposition – Fully Faithful Functors are Injective**
Let $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ be fully faithful, $U, V \in \mathrm{Obj}(\mathcal{C})$ such that $F(U) \cong F(V)$. Then $U \cong V$.

*Proof.* Let $f_1 \in \mathcal{D}(F(U), F(V))$ and $f_2 \in \mathcal{D}(F(V), F(U))$ such that $\mathbb{1}_{F(U)} = f_2 \circ f_1$ and $\mathbb{1}_{F(V)} = f_1 \circ f_2$. Then $f_1, f_2$ corresponds respectively to $g_1, g_2 \in \mathcal{C}(U, V), \mathcal{C}(V, U)$ through $F$. We thus have

$$F(g_2 \circ g_1) = F(g_2) \circ F(g_1) = f_2 \circ f_1 = \mathbb{1}_{F(U)} = F(\mathbb{1}_U)$$

which by $F$ fully faithful gives $g_2 \circ g_1 = \mathbb{1}_U$. Similarly, $g_1 \circ g_2 = \mathbb{1}_V$. □

**Definition – Natural Transformations**
Let $F, G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then a *natural transformation* $\eta$ *from* $F$ *to* $G$ is defined by the following data :

1. For all $U \in \mathrm{Obj}(\mathcal{C})$, $\eta_U : F(U) \xrightarrow{\mathcal{D}} G(U)$.
2. (Naturality) For all $U, V \in \mathrm{Obj}(\mathcal{C})$ and $f : U \xrightarrow{\mathcal{C}} V$, we have the following commutative diagram.

$$\begin{tikzcd} F(U) \arrow[r, "\eta_U"] \arrow[d, "F(f)"'] & G(U) \arrow[d, "G(f)"] \\ F(V) \arrow[r, "\eta_V"'] & G(V) \end{tikzcd}$$

**Definition – Category of Functors**
Let $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}\,(\mathbf{Cat})$. Then the *category of functors from $\mathcal{C}$ to $\mathcal{D}$*, denoted $\mathcal{D}^{\mathcal{C}}$, is defined by

1. $\mathrm{Obj}\left(\mathcal{D}^{\mathcal{C}}\right) := \mathbf{Cat}\,(\mathcal{C}, \mathcal{D})\,.$
2. For all $F, G \in \mathrm{Obj}\left(\mathcal{D}^{\mathcal{C}}\right)$, $\mathcal{D}^{\mathcal{C}}\,(F, G)\ :=$ the set of natural transformations from $F$ to $G$.
3. For all $F \in \mathrm{Obj}\left(\mathcal{D}^{\mathcal{C}}\right)$, $\mathbb{1}_F$ is the obvious thing.

**Definition – Equivalence of Categories**
Let $\mathcal{C}, \mathcal{D}$ be categories, $F \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$. Then $F$ is called an *equivalence of categories* when there exists $G \in \mathbf{Cat}(\mathcal{D}, \mathcal{C})$ such that $G \circ F \cong \mathbb{1}_{\mathcal{C}}$ and $F \circ G \cong \mathbb{1}_{\mathcal{D}}$.

**Definition – Essentially Surjective**
Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then $F$ is called *essentially surjective* when for all $X \in \mathrm{Obj}\,(\mathcal{D})$, there exists $U \in \mathrm{Obj}\,(\mathcal{C})$ such that $F(U) \cong X$.

**Proposition – Characterisation of Equivalence of Categories**
Let $F : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then $F$ is an equivalence of categories if and only if $F$ is fully faithful and essentially surjective.

# 3 Universal Morphisms

**Definition – Comma Category**
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $X \in \mathrm{Obj}\,(\mathcal{D})$. Then the *comma category $X \downarrow G$* is defined as follows.

1. $\mathrm{Obj}\,(X \downarrow G)$ consists of pairs $(U, u)$ where $U \in \mathrm{Obj}\,(\mathcal{C})$ and $u : X \xrightarrow{\mathcal{D}} G(U)$.
2. For $(U, u), (V, v) \in \mathrm{Obj}\,(X \downarrow G)$, $X \downarrow G\,((U, u), (V, v))$ consists of $f : U \xrightarrow{\mathcal{C}} V$ such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "v"'] & G(U) \arrow[d, "G(f)"] \\ & G(V) \end{tikzcd}$$

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$ and $U \in \mathrm{Obj}\,(\mathcal{C})$. Then the *comma category $F \downarrow U$* is defined as follows.

1. $\mathrm{Obj}\,(F \downarrow U)$ consists of pairs $(X, x)$ where $X \in \mathrm{Obj}\,(\mathcal{D})$ and $x : F(X) \xrightarrow{\mathcal{C}} U$.
2. For $(X,x), (Y,y) \in \mathrm{Obj}\,(F \downarrow U)$, $X \downarrow F\left((X,x),(Y,y)\right)$ consists of $g : X \xrightarrow{\mathcal{D}} Y$ such that

$$\begin{tikzcd} F(X) \arrow[d, "F(g)"'] \arrow[dr, "x"] & \\ F(Y) \arrow[r, "y"'] & U \end{tikzcd}$$

*Remark.* Here is a special case of the comma category worth noting.

**Definition – Over Category**
Let $\mathcal{C}$ be a category and $U \in \mathrm{Obj}\,(\mathcal{C})$. Then the *over category* $\mathcal{C} \downarrow U$ is defined as $\mathbb{1}_{\mathcal{C}} \downarrow U$.

Dually, the *under category* $U \downarrow \mathcal{C}$ is defined as $U \downarrow \mathbb{1}_{\mathcal{C}}$.

*Example* (*Over and Under Categories*). *1. Let $R \in \mathrm{Obj}\,(\mathbf{Ring})$. Then $R$-$\mathbf{Alg} = R \downarrow \mathbf{Ring}$.*

*2. Let $X \in \mathrm{Obj}\,(\mathbf{Top})$. Then we have the category of covering spaces of $X$ with is the subcategory of $\mathbf{Top} \downarrow X$ where objects are $(\tilde{X}, p)$ with $p$ a covering map.*

**Definition – Universal Morphism**
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $X \in \mathrm{Obj}\,(\mathcal{D})$. Then a *universal morphism from $X$ to $G$* is the following data.

1. An object $(F(X), \eta_X)$ of the comma category $X \downarrow G$.
2. (Universal Property) For all $(V, v) \in \mathrm{Obj}\,(X \downarrow G)$, there exists a unique morphism $(F(X), \eta_X) \xrightarrow{X \downarrow G} (V, v)$.

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$ and $U \in \mathrm{Obj}\,(\mathcal{C})$. Then a *universal morphism from $F$ to $U$* is the following data.

1. An object $(G(U), \varepsilon_U)$ of the comma category $F \downarrow U$.
2. (Universal Property)For all $(Y, y) \in \mathrm{Obj}\,(F \downarrow U)$, there exists a unique morphism $(Y, y) \xrightarrow{F \downarrow U} (G(U), \varepsilon_U)$.

**Proposition – Unique up to Unique Isomorphism**
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $X \in \mathrm{Obj}\,(\mathcal{D})$, $(U, u), (V, v) \in X \downarrow G$ both universal morphisms from $X$ to $G$. Then there exists a unique $f : (U, u) \xrightarrow{X \downarrow G} (V, v)$ and $g : (V, v) \xrightarrow{X \downarrow G} (U, u)$ such that $g \circ f = \mathbb{1}_{(U,u)}$ and $f \circ g = \mathbb{1}_{(V,v)}$. Thus, if a universal morphism exists, we say it is *unique up to unique isomorphism*.

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $U \in \mathrm{Obj}\,(\mathcal{C})$, $(X, x), (Y, y) \in F \downarrow U$ both universal morphisms from

$F$ to $U$. Then there exists a unique $f : (X,x) \xrightarrow{F\downarrow U} (Y,y)$ and $g : (Y,y) \xrightarrow{F\downarrow u} (X,x)$ such that $g \circ f = \mathbb{1}_{(X,x)}$ and $f \circ g = \mathbb{1}_{(Y,y)}$.

*Proof.* (Shorter proof that does not go through Yoneda).

By the universal property of $(U,u)$, There exists a unique $f : (U,u) \xrightarrow{X\downarrow G} (V,v)$. Similarly, there exists a unique $g : (Y,y) \xrightarrow{F\downarrow u} (X,x)$. But then $g \circ f : (U,u) \xrightarrow{X\downarrow G} (U,u)$. By applying the universal property of $(U,u)$ with itself, we see that $\mathbb{1}_{(U,u)}$ is the unique $(U,u) \xrightarrow{X\downarrow G} (U,u)$. In particular, we have $g \circ f = \mathbb{1}_{(U,u)}$. Similarly, we have $f \circ g = \mathbb{1}_{(V,v)}$. Since $f$ and $g$ are the *only* morphisms between $(U,u)$, $(V,v)$, they are *the* unique isomorphism between $(U,u)$ and $(V,v)$. □

*Remark – "Canonically Isomorphic".* It is common in category theory and maths at large to *equate* two objects that satisfy the same universal property, since they are not only isomorphic, but also isomorphic in a unique way. Some also call these *canonically isomorphic*.

**Proposition – Isomorphic to Universal implies Universal**
Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $X \in \mathrm{Obj}(\mathcal{D})$, $(U,u),(V,v) \in X \downarrow G$ where $(U,u) \cong_{X\downarrow G} (V,v)$ and $(U,u)$ is a universal morphism from $X$ to $G$. Then $(V,v)$ is a universal morphism from $X$ to $G$.

Dually, let $F : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $U \in \mathrm{Obj}(\mathcal{C})$, $(X,x),(Y,y) \in F \downarrow U$ where $(X,x) \cong_{F\downarrow U} (Y,y)$ and $(X,x)$ is a universal morphism from $F$ to $U$. Then $(Y,y)$ is a universal morphism from $F$ to $U$.

*Proof.* Let $f : (U,u) \xrightarrow[\sim]{X\downarrow G} (V,v)$. Let $(W,w) \in \mathrm{Obj}(X \downarrow G)$. Then $f$ induces a bijection between $X \downarrow G\left((U,u),(W,w)\right)$ and $X \downarrow G\left((V,v),(W,w)\right)$. Since the former is singleton, so is the latter.

The dual has a similar argument. □

# 4 Yoneda's Lemma

**Definition – Dual Categories**
Let $\mathcal{C} \in \mathrm{Obj}(\mathbf{Cat})$. Then the *dual category of* $\mathcal{C}$, denoted $\mathcal{C}^{op}$, is defined by :

1. $\mathrm{Obj}(\mathcal{C}^{op}) := \mathrm{Obj}(\mathcal{C})$.
2. For all $U, V \in \mathrm{Obj}(\mathcal{C}^{op})$, $\mathcal{C}^{op}(U,V) := \mathcal{C}(V,U)$.

**Definition – Contravariant Functors**
Let $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathbf{Cat})$. Then a *contravariant functor from* $\mathcal{C}$ *to* $\mathcal{D}$ is just a functor $\mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Functors $\mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ are henceforth called *covariant functors from* $\mathcal{C}$ *to* $\mathcal{D}$.

**Definition – Morphism Functor**
Let $\mathcal{C}$ be a category and $U \in \mathrm{Obj}(\mathcal{C})$. Then $h_U : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$ is defined as :

1. For all $V \in \mathrm{Obj}(\mathcal{C}^{op})$, $h_U(V) := \mathcal{C}(V,U)$.

2. For all $V, W \in \mathrm{Obj}\left(\mathcal{C}^{op}\right)$ and $f : V \xrightarrow{\mathcal{C}^{op}} W$, $h_U(f) : h_U(V) \to h_U(W), g \mapsto g \circ f$.

Similarly, $h^U : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$ is defined as :

1. For all $V \in \mathrm{Obj}\left(\mathcal{C}\right)$, $h^U(V) := \mathcal{C}\left(U, V\right)$.
2. For all $V, W \in \mathrm{Obj}\left(\mathcal{C}\right)$ and $f : V \xrightarrow{\mathcal{C}} W$, $h^U(f) : h^U(V) \to h^U(W), g \mapsto f \circ g$.

**Proposition – Morphism Functor is Functorial**

Let $\mathcal{C}$ be a category. Then $h_\star : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}^{op}}$. Similarly, $h^\star : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}}$.

*Remark – Functor of Points.* Because of its relevance in algebraic geometry, $h_U$ is called the *functor of points of* $U$.

**Proposition – Yoneda's Lemma**

Let $\mathcal{C}$ be a category. Then $h_\star : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}^{op}}$ is fully faithful. Since fully faithful functors are injective, $h_\star$ is called the *Yoneda embedding*.

More generally, for any $U \in \mathrm{Obj}\left(\mathcal{C}\right)$ and $F \in \mathrm{Obj}\left(\mathbf{Set}^{\mathcal{C}^{op}}\right)$, $\mathbf{Set}^{\mathcal{C}^{op}}\left(h_U, F\right)$ bijects with $F(U)$ via $s \mapsto s_U(\mathbb{1}_U)$ and this bijection is natural in both $U$ and $F$.

Dually, $h^\star : \mathcal{C}^{op} \xrightarrow{\mathbf{Cat}} \mathbf{Set}^{\mathcal{C}}$ is fully faithful and more generally, for any $U \in \mathrm{Obj}\left(\mathcal{C}^{op}\right)$ and $F \in \mathrm{Obj}\left(\mathbf{Set}^{\mathcal{C}}\right)$, $\mathbf{Set}^{\mathcal{C}}\left(h^U, F\right)$ naturally bijects with $F(U)$ via $s \mapsto s_U(\mathbb{1}_U)$.

*Proof.* We first prove the general statement. Let $U \in \mathrm{Obj}\left(\mathcal{C}\right)$ and $F \in \mathrm{Obj}\left(\mathbf{Set}^{\mathcal{C}^{op}}\right)$. To show injectivity, let $s, t \in \mathbf{Set}^{\mathcal{C}^{op}}\left(h_U, F\right)$ and assume $s_U(\mathbb{1}_U) = t_U(\mathbb{1}_U)$. To show $s = t$, let $W \in \mathrm{Obj}\left(\mathcal{C}\right)$, $f \in h_U(W)$ and consider the following commutative diagram,

$$
\begin{tikzcd}
h_U(U) \arrow[rrr, "s_U"] \arrow[ddd, "h_U(f)"'] & & & F(U) \arrow[ddd, "F(f)"] \\
& \mathbb{1}_U \arrow[r, mapsto] \arrow[d, mapsto] & s_U(\mathbb{1}_U) \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & s_W(f) = F(f)(s_U(\mathbb{1}_U)) & \\
h_U(W) \arrow[rrr, "s_W"'] & & & F(W)
\end{tikzcd}
$$

By considering an analogous for $t$, we get $s_W(f) = t_W(f)$. So $s_W = t_W$, and hence $s = t$. To show surjectivity, let $x \in F(U)$. Define $s \in \mathbf{Set}^{\mathcal{C}^{op}}\left(h_U, F\right)$ by for all $V \in \mathrm{Obj}\left(\mathcal{C}\right)$,

$$s_V : f \in h_U(V) \mapsto F(f)(x) \in F(V)$$

Then by the above diagram, $s_U(\mathbb{1}_U) = x$. This proves the desired bijection.

For naturality in the first component, let $f : U \xrightarrow{\mathcal{C}} V$. Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{op}}(h_U, F) \arrow[rrr] & & & F(U) \\
& s \circ h^f \arrow[r, mapsto] & (s \circ h^f)_U(\mathbb{1}_U) = s_V(f) = F(f) \circ s_U(\mathbb{1}_U) & \\
& s \arrow[r, mapsto] \arrow[u, mapsto] & s_U(\mathbb{1}_U) \arrow[u, mapsto] & \\
\mathbf{Set}^{\mathcal{C}^{op}}(h_V, F) \arrow[rrr] \arrow[uuu, "(\star \circ h^f)"] & & & F(V) \arrow[uuu, "F(f)"']
\end{tikzcd}$$

For naturality in the second component, let $\alpha : F \xrightarrow{\mathbf{Set}^{\mathcal{C}^{op}}} G$. Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{op}}(h_U, F) \arrow[rrr] \arrow[ddd, "(\alpha \circ \star)"'] & & & F(U) \arrow[ddd, "\alpha_U"] \\
& s \arrow[r, mapsto] \arrow[d, mapsto] & s_U(\mathbb{1}_U) \arrow[d, mapsto] & \\
& \alpha \circ s \arrow[r, mapsto] & (\alpha \circ s)_U(\mathbb{1}_U) = \alpha_U \circ s_U(\mathbb{1}_U) & \\
\mathbf{Set}^{\mathcal{C}^{op}}(h_U, G) \arrow[rrr] & & & G(U)
\end{tikzcd}$$

We thus have the desired result.

To show $h_\star$ is fully faithful, let $U, V \in \mathrm{Obj}(\mathcal{C})$ and apply the above to $F := h_V$. □

> **Definition – Representable Functors**
> Let $G : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ be a covariant functor. Then a *representation of* $G$ is a $(U, u) \in h^\star \downarrow G$ where $u : h^\star \xrightarrow[\sim]{\mathbf{Set}^{\mathcal{C}}} G$.
>
> Dually, let $F : \mathcal{C}^{op} \longrightarrow \mathcal{D}$ be a contravariant functor. Then a *representation of* $F$ is a $(U, u) \in h_\star \downarrow F$ where $u : h_\star \xrightarrow[\sim]{\mathbf{Set}^{\mathcal{C}^{op}}} F$.
>
> A functor (covariant or contravariant) that has a representation is called *representable*.

*Remark.* If a functor has a representation, Yoneda's lemma implies it is canonical. This is the next result.

Before this, we first relate universal morphisms to representable functors. This is important as it leads to the notion of *adjunction*.

> **Proposition – Universal iff Represents**
> Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $X \in \operatorname{Obj}(\mathcal{D})$, $(L(X), \eta_X) \in \operatorname{Obj}(X \downarrow R)$. Then the following are equivalent :
>
> 1. $(L(X), \eta_X)$ is a universal morphism from $X$ to $R$.
> 2. $L(X)$ represents the covariant functor $\mathcal{D}(X, R(\star))$ and $\mathbb{1}_{L(X)}$ corresponds to $\eta_X$.
>
> Dually, let $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, $U \in \operatorname{Obj}(\mathcal{C})$, $(R(U), \varepsilon_U) \in \operatorname{Obj}(L \downarrow U)$. Then the following are equivalent :
>
> 1. $(R(U), \varepsilon_U)$ is a universal morphism from $L$ to $U$.
> 2. $R(U)$ represents the contravariant functor $\mathcal{C}(L(\star), U)$ and $\mathbb{1}_{R(U)}$ corresponds to $\varepsilon_U$.

*Proof.* (Universal implies Represents) Let $(L(X), \eta_X)$ be a universal morphism from $X$ to $R$. Define the following natural transformation,

$$\begin{aligned} &h^{L(X)} \xrightarrow{\mathbf{Set}^{\mathcal{C}}} \mathcal{D}(X, R(\star)) := \\ &W \in \operatorname{Obj}(\mathcal{C}) \mapsto \left[ f \in h^{L(X)}(W) \mapsto R(f) \circ \eta_X \in \mathcal{D}(X, R(W)) \right] \end{aligned}$$

Then for every $W \in \operatorname{Obj}(\mathcal{C})$, this is an isomorphism between $h^{L(X)}(W)$ and $\mathcal{D}(X, R(W))$, and hence a natural isomorphism. Indeed, $\mathbb{1}_{L(X)}$ corresponds to $\eta_X$ under this natural isomorphism.

(Represents implies Universal) Let $\alpha : h^{L(X)} \xrightarrow{\mathbf{Set}^{\mathcal{C}}} \mathcal{D}(X, R(\star))$ be a natural isomorphism where at $L(X)$, $\alpha_{L(X)}(\mathbb{1}_{L(X)}) = \eta_X$. Let $(V, v) \in \operatorname{Obj}(X \downarrow R)$. For any $f : L(X) \xrightarrow{\mathcal{C}} V$, consider the following commutative diagram.

$$\begin{tikzcd}
h^{L(X)}(L(X)) \arrow[rrr, "\alpha_{L(X)}"] \arrow[ddd, "h^{L(X)}(f)"'] & & & \mathcal{D}(X, RL(X)) \arrow[ddd, "\mathcal{D}(X, R(f))"] \\
& \mathbb{1}_{L(X)} \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_{L(X)}(\mathbb{1}_{L(X)}) = \eta_X \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \alpha_V(f) = R(f) \circ \eta_X & \\
h^{L(X)}(V) \arrow[rrr, "\alpha_V"'] & & & \mathcal{D}(X, R(V))
\end{tikzcd}$$

Thus $f : (L(X), \eta_X) \xrightarrow{X \downarrow R} (V, v)$ if and only if $\alpha_V(f) = v$. Then $\alpha_V^{-1}(v)$ is the unique morphism $(L(X), \eta_X) \xrightarrow{X \downarrow R} (V, v)$. Since there exists a unique $(L(X), \eta_X) \xrightarrow{X \downarrow R} (V, v)$, $(L(X), \eta_X)$ is universal.

The dual equivalence has an analogous proof. □

**Proposition – Canonical Representation**
Let $G : \mathcal{C} \xrightarrow{\textbf{Cat}} \mathcal{D}$ and $(U, u) \in h^\star \downarrow G$. Then the following are equivalent :

1. $(U, u)$ is a representation of $G$.
2. $(U, u)$ is a universal morphism from $h^\star$ to $G$.

In particular, representations of $G$ are canonically isomorphic.

Dually, let $F : \mathcal{C}^{op} \xrightarrow{\textbf{Cat}} \mathcal{D}$ and $(V, v) \in h_\star \downarrow F$. Then the following are equivalent :

1. $(V, v)$ is a representation of $F$.
2. $(V, v)$ is a universal morphism from $h_\star$ to $F$.

In particular, representations of $F$ are canonically isomorphic.

*Proof.* (Representation implies Universal) Let $(W, w) \in \mathrm{Obj}\,(h^\star \downarrow G)$. Then $u^{-1} \circ w : h^W \xrightarrow{\textbf{Set}^{\mathcal{C}}} h^U$. By Yoneda's lemma, there exists a unique $u(W, w) : U \xrightarrow{\mathcal{C}} W$ such that $u^{-1} \circ w = h^{u(W,w)}$. Hence $u(W, w)$ is the unique morphism $(W, w) \xrightarrow{h^\star \downarrow G} U, u$.

(Universal implies Representation) By universal iff represents and Yoneda's lemma, we have the following diagram.

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}}\left(h^\star, G\right) \arrow[r, "V \in \mathrm{Obj}(\mathcal{C}) \mapsto \left[s \in \mathbf{Set}^{\mathcal{C}}\left(h^V, G\right) \mapsto s_V(\mathbb{1}_V)\right]", "\sim"'] & G \\
h^U \arrow[u, "V \in \mathrm{Obj}(\mathcal{C}) \mapsto \left[f \in h^U(V) \mapsto u \circ h^f\right]", "\sim"'] \arrow[ur, "u"] &
\end{tikzcd}$$

The claim is that the above commutes, and hence $u$ is an isomorphism. Let $V \in \mathrm{Obj}\,(\mathcal{C})$ and $f \in h^U(V)$. Then

$$(h^f \circ u)_V(\mathbb{1}_V) = u_V \circ \left(h^f\right)_V(\mathbb{1}_V) = u_V(f)$$

So the above diagram commutes.

For the dual, the argument is similar. □

# 5 Adjoint Functors

**Definition – Adjoint Functors**
Let $R : \mathcal{C} \xrightarrow{\textbf{Cat}} \mathcal{D}$. Then $R$ is a *right adjoint* when there exists $L : \mathrm{Obj}\,(\mathcal{D}) \rightarrow \mathrm{Obj}\,(\mathcal{C})$ and $\eta \in \Pi X \in \mathrm{Obj}\,(\mathcal{D}), \mathcal{D}\,(X, RL(X))$ such that for all $X \in \mathrm{Obj}\,(\mathcal{D})$, $(L(X), \eta(X))$ is a universal morphism from $X$ to $R$. In this case, $L$ is called the *left adjoint of* $R$.

Dually, let $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then $L$ is a *left adjoint* when there exists $R : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ and $\varepsilon \in \Pi U \in \mathrm{Obj}(\mathcal{C}), \mathcal{C}(LR(U), U)$ such that for all $U \in \mathrm{Obj}(\mathcal{C})$, $(R(U), \varepsilon(U))$ is a universal morphism from $L$ to $U$. In this case, $R$ is called the *right adjoint of* $L$.

**Definition – Product Category**
Let $\mathcal{C}, \mathcal{D}$ be categories. Then the *product category of* $\mathcal{C}, \mathcal{D}$ is denoted $\mathcal{C} \times \mathcal{D}$ and is defined as follows.

1. $\mathrm{Obj}(\mathcal{C} \times \mathcal{D}) := \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$.
2. For $(U, X), (V, Y) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $\mathcal{C} \times \mathcal{D}((U, X), (V, Y)) := \mathcal{C}(U, V) \times \mathcal{D}(X, Y)$.

**Proposition – Natural Transformations on Product Category**
Let $F, G : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, $\alpha \in \Pi(U, X) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D}), \mathcal{E}(F(U), G(X))$. Then the following are equivalent.

1. $\alpha : F \to G$.
2. For all $(U, X) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $\alpha(U, -) : F(U, -) \to G(U, -)$ and $\alpha(-, X) : F(-, X) \to G(-, X)$.

**Definition – Adjunction**
Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. We have the two functors $\mathcal{C}(L(\star), -), \mathcal{D}(\star, R(-)) : \mathcal{D}^{op} \times \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathbf{Set}$. Then $(L, R)$ is an *adjunction* when $\mathcal{C}(L(\star), -), \mathcal{D}(\star, R(-))$ are naturally isomorphic.

In this case, $R$ is called the *right adjoint of* $L$ and $L$ is called the *left adjoint of* $R$. The isomorphism is called the *adjunction isomorphism*. For all $f : L(X) \xrightarrow{\mathcal{C}} U$, the image of $f$ under the adjunction isomorphism is called the *adjunct of* $f$, denoted $f^{\perp}$. Similarly for $g : X \xrightarrow{\mathcal{D}} R(U)$, we have the *adjunct of* $g$, denoted $g^{\perp}$.

**Proposition – Universal Morphism Characterisation of Adjunction**
Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$. Then the following are equivalent :

1. $R$ is a right adjoint.
2. There exists $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$ such that $(L, R)$ is an adjunction.

Dually, let $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then the following are equivalent :

1. $L$ is a left adjoint.
2. There exists $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ such that $(L, R)$ is an adjunction.

*Proof.* ($\Rightarrow$) Let $R$ be a right adjoint, $L : \operatorname{Obj}(\mathcal{D}) \to \operatorname{Obj}(\mathcal{C})$, $\eta \in \Pi X \in \operatorname{Obj}(\mathcal{D}), \mathcal{D}(X, RL(X))$, for all $X \in \operatorname{Obj}(\mathcal{D})$, $(L(X), \eta(X))$ universal morphism from $X$ to $R$.

The universal properties at every $X \in \operatorname{Obj}(\mathcal{D})$ implies $L$ is functorial. By universal iff represents, for all $X \in \operatorname{Obj}(\mathcal{D})$, we have $\mathcal{C}(L(X), -) \cong \mathcal{D}(X, R(-))$ as functors $\mathcal{C} \to \mathbf{Set}$. Let $f : X \xrightarrow{\mathcal{D}} Y$ and $U \in \operatorname{Obj}(\mathcal{C})$. Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{C}(L(X), U) \arrow[rrr, "R(-)\circ\eta(X)"] & & & \mathcal{D}(X, R(U)) \\
& g \circ L(f) \arrow[r, mapsto] & R(g \circ L(f)) \circ \eta(X) = R(g) \circ \eta(Y) \circ f & \\
& g \arrow[u, mapsto] \arrow[r, mapsto] & R(g) \circ \eta(Y) \arrow[u, mapsto] & \\
\mathcal{C}(L(Y), U) \arrow[uuu, "h^{L(f)}"] \arrow[rrr, "R(-)\circ\eta(Y)"'] & & & \mathcal{D}(Y, R(U)) \arrow[uuu, "h^f"']
\end{tikzcd}$$

Thus the isomorphism $\mathcal{C}(L(X), -) \cong \mathcal{D}(X, R(-))$ is functorial in $X$, and hence an isomorphism between $\mathcal{C}(L(\star), -) \cong \mathcal{D}(\star, R(-))$.

($\Leftarrow$) Let $L : \operatorname{Obj}(\mathcal{D}) \xrightarrow{\mathbf{Cat}} \mathcal{C}$ such that $(L, R)$ is an adjunction. Then for each $X \in \operatorname{Obj}(\mathcal{D})$, $\mathcal{C}(L(X), -) \cong \mathcal{D}(X, R(-))$. Let $\eta(X)$ be the adjunct of $\mathbb{1}_{L(X)}$. By universal iff represents, $(L(X), \eta(X))$ is a universal morphism from $X$ to $R$.

The dual has a similar argument. $\square$

> **Proposition – Uniqueness of Adjoints**
> Let $R, R_1 : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$, $L, L_1 : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then
>
> 1. If $(L, R)$ and $(L, R_1)$ are both adjunctions, then $R \cong R_1$ as functors.
> 2. If $(L, R)$ and $(L_1, R)$ are both adjunctions, then $L \cong L_1$ as functors.

*Proof.* (1) Let $(L, R), (L, R_1)$ both be adjunctions. Let $f : U \xrightarrow{\mathcal{C}} V$. We have an isomorphism between the functors $\mathcal{D}(-, R(U))$ and $\mathcal{D}(-, R_1(U))$ for all $U \in \operatorname{Obj}(\mathcal{C})$. By Yoneda's lemma, these isomorphisms are equal to $h_{\alpha_U}$ for some unique morphism $\alpha_U : R(U) \xrightarrow{\mathcal{D}} R_1(U)$. So we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{D}(-, R(U)) \arrow[r, "h_{\alpha_U}", "\sim"'] \arrow[d, "h_{R(f)}"'] & \mathcal{D}(-, R_1(U)) \arrow[d, "h_{R_1(f)}"] \\
\mathcal{D}(-, R(V)) \arrow[r, "\sim", "h_{\alpha_V}"'] & \mathcal{D}(-, R_1(V))
\end{tikzcd}$$

Again by Yoneda, we have $R_1(f) \circ \alpha_U = \alpha_V \circ R(f)$. The fact that $h_{\alpha_U}$ is an isomorphism implies $\alpha_U$ is an isomorphism. Thus $\alpha$ is a natural isomorphism between $R, R_1$.

(2) Analogous. □

*Remark.* There is another characterisation of adjunctions.

**Proposition – Unit/Counit Characterisation of Adjunction**
Let $R : \mathcal{C} \xrightarrow{\mathbf{Cat}} \mathcal{D}$ and $L : \mathcal{D} \xrightarrow{\mathbf{Cat}} \mathcal{C}$. Then the following are equivalent :

1. (Morphism Isomorphism) $(R, L)$ is an adjunction.
2. (Unit-Counit) There exists $\eta : \mathbb{1}_{\mathcal{D}} \to RL$ and $\varepsilon : LR \to \mathbb{1}_{\mathcal{C}}$ such that

   (a) $\mathbb{1}_L = \varepsilon L \circ L\eta$, that is to say for all $X \in \mathrm{Obj}\,(D)$, we have the following commutative diagram.

$$\begin{tikzcd} L(X) \arrow[r, "L(\eta(X))"] \arrow[dr, "\mathbb{1}_{L(X)}"'] & LRL(X) \arrow[d, "\varepsilon(L(X))"] \\ & L(X) \end{tikzcd}$$

   (b) $\mathbb{1}_R = R\varepsilon \circ \eta R$, i.e. for all $U \in \mathrm{Obj}\,(\mathcal{C})$, we have the following commutative diagram.

$$\begin{tikzcd} R(U) \arrow[r, "\eta(R(U))"] \arrow[dr, "\mathbb{1}_{R(U)}"'] & RLR(U) \arrow[d, "R(\varepsilon(U))"] \\ & R(U) \end{tikzcd}$$

The above two equations are often called *triangle-identities*.

*Proof.* $(1 \Rightarrow 2)$ For all $X \in \mathrm{Obj}\,(\mathcal{D})$, the adjunction isomorphism gives an isomorphism of functors

$$\mathcal{C}\,(L(X), -) \cong \mathcal{D}\,(X, R(-))$$

Define $\eta(X) := \mathbb{1}^{\perp}_{L(X)}$. Then by universal iff represents, $(L(X), \eta(X))$ is a universal morphism from $X$ to $R$. We claim that $\eta : \mathbb{1}_{\mathcal{D}} \to RL$.

Let $f : X \xrightarrow{\mathcal{D}} Y$. Then by the universal property of $(L(X), \eta(X))$, we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "\eta(X)"] \arrow[d, "f"'] & RL(X) \arrow[d, "RL(f)"] \\ Y \arrow[r, "\eta(Y)"'] & RL(Y) \end{tikzcd}$$

i.e. $\eta$ is a natural transformation as desired. We similarly define $\varepsilon(U) := \mathbb{1}_{R(U)}^{\perp}$ for $U \in \operatorname{Obj}(\mathcal{C})$ and see that $\varepsilon : LR \to \mathbb{1}_{\mathcal{C}}$.

To prove $(a)$, let $X \in \operatorname{Obj}(\mathcal{D})$. Then

$$\mathbb{1}_{L(X)} = \left(\mathbb{1}_{L(X)}^{\perp}\right)^{\perp} = (\eta(X))^{\perp} = \varepsilon(L(X)) \circ L(\eta(X))$$

where the last equality follows from the universal property of $(RL(X), \varepsilon(L(X)))$. Similarly for $(b)$, we have for $U \in \operatorname{Obj}(\mathcal{C})$,

$$\mathbb{1}_{R(U)} = \left(\mathbb{1}_{R(U)}^{\perp}\right)^{\perp} = (\varepsilon(U))^{\perp} = R(\varepsilon(U)) \circ \eta(R(U))$$

where the last equality is by the universal property of $(LR(U), \eta(R(U)))$.

$(2 \Rightarrow 1)$ Let $(X, U) \in \operatorname{Obj}(\mathcal{D}^{op} \times \mathcal{C})$. Since $(L(X), \eta(X))$ is supposed to be a universal morphism from $X$ to $R$, we define the adjunction map to be

$$\begin{aligned} \mathcal{C}(L(X), U) &\xleftrightarrow{\perp} \mathcal{D}(X, R(U)) \\ f &\longmapsto R(f) \circ \eta(X) \\ \varepsilon(U) \circ L(g) &\longleftarrow\!\shortmid g \end{aligned}$$

Then for $f : L(X) \xrightarrow{\mathcal{C}} U$,

$$\begin{aligned} \left(f^{\perp}\right)^{\perp} &= \varepsilon(U) \circ L(f^{\perp}) = \varepsilon(U) \circ L\left(R(f) \circ \eta(X)\right) \\ &= \varepsilon(U) \circ LR(f) \circ L(\eta(X)) = f \circ \varepsilon(L(X)) \circ L(\eta(X)) = f \end{aligned}$$

Similarly, $\left(g^{\perp}\right)^{\perp} = g$. So $\perp$ is an isomorphism at all $(X, U)$.

It remains to show naturality. It suffices to show that the isomorphism is natural in both components. Let $f : X \xrightarrow{\mathcal{D}^{op}} Y$. Then we have the following diagram.

$$\begin{tikzcd} \mathcal{C}(L(X), U) \arrow[r, "\perp"] \arrow[d, "h^{L(f)}"'] & \mathcal{D}(X, R(U)) \arrow[d, "h^{f}"] \\ \mathcal{C}(L(Y), U) \arrow[r, "\perp"'] & \mathcal{D}(Y, R(U)) \end{tikzcd}$$

It follows from $\eta : \mathbb{1}_{\mathcal{D}} \to RL$ that the above commutes. Similarly, naturality of $\varepsilon$ implies naturality in the second component. Hence $\perp$ is a natural isomorphism as desired. $\square$

*Remark.* The following is a special case of adjunction that is worth noting.

> **Definition – Galois Connection**
> Let $I, J$ be partially ordered sets. Then $I, J$ can be seen as categories. A *monotone Galois connection between* $I, J$ is an adjunction between $I, J$. A *antitone Galois connection between* $I, J$ is an adjunction

between $I^{op}, J$.

*Remark.* The unit/counit characterisation of adjunctions shows that if $(R, L)$ is a Galois connection (mono or anti) between partially ordered sets $I, J$, then $R$ and $L$ are bijective on their images.

**Definition – Free Functors**

# 6 Limits and Colimits

**Definition – (Co)Diagrams**

Let $\mathcal{I}, \mathcal{C}$ be categories. Then an *$\mathcal{I}$-diagram in $\mathcal{C}$* is a covariant functor from $\mathcal{I}$ to $\mathcal{C}$. Dually, an *$\mathcal{I}$-codiagram* is a contravariant functor from $\mathcal{I}$ to $\mathcal{C}$, i.e. an $\mathcal{I}^{op}$-diagram.

*Remark.* Often, it is easier to take $\mathcal{I}$ to be a subcategory of $\mathcal{C}$.

**Definition – Constant (Co)Diagrams**

Let $\mathcal{I}, \mathcal{C}$ be categories and $U \in \operatorname{Obj}(\mathcal{C})$. Then define the *constant diagram* $\Delta(U)$ as follows.

1. For all $i \in \mathcal{I}$, $\Delta(U)(i) := U$.
2. For all $\phi : i \xrightarrow{\mathcal{I}} j$, $\Delta(U)(\phi) := \mathbb{1}_U$.

Dually, we have the *constant codiagram* $\Delta^{op}(U)$ defined as :

1. For all $i \in \operatorname{Obj}(\mathcal{I})$, $\Delta^{op}(U)(i) := U$.
2. For all $\phi : i \xrightarrow{\mathcal{I}^{op}} j$, $\Delta(U)(\phi) := \mathbb{1}_U$.

**Proposition – Functoriality of Constant (Co)Diagrams**

Let $\mathcal{I}, \mathcal{C}$ be categories. Then $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$. Dually, $\Delta^{op} : \mathcal{C} \to \mathcal{C}^{\mathcal{I}^{op}}$.

**Definition – (Co)Limits of (Co)Diagrams**

Let $\mathcal{I}, \mathcal{C}$ be categories, $X$ a $\mathcal{I}$-diagram in $\mathcal{C}$, and $Y$ a $\mathcal{I}$-codiagram in $\mathcal{C}$.

Then a *limit of $X$* is a universal morphism from $\Delta$ to $X$. If a limit of $X$ exists, it is canonical and referred to as *the* limit, denoted $(\varprojlim X, \pi_X)$.

Dually, a *colimit of $Y$* is a universal morphism from $Y$ to $\Delta^{op}$. If a colimit of $Y$ exists, it is canonical and referred to as *the* colimit, denoted with $(\varinjlim Y, \iota_Y)$.

*Remark.* Sometimes limits are also called *projective limits*, and colimits are called *injective limits*.

**Definition – (Co)Completeness**

Let $\mathcal{C}$ be a category. Then it is called *complete* when for all "small" categories $\mathcal{I}$ and diagrams $X : \mathcal{I} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, there exists the limit of $X$.

Dually, it is called *cocomplete* when for all "small" categories $\mathcal{I}$ and codiagrams $Y : \mathcal{I}^{op} \xrightarrow{\mathbf{Cat}} \mathcal{C}$, there exists the colimit of $Y$.

*Remark.* We now cover important examples of limits and colimits.

**Definition – Discrete Category**

For $I \in \mathrm{Obj}(\mathbf{Set})$, $I$ can be turned into a category by having elements as objects and the only morphisms being identity morphisms. Categories obtained in this way are called *discrete categories*.

*Remark.* Note that a discrete category $\mathcal{I}$ is canonically isomorphic to its dual. That is to say, there is a unique isomorphism of categories between $\mathcal{I}$ and $\mathcal{I}^{op}$. Consequently, $\mathcal{I}$-diagrams and $\mathcal{I}$-codiagrams are "the same".

**Definition – (Co)Products**

Let $\mathcal{C}$ be a category and $\mathcal{I}$ a discrete category.

Let $X$ be an $\mathcal{I}$-diagram in $\mathcal{C}$. Then the limit of $X$ is called the *product of* $X(i)$.

Dually, let $Y$ be an $\mathcal{I}$-codiagram in $\mathcal{C}$. Then the colimit of $Y$ is called the *coproduct of* $Y(i)$.

In the special case of $I = \varnothing$, the product is called the *final object of* $\mathcal{C}$. Dually, the coproduct is called the *initial object of* $\mathcal{C}$.

*Example* (*Final Objects*).

*Example* (*Initial Objects*).

*Example* (*Products*).

*Example* (*Coproducts*).

**Definition – (Co)Equalizers**

Let $\mathcal{C}$ be a category. Let $I$ be an arbitrary set and $\mathcal{I}$ be the following category.

$$\begin{tikzcd} 0 \arrow[loop above, "\mathbb{1}_0"] \arrow[r, "i"] & 1 \arrow[loop above, "\mathbb{1}_1"] \end{tikzcd}$$

where there is a morphism $i : 0 \xrightarrow{\mathcal{I}} 1$ for all $i \in I$.

Let $X$ be an $\mathcal{I}$-diagram in $\mathcal{C}$. Then the limit of $X$ is called the *equalizer of* $X(i)$*'s*. Dually, let $Y$ be an $\mathcal{I}$-codiagram in $\mathcal{C}$. Then the colimit of $Y$ is called the *coequalizer of* $Y(i)$*'s*.

*Example* (*Equalizers*).

*Example* (*Coequalizers*).

**Definition – Pullbacks and Pushouts**

Let $\mathcal{C}$ be a category, $U \in \mathrm{Obj}\,(\mathcal{C})$. Then a *pullback over* $U$ is a product in the category $\mathcal{C} \downarrow U$. Dually, a *pushout under* $U$ is a coproduct in the category $U \downarrow \mathcal{C}$.

Let $I$ be an arbitrary set and $\mathcal{I}$ the following category.

$$\begin{tikzcd} i \arrow[loop above, "\mathbb{1}_0"] \arrow[r, "\phi(i)"] & * \arrow[loop above, "\mathbb{1}_1"] \end{tikzcd}$$

1. $\mathrm{Obj}\,(\mathcal{I}) = I \sqcup \{*\}$.
2. For all $x \in \mathrm{Obj}\,(\mathcal{I})$, $\mathcal{I}\,(x, x) = \{\mathbb{1}_x\}$.
3. For all $i \in I$, $\mathcal{I}\,(i, *) = \{\phi(i)\}$.

Then a pullback over $U$ is equivalently the limit of an $\mathcal{I}$-diagram $X$ with $X(*) = U$. Dually, a pushout under $U$ is equivalently the colimit of an $\mathcal{I}$-codiagram $Y$ with $Y(*) = U$.

# 7 Completeness

**Proposition – Characterisation of Completeness, Cocompleteness**

**Proposition – Set Complete**

**Proposition – Top Complete**

**Proposition – Grp Complete**

**Proposition – Ring Complete**

**Proposition – $R$-Mod Complete**

**Proposition – Set-theoretic Characterisation of Limits and Colimits**
Let $X$ be an $\mathcal{I}$-shaped diagram in a category $\mathcal{C}$ and $(U, u) \in \mathrm{Obj}\,(\Delta \downarrow X)$. We have an $\mathcal{I}$-shaped diagram in $\mathbf{Set}^{\mathcal{C}^{op}}$ that is $h_X := h_\star \circ X$. We also have $h_{\Delta(U)} = \Delta(h_U)$ and the natural transformation $h_u : \Delta(h_U) \to h_X$. So $(h_U, h_u) \in \mathrm{Obj}\,(\Delta \downarrow h_X)$. Then the following are equivalent.

1. $(U, u)$ is a limit of $X$.
2. $(h_U, h_u)$ is a limit of $h_X$.

**Corollary – Right Adjoints commute with Limits, Left Adjoints commute with Colimits**

**Definition – Filtered Sets and Filtered Colimits**

**Proposition – Filtered Colimits commute with Finite Limits**

# 8 Abelian Categories

**Definition – Zero Objects**

**Definition – Kernels and Cokernels**