Derived categories of quasi-coherent sheaves

Ken Lee

March 2025

This is an exposition to define the unbounded derived stable ∞-category of quasi-coherent sheaves on a scheme using stable ∞-categories and prove that derived global sections can be computed by Cech cohomology, hopefully in an obvious way!

1 Derived stable ∞-categories for the working mathematician

The goal of this section is to answer :

Question : What should derived categories and derived functors be?

Idea : The derived category $\mathcal{D}(A)$ of an abelian category A should have the following properties.

1. $\mathcal{D}(A)$ has zero object
2. In $\mathcal{D}(A)$, every morphism has kernel and cokernel. Things will get confusing so let us rename them to fiber and cofiber, reserving the words kernel and cokernel for abelian categories. A sequence $M \to N \to Q$ in $\mathcal{D}(A)$ is called left exact when it is fiber square. Similarly for right exact and cofiber sequence.
3. In $\mathcal{D}(A)$, left exact iff right exact iff exact. These are called exact triangles.
4. Equipped with a fully faithful functor $A \to \mathcal{D}(A)$

All of above is satisfied by the category of chain complexes with morphisms up to chain homotopy, when suitably interpreted. The following makes the difference and gives $\mathcal{D}(A)$:

5. A sequence $0 \to M \to N \to Q \to 0$ in A should be short exact iff $M \to N \to Q$ is exact triangle in $\mathcal{D}(A)$.[1]

The properties (1) to (3) are word-for-word the definition of a *stable ∞-category*,[Lur17, Def. 1.1.1.9] modulo being precise about universal properties (zero object, fiber, cofiber). Property (4) will be the reason for *t-structures*. You are probably rolling your eyes: *Why must ∞-categories be involved?* Here are some reasons:

1. Abelian categories are too strict : Property (3) forces all morphisms to be epimorphisms. Given a morphism $f : M \to N$, we get a left exact sequence $\ker f \to M \to N$. If we have property (3) then

[1] **This feels very much like C category with site structure, $\mathrm{PSh}\, C$ has all colimits, now quotient to $\mathrm{Sh}\, C$ to force some coequalizers in C to be preserved under $C \to \mathrm{PSh}\, C$. Is there more to this? It would be nice to have a more canonical construction of $N_{dg}(\mathrm{Ch}\, A)$ for Grothendieck abelian A. This is not quite animation because A may not have enough compact projectives. (E.g. category of abelian sheaves on a topological space.)**

we are forced $\operatorname{Im} f \simeq N$. The "extra room" granted by seeing things as ∞-categories rather than 1-categories is what will make property (3) possible.

2. Triangulated categories are difficult to glue : we will return to this point later. Let us say for now that the theory of triangulated categories is literally a truncation of the theory of stable ∞-categories.[Lur17, Thm. 1.1.2.14]

Here are some general intuition for thinking about ∞-categories :

1. There is really only one idea : sets are seen as discrete spaces, then generalised to CW complexes *up to homotopy*. The rest are consequences of this. "CW complexes up to homotopy" are what people mean by *spaces*.
2. Given an ∞-category $\mathcal{C}$ and two objects x, y in it, there is a space $\operatorname{Map}_{\mathbb{C}}(x, y)$ whose points are morphisms $f : x \to y$ in $\mathcal{C}$.
3. Whenever you see "there exists unique" in 1-category theory, it is equivalent to asking for a set to be isomorphic to the singleton. In ∞-categories, you need to replace this with "the space is contractible". This underlies the entire theory of universal properties.

Regarding derived functors : Given an additive functor $F : A \to B$, there should be some induced functor $\mathcal{D}(A) \to \mathcal{D}(B)$. When F is say left exact, the induced functor should be left exact too. But it turns out for functors between stable ∞-categories, a functor is left exact iff right exact iff exact! [Lur17, Prop. 1.1.4.1] We will now make the following precise, but the answer to the first question is :

Derived categories and derived functors are what you get when you force left exact iff right exact iff exact.

Without further ado :

> **Proposition 1.1.** Let A be an additive category. Consider $\mathcal{K}(A) \in \mathrm{sSet}$ the dg nerve of the differential graded category of unbounded chain complexes in A. Then $\mathcal{K}(A)$ is a stable infinity category. **TODO : Fill in details about standard t-structure giving A as heart.**

Proof. [Lur17, Prop. 1.3.2.10] $\square$

[Lur17, Remark 1.3.1.12] shows that for $X, Y \in \mathcal{K}(A)$, we have an equivalence of spaces

$$\operatorname{Map}_{\mathcal{K}(A)}(X, Y) \simeq \mathrm{DK}(\tau^{\leq 0}\operatorname{Hom}(X, Y))$$

where $\operatorname{Hom}(X, Y) \in \mathrm{Ch}\,\mathbb{Z}$ is the hom complex, $\tau^{\leq 0}$ is truncation killing all cohomologies in positive degree, and DK is the Dold-Kan equivalence of 1-categories :

$$\mathrm{DK} : \mathrm{Ch}^{\leq 0}(\mathrm{Ab}) \xrightarrow{\sim} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab})$$

Applying [Lur17, Remark 1.2.3.14], the homotopy groups of the mapping space are the same as the cohomologies of the hom complex. Since $\mathcal{K}(A)$ is stable, the mapping spaces naturally extend to mapping spectra. We can access the negative homotopy groups by looping Y.

> **Definition 1.2.** [Lur17, Definition 1.3.4.1] Let A be an abelian category. Let W denote the collection of quasi-isomorphisms in $\mathcal{K}(A)$. We say an exact functor of stable ∞-categories $L : \mathcal{K}(A) \to \mathcal{D}(A)$ exhibits $\mathcal{D}(A)$ as the derived stable ∞-category of A when for any stable ∞-category $\mathcal{E}$, restriction

along L induces a fully faithful embeding

$$\mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}(A), \mathcal{E}) \to \mathrm{Fun}_{\mathrm{ex}}(\mathcal{K}(A), \mathcal{E})$$

where $\mathrm{Fun}_{\mathrm{ex}}$ means the ∞-category of exact functors and we require the essential image to be precisely consisting of exact functors which invert W.

This is equivalent to the following formulation, closer to the classical idea of Verdier quotients of triangulated categories :

Proposition 1.3. An exact functor $L : \mathcal{K}(A) \to \mathcal{D}(A)$ of stable ∞-categories exhibits $\mathcal{D}(A)$ as the derived stable ∞-category of A iff for all stable infinity categories $\mathcal{E}$, restriction along L induces a fully faithful embedding of infinity categories

$$\mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}(A), \mathcal{E}) \to \mathrm{Fun}_{\mathrm{ex}}(\mathcal{K}(A), \mathcal{E})$$

with essential image consisting precisely of functors where every acyclic X maps to a zero object.

Proof. Let $F : \mathcal{K}(A) \to \mathcal{E}$ be an exact functor. If F inverts quasi-isomorphisms, then $F(Q) = 0$ for any acyclic Q because $0 \to Q \in W$. Conversely, if F kills acyclics, then $f \in W$ implies its cofiber Q is acyclic by long exact sequence of cohomologies associated to the standard t-structure (or just direct computation), which implies $F(Q) = 0$ which implies $F(f)$ is invertible because exact functors preserve exact triangles. $\square$

Let $F : A \to B$ an additive functor of abelian categories. Suppose we have derived categories for A and B. The functor F induces an exact functor $\mathcal{K}(A) \to \mathcal{K}(B)$.

Question : Can we extend F along derived categories?

$$\begin{tikzcd} \mathcal{K}(A) \arrow[r, "F"] \arrow[d] & \mathcal{K}(B) \arrow[d] \\ \mathcal{D}(A) \arrow[r, dashed, "?"] & \mathcal{D}(B) \end{tikzcd}$$

In general, this won't be possible because F need not take quasi-isomorphisms to quasi-isomorphisms.

Idea : approximate $X \in \mathcal{D}(A)$ with objects from $\mathcal{K}(A)$.

There are two directions we can do : look at morphisms $LY \to X$ or look at morphisms $X \to LY$. Let us focus on the morphisms from the left. More precisely, consider the comma category $\mathcal{K}(A)_{/X}$ whose objects are pairs $(Y, LY \to X)$ where $Y \in \mathcal{K}(A)$ and morphisms are ones in $\mathcal{K}(A)$ commuting with the map to X after applying L. Consider the diagram in $\mathcal{D}(B)$:

$$\mathcal{K}(A)_{/X} \to \mathcal{K}(A) \xrightarrow{F} \mathcal{K}(B) \to \mathcal{D}(B)$$

Then the "best approximation of F at X from the left" is defined as

$$\text{“}RF(X) := \varinjlim_{(Y \in \mathcal{K}(A), LY \to X)} F(Y) \in \mathcal{D}(B)\text{”}$$

Doing this at every $X \in \mathcal{D}(A)$, we get the so-called *(pointwise) left Kan extension of F along $\mathcal{K}(A) \to \mathcal{D}(A)$*.[Lur25, Tag 0300] However, I put quotation marks because this runs into two issues :

1. The colimit may not exist in $\mathcal{D}(B)$
2. If RF is defined on terms of an exact triangle $M \to N \to Q$, we would like RF to take this to an exact triangle in $\mathcal{D}(B)$, i.e. we want RF to be an exact functor. This is not immediately clear from the formula.

Stable ∞-categories like $\mathcal{D}(B)$ are not far off from having all colimits; they already have finite colimits. For ∞-categories with finite colimits, there is a canonical way to formally add in all colimits whilst preserving the finite colimits already present : the ind-completion. We proceed as follows :

Definition 1.4. Consider the commutative diagram :

$$\begin{tikzcd} \mathcal{K}(A) \arrow[d] & \mathcal{K}(A) \arrow[l, "="'] \arrow[r, "F"] \arrow[d] & \mathcal{D}(B) \arrow[d] \\ \mathcal{D}(A) \arrow[r, "\mathrm{Lan}"'] & \operatorname{Ind}\mathcal{K}(A) \arrow[r, "\operatorname{Ind}F"'] & \operatorname{Ind}\mathcal{D}(B) \end{tikzcd}$$

where

1. each Yoneda embedding $\mathcal{K} \to \operatorname{PSh}\mathcal{K}$ factors through the inclusion $\operatorname{Ind}\mathcal{K} \subseteq \operatorname{PSh}\mathcal{K}$ and preserves finite colimits.
2. The ind-completions are stable. [Lur17, p. 1.1.3.6]
3. The ind-completions have arbitrary colimits because $\mathcal{K}(A), \mathcal{D}(B)$ have finite colimits.
4. $\mathcal{D}(A) \to \operatorname{Ind}\mathcal{K}(A)$ is the restriction of the Yoenda embedding. On objects, it takes $X \in \mathcal{D}(A)$ to $\operatorname{Hom}(L\_, X)$. Under the equivalence of presheaves of spaces and right fibrations, $\operatorname{Hom}(L\_, X)$ corresponds to precisely the diagram $\mathcal{K}(A)_{/X} \to \mathcal{K}(A)$. Since $\operatorname{Ind} \subseteq \operatorname{PSh}$ is stable under finite limits, we have that $\mathcal{D}(A) \to \operatorname{Ind}\mathcal{K}(A)$ is exact.

The composition of bottom row is thus an exact functor, which we call the *right derived functor RF of F*. We say RF is defined at $X \in \mathcal{D}(A)$ when $RF(X)$ is in the essential image of $\mathcal{D}(B)$. For $X \in \mathcal{K}(A)$, it is customary to write $RF(X)$ for $RF(LX)$.

There are still several problems :

- (Q1) How can we compute RF?
- (Q2) If A is not small, like the 1-category of abelian groups, then the homs of, say, the classical derived stable ∞-category $h\mathcal{D}(A)$ are too large to be sets!

For (Q1), we seek to find conditions so that the colimit diagram simplifies and lands in $\mathcal{D}(B)$. The most ideal situation would be if we can replace the diagram by constant diagram without changing the colimit. This happens iff the diagram has a final object, [Lur25, Tag 03LQ] so this is what we ask for!

Proposition 1.5. Let $X \in \mathcal{D}(A)$. Suppose the functor $\mathrm{Hom}_{\mathcal{D}(A)}(L\_, X)$ on $\mathcal{K}(A)$ is representable by some $I \in \mathcal{K}(A)$, i.e. we have an isomorphism

$$\mathrm{Hom}_{\mathcal{D}(A)}(L\_, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(A)}(\_, I)$$

In particular, the identity of I gives a point

$$\Delta^0 \to \mathcal{K}(A)_{/X}$$

This inclusion is right cofinal. Hence the right derived functor of F is defined at X and we have $RF(X) \simeq F(I)$.

Proof. Under the equivalence of contravariant functors into Spaces and right fibrations, the assumption is precisely an isomorphism between right fibrations

$$\begin{tikzcd} \mathcal{K}(A)_{/X} \arrow[rr, "\sim"'] \arrow[dr] & & \mathcal{K}(A)_{/I} \arrow[dl] \\ & \mathcal{K}(A) & \end{tikzcd}$$

By preservation of colimits under changing the diagram up to categorical equivalence, [Lur25, Tag 02N5] we now seek the colimit of the diagram $\mathcal{K}(A)_{/I} \to \mathcal{K}(A) \to \mathcal{D}(B) \to \mathrm{Ind}\,\mathcal{D}(B)$. We have the final object in $\mathcal{K}(A)/I$ given by $\mathbb{1} : I \to I$. The inclusion of a point is right cofinal iff the point is a final object. [Lur25, Tag 03LQ] Thus the desired colimit is computed as the image $F(I)$. [Lur25, Tag 02XW] $\square$

Punchline : to compute RF on all of $\mathcal{D}(A)$, it suffices to find a right adjoint to the localization functor $L : \mathcal{K}(A) \to \mathcal{D}(A)$.[1]

Of course, the dual story for *the left derived functor of* F is true by reversing all arrows, replacing colimits with limits, ind-completion with pro-completion. (Keeping in mind the dual of a stable ∞-category is again stable!)

It turns out there is a situation where one can find a *fully faithful* right adjoint to L, realising $\mathcal{D}(A)$ as a *full subcategory of* $\mathcal{K}(A)$. This general pattern of finding localizations as full subcategories is called *reflective localization*, or in model category theory called *Bousfield localization*. The dual notion when one can find a full faithful *left* adjoint to the localization functor L is called *coreflective localization*.

Definition 1.6. [Lur25, Tag 02G0]

Let $\mathcal{C}$ be an ∞-category and W a collection of morphisms. W is called *localizing* when the following are satisfied :

1. isomorphisms are in W
2. For any commuting triangle in C

[1] Dualize [Lur25, Tag 02FV] for precise statment about existence of right adjoint from pointwise representability.

$$\begin{tikzcd} X \arrow[rr, "c"] \arrow[dr, "a"'] & & Z \\ & Y \arrow[ur, "b"'] & \end{tikzcd}$$

with $b \in W$, then $a \in W$ iff $c \in W$.

3. For all $X \in \mathcal{C}$ there exists a morphism $X \to \widetilde{X}$ in W such that $\widetilde{X}$ is W-local, meaning for all $f \in W$, $\mathrm{Map}_C(f, \widetilde{X})$ is an equivalence of spaces.

Dually, W is called *co-localizing* when it satisfies (1) and (2) with a dual version of (3) : For all $X \in \mathcal{C}$ there exists $\widetilde{X} \to X$ in W with $\widetilde{X}$ W-co-local, meaning for all $f \in W$, $\mathrm{Map}_{\mathcal{C}}(\widetilde{X}, f)$ is an equivalence of spaces.

The significance of localizing / co-localizing collections of morphisms is the following :

Proposition 1.7 – Characterisation of (co)reflective localizations. [Lur25, Tag 02G3] Let $\mathcal{C}$ be an ∞-category and W a collection of morphisms. Let $\mathcal{I} \subseteq \mathcal{C}$ is the full subcategory of W-local objects. Then TFAE :

1. W is localizing.
2. $\mathcal{I}$ is reflective and W is precisely the collection of morphisms f such that $L(f)$.

In this case, $\mathcal{I} \subseteq \mathcal{C}$ has a left adjoint L and L exhibits $\mathcal{I}$ as the localization of $\mathcal{C}$ at W.

We have dual statements for when W is co-localizing.

Proof. (1) iff (2) is the equivalence of (a) and (c) in [Lur25, Tag 05Z2].

A full subcategory of reflective iff the inclusion admits a left adjoint.[Lur25, Tag 02FA]

L is then a localization functor. [Lur25, Tag 04JL] L exhibits $\mathcal{I}$ as the localization of $\mathcal{C}$ at the collection of morphisms f such that $L(f)$. [Lur25, Tag 04JH] This is precisely W by (2). $\square$

We will prove that when $\mathcal{C} = \mathcal{K}(A)$ for abelian A and W the collection of quasi-isomorphisms, then certain conditions make W localizing or co-localizing. Some terminology : in the case of $\mathcal{K}(A)$, W-local objects are called K-injectives and W-co-local objects are called K-projectives. [Stacks, Tag 070G] The are also respectively called homotopy injectives, homotopy projectives in [KS06, Section 14.3].

Proposition 1.8 – Derived categories via reflective localization. Let A be Grothendieck abelian, i.e. presentable abelian category with exact filtered colimits. Then the collection W of quasi-isomorphisms is localizing. Hence

1. $\mathcal{I}$ the full subcategory of K-injective objects is stable.
2. The inclusion $\mathcal{I} \subseteq \mathcal{K}(A)$ has a left adjoint L, and it exhibits $\mathcal{I}$ as a derived stable ∞-category of A.

Proof. The first two conditions for W being localizing is clear. We need to produce for each $X \in \mathcal{K}(A)$ a quasi-isomorphism $X \to I$ such that I is K-injective. We refer the reader to [KS06, Theorem 14.3.1]. It is clear that $\mathcal{I}$ is closed under translation and cofibers, and hence (1). 1.7 gives (2). $\square$

The size issue (Q2) is solved if when one can show the homs in $h\mathcal{I}$ are small. This is covered in [Lur17, Prop. 1.3.5.14]. Let us note that we can also have co-localizing situation.

> **Proposition 1.9 – Derived categories via coreflective localization.** Let A be abelian with small direct sums, which are also exact. (In particular, A Grothendieck satisfies this.) Suppose A has enough projectives. Then the collection W of quasi-isomorphisms is co-localizing. Hence
>
> 1. the full subcategory $\mathcal{P}$ of K-projectives is stable.
> 2. the inclusion $\mathcal{P} \subseteq \mathcal{K}(A)$ has a right adjoint L, and it exhibits $\mathcal{P}$ as a derived stable ∞-category of A.

Proof. Again, first two conditions of W being co-localizing is clear. [KS06, Theorem 14.4.3] produces for each $X \in \mathcal{K}(A)$ a quasi-isomorphism $P \to X$ with $P \in \mathcal{P}$, proving W is co-localizing. (1) follows because again P $\square$

Now, given Grothendieck abelian A with enough projectives, we can always take left and right derived functors! In this case, we have the following adjunction for any stable ∞-category $\mathcal{E}$:

$$\begin{tikzcd} \mathcal{P} \arrow[d, "\sim"'] \arrow[dr, shift left, "p"] & \\ \mathcal{D}(A) & \mathcal{K}(A) \arrow[l, "\pi"'] \arrow[ul, shift left] \arrow[dl, shift right] \arrow[ul, phantom, "\perp"] \arrow[dl, phantom, "\perp"] \\ \mathcal{I} \arrow[u, "\sim"] \arrow[ur, shift right, "i"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathrm{Fun}_{\mathrm{ex}}(\mathcal{K}(A), \mathcal{E}) \arrow[r, shift left=2, "F \mapsto RF = Fi"] \arrow[r, shift right=2, "F \mapsto LF = Fp"'] & \mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}(A), \mathcal{E}) \arrow[l, "\pi^*"' near start] \end{tikzcd}$$

where

1. we have *two* incarnations of the derived stable ∞-category $\pi : \mathcal{K}(A) \to \mathcal{D}(A)$. Notice that $\mathcal{P}$ and $\mathcal{I}$ are *not* equal as full subcategories of $\mathcal{K}$.
2. taking *left* derived functors by composing with K-projective replacement gives *right* adjoint to restricting functors along π, A.K.A. (pointwise) *right* Kan extensions. It is unfortunate that the left/right naming does not match.
3. taking *right* derived functors by composing with K-injective replacement gives a *left* adjoint to restricting functors along π, A.K.A. (pointwise) *left* Kan extension.

It is possible to be in the Grothendieck abelian case, with a priori not enough projectives. However, if you happen to have specific bounded above projective complexes, then you can still compute some values of left derived functors.

> **Proposition 1.10.** Let $F : A \to B$ be additive functor between abelian categories with derived stable ∞-categories. Suppose W is localizing for $\mathcal{K}(A)$ but assume F is *right exact*. Let $X \in \mathcal{D}(A)$ and $P \to X$ a quasi-isomorphism with a cohomologically bounded above complex of projective objects in A. Then $LF(X)$ is defined, and computes $F(P)$.

Proof. [Lur17, Lem. 1.3.2.20] shows that P is K-projective. Then

$$\mathrm{Hom}(P, \_) \xrightarrow{\sim} \mathrm{Hom}(P, L\_) \xrightarrow{\sim} \mathrm{Hom}(X, L\_)$$

where $L : \mathcal{K}(A) \to \mathcal{I} \simeq \mathcal{D}(A)$ is the localization functor, the first equivalence is P being K-projective, and the second is $L(\_)$ being K-injective. The result follows from the dual of 1.5. $\square$

One can ask for the comparison of RF to F :

> **Proposition 1.11.** Let $F : A \to B$ an additive functor of abelian categories. Assume that W in $\mathcal{K}(A)$ is localizing. Suppose we also have a derived stable ∞-category of B. Assume F is left exact. Then there is an induced isomorphism $H^0(RF(X)) \to F(X)$. [a]
>
> Similarly for left derived functors and enough projectives.
>
> [a]Converse is probably true, but I am uninterested.

Proof. Let I be the image of X in $\mathcal{D}(A)$ and $\eta : X \to I$ the unit morphism of the adjunction in $\mathcal{K}(A)$. Because $\mathcal{D}(A) \to \mathcal{K}(A)$ is fully faithful right adjoint, $L(\eta)$ is an isomorphism and hence η is a quasi-isomorphism. 1.7 Since $H^i X = X$ when $i = 0$ and zero otherwise, we can replace I with a complex concentrated in cohomological degrees ≥ 0, exact everywhere except $X = H^i(X) \simeq H^0(I)$. Then

$$RF(X) \simeq F(I)$$

as objects in $\mathcal{D}(B)$. Then $H^0(RF(X)) \simeq H^0(F(I)) = \mathrm{Ker}(FI^0 \to FI^1)$. Now use F left exact. $\square$

Question : These K-injective / K-projective resolutions seem quite opaque. Can we compute derived functors using more efficient resolutions?

To answer this, we introduce *acyclic resolutions*. This requires W to be co-localizing to the left derived functors and localizing to do right derived functors.

> **Proposition 1.12.** Let $F : A \to B$ a left exact additive functor between abelian categories. Assume we have derived stable ∞-categories. Further assume for A that W is co-localizing so LF is defined on all of $\mathcal{D}(A)$. Now let $X \in \mathcal{K}(A)$ and $P \to X$ be a quasi-isomorphism from a complex of *acyclics for* LF, meaning for $n \in \mathbb{Z}$, we have $LF(P_n) \to P_n$ is an isomorphism in $\mathcal{D}(B)$. Then $LF(X) \simeq F(P)$. We have dual results for right derived functors.

Proof. Let $\widetilde{P}$ be the image of X in $\mathcal{P}$. Then by the ∞-categorical adjunction and two-out-of-three for quasi-isomorphisms, we have a quasi-isomorphism $f : \widetilde{P} \to P$. This gives an exact triangle $F(\widetilde{P}) \to F(P) \to F(\mathrm{Cone}\, f)$. By the explicit construction of $\mathrm{Cone}\, f$ and the fact that both $\widetilde{P}$ and P are acyclic for LF, we have that $\mathrm{Cone}\, f$ is also acyclic for LF. By f being quasi-isomorphism implies $\mathrm{Cone}\, f$ is acyclic. It is an easy exercise to show that acyclicity of $\mathrm{Cone}\, f$ for LF now implies $F(\mathrm{Cone}\, f)$ is acyclic. Since this is the cofiber for $F(\widetilde{P}) \to F(P)$, we deduce that $LF(X) \simeq F(\widetilde{P}) \simeq F(P)$ in $\mathbb{D}(B)$. $\square$

The above implies, for example, one can compute $\_ \otimes^L_A \_$ in $\mathcal{D}(\mathrm{Mod}_A)$ for a commutative ring A using flat modules. We are also free to cook up any resolution as long as it is acyclic for the derived functor. Ultimately, this is the homological explanation for why Cech cohomology works.

TODO

1. Lifting adjunctions to derived adjunctions.
2. If W is co-localizing for $\mathcal{K}(A)$ and A has $\_ \otimes \_$ lifting to $\mathrm{Tot}(\_ \otimes \_)$ on $\mathcal{K}(A)$, then showing say $\mathcal{P}$ is closed under $\mathrm{Tot}(\_ \otimes \_)$ should give symmetric monoidal structure on $\mathcal{D}(A)$?

2 The stable ∞-category of quasi-coherent sheaves for a scheme

TODO

1. A scheme X is a functor which is a small colimit of affines. In fact, determined by right cofinal subdiagram of the affine opens.
2. Define $\mathrm{QCoh}\ X := \varprojlim_{\mathrm{Spec}\,A \to X} D(\mathrm{Mod}_A)$. Works because small limits of stable ∞-categories is again a stable ∞-category. This is easier than triangulated categories because stability is a *property* whilst a triangulated structure on an additive category is *data*.
3. $\mathrm{QCoh}\ X \simeq \varprojlim_{U \text{ affine open } \subseteq X} D(\mathrm{Mod}_{\mathcal{O}(U)})$ and Zariski localization is flat so the t-structure on affine opens define a t-structure on $\mathrm{QCoh}\ X$. Specifically, $(\mathrm{QCoh}\ X)^{\leq 0} := \varprojlim_U D^{\leq 0}(\mathrm{Mod}_{\mathcal{O}(U)})$.
4. Prove $(\mathrm{QCoh}\ X)^{\heartsuit} \simeq \varprojlim_U \mathrm{Mod}_{\mathcal{O}(U)}$ which is the usual definition of quasi-coherent sheaves. Also prove $D((\mathrm{QCoh}\ X)^{\heartsuit}) \simeq \mathrm{QCoh}\ X$.

3 Cech cohomology and more generally flat base change

TODO

1. Affines having no higher cohomology means we have "a good approximation of a scheme by cohomological simplicies" in the same sense as if we built a CW complex by gluing simplicies. Explain how taking tubular neighbourhoods of faces gives a Cech cover of CW complex.
2. Cech complex being resolution is just Zariski descent. Previous point implies Cech complex is acyclic resolution for derived direct image. Hence Cech cohomology computes higher direct images by 1.12.
3. Flat base change for qcqs morphisms is just realising Cech complex only requires interesction of affines to be covered by finitely many affines and computes derived direct image for base change because of flatness assumption.

References

[KS06] M. Kashiwara and P. Schapira. *Categories and Sheaves*. Springer Berlin, Heidelberg, 2006.
[Lur17] J. Lurie. "Higher algebra". Unpublished. Available online at https://www.math.ias.edu/~lurie/. Sept. 2017.
[Lur25] J. Lurie. *Kerodon*. https://kerodon.net. 2025.
[Stacks] T. Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2018.