Notes on Model Categories

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Winter 2021

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1 Homotopical Categories

Definition – Homotopical Categories

A *homotopical category* consists of the following data :

- a category M
- a class of morphisms W of M
- W contains identity morphisms and satisfies *two-out-of-six*

Morphisms in W are called *weak equivalences*. Sometimes W is implicit and we simply say M is a homotopical category.

Example. 1. *Given any category M, the class of isomorphisms satisfies two-out-of-six.*

2. *Let $M = \mathbf{Cat}$ the category of (small) categories. Declare W to be the class of categorical-equivalences. Then W satisfies two-out-of-six by application of the previous example on the "Π_0" of categories and maps on hom sets.*

3. *Let A be an abelian category and $\mathrm{Ch}\,A$ the category of (unbounded) complexes in A. Declare W to be*

the class of quasi-isomorphisms*, i.e. the morphisms which induce isomorphisms on cohomology. Then* $(\mathrm{Ch}\,A, W)$ *is homotopical by the previous example.*

This includes the examples : the category $B\mathbf{Mod}$ *of (left) modules over an associative ring* B*, the catgeory* $\mathrm{QCoh}X$ *of quasi-coherent sheaves on a scheme* X*.*

4. *Let* $M = \mathbf{Top}$ *and declare* W *to be the class of morphisms which induce an isomophism on* π_0 *and* $\pi_{>0}$ *at all points. Then* $(\mathbf{Top}, W)$ *is again homotopical by the first example.*

Lemma. Let M *be a category and* W *a class of morphisms of* M *satisfying two-out-of-six. Then* W *satisfies two-out-of-three.*

Proof. ■

Proposition – 1-Categorical Localisation

Let (M, W) be a homotopical category. Define the category $M[W^{-1}]$ by the following data :

- objects of $M[W^{-1}]$ are the same as those of M.
- $M[W^{-1}](x, y)$ consists of "fractions of morphisms" $(f_0/g_0)\cdots(f_n/g_n)$ where the g_i are in W, quotiented out by removing any "f/g" with $f, g \in W$ and $\mathbb{1}/\mathbb{1}$.

  These are also sometimes called *zig-zags*.
- Morphisms compose by "multiplication". Checking associativity is tedious.

There is a functor $M \to M[W^{-1}]$ that is identity on objects and maps each morphism f to the equivalence class of $f/\mathbb{1}$.

For any category N, define $\mathbf{Cat}_W(M, N)$ to be the full subcategory of functors $M \to N$ *inverting* W, i.e. maps morphisms in W to isomorphisms. For any functor $l : M \to L(M)$, we say l *exhibits* $L(M)$ *as a (weak) localisation of* M *at* W when l inverts W and or any category N we have a categorical equivalence :

$$\_ \circ l : \mathbf{Cat}(L(M), N) \xrightarrow{\sim} \mathbf{Cat}_W(M, N)$$

Then $M \to M[W^{-1}]$ exhibits $M[W^{-1}]$ as a (weak) localisation of M at W. Furthermore, $M \to M[W^{-1}]$ is an epimorphism.

If W is implicit, we use $\mathbf{Ho}M$ to denote $M[W^{-1}]$.

Proof. Nothing unexpected. The morphism $M \to M[W^{-1}]$ is epi because for any commuting triangle

$$\begin{tikzcd} M \arrow[d] \arrow[dr, "T"] & \\ M[W^{-1}] \arrow[r, "\tilde{T}"'] & N \end{tikzcd}$$

$\tilde{T}$ sends a zig-zag $(f_0/g_0)\cdots(f_n/g_n)$ to $T(f_0)T(g_0)^{-1}\cdots T(f_n)T(g_n)^{-1}$, and hence is determined by T. □

Remark. Some motivations for *model structures* via issues with the 1-categorical localisation :

- Using ZFC set theory as foundations, a common issue is that when M has a proper class of objects (e.g. $M = \mathbf{Set}, \mathbf{Grp}, \mathbf{Ring}$ etc), the homs of $M[W^{-1}]$ are also proper classes.
- The homs of $M[^{-1}]$ are hard to get your hands on.

Definition – Homotopical Functors
Let (M, W_M) and (N, W_N) be homotopical categories. Then a functor $F : M \to N$ is called *homotopical* when $FW_M \subseteq W_N$.

Example.
Let (M, W) be a homotopical category and $F : M \to M$ together with $\alpha : \mathbb{1} \to F$ with components in W. Then F is homotopical by W satisfying two-out-of-three.

Counter Example.
A list of functors between homotopical categories which we wish were homotopical :

1. *Let I be a (small) category and (M, W) a homotopical category. Suppose M has colimits of I-diagrams, i.e. we have a functor $\varinjlim_I : M^I \to M$ left adjoint to the constant diagram functor $M \to M^I$. We can give M^I a class of weak equivalences by declaring weak equivalences to be the ones that are component-wise in W.*

   It would be nice if $\varinjlim_I : M^I \to M$ is homotopical since that would say "colimits are invariant under W". Unfortunately, this is not true in general.

   Take $M = \mathbf{Top}$ and I the shape of the diagram for a pushout. A counterexample is then given by the following morphism of diagrams :

$$\begin{tikzcd}
D^{n+1} \arrow[d] & S^n \arrow[l] \arrow[r] \arrow[d, dashed, "\nparallel"] & D^{n+1} \arrow[d] \\
\bullet & S^n \arrow[l] \arrow[r] & \bullet
\end{tikzcd}$$

   All vertical morphisms are weak-equivalences in $\mathbf{Top}$ but the induced map $S^{n+1} \to \bullet$ is certainly not a weak equivalence.

2. *TODO : show how non-exactness of functors from homological algebra like tensor and hom can be expressed as non-homotopicalness of the functors.*

2 Derived Functors

Remark. Digestion of the exposition in Riehl's *Categorical Homotopy Theory*.

Definition

Let M, N be homotopical categories and $F : M \to N$ a functor.

Let $\gamma : M \to \mathbf{Ho}M$ and $\delta : N \to \mathbf{Ho}N$ be the obvious functors.

Then a *total left derived functor of* F is defined as a *right* Kan extension of δF along γ. Unpacking the definition, this consists of the following data :

- a functor $\mathbf{L}F : \mathbf{Ho}M \to \mathbf{Ho}N$
- a natural transformation $\lambda : \mathbf{L}F\gamma \to \delta F$ such that

$$\_\,\gamma : \mathrm{Mor}(\_, \mathbf{L}F) \xrightarrow{\cong} \mathrm{Mor}(\_, \delta F)$$

i.e. "$\mathbf{L}F$ is the left-closest extension of δF along γ". Using the equivalence $\mathbf{Cat}(\mathbf{Ho}M, \mathbf{Ho}N) \simeq \mathbf{Cat}_W(M, \mathbf{Ho}N)$, the above is equivalent to specifying a universal morphism $\lambda : \mathbf{L}F \to \delta F$ from $\mathbf{Cat}_W(M, \mathbf{Ho}\,N)$ to δF.

Often, it is possible to improve the situation by finding a homotopical functor $\mathbb{L}F : M \to N$ with $\lambda : \mathbb{L}F \to F$ such that $\delta\mathbb{L}F$ in $\mathbf{Cat}(M, \mathbf{Ho}\,N)$ corresponds to a total left derived functor of F.

$$\mathbf{Cat}_W(M, \mathbf{Ho}N)(\_, \delta\mathbb{L}F) \xrightarrow{\cong} \mathbf{Cat}(M, \mathbf{Ho}N)(\_, \delta F)$$

We call such $(\mathbb{L}F, \lambda)$ a *left derived functor* of F.

These definitions dualise to give *total right derived functors* and *right derived functors*.

Remark. The following result gives a way of constructing derived functors.

Proposition – Derived Functors via Deformations

Let M, N be homotopical categories and $F : M \to N$ a functor (not necessarily homotopical).

Define a *left deformation* of F to consist of the following data :

- a functor $Q : M \to M$.
- a natural transformation $q : Q \to \mathbb{1}_M$ with components in weak equivalences of M.
- we require that $F : M_Q \to N$ is homotopical where M_Q is the full subcategory of M consisting of objects in the image of Q, turned into a homotopical category via endowing it with weak equivalences from M.

We have the following :

1. Let (Q, q) be a left deformation of F. (FQ, Fq) gives a left derived functor for F. Furthermore, this is in fact an *absolute* right Kan extension of $M \to N \to \mathbf{Ho}N$ along $M \to \mathbf{Ho}\,M$.
2. Suppose $F \dashv G : M \rightleftharpoons N$ is an adjunction, such that we have a total left derived functor $(\mathbf{L}F, \lambda)$ of F and a total right derived functor $(\mathbf{R}G, \rho)$ of G where both total derived functors are absolute Kan extensions. Then we have an adjunction

$$\begin{tikzcd}
\mathbf{Ho}\, M \arrow[r, shift left=2, "\mathbf{L}F"] \arrow[r, phantom, "\perp"] & \mathbf{Ho}\, N \arrow[l, shift left=2, "\mathbf{R}G"]
\end{tikzcd}$$

that is compatible with localisation in the sense that we have the commuting square :

$$\begin{tikzcd}
N(Fm, n) \arrow[r, "\cong"] \arrow[d] & M(m, Gn) \arrow[d] \\
\mathbf{Ho}\, N(Fm, n) \arrow[d, "_\lambda_m"'] & \mathbf{Ho}\, M(m, Gn) \arrow[d, "\rho_{n_}"] \\
\mathbf{Ho}\, N(\mathbf{L}Fm, n) \arrow[r, "\cong"'] & \mathbf{Ho}\, M(m, \mathbf{R}Gn)
\end{tikzcd}$$

Proof. (1)

Let $\delta : N \to \mathbf{Ho}N$ be the obvious functor. We directly show that $(\delta FQ, \delta Fq)$ gives an absolute right Kan extension of δF along $M \to \mathbf{Ho}\, M$.

Let $H : \mathbf{Ho}\, N \to E$. Let $G \in \mathbf{Cat}_W(M, E)$ where W is the class of weak equivalences of M. We need to show a bijection :

$$G \to H\delta FQ \quad || \quad G \to H\delta F$$

via composing with $H\delta Fq$. Let $\alpha : G \to H\delta F$. We give a unique $\alpha_0 : G \to H\delta FQ$ such that $\alpha = (H\delta Fq)\alpha_0$. Suppose we have such an α_0. By "restricting" the commuting triangle $\alpha = (H\delta Fq)\alpha_0$ along $q : Q \to \mathbb{1}_M$, we have the following commutative diagram :

$$\begin{tikzcd}
 & H\delta FQ^2 \arrow[r, "H\delta FQq"] \arrow[d] & H\delta FQ \arrow[d, "H\delta Fq"] \\
GQ \arrow[ur, "\alpha_0 Q"] \arrow[urr, "\alpha Q"] \arrow[d, "Gq"'] & H\delta FQ \arrow[r, "H\delta Fq"] & H\delta F \\
G \arrow[ur, "\alpha_0"] \arrow[urr, "\alpha"] & &
\end{tikzcd}$$

Now the point is that since q has components in weak equivalences of M, we have Gq is an isomorphism since G maps weak equivalences to isomoprhisms, and $H\delta FQq$ is also an isomorphism since Q is homotopical and F is homotopical when restricted to M_Q, and H preserves isomorphisms. We can thus solve for α_0 uniquely as

$$\alpha_0 = (H\delta FQq)(\alpha_0 Q)(Gq)^{-1} = (H\delta FQq)(H\delta FQq)^{-1}(\alpha Q)(Gq)^{-1} = (\alpha Q)(Gq)^{-1}$$

(2) Purely formal. Indefinitely postponed.

$\square$

Remark. The above proposition gives another reason for model structures : given a not necessarily homotopical functor $F : M \to N$ between homotopical categories, a suitable model structure on M will provide deformations for F.

3 Definitions of Model Structures

Remark. Digestion of the exposition on Joyal's nCatLab.

Definition – Model Categeories, Weak factorisation systems, Lifting Properties

Let $\mathcal{E}$ be a category. For (L, R) a pair of classes of morphisms of $\mathcal{E}$, (L, R) is called a *weak factorisation system* when the following are true :

- any $f \in \mathcal{E}^{\to}$ factorises as

$$\begin{tikzcd} A \arrow[r, "l \in L"] \arrow[dr, "f"'] & C \arrow[d, "r \in R"] \\ & B \end{tikzcd}$$

  (possibly non-uniquely)

- L is exactly the morphisms that have *left-lifting-property* against all morphisms in R.
- R is exactly the morphisms that have *right-lifting-property* against all morphisms in L.

Let $(\mathcal{E}, \mathscr{W})$ be a homotopical category where $\mathcal{E}$ is finitely complete and finitely cocomplete. A *model structure* is a pair $\mathscr{C}, \mathscr{F}$ of classes of families of morphisms in $\mathcal{E}$ such that :

- $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ is a weak factorisation system for $\mathcal{E}$.
- $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$ is a weak factorisation system for $\mathcal{E}$.

A *model category* is a finitely complete and finitely cocomplete homotopical category equipped with a model structure.

Definition – Homotopy Category, Weak Equivalences/Acyclic Morphisms, (Co)Fibrations, (Co)Fibrant Objects

Let $(\mathcal{E}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ be a model category.

- morphisms in $\mathscr{W}$ are called *acyclic*. Sometimes, they are also called *trivial*.
- morphisms in $\mathscr{C}$ are called *cofibrations*
- morphisms in $\mathscr{F}$ are called *fibrations*
- objects X where $X \to 1$ is a fibration are called *fibrant*.
- objects X where $\varnothing \to X$ is a cofibration are called *cofibrant*.

– objects that are both fibrant and cofibrant, we call *fibrant-cofibrant*.

Remark. Many authors (such as Hovey) require the two factorisations to be *functorial*. This is usually satisfied in practice (in particular, whenever we have a cofibrantly generated model structure).

Proposition

Let $(\mathcal{E}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ be a model category with functorial factorisations. The following square of categories consists of equivalences :

$$\begin{tikzcd} \mathbf{Ho}\,\mathcal{E}_{fc} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathbf{Ho}\,\mathcal{E}_f \arrow[d, "\simeq"] \\ \mathbf{Ho}\,\mathcal{E}_c \arrow[r, "\simeq"'] & \mathbf{Ho}\,\mathcal{E} \end{tikzcd}$$

Proof. The functorial factorisation $(\mathscr{W} \cap \mathscr{C}, \mathscr{F})$ gives a right deformation of $\mathcal{E}$ to $\mathcal{E}_f$ and hence a quasi-inverse to $\mathbf{Ho}\,\mathcal{E}_f \to \mathbf{Ho}\,\mathcal{E}$. Now the functorial factorisation $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$ similarly gives a left deformation of $\mathcal{E}_f$ to $\mathcal{E}_{fc}$. The rest is analogous. □

Proposition – Elementary results

Let $(\mathcal{E}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ be a model category.

1. $\mathscr{C}, \mathscr{C} \cap \mathscr{W}, \mathscr{F}, \mathscr{F} \cap \mathscr{W}$ closed under composition and retracts.
2. $\mathscr{F}, \mathscr{F} \cap \mathscr{W}$ closed under base change and product.
3. $\mathscr{C}, \mathscr{C} \cap \mathscr{W}$ closed under cobase change and coproduct.
4. $\mathscr{C} \cap \mathscr{W} \cap \mathscr{F}$ is precisely the isomorphisms.
5. (Tierney) $\mathscr{W}$ closed under retracts. (Slightly non-trivial.) In particular, identity morphisms are in $\mathscr{W}$.

Proof. Besides that last result by Tierney, the rest is rather formal. □

4 Computing the Homotopy Category of a Model Category

Remark. In this section we fulfil the promise of computing the homotopy category of a $(\mathcal{E}, \mathscr{W})$ given a model structure $(\mathscr{C}, \mathscr{F})$. We follow the exposition of Joyal's nCatLab. Throughout this section, we fix a model catgory $(\mathcal{E}, \mathscr{W}, \mathscr{C}, \mathscr{F})$.

We have already seen $\mathbf{Ho}\,\mathcal{E} \simeq \mathbf{Ho}\,\mathcal{E}_{fc}$. So the goal is to give a nice description of morphisms in $\mathbf{Ho}\,\mathcal{E}_{fc}$.

Definition – Cylinder Object, Left Homotopy , Path Object , Right Homotopy

Let $A \in \mathcal{E}$. A *cylinder object for A* is a factoring [a]

$$\begin{tikzcd} A+A \arrow[r, "\in\mathscr{C}"] \arrow[dr, "\mathbb{1}+\mathbb{1}"'] & IA \arrow[d, "\in\mathscr{W}"] \\ & A \end{tikzcd}$$

(not necessarily unique).

Let $f, g : A \to X$ in $\mathcal{E}$. Then a *left homotopy*, $h : f \xrightarrow{l} g$, is defined as a factoring with respect to a cylinder object IA of A :

$$\begin{tikzcd} A+A \arrow[r] \arrow[dr, "f+g"'] & IA \arrow[d, "h"] \\ & X \end{tikzcd}$$

The above definition dualises. Let $X \in \mathcal{E}$. A *path object for* X is a factoring

$$\begin{tikzcd} X \arrow[d, "\in\mathscr{W}"'] \arrow[dr, "(\mathbb{1},\mathbb{1})"] & \\ PX \arrow[r, "\in\mathscr{F}"'] & X \times X \end{tikzcd}$$

Let $f, g : A \to X$ in $\mathcal{E}$. Then a *right homotopy* $h : f \xrightarrow{r} g$ is defined as a a factoring of (f, g) with respect to a path object PX of X :

$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, "(f,g)"] & \\ PX \arrow[r] & X \times X \end{tikzcd}$$

[a]We don't require $IA \to A$ to be a fibration because in computations it is useful not to. For example, in simplicial sets $X \times \Delta^1 \to X$ is in general not a fibration.

Proposition – The Homotopy Category of a Model Category

Let $A, X \in \mathcal{E}$. Then we have the following :

1. Given A is cofibrant, the left homotopy relation on $\mathcal{E}(A, X)$ is an equivalence relation. Letting $\pi^l(A, X)$ denote the quotient by left homotopy equivalence, we have a functor $\pi^l(A, \_) : \mathcal{E} \to \mathbf{Set}$.

   Dually, given X is fibrant, the right homotopy relation on $\mathcal{E}(A, X)$ is an equivalence relation. Letting $\pi^r(A, X)$ denote the quotient by left homotopy equivalence, we have a functor $\pi^r(\_, X) : \mathcal{E}^{op} \to \mathbf{Set}$.

2. Given A is cofibrant, the left homotopy relation on $\mathcal{E}(A, X)$ implies the right homotopy relation.

   Dually, given X is fibrant, the right homotopy relation on $\mathcal{E}(A, X)$ implies the left homotopy relation.

   Hence for A cofibrant and X fibrant, the two homotopy relations coincide. Letting $\pi(A, X) := \pi^r(A, X) = \pi^l(A, X)$, we obtain a functor $\pi : \mathcal{E}_c^{op} \times \mathcal{E}_f \to \mathbf{Set}$.

3. Let $\pi\mathcal{E}_{fc}$ denote the category defined by the data :

   – objects are fibrant-cofibrant objects of $\mathcal{E}$

   – The functor $\pi : \mathcal{E}_c^{op} \times \mathcal{E}_f \to \mathbf{Set}$ restricts to a functor $\pi : \mathcal{E}_{fc}^{op} \times \mathcal{E}_{fc} \to \mathbf{Set}$. We use this for the morphism functor.

   There is an obvious functor $\mathcal{E}_{fc} \to \pi\mathcal{E}_{fc}$. The result is that this exhibits $\pi\mathcal{E}_{fc}$ as the localisation of $\mathcal{E}_{fc}$ at weak equivalences. Furthermore, a morphism $f : A \to X$ in $\mathcal{E}_{fc}$ is a weak equivalence iff it is an isomorphism in $\pi\mathcal{E}_{fc}$.

4. Given the following square consists of equivalences

$$\begin{tikzcd} \mathbf{Ho}\,\mathcal{E}_{fc} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathbf{Ho}\,\mathcal{E}_f \arrow[d, "\simeq"] \\ \mathbf{Ho}\,\mathcal{E}_c \arrow[r, "\simeq"'] & \mathbf{Ho}\,\mathcal{E} \end{tikzcd}$$

   we have that a morphism f in $\mathcal{E}$ is acyclic iff it is inverted in $\mathbf{Ho}\,\mathcal{E}$.

Proof. (1) Let A be cofibrant. The left homotopy relation is already reflexive and symmetric. To show transitivity, suppose we have two left homotopies :

$$\begin{tikzcd} A + A \arrow[r, tail, "i"] \arrow[dr, "{f,g}"'] & IA \arrow[d, "H"', "\sim"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} A + A \arrow[r, tail, "j"] \arrow[dr, "{g,h}"'] & JA \arrow[d, "H_1"', "\sim"] \\ & X \end{tikzcd}$$

The key is that we can glue the two cylinders IA and JA together by the "end side of IA" and the "start side of JA" to form another cylinder KA whose start is f and end is h. More concretely, we define KA by the following pushout :

$$\begin{tikzcd} A \arrow[r, "j_0"] \arrow[d, "i_1"'] & JA \arrow[d] \\ IA \arrow[r] & KA \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Define $k : A + A \to KA$ by setting $k_0 := A \to IA \to KA$ via i_0 and $k_1 := A \to JA \to KA$ via j_1. Then the left homotopies H, H_1 glue to give a morphism $H_2 : KA \to X$ such that $H_2 k_0 = f$ and $H_2 k_1 = h$. It remains to show that KA is a cylinder object for A via $k : A + A \to KA$ and $KA \to A$.

First, let's show $KA \to A$ is in $\mathscr{W}$. This morphism is defined by the diagram :

$$\begin{tikzcd}
A \arrow[r, "j_0"] \arrow[d, "i_1"'] & JA \arrow[d] \arrow[ddr, bend left, "\sim"] & \\
IA \arrow[r] \arrow[drr, bend right, "\sim"'] & KA \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] & \\
 & & A
\end{tikzcd}$$

By $\mathscr{W}$ having two-out-of-three, it suffices to show that $JA \to KA$ is in $\mathscr{W}$. Since $\mathscr{W} \cap \mathscr{C}$ is closed under cobase change, it suffices to show that the side inclusion $i_0 : A \to IA$ is an acyclic cofibration.

> *Lemma. Given a cylinder object (IA, i, σ) of a cofibrant A, we have $i_0 : A \to IA$ in $\mathscr{W} \cap \mathscr{C}$.*
>
> *There is a dual result for path objects of fibrant objects.*
>
> *Proof.* It is acyclic by two-out-of-three for $\mathscr{W}$. And it is a cofibration because it is the composition $A \to A + A \to IA$ where $i : A + A \to IA$ is in $\mathscr{C}$ by definition of cylinder objects and $A \to A + A$ is the cobase change of $\varnothing \to A$, a cofibration as assumed. ■

Next, we show $k : A + A \to KA$ is a cofibration. This morphism fits into the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[d] & \varnothing \arrow[l, tail] \arrow[d] \\
A + A + A + A \arrow[r, "\mathbb{1},(\mathbb{1},\mathbb{1}),\mathbb{1}"] \arrow[d, tail, "i,j"'] & A + A + A \arrow[d, tail, "i_0,_,j_1"'] \arrow[ur, phantom, "\urcorner", very near start] & A + A \arrow[l, tail] \arrow[dl, tail, "k"] \\
IA + JA \arrow[r] & KA \arrow[ul, phantom, "\ulcorner", very near start] &
\end{tikzcd}$$

(2) It suffices to prove the following stronger fact :

> *Lemma. Let (IA, i, σ) be a cylinder object for A and (PX, ρ, e) a path object for X. Suppose we have an "open box of homotopies"*
>
> $$\begin{tikzcd}
> f_{00} \arrow[r, "h_0"] \arrow[d, "v_0"'] & f_{01} \\
> f_{10} \arrow[r, "h_1"'] & f_{11}
> \end{tikzcd}$$
>
> *where $f_{jk} : A \to X$, $v_0 : f_{00} \underset{r}{\longrightarrow} f_{10}$ and $h_j : f_{j0} \underset{l}{\longrightarrow} f_{j1}$. Suppose A is cofibrant. Then there exists*

$H : IA \to PX$ such that $e_k H i_j = f_{jk}$. Pictorially, we can "complete the box with a double homotopy*".*

$$\begin{tikzcd}
f_{00} \arrow[r, "h_0"] \arrow[d, "v_0"'] \arrow[dr, "H"] & f_{01} \arrow[d, "v_1"] \\
f_{10} \arrow[r, "h_1"'] & f_{11}
\end{tikzcd}$$

Proof. The lemma can be formulated as finding a solution to the following lifting problem

$$\begin{tikzcd}
A \arrow[r, "v_0"] \arrow[d, tail, "i_0"', "\sim"] & PX \arrow[d, two heads] \\
IA \arrow[r, "{h_0, h_1}"'] & X \times X
\end{tikzcd}$$

But we know that given A cofibrant, we have $i_0 \in \mathscr{W} \cap \mathscr{C}$. Hence we have a solution since fibrations right lift against acyclic cofibrations. ■

Now, given A cofibrant and a left homotopy $h_1 : f \underset{l}{\longrightarrow} g$, we get a right homotopy $v_1 : f \underset{r}{\longrightarrow} g$ by completing the following open box of homotopies

$$\begin{tikzcd}
f \arrow[r] \arrow[d] \arrow[dr, dashed, "H"] & f \arrow[d, dashed, "v_1"] \\
f \arrow[r, "h_1"'] & g
\end{tikzcd}$$

where the unnamed homotopies are from reflexivity of the two homotopy relations.

(3) First, we show that $\mathcal{E}_{fc} \to \pi\mathcal{E}_{fc}$ inverts weak equivalences. To that end, we prove a lemma which can be used to show when a functor from cofibrant objects is homotopical.

Lemma (Kenneth Brown). Let $F : \mathcal{E}_c \to C$ where C is a category with a class of morphisms W satisfying identities and two-out-of-three. Suppose $F(\mathscr{W} \cap \mathscr{C}) \subseteq W$. Then $F(\mathscr{W}) \subseteq W$.

There is a dual result for $F : \mathcal{E}_f \to C$ with $F(\mathscr{W} \cap \mathscr{C}) \subseteq W$.

Proof. The new idea is that of a *mapping cylinder*. Given a morphism $f : A \to B$, a *mapping cylinder factorisation* of f is a factorisation

$$\begin{tikzcd}
A + B \arrow[d, tail] \arrow[dr, "{f, \mathbb{1}}"] & \\
C(f) \arrow[r, "\sim"'] & B
\end{tikzcd}$$

Let $f : A \to B$ in $\mathcal{E}_c$ be acyclic. Via the mapping cylinder factorisation above and the fact that A is cofibrant, we have

$$\begin{tikzcd} A \arrow[d, tail, "i_A"'] \arrow[dr, "f"] & \\ C(f) \arrow[r, "q_B"'] & B \end{tikzcd}$$

For $F(f) \in W$, it suffices that $F(i_A), F(q_B) \in W$. We have $F(i_A)$ because it is an acyclic cofibration by two-out-of-three with $q_B, f \in \mathscr{W}$. For $F(q_B)$, note that since we have

$$\begin{tikzcd} B \arrow[d, tail, "i_B"', "\sim"] \arrow[dr, "\mathbb{1}"] & \\ C(f) \arrow[r, "q_B"'] & B \end{tikzcd}$$

it suffices to show that $F(i_B) \in W$ by two-out-of-three for W. Indeed we have $F(i_B) \in F(\mathscr{W} \cap \mathscr{C}) \subseteq W$ by assumption.

That technically completes the proof, but let us make a remark on how to make mapping cylinders for intuition's sake. Such a factorisation can be obtained from a cylinder object (IA, i, σ) of A when it is cofibrant by the pushout

$$\begin{tikzcd} A + A \arrow[r, tail, "i"] \arrow[d, "\mathbb{1}, f"'] & IA \arrow[d] \arrow[ddr, bend left, "f\sigma"] & \\ A + B \arrow[r, tail] \arrow[drr, bend right, "f, \mathbb{1}"'] \arrow[ur, phantom, "\ulcorner"] & C(f) \arrow[dr, dashed, "q_B"] & \\ & & B \end{tikzcd}$$

To show $q_B \in \mathscr{W}$, it suffices to show that the composition $i_B : B \to A + B \to C(f)$ is acyclic. This fits into the commutative diagram of pushout squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A + A \arrow[r, tail, "i"] \arrow[d, "\mathbb{1}, f"] & IA \arrow[d] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A + B \arrow[r, tail] \arrow[ur, phantom, "\ulcorner"] & C(f) \end{tikzcd}$$

So we see that $B \to C(f)$ is a cobase change of $i_0 : A \to IA$. Since A is cofibrant, i_0 is an acyclic cofibration and hence so is $B \to C(f)$. ■

We will use Yoneda's lemma to show $\mathcal{E}_{fc} \to \pi\mathcal{E}_{fc}$ inverts weak equivalences. Specifically, we show the following :

> *Lemma. Let A be cofibrant. Then $\pi^l(A, \_) : \mathcal{E}_f \to \mathbf{Set}$ inverts weak equivalences. In particular, for an acyclic fibration $f : X \to S$ where S is cofibrant, the set of sections of f is singleton up to left homotopy.*
>
> *Proof.* By the lemma of Kennth Brown, it suffices to show that $\pi^l(A, \_)$ inverts acyclic fibrations. So let $f : X \to Y$ be an acyclic fibration. Then surjectivity of $\pi^l(A, f)$ is equivalent to solving lifting problems of the form :
>
> $$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "\sim"] \\ A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$
>
> which we have due to A being cofibrant and the WFS $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$. On the other hand injectivity of $\pi^l(A, f)$ is equivalent to solving lifting problems of the form :
>
> $$\begin{tikzcd} A + A \arrow[r] \arrow[d, tail, "i"'] & X \arrow[d, two heads, "\sim"] \\ IA \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$
>
> which we also have by the same reason. ∎

We have thus proved that $\mathcal{E}_{fc} \to \pi\mathcal{E}_{fc}$ inverts weak equivalences. So by the UP of localisation, we have a commuting triangle :

$$\begin{tikzcd} \mathcal{E}_{fc} \arrow[d] \arrow[dr] & \\ \mathbf{Ho}\,\mathcal{E}_{fc} \arrow[r, "T"'] & \pi\mathcal{E}_{fc} \end{tikzcd}$$

The idea here is that homotopic maps are identified under functors which invert weak equivalences.

> *Lemma. Let $F : \mathcal{E} \to C$ be a functor.*
>
> 1. *If F inverts weak equivalences, then F identifies left homotopic morphisms.*
> 2. *This is also true for functors $F : \mathcal{E}_c \to C$*
> 3. *It is furthermore also true for $F : \mathcal{E}_{fc} \to C$.*
>
> *Proof.* (1) The key is that for any cylinder object IA of an object A, "the cylinder collapses". More precisely, the morphism $IA \to A$ gets inverted under F, and so the two inclusions $i_0, i_1 : A \to IA$ get identified.

(2) To reuse the above argument, we need only to ensure any cylinder object IA of cofibrant A is also cofibrant. But this is true since the composition $\varnothing \to A \to A+A \to IA$ consists of cofibrations.

(3) We can try again to reuse the above argument, in which case we need to prove that any cylinder object IA of a fibrant-cofibrant object A is also fibrant-cofibrant. This is false in general. [a] Instead, we proceed to show that we only need the existence of *one* fibrant-cofibrant cylinder object IA for each A, which indeed exists by $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$-factoring $A + A \to A$.

> *Lemma. Let $A, X \in \mathcal{E}$ where X is fibrant. Then the left homotopy relation on $\mathcal{E}(A, X)$ can be defined with respect to a single cylinder object of A.*
>
> *Proof.* Let $f, g : A + A \to X$, $(IA, i, \sigma), (JA, j, \tau)$ cylinder objects of A and $H : IA \to X$ a left homotopy from f to g. By the WFS $(\mathscr{W} \cap \mathscr{C}, \mathscr{F})$, the fact that X is fibrant says that we can change IA along any acyclic cofibration $IA \to I'A$.
>
> $$\begin{tikzcd} IA \arrow[r, "H"] \arrow[d, tail, "\sim"'] & X \arrow[d] \\ I'A \arrow[r] \arrow[ur, dashed, "H'"] & \bullet \end{tikzcd}$$
>
> So we factor $\sigma : IA \to A$ into an acyclic cofibration followed by a fibraton to get the following commutative diagram :
>
> $$\begin{tikzcd} A + A \arrow[r, tail] \arrow[d, tail] & IA \arrow[r, tail, "\sim"] \arrow[d, "\sim"] & I'A \arrow[d, two heads] \\ JA \arrow[r, "\sim"'] \arrow[urr, dashed] & A \arrow[r, "\mathbb{1}"'] & A \end{tikzcd}$$
>
> The outter square is a lifting problem to which a solution would give a left homotopy $\tilde{H} : JA \to X$ from f to g. But acyclicity of $IA \to A$ implies $I'A \to A$ is an acyclic fibration by two-out-of-three for $\mathscr{W}$, and so we indeed have a solution. ∎
>
> [a] In simplicial sets, $X + X \to X \times \Delta^1 \to X$ is the standard cylinder object to take.
> But one can see that for $X = \Delta^0$, $\Delta^1 \to \Delta^0$ is not an acyclic fibration, i.e. Δ^1 is not a Kan complex.

∎

The above result means that we have a well-defined functor U making the commutative triangle :

$$\begin{tikzcd} \mathcal{E}_{fc} \arrow[r] \arrow[dr] & \pi\mathcal{E}_{fc} \arrow[d, "U"] \\ & \mathbf{Ho}\,\mathcal{E}_{fc} \end{tikzcd}$$

The claim is that U, T are inverses. For this, it suffices that both $\mathcal{E}_{fc} \to \mathbf{Ho}\,\mathcal{E}_{fc}$ and $\mathcal{E}_{fc} \to \pi\mathcal{E}_{fc}$ are epimorphisms of categories. The former we have seen before. For the latter, note that $\mathcal{E}_{fc} \to \pi\mathcal{E}_{fc}$ is full and bijective on objects. This concludes $\pi\mathcal{E}_{fc} \cong \mathbf{Ho}\,\mathcal{E}_{fc}$ as categories under $\mathcal{E}_{fc}$.

Finally, let us show that a morphism f in $\mathcal{E}_{fc}$ is acyclic iff it it inverted in $\pi\mathcal{E}_{fc}$. The forward direction has been proved. For the converse, we first note that suffices to do the case that f is a fibration : factor $f = pu$ where $u \in \mathscr{W} \cap \mathscr{C}$ and $p \in \mathscr{F}$. Then we know already that u is inverted in $\pi\mathcal{E}_{fc}$. So by two-out-of-three for $\mathscr{W}$ and isomorphisms (in $\pi\mathcal{E}_{fc}$), we see that $f \in \mathscr{W}$ iff $p \in \mathscr{W}$ and f is inverted in $\pi\mathcal{E}_{fc}$ iff p is.

So WLOG assume $f : X \to Y$ is a fibration which is inverted in $\pi\mathcal{E}_{fc}$. This means we have a $g : Y \to X$ with $fg \sim \mathbb{1}_Y$ and $gf \sim \mathbb{1}_X$. Notice that from X cofibrant, acyclicity of morphisms out of X is preserved along left homotopies. Since $\mathbb{1}_X \in \mathscr{W}$, we have $gf \in \mathscr{W}$. Now *if* g was a "true" section of f, i.e. $fg = \mathbb{1}_Y$, then we would have f as a retract of gf.

$$
\begin{tikzcd}
X \arrow[r, "\mathbb{1}"] \arrow[d, "f"'] & X \arrow[r, "\mathbb{1}"] \arrow[d, dashed, "gf"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

And hence we would be able to conclude that f is acyclic since $\mathscr{W}$ is closed under retracts.

To that end, we seek to replace g with some s where $fs = \mathbb{1}_Y$. To ensure that $sf \in \mathscr{W}$, it suffices to keep $g \sim s$ for then $sf \sim gf \sim \mathbb{1}_X$. We can indeed find such $s : Y \to X$ as $s = Hi_1$ where H is a solution of the lifting problem

$$
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, tail, "i_0"'] & X \arrow[d, two heads, "f"] \\
IY \arrow[r] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}
$$

where the bottom morphism exhibits $fg \sim \mathbb{1}_Y$.

(4) Although we used functorial factorisations to obtain the square of equivalence of homotopy categories, we won't need functoriality of factorisations to show the desired result.

Let $f : X \to Y$ in $\mathcal{E}$. Then by repeated application of factorisations into $(\mathscr{W} \cap \mathscr{C}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$, we obtain the commutative diagram :

$$
\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, tail, "\sim"'] & Y \arrow[d, tail, "\sim"] & \\
 & RX \arrow[r, two heads, "f_1"] & RY \arrow[r, two heads] & \bullet \\
\varnothing \arrow[r, tail] & WX \arrow[r, tail, "f_2"] \arrow[u, two heads, "\sim"] & WY \arrow[u, two heads, "\sim"'] &
\end{tikzcd}
$$

This is exactly what we would get from functorial factorisations so not much surprises here. We have RX, RY fibrant and WX, WY fibrant-cofibrant. Now we see that $f \in \mathscr{W}$ iff $f_1 \in \mathscr{W}$ iff $f_2 \in \mathscr{W}$ iff f_2 is inverted in $\mathbf{Ho}\,\mathcal{E}_{fc}$ iff f_1 is inverted in $\mathbf{Ho}\,\mathcal{E}_f$ iff f is inverted in $\mathbf{Ho}\,\mathcal{E}$ as desired.

□

5 Equivalence of Model Categories

Remark. From the Ken Brown lemma, we see that for a functor $F : M \to N$ between model categories, it is homotopical on M_c if F maps acyclic cofibrations between cofibrant objects to weak equivalences. In the presence of a functorial factorisations in the model structure of M, we can then compute the left derived functor of F. This partially motivates the following definition.

Definition

Let $F : M \to N$ be a functor between model categories. Then F is called *left Quillen* when $F(\mathscr{C}) \subseteq \mathscr{C}$ and $F(\mathscr{W} \cap \mathscr{C}) \subseteq \mathscr{W} \cap \mathscr{C}$.

The above dualises to *right Quillen* functors.

Proposition – Quillen Adjunctions

Let $F \dashv G : M \leftrightarrow N$ be an adjunction where F is left Quillen and G is right Quillen. Assume that M, N have functorial factorisations. Then the total left derived functor of F and the total right derived functor of G exists, together forming an adjunction at the level of homotopy categories.

$$\begin{tikzcd} \mathbf{Ho}\,M \arrow[r, shift left=1.5ex, "\mathbf{L}F"] \arrow[r, phantom, "\perp"] & \mathbf{Ho}\,N \arrow[l, shift left=1.5ex, "\mathbf{R}G"] \end{tikzcd}$$

Proof. Direct application of derived functors via deformations. □

6 How to give Model Structures - Cofibrant Generation

TODO