Infinity Categories via Homotopical Segal Condition

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2021 Autumn

Contents

1 Segal Condition - Categories as simplicial sets equipped with a path composition operation

Proposition – Segal Condition

Let **Cat** be the (1-)category of (small) categories. Define the *nerve functor* $N : \mathbf{Cat} \to \mathbf{sSet}$ as the nerve along the forgetful functor $\Delta \to \mathbf{Cat}$.

Let $h : \mathbf{sSet} \to \mathbf{Cat}$ by sending X to the *homotopy category of* X, denoted hX, defined as follows : [a]

– objects of hX are points of X

– For simplicity, call sequences of composable edges in X *chains* in X. Given two chains γ, δ in X, we say γ is a *1-step simplification of* δ when δ can by obtained from γ by either :

  – replacing an edge h in γ by a length two chain f, g whenever we have a triangle $\sigma : \Delta^2 \to X$ of the form :

$$\begin{tikzcd} \ldots \arrow[r] & \bullet \arrow[r, "f"'] & \bullet \arrow[r, "g"'] & \bullet & \ldots & \rightsquigarrow & \ldots \arrow[r] & \bullet \arrow[rr, "h"] \arrow[dr, "f"'] & & \bullet \arrow[r] & \ldots \\ & & & & & & & & \bullet \arrow[u, phantom, "\overset{\sigma}{\Uparrow}"] \arrow[ur, "g"'] & & \end{tikzcd}$$

  – inserting a constant edge in γ

$$\ldots \longrightarrow \bullet \xrightarrow{id} \bullet \longrightarrow \ldots \quad \rightsquigarrow \quad \ldots \longrightarrow \bullet \longrightarrow \ldots$$

(This is actually covered by the previous case via degenerate triangles.)

We say γ is a *simplification* of δ when there's a finite sequence of 1-step simplifications from δ to γ.

We declare $\gamma \sim \delta$ when there exists a chain ε that simplifies to both γ and δ. This defines an equivalence relation on chains in X and we define morphisms in hX to be chains in X up to this equivalence.

- Composition of morphisms is given by concatenation of chains. This respects simplifications and hence is well-defined on hX.
- Associativity of composition comes from associativity of concatenation of chains.
- For every point x in X, the equivalence class of the constant edge id_x works as the identity morphism of x in hX.

Then

1. (Segal-Condition)[b] a simplicial set X is in the essential image of N if and only if for all $n \in \mathbb{N}$, we have a bijection

$$\mathbf{sSet}(\Delta^{n+2}, X) \xrightarrow{\cong} \mathbf{sSet}(I^{n+2}, X)$$

where I^k is the *spine of* Δ^k, the sub-simplicial set of Δ^k generated by the edges "$0 \to 1, 1 \to 2, \ldots, k-1 \to k$".

2. N is fully faithful. Henceforth, we can see $\mathbf{Cat}$ as a subcategory of $\mathbf{sSet}$.

3. Given $X \in \mathbf{sSet}$, we have a natural morphism $h : X \to N(hX)$ defined by Yoneda's lemma and the following commutative square natural in the first argument :

$$\begin{tikzcd}
\mathbf{sSet}(\Delta^n, X) \arrow[r] \arrow[d, "\cong"'] & \mathbf{sSet}(\Delta^n, N(hX)) \arrow[d, "\cong"] \\
\mathbf{sSet}(I^n, X) \arrow[r] & \mathbf{sSet}(I^n, N(hX))
\end{tikzcd}$$

where the bottom horizontal map is given by realising edges in X as morphisms in hX. Precomposition with h gives a bijection

$$\mathbf{Cat}(hX, C) \xrightarrow{\cong} \mathbf{sSet}(X, NC)$$

functorial in X and C, defining an adjunction $h \dashv N$. [c]

Hence $\mathbf{Cat}$ is equivalent to a full reflective subcategory of $\mathbf{sSet}$. In particular, since $\mathbf{sSet}$ is complete and cocomplete, so is $\mathbf{Cat}$ and its limits are computed in $\mathbf{sSet}$ and colimits computed by taking the reflection of the colimit in $\mathbf{sSet}$. Since every inclusion of a full reflective subcategory is monadic, we can think of categories as simplicial sets equipped with an operation to compose edges.

[a]Land 1.2.5.
[b]Land 1.1.52.
[c]Land 1.2.18.

Proof. (1) Nerves of categories satisfy the Segal condition because they have a composition operation.

Now let $X \in \mathbf{sSet}$ satisfy the Segal condition. The idea is that the Segal condition is enough to recover all the higher simplicies in X from $N(hX)$. From the definition of $X \to N(hX)$, we also see that to prove it is an isomorphism, it suffices for $\mathbf{sSet}(I^n, X) \cong \mathbf{sSet}(I^n, N(hX))$. Since the map takes a length n chain in X to a length n chain in $N(hX)$, it suffices to show $\mathbf{sSet}(I^1, X) \cong \mathbf{sSet}(I^1, N(hX))$. For surjectivity, note that any chain in X has a simplification to a single edge by the surjectivity part of the Segal condition.

For injectivity, let $p, q : \Delta^1 \to X$ such that $[p] = [q] : \Delta^1 \to N(hX)$. This means there exists $\gamma : I^n \to X$ with $n \geq 1$ that simpifies both to p *and* to q, in possibly different ways. However, we know that $\gamma : I^n \to X$ extends to a $\tilde{\gamma} : \Delta^n \to X$ by the surjectivity part of the Segal condition. Now we inspect how γ can simplify to p.

$$\begin{tikzcd}
I^1 \arrow[d] \arrow[rrrrd, "p"] & \cdots \arrow[l, leftsquigarrow] \arrow[d] & I^{n-1} \arrow[l, leftsquigarrow] \arrow[d] \arrow[rrd] & I^n \arrow[l, leftsquigarrow] \arrow[d] \arrow[rd, "\gamma"] & \\
 & & & & X \\
{[0 \to n]} \arrow[r] \arrow[rrrru] & \cdots \arrow[r] & \Delta^{n-1} \arrow[r] & \Delta^n \arrow[ru, "\tilde{\gamma}"'] &
\end{tikzcd}$$

The top row shows a sequence of 1-step simplifications from γ to p. Each step is witness by a triangle in X composing two edges from the previous step. However, such triangles must in fact be 2-dimensional faces of $\tilde{\gamma}$ by the injectivity part of the Segal condition at $n = 2$. So we see that each step $I^k \to X$ must in fact factor through a sub-simplex $\Delta^k \to \Delta^n$ containing the edge $0 \to n$. Simplifying all the way down, we see that p must be the restriction of $\tilde{\gamma}$ to the edge $0 \to n$. By the same argument, this must also be q and hence $p = q$.

(2) This can be proved either by showing a bijection $\mathbf{Cat}(C, D) \cong \mathbf{sSet}(NC, ND)$ functorial in C, D or showing that $hNC \cong C$ functorially in C. We will show the former. Land chooses to show the latter (See Land - Corollary 1.2.14).

Each morphism $NC \to ND$ defines a functor $C \to D$ by following where edges go. Composition is preserved because of the Segal condition at $n = 2$. Identity morphisms are preserved because image of these are precisely the constant edges in the nerve and image of constant edges are constant edges One can recover the original morphism $NC \to ND$ from the functor $C \to D$ via the following commutative square natural in the first argument :

$$\begin{tikzcd}
\mathbf{sSet}(\Delta^n, NC) \arrow[r] \arrow[d, "\cong"'] & \mathbf{sSet}(\Delta^n, ND) \arrow[d, "\cong"] \\
\mathbf{sSet}(I^n, NC) \arrow[r] & \mathbf{sSet}(I^n, ND)
\end{tikzcd}$$

where the vertical bijections are due to the Segal condition. The above square also gives a way of extending any functor $C \to D$ to a morphism of nerves. This gives a bijection $\mathbf{sSet}(NC, ND) \cong \mathbf{Cat}(C, D)$ and is easily seen to be functorial in C, D.

(3) It suffices to show that for every $C \in \mathbf{Cat}$ the map

$$\mathbf{sSet}(N(hX), NC) \to \mathbf{sSet}(X, NC)$$

is bijective.

For injectivity, let $F, G : hX \to C$ be functors such that $Fh = Gh$. We can see that F, G agree on objects. To see the same for morphisms, let p be a morphism in hX. Then there exists $\gamma : I^n \to X$ representing p. This means that in hX, $h\gamma$ composes to p. Then $Fp = Fh\gamma = Gh\gamma = Gp$.

For surjectivity, let $F : X \to NC$. Define a functor $\tilde{F} : hX \to C$ as follows :

- on objects, do what F does to points.
- Map chains in X to morphisms in C by taking image under F then using the composition operation of C. Simplifications $\delta \rightsquigarrow \gamma$ are mapped to the same morphism in C by associativity of composition in C, and hence gives a way of mapping morphisms in hX to morphisms in C.
- Compositions are preserved by the Segal condition on NC. Identity morphisms are preserved since constant paths in C are exactly identity morphisms.

Viewing $\mathbf{Cat}$ as a subcategory of $\mathbf{sSet}$, we see that we need to show $\tilde{F}h = F$. By the density theorem, it suffices to check for each $x : \Delta^n \to X$ that $\tilde{F}hx = Fx$. But by the Segal condition, it suffices to check $\tilde{F}hx = Fx$ on the spine I^n. The spine is determined by its n edges, so it suffices to check $\tilde{F}hx = Fx$ for $x : \Delta^1 \to X$. But this is true since x represents hx.

□

2 Motivating Lifting Inner Horns - Simpler Description of Homotopy Categories

Proposition – Better Description of Homotopy Category [a]

Let $X \in \mathbf{sSet}$. Suppose X lifts against inner 2-horns and inner 3-horns. We can then define a category πX as follows :

- objects of piX are points of X.
- For $x, y : \Delta^0 \to X$, the following two relations on $X(x, y)$ are the same :
  - $f \sim g$ when there exists a triangle in X of the following form :

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "g"'] & \bullet \arrow[d, "id"] \arrow[dl, phantom, "\Swarrow"] \\ & \bullet \end{tikzcd}$$

  - $f \sim g$ when there exists a triangle in X of the following form :

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "id"'] \arrow[dr, phantom, "\Rightarrow"] & \bullet \\ \bullet \arrow[ur, "g"'] & \end{tikzcd}$$

  In fact, this defines an equivalence relation. Define $\pi X(x, y)$ to be the quotient of $X(x, y)$ by it.

- For $I^2 \to X$ representing two composable edges f, g, define *composites* of f, g to be extensions of $I^2 \to X$ to all of Δ^2. Then for any pair of composable edges in X, a composite exists and is unique up to the equivalence, hence defining a composition operation for πX.

Then $hX \cong \pi X$.

[a]This is Land 1.2.9, 1.2.10, 1.2.11 combined.

Proof. (*The two relations*)

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "id"'] \arrow[dr, phantom, "\Rightarrow"] & \bullet \\ \bullet \arrow[ur, "g"'] & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & & \bullet \arrow[dd, "id"] & \\ \bullet \arrow[r, "id"] \arrow[urr, "f"] \arrow[drr, "g"'] & \bullet \arrow[ur, "f"] \arrow[dr, "f"'] & & \Lambda^3_1 \\ & & \bullet & \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "g"'] \arrow[dr, phantom, shift left, "\Leftarrow"] & \bullet \arrow[d] \\ & \bullet \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \bullet \arrow[dr, "f"] \arrow[drr, "f"] & & & \\ & \bullet \arrow[r, "id"] & \bullet & \Lambda^3_2 \\ \bullet \arrow[uu, "id"] \arrow[ur, "f"] \arrow[urr, "g"'] & & & \end{tikzcd}$$

(*Equivalence Relation*) Symmetry and transitivity are given respectively by :

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "g"'] \arrow[dr, phantom, shift left, "\Leftarrow"] & \bullet \arrow[d, "id"] \\ & \bullet \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & & \bullet & \\ \bullet \arrow[r, "f"] \arrow[urr, "f"] \arrow[drr, "g"'] & \bullet \arrow[ur, "id"] \arrow[dr, "id"'] & & \Lambda^3_1 \\ & & \bullet \arrow[uu, "id"'] & \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[dr, "h"'] \arrow[dr, phantom, shift left, "\Leftarrow"] & \bullet \arrow[d, "id"] \\ & \bullet \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \bullet \arrow[dr, "id"] \arrow[drr, "id"] & & & \\ & \bullet \arrow[r, "id"] & \bullet & \Lambda^3_2 \\ \bullet \arrow[uu, "f"] \arrow[ur, "g"] \arrow[urr, "h"'] & & & \end{tikzcd}$$

(*Composites*) Existence of composites in X is precisely the fact that X lifts against the inner 2-horn. For uniqueness of composites up to equivalence, use :

$$\begin{tikzcd}
 & & \bullet \arrow[dd, "id"] \\
\bullet \arrow[r, "f"] \arrow[urr, "h"] \arrow[drr, "h'"'] & \bullet \arrow[ur, "g"] \arrow[dr, "g"'] & \\
 & & \bullet
\end{tikzcd} \qquad \Lambda^3_1$$

(*Associativity of Composition*) Let f, g, h be a triple of composable edges in X. Then we have

$$\begin{tikzcd}
 & & \bullet \arrow[dd, "h"] \\
\bullet \arrow[r, "f"] \arrow[urr, "gf"] \arrow[drr, "(hg)f"'] & \bullet \arrow[ur, "g"] \arrow[dr, "hg"'] & \\
 & & \bullet
\end{tikzcd} \qquad \Lambda^3_1$$

where gf is a composite of f, g and hg is a composite of g, h, and $(hg)f$ is a composite of hg with f. Filling the above inner horn shows that $(hg)f$ is in fact also a composite for h with gf. By uniqueness of composites up to equivalence, this proves associativity of composition.

($hX \cong \pi X$) From the universal property of $h : X \to N(hX)$, we have a commuting triangle of simplicial sets :

$$\begin{tikzcd}
X \arrow[d, "h"'] \arrow[dr] & \\
N(hX) \arrow[r] & N(\pi X)
\end{tikzcd}$$

By fully faithfulness of N, we need to show the horizontal morphism is an isomorphism. It is bijective at the level of objects. So we look at the level of morphisms. Let x, y be points of X. Then

$$\begin{tikzcd}
\mathbf{sSet}(\Delta^1, X) \arrow[d, "h"'] \arrow[dr] & \\
hX(x, y) \arrow[r] & \pi X(x, y)
\end{tikzcd}$$

The diagonal map is surjective by definition, hence so is the bottom map. The vertical map is also surjective : 1-step simplifications are obtained by composites of pairs of morphisms. So to show injectivity of the bottom map, we can take two $f, g \in hX(x, y)$, lift them to edges in X then quotient to $\pi X(x, y)$. If they are equal in $\pi X(x, y)$, then there is a triangle in X witnessing their equivalence. The spine of this triangle simplifies both to f and g as edges in X, and thus $f = g \in hX(x, y)$ as desired. □

Intuition. Let $X \in \mathbf{sSet}$ and $(f, g) : \Lambda^2_1 \to X$ a pair of composable edges. Then the "space" of composites of f with g can be defined by the following cartesian square :

$$\begin{tikzcd} \mathrm{Comp}(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\Delta^2} \arrow[d] \\ \Delta^0 \arrow[r, "(f,g)"'] & X^{\Lambda^2_1} \end{tikzcd}$$

In the above proposition, we saw that being able to lift against inner 3-horns implies that $\pi_0^{\Delta}\mathrm{Comp}(f, g)$ is singleton. The incarnation of infinity categories as *quasi-categories* will say that the family $X^{\Delta^2} \to X^{\Lambda^2_1}$ is "homotopically trivial". In particular, we will much later show that $\mathrm{Comp}(f, g)$ is a contractible Kan complex.

Proposition – Equivalent Definitions of Infinity Categories (Preview) [a]

Let X be a simplicial set. Define the following classes of morphisms of simplicial sets :

- inner horns $\mathrm{IH} = \{\Lambda^n_k \to \Delta^n \mid n \geq 2, 0 < k < n\}$.
- cell inclusions $\mathrm{Cell} = \{\partial\Delta^n \to \Delta^n \mid n \geq 0\}$

Then the following are equivalent :

- X lifts against IH
- (Homotopy Unique Composition) $X^{\Delta^2} \to X^{\Lambda^2_1}$ lifts against Cell.
- (Homotopical Segal Condition) For $n \geq 2$, $X^{\Delta^n} \to X^{I^n}$ lifts against Cell.

Call X an *infinity category* when it satisfies any (and thus all) of the above.

[a] Land - 1.3.34

Example.

- *Infinity groupoids $S(X)$ of topological spaces X, i.e. nerves of topological spaces along the standard realisation of simplicies $\mathbb{\Delta} \to$* **Top**.
- *Nerves of 1-categories.*
- *Later we will be able to turn categories enriched over $\mathbb{Z}$-complexes into infinity categories via a procedure called the* dg-nerve*.*

  This will give examples from homological algebra and algebraic geometry : the dg-nerve $N_{dg}(\mathrm{Ch}A)$ of (unbounded) complexes in a (Grothendieck) abelian category.
- *In order to take "higher" (co)limits of infinity categories, we will have to make an infinity category $\mathbf{Cat}_\infty$ of (small) infinity categories. More important yet : we will want to make "families of infinity categories over a base infinity category B" which vary contravariantly along B. These should corre-*

spond to functors $F : B^{op} \to \mathbf{Cat}_\infty$. This is the subject of Lurie's straightening-unstraightening theorem.