Derived categories of quasi-coherent sheaves

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March 2025

This is an exposition to define the unbounded derived stable ∞-category of quasi-coherent sheaves on a scheme using stable ∞-categories and prove that derived global sections can be computed by Cech cohomology, hopefully in an obvious way!

1 Derived stable ∞-categories for the working mathematician

The goal of this section is to answer :

Question : What should derived categories and derived functors be?

Idea : The derived category $\mathcal{D}(A)$ of an abelian category A should have the following properties.

1. $\mathcal{D}(A)$ has zero object
2. In $\mathcal{D}(A)$, every morphism has kernel and cokernel. Things will get confusing so let us rename them to fiber and cofiber, reserving the words kernel and cokernel for abelian categories. A sequence $M \to N \to Q$ in $\mathcal{D}(A)$ is called left exact when it is fiber square. Similarly for right exact and cofiber sequence.
3. (Key) In $\mathcal{D}(A)$, left exact iff right exact iff exact. These are called exact triangles.
4. Equipped with a fully faithful functor $A \to \mathcal{D}(A)$

All of above is satisfied by the category of chain complexes with morphisms up to chain homotopy, when suitably interpreted. The following makes the difference and gives $\mathcal{D}(A)$:

5. A sequence $0 \to M \to N \to Q \to 0$ in A should be short exact iff $M \to N \to Q$ is exact triangle in $\mathcal{D}(A)$.[1]
6. Regarding derived functors : Given an additive functor $F : A \to B$, there should be some induced functor $\mathcal{D}(A) \to \mathcal{D}(B)$. When F is say left exact, the induced functor should be left exact too. But it turns out for functors between stable ∞-categories, a functor is left exact iff right exact iff exact! [Lur17, Prop. 1.1.4.1]

[1] **This feels very much like C category with site structure, $\mathrm{PSh}\, C$ has all colimits, now quotient to $\mathrm{Sh}\, C$ to force some coequalizers in C to be preserved under $C \to \mathrm{PSh}\, C$. Is there more to this? It would be nice to have a more canonical construction of $N_{dg}(\mathrm{Ch}\, A)$ for Grothendieck abelian A. This is not quite animation because A may not have enough compact projectives. (E.g. category of abelian sheaves on a topological space.)**

The properties (1) to (4) are word-for-word the definition of a *stable ∞-category*,[Lur17, Def. 1.1.1.9] modulo being precise about universal properties (zero object, fiber, cofiber). You are probably rolling your eyes : *Why must ∞-categories be involved?* Here are some reasons :

1. Abelian categories are not enough : The key property forces all morphisms to be epimorphisms. Given a morphism $f : M \to N$, we get a left exact sequence $\ker f \to M \to N$. If we have property (3) then we are forced $\operatorname{Im} f \simeq N$. The "extra room" granted by seeing things as ∞-categories rather than 1-categories is what will make property (3) possible.

2. Triangulated categories are difficult to glue : we will return to this point later. Let us say for now that the theory of triangulated categories is literally a truncation of the theory of stable ∞-categories.[Lur17, Thm. 1.1.2.14]

Here are some general intuition for thinking about ∞-categories :

1. There is really only one idea : sets are seen as discrete spaces, then generalised to CW complexes *up to homotopy*. The rest are consequences of this. "CW complexes up to homotopy" are what people mean by *spaces*.

2. Given an ∞-category $\mathcal{C}$ and two objects x, y in it, there is a space $\operatorname{Map}_{\mathbb{C}}(x, y)$ whose points are morphisms $f : x \to y$ in $\mathcal{C}$.

3. Whenever you see "there exists unique" in 1-category theory, it is equivalent to asking for a set to be isomorphic to the singleton. In ∞-categories, you need to replace this with "the space is contractible". This underlies the entire theory of universal properties.

Without further ado :

> **Proposition 1.1.** Let A be an additive category. Consider $\mathcal{K}(A) \in \mathrm{sSet}$ the dg nerve of the differential graded category of unbounded chain complexes in A. Then $\mathcal{K}(A)$ is a stable infinity category. **TODO : Fill in details about standard t-structure giving A as heart.**

Proof. [Lur17, Prop. 1.3.2.10] □

[Lur17, Remark 1.3.1.12] shows that for $X, Y \in \mathcal{K}(A)$, we have an equivalence of spaces

$$\operatorname{Map}_{\mathcal{K}(A)}(X, Y) \simeq \mathrm{DK}(\tau^{\leq 0}\operatorname{Hom}(X, Y))$$

where $\operatorname{Hom}(X, Y) \in \mathrm{Ch}\,\mathbb{Z}$ is the hom complex, $\tau^{\leq 0}$ is truncation killing all cohomologies in positive degree, and DK is the Dold-Kan equivalence of 1-categories :

$$\mathrm{DK} : \mathrm{Ch}^{\leq 0}(\mathrm{Ab}) \xrightarrow{\sim} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab})$$

Applying [Lur17, Remark 1.2.3.14], the homotopy groups of the mapping space are the same as the cohomologies of the hom complex. Since $\mathcal{K}(A)$ is stable, the mapping spaces naturally extend to mapping spectra. We can access the negative homotopy groups by looping Y.

> **Definition 1.2.** [Lur17, Definition 1.3.4.1] Let A be an abelian category. Let W denote the collection of quasi-isomorphisms in $\mathcal{K}(A)$. We say an exact functor of stable ∞-categories $L : \mathcal{K}(A) \to \mathcal{D}(A)$ exhibits $\mathcal{D}(A)$ as the derived stable ∞-category of A when for any stable ∞-category $\mathcal{E}$, restriction

along L induces a fully faithful embeding

$$\mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}(A), \mathcal{E}) \to \mathrm{Fun}_{\mathrm{ex}}(\mathcal{K}(A), \mathcal{E})$$

where $\mathrm{Fun}_{\mathrm{ex}}$ means the ∞-category of exact functors and we require the essential image to be precisely consisting of exact functors which invert W.

This is equivalent to the following formulation, closer to the classical idea of Verdier quotients of triangulated categories :

Proposition 1.3. An exact functor $L : \mathcal{K}(A) \to \mathcal{D}(A)$ of stable ∞-categories exhibits $\mathcal{D}(A)$ as the derived stable ∞-category of A iff for all stable infinity categories $\mathcal{E}$, restriction along L induces a fully faithful embedding of infinity categories

$$\mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}(A), \mathcal{E}) \longrightarrow \mathrm{Fun}_{\mathrm{ex}}(\mathcal{K}(A), \mathcal{E})$$

with essential image consisting precisely of functors where every acyclic X maps to a zero object.

Proof. Let $F : \mathcal{K}(A) \to \mathcal{E}$ be an exact functor. If F inverts quasi-isomorphisms, then $F(Q) = 0$ for any acyclic Q because $0 \to Q \in W$. Conversely, if F kills acyclics, then $f \in W$ implies its cofiber Q is acyclic by long exact sequence of cohomologies associated to the standard t-structure (or just direct computation), which implies $F(Q) = 0$ which implies $F(f)$ is invertible because exact functors preserve exact triangles. $\square$

Let $F : A \to B$ an additive functor of abelian categories. Suppose we have derived categories for A and B. The functor F induces an exact functor $\mathcal{K}(A) \to \mathcal{K}(B)$.

Question : Can we extend F along derived categories?

$$\begin{tikzcd} \mathcal{K}(A) \arrow[r, "F"] \arrow[d] & \mathcal{K}(B) \arrow[d] \\ \mathcal{D}(A) \arrow[r, dashed, "?"] & \mathcal{D}(B) \end{tikzcd}$$

In general, this won't be possible because F need not take quasi-isomorphisms to quasi-isomorphisms.

Idea : approximate $X \in \mathcal{D}(A)$ with objects from $\mathcal{K}(A)$.

There are two directions we can do : look at morphisms $LY \to X$ or look at morphisms $X \to LY$. Let us focus on the morphisms from the left. More precisely, consider the comma category $\mathcal{K}(A)_{/X}$ whose objects are pairs $(Y, LY \to X)$ where $Y \in \mathcal{K}(A)$ and morphisms are ones in $\mathcal{K}(A)$ commuting with the map to X after applying L. Consider the diagram in $\mathcal{D}(B)$:

$$\mathcal{K}(A)_{/X} \to \mathcal{K}(A) \xrightarrow{F} \mathcal{K}(B) \to \mathcal{D}(B)$$

Then the "best approximation of F at X from the left" is defined as

$$\text{“}RF(X) := \varinjlim_{(Y \in \mathcal{K}(A), LY \to X)} F(Y) \in \mathcal{D}(B)\text{”}$$

Doing this at every $X \in D(A)$, we get the so-called (*pointwise*) *left Kan extension of F along* $\mathcal{K}(A) \to \mathcal{D}(A)$.[Lur25, Tag 0300] However, I put quotation marks because this runs into two issues :

1. The colimit may not exist in $\mathcal{D}(B)$
2. If RF is defined on terms of an exact triangle $M \to N \to Q$, we would like RF to take this to an exact triangle in $\mathcal{D}(B)$, i.e. we want RF to be an exact functor. This is not immediately clear from the formula.

Stable ∞-categories like $\mathcal{D}(B)$ are not far off from having all colimits; they already have finite colimits. For ∞-categories with finite colimits, there is a canonical way to formally add in all colimits whilst preserving the finite colimits already present : the ind-completion. We proceed as follows :

Definition 1.4. Consider the commutative diagram :

$$\begin{tikzcd}
\mathcal{K}(A) \arrow[d] & \mathcal{K}(A) \arrow[l, "="'] \arrow[r, "F"] \arrow[d] & \mathcal{D}(B) \arrow[d] \\
\mathcal{D}(A) \arrow[r, "\mathrm{Lan}"'] & \operatorname{Ind}\mathcal{K}(A) \arrow[r, "\operatorname{Ind}F"'] & \operatorname{Ind}\mathcal{D}(B)
\end{tikzcd}$$

where

1. each Yoneda embedding $\mathcal{K} \to \operatorname{PSh}\mathcal{K}$ factors through the inclusion $\operatorname{Ind}\mathcal{K} \subseteq \operatorname{PSh}\mathcal{K}$ and preserves finite colimits.
2. The ind-completions are stable. [Lur17, p. 1.1.3.6]
3. The ind-completions have arbitrary colimits because $\mathcal{K}(A), \mathcal{D}(B)$ have finite colimits.
4. $\mathcal{D}(A) \to \operatorname{Ind}\mathcal{K}(A)$ is the restriction of the Yoenda embedding. On objects, it takes $X \in \mathcal{D}(A)$ to $\operatorname{Hom}(L\_, X)$. Under the equivalence of presheaves of spaces and right fibrations, $\operatorname{Hom}(L\_, X)$ corresponds to precisely the diagram $\mathcal{K}(A)_{/X} \to \mathcal{K}(A)$. Since $\operatorname{Ind} \subseteq \operatorname{PSh}$ is stable under finite limits, we have that $\mathcal{D}(A) \to \operatorname{Ind}\mathcal{K}(A)$ is exact.

The composition of bottom row is thus an exact functor, which we call the *right derived functor RF of F*. We say RF is defined at $X \in \mathcal{D}(A)$ when $RF(X)$ lands in $\mathcal{D}(B)$.

There are still several problems :

- (Q1) How can we compute RF?
- (Q2) If A is not small, like the 1-category of abelian groups, then the homs of, say, the classical derived stable ∞-category $h\mathcal{D}(A)$ are too large to be sets!

For (Q1), we seek to find conditions so that the colimit diagram simplifies and lands in $\mathcal{D}(B)$. The most ideal situation would be if we can replace the diagram by constant diagram without changing the colimit. This happens iff the diagram has a final object, [Lur25, Tag 03LQ] so this is what we ask for!

Proposition 1.5. Let $X \in \mathcal{D}(A)$. Suppose the functor $\mathrm{Hom}_{\mathcal{D}(A)}(L\_, X)$ on $\mathcal{K}(A)$ is representable by some $I \in \mathcal{K}(A)$, i.e. we have an isomorphism

$$\mathrm{Hom}_{\mathcal{D}(A)}(L\_, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(A)}(\_, I)$$

In particular, the identity of I gives a point

$$\Delta^0 \to \mathcal{K}(A)_{/X}$$

This inclusion is right cofinal. Hence the right derived functor of F is defined at X and we have $RF(X) \simeq F(I)$.

Proof. Under the equivalence of contravariant functors into Spaces and right fibrations, the assumption is precisely an isomorphism between right fibrations

$$\begin{tikzcd} \mathcal{K}(A)_{/X} \arrow[rr, "\sim"'] \arrow[dr] & & \mathcal{K}(A)_{/I} \arrow[dl] \\ & \mathcal{K}(A) & \end{tikzcd}$$

By preservation of colimits under changing the diagram up to categorical equivalence, [Lur25, Tag 02N5] we now seek the colimit of the diagram $\mathcal{K}(A)_{/I} \to \mathcal{K}(A) \to \mathcal{D}(B) \to \mathrm{Ind}\,\mathcal{D}(B)$. We have the final object in $\mathcal{K}(A)/I$ given by $\mathbb{1} : I \to I$. The inclusion of a point is right cofinal iff the point is a final object. [Lur25, Tag 03LQ] Thus the desired colimit is computed as the image $F(I)$. [Lur25, Tag 02XW] $\square$

Punchline : to compute RF on all of $\mathcal{D}(A)$, it suffices to find a right adjoint to the localization functor $L : \mathcal{K}(A) \to \mathcal{D}(A)$.[1]

Of course, the dual story for *the left derived functor of* F is true by reversing all arrows, replacing colimits with limits, ind-completion with pro-completion. (Keeping in mind the dual of a stable ∞-category is again stable!)

It turns out there is a situation where one can find a *fully faithful* right adjoint to L, realising $\mathcal{D}(A)$ as a *full subcategory of* $\mathcal{K}(A)$. This general pattern of finding localizations as full subcategories is called *reflective localization*, or in model category theory called *Bousfield localization*. The dual notion when one can find a full faithful *left* adjoint to the localization functor L is called *coreflective localization*.

Definition 1.6. [Lur25, Tag 02G0]

Let $\mathcal{C}$ be an ∞-category and W a collection of morphisms. W is called *localizing* when the following are satisfied :

1. isomorphisms are in W
2. For any commuting triangle in C

[1] Dualize [Lur25, Tag 02FV] for precise statment about existence of right adjoint from pointwise representability.

$$\begin{tikzcd} X \arrow[rr, "c"] \arrow[dr, "a"'] & & Z \\ & Y \arrow[ur, "b"'] & \end{tikzcd}$$

with $b \in W$, then $a \in W$ iff $c \in W$.

3. For all $X \in \mathcal{C}$ there exists a morphism $X \to \widetilde{X}$ in W such that $\widetilde{X}$ is W-local, meaning for all $f \in W$, $\mathrm{Map}_C(f, \widetilde{X})$ is an equivalence of spaces.

Dually, W is called *co-localizing* when it satisfies (1) and (2) with a dual version of (3) : For all $X \in \mathcal{C}$ there exists $\widetilde{X} \to X$ in W with $\widetilde{X}$ W-co-local, meaning for all $f \in W$, $\mathrm{Map}_{\mathcal{C}}(\widetilde{X}, f)$ is an equivalence of spaces.

The significance of localizing / co-localizing collections of morphisms is the following :

Proposition 1.7. [Lur25, Tag 02G3] Let $\mathcal{C}$ be an ∞-category and W a collection of morphisms. Let $\mathcal{I} \subseteq \mathcal{C}$ is the full subcategory of W-local objects. Then TFAE :

1. W is localizing.
2. $\mathcal{I}$ is reflective and W is precisely the collection of morphisms f such that $L(f)$.

In this case, $\mathcal{I} \subseteq \mathcal{C}$ has a left adjoint L and L exhibits $\mathcal{I}$ as the localization of $\mathcal{C}$ at W.

We have dual statements for when W is co-localizing.

Proof. (1) iff (2) is the equivalence of (a) and (c) in [Lur25, Tag 05Z2].

A full subcategory of reflective iff the inclusion admits a left adjoint.[Lur25, Tag 02FA]

L is then a localization functor. [Lur25, Tag 04JL] L exhibits $\mathcal{I}$ as the localization of $\mathcal{C}$ at the collection of morphisms f such that $L(f)$. [Lur25, Tag 04JH] This is precisely W by (2). $\square$

We will prove that when $\mathcal{C} = \mathcal{K}(A)$ for abelian A and W the collection of quasi-isomorphisms, then certain conditions make W localizing or co-localizing. Some terminology : in the case of $\mathcal{K}(A)$, W-local objects are called K-injectives and W-co-local objects are called K-projectives. [Stacks, Tag 070G]

Proposition 1.8 – Derived categories via injective resolution. Let A be Grothendieck abelian, i.e. presentable abelian category with exact filtered colimits. Then the collection W of quasi-isomorphisms is localizing. Hence

1. the inclusion $\mathcal{I} \subseteq \mathcal{K}(A)$ of the full subcategory of K-injective objects has a left adjoint L.
2. $\mathcal{I}$ is stable. Hence L exhibits it as a derived stable ∞-category of A.

Proof. The first two conditions for W being localizing is clear. We need to produce for each $X \in \mathcal{K}(A)$ a quasi-isomorphism $X \to I$ such that I is K-injective. And now enters the non-formal input : there is a left proper combinatorial model structure on $\mathrm{Ch}\, A$. [Lur17, Lem. 1.3.5.3] It follows that every $X \in \mathcal{K}(A)$ admits a trivial cofibration $X \to I$ with I fibrant. [Lur17, Lem. 1.3.5.14] shows that fibrant I lie in $\mathcal{I}$, completing the proof that W is localizing. 1.7 gives (1) and shows that we have a derived stable ∞-category of A provided we prove $\mathcal{I}$ is stable. [Lur17, Prop. 1.3.5.9] proves precisely this. $\square$

The size issue (Q2) is solved if when one can show the homs in $h\mathcal{I}$ are small. This is covered in [Lur17, Prop. 1.3.5.14].

2 The stable ∞-category of quasi-coherent sheaves for a scheme

TODO

1. A scheme X is a functor which is a small colimit of affines. In fact, determined by right cofinal subdiagram of the affine opens.
2. Define $\mathrm{QCoh}\ X := \varprojlim_{\mathrm{Spec}\ A \to X} D(\mathrm{Mod}_A)$. Works because small limits of stable ∞-categories is again a stable ∞-category. This is easier than triangulated categories because stability is a *property* whilst a triangulated structure on an additive category is *data*.
3. $\mathrm{QCoh}\ X \simeq \varprojlim_{U \text{ affine open } \subseteq X} D(\mathrm{Mod}_{\mathcal{O}(U)})$ and Zariski localization is flat so the t-structure on affine opens define a t-structure on $\mathrm{QCoh}\ X$. Specifically, $(\mathrm{QCoh}\ X)^{\leq 0} := \varprojlim_U D^{\leq 0}(\mathrm{Mod}_{\mathcal{O}(U)})$.
4. Prove $(\mathrm{QCoh}\ X)^{\heartsuit} \simeq \varprojlim_U \mathrm{Mod}_{\mathcal{O}(U)}$ which is the usual definition of quasi-coherent sheaves.

3 Cech cohomology and more generally flat base change

TODO

1. Cech complex being resolution is just Zariski descent.
2. Affines having no higher cohomology means we have "a good approximation of a scheme by cohomological simplicies" in the same sense as if we built a CW complex by gluing simplicies.
3. Crucially use the fact that the *raison d'être* of derived functors between derived categories is that everything is always *exact*.
4. **Systematic handling when cover size is enlarged is what leads to spectral sequence?**
5. Flat base change for qcqs morphisms is just realising Cech complex only requires interesction of affines to be covered by finitely many affines and computes derived direct image for base change because of flatness assumption.

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