Derived categories of quasi-coherent sheaves

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This is an exposition to define the unbounded derived stable ∞-category of quasi-coherent sheaves on a scheme using stable ∞-categories and prove that derived global sections can be computed by Cech cohomology, hopefully in an obvious way!

1 Derived stable ∞-categories for the working mathematician

The goal of this section is to answer :

Question : What should derived categories and derived functors be?

Idea : The derived category $\mathcal{D}(A)$ of an abelian category A should have the following properties.

1. $\mathcal{D}(A)$ has zero object
2. In $\mathcal{D}(A)$, every morphism has kernel and cokernel. Things will get confusing so let us rename them to fiber and cofiber, reserving the words kernel and cokernel for abelian categories. A sequence $M \to N \to Q$ in $\mathcal{D}(A)$ is called left exact when it is fiber square. Similarly for right exact and cofiber sequence.
3. In $\mathcal{D}(A)$, left exact iff right exact iff exact. These are called exact triangles.
4. Equipped with a fully faithful functor $A \to \mathcal{D}(A)$

All of above is satisfied by the category of chain complexes with morphisms up to chain homotopy, when suitably interpreted. The following makes the difference and gives $\mathcal{D}(A)$:

5. A sequence $0 \to M \to N \to Q \to 0$ in A should be short exact iff $M \to N \to Q$ is exact triangle in $\mathcal{D}(A)$.[1]

The properties (1) to (3) are word-for-word the definition of a *stable ∞-category*,[Lur17, Def. 1.1.1.9] modulo being precise about universal properties (zero object, fiber, cofiber). Property (4) will be the reason for *t-structures*. You are probably rolling your eyes: *Why must ∞-categories be involved?* Here are some reasons:

1. Abelian categories are too strict : Property (3) forces all morphisms to be epimorphisms. Given a morphism $f : M \to N$, we get a left exact sequence $\ker f \to M \to N$. If we have property (3) then

[1] **This feels very much like C category with site structure, $\mathrm{PSh}\, C$ has all colimits, now quotient to $\mathrm{Sh}\, C$ to force some coequalizers in C to be preserved under $C \to \mathrm{PSh}\, C$. Is there more to this? It would be nice to have a more canonical construction of $N_{dg}(\mathrm{Ch}\, A)$ for Grothendieck abelian A. This is not quite animation because A may not have enough compact projectives. (E.g. category of abelian sheaves on a topological space.)**

we are forced $\operatorname{Im} f \simeq N$. The "extra room" granted by seeing things as ∞-categories rather than 1-categories is what will make property (3) possible.

2. Triangulated categories are difficult to glue : we will return to this point later. Let us say for now that the theory of triangulated categories is literally a truncation of the theory of stable ∞-categories.[Lur17, Thm. 1.1.2.14]

Here are some general intuition for thinking about ∞-categories :

1. There is really only one idea : sets are seen as discrete spaces, then generalised to CW complexes *up to homotopy*. The rest are consequences of this. "CW complexes up to homotopy" are what people mean by *spaces*.
2. Given an ∞-category $\mathcal{C}$ and two objects x, y in it, there is a space $\mathrm{Map}_{\mathbb{C}}(x, y)$ whose points are morphisms $f : x \to y$ in $\mathcal{C}$.
3. Whenever you see "there exists unique" in 1-category theory, it is equivalent to asking for a set to be isomorphic to the singleton. In ∞-categories, you need to replace this with "the space is contractible". This underlies the entire theory of universal properties.

Regarding derived functors : Given an additive functor $F : A \to B$, there should be some induced functor $\mathcal{D}(A) \to \mathcal{D}(B)$. When F is say left exact, the induced functor should be left exact too. But it turns out for functors between stable ∞-categories, a functor is left exact iff right exact iff exact! [Lur17, Prop. 1.1.4.1] We will now make the following precise, but the answer to the first question is :

Derived categories and derived functors are what you get when you force left exact iff right exact iff exact.

Without further ado :

> **Proposition 1.1.** Let A be an additive category. Consider $\mathcal{K}(A) \in \mathrm{sSet}$ the dg nerve of the differential graded category of unbounded chain complexes in A. Then $\mathcal{K}(A)$ is a stable infinity category. In particular, the cofiber of a morphism $f : M \to N$ in $\mathcal{K}(A)$ is given by the mapping cone construction $\mathrm{Cone}\, f$. [Lur17, Remark 1.3.2.17]

Proof. [Lur17, Prop. 1.3.2.10] □

The homotopy groups of mapping spaces in $\mathcal{K}(A)$ is given to us by the Dold-Kan corresondence. [Lur17, Remark 1.3.1.12] shows that for $X, Y \in \mathcal{K}(A)$, we have an equivalence of spaces

$$\mathrm{Map}_{\mathcal{K}(A)}(X, Y) \simeq \mathrm{DK}(\tau^{\leq 0}\mathrm{Hom}(X, Y))$$

where $\mathrm{Hom}(X, Y) \in \mathrm{Ch}\,\mathbb{Z}$ is the hom complex, $\tau^{\leq 0}$ is truncation killing all cohomologies in positive degree, and DK is the Dold-Kan equivalence of 1-categories :

$$\mathrm{DK} : \mathrm{Ch}^{\leq 0}(\mathrm{Ab}) \xrightarrow{\sim} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab})$$

Applying [Lur17, Remark 1.2.3.14], the homotopy groups of the mapping space are the same as the cohomologies of the hom complex. Since $\mathcal{K}(A)$ is stable, the mapping spaces naturally extend to mapping spectra. We can access the negative homotopy groups by looping Y.

The truncation $h\mathcal{K}(A)$ is the familiar 1-category of chain complexes with morphisms up to chain homotopy. We have a fully faithful functor $A \to \mathcal{K}(A)$. It is worth noting that for a morphism $f : M \to N$ in A, the mapping cone $\mathrm{Cone}(f)$ is the complex $M \to N$ situated in cohomological degrees $[-1, 0]$. This in particular implies that for a short exact sequence $0 \to M \to N \to Q \to 0$, we have a quasi-isomorphism

$$\mathrm{Cone}(f) \xrightarrow{\text{qis}} Q$$

This is generally *not* an isomorphism up to chain homotopy. Indeed, since Q as a complex is only in degree zero, any self-chain-map is null-homotopic iff it is zero. So a morphism $Q \to \mathrm{Cone}(f)$ giving an inverse up to chain homotopy must be a splitting of the short exact sequence, which does not generally exist. In other words, the embedding $A \to \mathcal{K}(A)$ does not send cokernels to cofibers. Motivated by the mapping cone, we are forced to invert all quasi-isomorphisms.

> **Definition 1.2.** [Lur17, Definition 1.3.4.1] Let A be an abelian category. Let W denote the collection of quasi-isomorphisms in $\mathcal{K}(A)$. We say an exact functor of stable ∞-categories $\alpha : \mathcal{K}(A) \to \mathcal{D}(A)$ exhibits $\mathcal{D}(A)$ as the derived stable ∞-category of A when for any stable ∞-category $\mathcal{E}$, restriction along α induces a fully faithful embeding
>
> $$\mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}(A), \mathcal{E}) \to \mathrm{Fun}_{\mathrm{ex}}(\mathcal{K}(A), \mathcal{E})$$
>
> where $\mathrm{Fun}_{\mathrm{ex}}$ means the ∞-category of exact functors and we require the essential image to be precisely consisting of exact functors which invert W.[a]
>
> [a] I would like to define $\mathcal{D}(A)$ as the localization of $\mathcal{K}(A)$ with respect to W, which always exists though may not be locally small, and prove that it must be stable and $\mathcal{K}(A) \to \mathcal{D}(A)$ is exact. However, I have not been able to do this.

This is equivalent to the following formulation, closer to the classical idea of Verdier quotients of triangulated categories :

> **Proposition 1.3.** An exact functor $\alpha : \mathcal{K}(A) \to \mathcal{D}(A)$ of stable ∞-categories exhibits $\mathcal{D}(A)$ as the derived stable ∞-category of A iff for all stable infinity categories $\mathcal{E}$, restriction along α induces a fully faithful embedding of infinity categories
>
> $$\mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}(A), \mathcal{E}) \to \mathrm{Fun}_{\mathrm{ex}}(\mathcal{K}(A), \mathcal{E})$$
>
> with essential image consisting precisely of functors where every acyclic X maps to a zero object.

Proof. Let $F : \mathcal{K}(A) \to \mathcal{E}$ be an exact functor. If F inverts quasi-isomorphisms, then $F(Q) = 0$ for any acyclic Q because $0 \to Q \in W$. Conversely, if F kills acyclics, then $f \in W$ implies its cofiber Q is acyclic by long exact sequence of cohomologies associated to the standard t-structure (or just direct computation), which implies $F(Q) = 0$ which implies $F(f)$ is invertible because exact functors preserve exact triangles. □

Let $F : A \to B$ an additive functor of abelian categories. Suppose we have derived categories for A and B. The functor F induces an exact functor $\mathcal{K}(A) \to \mathcal{K}(B)$.

Question : Can we extend F along derived categories?

$$\begin{tikzcd}
\mathcal{K}(A) \arrow[r, "F"] \arrow[d, "\alpha"] & \mathcal{K}(B) \arrow[d, "\beta"] \\
\mathcal{D}(A) \arrow[r, dashed, "?"] & \mathcal{D}(B)
\end{tikzcd}$$

In general, this won't be possible because F need not take quasi-isomorphisms to quasi-isomorphisms.

Idea : approximate $X \in \mathcal{D}(A)$ with objects from $\mathcal{K}(A)$.

There are two directions we can do : look at morphisms $\alpha Y \to X$ or look at morphisms $X \to \alpha Y$. Let us focus on the morphisms from the left. More precisely, consider the comma category $\mathcal{K}(A)_{/X}$ whose objects are pairs $(Y, \alpha Y \to X)$ where $Y \in \mathcal{K}(A)$ and morphisms are ones in $\mathcal{K}(A)$ commuting with the map to X after applying α. Consider the diagram in $\mathcal{D}(B)$:

$$\mathcal{K}(A)_{/X} \to \mathcal{K}(A) \xrightarrow{F} \mathcal{K}(B) \to \mathcal{D}(B), (Y, \alpha Y \to X) \mapsto \beta F(Y)$$

Then the "best approximation of F at X by probing from the left" is defined as

$$\text{"}RF(X) := \varinjlim_{(Y \in \mathcal{K}(A), \alpha Y \to X)} \beta F(Y) \in \mathcal{D}(B)\text{"}$$

Doing this at every $X \in \mathcal{D}(A)$, we get the so-called *(pointwise) left Kan extension of F along* $\mathcal{K}(A) \to \mathcal{D}(A)$.[Lur25, Tag 0300] However, I put quotation marks because this runs into two issues :

1. The colimit may not exist in $\mathcal{D}(B)$
2. If RF is defined on terms of an exact triangle $M \to N \to Q$ in $\mathcal{D}(A)$, we would like RF to take this to an exact triangle in $\mathcal{D}(B)$, i.e. we want RF to be an exact functor. This is not immediately clear from the formula.

Stable ∞-categories like $\mathcal{D}(B)$ are not far off from having all colimits; they already have finite colimits. For ∞-categories with finite colimits, there is a canonical way to formally add in all colimits whilst preserving the finite colimits already present : the ind-completion. We proceed as follows :

> **Definition 1.4.** Let $F : A \to B$ be an additive functor of abelian categories. Assume that we have derived stable ∞-categories for A and B. Consider the commutative diagram :
>
> $$\begin{tikzcd}
\mathcal{K}(A) \arrow[d, "L"'] & \mathcal{K}(A) \arrow[l, "="'] \arrow[r, "F"] \arrow[d] & \mathcal{D}(B) \arrow[d] \\
\mathcal{D}(A) \arrow[r, "\mathrm{Lan}"'] & \mathrm{Ind}\,\mathcal{K}(A) \arrow[r, "\mathrm{Ind}\,F"'] & \mathrm{Ind}\,\mathcal{D}(B)
\end{tikzcd}$$
>
> where
>
> 1. each Yoneda embedding $\mathcal{K} \to \mathrm{PSh}\,\mathcal{K}$ factors through the inclusion $\mathrm{Ind}\mathcal{K} \subseteq \mathrm{PSh}\,\mathcal{K}$ and preserves finite colimits.
> 2. The ind-completions are stable. [Lur17, p. 1.1.3.6]

3. The ind-completions have arbitrary colimits because $\mathcal{K}(A), \mathcal{D}(B)$ have finite colimits.

4. $\mathcal{D}(A) \to \operatorname{Ind}\mathcal{K}(A)$ is the restriction of the Yoneda embedding. On objects, it takes $X \in \mathcal{D}(A)$ to $\operatorname{Hom}(\alpha(\_), X)$. Under the equivalence of presheaves of spaces and right fibrations, $\operatorname{Hom}(\alpha(\_), X)$ corresponds to precisely the diagram $\mathcal{K}(A)_{/X} \to \mathcal{K}(A)$. Since $\operatorname{Ind} \subseteq \operatorname{PSh}$ is stable under finite limits, we have that $\mathcal{D}(A) \to \operatorname{Ind}\mathcal{K}(A)$ is exact.

The composition of bottom row is thus an exact functor, which we call the *right derived functor RF of F*. For $X \in \mathcal{D}(A)$, we say *RF computes at X* when $RF(X)$ is in the essential image of $\mathcal{D}(B)$. For $X \in \mathcal{K}(A)$, it is customary to write $RF(X)$ for $RF(\alpha X)$.

We have the dual version for *left derived functor LF of F* by using pro-completion.[a]

[a] Stable because dual of stable is stable and ind of stable is stable.

We emphasize that the above is well-defined *as soon as* we have existence of the derived stable ∞-categories. In particular, if you've done homological algebra before already, we *do not* need existence of projective resolutions, nor injective resolutions for derived functors to be well-defined. This is why we prefer to say "RF computes at X" rather than the more conventional terminology "RF is defined at X". There are still several problems :

- (Q1) When does RF compute and how can we find the value?
- (Q2) If A is not small, like the 1-category of abelian groups, then the homs of, say, the classical derived stable ∞-category $h\mathcal{D}(A)$ are too large to be sets!

For (Q1), there are two approaches :

1. For general $X \in \mathcal{D}(A)$, seek conditions so that the colimit diagram simplifies and lands in $\mathcal{D}(B)$.
2. For $X \in A$ seen as an object in $\mathcal{D}(A)$, we can try to describe $RF(X)$ in terms of other $RF(Y)$ which we can compute.

Let us talk about approach (1) and return to approach (2) later. The most ideal situation would be if we can replace the diagram $\mathcal{K}(A)_{/X} \to \mathcal{D}(B)$ by constant diagram without changing the colimit. This happens iff the source of the diagram has a final object, [Lur25, Tag 03LQ] so this is what we ask for!

Proposition 1.5. Let $F : A \to B$ be an additive functor between abelian categories which admit derived stable ∞-categories. Let $X \in \mathcal{D}(A)$. Suppose the functor $\operatorname{Hom}_{\mathcal{D}(A)}(\alpha(\_), X)$ on $\mathcal{K}(A)$ is representable by some $I \in \mathcal{K}(A)$, i.e. we have an isomorphism

$$\operatorname{Hom}_{\mathcal{D}(A)}(\alpha(\_), X) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{K}(A)}(\_, I)$$

In particular, the identity of I gives a point

$$\Delta^0 \to \mathcal{K}(A)_{/X}$$

This inclusion is right cofinal. Hence RF computes at X and $RF(X) \simeq F(I)$.

Proof. Under the equivalence of contravariant functors into Spaces and right fibrations, the assumption is precisely an isomorphism between right fibrations

$$\begin{tikzcd} \mathcal{K}(A)_{/X} \arrow[rr, "\sim"'] \arrow[dr] & & \mathcal{K}(A)_{/I} \arrow[dl] \\ & \mathcal{K}(A) & \end{tikzcd}$$

By preservation of colimits under changing the diagram up to categorical equivalence, [Lur25, Tag 02N5] we now seek the colimit of the diagram $\mathcal{K}(A)_{/I} \to \mathcal{K}(A) \to \mathcal{D}(B) \to \operatorname{Ind}\mathcal{D}(B)$. We have the final object in $\mathcal{K}(A)/I$ given by $\mathbb{1} : I \to I$. The inclusion of a point is right cofinal iff the point is a final object. [Lur25, Tag 03LQ] Thus the desired colimit is computed as the image $F(I)$. [Lur25, Tag 02XW] $\square$

Punchline : For RF to compute on all of $\mathcal{D}(A)$, it suffices to find a right adjoint to the localization functor $\alpha : \mathcal{K}(A) \to \mathcal{D}(A)$.[1]

Of course, the dual story for the left derived functor of F is true by reversing all arrows, replacing colimits with limits, ind-completion with pro-completion. (Keeping in mind the dual of a stable ∞-category is again stable!)

It turns out there is a situation where one can find a *fully faithful* right adjoint to α, realising $\mathcal{D}(A)$ as a *full subcategory of* $\mathcal{K}(A)$. This general pattern of finding localizations as full subcategories is called *reflective localization*, or in model category theory called *Bousfield localization*. The dual notion when one can find a full faithful *left* adjoint to the localization functor α is called *coreflective localization*.

Definition 1.6. [Lur25, Tag 02G0]

Let $\mathcal{C}$ be an ∞-category and W a collection of morphisms. W is called *localizing* when the following are satisfied :

1. isomorphisms are in W
2. For any commuting triangle in C

$$\begin{tikzcd} X \arrow[rr, "c"] \arrow[dr, "a"'] & & Z \\ & Y \arrow[ur, "b"'] & \end{tikzcd}$$

   with $b \in W$, then $a \in W$ iff $c \in W$.
3. For all $X \in \mathcal{C}$ there exists a morphism $X \to \widetilde{X}$ in W such that $\widetilde{X}$ is W-local, meaning for all $f \in W$, $\operatorname{Map}_C(f, \widetilde{X})$ is an equivalence of spaces.

Dually, W is called *co-localizing* when it satisfies (1) and (2) with a dual version of (3) : For all $X \in \mathcal{C}$ there exists $\widetilde{X} \to X$ in W with $\widetilde{X}$ W-co-local, meaning for all $f \in W$, $\operatorname{Map}_{\mathcal{C}}(\widetilde{X}, f)$ is an equivalence of spaces.

The significance of localizing / co-localizing collections of morphisms is the following :

[1]Dualize [Lur25, Tag 02FV] for precise statment about existence of right adjoint from pointwise representability.

Proposition 1.7 – Characterisation of (co)reflective localizations. Let $\mathcal{C}$ be an ∞-category.

1. For localizing collection W of morphisms, the full subcategory $\mathcal{I}$ of W-local objects in $\mathcal{C}$ is a replete reflective subcategory.
2. Given a replete reflective subcategory $\mathcal{I} \subseteq \mathcal{C}$, there exists a left adjoint $\alpha : \mathcal{C} \to \mathcal{I}$ to the inclusion, then the set of morphisms f such that $\alpha(f)$ is an isomorphism is a localizing collection of morphisms in $\mathcal{C}$.

This gives a bijection between

1. localizing collections W of morphisms in $\mathcal{C}$
2. replete reflective subscategories of $\mathcal{C}$.

Consequently, if W localizing, a left adjoint $\alpha : \mathcal{C} \to \mathcal{I}$ to the inclusion $\mathcal{I} \subseteq \mathcal{C}$ of W-local objects exhibits $\mathcal{I}$ as the localization of $\mathcal{C}$ at W.

We have dual statements for when W is co-localizing.

Proof. For the bijection, see [Lur25, Tag 02G8]. For the "consequently", see [Lur25, Tag 02G3]. □

We will prove that when $\mathcal{C} = \mathcal{K}(A)$ for abelian A and W the collection of quasi-isomorphisms, then certain conditions make W localizing or co-localizing. Some terminology : in the case of $\mathcal{K}(A)$, W-local objects are called K-injectives and W-co-local objects are called K-projectives. [Stacks, Tag 070G] They are also respectively called homotopy injectives, homotopy projectives in [KS06, Section 14.3].

Proposition 1.8 – Derived categories via reflective localization. Let A be Grothendieck abelian, i.e. presentable abelian category with exact filtered colimits. Then the collection W of quasi-isomorphisms is localizing. Hence

1. $\mathcal{I}$ the full subcategory of K-injective objects is stable.
2. The inclusion $\mathcal{I} \subseteq \mathcal{K}(A)$ has a left adjoint α, and it exhibits $\mathcal{I}$ as a derived stable ∞-category of A.

Proof. The first two conditions for W being localizing is clear. We need to produce for each $X \in \mathcal{K}(A)$ a quasi-isomorphism $X \to I$ such that I is K-injective. We refer the reader to [KS06, Theorem 14.3.1]. It is clear that $\mathcal{I}$ is closed under translation and cofibers, and hence (1). 1.7 gives (2). □

The size issue (Q2) is solved if when one can show the homs in $h\mathcal{I}$ are small. This is covered in [Lur17, Prop. 1.3.5.14]. Let us note that we can also have co-localizing situation.

Proposition 1.9 – Derived categories via coreflective localization. Let A be abelian with small direct sums, which are also exact. (In particular, A Grothendieck satisfies this.) Suppose A has enough projectives. Then the collection W of quasi-isomorphisms is co-localizing. Hence

1. the full subcategory $\mathcal{P}$ of K-projectives is stable.
2. the inclusion $\mathcal{P} \subseteq \mathcal{K}(A)$ has a right adjoint α, and it exhibits $\mathcal{P}$ as a derived stable ∞-category of A.

Proof. Again, first two conditions of W being co-localizing is clear. [KS06, Theorem 14.4.3] produces for each $X \in \mathcal{K}(A)$ a quasi-isomorphism $P \to X$ with $P \in \mathcal{P}$, proving W is co-localizing. (1) follows because again P $\square$

Now, given Grothendieck abelian A with enough projectives, we can always take left and right derived functors. This covers the case of $A = \mathrm{Mod}_R$ the category of left modules over an associative ring R with unity which will suffice for this article. In general, when the class W of quasi-isomorphisms in $\mathcal{K}(A)$ is both localizing and co-localizing, we have the following adjunction for any stable ∞-category $\mathcal{E}$:

$$
\begin{tikzcd}
\mathcal{P} \arrow[d, "\sim"'] \arrow[dr, shift left, "p"] & \\
\mathcal{D}(A) & \mathcal{K}(A) \arrow[l, "\alpha"'] \arrow[ul, shift left] \arrow[ul, phantom, "\perp"] \arrow[dl, shift left] \arrow[dl, phantom, "\perp"] \\
\mathcal{I} \arrow[u, "\sim"] \arrow[ur, shift right, "i"'] &
\end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd}
\mathrm{Fun}_{\mathrm{ex}}(\mathcal{K}(A), \mathcal{E}) \arrow[r, shift left=2, "F \mapsto RF = Fi"] \arrow[r, phantom, shift left=1, "\perp"] \arrow[r, phantom, shift right=1, "\perp"] \arrow[r, shift right=2, "F \mapsto LF = Fp"'] & \mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}(A), \mathcal{E}) \arrow[l, "\alpha^*"]
\end{tikzcd}
$$

where

1. we have *two* incarnations of the derived stable ∞-category $\alpha : \mathcal{K}(A) \to \mathcal{D}(A)$. Notice that $\mathcal{P}$ and $\mathcal{I}$ are *not* equal as full subcategories of $\mathcal{K}$.
2. taking *left* derived functors by composing with K-projective replacement gives *right* adjoint to restricting functors along α, A.K.A. (pointwise) *right* Kan extensions. It is unfortunate that the left/right naming does not match.
3. taking *right* derived functors by composing with K-injective replacement gives a *left* adjoint to restricting functors along α, A.K.A. (pointwise) *left* Kan extension.

One can ask for the comparison of $RF(X)$ to $F(X)$ when $X \in A$. We have the following.

> **Proposition 1.10.** Let $F : A \to B$ an additive functor of abelian categories. Assume we have derived stable ∞-categories. Let $X \in A$ and $X \to I$ be a quasi-isomorphism such that $RF(X) \simeq F(I)$. If F is left exact, there is an induced isomorphism $H^0(RF(X)) \to F(X)$ in B. [a]
>
> Similarly for having quasi-isomorphism $P \to X$ such that $LF(X) \simeq F(P)$. If F is right exact then the induced $H^0(LF(X)) \to F(X)$ is an isomorphism in B.
>
> [a] Converse is probably true, but I am uninterested.

Proof. Since $H^i X = X$ when $i = 0$ and zero otherwise, up to quasi-isomorphism we can replace I with a complex concentrated in cohomological degrees ≥ 0, exact everywhere except $X = H^i(X) \simeq H^0(I)$. Then $H^0(RF(X)) \simeq H^0(F(I)) = \mathrm{Ker}(FI^0 \to FI^1)$. Now use F left exact. $\square$

Let us return to approach (2) for computing derived functors. In all practical cases, one does start with $X \in A$ and the following gives practical way of computing derived functors.

Proposition 1.11 – Computing derived functors by finite acyclic resolutions. Let $F : A \to B$ be an additive functor between abelian categories. Assume that we have derived stable ∞-categories for A and B. Let $X \in A$, which by abuse of notation we also treat as an object in $\mathcal{D}(A)$. Suppose we have a quasi-isomorphism $P \to X$ where

1. P is bounded in cohomological degrees $[-N, 0]$ for some $N \geq 0$.
2. the terms P^n in P are *acyclic for LF*, i.e. for all $n \in \mathbb{Z}$, LF is defined at P^n and computes $LF(P^n) \simeq F(P^n)$.

Then LF computes at X and $LF(X) \simeq F(P)$.

We have dual results for left exact F and existence of quasi-isomorphism $X \to I$ where I is bounded in cohomological degrees $[0, N]$ for some $0 \leq N$ and has terms I^n which are *acyclic for RF*.

Proof. To be clear, LF is an exact functor $\mathcal{D}(A) \to \mathrm{Pro}(\mathcal{D}(B))$. Proceeds by induction on N.

- $N = 0$ case is precisely the assumption that P^0 is acyclic for LF.
- Assume we have the result for $N = k$. Let $\widetilde{P}$ be the complex P but we remove the term $P^{-(k+1)}$. The point is that we have an exact triangle in $\mathcal{D}(A)$

$$P^{-(k+1)} \to \widetilde{P} \to X$$

Applying LF and the induction hypothesis on $\widetilde{P}$, we have an exact triangle in $\mathrm{Pro}(\mathcal{D}(B))$

$$F(P^{-(k+1)}) \to F(\widetilde{P}) \to LF(X)$$

The functor $\mathcal{K}(B) \to \mathrm{Pro}\,\mathcal{D}(B)$ is exact, so it suffices to compute the cofiber of $F(P^{-(k+1)}) \to F(\widetilde{P})$ in $\mathcal{K}(B)$. The mapping cone computes the cofiber in $\mathcal{K}(B)$ and explicit formulas show that it is $F(P)$. We thus have $LF(X) \simeq F(P)$, completing for proof for $N = k + 1$.

$\square$

With more work, one can make acyclic resolutions work for bounded below complex in the case of right derived functors and bounded above complexes in the case of left derived functors. Since we will not encounter this situation in this article, we refer the reader to [Stacks, Tag 05TA].

We have the following two ways of producing acyclic objects for left/right derived functors.

Proposition 1.12. Let $F : A \to B$ additive between abelian categories with derived stable ∞-categories. Suppose F is exact. Then

1. if W is localizing, then for any $X \in \mathcal{K}(A)$ the induced morphism $F(X) \to RF(X)$ is an isomorphism in $\mathcal{D}(B)$.
2. if W is co-localizing then for any $X \in \mathcal{K}(A)$ the induced morphism $LF(X) \to F(X)$ is an isomorphism in $\mathcal{D}(B)$.

Proof. Suppose W is localizing. Since $\mathcal{K}(A) \to \mathcal{D}(A)$ has a fully right adjoint, by 1.7 the unit $X \to I$ of the adjunction at X must be in W. Exactness of F implies $F(X) \to F(I)$ is still a quasi-isomorphism. 1.5 implies $RF(X) \simeq F(I) \simeq F(X)$ in $\mathcal{D}(B)$. The dual statement for LF is similar. $\square$

The other way is by right bounded complexes of projective objects and left bounded complexes of injective objects.

Proposition 1.13. Let $F : A \to B$ additive between abelian categories with derived stable ∞-categories. Let $I \in A$ be injective and $P \in A$ be projective. If W is localizing or co-localizing, then I is acyclic for RF and P is acyclic for LF.

Proof. The proof of [Lur17, Prop. 1.3.2.20] shows that for projective P and any quasi-isomorphism $f : X \to Y$ in $\mathcal{K}(A)$, we have a commutative square

$$\begin{tikzcd} H^i(\underline{\mathrm{Hom}}(P,X)) \arrow[r, "H^i(\underline{\mathrm{Hom}}(P,f))"] \arrow[d, "\sim"'] & H^i(\underline{\mathrm{Hom}}(P,Y)) \arrow[d, "\sim"] \\ \mathrm{Hom}(P, H^i(X)) \arrow[r, "\mathrm{Hom}(P,H^i(f))"'] & \mathrm{Hom}(P, H^i(Y)) \end{tikzcd}$$

where projectivity of P is used for the vertical arrows being isomorphisms. Hence P is W-co-local. The dual argument works to show injective I is W-local. If W is localizing, then $RF(X) \simeq F(I)$ by 1.5. Let $\alpha : \mathcal{K}(A) \to \mathcal{I} \simeq \mathcal{D}(A)$ be the localization functor with $\mathcal{I} \subseteq \mathcal{K}(A)$ as a fully faithful right adjoint. Then 1.7 implies the unit $\_ \to L(\_)$ must be a quasi-isomorphism. So we have the following equivalences

$$\mathrm{Hom}(P, \_) \xrightarrow{\sim} \mathrm{Hom}(P, \alpha(\_)) \xrightarrow{\sim} \mathrm{Hom}(\alpha(P), L(\_))$$

where the first step uses P is W-co-local and the second is the adjunction. By 1.5, we have $LF(P) \simeq F(P)$.

If W is instead co-localizing, let $\alpha : \mathcal{K}(A) \to \mathcal{P} \simeq \mathcal{D}(A)$ be the localization functor with $\mathcal{P} \subseteq \mathcal{K}(A)$ as a fully faithful left adjoint. The argument is completely dual. □

We talk about deriving adjunctions.

Proposition 1.14. Let $F \dashv G : A \rightleftarrows B$ be an adjunction of abelian categories with additive functors. Suppose we have derived stable ∞-categories $\alpha : \mathcal{K}(A) \to \mathcal{D}(A)$, $\beta : \mathcal{K}(B) \to \mathcal{D}(B)$. Let $X \in \mathcal{D}(A)$ and $P \in \mathcal{K}(A)$ with an isomorphism

$$\mathrm{Hom}(X, \alpha(\_)) \simeq \mathrm{Hom}(P, \_)$$

and similarly $Y \in \mathcal{D}(B)$ and $I \in \mathcal{D}(B)$ with an isomorphism

$$\mathrm{Hom}(\beta(\_), Y) \simeq \mathrm{Hom}(\_, I)$$

Then we have an equivalence

$$\mathrm{Hom}(LF(X), Y) \simeq \mathrm{Hom}(\beta F(P), Y) \simeq \mathrm{Hom}(F(P), I) \simeq \mathrm{Hom}(P, G(I)) \simeq \mathrm{Hom}(X, \alpha G(I)) \simeq \mathrm{Hom}(X, RG(Y))$$

Consequently, if W is co-localizing for $\mathcal{K}(A)$ and localizing for $\mathcal{K}(B)$, then we have an adjunction $RF \dashv RG : \mathcal{D}(A) \rightleftarrows \mathcal{D}(B)$.

Proof. If quasi-isomorphisms are co-localzing for $\mathcal{K}(A)$ and localizing for $\mathcal{K}(B)$ then fixing Y, the equivalence we gave $\mathrm{Hom}(LF(\_), Y) \simeq \mathrm{Hom}(\_, RG(Y))$ is functorial and hence gives an adjunction by [Lur25, Tag 02FX]. $\square$

TODO

1. If W is co-localizing for $\mathcal{K}(A)$ and A has $\_ \otimes \_$ lifting to $\mathrm{Tot}(\_ \otimes \_)$ on $\mathcal{K}(A)$, then showing say $\mathcal{P}$ is closed under $\mathrm{Tot}(\_ \otimes \_)$ should give symmetric monoidal structure on $\mathcal{D}(A)$?

2 The stable ∞-category of quasi-coherent sheaves for a scheme

TODO

1. A scheme X is a functor which is a small colimit of affines. In fact, determined by right cofinal subdiagram of the affine opens.
2. Define $\mathrm{QCoh}\ X := \varprojlim_{\mathrm{Spec}\ A \to X} D(\mathrm{Mod}_A)$. Works because small limits of stable ∞-categories is again a stable ∞-category. This is easier than triangulated categories because stability is a *property* whilst a triangulated structure on an additive category is *data*.
3. $\mathrm{QCoh}\ X \simeq \varprojlim_{U \text{ affine open } \subseteq X} D(\mathrm{Mod}_{\mathcal{O}(U)})$ and Zariski localization is flat so the t-structure on affine opens define a t-structure on $\mathrm{QCoh}\ X$. Specifically, $(\mathrm{QCoh}\ X)^{\leq 0} := \varprojlim_U D^{\leq 0}(\mathrm{Mod}_{\mathcal{O}(U)})$.
4. Prove $(\mathrm{QCoh}\ X)^{\heartsuit} \simeq \varprojlim_U \mathrm{Mod}_{\mathcal{O}(U)}$ which is the usual definition of quasi-coherent sheaves. Also prove $D((\mathrm{QCoh}\ X)^{\heartsuit}) \simeq \mathrm{QCoh}\ X$.

3 Cech cohomology and more generally flat base change

TODO

1. Affines having no higher cohomology means we have "a good approximation of a scheme by cohomological simplicies" in the same sense as if we built a CW complex by gluing simplicies. Explain how taking tubular neighbourhoods of faces gives a Cech cover of CW complex.
2. Cech complex being resolution is just Zariski descent. Previous point implies Cech complex is acyclic resolution for derived direct image. Hence Cech cohomology computes higher direct images by 1.11.
3. Flat base change for qcqs morphisms is just realising Cech complex only requires interesction of affines to be covered by finitely many affines and computes derived direct image for base change because of flatness assumption.

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