

Infinity Categories via Homotopical Segal Condition

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1 Segal Condition - Categories as simplicial sets equipped with a path composition operation

Proposition – Segal Condition

Let \mathbf{Cat} be the (1-)category of (small) categories. Define the *nerve functor* $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$ as the nerve along the forgetful functor $\mathbb{A} \rightarrow \mathbf{Cat}$.

Let $h : \mathbf{sSet} \rightarrow \mathbf{Cat}$ by sending X to the *homotopy category of X* , denoted hX , defined as follows : ^a

- objects of hX are points of X
- For simplicity, call sequences of composable edges in X *chains* in X . Given two chains γ, δ in X , we say γ is a *1-step simplification of δ* when δ can be obtained from γ by either :
 - replacing an edge h in γ by a length two chain f, g whenever we have a triangle $\sigma : \Delta^2 \rightarrow X$ of the form :

$$\dots \longrightarrow \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \longrightarrow \dots \quad \rightsquigarrow \quad \dots \longrightarrow \bullet \xrightarrow{h} \bullet \longrightarrow \dots$$

- inserting a constant edge in γ

$$\dots \longrightarrow \bullet \xrightarrow{id} \bullet \longrightarrow \dots \quad \rightsquigarrow \quad \dots \longrightarrow \bullet \longrightarrow \dots$$

(This is actually covered by the previous case via degenerate triangles.)

We say γ is a *simplification* of δ when there's a finite sequence of 1-step simplifications from δ to γ .

We declare $\gamma \sim \delta$ when there exists a chain ε that simplifies to both γ and δ . This defines an equivalence relation on chains in X and we define morphisms in hX to be chains in X up to this equivalence.

- Composition of morphisms is given by concatenation of chains. This respects simplifications and hence is well-defined on hX .
- Associativity of composition comes from associativity of concatenation of chains.
- For every point x in X , the equivalence class of the constant edge id_x works as the identity morphism of x in hX .

Then

1. (Segal-Condition)^b a simplicial set X is in the essential image of N if and only if for all $n \in \mathbb{N}$, we have a bijection

$$\mathbf{sSet}(\Delta^{n+2}, X) \xrightarrow{\cong} \mathbf{sSet}(I^{n+2}, X)$$

where I^k is the *spine* of Δ^k , the sub-simplicial set of Δ^k generated by the edges " $0 \rightarrow 1, 1 \rightarrow 2, \dots, k-1 \rightarrow k$ ".

2. N is fully faithful. Henceforth, we can see \mathbf{Cat} as a subcategory of \mathbf{sSet} .
3. Given $X \in \mathbf{sSet}$, we have a natural morphism $h : X \rightarrow N(hX)$ defined by Yoneda's lemma and the following commutative square natural in the first argument :

$$\begin{array}{ccc} \mathbf{sSet}(\Delta^n, X) & \longrightarrow & \mathbf{sSet}(\Delta^n, N(hX)) \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{sSet}(I^n, X) & \longrightarrow & \mathbf{sSet}(I^n, N(hX)) \end{array}$$

where the bottom horizontal map is given by realising edges in X as morphisms in hX . Pre-composition with h gives a bijection

$$\mathbf{Cat}(hX, C) \xrightarrow{\cong} \mathbf{sSet}(X, NC)$$

functorial in X and C , defining an adjunction $h \dashv N$.^c

Hence \mathbf{Cat} is equivalent to a full reflective subcategory of \mathbf{sSet} . In particular, since \mathbf{sSet} is complete and cocomplete, so is \mathbf{Cat} and its limits are computed in \mathbf{sSet} and colimits computed by taking the reflection of the colimit in \mathbf{sSet} . Since every inclusion of a full reflective subcategory is monadic, we can think of categories as simplicial sets equipped with an operation to compose edges.

^aLand 1.2.5.

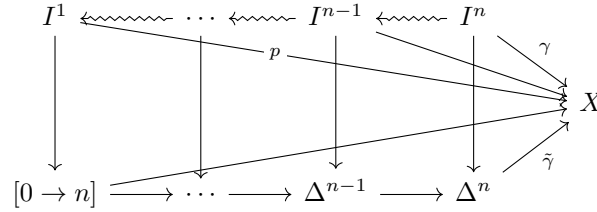
^bLand 1.1.52.

^cLand 1.2.18.

Proof. (1) Nerves of categories satisfy the Segal condition because they have a composition operation.

Now let $X \in \mathbf{sSet}$ satisfy the Segal condition. The idea is that the Segal condition is enough to recover all the higher simplices in X from $N(hX)$. From the definition of $X \rightarrow N(hX)$, we also see that to prove it is an isomorphism, it suffices for $\mathbf{sSet}(I^n, X) \cong \mathbf{sSet}(I^n, N(hX))$. Since the map takes a length n chain in X to a length n chain in $N(hX)$, it suffices to show $\mathbf{sSet}(I^1, X) \cong \mathbf{sSet}(I^1, N(hX))$. For surjectivity, note that any chain in X has a simplification to a single edge by the surjectivity part of the Segal condition.

For injectivity, let $p, q : \Delta^1 \rightarrow X$ such that $[p] = [q] : \Delta^1 \rightarrow N(hX)$. This means there exists $\gamma : I^n \rightarrow X$ with $n \geq 1$ that simplifies both to p and to q , in possibly different ways. However, we know that $\gamma : I^n \rightarrow X$ extends to a $\tilde{\gamma} : \Delta^n \rightarrow X$ by the surjectivity part of the Segal condition. Now we inspect how γ can simplify to p .



The top row shows a sequence of 1-step simplifications from γ to p . Each step is witness by a triangle in X composing two edges from the previous step. However, such triangles must in fact be 2-dimensional faces of $\tilde{\gamma}$ by the injectivity part of the Segal condition at $n = 2$. So we see that each step $I^k \rightarrow X$ must in fact factor through a sub-simplex $\Delta^k \rightarrow \Delta^n$ containing the edge $0 \rightarrow n$. Simplifying all the way down, we see that p must be the restriction of $\tilde{\gamma}$ to the edge $0 \rightarrow n$. By the same argument, this must also be q and hence $p = q$.

(2) This can be proved either by showing a bijection $\mathbf{Cat}(C, D) \cong \mathbf{sSet}(NC, ND)$ functorial in C, D or showing that $hNC \cong C$ functorially in C . We will show the former. Land chooses to show the latter (See Land - Corollary 1.2.14).

Each morphism $NC \rightarrow ND$ defines a functor $C \rightarrow D$ by following where edges go. Composition is preserved because of the Segal condition at $n = 2$. Identity morphisms are preserved because image of these are precisely the constant edges in the nerve and image of constant edges are constant edges. One can recover the original morphism $NC \rightarrow ND$ from the functor $C \rightarrow D$ via the following commutative square natural in the first argument :

$$\begin{array}{ccc} \mathbf{sSet}(\Delta^n, NC) & \longrightarrow & \mathbf{sSet}(\Delta^n, ND) \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{sSet}(I^n, NC) & \longrightarrow & \mathbf{sSet}(I^n, ND) \end{array}$$

where the vertical bijections are due to the Segal condition. The above square also gives a way of extending any functor $C \rightarrow D$ to a morphism of nerves. This gives a bijection $\mathbf{sSet}(NC, ND) \cong \mathbf{Cat}(C, D)$ and is easily seen to be functorial in C, D .

(3) It suffices to show that for every $C \in \mathbf{Cat}$ the map

$$\mathbf{sSet}(N(hX), NC) \rightarrow \mathbf{sSet}(X, NC)$$

is bijective.

For injectivity, let $F, G : hX \rightarrow C$ be functors such that $Fh = Gh$. We can see that F, G agree on objects. To see the same for morphisms, let p be a morphism in hX . Then there exists $\gamma : I^n \rightarrow X$ representing p . This means that in hX , $h\gamma$ composes to p . Then $Fp = Fh\gamma = Gh\gamma = Gp$.

For surjectivity, let $F : X \rightarrow NC$. Define a functor $\tilde{F} : hX \rightarrow C$ as follows :

- on objects, do what F does to points.
- Map chains in X to morphisms in C by taking image under F then using the composition operation of C . Simplifications $\delta \rightsquigarrow \gamma$ are mapped to the same morphism in C by associativity of composition in C , and hence gives a way of mapping morphisms in hX to morphisms in C .
- Compositions are preserved by the Segal condition on NC . Identity morphisms are preserved since constant paths in C are exactly identity morphisms.

Viewing \mathbf{Cat} as a subcategory of \mathbf{sSet} , we see that we need to show $\tilde{F}h = F$. By the density theorem, it suffices to check for each $x : \Delta^n \rightarrow X$ that $\tilde{F}hx = Fx$. But by the Segal condition, it suffices to check $\tilde{F}hx = Fx$ on the spine I^n . The spine is determined by its n edges, so it suffices to check $\tilde{F}hx = Fx$ for $x : \Delta^1 \rightarrow X$. But this is true since x represents hx .

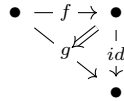
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2 Motivating Lifting Inner Horns - Simpler Description of Homotopy Categories

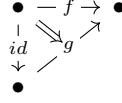
Proposition – Better Description of Homotopy Category ^a

Let $X \in \mathbf{sSet}$. Suppose X lifts against inner 2-horns and inner 3-horns. We can then define a category πX as follows :

- objects of πX are points of X .
- For $x, y : \Delta^0 \rightarrow X$, the following two relations on $X(x, y)$ are the same :
 - $f \sim g$ when there exists a triangle in X of the following form :



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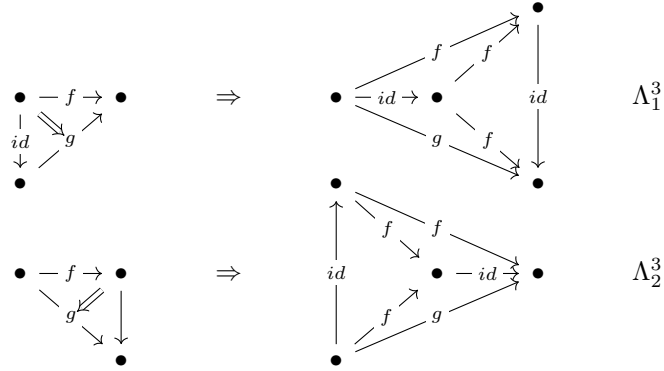
In fact, this defines an equivalence relation. Define $\pi X(x, y)$ to be the quotient of $X(x, y)$ by it.

- For $I^2 \rightarrow X$ representing two composable edges f, g , define *composites* of f, g to be extensions of $I^2 \rightarrow X$ to all of Δ^2 . Then for any pair of composable edges in X , a composite exists and is unique up to the equivalence, hence defining a composition operation for πX .

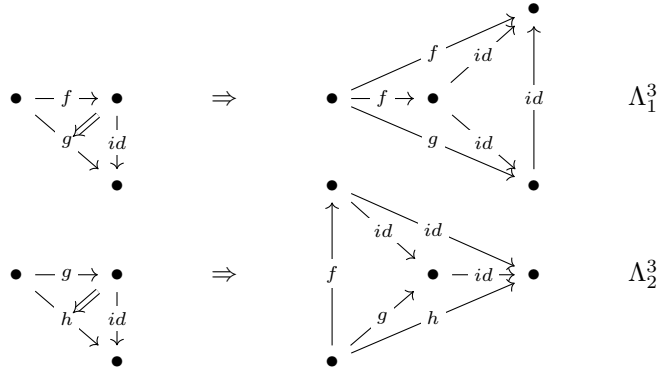
Then $hX \cong \pi X$.

^aThis is Land 1.2.9, 1.2.10, 1.2.11 combined.

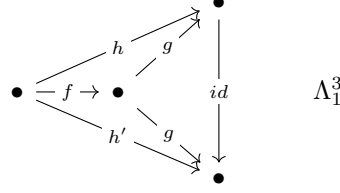
Proof. (The two relations)



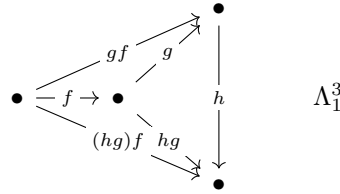
(Equivalence Relation) Symmetry and transitivity are given respectively by :



(Composites) Existence of composites in X is precisely the fact that X lifts against the inner 2-horn. For uniqueness of composites up to equivalence, use :



(Associativity of Composition) Let f, g, h be a triple of composable edges in X . Then we have



where gf is a composite of f, g and hg is a composite of g, h , and $(hg)f$ is a composite of hg with f . Filling the above inner horn shows that $(hg)f$ is in fact also a composite for h with gf . By uniqueness of composites up to equivalence, this proves associativity of composition.

($hX \cong \pi X$) From the universal property of $h : X \rightarrow N(hX)$, we have a commuting triangle of simplicial sets :

$$\begin{array}{ccc} X & & \\ h \downarrow & \searrow & \\ N(hX) & \longrightarrow & N(\pi X) \end{array}$$

By fully faithfulness of N , we need to show the horizontal morphism is an isomorphism. It is bijective at the level of objects. So we look at the level of morphisms. Let x, y be points of X . Then

$$\begin{array}{ccc} \mathbf{sSet}(\Delta^1, X) & & \\ h \downarrow & \searrow & \\ hX(x, y) & \longrightarrow & \pi X(x, y) \end{array}$$

The diagonal map is surjective by definition, hence so is the bottom map. The vertical map is also surjective : 1-step simplifications are obtained by composites of pairs of morphisms. So to show injectivity of the bottom map, we can take two $f, g \in hX(x, y)$, lift them to edges in X then quotient to $\pi X(x, y)$. If they are equal

in $\pi X(x, y)$, then there is a triangle in X witnessing their equivalence. The spine of this triangle simplifies both to f and g as edges in X , and thus $f = g \in hX(x, y)$ as desired. \square

Intuition. Let $X \in \mathbf{sSet}$ and $(f, g) : \Lambda_1^2 \rightarrow X$ a pair of composable edges. Then the “space” of composites of f with g can be defined by the following cartesian square :

$$\begin{array}{ccc} \mathrm{Comp}(f, g) & \longrightarrow & X^{\Delta^2} \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{(f, g)} & X^{\Lambda_1^2} \end{array}$$

In the above proposition, we saw that being able to lift against inner 3-horns implies that $\pi_0^{\Delta} \mathrm{Comp}(f, g)$ is singleton. The incarnation of infinity categories as *quasi-categories* will say that the family $X^{\Delta^2} \rightarrow X^{\Lambda_1^2}$ is “homotopically trivial”. In particular, we will much later show that $\mathrm{Comp}(f, g)$ is a contractible Kan complex.

Proposition – Equivalent Definitions of Infinity Categories (Preview) ^a

Let X be a simplicial set. Define the following classes of morphisms of simplicial sets :

- inner horns $\mathrm{IH} = \{\Lambda_k^n \rightarrow \Delta^n \mid n \geq 2, 0 < k < n\}$.
- cell inclusions $\mathrm{Cell} = \{\partial \Delta^n \rightarrow \Delta^n \mid n \geq 0\}$

Then the following are equivalent :

- X lifts against IH
- (Homotopy Unique Composition) $X^{\Delta^2} \rightarrow X^{\Lambda_1^2}$ lifts against Cell .
- (Homotopical Segal Condition) For $n \geq 2$, $X^{\Delta^n} \rightarrow X^{I^n}$ lifts against Cell .

Call X an *infinity category* when it satisfies any (and thus all) of the above.

^aLand - 1.3.34

Example. – Infinity groupoids $S(X)$ of topological spaces X , i.e. nerves of topological spaces along the standard realisation of simplicies $\Delta \rightarrow \mathbf{Top}$.

- Nerves of 1-categories.
- Later we will be able to turn categories enriched over \mathbb{Z} -complexes into infinity categories via a procedure called the **dg-nerve**.

This will give examples from homological algebra and algebraic geometry : the dg-nerve $N_{\mathrm{dg}}(\mathrm{Ch}A)$ of (unbounded) complexes in a (Grothendieck) abelian category.

- In order to take “higher” (co)limits of infinity categories, we will have to make an infinity category \mathbf{Cat}_{∞} of (small) infinity categories. More important yet : we will want to make “families of infinity categories over a base infinity category B ” which vary contravariantly along B . These should corre-

respond to functors $F : B^{op} \rightarrow \mathbf{Cat}_\infty$. This is the subject of Lurie's straightening-unstraightening theorem.