Functor of points done right

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This documents my personal journey of understanding the foundations of modern algebraic geometry completely from the functorial point of view. At the risk of sounding controversial, I believe this is the way involving *least* amount of technology, however requires developing the most intuition.

Problems with courses in modern algebraic geometry I've seen :

1. don't get to schemes, leaving students unable to access modern AG
2. amount of theory of sheaves on topological spaces is demotivating. I want sheaves to appear as requiring coverings to be effective epimorphisms.
3. With locally ringed spaces, it feels difficult to write down a scheme. New students often mistakenly conflate a scheme with its underlying topological space. The point that "schemes = commutative algebra + descent" is clouded.
4. takes too long to get to modern treatment of varieties
5. With locally ringed spaces, quasi-coherent sheaves live in a larger category of topological sheaves. This obscures the idea that "quasi-coherent sheaves = a module in any chart compatibly" and also takes longer to develop technically speaking.
6. too many properties of morphisms. Hard for students to get a handle on.
7. sheaf cohomology : is there really a need to see abstract sheaf cohomology when seeing schemes for first time?

When corrected motivated, the functor of points approach solves (1), (2), (3), (4), (5). Point (7) is debatable. Point (6) may be better if there was a specific purpose for learning scheme theory? E.g. if one focuses on curves?

Places where functors shine :

1. a scheme is really nothing more than "affine schemes quotient along opens".
2. definition of pullback of quasi-coherent sheaves
3. the associated topological space of a scheme is *secondary*, so it avoids confusion.
4. definition of tangent bundle
5. formal schemes and ind-schemes are easy to introduce and compute with
6. the generalization to algebraic spaces is conceptually explained as replacing quotients along open immersions to quotients along étale morphisms

7. the generalization to stacks is conceptually explained as using groupoids to describe "non-propositional equivalence relations".

Some questions for myself :

1. Maths where schemes are indispensible?
   - Modular forms to Galois reps. Modular curves are usually described complex analytically. But the moduli description makes them over $\mathbb{Q}$. Now étale cohomologies naturally carry action from absolute Galois group.

1 Affine schemes

In this chapter, we develop intuition for commutative algebra as geometry.

1.1 Affine space

Slogan : *Commutative algebras as mathematical formalisation of physical systems*

The above idea is from [Nes21].

The idea of scientific empiricisms is that a *physical system* nothing more than what we can *observe*. Observations are made with *measuring devices*. Measuring devices give for each *state* a number. Numbers are elements of a *field*. Given that numbers are elements of a field K, one sees that measuring devices can be added, subtracted, scaled, multiplied, but **not** divided because they may give zero for certain states. We thus arrive at the concept of a commutative ring.

We think of a commutative k-algebra A as defining a physical system by the measuring devices we can use on it. When we want to think to A as a physical system, we refer to it as $\mathrm{Spec}\, A$. This is nice because this is one idea away from schemes: schemes add the idea of allowing measuring devices which can only give values on certain states, i.e. *local* measuring devices. One also adds the idea that local measuring devices can be *patched together* along states which they share equal values on. Unless stated otherwise, all algebras are algebras over k and commutative unital.

A single state in $\mathrm{Spec}\, A$ is defined as a morphism of k-algebras $A \to k$. Intuitively, the role of a single state is give for each measuring device a number.

Examples of algebras and their physical interpretation : $k[x_1, \dots, x_n]$ represents affine n-space, k represents the system where there is only one state. If one thinks of all states as positions, then it makes sense to confuse the word "physical system" with the word "space". This is intuitively why one might call $\mathrm{Spec}\, A$ a space.

A *map of spaces* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ should be determined by how measuring devices on $\mathrm{Spec}\, A$ give rise to measuring devices on $\mathrm{Spec}\, B$, i.e. a morphism of k-algebras $A \to B$. We take this as the definition.

One can also think of a map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ as a family of states in $\mathrm{Spec}\, A$ parameterised by $\mathrm{Spec}\, B$. E.g. $\mathbb{A}^1 \to \mathrm{Spec}\, A$ is a line of states in $\mathrm{Spec}\, A$.

Let us make formal definitions.

Definition

Throughout, we fix a field k. Define the *category* Aff_k *of affine schemes over* k as the opposite category of the category of commutative algebras over k.

For a k-algebra A, we use $\operatorname{Spec} A$ to denote the correponding object in Aff_k. We refer to $\operatorname{Spec} A$ as the *affine scheme associated to* A. Conversely, given $S \in \mathrm{Aff}_k$, we write $\mathcal{O}(S)$ for the corresponding k-algebra. We refer to $\mathcal{O}(S)$ as the *coordinate ring of* S. We refer to elements $f \in \mathcal{O}(S)$ as *functions on* S.

For a morphism $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ in Aff_k, the corresponding map $A \to B$ will be denote

$$\varphi^* : A \to B, f \mapsto f\varphi$$

We refer to φ^* as *pulling back functions on* $\operatorname{Spec} A$ *along* $\operatorname{Spec} B$.

Remark for readers concerned about having a base field k : the theory of schemes importantly does *not* need a base field k. Historically, one of the reasons schemes came about is because of the desire to apply geometric arguments to problems in number theory. (See Weil conjectures.) In such applications, it is important to work with *all* commutative rings, or in other words $\mathbb{Z}$-algebras. In the "physics" motivation we gave above, we fixed a base field k only for narrative.

Definition

Let X be an affine scheme over k. A *k-point of X* is defined as a morphism of affine schemes $\operatorname{Spec} k \to X$. We write $X(k)$ for the set of k-points of X.

More generally, for affine scheme S an *S-point of X* is defined as a morphism of affine schemes $S \to X$. We write $X(S)$ for the set of S-points of X.

Example (*k-points do not determine affine schemes*).
Consider the morphism of affine schemes over k

$$0 : \operatorname{Spec} k \to \operatorname{Spec} k[t]/(t^2)$$

corresponding to the algebra morphism $k[t]/(t^2) \to k$ given by setting $t = 0$. Note that tautologically, 0 is a k-point of $\operatorname{Spec} k[t]/(t^2)$. This is in fact the unique k-point because $t^2 = 0$ in $k[t]/(t^2)$ so any k-algebra map $k[t]/(t^2) \to k$ must send $t \mapsto 0$.

There is only one k-point of $\operatorname{Spec} k$ because the only k-algebra map $k \to k$ is the identity. Composing with 0 gives the k-point 0 of $\operatorname{Spec} k[t]/(t^2)$. So the morphism 0 induces a bijection on k-points :

$$\bullet = (\operatorname{Spec} k)(k) \xrightarrow{\sim} (\operatorname{Spec} k[t]/(t^2))(k) = \{0\}$$

However 0 is not an isomorphism of affine schemes over k. Indeed, $k[t]/(t^2) \to k$ is not an isomorphism of k-algebras.

Example (*k-points can be empty*).
Let $k = \mathbb{Q}$. Consider $\varnothing = \operatorname{Spec} 0$ where 0 is the zero ring. Then $\varnothing$ has no k-points because $1 \neq 0$ in any field.

Consider $S = \operatorname{Spec} \mathbb{Q}[t]/(t^2 + 1)$. Then S has no k-points because $\mathbb{Q}$ does not contain an element i such that $i^2 + 1 = 0$.

So T, S both have empty sets of k-points! Of course, the two are not isomorphic as affine schemes over k because $\mathbb{Q}[t]/(t^2+1) \simeq \mathbb{Q}(i)$ as $\mathbb{Q}$-algebras and in particular not the zero ring.

Definition
Affine n-space over k is defined as the affine scheme $\mathbb{A}^n_k$ corresponding to the k-algebra $k[x_1, \ldots, x_n]$.

In the theory of smooth manifolds, we have a bijection between smooth functions and manifold morphisms to $\mathbb{R}$

$$C^\infty(M) \simeq \mathrm{Mfd}(M, \mathbb{R})$$

which is functorial in manifolds M. The terminology is that $\mathbb{R}$ *classifies* smooth functions or that $\mathbb{R}$ is a *moduli space* for smooth functions. We have an analogous statement in algebraic geometry.

Proposition – Universal property of the affine line
There exists a bijection

$$\mathcal{O}(S) \simeq \mathrm{Aff}_k(S, \mathbb{A}^1_k)$$

functorial in $S \in \mathrm{Aff}_k$. Given $f \in \mathcal{O}(S)$, we often say that *by the universal property of $\mathbb{A}^1_k$, f gives a map $S \to \mathbb{A}^1_k$*. We will usually use f again to denote the morphism $S \to \mathbb{A}^1_k$.

In our setup of algebraic geometry, the ring of functions on a space comes first before the space so this statement will be more or less a tautology.

Proof. We describe the two directions :

1. Given $f \in \mathcal{O}(S)$, define $k[t] \to \mathcal{O}(S)$ by $t \mapsto f$. This defines a morphism $S \to \mathbb{A}^1_k$ in Aff_k.
2. Given $\varphi : S \to \mathbb{A}^1_k$ in Aff_k, we obtain an element $\varphi^*(t) = t\varphi \in \mathcal{O}(S)$.

One can show the above are inverses and functorial in S. $\square$

Under this identitification, for $\varphi : T \to S$ in Aff_k, the k-algebra map φ^* really does correspond to precomposition of morphisms $S \to \mathbb{A}^1_k$ with φ.

> *Exercise* ($\mathrm{Spec}\, k$ *is the final object of* Aff_k). *Show that every affine scheme over k has a unique map to* $\mathrm{Spec}\, k$. *In the language of category theory, we say that* $\mathrm{Spec}\, k$ *is a* final object *of* Aff_k. *This is analogous to how the singleton set is the final object in the category of sets. In other words, one should think of* $\mathrm{Spec}\, k$ *as "the singleton space" in the category* Aff_k.

1.2 Closed immersion

One of the easiest way to create new spaces from given spaces is by taking "subspaces". The first of these kind of spaces we will consider is *closed embedding*. [1] The idea is simple : given $f \in A$ a k-algebra, we want to consider the "subspace of $\mathrm{Spec}\, A$ where $f = 0$".

[1] This is more standardly called *closed immersion*. I do not like the terminology because for those who have touched manifolds, the word "immersion" brings too much preconceptions.

Definition

Let A be a k-algebra. Given an element $f \in A$, we define the *zero locus of* f to be the affine scheme $V(f) := \operatorname{Spec} A/(f)$. The quotient map $A \to A/(f)$ corresponds to a morphism of affine scheme over k

$$i : \operatorname{Spec} A/(f) \to \operatorname{Spec} A$$

More generally, given a morphism $i : Z \to X$ of affine schemes over k, we call it a *closed embedding* when the corresponding algebra map $\mathcal{O}(X) \to \mathcal{O}(Z)$ is surjective. For such a morphism, the kernel I of $\mathcal{O}(X) \to \mathcal{O}(Z)$ is called *the ideal of* i. We refer to Z as the *zero locus of* I.

A *closed subspace* of $X \in \mathrm{Aff}_k$ is an isomorphism class of closed embeddings into X.

Proposition

Let $S \in \mathrm{Aff}_k$. Then we have an order reversing bijection between the set of closed embeddings into S and the set of ideals of $\mathcal{O}(S)$ given by taking zero locus and taking ideals.

Proof. A tautology. □

Proposition – Universal property of zero locus

Let $i : Z \to X$ be a closed embedding in Aff_k and let I be its ideal. Then for any morphism $\varphi : T \to S$ in Aff_k, there is a factoring

$$\begin{tikzcd} T \arrow[r, dashed] \arrow[dr, "\varphi"'] & Z \arrow[d, "i"] \\ & S \end{tikzcd}$$

if and only if under the k-algebra map $\varphi^* : \mathcal{O}(S) \to \mathcal{O}(T)$, the ideal I is sent to zero. Furthermore, such a factoring is unique when it exists.

Proof. After dualizing affine schemes over k to k-algebras, this is precisely the universal property of the quotient map $\mathcal{O}(S) \to \mathcal{O}(S)/I$. □

In what sense is a closed embedding $i : Z \to X$ the "injection of a subspace"? For this we review subsets of sets.

Proposition – Characterisation of monomorphisms in sets

Let $i : Z \to X$ be a map of sets. Then i is an isomorphism onto a subset of X iff for all maps $x, x_1 : T \to Z$ such that $i(x) = i(x_1)$, we have $x = x_1$.

Proof. Exercise. □

The categorical property which generalises the above phenomenon is the notion of a *monomorphism*. In fact, part of the statement of the universal property of zero locus is that the closed embedding is a monomorphism in Aff_k.

1.3 Basic opens

From the perspective of topology, the obvious thing to consider after closed embeddings is the notion of an *open embedding*. This turns out to be quite subtle and will in fact lead us directly to schemes. For now, we introduce the more restrictive notion of a *basic open*. The idea is again simple : given an element $f \in A$ of a k-algebra, we want to consider the "locus of $\operatorname{Spec} A$ where f is *invertible*".

Definition

Let $X \in \mathrm{Aff}_k$ and $f \in \mathcal{O}(X)$. Define the *basic open associated to* f as

$$D(f) := \operatorname{Spec} \mathcal{O}(X)[1/f]$$

where $\mathcal{O}(X)[1/f] := \mathcal{O}(X)[x]/(xf - 1)$ the $\mathcal{O}(X)$-algebra obtained by freely adjoining an inverse to f. The k-algebra map $\mathcal{O}(X) \to \mathcal{O}(X)[1/f]$ defines a morphism

$$j : D(f) \to X$$

We refer to j as *the inclusion of a basic open*.

More generally, a basic open embedding of X is a morphism $j : U \to X$ in Aff_k isomorphic to the inclusion of a basic open.

Example.
The quintessential example of a basic open embedding is the morphism

$$\mathbb{A}^1_k \setminus \{0\} := D(t) \to \mathbb{A}^1_k$$

where $D(t) = \operatorname{Spec} k[t, t^{-1}]$ and the above morphism corresponds to the k-algebra map $k[t] \to k[t, t^{-1}]$. We will see later that in a precise sense this is the universal basic open embedding*. Note that we have also introduced suggestive notation of $D(t)$ as $\mathbb{A}^1_k \setminus \{0\}$. We will be able to explain this in the next section after introducing* fiber products*.*

Proposition – Universal property of basic opens

Let $X \in \mathrm{Aff}_k$ and $f \in \mathcal{O}(X)$. Let $j : D(f) \to X$ denote the associated basic open embedding. Then for all $s : T \to X$ in Aff_k, we have a factoring

$$\begin{tikzcd} T \arrow[r, dashed] \arrow[dr, "s"'] & D(f) \arrow[d, "j"] \\ & X \end{tikzcd}$$

iff under the k-algebra map $s^* : \mathcal{O}(X) \to \mathcal{O}(T)$, the element f becomes invertible. Furthermore, such a factoring is unique when it exists.

Proof. This is just a restatement of the universal property of the $\mathcal{O}(X)$-algebra map $\mathcal{O}(X) \to \mathcal{O}(X)[1/f]$. $\square$

Again, the uniqueness part of the universal property of basic opens says that $j : D(f) \to X$ is a monomorphism in Aff_k so it makes sense to think about j as the isomorphism onto a subspace of X.

1.4 Fiber product

We give intuition for fiber products of affine schemes. To do this, we first give intuition for fiber products of sets. A fiber product generalises the following notions :

1. Given a map $f : X \to Y$ of sets, then subsets V of Y pullback to subsets $f^{-1}V$ of X.
2. Given two sets X, Y , we can form the product $X \times Y$.

To describe fiber products, it is convenient to realise the following tautology : a map of sets $X \to S$ can equivalently be thought of as a family of sets X_s parameterised by points $s \in S$ where X_s are the fibers of the map above s. This is called the *relative point of view* and it was first realized by Grothendieck. To emphasize that when we are thinking of $X \to S$ as a family of sets over S, one often draws the arrow *vertically*.

$$\begin{array}{c} X \\ \downarrow \\ S \end{array}$$

With this, we are ready for the fiber product.

Definition – Fiber product of sets
Let S be a set and $X \to S, Y \to S$ be families of sets over S. We define their *fiber product* as

$$X \times_S Y := \coprod_{s \in S} X_s \times Y_s$$

This has a natural projection to S making it a family of sets over S with fiber over s giving $X_s \times Y_s$.

Exercise. Show that we can alternatively define

$$X \times_S Y := \{(x, y) \in X \times Y \mid x = y \text{ when projected to } S\}$$

Projection into the X and Y factors gives maps

$$X \leftarrow X \times_S \to Y$$

By definition, the two are equal when composed with $X \to S, Y \to S$ respectively and thus defines a *morphism* $X \times_S Y \to S$.

Proposition – Universal property of fiber product of sets
Let S be a set and $X \to S, Y \to S$ be families of sets over S. Define

$$X \times_S Y \to X$$

by doing the projection $X_s \times Y_s \to X_s$ for each fiber above $s \in S$. We can similarly define $X \times_S Y \to Y$. Then we have a commutative square

$$\begin{tikzcd}
X & X \times_S Y \arrow[l] \arrow[d] \\
S \arrow[from=u] & Y \arrow[l]
\end{tikzcd}$$

and furthermore, for any $T \to S$ together with maps $x : T \to X$ and $y : T \to Y$ over S, there exists a unique map $(x, y) : T \to X \times_S Y$ commuting with the two projections to X, Y.

$$\begin{tikzcd}
 & & T \arrow[dll, bend right, "x"'] \arrow[dl, dashed, "{(x,y)}"'] \arrow[ddl, bend left, "y"] \\
X \arrow[d] & X \times_S Y \arrow[l] \arrow[d] & \\
S & Y \arrow[l] &
\end{tikzcd}$$

Proof. An easy exercise. □

Given a commutative square

$$\begin{tikzcd}
X \arrow[d] & T \arrow[l] \arrow[d] \\
S & Y \arrow[l]
\end{tikzcd}$$

such that the induced map $T \to X \times_S Y$ is an isomorphism, we denote

$$\begin{tikzcd}
X \arrow[d] & T \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
S & Y \arrow[l]
\end{tikzcd}$$

and say that we have a *cartesian square*.

Using the universal property, it is easy to verify the following. (It is also easy to directly provide an isomorphism.)

> *Example* (*Pullback of subsets as fiber products*).
> *Let $f : X \to Y$ be a map of sets. For the inclusion of a subset $V \subseteq Y$, we have an isomorphism of sets over X*
>
> $$f^{-1}V \simeq X \times_Y V$$

> *Example* (*Products as fiber products*).
> *For $X \to S$ and $Y \to S$ where S is singleton, we have*
>
> $$X \times_S X \simeq X \times Y$$

To deal with fiber products in a general category such as Aff_k, one reverses the above and defines fiber products by the universal property. First, we introduce terminology for the relative point of view.

> **Definition**
>
> Let S be an affine scheme over k. Then *an affine scheme over S* is defined as a morphism $T \to S$ in Aff_k.

Meta remark : The "logically correct" thing to do that we could have done at the beginning of our theory is to start by defining the category of affine schemes Aff as opposite to the category of commutative rings, not over any base field in particular. Then for every field k, the category of affine schemes over $\operatorname{Spec} k$ is equivalent to what we have called Aff_k. Of course, this was inconvenient for our narrative of commutative algebras as formalising physical systems which is why we started with Aff_k. We hope the reader will forgive us. We will henceforth work with Aff as our basic category of affine schemes. All the results of closed embeddings and basic opens still hold without a base field by the same arguments. Affine n-space $\mathbb{A}^n$ is defined instead as $\mathbb{Z}[x_1, \ldots, x_n]$ and its universal property is similarly proved.

> **Definition – Fiber products in Aff**
>
> Let $X \to S, Y \to S$ be affine schemes over an affine scheme S. A *fiber product of X, Y over S* is an affine scheme $X \times_S Y$ together with morphisms $X \times_S Y \to X, Y$ such that we have a commutative square
>
> $$\begin{array}{ccc} X & \longleftarrow & X \times_S Y \\ \big\downarrow & & \big\downarrow \\ S & \longleftarrow & Y \end{array}$$
>
> and furthermore, for any $T \to S$ together with maps $x : T \to X$ and $y : T \to Y$ over S, there exists a unique map $(x, y) : T \to X \times_S Y$ commuting with the two projections to X, Y.

$$\begin{tikzcd}
& & T \arrow[dl, dashed, "{(x,y)}"'] \arrow[dll, bend right, "x"'] \arrow[dl, phantom] \arrow[d, bend left, "y"] \\
X & X \times_S Y \arrow[l] \arrow[d] & \\
S \arrow[from=u] & Y \arrow[l] &
\end{tikzcd}$$

We give examples of fiber products.

Example (*Zero locus is the preimage of zero*).
Let i *denote the closed embedding*

$$\{0\} \to \mathbb{A}^1$$

corresponding to the quotient map $\mathbb{Z}[t] \to \mathbb{Z}[t]/(t)$. *Let* $f \in A$ *where* A *is a ring. Reinterpreting* f *as a map* $\operatorname{Spec} A \to \mathbb{A}^1$, *we can consider the fiber product :*

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[d, "f"'] & ? \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathbb{A}^1 & \{0\} \arrow[l, "i"]
\end{tikzcd}$$

Unraveling the definition of fiber products, we are trying to find a ring B *together with a commutative diagram*

$$\begin{tikzcd}
A \arrow[r] & B \\
\mathbb{Z}[t] \arrow[u, "t \mapsto f"] \arrow[r, "t \mapsto t"'] & \mathbb{Z}[t]/(t) \arrow[u]
\end{tikzcd}$$

such that for all algebras C *with maps* $A \to C$ *and* $\mathbb{Z}[t]/(t) \to C$ *such that both agree when restricted to* $\mathbb{Z}[t]$, *there exists a unique map* $B \to C$ *making the following diagram commute :*

$$\begin{tikzcd}
& & C \\
A \arrow[r] \arrow[urr, bend left] & B \arrow[ur, dashed, "\exists!"] & \\
\mathbb{Z}[t] \arrow[u, "t \mapsto f"] \arrow[r, "t \mapsto t"'] & \mathbb{Z}[t]/(t) \arrow[u] \arrow[uur, bend right] &
\end{tikzcd}$$

The fact that the square for B *commutes is equivalent to the fact that under* $A \to B$, *the element* f *becomes zero. The map* $\mathbb{Z}[t]/(t) \to B$ *is completely determined by the map* $\mathbb{Z}[t] \to A \to B$.

Similarly, the map $\mathbb{Z}[t]/(t) \to C$ is uniquely determined by the map $\mathbb{Z}[t] \to A \to C$ and exists iff $\mathbb{Z}[t] \to A \to C$ sends t to zero in C, which happens iff $f \in A$ is sent to zero in C. This shows that the universal property of B as a fiber product is equivalent to the universal property of $A \to B$, which is plainly the universal property of the quotient map $A \to A/(f)$. This shows that we have a cartesian square

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d, "f"'] & \operatorname{Spec} A/(f) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ \mathbb{A}^1 & \{0\} \arrow[l, "i"] \end{tikzcd}$$

In other words, the zero locus of f is the preimage of 0 in $\mathbb{A}^1$ under the reinterpretation of f as a map $\operatorname{Spec} A \to \mathbb{A}^1$.

Example (*Basic open is the preimage of* $\mathbb{A}^1 \setminus \{0\}$).
Let j denote the inclusion of the basic open

$$\mathbb{A}^1 \setminus \{0\} := D(t) \to \mathbb{A}^1$$

corresponding to the ring map $\mathbb{Z}[t] \to \mathbb{Z}[t, t^{-1}]$. Let $f \in A$ where A is a ring. Reinterpreting f as a map $\operatorname{Spec} A \to \mathbb{A}^1$, *one can similarly show that we have a cartesian square :*

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d, "f"'] & D(f) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ \mathbb{A}^1 & D(t) \arrow[l, "j"] \end{tikzcd}$$

Example.
An easy exercise is to show that $\mathbb{A}^2$ with its two projections to $\mathbb{A}^1$ is a fiber product of $\mathbb{A}^1, \mathbb{A}^1$ over $\operatorname{Spec} \mathbb{Z}$.

Example (*A family of pairs of points*).
For this example, let us work over an algebraically closed field k, which means we are working in Aff_k. *We will further assume k has characteristic not equal to two. Consider the following morphism*

$$\varphi : \mathbb{A}^1_k \to \mathbb{A}^1_k, t \mapsto t^2$$

corresponding to the k-algebra map $k[t] \to k[t], t \mapsto t^2$. We have an identitification :

$$\mathbb{A}^1_k(k) \simeq k, \lambda \mapsto t(\lambda)$$

For each $\lambda \in k$ let λ also denote the morphism $\operatorname{Spec} k \to \mathbb{A}^1_k$ *of affine schemes over k. Let us investigate the fibers of φ over λ, or in other words, give a fiber product*

$$\begin{tikzcd}
\mathbb{A}^1_k \arrow[d, "t^2"'] & \operatorname{Spec} A_\lambda \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathbb{A}^1_k & \operatorname{Spec} k \arrow[l, "\lambda"]
\end{tikzcd}$$

Unraveling the definition of fiber products, one can see that A_λ *is supposed to have an element* f *with* $f^2 - \lambda = 0$ *and* A_λ *is supposed to be the universal such* k*-algebra.* $A_\lambda = k[t]/(t^2 - \lambda)$ *does the trick. Indeed, we have a commutative square*

$$\begin{tikzcd}
k[t] \arrow[r] & k[t]/(t^2 - \lambda) \\
k[t] \arrow[u, "t \mapsto t^2"] \arrow[r, "t \mapsto \lambda"'] & k \arrow[u]
\end{tikzcd}$$

and the universal property is equivalent to the universal property of the quotient map $k[t] \to k[t]/(t^2 - \lambda)$.

The interesting thing here is that for $\lambda \neq 0$, *since* k *is algebraically closed and characteristic not two, we have a factoring*

$$t^2 - \lambda = (t - a)(t + a)$$

with $a \neq -a \in k$. *By the chinese remainder theorem, we then have*

$$k[t]/(t^2 - \lambda) \simeq k[t]/(t - a) \times k[t]/(t + a)$$

From this it follows that $\operatorname{Spec} A_\lambda$ *has exactly two* k*-points corresponding to* $\pm a$.

When $\lambda = 0$ *we do not have such a factoring. We have*

$$\operatorname{Spec} A_\lambda = \operatorname{Spec} k[t]/(t^2)$$

which only has a single k*-point corresponding to* $t = 0$. *So the map*

$$\begin{tikzcd}
\mathbb{A}^1_k \arrow[d, "t^2"] \\
\mathbb{A}^1_k
\end{tikzcd}$$

behaves almost like a family of two points at the level of k*-points. It only fails at the origin.* ***However****, if instead of counting the number of* k*-points we counted the* dimension *of* A_λ *as a vector space over* k, *then we would get a constant number two. Indeed, we will see much later that this is a consequence of the fact*

$$k[t] = k[t^2] \oplus t k[t^2]$$

as modules over $k[t^2]$ and hence $\varphi : k[t] \to k[t]$ exhibits the latter as a free rank 2 module over $k[t]$. So in this sense, we do have a family of two points; it's just that sometimes the two points are "on top of each other". In algebraic geometry, this is called a double point*. This shows that* nilpotents *are useful for keeping track of "degenerate cases" such as $\lambda = 0$ in this example.*

For fiber products in general, we have the following :

Proposition – Fiber products exist in affine schemes.
Let $A \to B, A \to C$ be algebra maps. Then a fiber product of $\operatorname{Spec} B, \operatorname{Spec} C$ over $\operatorname{Spec} A$ is given by $\operatorname{Spec}(B \otimes_A C)$ where $B \otimes_A C$ is the tensor product of B and C over A as rings.

Proof. This is a restatement of universal property of the tensor product of rings. □

We have secretly danced around an important question :

Given two fiber products of the same diagram, are they isomorphic?

More generally, we have seen many examples of objects with universal properties by now.

Given two objects with the same universal properties, are they isomorphic?

The answer is given by *Yoneda's lemma* but we will delay the discussion to a later section so that we can give more examples of "algebra capturing geometry".

1.5 Finite disjoint union

In the previous section, we saw that for k algebraically closed characteristic not two, we have

$$k[t]/(t^2 - \lambda) \simeq k[t]/(t - a) \times k[t]/(t + a)$$

for $\lambda \in k^\times$ and some $a \in k$ with $a^2 = \lambda$. The k-algebras $k[t]/(t \pm a)$ individually are isomorphic to k and we saw that the k-points of the affine scheme associated to the above ring had exactly two elements. This is an example of *finite disjoint unions of affine schemes*. Once again, the notion of finite disjoint union in Aff can be understood by

1. considering finite disjoint union in the category of sets
2. extracting the universal property in the example of sets
3. using the universal property to define finite disjoint union of affine schemes.

Since we have done this a few times now, we skip straight to the answer.

Definition
Let $S_1, \ldots, S_n$ be a finite collection of affine schemes. Then a *disjoint union of* $S_1, \ldots, S_n$ is defined as an affine scheme S together with morphisms $\iota_k : S_k \to S$ such that for all affines schemes T together with morphism $\tau_k : S_k \to T$ there exists a unique morphism $S \to T$ making the following diagram commute :

$$\begin{tikzcd}
S_1 \arrow[dr, "\iota_1"'] \arrow[drr, bend left, "\tau_1"] & & \\
\vdots & S \arrow[r, dashed, "\exists!"] & T \\
S_n \arrow[ur, "\iota_n"] \arrow[urr, bend right, "\tau_n"'] & &
\end{tikzcd}$$

Proposition – Existence of finite disjoint union
Let $A_1, \ldots, A_n$ be a finite collection of rings. Let $A := \prod_{l=1}^n A_l$. Then a disjoint union of $\operatorname{Spec} A_1, \ldots, \operatorname{Spec} A_n$ is given by $\operatorname{Spec} A$ where the maps $\operatorname{Spec} A_i \to \operatorname{Spec} A$ come from the projections $A \to A_k$. Furthermore, the morphisms $\operatorname{Spec} A_k \to \operatorname{Spec} A$ are both closed embeddings and basic open embeddings.

Note that this makes sense intuitively : a function on $\operatorname{Spec} A \sqcup \operatorname{Spec} B$ should be a pair of functions, one on $\operatorname{Spec} A$ and one on $\operatorname{Spec} B$, i.e. an element of $A \times B$.

Proof. For the existence, down to the universal property of the product of rings.

The fact that $\operatorname{Spec} A_k \to \operatorname{Spec} A$ is a closed embedding comes down to the fact that the projection $A \to A_k$ is surjective.

Let $e_k \in A$ denote the element that is 1 in the k-th component and zero everywhere else. Then the kernel of the projection to A_k is precisely the elements f such that $fe_k = 0$. It follows that for any ring map $A \to B$ which inverts e_k, the kernel of the projection $A \to A_k$ is annihilated so $A \to B$ factors through $A \to A_k$ which proves that $\operatorname{Spec} A_k \to \operatorname{Spec} A$ is a basic open of $\operatorname{Spec} A$ for e_k. □

We also have a criterion.

Proposition
Let $S \in \mathrm{Aff}$. Then S is a disjoint union of two affine schemes iff there exists an idempotent $e \in \mathcal{O}(S)$.

Intuitively, a idempotent e is the constant function with value 1 on a closed and open subspace and $1 - e$ is the corresponding constant function with value 1 on the complement.

Proof. Exercise. Hint : See footnote.[1] □

One may be wondering why we are sticking to finite disjoint unions when the proof of the existence of disjoint unions does not seem to use finiteness. The following exercise demonstrates infinite product of rings "computes the wrong thing".

> *Exercise. Define the affine scheme $S := \operatorname{Spec} \mathbb{F}_2^{\mathbb{N}}$ over $\mathbb{F}_2$. Show that S has a $\mathbb{F}_2$-point which does not come from any of the projections $\mathbb{F}_2^{\mathbb{N}} \to \mathbb{F}_2$.*

[1] Use the chinese remainder theorem.

1.6 Basic Zariski coverings

To motivate coverings, let us first review what we know about surjections in sets!

Proposition – Surjections in sets
Let $q : \mathcal{U} \to X$ of sets. Then the following are equivalent :

1. q is surjective
2. (Effective epimorphism) Consider $\mathcal{U} \times_X \mathcal{U}$ whose points are pairs of points $(a, b) \in \mathcal{U} \times \mathcal{U}$ such that $qa = qb$.
$$\mathcal{U} \times_X \mathcal{U} \rightrightarrows \mathcal{U} \to X$$
is the coequalizer diagram. In concrete terms, this says "a map out of X is the same as a map out of $\mathcal{U}$ which is constant along fibers of $\mathcal{U}$".

We can play the same game in Aff, or dually in **CRing**. Specifically, given a commutative ring A and a finite set of elements $f_1, \dots, f_n \in A$, we can form

$$\mathcal{U} := \coprod_{i=1}^{n} D(f_i) = \coprod_{i=1}^{n} \mathrm{Spec}\, A[1/f_i] = \mathrm{Spec} \prod_{i=1}^{n} A[1/f_i]$$

Letting $X := \mathrm{Spec}\, A$, one can show that a fiber product of $\mathcal{U}$ with itself over X is given by

$$\mathcal{U} \times_X \mathcal{U} := \mathrm{Spec} \prod_{i,j=1}^{n} A[1/f_i f_j]$$

It then makes sense to ask for when the following is a coequalizer diagram in Aff

$$\mathcal{U} \times_X \mathcal{U} \rightrightarrows \mathcal{U} \to X$$

Or dually, we can ask when the following diagram in **CRing** is a equalizer

$$A \to \prod_{i=1}^{n} A[1/f_i] \rightrightarrows \prod_{i,j=1}^{n} A[1/f_i f_j]$$

The idea is that if we view $\mathrm{Spec}\, A$ as a physical system, the basic opens $D(f_1), \dots, D(f_n)$ cover $\mathrm{Spec}\, A$ precisely when given any state $A \to K$ with measurements in a field K, at least one of the measuring devices f_i give a non-zero measurement. This leads to the following criterion.

Proposition – Criterion for partition of unity using states
Let $f_1, \dots, f_n \in A$ be elements of a ring. Then the following are equivalent :

1. for all fields K and maps of affine schemes $x : \mathrm{Spec}\, K \to \mathrm{Spec}\, A$, there exists some f_i such that $f_i(x) \in K^\times$.
2. the elements $f_1, \dots, f_n$ gives a *partition of unity*, meaning $(f_1, \dots, f_n) = (1)$.

Proof. (1 implies 2) This is the standard application of Zorn's lemma. Assuming for a contradiction that $(f_1, \dots, f_n)$ does not generate the unit ideal, one can use Zorn's lemma to give a maximal ideal $\mathfrak{m}$ of A containing $(f_1, \dots, f_n)$. Using $K := A/\mathfrak{m}$ gives a contradiction. (2 implies 1) Suppose there exists a linear combination $\sum_{i=1}^n \lambda_i f_i = 1$ inside A. Let K be a field and $x : \operatorname{Spec} K \to \operatorname{Spec} A$. Then we cannot have $f_i(x) = 0$ for all i because that would imply $1 = 0$ inside K. Thus some $f_i(x)$ must be non-zero and hence a unit in K. $\square$

> **Definition – Basic Zariski covers**
> We call $\operatorname{Spec} B \to \operatorname{Spec} A$ a *basic Zariski covering* when there exists $f_1, \dots, f_n \in A$ such that
> $$B \simeq \prod_{i=1}^n A[1/f_i]$$
> as A-algebras and any (and thus both) of the conclusions of the previous proposition holds.

We now show that basic Zariski coverings indeed give effective coequalizers we discussed at the start.

> **Proposition – Fundamental lemma of basic Zariski covers**
> Let $f_1, \dots, f_n \in A$ and B be the A-algebra $\prod_{i=1}^n A[1/f_i]$. Then $(f_1, \dots, f_n) = (1)$ iff for all A-modules M
> $$M \to M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$
> is a equalizer diagram in $A\mathbf{Mod}$.

Proof. ($\Leftarrow$) We want $A/(f_1, \dots, f_n) = 0$. Applying $M = A/(f_1, \dots, f_n)$ to the equalizer diagram, it suffices to show $(A/(f_1, \dots, f_n))[1/f_i] = 0$ for each i. This follows since $0 = f_i$ would be a unit.

($\Rightarrow$) Note that we have isomorphisms of B modules and $B \otimes_A B$ modules

$$M \otimes_A B \simeq \prod_{i=1}^n M[1/f_i] \qquad M \otimes_A B \otimes_A B \simeq M \otimes_A \left(\prod_{i,j=1}^n A[1/f_i f_j] \right) \simeq \prod_{i,j=1}^n M[1/f_i f_j]$$

so we equivalently asking for the following diagram to be an equalizer diagram :

$$M \to \prod_{i=1}^n M[1/f_i] \rightrightarrows \prod_{i,j=1}^n M[1/f_i f_j]$$

One can prove this by direct computation. (Hint : see footnote.[1]) There is an alternative proof which generalises to *fpqc coverings*. The idea is :

1. B is a *faithfully flat* A-algebra. This means $\_ \otimes_A B$ is left exact and for $M \in A\mathbf{Mod}$, $M \otimes_A B \simeq 0$ implies $M = 0$.
2. B faithfully flat over A implies that to show left exactness sequences in $A\mathbf{Mod}$, it suffices to show left exactness after $\_ \otimes_A B$ in $B\mathbf{Mod}$.

[1]Prove that $(f_1, \dots, f_n) = (1)$ actually implies $(f_1^N, \dots, f_n^N) = (1)$ for any $N > 0$.

3. After $\_ \otimes_A B$ the equalizer fork becomes a split equalizer diagram and hence is left exact.

$\square$

In chapter 3 when we see modules over a ring A as generalising vector bundles on $\operatorname{Spec} A$, we will be able to see the above proposition as saying $f_1, \ldots, f_n$ gives a partition of unity iff generalised vector bundles on $\operatorname{Spec} A$ are determined by their restrictions to $\mathcal{U} := \coprod_{i=1}^n D(f_i)$.

2 Schemes

2.1 Yoneda's lemma and presheaves

Before we discuss an example of "non-affine phenomenon", which will lead us to the definition of schemes, let us give the promised discussion on *uniqueness* of objects equipped with universal properties. This will nicely lead us to consider an enlargement of Aff in which schemes will eventually reside. The reader familiar with category theory can skip this section.

To recap, we have seen the following examples of objects with universal properties :

1. The affine lines $\mathbb{A}^1$ has the universal property that morphisms into it correspond to elements of coordinate rings. In other words, we have an isomorphism

$$\mathrm{Aff}(S, \mathbb{A}^1) \simeq \mathcal{O}(S)$$

   functorial in $S \in \mathrm{Aff}$.

2. Given a closed embedding $i : Z \to X$ in Aff with ideal I_Z, a morphism $S \to X$ factors through i iff I_Z pullsback to zero on S and the factoring is unique when it exists. This can be phrased as saying composition with i gives a bijection

$$\mathrm{Aff}(S, Z) \simeq \{x \in \mathrm{Aff}(S, X) \mid \forall\, f \in I_Z, f(x) = 0\}$$

   functorial in $S \in \mathrm{Aff}_k$. This is a nice rephrasing because the latter set is a subset of $\mathrm{Aff}(S, X)$ which aligns with the idea that i realises Z as a "subspace" of X.

3. Let $X \in \mathrm{Aff}$ and $f \in \mathcal{O}(X)$. Then the universal property of $j : D(f) \to X$ can phrased as saying the composition with j gives a bijection

$$\mathrm{Aff}(S, D(f)) \simeq \left\{x \in \mathrm{Aff}(S, X) \mid f(x) \in \mathcal{O}(S)^\times\right\}$$

   functorial in $S \in \mathrm{Aff}$. Again, the latter is a subset of $\mathrm{Aff}(S, X)$ which is nice because one thinks of j as realising $D(f)$ as a "subspace" of X.

4. Let X, Y be affine schemes over an affine scheme S. Given a fiber product W of X, Y over S with projections $a : W \to X, b : W \to Y$, note that composition by a and b gives

$$\mathrm{Aff}(T, W) \to \mathrm{Aff}(T, X)$$

$$\mathrm{Aff}(T, W) \to \mathrm{Aff}(T, Y)$$

functorial in $T \in \mathrm{Aff}$. Furthermore, the assumption that $W \to X \to S$ equals $W \to Y \to S$ implies that we have a map of sets

$$\mathrm{Aff}(T, W) \to \mathrm{Aff}(T, X) \times_{\mathrm{Aff}(T,S)} \mathrm{Aff}(T, Y)$$

functorial in $T \in \mathrm{Aff}$. The universal property of W as a fiber product then precisely says that the above map functorial in T is a bijection.

The finite disjoint union and basic Zariski coverings are of a dual nature since their universal property is not about mapping *into* them but rather mapping *out*. We will discuss these in the next section.

Back to the four examples above, the common pattern is that they are all of the form of

$$\mathrm{Aff}(\_, \mathcal{X}) \simeq F(\_)$$

where

- F is a functor $\mathrm{Aff}^{\mathrm{op}} \to \mathbf{Set}$ describing the desired property
- $\mathcal{X}$ is an affine scheme and $\mathrm{Aff}(\_, \mathcal{X})$ is the functor $\mathrm{Aff}^{\mathrm{op}} \to \mathbf{Set}, S \mapsto \mathrm{Aff}(S, \mathcal{X})$.
- the above isomorphism exhibits $\mathcal{X}$ as being the *universal* affine scheme with the desired property speciﬁced by F.

Our goal is to show that in the above situation, the affine scheme $\mathcal{X}$ is unique is some appropriate sense. Now that we have enough examples, we make the formal development.

Definition

The category of *presheaves* (*on* Aff) is defined as the category of functors $\mathrm{Aff}^{\mathrm{op}} \to \mathbf{Set}$.

For each $S \in \mathrm{Aff}$, we have the presheaf

$$\underline{S} : \mathrm{Aff}^{\mathrm{op}} \to \mathbf{Set}, T \mapsto S(T)$$

This defines a functor called the *Yoneda embedding*

$$\mathrm{Aff} \to \mathrm{PSh}\,\mathrm{Aff}, S \mapsto \underline{S}$$

Hopefully through the four examples we have seen now, one can see that intuitively a presheaf X is supposed to be "something affine schemes map into". In more details,

- For each $S \in \mathrm{Aff}$, one should think of the set $X(S)$ as "the set of morphisms $S \to X$".
- Given $t : T \to S$ in Aff, we should be able to "restrict morphisms $S \to X$ to get morphisms $T \to X$". In other words, we should be given a map of sets

$$X(S) \to X(T)$$

  which we will denote with $x \mapsto xt$.
- If $t = \mathbb{1}_S : S \to S$, then we should have

$$X(S) \to X(S), x \mapsto x\mathbb{1}_S = x$$

– Functoriality of X expresses the idea that iterated restriction should be equal to restricting along the composition.

– A morphism of presheaves $\varphi : X \to Y$ should be the data of turning maps $S \to X$ from $S \in \mathrm{Aff}$ into $S \to Y$. In other words, for every $S \in \mathrm{Aff}$ a map

$$\varphi : X(S) \to Y(S)$$

Naturality expresses the idea that given $t : T \to S$ in Aff, composing $x : S \to X$ by φ then restricting along t should be equal to first restricting along T and then composing with φ.

$$(\varphi x)t = \varphi(xt)$$

– Each affine S is clearly something that affines can map to. This is what $\underline{S}$ records.

With the above intuition, one obvious question to ask is whether we have an isomorphism

$$(\mathrm{PSh}\,\mathrm{Aff})(\underline{S}, X) \simeq X(S)$$

for a presheaf X and affine S. Given a morphism $\varphi : \underline{S} \to X$ which we see as a map from $S \to X$ as presheaves, we should be able to get a map $S \to X$ as a point in $X(S)$. The obvious thing to do is to "compose φ with the identity of S". This is precisely *Yoneda's lemma*.

> **Proposition – Yoneda's lemma**
> We have a bijection
> $$(\mathrm{PSh}\,\mathrm{Aff})(\underline{S}, X) \simeq X(S), \varphi \mapsto \varphi \mathbb{1}_S$$
> functorial in $X \in \mathrm{PSh}\,\mathrm{Aff}$ and $S \in \mathrm{Aff}$. We will henceforth confuse the two sets.

Proof. (Functoriality in S) Given $t : T \to S$, we have

$$\begin{tikzcd}
\varphi \underline{t} \arrow[rrr] & & & (\varphi\, \underline{t})(\mathbb{1}_T) = \varphi(t) = (\varphi \mathbb{1}_S)t \\
& \mathrm{PSh}(\underline{T}, X) \arrow[r] & X(T) & \\
& \mathrm{PSh}(\underline{S}, X) \arrow[r] \arrow[u] & X(S) \arrow[u] & \\
\varphi \arrow[rrr] \arrow[uuu] & & & \varphi \mathbb{1}_S \arrow[uuu]
\end{tikzcd}$$

(Functoriality in X) Given $\varphi : X \to Y$, we have

$$
\begin{tikzcd}
x \arrow[rrr] \arrow[ddd] & & & x\mathbb{1}_S \arrow[ddd] \\
& \mathrm{PSh}(\underline{S}, X) \arrow[r] \arrow[d] & X(S) \arrow[d] & \\
& \mathrm{PSh}(\underline{S}, Y) \arrow[r] & Y(S) & \\
\varphi x \arrow[rrr] & & & (\varphi x)\mathbb{1}_S = \varphi(x\mathbb{1}_S)
\end{tikzcd}
$$

(Inverse) Given $x \in X(S)$, we construct $\underline{x} : \underline{S} \to X$ as follows : for each $t \in \underline{S}(T) = S(T)$ define $\underline{x}t := xt$ by functoriality of X. Naturality of $\underline{x}$: For $r : U \to T$ in Aff, we have for every $t \in \underline{S}(T) = S(T)$

$$(\underline{x}t)r = (xt)r = x(tr) = \underline{x}(tr)$$

where the middle equality is functoriality of X. Clearly, $\underline{x}\mathbb{1}_S = x$. And conversely given $\varphi : \underline{S} \to X$, we have that for all points $t \in \underline{S}(T) = S(T)$,

$$\varphi t = \varphi(\mathbb{1}_S t) = (\varphi \mathbb{1}_S)t = \underline{\varphi \mathbb{1}_S} t$$

$\square$

Yoneda's lemma implies that :

> **Proposition**
> The Yoneda embedding $\mathrm{Aff} \to \mathrm{PSh}\,\mathrm{Aff}$ is fully faithful. We henceforth confuse S with $\underline{S}$ for $S \in \mathrm{Aff}$ and view Aff as a full subcategory of $\mathrm{PSh}\,\mathrm{Aff}$.

Yoneda's lemma also justifies the idea that points of a presheaf X are precisely maps from affines in X. A presheaf should be determined by its points and thus the following theorem surfaces.

> **Proposition – Density theorem A.K.A. function extensionality for presheaves**
> Let $X, Y \in \mathrm{PSh}\,\mathrm{Aff}$ Then
>
> $$\begin{aligned}(\mathrm{PSh}\,\mathrm{Aff})(X, Y) &= \left\{ (\varphi x) \prod_{S \in \mathrm{Aff}} \prod_{x \in X(S)} Y(S) \mid \text{ for all } t : T \to S \text{ in } \mathrm{Aff} \text{ we have } (\varphi x)t = \varphi(xt) \right\} \\ &\simeq \left\{ (\varphi x) \prod_{x : S \to X} \mathrm{PSh}(S, Y) \mid \text{ for all } t : T \to S \text{ in } \mathrm{Aff} \text{ we have } (\varphi x)t = \varphi(xt) \right\}\end{aligned}$$
>
> where $S \to X$ ranges over all morphisms from all affines into X.

This expresses X as the colimit of the diagram Aff/X in $\mathrm{PSh}\,\mathrm{Aff}$.

Proof. The isomorphism is given by Yoneda's lemma. □

To talk about uniqueness of objects equipped with universal properties, we need the notion of a final object in a category.

> **Proposition – Final objects and their uniqueness**
> Let $\mathcal{C}$ be a category and A an object of $\mathcal{C}$. A is called a final object when every object in $\mathcal{C}$ has a *unique* morphism to A. Suppose A, B are both final objects over $\mathcal{C}$. Then there exists a unique isomorphism $A \simeq B$.

Proof. By finality of A there exists a unqiue map $a : B \to A$. Similarly there exists a unique map $b : A \to B$. The composition $ab : A \to A$ and the identity of A both give maps from A to itself so we must have $ab = \mathbb{1}_A$. Similarly $ba = \mathbb{1}_B$. □

We are ready.

> **Proposition – Uniqness of objects defined by universal properties**
> Let $X \in \mathrm{PSh}$, $S \in \mathrm{Aff}$ and $x : S \to X$. The following are equivalent :
>
> 1. x is an isomorphism of presheaves
> 2. $x : S \to X$ is final in the category Aff/X
>
> When the above is true we say that x exhibits S as a representative of X It follows that representatives (S, x) are unique up to unique isomorphism in the sense that given another representative $(\tilde{S}, \tilde{x})$ there exists a unique isomorphism $S \simeq \tilde{S}$ such that we have a commuting triangle
>
> $$\begin{tikzcd} & S \arrow[d, "x"] \\ \tilde{S} \arrow[ur, "\sim"] \arrow[r, "\tilde{x}"'] & X \end{tikzcd}$$

Proof. (1 to 2) Clear. (2 to 1) We need to construct an inverse $s : X \to S$ to x. We do it by the density theorem of presheaves. For a point $t : T \to X$ where $T \in \mathrm{Aff}$, by finality of (S, x) there exists a unique map $st : T \to S$ such that $x(st) = t$. We need to show that given $u : U \to T$ in Aff and $t : T \to X$ we have

$$s(tu) = (st)u$$

By uniqueness of the finality of (S, x), it suffices to show $x(s(tu)) = x((st)u)$. We have

$$\begin{aligned} x(s(tu)) &= tu && \text{by def of } s(tu) \\ &= (x(st))u && \text{by def of } st \\ &= x((st)u) && \text{by naturality of } x \end{aligned}$$

This defines a morphism $s : X \to S$ which by definition satisfies $xs = \mathbb{1}_X$. It remains to check $sx = \mathbb{1}_S$. By uniqueness of finality of (S, x) it suffices to check that $x(sx) = x$. This is true by definition of sx.

□

That concludes the discussion on uniqueness of objects defined by universal properties. For the rest of this section, we give some features of the presheaf category $\mathrm{PSh}\,\mathrm{Aff}$. First, it interacts well with relative point of view.

Proposition

For $S \in \mathrm{PSh}\,\mathrm{Aff}$, the Yoneda embedding $\mathrm{Aff} \to \mathrm{PSh}\,\mathrm{Aff}$ gives a fully faithful functor

$$\mathrm{Aff}/S \to (\mathrm{PSh}\,\mathrm{Aff})/S$$

This extends to an equivalence

$$\mathrm{PSh}(\mathrm{Aff}/S) \simeq (\mathrm{PSh}\,\mathrm{Aff})/S$$

In particular, for any algebra k, $(\mathrm{PSh}\,\mathrm{Aff})/\operatorname{Spec} k \simeq \mathrm{PSh}(\mathrm{Aff}_k)$.

Proof. For $X \in \mathrm{PSh}(\mathrm{Aff}/S)$, use the density theorem to write it as the colimit of affines over S. Taking this diagram and take the colimit in $\mathrm{PSh}\,\mathrm{Aff}$. This defines a presheaf equipped with a morphism to S. This procedure is functorial in X. For a quasi-inverse functor, simply do the reverse. Take a presheaf X with a morphism to S, write X as the colimit of affines using the density theorem. These affines map to S via $X \to S$ and so define objects in Aff/S. Finally take the colimit in $\mathrm{PSh}(\mathrm{Aff}/S)$. $\square$

As another show case of the power of the Yoneda embedding. Let us prove the following fact which we will need next section. The yoga is that once you know the proof in the category of sets, you know the proof in any presheaf category and hence any category.

Proposition – Criterion for factoring through monomorphisms

The following are true :

1. Let $f : X \to Y$ be a map of sets and $V \subseteq Y$ a subset of Y. Then f factors through V iff there exists a map $X \to V$ which exhibits $X \simeq X \times_Y V$.
2. The same in Aff, replacing subset with monomorphism.

Proof. (1) is an easy exercise. (2) We have already seen that fiber product of presheaves is computed "pointwise" : for presheaves X, Y over a presheaf S we have isomorphism functorial in affine T

$$(X \times_S Y)(T) \simeq X(T) \times_{S(T)} Y(T)$$

Similarly, a morphism of presheaves is a monomorphism iff it is pointwise a monomorphism. It follows that (1) implies (2). $\square$

2.2 Motivation of non-affine spaces

There are two examples of spaces we want which are non-affine.

1. $\mathbb{A}^2 \setminus \{0\}$, the complement of the closed embedding $i : \{0\} \to \mathbb{A}^2$ corresponding to the quotient map $\mathbb{Z}[x, y] \to \mathbb{Z}[x, y]/(x, y)$.
2. Projective space $\mathbb{P}^1$. This should be obtained as the coequalizer

$$D(t) \rightrightarrows \mathbb{A}^1 + \mathbb{A}^1 \to \mathbb{P}^1$$

where we have $D(t) \to \mathbb{A}^1$ via $t \mapsto t$ and $t \mapsto 1/t$.

Let us discuss (1) first. We will discuss (2) in the next section. The punchline is :

Considering complements of closed embeddings forces us into the larger category of presheaves.

Definition – Complements

Let $i : Z \to X$ be a morphism in Aff. Define the *complement of* i as the subpresheaf of X

$$(X \setminus Z)(T) := \{x \in X(T) \,|\, T \times_X Z = \varnothing\}$$

We say a map $j : U \to X$ in Aff is an affine complement of i when j induces an isomorphism

$$U \simeq X \setminus Z$$

We say i has an complement representable by an affine scheme.

From the previous section, it is clear that if an affine complement exists, it is unique up to isomorphism. What is not clear is that affine complements exist. In fact, it is not true.

Proposition

The closed embedding $i : \{0\} \to \mathbb{A}^2$ has no affine complement.

Proof. Suppose $j : \operatorname{Spec} C \to \mathbb{A}^2$ is an affine complement of i. The idea is that $\operatorname{Spec} C$ should be covered by $D(x), D(y)$. Let $j = (f, g)$ where $f, g \in C$. We compute the pullback of $D(x)$ along $f : \operatorname{Spec} C \to \mathbb{A}^2$. Note that since x invertible in $k[x, y, 1/x]$, the inclusion $D(x) \to \mathbb{A}^2$ factors through j.

$$\begin{tikzcd} \operatorname{Spec} C \arrow[d, "j"'] & \\ \mathbb{A}^2 & D(x) \arrow[l] \arrow[ul, dashed, "s"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \operatorname{Spec} C \arrow[d, "j"'] & D(x) \arrow[l, "s"'] \arrow[d, "\mathrm{id}"] \arrow[dl, phantom, "\llcorner"] \\ \mathbb{A}^2 & D(x) \arrow[l] \end{tikzcd}$$

The universal property of $j : \operatorname{Spec} C \to \mathbb{A}^2$ implies that $\operatorname{Spec} C \to \mathbb{A}^2$ is a monomorphism. Using the final proposition of the previous section, we obtain $\operatorname{Spec} C[1/f] = \operatorname{Spec} C \times_{\mathbb{A}^2} D(x) \simeq D(x)$. The same goes for $D(y)$ and $D(xy)$.

Now, by assumption $C/(f, g) = 0$ i.e. $(f, g) = (1)$ so $D(f) + D(g)$ gives a Zariski cover of $\operatorname{Spec} C$. This makes the top induced morphism in the following commutative diagram an isomorphism.

$$\begin{tikzcd} C \arrow[r, "\sim"] & C[1/f] \times_{C[1/fg]} C[1/g] \\ k[x,y] \arrow[u, "x,y \mapsto f,g"] \arrow[r, "\sim"] & k[x, y, 1/x] \times_{k[x,y,1/x,1/y]} k[x, y, 1.y] \arrow[u, "\sim"'] \end{tikzcd}$$

The right morphism is an isomorphism by the previous paragraph and a simple computation shows the bottom morphism is also an isomorphism. It follows that j is an isomorphism. But this is a contradiction since taking any field k

$$\begin{tikzcd} (\mathrm{Spec}\ C)(k) \arrow[d, "j"', "\sim"] & k^2 \setminus \{0\} \arrow[l, "\sim"'] \arrow[d, "\subseteq"', "\nsim"] \\ \mathbb{A}^2(k) & k^2 \arrow[l, "\sim"] \end{tikzcd}$$

□

This shows that if we want *open embeddings* to make sense beyond basic opens, we need to allow "non-affine spaces" i.e. presheaves. More generally,

> **Definition – Open embedding with affine target**
> Let $I \subseteq A$ be an ideal. Then the complement of $\mathrm{Spec}\, A/I \to \mathrm{Spec}\, A$ is defined as the subfunctor of the Yoneda embedding of $\mathrm{Spec}\, A$ consisting of maps $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ such that $B/IB = 0$.
>
> A map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is called an *open embedding* when it is the complement of a closed embedding into $\mathrm{Spec}\, A$.

We give some more properties of the presheaf category over Aff.

> **Proposition**
> The following are true :
>
> 1. $\mathrm{PSh}\,\mathrm{Aff}$ has all limits and it is computed "pointwise" i.e. for a diagram $X : I \to \mathrm{PSh}\,\mathrm{Aff}$ we have functorially in $S \in \mathrm{Aff}$
> $$(\varprojlim_{i \in I} X_i)(S) \xrightarrow{\sim} \varprojlim_{i \in I} X_i(S)$$
> The same is true for colimits.
> 2. All epimorphisms are effective epimorphisms, i.e. "a surjection is the quotient of its source by the equivalence relation of being in same fibers".
> 3. Limits and colimits of subobjects are computed pointwise by intersection and union.
> 4. The Yoneda embedding preserves limits. However it does *not* preserve colimits.

None of the above are hard to prove, but let us say something about the Yoneda embedding not preserving colimits. At first, this may appear to be a bad thing however this is in fact desireable. Indeed, throughout the proof that $\mathbb{A}^2 \setminus \{0\}$ cannot be affine, we proved that we have an coequalizer diagram

$$D(xy) \rightrightarrows D(x) + D(y) \to \mathbb{A}^2$$

inside the category Aff. This is bad because this coequalizer should be $\mathbb{A}^2 \setminus \{0\}$ and we do not want the inclusion $\mathbb{A}^2 \setminus \{0\} \subseteq \mathbb{A}^2$ to be an isomorphism!

Let us investigate further. Note that the inclusions $D(x), D(y) \subseteq \operatorname{Spec} A$ factor through $\mathbb{A}^2 \setminus \{0\}$ so it makes sense to ask whether the following diagram in $\operatorname{PSh} \operatorname{Aff}$ is a coequalizer diagram

$$D(xy) \rightrightarrows D(x) + D(y) \to \mathbb{A}^2 \setminus \{0\}$$

> **Proposition**
>
> Let U denote the coequalizer of $D(xy) \rightrightarrows D(x) + D(y)$ inside $\operatorname{PSh} \operatorname{Aff}$. This is a subobject of $\mathbb{A}^2$ but it is *not* isomorphic to $\mathbb{A}^2 \setminus \{0\}$ as subobjects of $\mathbb{A}^2$.

Proof. Because $D(xy), D(x), D(y)$ are all subfunctors of $\mathbb{A}^2$, the coequalizer is simply the union. In other words, for $\operatorname{Spec} A \in \operatorname{Aff}$

$$U(A) = \left\{(f, g) \in A^2 \mid f \in A^\times \text{ or } g \in A^\times\right\}$$

Since $D(x), D(y) \subseteq \mathbb{A}^2 \setminus \{0\}$ we have an induced inclusion $U \subseteq \mathbb{A}^2 \setminus \{0\}$. However this is not an equality. Indeed for $A = k[t]$, $f = t, g = 1 - t$ we have that neither $t, 1 - t$ are units but $(1) = (t, 1 - t)$. □

One way of understanding the above phenomenon is that given $(f, g) : \operatorname{Spec} A \to \mathbb{A}^2 \setminus \{0\}$ we have a basic Zariski cover

$$D(fg) \rightrightarrows D(f) + D(g) \to \operatorname{Spec} A$$

which is a coequalizer diagram in Aff, however U fails to recognise $\operatorname{Spec} A$ as the coequalizer. Indeed, with the example in the proof, we have a map $D(f) + D(g) \to U$ which agrees on the intersection $D(fg)$ but fails to descend to a map $\operatorname{Spec} A \to U$. More precisely, let $S_{\{f,g\}}$ be the coequalizer of $D(fg) \rightrightarrows D(f) + D(g)$ inside $\operatorname{PSh} \operatorname{Aff}$. This induces a map

$$S_{\{f,g\}} \to \operatorname{Spec} A$$

and U fails to recognise this as an isomorphism. It turns out, trying to force U to see this map as an isomorphism produces $\mathbb{A}^2 \setminus \{0\}$. Making this idea rigorous leads us to the notion of a *Zariski sheaf*. We first abstract the above example.

> **Proposition – Zariski sieves**
>
> Let $X = \operatorname{Spec} A \in \operatorname{PSh} \operatorname{Aff}$ and let $I \subseteq A$ be finite with $(I) = A$. Define the Zariski sieve associated I as the subfunctor $S_I \subseteq \operatorname{Spec} A$ of points $x : \operatorname{Spec} B \to \operatorname{Spec} A$ such that there exists a factoring for some $f \in I$
>
> $$\operatorname{Spec} B \to \operatorname{Spec} A[1/f] \to \operatorname{Spec} A$$
>
> Then we have a coequalizer diagram of presheaves
>
> $$\mathcal{U}_I \times_X \mathcal{U}_I \rightrightarrows \mathcal{U}_I \to S_I$$
>
> where $\mathcal{U}_I := \coprod_{f \in I} \operatorname{Spec} A[1/f]$.

Proof. Colimits of presheaves are computed fiberwise. We are thus reduced to showing

$$\mathcal{U}_I(B) \times_{X(B)} \mathcal{U}_I(B) \rightrightarrows \mathcal{U}_I(B) \to S_I(B)$$

is a coequalizer diagram for each $\operatorname{Spec} B \in \operatorname{Aff}$. This follows because $S_I(B)$ is precisely the image of $U_I(B)$ and for sets, the image of a map is the coequalizer of its kernel pair. □

Proposition – Zariski sheaves

Define $\mathrm{Sh}_{\mathrm{Zar}}\,\mathrm{Aff}$ to be the full subcategory of $\mathrm{PSh}\,\mathrm{Aff}$ of presheaves X such that for any Zariski sieve inclusion $S_I \to \mathrm{Spec}\,A$ we have an isomorphism

$$(\mathrm{PSh}\,\mathrm{Aff})(\mathrm{Spec}\,A, Y) \xrightarrow{\sim} (\mathrm{PSh}\,\mathrm{Aff})(S_I, Y)$$

Such presheaves are called *Zariski sheaves*. [a] Then

1. We have an adjunction

$$\begin{tikzcd} \mathrm{Sh}_{\mathrm{Zar}}\mathrm{Aff} \arrow[r, shift right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathrm{PShAff} \arrow[l, shift right, "L"'] \end{tikzcd}$$

2. the left adjoint L is commutes with finite limits, equivalently sends the final object to the final object and preserves (finite) fiber products.
3. for all presheaves X and $x : S \to LX$ where S is affine, there exists a basic Zariski cover $\mathcal{U}$ of S with a map $\tilde{x} : \mathcal{U} \to X$ such that we have a commuting square

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\tilde{x}"] \arrow[d] & X \arrow[d] \\ S \arrow[r, "x"'] & LX \end{tikzcd}$$

[a] [Stacks, Tag 020W] shows that we have a good definition.

The functor L is called (Zariski) sheafification. Note that the lemma of Zariski covers shows that all affine schemes are Zariski sheaves.

To complete the example of $\mathbb{A}^2 \setminus \{0\}$, we have

Proposition

The inclusion $j : U \subseteq \mathbb{A}^2 \setminus \{0\}$ exhibits the latter as the Zariski sheafification of U.

Proof. Let $Y \in \mathrm{Sh}\,\mathrm{Aff}$. We need to show that $\mathbb{A}^2 \setminus \{0\}$ is a Zariski sheaf and j induces

$$\mathrm{Sh}_{\mathrm{Zar}}\,\mathrm{Aff}(\mathbb{A}^2 \setminus \{0\}, Y) \xrightarrow{\sim} \mathrm{PSh}\,\mathrm{Aff}(U, Y)$$

(injectivity) Let $\varphi, \psi : \mathbb{A}^2 \setminus \{0\} \to Y$ such that $\varphi|_U = \psi|_U$. To show $\varphi = \psi$, the uniqueness part of the density theorem says it suffices to show that for all $(f, g) : \mathrm{Spec}\,A \to \mathbb{A}^2 \setminus \{0\}$ we have $\varphi(f, g) = \psi(f, g)$. We have the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\,A \arrow[d, "\varphi(f,g)"'] & D(f) + D(g) \arrow[l] \arrow[d] & D(fg) \arrow[l, shift left] \arrow[l, shift right] \\ Y & U \arrow[l] & \end{tikzcd}$$

Since Y is a Zariski sheaf, it sees $\operatorname{Spec} A$ as the coequalizer of $D(fg) \rightrightarrows D(f) + D(g)$. In particular, $\varphi(f,g)$ is determined by its restrictions to $D(f), D(g)$. But $\varphi(f,g)|_{D(f)} = (\varphi|_U)|_{D(f)}$ and similarly for $D(g)$. The same goes for $\psi(f,g)$ and thus $\varphi = \psi$.

(surjectivity) Assume $\psi : U \to Y$ with the goal of showing this extends to a map $\mathbb{A}^2 \setminus \{0\} \to Y$. Again by the existence part of the density theorem, the data of the such a map is equivalent to the following

1. for each point $(f,g) : \operatorname{Spec} A \to \mathbb{A}^2 \setminus \{0\}$ we give a point $\tilde{\psi}(f,g) : \operatorname{Spec} A \to Y$
2. for each $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ we have
$$\tilde{\psi}(f,g)\varphi = \tilde{\psi}(f\varphi, g\varphi)$$
3. When (f,g) lies in U, we want $\tilde{\psi}(f,g) = \psi(f,g)$.

Let (f,g) be an A-point of $\mathbb{A}^2 \setminus \{0\}$. Again since Y is a Zariski sheaf, it recognises $\operatorname{Spec} A$ as the coequalizer of $D(fg) \rightrightarrows D(x) + D(y)$. So to define $\tilde{\psi}(f,g)$ is equivalent to defining an element of $Y(A[1/f]) \times_{Y(A[1/fg])} Y(A[1/g])$. Again note that $D(f), D(g) \to \mathbb{A}^2 \setminus \{0\}$ factor through U and agree on $D(fg)$. So we use the element
$$\left(\psi\left((f,g)|_{D(f)}\right), \psi\left((f,g)|_{D(g)}\right)\right)$$
We leave (2) and (3) as an exercise.

(Zariski sheaf) A map $\operatorname{Spec} A \to \mathbb{A}^2 \setminus \{0\}$ is the same as $(f,g) \in A$ such that $(f,g) = (1)$. This is the same as an A-module map $A^2 \to A$ which is a surjection. Given a basic Zariski cover U of $\operatorname{Spec} A$, Zariski descent says the set of maps $A^2 \to A$ bijects by restriction to the set of maps $\mathcal{O}(U)^2 \to \mathcal{O}(U)$ compatible with descent data. Since Zariski descent is an equivalence of abelian categories, and cokernel on the cover is computed in each open, this shows that surjectivity of an A-module map is *Zariski local*, which completes the proof. ☐

The above proof that $\mathbb{A}^2 \setminus \{0\}$ is a Zariski sheaf is an example of computation with *quasi-coherent sheaves*. We will come back to this in chapter 3.

The category of Zariski sheaves is designed to force Zariski covers to be effective epimorphisms. We have the following useful criterion for checking Zariski epimorphisms between general sheaves.

> **Proposition**
>
> Let $p : X \to Y$ be an map in $\mathrm{Sh}_{\mathrm{Zar}}\,\mathrm{Aff}$. Then p is an epimorphism iff it is an effective epimorphism iff "it locally has sections" i.e. for all points $y : \operatorname{Spec} A \to Y$ there exists a basic Zariski cover $\mathcal{U}$ of $\operatorname{Spec} A$ such that there exists a commutative diagram
>
> $$\begin{tikzcd} \mathcal{U} \arrow[r, dashed, "\exists x"] \arrow[d] & X \arrow[d, "p"] \\ \operatorname{Spec} A \arrow[r, "y"'] & Y \end{tikzcd}$$

Proof. Let $\operatorname{Im} p$ denote the subfunctor of Y consisting of points $y \in Y(A)$ which admit local lifts across p. Whether p is an epimorphism or not is about maps out of Y. Therefore we need a way of detecting whether $\operatorname{Im} p = Y$ using a map out of Y. The idea is to use indicator functions A.K.A. the existence of a subobject classifier. We omit the proof since it requires no geometric intuition.

□

2.3 Definition

Slogan : *a scheme is the quotient of affines schemes along open immersions*. For this, we need to define open immersions between Zariski sheaves.

> **Definition**
>
> Let $j : X \to Y$ be a map of presheaves. Then j is called an open immersion when for all points $y : \operatorname{Spec} A \to Y$, the pullback $\operatorname{Spec} A \times_Y X \to \operatorname{Spec} A$ is an open immersion.

We can now define schemes.

> **Definition**
>
> Let X be a Zariski sheaf. An affine Zariski cover / atlas of X is defined as a set of open immersions $\{j_i : \operatorname{Spec} A_i \to X\}_{i \in I}$ such that the Zariski disjoint union gives a Zariski epimorphism
>
> $$\coprod_{i \in I} \operatorname{Spec} A_i \twoheadrightarrow X$$
>
> A scheme X is a Zariski sheaf that admits an affine Zariski atlas.

The following justifies the slogan of this section.

> **Proposition − Scheme = quotient of affines along open immersions**
>
> Let X be a Zariski sheaf and let $\{j_i : \operatorname{Spec} A_i \to X\}_{i \in I}$ be a set of maps. Let $\mathcal{U} := \coprod_{i \in I} \operatorname{Spec} A_i$ be the Zariski disjoint union. Then the following are equivalent :
>
> 1. $\{j_i : \operatorname{Spec} A_i \to X\}_{i \in I}$ is a Zariski affine atlas for X.
> 2. There is an equivalence relation $\mathcal{R}$ on $\mathcal{U}$ such that
>   - the projections $\mathcal{R} \rightrightarrows \mathcal{U}$ are open embeddings
>   - The map $\mathcal{U} \to X$ exhibits X as the Zariski coequalizer of $\mathcal{R} \rightrightarrows \mathcal{U}$.

The proof requires some basic knowledge of the category of Zariski sheaves $\operatorname{Sh}_{\mathrm{Zar}} \operatorname{Aff} \subseteq \operatorname{PSh} \operatorname{Aff}$.

Proof. (1) implies (2) is clear.

Let $\mathcal{U}, \mathcal{R}$ be as in the situation (2). It suffices to show for each affine scheme summand $U = \operatorname{Spec} A_i$ of $\mathcal{U}$ the map $j_U : U \to \mathcal{U} \to X$ is an open embedding. Let $\tilde{X}$ be the coequaliser of $\mathcal{R} \rightrightarrows \mathcal{U}$ in the category of presheaves. Note that the Zariski sheafification of the coequaliser sequence for $\tilde{X}$ is the coequaliser sequence for X because Zariski sheafification is a left exact left adjoint. We need the following.

> *Lemma. Let $f : X \to Y$ be a morphism of presheaves which is an open immersion. Then*
>
> 1. *if Y is a Zariski sheaf, so is X.*
> 2. *if f is an open embedding then so is its Zariski sheafification.*

Proof. (1) Exercise.

(2) Suppose we have $S \to L(Y)$ with S affine. We need to show $S \times_{L(Y)} L(X) \to S$ is an open embedding. It suffices to do so after passing to some Zariski cover $\tilde{S} \to S$. One of the properties of Zariski sheafification is that we can find a Zariski cover $\tilde{S} \to S$ such that $\tilde{S} \to S \to L(Y)$ factors as $\tilde{S} \to Y \to L(Y)$. The rest follows from (1) and the fact that Zariski sheafification L is left exact.

$$\begin{tikzcd}
L(\tilde{S} \times_Y X) \arrow[d, phantom, "\simeq"] & \simeq & \tilde{S} \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \arrow[dr] & \\
\tilde{S} \times_{L(Y)} L(X) & & \tilde{S} \arrow[r, dashed] \arrow[dr, dashed] & Y \arrow[dr] & L(X) \arrow[d] \\
& & & S \arrow[r] & L(Y)
\end{tikzcd}$$

■

So it suffices to show $U \to \tilde{X}$ is an open embedding. Let $T \to \tilde{X}$ be a general point where T is affine. By the way colimits of presheaves are computed, we can lift $T \to \tilde{X}$ to a map $T \to \mathcal{U}$. We then have the cartesian squares :

$$\begin{tikzcd}
T \times_{\tilde{X}} U \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathcal{U} \times_{\tilde{X}} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\
& \mathcal{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \arrow[d] \\
T \arrow[r] & \mathcal{U} \arrow[r] & \tilde{X}
\end{tikzcd}$$

For the left vertical morphism to be an open embedding, it suffices that the middle two vertical morphisms are open embeddings. By assumption, $\mathcal{R} \to \mathcal{U}$ is an open embedding. It remains to prove that the inclusion $U \to \mathcal{U}$ is an open embedding. Using the lemma above again, it suffices to do so for the coproduct as presheaves, where the result is then clear.

□

Example.
Our example $\mathbb{A}^2 \setminus \{0\}$ which motivated Zariski sheaves is a scheme with atlas given by $D(x) + D(y)$.

Example.
Projective 1-space $\mathbb{P}^1$ is defined by the equivalence relation

$$\mathbb{A}^1 \setminus 0 \underset{1/t}{\overset{t}{\rightrightarrows}} \mathbb{A}^1 \coprod \mathbb{A}^1 \longrightarrow \mathbb{P}^1$$

After we see how to interpret modules from algebra as vector bundles in geometry, we will have another definition of $\mathbb{P}^1$ and $\mathbb{P}^n$ in general.

2.4 Associated topological space

Although schemes are built out of commutative rings and pictures can be drawn without reference to topological spaces, it can sometimes be useful to have a topological space. In this section, we show how to associate a topological space $|X|$ to a scheme X. The reader is warned that the procedure $X \rightsquigarrow |X|$ loses *a lot* of information. One *should not* conflate X with $|X|$.

Definition

For $A \in \mathbf{CRing}$, define

$$|\mathrm{Spec}\, A| := \{A \to K \mid K \text{ field}\} / \sim$$

where the equivalence relation is generated by $(A \to K) \sim (A \to L)$ when the first factors through the second. This forms a functor to $\mathrm{Aff} \to \mathbf{Set}$. For $f \in A$, because $A \to A_f$ is an epimorphism it follows that $|D(f)| \to |\mathrm{Spec}\, A|$ is injective, which we view as a subset. Let the opens in $|\mathrm{Spec}\, A|$ be generated by subsets of the form $|D(f)|$ for $f \in A$. Then this forms a functor

$$|\_| : \mathrm{Aff} \to \mathbf{Top}$$

Remark – Worries. For those who have already done some modern algebraic geometry, you may be wondering where the prime ideals are, since $|\mathrm{Spec}\, A|$ is more standardly defined as the set of prime ideals of A. A perhaps more serious question is whether under our definition, $|\mathrm{Spec}\, A|$ is even a set at all?

Both these questions are answered by the following lemma which realizes our idea that for $p \in |\mathrm{Spec}\, A|$, $\mathrm{ev}_p : A \to \kappa(p)$ is the minimal representative of p.

> *Lemma (Evaluation). Let $A \in \mathbf{Ring}$, $p \in |\mathrm{Spec}\, A|$ be a point. Then there exists $\mathrm{ev}_p : A \to \kappa(p)$ where $\kappa(p)$ is a field that is minimal in the equivalence class p, i.e. every other $A \to K$ representing p factors uniquely through ev_p. ev_p is thus unique up to unique isomorphism in the class p. In fact, ev_p is an epimorphism of rings. Its codomain $\kappa(p)$ is called the* residue field *at p.*
>
> *Hence, $|\mathrm{Spec}\, A|$ bijects with the set of prime ideals of A.*

Proof. For any $\mathrm{ev}_K : A \to K$ representing p, the ideal $I = \ker \mathrm{ev}_K$ is independent of K by the equivalence relation and ring morphisms from fields being injective. Define $\mathrm{ev}_p : A \to \kappa(p)$ with $\kappa(p) :=$ Frac A/I. The the UP of quotients and fields of fractions implies the minimality of ev_p as a representative of p. ev_p is indeed epi and $I = \ker \mathrm{ev}_p$.

Prime ideals inject into $|\mathrm{Spec}\, A|$ via $(p) \mapsto$ Frac $A/(p)$. Now for $p \in |\mathrm{Spec}\, A|$, we already saw that $\kappa(p) =$ Frac $A/\ker \mathrm{ev}_p$. $\ker \mathrm{ev}_p$ is thus a prime ideal mapping to p, proving surjectivity. $\square$

Definition
Let X be a scheme. Then define

$$|X| := \varinjlim_{U} |U|$$

where U runs through all affine opens of X. This upgrades our previous functor to

$$|\_| : \mathrm{Sch} \to \mathbf{Top}$$

Proposition
Let $\mathcal{U}$ be a Zariski affine cover of a scheme X. Then we have a coequalizer diagram

$$|\mathcal{U} \times_X \mathcal{U}| \rightrightarrows |\mathcal{U}| \to |X|$$

3 Quasi-coherent sheaves

3.1 Examples of modules as general vector bundles

We have seen how to interpret commutative algebras as spaces. The focus of this chapter is to extend this analogy to *modules*. Slogan :

Modules are $($*dual to*$)$ *generalised vector bundles*

For intuition, let us work over an algebraically closed field k. The simplest example of the slogan is the interpretation of a finite dimensional k-vector space V as a vector bundle over $\operatorname{Spec} k$.

Proposition
Let $V \in \mathbf{Vec}_k^c$ be a finite dimensional k-vector space. Define the presheaf $\underline{V}$ on Aff by

$$\underline{V}(S) := V \otimes_k \mathcal{O}(S)$$

Then

$$\underline{V} \simeq \operatorname{Spec} \operatorname{Sym}_k V^\vee$$

where $V^\vee := \underline{\mathrm{Hom}}_k(V, k)$.

This fits with the intuition that $V^\vee$ is the collection of *linear* functions on V so $\operatorname{Sym}_k V^\vee$ is the collection of functions on V and hence $\underline{V}$ is affine scheme capturing V. Elements of $V \otimes_k (S)$ are weighted sums of vectors in V by measuring devices on S. Given a state $x : \operatorname{Spec} k \to S$, we can evaluate the weighted sum to obtain a single vector in V.

Proof.

$$V \otimes_k \mathcal{O}(S) \simeq \mathbf{Vec}_k(V^\vee, \mathcal{O}(S)) \simeq \mathrm{Aff}_k(S, \operatorname{Spec} \operatorname{Sym}_k V^\vee)$$

$\square$

3.2 Descent

Descent of modules formalizes the idea that under certain conditions, one can view $p : \operatorname{Spec} B \to \operatorname{Spec} A$ as being a "surjection" in the sense that A-modules can be thought of as B-modules equipped identifications of fibers along fibers of p. We begin with motivation from sets again.

Proposition

Let $p : \mathcal{U} \to X$ be a surjection of sets. Define functors

$$\begin{tikzcd} \mathrm{Set}/X \arrow[r, "p^*"] & \mathrm{Set}/\mathcal{U} \arrow[l, bend right, "p_!"'] \arrow[l, bend left, "p_*"] \end{tikzcd}$$

1. p^* sends $E \to X$ to $E \times_X \mathcal{U}$
2. $p_!$ sends $F \to \mathcal{U}$ to $F \to X$
3. p_* sends $F \to \mathcal{U}$ to the X-family $x \mapsto \prod_{u \in p^{-1}(x)} F_u$.

Consider the following diagram :

$$\begin{tikzcd} \mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U} \arrow[r, dashed, shift left=2, "p_{01}"] \arrow[r, dashed, "p_{12}"] \arrow[r, dashed, shift right=2, "p_{02}"'] & \mathcal{U} \times_X \mathcal{U} \arrow[r, dashed, shift left, "p_0"] \arrow[r, dashed, shift right, "p_1"'] & \mathcal{U} \end{tikzcd}$$

where we have the projections

- $p_i : \mathcal{U} \times_X \mathcal{U} \to \mathcal{U}, (x_0, x_1) \mapsto x_i$
- $p_{ij} : \mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U} \to \mathcal{U} \times_X \mathcal{U}, (x_0, x_1, x_2) \mapsto (x_i, x_j)$

Note that pullback along these projections correspond to restricting to double and triple intersections. Define $(\mathbf{Set}/\mathcal{U})^{\mathcal{U} \times_X \mathcal{U}}$ as the category where :

- An object of $(\mathbf{Set}/\mathcal{U})^{\mathcal{U} \times_X \mathcal{U}}$ is a family $\mathcal{F} \in \mathbf{Set}/\mathcal{U}$ equipped with a *transition map*

  $$\phi : p_0^*\mathcal{F} \cong p_1^*\mathcal{F}, (u_0, u_1, v \in F_{u_0}) \mapsto (u_0, u_1, \phi(u_0, u_1, v) \in F_{u_1})$$

  in $\mathbf{Set}/\mathcal{U} \times_X \mathcal{U}$ such that

  1. (reflexivity) $\phi = \mathbb{1}$ when restricted to the diagonal $\Delta : \mathcal{U} \to \mathcal{U} \times_X \mathcal{U}$. i.e.

     $$\phi(u_0, u_0, v) = v$$

  2. (transitivity) the following diagram commute :

$$\begin{tikzcd}[column sep=small]
& p_{01}^* p_1^* \mathcal{F} \arrow[dr, "\sim"] & \\
p_{01}^* p_0^* \mathcal{F} \arrow[ur, "p_{01}^*(\phi)"] \arrow[d, "\sim"'] & & p_{12}^* p_0^* \mathcal{F} \arrow[d, "p_{12}^*(\phi)"] \\
p_{02}^* p_0^* \mathcal{F} \arrow[dr, "p_{02}^*(\phi)"'] & & p_{12}^* p_1^* \mathcal{F} \arrow[dl, "\sim"] \\
& p_{02}^* p_1^* \mathcal{F} &
\end{tikzcd}$$

i.e. $\phi(u_1, u_2, \phi(u_0, u_1, v)) = \phi(u_0, u_2, v)$.

- a morphism $\eta : (\mathcal{F}, \phi) \to (\mathcal{G}, \psi)$ is a morphism $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Set}/\mathcal{U}$ such that the following commutes :

$$\begin{tikzcd}
p_0^* \mathcal{F} \arrow[r, "\phi"] \arrow[d, "p_0^*(\eta)"'] & p_1^* \mathcal{F} \arrow[d, "p_1^*(\eta)"] \\
p_0^* \mathcal{G} \arrow[r, "\psi"'] & p_1^* \mathcal{G}
\end{tikzcd}$$

Then we have an equivalence between three categories :

$$\mathbf{Set}/X \simeq \mathbf{Mod}_T \simeq (\mathbf{Set}/\mathcal{U})^{\mathcal{U} \times_X \mathcal{U}}$$

where $\mathbf{Mod}_T$ is the category of algebras over the monad $T := p^* p_!$ under which the following adjunctions coincide :

- $p_! \dashv p^*$ for $\mathbf{Set}/X$
- free $\dashv$ forget for $\mathbf{Mod}_T$
- $L \dashv$ forget for $(\mathbf{Set}/\mathcal{U})^{\mathcal{U} \times_X \mathcal{U}}$ where L is explicit but I am too lazy right now to describe.

A key example of a surjection $p : \mathcal{U} \to X$ is when $\mathcal{U} = \coprod_{U \in I} U$ where I is a collection of subsets of X which covers X. In this case $\mathcal{U} \times_X \mathcal{U} \simeq \coprod_{U,V \in I} U \cap V$ tracks the pairwise intersections.

Proof. Let us make the comparison $\mathbf{Mod}_T \simeq (\mathbf{Set}/\mathcal{U})^{\mathcal{U} \times_X \mathcal{U}}$ first. Note that

$$\begin{tikzcd}
\mathcal{U} \arrow[d] & \mathcal{U} \times_X \mathcal{U} \arrow[l, "p_1"'] \arrow[d, "p_0"] \arrow[dl, phantom, "\llcorner"] & F \times_X \mathcal{U} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\
X & \mathcal{U} \arrow[l] & F \arrow[l]
\end{tikzcd}$$

Concretely, $F \times_X \mathcal{U}$ consists of $(v \in F_{u_0}, u_1)$ where $p(u_0) = p(u_1)$ and the projection $F \times_X \mathcal{U} \to \mathcal{U}$ is

$(v, u_0, u_1) \mapsto u_1$.

Now let's work out the multiplication $\mu : T^2 \to T$ of the monad. We have

$$T^2 F = (F \times_X \mathcal{U}) \times_X \mathcal{U}$$

The projection $(F \times_X \mathcal{U}) \times_X \mathcal{U}$ is $(v \in F_{u_0}, u_1, u_2) \mapsto u_2$. So the multiplication $T^2F \mapsto TF$ is

$$(v \in F_{u_0}, u_1, u_2) \mapsto (v \in F_{u_0}, u_2)$$

We can also compute the unit $F \to TF$ as

$$v \in F_{u_0} \mapsto (v \in F_{u_0}, u_0)$$

and verify the unit laws :

$$\begin{tikzcd} F \times_X \mathcal{U} \arrow[r] \arrow[d] & (F \times_X \mathcal{U}) \times_X \mathcal{U} \arrow[d] \\ (F \times_X \mathcal{U}) \times_X \mathcal{U} \arrow[r] & F \times_X \mathcal{U} \end{tikzcd}$$

– "unit then multiply" is $(v \in F_{u_0}, u_1) \mapsto (v \in F_{u_0}, u_0, u_1) \mapsto (v \in F_{u_0}, u_1)$

– "multiply then unit" is $(v \in F_{u_0}, u_1) \mapsto (v \in F_{u_0}, u_1, u_1) \mapsto (v \in F_{u_0}, u_1)$.

Associativity is the fact that given $(v \in F_{u_0}, u_1, u_2, u_3)$, first forgeting u_2 then u_1 is the same as first forgetting u_1 then u_2.

We are ready for modules over T. A map $\phi : F \times_X \mathcal{U} \to F$ over $\mathcal{U}$ sends $(v \in F_{u_0}, u_1) \mapsto \phi(v, u_1) \in F_{u_1}$. In other words, ϕ transports $v \in F_{u_0}$ to the fiber F_{u_1} given that $u_0 = u_1$ in X. This gives F a T-module structure when we have

– (associativity) $\phi(\phi(v \in F_{u_0}, u_1), u_2) = \phi(v \in F_{u_0}, u_2)$

– (unity) $\phi(v \in F_{u_0}, u_0) = v$.

Here is the tautology : ϕ is equivalent to giving a map $\widetilde{\phi} : F_{u_0} \to F_{u_1}$ for every pair $p(u_0) = p(u_1)$. In fancy terms, we have $p^* p_! \simeq (p_0)_! p_1^*$ which implies

$$(\mathbf{Set}/\mathcal{U})(p^* p_! F, F) \simeq (\mathbf{Set}/\mathcal{U} \times_X \mathcal{U})(p_0^* F, p_1^* F)$$

The point now is that

– associativity for ϕ corresponds to transitivity of $\widetilde{\phi}$

– unity for ϕ corresponds to reflexivity of $\widetilde{\phi}$.

What about symmetry of $\widetilde{\phi}$? This is implied because

$$\phi(\phi(v \in F_{u_0}, u_1), u_0) = \phi(v \in F_{u_0}, u_0) = v$$

This proves the equivalence $\mathbf{Mod}_T \simeq (\mathbf{Set}/\mathcal{U})^{\mathcal{U} \times_X \mathcal{U}}$. One can see that the forgetful functor $\mathbf{Mod}_T \to \mathbf{Set}/\mathcal{U}$ corresponds to the forgetful functor $(\mathbf{Set}/\mathcal{U})^{\mathcal{U} \times_X \mathcal{U}} \to \mathbf{Set}/\mathcal{U}$.

The comparison $\mathbf{Set}/X \simeq \mathbf{Mod}_T$ comes from the monadicity theorem. Let $\mathrm{Glue} : \mathbf{Mod}_T \to \mathbf{Set}/X$ denote the inverse constructed in the proof of the monadicity theorem. Indeed, all the categories involved are locally small and we have a triple of adjoints $p_! \dashv p^* \dashv p_*$ so p^* preserves small colimits. The fact that p is surjective implies that p^* is conservative.[1] Opening the proof of the monadicity theorem, the key part is that for any T-module $\phi : TF \to F$, we have the (split) coequalizer in $\mathbf{Mod}_T$:

$$\begin{tikzcd} T^2F \arrow[r, shift left, "\mu_F"] \arrow[r, shift right, "T(\phi)"'] & TF \arrow[r, "\phi"] & F \end{tikzcd}$$

Concretely in our situation,

- $\mu_F : (v \in F_{u_0}, u_1, u_2) \mapsto (v \in F_{u_0}, u_2)$
- $T(\phi) : (v \in F_{u_0}, u_1, u_2) \mapsto (\phi(v \in F_{u_0}, u_1) \in F_{u_1}, u_2)$

The functor Glue is defined by forming the coequalizer diagram in $\mathbf{Set}/X$

$$\begin{tikzcd} p_!p^*p_!F \arrow[r, shift left, "\nu_F"] \arrow[r, shift right, "p_!(\phi)"'] & p_!F \arrow[r] & \mathrm{Glue}(F, \phi) \end{tikzcd}$$

where

- $\nu_V : (v \in F_{u_0}, u_1, x) \mapsto (v, x)$
- $p_!(\phi) : (v \in F_{u_0}, u_1, x) \mapsto (\phi(v, u_1), x)$

In other words, for each $x \in X$, the fiber $\mathrm{Glue}(F, \phi)_x$ is obtained by identifying all F_u with $p(u) = x$ using the transition map ϕ. □

We now apply the above idea to standard Zariski covers $(f_1, \ldots, f_n) = A$.

$$p : \coprod_{i=1}^{n} D(f_i) \to \mathrm{Spec}\, A$$

Here the adjunction is

$$\begin{tikzcd} (A\mathbf{Mod})^{\mathrm{op}} \arrow[r, shift right, "B\otimes_A -"'] \arrow[r, phantom, "\perp"] & (B\mathbf{Mod})^{\mathrm{op}} \arrow[l, shift right, "\text{forget}"'] \end{tikzcd}$$

where $B = \prod_{i=1}^{n} A[1/f_i]$.

> **Definition – Descent data**
> Let $A \to B$ be an algebra map. Define $(B\mathbf{Mod})^{B\otimes_A B}$ as the category with

[1] In fact, in this situation p being surjective is equivalent to conservativity of p^*.

- objects (N, ϕ) where $N \in B\mathbf{Mod}$ and $\phi \in (B \otimes_A B)\mathbf{Mod}(B \otimes_A N, N \otimes_A B)$ such that we have the *cocycle condition*, i.e. the following triangle commutes

$$\begin{tikzcd} & B \otimes_A N \otimes_A B \arrow[dd, "\phi_{12}"] \\ N \otimes_A B \otimes_A B \arrow[ur, "\phi_{01}"] \arrow[dr, "\phi_{02}"'] & \\ & B \otimes_A B \otimes_A N \end{tikzcd}$$

  where

  - $\phi_{01} = \phi \otimes \mathbb{1}_B$
  - $\phi_{12} = \mathbb{1}_B \otimes \phi$
  - ϕ_{02} = "swap 0-th and 2nd factor using ϕ", which is $n \otimes f \otimes g \mapsto \sum_i a_i \otimes f \otimes n_i$ where $\phi(n \otimes g) = \sum_i a_i \otimes n_i$.

- a morphism $(N, \phi) \to (N_1, \phi_1)$ is a morphism

Proposition – Descent for standard Zariski covers

3.3 Quasi-coherent sheaves

We define quasi-coherent sheaves on a scheme X to be such that it is equivalent to having modules over an affine open cover plus identification on pairwise intersections. In practice, Zariski affine covers of schemes will be such that the intersection of two affine opens will always be affine again.

Definition – Quasi-coherent sheaves w.r.t. Zariski affine cover

Let X be a scheme and $\mathcal{U} = \coprod_{i \in I} \operatorname{Spec} A_i$ a Zariski affine cover where the pairwise intersections are affine. Consider the following diagram :

$$\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U} \underset{p_{02}}{\overset{p_{01}}{\underset{p_{12}}{\rightrightarrows}}} \mathcal{U} \times_X \mathcal{U} \underset{p_1}{\overset{p_0}{\rightrightarrows}} \mathcal{U}$$

where we have the projections

- $p_i : \mathcal{U} \times_X \mathcal{U} \to \mathcal{U}, (x_0, x_1) \mapsto x_i$
- $p_{ij} : \mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U} \to \mathcal{U} \times_X \mathcal{U}, (x_0, x_1, x_2) \mapsto (x_i, x_j)$

By our assumption on X, all schemes are disjoint unions of affine opens of X. Momentarily, let

$$\mathrm{QCoh}\,\mathcal{U} := \prod_{i \in I} A_i\mathbf{Mod}$$

and similarly for $\mathrm{QCoh}(\mathcal{U} \times \mathcal{U}), \mathrm{QCoh}(\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U})$. For $p : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ any of the above projections, we use p^* to denote the functor $B \otimes_A \_$. Then define the *category* $\mathrm{QCoh}(X, \mathcal{U})$ *of quasi-coherent sheaves on* X *w.r.t.* $\mathcal{U}$ as follows :

– An object of $\mathrm{QCoh}(X, \mathcal{U})$ is a set of modules $\mathcal{F} \in \mathrm{QCoh}(\mathcal{U})$ equipped with a *transition map* $\phi : p_0^*\mathcal{F} \cong p_1^*\mathcal{F}$ in $\mathrm{QCoh}(\mathcal{U} \times_X \mathcal{U})$ such that

1. $\phi = \mathbb{1}$ when restricted to the diagonal $\Delta : \mathcal{U} \to \mathcal{U} \times_X \mathcal{U}$
2. the following diagram commute :

$$\begin{tikzcd}
 & p_{01}^*p_1^*\mathcal{F} \arrow[dr, "\sim"] & \\
p_{01}^*p_0^*\mathcal{F} \arrow[ur, "p_{01}^*(\phi)"] \arrow[d, "\sim"'] & & p_{12}^*p_0^*\mathcal{F} \arrow[d, "p_{12}^*(\phi)"] \\
p_{02}^*p_0^*\mathcal{F} \arrow[dr, "p_{02}^*(\phi)"'] & & p_{12}^*p_1^*\mathcal{F} \arrow[dl, "\sim"] \\
 & p_{02}^*p_1^*\mathcal{F} &
\end{tikzcd}$$

– a morphism $\eta : (\mathcal{F}, \phi) \to (\mathcal{G}, \psi)$ is a morphism $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{QCoh}\,\mathcal{U}$ such that the following commutes : [a]

$$\begin{tikzcd}
p_0^*\mathcal{F} \arrow[r, "\phi"] \arrow[d, "p_0^*(\eta)"'] & p_1^*\mathcal{F} \arrow[d, "p_1^*(\eta)"] \\
p_0^*\mathcal{G} \arrow[r, "\psi"'] & p_1^*\mathcal{G}
\end{tikzcd}$$

[a] This may remind you of the definition of equivariant maps between set endowed with an action of a group G. This is not a coincidence : $\mathrm{QCoh}(X, \mathcal{U})$ is effectively quasi-coherent sheaves on $\mathcal{U}$ equipped with an action of the *groupoid* $\mathcal{U} \times_X \mathcal{U} \rightrightarrows \mathcal{U}$.

Example.
Let $U_0 := \mathrm{Spec}\, k[t], U_1 = \mathrm{Spec}\, k[t^{-1}]$ *be the standard affine opens of* $\mathbb{P}^1_k$. *Then* $U_{01} := U_0 \times_{\mathbb{P}^1_k} U_1 \simeq \mathrm{Spec}\, k[t, t^{-1}]$.

The above definition is how one goes about giving examples of quasi-coherent sheaves on non-affine schemes in practice. For theoretical purposes, we need to have a definition that is independent of the choice of Zariski affine cover and then prove that it is equivalent to the above.

Definition – Quasi-coherent sheaves on a functor

Let X be a scheme. Then a quasi-coherent sheaf on X consists of the following data :

- for each $x : \operatorname{Spec} A \to X$, an A-module $\mathcal{F}_x$.
- (transition map) for each $f : \operatorname{Spec} B \to \operatorname{Spec} A$ and $x : \operatorname{Spec} A \to X$, a morphism of A-modules

$$\mathcal{F}_x \to \mathcal{F}_{xf}$$

  that is identity when f is. We call the map $\mathcal{F}_x \to \mathcal{F}_{xf}$ the *transition map associated to f*.
- (transitivity) We require the maps given in the previous point to satisfy that for any commuting triangle on the left,

$$\begin{tikzcd} \operatorname{Spec} C \arrow[d, "g"'] \arrow[dr] & & \rightsquigarrow & \mathcal{F}_{xfg} & \\ \operatorname{Spec} B \arrow[r, "f"'] & \operatorname{Spec} A & & \mathcal{F}_{xf} \arrow[u] & \mathcal{F}_x \arrow[l] \arrow[ul] \end{tikzcd}$$

  we get a commuting triangle on the right of A-modules.
- (quasi-coherence) The transition map associated to $f : \operatorname{Spec} B \to \operatorname{Spec} A$ for any point $x : \operatorname{Spec} A \to X$ induces an isomorphism $\mathcal{F}_x \otimes_A B \simeq \mathcal{F}_{xf}$.

Let $\mathcal{F}, \mathcal{G}$ be two quasi-coherent sheaves on X. Then a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ consists of the data :

- for each $x : \operatorname{Spec} A \to X$, a morphism of A-modules

$$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$$

- for each $f : \operatorname{Spec} B \to \operatorname{Spec} A$ and $x : \operatorname{Spec} A \to X$, we have the commutative diagram in A-modules

$$\begin{tikzcd} \mathcal{F}_{xf} \arrow[r, "\varphi_{xf}"] & \mathcal{G}_{xf} \\ \mathcal{F}_x \arrow[u] \arrow[r, "\varphi_x"'] & \mathcal{G}_x \arrow[u] \end{tikzcd}$$

  where the vertical morphisms are the transition maps of $\mathcal{F}$ and $\mathcal{G}$ associated to f.

We write $\operatorname{QCoh} X$ for the category of quasi-coherent sheaves on X.

Note that a priori, a quasi-coherent sheaf consists of infinite amount of data. [1] We first note :

[1]For those worried about set-theoretic issues, we do indeed have a problem here. For any functor X, the category Aff/X is too large to be a set. So as of now, $\operatorname{QCoh} X$ has a proper class of objects and even for $\mathcal{F}, \mathcal{G} \in \operatorname{QCoh} X$, the collection $(\operatorname{QCoh} X)(\mathcal{F}, \mathcal{G})$ is also a proper class.

Proposition

For $X = \operatorname{Spec} A$,
$$\operatorname{QCoh} X \simeq A\mathbf{Mod}$$

Proof. Exercise. □

We now justify our motivation that we can compute quasi-coherent sheaves from affine open covers.

Proposition

Let X be a scheme. Consider $\operatorname{QCoh}_{\mathrm{Aff}} X$ which is defined as the same as $\operatorname{QCoh} X$ except we only look at $x : \operatorname{Spec} A \to X$ which is an open of X. Also let $\mathcal{U}$ be a Zariski affine cover of X where pairwise intersections are affine. Then the restriction functors are equivalences of categories.[a]

$$\begin{tikzcd} \operatorname{QCoh} X \arrow[r, "\sim"] \arrow[dr, "\sim"'] & \operatorname{QCoh}_{\mathrm{Aff}} X \arrow[d, "\sim"] \\ & \operatorname{QCoh}(X, \mathcal{U}) \end{tikzcd}$$

[a] In particular, all set-theoretic issues are resolved. $\operatorname{QCoh}_{\mathrm{Aff}} X$ has a set of objects and for $\mathcal{F}, \mathcal{G} \in \operatorname{QCoh}_{\mathrm{Aff}} X$, the collection $(\operatorname{QCoh}_{\mathrm{Aff}} X)(\mathcal{F}, \mathcal{G})$ is a set.

Proof. We only construct a quasi-inverse to the diagonal arrow. A quasi-inverse to the horizontal map is similar using a choice of arbitrary Zariski affine cover of X.

(Objects) Let $(\mathcal{F}, \phi) \in \operatorname{QCoh}(X, \mathcal{U})$. Let $x : \operatorname{Spec} A \to X$. By assumption of $\mathcal{U} \to X$ being a Zariski epimorphism, there exists a standard Zariski cover $\mathcal{U}_A$ of $\operatorname{Spec} A$ and a commuting diagram :

$$\begin{tikzcd} \mathcal{U}_A \arrow[r, dashed, "\exists"] \arrow[d] & \mathcal{U} \arrow[d] \\ \operatorname{Spec} A \arrow[r] & X \end{tikzcd}$$

For such a $\mathcal{U}_A$, we have an explicit inverse functor given by Zariski descent of modules :

$$\operatorname{QCoh}(\operatorname{Spec} A, \mathcal{U}_A) \xrightarrow{\mathrm{Glue}_{\mathcal{U}_A}} A\mathbf{Mod}$$

$$(\mathcal{F}|_{\mathcal{U}_A}, \phi|_{\mathcal{U}_A}) \longmapsto \operatorname{Eq}\left(\prod_{V \in \mathcal{U}_A} \mathcal{F}(V) \rightrightarrows \prod_{V_0, V_1 \in \mathcal{U}_A} \mathcal{F}(V_0 \cap V_1)\right)$$

And hence gives
$$\mathrm{Glue}_A := \mathrm{Glue}_{\mathcal{U}_A} : \operatorname{QCoh}(X, \mathcal{U}) \to A\mathbf{Mod}$$

However, in order to get a quasi-coherent sheaf on X we also need for each inclusion of affine opens $\operatorname{Spec} B \to \operatorname{Spec} A \to X$ a morphism of A-modules

$$\operatorname{Glue}_A(\mathcal{F}, \phi) \to \operatorname{Glue}_B(\mathcal{F}, \phi)$$

satisfying identity, transitivity, and quasi-coherence. So we cannot depend on the choice of $\mathcal{U}_A$. We thus redefine Glue_A by using all of them at once :

$$\operatorname{Glue}_A : \operatorname{QCoh}(X, \mathcal{U}) \to A\mathbf{Mod}, (\mathcal{F}, \phi) \mapsto \varinjlim_{\mathcal{V}} \operatorname{Glue}_{\mathcal{U}_A}(\mathcal{F}|_{\mathcal{U}_A}, \phi|_{\mathcal{U}_A})$$

where the colimit is over the filtered diagram of standard Zariski covers $\mathcal{U}_A$ of $\operatorname{Spec} A$ factoring through $\mathcal{U}$. [1] Now we get induced transition maps which satisfies identity and transitivity. It remains to check quasi-coherence. The point is that the given projection

$$\operatorname{Glue}_A(\mathcal{F}, \phi) \xrightarrow{\sim} \operatorname{Glue}_{\mathcal{U}_A}(\mathcal{F}|_{\mathcal{U}_A}, \phi|_{\mathcal{U}_A})$$

is still an isomorphism for any standard Zariski cover $\mathcal{U}_A$ factoring through $\mathcal{U}$ because all the transition maps of the filtered diagram are isomorphisms. Furthermore, for inclusions of $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine opens of X, the base change $\mathcal{U}_B$ of $\mathcal{U}_A$ to $\operatorname{Spec} B$ is another standard Zariski cover. So it follows that

$$\begin{tikzcd}
\operatorname{Glue}_A(\mathcal{F}, \phi) \arrow[r] \arrow[d, "\sim"'] & \operatorname{Glue}_B(\mathcal{F}, \phi) \arrow[d, "\sim"'] \\
\operatorname{Glue}_{\mathcal{U}}(\mathcal{F}|_{\mathcal{U}_A}, \phi|_{\mathcal{U}_A}) \arrow[r] & \operatorname{Glue}_{\mathcal{U}}(\mathcal{F}|_{\mathcal{U}_B}, \phi|_{\mathcal{U}_B})
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
B \otimes_A \operatorname{Glue}_A(\mathcal{F}, \phi) \arrow[r] \arrow[d, "\sim"'] & \operatorname{Glue}_B(\mathcal{F}, \phi) \arrow[d, "\sim"] \\
B \otimes_A \operatorname{Glue}_{\mathcal{U}}(\mathcal{F}|_{\mathcal{U}_A}, \phi|_{\mathcal{U}_A}) \arrow[r, "\sim"'] & \operatorname{Glue}_{\mathcal{U}}(\mathcal{F}|_{\mathcal{U}_B}, \phi|_{\mathcal{U}_B})
\end{tikzcd}$$

So $\operatorname{Glue}_A(\mathcal{F}, \phi)$ assembles across all affine opens of X to give a quasi-coherent sheaf $\operatorname{Glue}(\mathcal{F}, \phi)$ on X.

(Morphisms) Clear from the construction of Glue.

$(\operatorname{QCoh}(X, \mathcal{U}) \to \operatorname{QCoh}(X, \mathcal{U})$ isomorphic to identity$)$ Clear by construction.

$(\operatorname{QCoh} X \to \operatorname{QCoh} X$ isomorphic to identity$)$ Also clear from construction. $\square$

Proposition

Let X be a scheme. Then

1. $\operatorname{QCoh} X$ is an abelian category with small colimits.
2. All small colimits and finite limits are computed on affine opens.
3. fiberwise tensor product endows $\operatorname{QCoh} X$ with a symmetric monoidal structure.

Proof. Small colimits are computed fiberwise. This only relies on the fact that tensor product preserves small colimits. For kernels to exist, one must compute on affine opens. Then the key fact is that for $\operatorname{Spec} B \to \operatorname{Spec} A$ an open embedding, the functor $B \otimes_A \_$ is flat. $\square$

[1] The lesson here is that when you do not want to make a choice and all choices are equivalent by induced maps, just take the limit or colimit of the choices. This gives something isomorphic to any choice but is canonical.

3.4 Induced functors on quasi-coherent sheaves

Definition

Let $f : X \to Y$ in $\mathrm{PSh}\,\mathrm{Aff}$. Define

$$f^* : \mathrm{QCoh}\, Y \to \mathrm{QCoh}\, X$$

by $\mathcal{F} \mapsto (x \in X(A) \mapsto \mathcal{F}_{fx})$. This is symmetric monoidal with respect to the tensor product of quasi-coherent sheaves.

Proposition

Let $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$. Then under the identification $\mathrm{QCoh}\,\mathrm{Spec}\, A = A\mathbf{Mod}$ and $\mathrm{QCoh}\,\mathrm{Spec}\, B = B\mathbf{Mod}$, we have

$$f^* M \simeq B \otimes_A M$$

Proof. Exercise. □

Proposition

Let $f : X \to Y$ be a morphism between schemes. Then there exists an adjunction

$$f^* \dashv f_* : \mathrm{QCoh}\, X \leftrightarrows \mathrm{QCoh}\, Y$$

Proof. This is abstract non-sense and has no geometric content. The fact that X, Y are schemes ensure that $\mathrm{QCoh}\, X$, $\mathrm{QCoh}\, Y$ are equivalent to small categories so that one can apply the adjoint functor theorem to f^*. The reason that f^* preserves small colimits is that they are computed fiberwise. □

We will show how to compute sections of pushforward on affine opens. This requires a technical assumption of quasi-compact quasi-separated-ness which is covered in the next section.

Proposition – Base change for affine opens

Let $f : X \to Y$ be a qcqs morphism between schemes. Let $j : U \subseteq Y$ be an affine open and $\mathcal{F} \in \mathrm{QCoh}\, X$. Then

$$(f_* \mathcal{F})(U) \simeq \mathcal{F}(f^{-1}(U))$$

Since U is affine, we are equivalently proving $j^* f_* \simeq (f_1)_* j_1^*$ which is an example of base change.

Proof. Proved next section. □

3.5 Concrete description of pushforward

In this section, we prove the concrete description of pushforward of quasi-coherent sheaves along a qcqs morphism $f : X \to Y$ between schemes.

Let us first consider the case when $Y = \mathrm{Spec}\, A$ an affine.

Proposition – Pushforward with affine target
The following three functors $\mathrm{QCoh}\,X \to \mathrm{QCoh}(\mathrm{Spec}\,A)$ are isomorphic :

1. (Abstract pushforward) the pushforward f_*
2. (Global sections) For each $\mathcal{F} \in \mathrm{QCoh}\,X$, take the A-module
$$\mathcal{F}(X) := (\mathrm{QCoh}\,X)(\mathcal{O}_X, \mathcal{F})$$
3. (Practical) For an affine Zariski cover $(U_i)_{i\in I}$ of X with $U_{ij} := U_i \cap U_j$ affine again, take the A-module
$$\mathrm{Eq}\left(\prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I}\mathcal{F}(U_{ij})\right)$$

Proof. (1 isomorphic 2)
$$f_*\mathcal{F} \simeq A\mathbf{Mod}(A, f_*\mathcal{F}) \simeq (\mathrm{QCoh}\,X)(f^*A, \mathcal{F}) \simeq (\mathrm{QCoh}\,X)(\mathcal{O}_X, \mathcal{F}) =: \mathcal{F}(X)$$

(2 isomorphic 3) Let $\mathcal{U} := \coprod_{i\in I} U_i$. Then we use
$$\mathrm{QCoh}\,X \simeq \mathrm{QCoh}(X,\mathcal{U})$$
Under this equivalence, $\mathcal{O}_X \in \mathrm{QCoh}\,X$ is sent to the descent data $(\mathcal{O}(U_i))_{i\in I}$ with the natural cocycle data. Then the A-module $(\mathrm{QCoh}\,X)(\mathcal{O}_X, \mathcal{F})$ is isomorphic to the A-module of morphisms of cocycle data
$$(\mathcal{O}(U_i))_{i\in I} \to (\mathcal{F}(U_i))_{i\in I}$$
Such a map is equivalent to choosing a compatible system of sections of $\mathcal{F}$ on the opens U_i, i.e. an element of the desired equalizer. □

Most practical situations fall in the case of (3). For theoretical purposes, we can relax the condition of $U_i \cap U_i$ being affine by choosing an affine Zariski cover $(U_{ij}^k)_{k\in I_{i,j}}$ of $U_i \cap U_j$. With that, one can define $f_*\mathcal{F}$ by the equaliser diagram in $\mathrm{QCoh}\,A$,
$$f_*\mathcal{F} \to \prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I}\prod_{k\in I_{i,j}}\mathcal{F}(U_{ij}^k)$$
Let us call the data of $((U_i)_{i\in I}, (U_{ij}^k)_{k\in I_{ij}})$ a *Zariski presentation* of X because it "presents" X as a coequalizer of affines
$$\coprod_{i,j\in I}\coprod_{k\in I_{ij}} U_{ij}^k \rightrightarrows \coprod_{i\in I} U_i \to X$$
Let us be more ambitious now and consider $f : X \to Y$ a morphism between schemes. By the definition of $\mathrm{QCoh}\,Y$, we must produce for each $y : \mathrm{Spec}\,A \to Y$ a module $y^*f_*\mathcal{F} \in \mathrm{QCoh}\,A$ equipped with quasi-coherent pullback when changing y. Given $y : \mathrm{Spec}\,A \to Y$, one can form the pullback square :

$$\begin{tikzcd} X_y \arrow[r, "\tilde{y}"] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ \mathrm{Spec}\,A \arrow[r, "y"'] & Y \end{tikzcd}$$

which suggests a way of forming $y^* f_* \mathcal{F}$ is by using $\tilde{f}_* \tilde{y}^* \mathcal{F}$ since we know how to deal with $\tilde{f}_*$ by the proposition we just proved. The main issue is the *quasi-coherence*.

1. With the abstract definition of $\tilde{f}_*$ as the right adjoint of $\tilde{f}^*$ it is hard to say anything because when we have a change in source $\operatorname{Spec} B \to \operatorname{Spec} A$, we need to $\_ \otimes_A B$ which is a left adjoint. Categorically we cannot say much about how left and right adjoints interact.

2. With the global sections definition, it is again not clear how this interacts with $\_ \otimes_A B$.

3. The definition with
$$\tilde{f}_* \tilde{y}^* \mathcal{F} \to \prod_{i \in I} (\tilde{y}^* \mathcal{F})(U_i) \rightrightarrows \prod_{i,j \in I} \prod_{k \in I_{ij}} (\tilde{y}^* \mathcal{F})(U_{ij}^k)$$
where $((U_i)_{i \in I}, (U_{ij}^k)_{k \in I_{ij}})$ is Zariski presentation of X_y has two orthogonal issues :
   (a) Requires choosing a Zariski presentation of X_y and thus will not be functorial in y.
   (b) $\_ \otimes_A B$ may not preserve equalisers.
   (c) $\_ \otimes_A B$ may not commute with the arbitrary products defining the equaliser.

(a) can be solved by taking filtered colimit over all Zariski presentations ordered by refinement. (b) can be solved by using the equivalence
$$\operatorname{QCoh} Y \simeq \operatorname{QCoh}_{\mathrm{Aff}} Y$$
because for $\operatorname{Spec} B \subseteq \operatorname{Spec} A$ an inclusion of affine opens of Y the algebra map $A \to B$ is flat. (c) is a serious issue because $\_ \otimes_A B$ *never* commutes with taking infinite products. For this to work, we need I to be finite and each I_{ij} also to be finite so that products are equivalent to direct sums. This leads us to the notion of *quasi-compact quasi-separatedness*.

> **Definition**
> Let X be a scheme and $f : X \to Y$ be a morphism of schemes. Then X is called a quasi-compact when it admits a *finite* Zariski affine atlas. Relatively, f is called quasi-compact when for all $y : \operatorname{Spec} A \to Y$, $f^{-1}(y)$ is qc.
>
> X is called quasi-separated when $\Delta : X \to X \times X$ is quasi-compact. Relatively, f is called quasi-separated when $\Delta_f : X \to X \times_Y X$ is qc.
>
> One often asummes quasi-compact and quasi-separated together, which we will abbreviate to qcqs.

The rough analogy is

M A-module	X scheme
M f.g.	X q.c.
M coherent	X q.c.q.s

Proposition
Let $f : X \to Y$ be a morphism of schemes. Then f is qcqs iff for all $y : S \to Y$ with S affine, the scheme X_y is qcqs.

Proof. (Forward) $T := X \times_Y S$ is qc by definition. To show qs, note that we have a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \times T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \times T \arrow[d, hook] \\ X \arrow[r] & X \times_Y X \arrow[r, hook] & X \times X \end{tikzcd}$$

where

1. The right square is cartesian because $X \times_Y X \to X \times X$ is a monomorphism of presheaves and $T \times T \to X \times T$ factors through $X \times_Y X$.
2. The outter square is cartesian because $T \to X$ is a monomorphism.
3. Hence the left square is cartesian.

Since $X \to X \times_Y X$ is qc, we have $T \to T \times T$ is qc as desired.

(Reverse) Again, it is qs that is non-trivial. Goal : for $S \to X \times_Y X$ with S affine, show $U := X \times_{X \times_Y X} S$ is qc. Given $S \to X \times_Y X$, we have single map $S \to Y$. Let $T := X \times_S Y$. Then we have

$$\begin{tikzcd} X \arrow[d] & T \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & U \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ X \times_Y X & T \times T \arrow[l] & S \arrow[l] \end{tikzcd}$$

where

1. the left square is cartesian as in the proof of the forward direction
2. the bottom row comes from $S \to X \times_S X$ factoring through $T \times T$
3. the outter square is cartesian by definition of U.

It follows that the right square is cartesian. By assumption, $T \to T \times T$ is qc so $U \to S$ is qc and hence U is qc. $\square$

Proposition – Base change for affine opens
Let $f : X \to Y$ be a qcqs morphism between schemes. Let $j : U \subseteq Y$ be an affine open and $\mathcal{F} \in \mathrm{QCoh}\, X$. Then

$$(f_* \mathcal{F})(U) \simeq \mathcal{F}(f^{-1}(U))$$

Proof. As we discussed. f qcqs implies that for each $y : S \to Y$ with S affine, the fiber X_y admits a *finite* Zariski presentation. Then the pushforward $X_y \to S$ can be computed by taking global sections w.r.t. any finite Zariski presentation of X. To make this functorial in y, we take the filtered colimit across all finite Zariski presentations ordered by refinement. Then the fact that the products in the equalizer defining global sections are finite and the fact that restricting along affine opens are flat implies that we have defined a quasi-coherent sheaf on Y. The fact that this is a right adjoint to f^* follows from our computation of the pushforward in the case of having an affine target. □

4 Properties of Morphisms

4.1 Vector bundles

4.2 Affine morphisms

4.3 Scheme theoretic image

4.4 Finite type, finitely presented morphisms

4.5 Universally closed, integral, finite morphisms

5 Varieties

5.1 Integral Schemes and Rational Functions

Proposition

Let X be a scheme. Then the following are equivalent :

1. X is nonempty and every nonempty affine open of X is an integral domain.
2. There exists an affine open cover $\mathcal{U}$ of X such that each $U \in \mathcal{U}$ is an integral domain and for $U, V \in \mathcal{U}$ we have $U \cap V \neq \varnothing$.
3. $|X|$ is irreducible and X is reduced.

Definition

A scheme X is called a *domain* when it satisfies any (and thus all) of the conditions in the previous proposition.

Proposition

Let $X = \operatorname{Spec} A$ where A is an integral domain. Consider $\varnothing \neq U \subseteq X$ an open. Then $A \to \mathcal{O}(U)$ is injective.

Thus in general, when X is a domain, restriction of local functions along affine opens is injective.

Proof. Let $f \in A$ and assume it is zero in $\mathcal{O}(U)$. Then there exists $g \in A$ with $\varnothing \neq D(g) \subseteq U$. By assumption, f must also be zero when restricted to $D(g)$, i.e. it is in the kernel of $A \to A[1/g]$. This implies the existence of $n \geq 0$ with $g^n f = 0$. The condition $\varnothing \neq D(g)$ implies $0 \neq g$. Hence by A being integral domain, $f = 0$. □

Definition
Let X be a domain. Define the function field of X to be

$$K(X) := \varinjlim_{\varnothing \neq U \text{ affine open } \subseteq X} \mathcal{O}(U)$$

Note that for a nonempty affine open U, the map $\mathcal{O}(U) \to K(X)$ is injective because all transition maps of the filtered diagram are injective. One can thus safely think about $K(X)$ as the "union" of $\mathcal{O}(U)$'s.

Proposition
Let X be a domain. Then $K(X)$ is a field.

Proof. Exercise. □

5.2 Algebraic curves

6 Smoothness

6.1 The start of calculus : Tangent vectors

Let $\operatorname{Spec} A$ be a physical system and a_0 a single state in it. Suppose a_0 actually lies in a line of states a,

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "a_0"] \arrow[d, "0"'] & \operatorname{Spec} A \\ \mathbb{A}^1 \arrow[ur, "a"'] & \end{tikzcd}$$

Then given a measuring device f on $\operatorname{Spec} A$, we get

$$f(a) = f_0 + f_1 t + f_2 t^2 + \cdots \in \mathcal{O}(\mathbb{A}^1)$$

where t is the standard meausring device on $\mathbb{A}^1$. Note that $f_0 = f(a_0)$. This says the way the value of f changes over the family of states a is described as a polynomical expression in the values of t.

The idea of a *tangent vector at the state a_0 in the direction of a* is we want only the *first order change of f along a*. With highschool intuition, we write

$$\left(\frac{\partial}{\partial a}\right)_{a_0} f = \frac{f(a) - f(a_0)}{t} \text{ ignore } t^2 \text{ and above}$$

But we see that this is actually formal! This is the coefficient f_1. This gives the first definition of a tangent vector.

Definition – Deformation perspective of tangent vectors
A tangent vector at a_0 is an extension

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "a_0"] \arrow[d, "0"'] & \operatorname{Spec} A \\ \operatorname{Spec} k[t]/(t^2) \arrow[ur, dashed, "a"'] & \end{tikzcd}$$

For such an extension, we will write

$$f(a) = f(a_0) + \left(\left(\frac{\partial}{\partial a} \right)_{a_0} f \right) t$$

More generally, a vector field along a map $a_0 : \operatorname{Spec} B \to \operatorname{Spec} A$ is an extension

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "a_0"] \arrow[d, "0"'] & \operatorname{Spec} A \\ \operatorname{Spec} B[t]/(t^2) \arrow[ur, dashed, "a"'] & \end{tikzcd}$$

Cons of this perspective :

- the k-vector space structure of tangent spaces is not clear. It is only clearly pointed. [1] At this point, this is too much work for something that should be trivial. This *is* however useful for defining tangent bundle of stacks because it is less obvious to generalise algebraic derivations.

Pros of this perspective :

- it is closest to highschool intuition : that of first order change.
- the definition of the tangent bundle of $\operatorname{Spec} A$ is intuitive. It is the space $T \operatorname{Spec} A$ such that

$$\mathrm{Aff}(\operatorname{Spec} \_, T \operatorname{Spec} A) \simeq \mathrm{Aff}(\operatorname{Spec}(\_[t]/(t^2)), \operatorname{Spec} A)$$

  This exists a priori in $\mathrm{PSh}\,\mathrm{Aff}$.
- it is clear that tangent vectors pushforward; this is a direct consequence of composing maps.
- the relative definition is also clear. Consider the example of $\pi : \mathbb{A}^2 \to \mathbb{A}^1, (x, y) \mapsto x$ and finding a tangent vectors in $\mathbb{A}^2$ relative to π at points. You will discover that the tangent vectors are forced to be

[1] Actually, a k-vector space structure is present, but it is non-trivial to check. The key is

$$\frac{k[x]}{(x^2)} \times_k \frac{k[y]}{(y^2)} \simeq \frac{k[x, y]}{(x, y)^2}$$

Then the k-vector space structure on (t) inside $k[t]/(t^2)$ makes $k[t]/(t^2)$ into a *"k-vector space object in k-algebras"*. [2] Making this into spaces, we want $\operatorname{Spec} \frac{k[x,y]}{(x,y)^2}$ to be the pushout of $\operatorname{Spec} k[x]/(x^2) \leftarrow \operatorname{Spec} k \to k[y]/(y^2)$ at least in the category of k-schemes. One of showing this is to see that any map into a k-scheme must factor through an affine open and thus reduces to the affine case where it is true. This endows the k-valued points of the tangent bundle with a k-vector space structure. To get this for B-valued points where B is any k-algebra, one must repeat the above for $B[t]/(t^2)$. The non-trivial part is showing $\operatorname{Spec} B[x, y]/(x, y)^2$ is the desired pushout in the category of schemes. This is [**stacks**].

vertical, i.e. lie within the fiber of π. This leads to the general definition : Given a map $X \to S$ of affine schemes, a vector field along a map $T \to X$ is a solution to the lifting problem :

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ T \times_k k[t]/(t)^2 \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

– Given a sequence of affine schemes $X \to Y \to S$, we get $T(X/S) \to T(Y/S)$ where $T(\_/S)$ denotes tangent bundle relative to S. There is also $T(X/Y)$ consisting of tangent vectors in X which lie in the fibers of the map $X \to Y$. Since fibers of $X \to Y$ lie in fibers of $X \to S$, we obtain

$$T(X/Y) \subseteq T(X/S) \to T(Y/S)$$

Looking at the fibers as pointed sets, one can see that $T(X/Y)$ is in the "kernel" of $T(X/S) \to T(Y/S)$. Conversely, any tangent vector of X relative S which dies under projection to Y must tautologically lie in $T(X/Y)$. So the above is "left exact", if only we are able to put these three spaces inside some abelian category. This is one of the pros of the next definition of tangent vectors.

One can see that the data of a is equivalent to specifying the linear operator $\left(\frac{\partial}{\partial a}\right)_{a_0} \in k\mathbf{Mod}(A, k)$. Following one's nose, we see that such linear operators ∂ satisfy the Leibniz rule.

$$\partial(fg) = \partial(f)g(a_0) + f(a_0)\partial(g)$$

These are called *k-derivations*. Let $\mathrm{Der}_k(A, k) \subseteq k\mathbf{Mod}(A, k)$ denote the subset of k-derivations. One can show $\mathrm{Der}_k(A, k)$ bijects with the set tangent vectors in our first definition, giving a second definition of tangent vectors.

> **Definition – Pragmatic perspective on tangent vectors**
> A tangent vector at state a_0 is a k-derivation from A to k_{a_0}.
>
> More generally, a vector field along $a_0 : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is a k-derivation $A \to B$ where B is an A-module via its A-algebra structure.

Pros of the this perspective :

– Computable! E.g. show that the tangent space of any point in $\mathbb{A}^n$ is an n-dimensional vector space.

– Makes obvious the k-vector space structure of tangent spaces. We can do even better. Given $a : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$,

$$\mathrm{Aff}_A(\mathrm{Spec}\, B, T\, \mathrm{Spec}\, A) \simeq \mathrm{Der}_k(A, B)$$

shows that local sections of the tangent bundle $T\, \mathrm{Spec}\, A$ have an A-module structure.

– Let $X \to Y \to S$ be maps of affine schemes and $R \to A \to B$ be the corresponding maps of algebras of functions. Then the sequence of spaces

$$T(X/Y) \subseteq T(X/S) \to T(Y/S)$$

corresponds to the sequence

$$0 \to \mathrm{Der}_A(B, \_) \stackrel{(1)}{\to} \mathrm{Der}_R(B, \_) \stackrel{(2)}{\to} \mathrm{Der}_R(A, \_)$$

1. "fibers of $X \to Y$ lie in fibers of $X \to S$ so relative tangents of X to Y are also relative tangents of X to S" is precisely the injection (1).
2. "relative tangents of $X \to Y$ die when pushforward to Y" is precisely the fact that at (2) the image lies in the kernel.
3. "relative tangents of $X \to S$ whose pushforward to Y is zero are relative tangents of $X \to Y$" precisely says that at (2) the kernel lies in the image.

So far, we have seen how to differentiate $f \in A$ with respect to some state $a \in \mathrm{Spec}\, A$ using a tangent vector. In other words, we have the inclusion of A-modules

$$\mathrm{Der}_k(A, \kappa(a)) \to k\mathbf{Mod}(A, k)$$

But the duality of space and function via evaluation gives us

$$\begin{aligned} A &\to k\mathbf{Mod}(\mathrm{Der}_k(A, \kappa(a)), k) \\ f &\mapsto \left(\left.\frac{\partial}{\partial v}\right|_a \mapsto \left.\frac{\partial}{\partial v}\right|_a f \right) =: (df)_a \end{aligned}$$

In the example of $A = k[x]$, one sees that

$$(df)_a(X) = X(f) = f'(a)X(x) = f'(a)(dx)_a(X)$$

in other words

$$(df)_a = f'(a)(dx)_a$$

> **Definition – Algebraic one forms as dual of vector fields**
> Let $R \to A$ be a map of algebras. Then a *module of differentials of A relative R* is an A-module $\Omega_{A/R}$ representing the functor on A-modules $\mathrm{Der}_R(A, \_)$. In other words, $\Omega_{A/R}$ is equipped with an isomorphism
>
> $$A\mathbf{Mod}(\Omega_{A/R}, \_) \simeq \mathrm{Der}_R(A, \_)$$

This definition can be seen as the algebraic formulation of the idea that one forms are dual of tangent vectors. This is somewhat unsatisfying because it would be nice to have a way of thinking about one forms independent of tangent vectors.

Given a point $a : A \to k$, we have the following interpretation.

$$(df)_a := f - f(a) \bmod I(a)^2$$

To get one forms with domains larger than a point, we need to generalise the above formula to $a : A \to B$. This is an issue because there is no canonical copy of B inside A for general B unlike $B = k$.

Let I_Δ be the kernel of $A \otimes A \to A, f \otimes g \mapsto fg$. Consider I_Δ as an A-module via the left component of A. If $f_0 \otimes g_0 + f_1 \otimes g_1$ is in I_Δ, then

$$f_0 g_0 + f_1 g_1 = 0$$

It follows that

$$f_0 \otimes g_0 + f_1 \otimes g_1 = f_0(1 \otimes g_0 - g_0 \otimes 1) + f_1(1 \otimes g_1 - g_1 \otimes 1)$$

This generalises to $f_0 g_0 + \cdots + f_n g_n$ in I_Δ and hows that I_Δ is generated as a left A-module by elements of the form

$$1 \otimes f - f \otimes 1$$

For the example of $A = k[t]$ and $A \otimes A = k[x, y]$, elements like this look like

$$f(y) - f(x)$$

This inspires the choice of notation

$$\Delta f := 1 \otimes f - f \otimes 1$$

which in turn inspires the definition

$$df := \Delta f \bmod I_\Delta^2$$

Let's check this is k-linear. For $\lambda \in k$ and $f \in A$, we have

$$d(\lambda f) = 1 \otimes \lambda f - \lambda f \otimes 1 = \lambda(1 \otimes f - f \otimes 1) = \lambda(df)$$

So k-linearity of d comes from the fact that k is allowed to pass between the two sides in $A \otimes A$.

> **Definition – Algebraic one forms as linear change**
> For a map of algebras $R \to A$, let I_Δ be the kernel of multiplication $A \otimes_R A \to A$. Then define
>
> $$\Omega_{A/R} := I_\Delta / I_\Delta^2$$
>
> This is a left A-module via $A \otimes 1 \subseteq A \otimes A$. Define
>
> $$d : A \to \Omega_{A/R}, f \mapsto (1 \otimes f - f \otimes 1) \bmod I_\Delta^2$$
>
> This map is only R-linear.

Our computation earlier shows that $\Omega_{A/R}$ is generated as an A-module by the image of d.

Relation between the two definitions? Something something universal square zero extension.

7 Group actions

7.1 Fundamental vector fields

> **Definition – Fundamental vector fields A.K.A. infinitesimal action**
> Let G be an algebraic group over k acting on a k-space X. Let $\mathfrak{g}$ denote the Lie algebra of G. Then given $\delta \in \mathfrak{g} = G(k[\varepsilon])$, we obtain a global vector field X_δ on X by
>
> $$k[\varepsilon] \times_k X \overset{(\delta, \mathrm{id})}{\to} G \times X \overset{\mathrm{act}}{\to} X$$

X_δ is called the *fundamental vector field associated to δ.*

We obtain a morphism of Lie algebras

$$\mathfrak{g} \to \Gamma(X, T_X)$$

In fact, for any affine open $U \subseteq X$, due to smoothness of U we have that fundamental vector fields on U must land inside U :

$$\begin{tikzcd} U \arrow[r, "\mathbb{1}"] \arrow[d] & U \arrow[d] \\ k[\epsilon] \times_k U \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

So we get for every affine open U,

$$\mathfrak{g} \to \Gamma(U, T_X)$$

7.2 Connections on G-torsors

Let's review the three definitions of connections on vector bundles in differential geometry. Let $\pi : E \to B$ be a rank m vector bundle over an n-dimensional manifold B.

Definition

The *vertical bundle of E* is defined as the kernel of vector bundles on E :

$$0 \to VE \to TE \xrightarrow{D\pi} TB \times_B E \to 0$$

Let's see what this sequence looks like locally on E. Locally on B, we have an isomorphism of vector bundles over spaces

$$\begin{tikzcd} E \arrow[r, "(x,a)", "\sim"'] \arrow[d] & \mathbb{R}^n \times \mathbb{R}^m \arrow[d] & (x,a) \arrow[d, mapsto] \\ B \arrow[r, "\sim", "x"'] & \mathbb{R}^n & x \end{tikzcd}$$

Then the above SES looks like at each $p \in E$ with $b = \pi(p)$

$$0 \to \left\langle \frac{\partial}{\partial a^j}_p \right\rangle \to \left\langle \frac{\partial}{\partial a^j}_p, \frac{\partial}{\partial x^k}_p \right\rangle \to \left\langle \frac{\partial}{\partial x^k}_b \right\rangle \to 0$$

Definition – Family of Horizontal subspaces

A family of horizontal subspace is a splitting of the vertical bundle short exact sequence.

From linear algebra, giving a splitting $0 \to V \to W \to W/V \to 0$ is the same as giving a retraction of $V \to W$. Using this, let's describe a choice of family of horizontal subspaces in terms of concrete data. A retraction of $0 \to \left\langle \frac{\partial}{\partial a^j}_p \right\rangle \to \left\langle \frac{\partial}{\partial a^j}_p, \frac{\partial}{\partial x^k}_p \right\rangle$ is equivalent to giving cotangent vectors $\theta^1_p, \ldots, \theta^m_p$ in $T^*_p E$ such that

$$\theta^i_p \frac{\partial}{\partial a^j}_p = \delta^i_j \frac{\partial}{\partial a^j}_p$$

This is equivalent to the condition

$$\theta^j = (da^j)_p + A^j_k(p)(dx^k)_p$$

> **Definition – Linear family of horizontal spaces**
> Let $r : TE \to VE$ be a family of horizontal subspaces. For $\lambda \in \mathbb{R}$ let $\lambda : E \to E$ be the fiberwise scaling by λ map. This in particular induces $d\lambda : TE \to TE$. Then r is called *linear* when for all $e \in E$ and scalars λ we have
>
> $$r_{\lambda e}(d\lambda)_e = r_e$$

For some reason, differential geometers like working with vector bundles by their total spaces whilst algebraic geometers like working with them through their quasi-coherent sheaf of sections.

Let $X = \operatorname{Spec} A$ where A is a commutative algebra over a field k. Let G be an algebraic group over k. We assume we already know what a principal G-bundle $\pi : P \to X$ means. Considering the sequence of spaces $P \to X \to \operatorname{Spec} k$, we obtain the exact sequence

$$0 \to T_{P/X} \to T_P \to \pi^* T_X$$

Let us compute $T_{P/X}$. For this, I found it easiest to consider it as a space over P and compute its points. Let $y : U \to P$ be a general point of P where U is an affine. Let $\tilde{U} := U \times_k \operatorname{Spec} k[\epsilon]$. Maps from y to $T_{P/X}$ are lifts $\tilde{y}$ satisfying

$$\begin{tikzcd}
U \arrow[r, "y"] \arrow[d, "\subseteq"'] & P \arrow[d, "\pi"] \\
\tilde{U} \arrow[r, "x=\pi(y)"'] \arrow[ur, dashed, "\tilde{y}"] & X
\end{tikzcd}$$

Note that $\tilde{U}$ has a retraction to U which gives rise to the zero tangent vector at y. By abuse of notation, we also use y to denote the zero tangent vector $\tilde{U} \to P$ at y. Giving a general relative tangent vector at y then amounts to a lift

$$\begin{tikzcd}
 & P \\
\tilde{U} \arrow[ur, "y"] \arrow[r, dashed, "(y,\tilde{y})"'] & P \times_X P \arrow[u, "\text{fst}"']
\end{tikzcd}$$

However by assumption $P \times_X P \simeq P \times G$, so one must have $\tilde{y} = gy$ for some unique $g \in G(\tilde{U}) = (\operatorname{Lie} G)(U)$. The above is all functorial in y and hence we deduce that as spaces over P we have

$$T_{P/X} \simeq P \times \operatorname{Lie} G$$

In fact, the reverse map is the map producing fundamental vector fields we previously saw.

Note that $P \to X$ is formally smooth. Writing everything as quasi-coherent sheaves on P we then have the SES

$$0 \to \mathcal{O}_P \otimes \operatorname{Lie} G \to T_P \to \pi^* T_X \to 0$$

We can descend this SES to X by modding out G. Formally, we have descent along G-bundles

$$(\operatorname{QCoh} P)^G \underset{\pi_*(\_^G)}{\overset{\pi^*}{\underset{\sim}{\leftrightarrows}}} \operatorname{QCoh} X$$

So we obtain the SES in $\operatorname{QCoh} X$,

$$0 \to \pi_*(\mathcal{O}_P \otimes \operatorname{Lie} G)^G \to \pi_*(T_P^G) \to T_X \to 0$$

A *connection on* P is a splitting of this SES in $\operatorname{QCoh} X$. Computationally, one should give such data on P. One way of doing this is to give a G-equivariant retraction of T_P to $\mathcal{O}_P \otimes \operatorname{Lie} G$. This amounts to giving a G-equivariant $(\operatorname{Lie} G)$-valued one form ω such that for any element $\delta \in \operatorname{Lie} G$,

$$\omega(X_\delta) = \delta$$

[Mic, Section 19.1]

Exercise (which I have not done) : boil this down for $G = \mathrm{GL}_n$.

Suppose one is given a connection on P as a section $\nabla : T_X \to \pi_*(T_P^G)$. The connection is called *flat* if this is a morphism of Lie algebroids.

Relation of D-modules and differential equations : Let $(\mathcal{E}, \nabla)$ be a vector bundle on a smooth curve X and ∇ a connection. Then on an open U of X on which $\mathcal{E}$ admits trivialising sections $e_1, \ldots, e_n$, we have for any section s on U,

$$\nabla s = \nabla(s^i e_i) = ds^i \otimes e_i + s^i \nabla e_i = ds^i \otimes e_i + s^i A_i^j \otimes e_j = ds + As$$

Therefore looking for horizontal sections $\nabla s = 0$ is equivalent to solving the n 1-dimensional ordinary differential equation $ds = -As$ and then pasting solutions together to all of X.

8 References

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