# Functor of points done right

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## Contents

This documents my personal journey of understanding the foundations of modern algebraic geometry completely from the functorial point of view. At the risk of sounding controversial, I believe this is the way involving *least* amount of technology, however requires developing the most intuition.

Problems with courses in modern algebraic geometry I've seen :

1. don't get to schemes, leaving students unable to access modern AG
2. amount of theory of sheaves on topological spaces is demotivating. I want sheaves to appear as requiring coverings to be effective epimorphisms.
3. With locally ringed spaces, it feels difficult to write down a scheme. New students often mistakenly conflate a scheme with its underlying topological space. The point that "schemes = commutative algebra + descent" is clouded.
4. takes too long to get to modern treatment of varieties
5. With locally ringed spaces, quasi-coherent sheaves leave in a larger category of topological sheaves. This obscures the idea that "quasi-coherent sheaves = a module in any chart compatibly".
6. too many properties of morphisms. Hard for students to get a handle on.
7. sheaf cohomology : is there really a need to see abstract sheaf cohomology when seeing schemes for first time?

When corrected motivated, the functor of points approach solves (1), (2), (3), (4), (5). Point (7) is debatable. Point (6) may be better if there was a specific purpose for learning scheme theory? E.g. if one focuses on curves?

Places where functors shine :

1. definition of tangent bundle
2. formal schemes and ind-schemes are easy to introduce and compute with
3. the generalization to algebraic spaces is conceptually explained as replacing quotients along open immersions to quotients along étale morphisms
4. the generalization to stacks is conceptually explained as using groupoids to describe "non-propositional equivalence relations".

Some questions for myself :

1. Maths where schemes are indispensible?

– Modular forms to Galois reps. Modular curves are usually described complex analytically. But the moduli description makes them over $\mathbb{Q}$. Now étale cohomologies naturally carry action from absolute Galois group.

# 1 Affine schemes

## 1.1 Affine space

Slogan : *Commutative algebras as mathematical formalisation of physical systems*

The above idea is from [Nes21].

*Numbers are elements of a field*. We fix a field $k$. For intuition, $k = \mathbb{R}$. Unless stated otherwise, all algebras are algebras over $k$ and commutative unital.

A *physical system* nothing more than what we can *observe*. Observations are made with *measuring devices*. Measuring devices give for each *state* a number. Given that numbers are elements of a field $k$, one sees that measuring devices can be added, subtracted, scaled, multiplied, but **not** divided. We thus arrive at the concept of a commutative $k$-algebra.

We think of a commutative $k$-algebra $A$ as defining a physical system by the measuring devices we can use on it. When we want to think to $A$ as a physical system, we refer to it as $\operatorname{Spec} A$. This is nice because this is one idea away from schemes: schemes add the idea of allowing measuring devices which can only give values on certain states, i.e. *local* measuring devices. One also adds the idea that local measuring devices can be *patched together* along states which they share equal values on.

A single state in $\operatorname{Spec} A$ is defined as a morphism of $k$-algebras $A \to k$. Intuitively, the role of a single state is give for each measuring device a number.

Examples of algebras and their physical interpretation : $k[x_1, \ldots, x_n]$ represents affine $n$-space, $k$ represents the system where there is only one state. If one thinks of all states as positions, then it makes sense to confuse the word "physical system" with the word "space". This is intuitively why one might call $\operatorname{Spec} A$ a space.

A *map of spaces* $\operatorname{Spec} B \to \operatorname{Spec} A$ should be determined by how measuring devices on $\operatorname{Spec} A$ give rise to measuring devices on $\operatorname{Spec} B$, i.e. a morphism of $k$-algebras $A \to B$. We take this as the definition.

One can also think of a map $\operatorname{Spec} B \to \operatorname{Spec} A$ as a family of states in $\operatorname{Spec} A$ parameterised by $\operatorname{Spec} B$. E.g. $\mathbb{A}^1 \to \operatorname{Spec} A$ is a line of states in $\operatorname{Spec} A$.

### 1.2 Closed embeddings

### 1.3 Standard opens

### 1.4 Fiber product

### 1.5 Finite disjoint union

### 1.6 Standard Zariski coverings

# 2 Non-affine spaces

## 2.1 Motivation of non-affine spaces

Let us work over a field $k$. There are two examples of spaces we want which are non-affine.

1. $\mathbb{A}^2 \setminus \{0\}$, the complement of the closed immersion $\operatorname{Spec} k[x,y]/(x,y) \to \mathbb{A}^2$.
2. Projective space $\mathbb{P}^1$. This should be obtained as the coequalizer

$$\mathbb{G}_m \rightrightarrows \mathbb{A}^1 + \mathbb{A}^1 \to \mathbb{P}^1$$

where we have $\mathbb{G}_m \to \mathbb{A}^1$ via $t \mapsto t$ and $t \mapsto 1/t$.

Let us discuss (1) first. We will discuss (2) in the next section.

**Definition – Tentative definition of open complement**
Let $i : \operatorname{Spec} B \to \operatorname{Spec} A$ be a closed immersion. Then we say a map $j : \operatorname{Spec} C \to \operatorname{Spec} A$ is an affine open complement of of $i$ when $j$ is universal amongst $j_1 : \operatorname{Spec} C_1 \to \operatorname{Spec} A$ such that $B \otimes_A C_1 = 0$.

It is clear that if an affine open complement exists, it is unique up to isomorphism. What is not clear is that affine open complements exist. In fact, it is not true.

**Proposition**
The closed immersion $i : \{0\} \to \mathbb{A}^2$ has no affine open complement.

*Proof.* Suppose $j : \operatorname{Spec} C \to \mathbb{A}^2$ is an affine open complement of $i$. The idea is that $\operatorname{Spec} C$ should be covered by $D(x), D(y)$. Given $(f,g) : \operatorname{Spec} R \to \mathbb{A}^2$, we have

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[d, "{(f,g)}"'] & D(f) + D(g) \arrow[l] \arrow[d] & D(fg) \arrow[l, shift left] \arrow[l, shift right] \arrow[d] \\
\mathbb{A}^2 & D(x) + D(y) \arrow[l] & D(xy) \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

where $D(f) \simeq \operatorname{Spec} R \times_{\mathbb{A}^2} D(x)$ because $R[1/f] \simeq R \otimes_{k[x,y]} k[x,y,1/x]$ and similarly with $g$. It follows that $(f,g)$ factors through $j$ iff $(1) = (f,g) \subseteq R$ iff the induced map

$$R \to R[1/f] \times_{R[1/fg]} R[1/g]$$

is an isomorphism.

Now note the fact that the inclusions $D(x), D(y) \to \mathbb{A}^2$ factor through $j$. This makes $D(x)$ the fiber product $\operatorname{Spec} C \times_{\mathbb{A}^2} D(x)$ and similarly for $D(y)$. By the above, we now have

$$\begin{tikzcd} C \arrow[r, "\sim"] & k[x,y,1/x] \times_{k[x,y,1/x,1/y]} k[x,y,1/y] \\ k[x,y] \arrow[u, "j"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

from which it follows that $j$ is an isomorphism. But this is a contradiction since

$$\begin{tikzcd} (\text{Spec } C)k \arrow[d, "j"', "\sim"] & k^2 \setminus \{0\} \arrow[l, "\sim"'] \arrow[d, "\subseteq"', "\nsim"] \\ \mathbb{A}^2(k) & k^2 \arrow[l, "\sim"] \end{tikzcd}$$

□

This shows that *open immersions* in general do not have an affine source even if the target is affine.

In the above example, we can actually see a possible definition of $\mathbb{A}^2 \setminus \{0\}$. Namely, we can define it as the *subfunctor of* $\underline{\mathrm{Hom}}(k[x,y], \_)$ *such that* $x, y$ *generate the unit ideal*. More generally,

**Definition – Open immersions with affine target**
Let $I \subseteq A$ be an ideal. Then the open complement of $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is defined as the subfunctor of the Yoneda embedding of $\operatorname{Spec} A$ consisting of maps $\operatorname{Spec} B \to \operatorname{Spec} A$ such that $B/IB = 0$.

A map $\operatorname{Spec} B \to \operatorname{Spec} A$ is called an open immersion when it is the open complement of a closed immersion into $\operatorname{Spec} A$.

The above discussion suggests that in general we can define candidate spaces first in $\mathrm{PSh}\,\mathrm{Aff}$ then ask is it representable by an affine. $\mathrm{PSh}\,\mathrm{Aff}$ has many desireable properties.

**Proposition**
The following are true :

1. (Yoneda lemma) we have a fully faithful functor $\mathrm{Aff} \to \mathrm{PSh}\,\mathrm{Aff}$ given by the Yoneda embedding so we can WLOG we view $\mathrm{Aff}$ as a full subcategory of $\mathrm{PSh}\,\mathrm{Aff}$.
2. $\mathrm{PSh}\,\mathrm{Aff}$ has all limits and it is computed "pointwise" i.e. for a diagram $X : I \to \mathrm{PSh}\,\mathrm{Aff}$ we have functorially in $S \in \mathrm{Aff}$
$$(\varprojlim_{i \in I} X_i)(S) \xrightarrow{\sim} \varprojlim_{i \in I} X_i(S)$$
The same is true for colimits.

3. (Density theorem / "function extensionality for presheaves") Let $X \in \mathrm{PSh}\,\mathrm{Aff}$. Consider the diagram $\mathrm{Aff}/X \to \mathrm{PSh}\,\mathrm{Aff}$ of points of $X$. Then $\varinjlim_{\mathrm{Aff}/X} \mathrm{Spec}\, A \xrightarrow{\sim} X$.

   We call objects of $\mathrm{Aff}/X$ points of $X$. The above says "$X$ is the colimit of its points".
4. All epimorphisms are effective epimorphisms, i.e. "a surjection is the quotient of its source by the equivalence relation of being in same fibers".
5. Limits and colimits of subobjects are computed pointwise by intersection and union.
6. The Yoneda embedding preserves limits. However it does *not* preserve colimits.
7. $\mathrm{PSh}\,\mathrm{Aff}$ is cartesian closed i.e. we can consider "mapping spaces".

None of the above are hard to prove, but let us say something about the Yoneda embedding not preserving colimits. At first, this may appear to be a bad thing however this is in fact desireable. Indeed, throughout the proof that $\mathbb{A}^2 \setminus \{0\}$ cannot be affine, we proved that we have an coequalizer diagram

$$D(xy) \rightrightarrows D(x) + D(y) \to \mathbb{A}^2$$

inside the category $\mathrm{Aff}$. This is bad because this coequalizer should be $\mathbb{A}^2 \setminus \{0\}$ and we do not want the inclusion $\mathbb{A}^2 \setminus \{0\} \subseteq \mathbb{A}^2$ to be an isomorphism!

Let us investigate further. Note that the inclusions $D(x), D(y) \subseteq \mathrm{Spec}\, A$ factor through $\mathbb{A}^2 \setminus \{0\}$ so it makes sense to ask whether the following diagram in $\mathrm{PSh}\,\mathrm{Aff}$ is a coequalizer diagram

$$D(xy) \rightrightarrows D(x) + D(y) \to \mathbb{A}^2 \setminus \{0\}$$

**Proposition**

Let $U$ denote the coequalizer of $D(xy) \rightrightarrows D(x) + D(y)$ inside $\mathrm{PSh}\,\mathrm{Aff}$. This is a subobject of $\mathbb{A}^2$ but it is *not* isomorphic to $\mathbb{A}^2 \setminus \{0\}$ as subobjects of $\mathbb{A}^2$.

*Proof.* Because $D(xy), D(x), D(y)$ are all subfunctors of $\mathbb{A}^2$, the coequalizer is simply the union. In other words, for $\mathrm{Spec}\, A \in \mathrm{Aff}$

$$U(A) = \left\{(f, g) \in A^2 \mid f \in A^\times \text{ or } g \in A^\times\right\}$$

Since $D(x), D(y) \subseteq \mathbb{A}^2 \setminus \{0\}$ we have an induced inclusion $U \subseteq \mathbb{A}^2 \setminus \{0\}$. However this is not an equality. Indeed for $A = k[t], f = t, g = 1 - t$ we have that neither $t, 1 - t$ are units but $(1) = (t, 1 - t)$. $\square$

One way of understanding the above phenomenon is that given $(f, g) : \mathrm{Spec}\, A \to \mathbb{A}^2 \setminus \{0\}$ we have a basic Zariski cover

$$D(fg) \rightrightarrows D(f) + D(g) \to \mathrm{Spec}\, A$$

which is a coequalizer diagram in $\mathrm{Aff}$, however $U$ fails to recognise $\mathrm{Spec}\, A$ as the coequalizer. Indeed, with the example in the proof, we have a map $D(f) + D(g) \to U$ which agrees on the intersection $D(fg)$ but fails to descend to a map $\mathrm{Spec}\, A \to U$. More precisely, let $S_{\{f,g\}}$ be the coequalizer of $D(fg) \rightrightarrows D(f) + D(g)$ inside $\mathrm{PSh}\,\mathrm{Aff}$. This induces a map

$$S_{\{f,g\}} \to \mathrm{Spec}\, A$$

and $U$ fails to recognise this as an isomorphism. It turns out, trying to force $U$ to see this map as an isomorphism produces $\mathbb{A}^2$. Making this idea rigorous leads us to the notion of a *Zariski sheaf*.

We first abstract the above example.

**Proposition – Zariski sieves**

Let $X = \operatorname{Spec} A \in \operatorname{PSh}\operatorname{Aff}$ and let $I \subseteq A$ be finite with $(I) = A$. Define the Zariski sieve associated $I$ as the subfunctor $S_I \subseteq \operatorname{Spec} A$ of points $x : \operatorname{Spec} B \to \operatorname{Spec} A$ such that there exists a factoring for some $f \in I$

$$\operatorname{Spec} B \to \operatorname{Spec} A[1/f] \to \operatorname{Spec} A$$

Then we have a coequalizer diagram of presheaves

$$\mathcal{U}_I \times_X \mathcal{U}_I \rightrightarrows \mathcal{U}_I \to S_I$$

where $\mathcal{U}_I := \coprod_{f \in I} \operatorname{Spec} A[1/f]$.

*Proof.* Colimits of presheaves are computed fiberwise. We are thus reduced to showing

$$\mathcal{U}_I(B) \times_{X(B)} \mathcal{U}_I(B) \rightrightarrows \mathcal{U}_I(B) \to S_I(B)$$

is a coequalizer diagram for each $\operatorname{Spec} B \in \operatorname{Aff}$. This follows because $S_I(B)$ is precisely the image of $U_I(B)$ and for sets, the image of a map is the coequalizer of its kernel pair. $\square$

**Proposition – Zariski sheaves**

Define $\operatorname{Sh}_{\text{Zar}}\operatorname{Aff}$ to be the full subcategory of $\operatorname{PSh}\operatorname{Aff}$ of presheaves $X$ such that for any Zariski sieve inclusion $S_I \to \operatorname{Spec} A$ we have an isomorphism

$$(\operatorname{PSh}\operatorname{Aff})(\operatorname{Spec} A, Y) \xrightarrow{\sim} (\operatorname{PSh}\operatorname{Aff})(S_I, Y)$$

Such presheaves are called *Zariski sheaves*.

Then have an adjunction

$$\begin{tikzcd} \operatorname{Sh}_{\text{Zar}}\operatorname{Aff} \arrow[r, phantom, "\perp"] \arrow[r, shift right, "\subseteq"'] & \operatorname{PSh}\operatorname{Aff} \arrow[l, shift right, "L"'] \end{tikzcd}$$

such that

- (Universal property) the left adjoint $L$ realises the category of Zariski sheaves as the localisation of the category of presheaves by inclusions of Zariski sieves.
- the left adjoint $L$ is left exact.

The functor $L$ is called (Zariski) sheafification. Note that the lemma of Zariski covers shows that all affine schemes are Zariski sheaves.

To complete the example of $\mathbb{A}^2 \setminus \{0\}$, we have

**Proposition**
The inclusion $j : U \subseteq \mathbb{A}^2 \setminus \{0\}$ exhibits the latter as the Zariski sheafification of $U$.

*Proof.* Let $Y \in \mathrm{PSh}\,\mathrm{Aff}$. We need to show that $j$ induces

$$\mathrm{PSh}\,\mathrm{Aff}(\mathbb{A}^2 \setminus \{0\}, Y) \xrightarrow{\sim} \mathrm{PSh}\,\mathrm{Aff}(U, Y)$$

(injectivity) Let $\varphi, \psi : \mathbb{A}^2 \setminus \{0\} \to Y$ such that $\varphi|_U = \psi|_\psi$. To show $\varphi = \psi$, the uniqueness part of the density theorem says it suffices to show that for all $(f, g) : \mathrm{Spec}\, A \to \mathbb{A}^2 \setminus \{0\}$ we have $\varphi(f, g) = \psi(f, g)$. We have the commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}\, A \arrow[d, "\varphi(f,g)"'] & D(f) + D(g) \arrow[l] \arrow[d] & D(fg) \arrow[l, shift left] \arrow[l, shift right] \\
Y & U \arrow[l] &
\end{tikzcd}$$

Since $Y$ is a Zariski sheaf, it sees $\mathrm{Spec}\, A$ as the coequalizer of $D(fg) \rightrightarrows D(f) + D(g)$. In particular, $\varphi(f, g)$ is determined by its restrictions to $D(f), D(g)$. But $\varphi(f, g)|_{D(f)} = (\varphi(f, g)|_U)|_{D(f)}$ and similarly for $D(g)$. The same goes for $\psi(f, g)$ and thus $\varphi = \psi$.

(surjectivity) Assume $\psi : U \to Y$ with the goal of showing this extends to a map $\mathbb{A}^2 \setminus \{0\} \to Y$. Again by the existence part of the density theorem, the data of the such a map is equivalent to the following

1. for each point $(f, g) : \mathrm{Spec}\, A \to \mathbb{A}^2 \setminus \{0\}$ we give a point $\tilde{\psi}(f, g) : \mathrm{Spec}\, A \to Y$
2. for each $\varphi : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ we have

$$\tilde{\psi}(f, g)\varphi = \tilde{\psi}(f\varphi, g\varphi)$$

3. When $(f, g)$ lies in $U$, we want $\tilde{\psi}(f, g) = \psi(f, g)$.

Let $(f, g)$ be an $A$-point of $\mathbb{A}^2 \setminus \{0\}$. Again since $Y$ is a Zariski sheaf, it recognises $\mathrm{Spec}\, A$ as the coequalizer of $D(fg) \rightrightarrows D(x) + D(y)$. So to define $\tilde{\psi}(f, g)$ is equivalent to defining an element of $Y(A[1/f]) \times_{Y(A[1/fg])} Y(A[1/g])$. Again note that $D(f), D(g) \to \mathbb{A}^2 \setminus \{0\}$ factor through $U$ and agree on $D(fg)$. So we use $(f|_{D(f)}, g|_{D(g)})$ as our element. We leave (2) and (3) as an exercise. □

The category of Zariski sheaves is designed to force Zariski covers to be effective epimorphisms. We have the following useful criterion for checking Zariski epimorphisms between general sheaves.

**Proposition**
Let $p : X \to Y$ be an map in $\mathrm{Sh}_{\mathrm{Zar}}\,\mathrm{Aff}$. Then $p$ is an epimorphism iff it is an effective epimorphism iff "it locally has sections" i.e. for all points $y : \mathrm{Spec}\, A \to Y$ there exists a basic Zariski cover $\mathcal{U}$ of $\mathrm{Spec}\, A$ such that there exists a commutative diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, dashed, "\exists x"] \arrow[d] & X \arrow[d, "p"] \\ \operatorname{Spec} A \arrow[r, "y"'] & Y \end{tikzcd}$$

*Proof.* Let $\operatorname{Im} p$ denote the subfunctor of $Y$ consisting of points $y \in Y(A)$ which admit local lifts across $p$. Whether $p$ is an epimorphism or not is about maps out of $Y$. Therefore we need a way of detecting whether $\operatorname{Im} p = Y$ using a map out of $Y$. The idea is to use indicator functions A.K.A. the existence of a subobject classifier. We omit the proof since it requires no geometric intuition.

□

## 2.2 Schemes

Slogan : *a scheme is the quotient of affines schemes along open immersions*. For this, we need to define open immersions between Zariski sheaves.

**Definition**

Let $j : X \to Y$ be a map of Zariski sheaves. Then $j$ is called an open immersion when for all points $y : \operatorname{Spec} A \to Y$, the pullback $\operatorname{Spec} A \times_Y X \to \operatorname{Spec} A$ is an open immersion.

We can now define schemes.

**Definition**

Let $X$ be a Zariski sheaf. An affine Zariski cover / atlas of $X$ is defined as a set of open immersions $\{j_i : \operatorname{Spec} A_i \to X\}_{i \in I}$ such that the Zariski disjoint union gives a Zariski epimorphism

$$\coprod_{i \in I} \operatorname{Spec} A_i \twoheadrightarrow X$$

A scheme $X$ is a Zariski sheaf that admits an affine Zariski atlas.

The following gives another way of thinking about schemes.

**Proposition**

Let $X$ be a Zariski sheaf. Then the following are equivalent :

1. $X$ has an affine Zariski atlas.
2. There exists a coproduct of affine schemes $\mathcal{U}$ as Zariski sheaves and an equivalence relation $\mathcal{R}$ on $\mathcal{U}$ such that
   - $\mathcal{R}$ is a coproduct of affine schemes as Zariski sheaves
   - the projections $\mathcal{R} \rightrightarrows \mathcal{U}$ are open embeddings
   - $X$ is the Zariski coequalizer of $\mathcal{R} \rightrightarrows \mathcal{U}$. (Equivalently the Zariski sheafification of the presheaf coequalizer.)

The proof requires some basic knowledge of the category of Zariski sheaves $\mathrm{Sh}_{\mathrm{Zar}}\,\mathrm{Aff} \subseteq \mathrm{PSh}\,\mathrm{Aff}$.

*Proof.* (1) implies (2) is clear.

Let $\mathcal{U}, \mathcal{R}$ be as in the situation (2). It suffices to show for each affine scheme summand $U$ of $\mathcal{U}$ the map $j_U : U \to \mathcal{U} \to X$ is an open embedding. Let $\tilde{X}$ be the coequaliser of $\mathcal{R} \rightrightarrows \mathcal{U}$ in the category of presheaves. Note that the Zariski sheafification of the coequaliser sequence for $\tilde{X}$ is the coequaliser sequence for $X$ because Zariski sheafification is a left exact left adjoint. We need the following.

> *Lemma. For any morphism of presheaves $f : X \to Y$, if $f$ is an open embedding then so is its Zariski sheafification.*
>
> *Proof.* Suppose we have $S \to L(Y)$ with $S$ affine. We need to show $S \times_{L(Y)} L(X) \to S$ is an open embedding. It suffices to do so after passing to some Zariski cover $\tilde{S} \to S$. One of the properties of Zariski sheafification is that we can find a Zariski cover $\tilde{S} \to S$ such that $\tilde{S} \to S \to L(Y)$ factors as $\tilde{S} \to Y \to L(Y)$. The rest follows from the fact that Zariski sheafification $L$ is left exact.
>
> $$\begin{tikzcd} L(\tilde{S} \times_Y X) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\simeq"] & \tilde{S} \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \arrow[dr] & \\ \tilde{S} \times_{L(Y)} L(X) & \tilde{S} \arrow[r, dashed] \arrow[dr, dashed] & Y \arrow[dr] & L(X) \arrow[d] \\ & & S \arrow[r] & L(Y) \end{tikzcd}$$
>
> ■

So it suffices to show $U \to \tilde{X}$ is an open embedding. Let $T \to \tilde{X}$ be a general point where $T$ is affine. By the way colimits of presheaves are computed, we can lift $T \to \tilde{X}$ to a map $T \to \mathcal{U}$. We then have the cartesian squares :

$$\begin{tikzcd} T \times_{\tilde{X}} U \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathcal{U} \times_{\tilde{X}} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\ & \mathcal{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \arrow[d] \\ T \arrow[r] & \mathcal{U} \arrow[r] & \tilde{X} \end{tikzcd}$$

For the left vertical morphism to be an open embedding, it suffices that the middle two vertical morphisms are open embeddings. By assumption, $\mathcal{R} \to \mathcal{U}$ is an open embedding. It remains to prove that the inclusion $U \to \mathcal{U}$ is an open embedding. Using the lemma above again, it suffices to do so for the coproduct as presheaves, where the result is then clear.

□

*Example.*
*Our example $\mathbb{A}^2 \setminus \{0\}$ which motivated Zariski sheaves is a scheme with atlas given by $D(x) + D(y)$.*

*Example.*
*Projective 1-space $\mathbb{P}^1$ is defined by the equivalence relation*

$$\mathbb{A}^1 \setminus 0 \underset{1/t}{\overset{t}{\rightrightarrows}} \mathbb{A}^1 \coprod \mathbb{A}^1 \longrightarrow \mathbb{P}^1$$

### 2.3 Examples of describing things by functors

## 3 Properties of Morphisms

### 3.1 Closed embeddings

### 3.2 Vector Bundles

### 3.3 Affine morphisms

## 4 Varieties

### 4.1 Dimension Theory

### 4.2 Finite Type Morphisms

### 4.3 Integral Schemes and Rational Functions

## 5 Quasi-coherent sheaves

### 5.1 Examples of modules as general vector bundles

### 5.2 Quasi-coherent sheaves on affine schemes

### 5.3 Quasi-coherent sheaves on a scheme

*First do descent of bundles of sets along surjections $\mathcal{U} \to X$ of sets.*

In practice, all schemes will have affine diagonal, i.e. the intersection of two affine opens will always be affine again.

**Definition – Quasi-coherent sheaves w.r.t. Zariski affine cover**
Let $X$ be a scheme with affine diagonal and $\mathcal{U}$ a Zariski affine cover. Consider the following diagram :

$$\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U} \begin{array}{c} \xrightarrow{p_{01}} \\ \xrightarrow{p_{12}} \\ \xrightarrow{p_{02}} \end{array} \mathcal{U} \times_X \mathcal{U} \begin{array}{c} \xrightarrow{p_0} \\ \xrightarrow{p_1} \end{array} \mathcal{U}$$

where we have the projections

- $p_i : \mathcal{U} \times_X \mathcal{U} \to \mathcal{U}, (x_0, x_1) \mapsto x_i$
- $p_{ij} : \mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U} \to \mathcal{U} \times_X \mathcal{U}, (x_0, x_1, x_2) \mapsto (x_i, x_j)$

By our assumption on $X$, all schemes are disjoint unions of affine opens of $X$. We already know what quasi-coherent sheaves look like on $\mathcal{U}, \mathcal{U} \times_X \mathcal{U}, \mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}$. For example, $\mathrm{QCoh}\,\mathcal{U} \simeq \prod_{U \in \mathcal{U}} \mathcal{O}(U)\mathbf{Mod}$. Note that pullback along these projections correspond to restricting to double and triple intersections.

Then define the *category* $\mathrm{QCoh}(X, \mathcal{U})$ *of quasi-coherent sheaves on* $X$ *w.r.t.* $\mathcal{U}$ as follows :

- An object of $\mathrm{QCoh}(X, \mathcal{U})$ is a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}\,\mathcal{U}$ equipped with a *transition map* $\phi : p_0^* \mathcal{F} \cong p_1^* \mathcal{F}$ in $\mathrm{QCoh}(\mathcal{U} \times_X \mathcal{U})$ such that
  1. $\phi = \mathbb{1}$ when restricted to the diagonal $\Delta : \mathcal{U} \to \mathcal{U} \times_X \mathcal{U}$
  2. the following diagram commute :

$$\begin{tikzcd}
 & p_{01}^* p_1^* \mathcal{F} \arrow[dr, "\sim"] & \\
p_{01}^* p_0^* \mathcal{F} \arrow[ur, "p_{01}^*(\phi)"] \arrow[d, "\sim"'] & & p_{12}^* p_0^* \mathcal{F} \arrow[d, "p_{12}^*(\phi)"] \\
p_{02}^* p_0^* \mathcal{F} \arrow[dr, "p_{02}^*(\phi)"'] & & p_{12}^* p_1^* \mathcal{F} \arrow[dl, "\sim"] \\
 & p_{02}^* p_1^* \mathcal{F} &
\end{tikzcd}$$

- a morphism $\eta : (\mathcal{F}, \phi) \to (\mathcal{G}, \psi)$ is a morphism $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{QCoh}\,\mathcal{U}$ such that the following commutes : [a]

$$\begin{tikzcd}
p_0^* \mathcal{F} \arrow[r, "\phi"] \arrow[d, "p_0^*(\eta)"'] & p_1^* \mathcal{F} \arrow[d, "p_1^*(\eta)"] \\
p_0^* \mathcal{G} \arrow[r, "\psi"'] & p_1^* \mathcal{G}
\end{tikzcd}$$

[a] This may remind you of the definition of equivariant maps between set endowed with an action of a group $G$. This is not a coincidence : $\mathrm{QCoh}(X, \mathcal{U})$ is effectively quasi-coherent sheaves on $\mathcal{U}$ equipped with an action of the *groupoid* $\mathcal{U} \times_X \mathcal{U} \rightrightarrows \mathcal{U}$.

The above definition is how one goes about giving examples of quasi-coherent sheaves on non-affine schemes in practice. For theoretical purposes, we need to have a definition that is independent of the choice of Zariski affine cover and then prove that it is equivalent to the above.

**Definition – Quasi-coherent sheaves on presheaves**

> **Proposition**
> Let $X$ be a scheme with affine diagonal and $\mathcal{U}$ a Zariski affine cover. Then the restriction functor
> $$\mathrm{QCoh}\, X \to \mathrm{QCoh}(X, \mathcal{U})$$
> is an equivalence of categories.

*Proof.* Let $R$ denote the restriction functor. We are doing an example of fpqc descent.

(Faithful) For faithfullness, reduce to requiring for $\mathcal{F} \in \mathrm{QCoh}\, X$, $R\mathcal{F} = 0$ implies $\mathcal{F} = 0$. Comes down to $\mathcal{U}$ Zariski cover of $X$. This is conservativity part of fpqc descent.

(Fullness) Let $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}\, X$ and $\phi : R\mathcal{F} \to R\mathcal{G}$. We seek to construct $\mathcal{F} \to \mathcal{G}$ which restricts to $\phi$. Let $x : \mathrm{Spec}\, A \to X$. Goal : construct $\mathcal{F}_x \to \mathcal{G}_x$ in $A\mathbf{Mod}$. Idea : $\mathcal{U}$ Zariski affine cover of X and X affine diagonal gives Zariski affine cover of $\mathrm{Spec}\, A$. We are now in fpqc descent of affine schemes. For each $U \in \mathcal{U}$, denote $U_A := \mathrm{Spec}\, A \times_X U$ and the pullback of $\mathcal{F}$ to $U_A$ as $\mathcal{F}(U_A)$. The point of fpqc descent is that "$\mathcal{F}_x$ is equivalent to gluing data on a cover of $\mathrm{Spec}\, A$". More concretely, $\mathcal{F}_x$ can be realised by the equalizer diagram :

$$0 \to \mathcal{F}_x \to \prod_{U \in \mathcal{U}} \mathcal{F}(U_A) \rightrightarrows \prod_{U, V \in \mathcal{U}} \mathcal{F}(U_A \cap V_A)$$

The data of $\phi$ then gives a morphism $\mathcal{F}_x \to \mathcal{G}_x$ functorially in $x$. One can check that this does nothing when $\mathrm{Spec}\, A$ is one of the affine opens in the cover $\mathcal{U}$ as desired.

(Essential Surjectivity) Take $\mathcal{F} \in \mathrm{QCoh}(X, \mathcal{U})$. Goal : construct $\mathcal{F}_x$ for every $x : \mathrm{Spec}\, A \to X$ functorially and quasi-coherent.

Sub-goal : fix $x : \mathrm{Spec}\, A \to X$. Construct $\mathcal{F}_x \in A\mathbf{Mod}$ first. This time, we *define* $\mathcal{F}_x$ by the equalizer diagram in $A\mathbf{Mod}$

$$0 \to \mathcal{F}_x \to \prod_{U \in \mathcal{U}} \mathcal{F}(U_A) \rightrightarrows \prod_{U, V \in \mathcal{U}} \mathcal{F}(U_A \cap V_A)$$

Sub-goal : now functoriality. We are given

$$\mathrm{Spec}\, B \xrightarrow{f} \mathrm{Spec}\, A \xrightarrow{x} X$$

We need to give a morphism $\mathcal{F}_x \to \mathcal{F}_{xf}$ in $A\mathbf{Mod}$. The forgetful functor $B\mathbf{Mod} \to A\mathbf{Mod}$ preserves limits, so the UP of $\mathcal{F}_{xf}$ induces the dashed morphism in the commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{xf} \arrow[r] & \prod_{U \in \mathcal{U}} \mathcal{F}(U_B) \arrow[r, shift left] \arrow[r, shift right] & \prod_{U,V \in \mathcal{U}} \mathcal{F}(U_B \cap V_B) \\
0 \arrow[r] & \mathcal{F}_x \arrow[r] \arrow[u, dashed] & \prod_{U \in \mathcal{U}} \mathcal{F}(U_A) \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & \prod_{U,V \in \mathcal{U}} \mathcal{F}(U_A \cap V_A) \arrow[u]
\end{tikzcd}$$

The compatibility of compositions $\mathrm{Spec}\, C \to \mathrm{Spec}\, B \to \mathrm{Spec}\, A \to X$ comes from the UP of equalizers.

Sub-goal : Quasi-coherence. We need to show that the morphisms $\mathcal{F}_x \to \mathcal{F}_{xf}$ induces $\mathcal{F}_x \otimes_A B \cong \mathcal{F}_{xf}$ in $B\mathbf{Mod}$. To prove this morphisms induces $\mathcal{F}_x \otimes_A B \cong \mathcal{F}_{xf}$, it suffices to check it is an isomorphism on each open $U_B$ in the cover of $\mathrm{Spec}\, B$. Tensoring $\mathcal{F}_x \otimes_A B \to \mathcal{F}_{xf}$ by $\mathcal{O}(U_B)$, we obtain the commutative diagram in $\mathrm{QCoh}\, U_B$ :

$$\begin{tikzcd}
\mathcal{F}_{xf} \otimes_B \mathcal{O}(U_B) \arrow[rr, "\cong"] & & \mathcal{F}(U_B) \\
\mathcal{F}_x \otimes_A B \otimes_B \mathcal{O}(U_B) \arrow[r, "\cong"'] \arrow[u] & \mathcal{F}(U_A) \otimes_{\mathcal{O}(U_A)} \mathcal{O}(U_B) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

□

## 5.4 Pullback of quasi-coherent sheaves

## 5.5 Pushforward along affine morphisms

## 5.6 Technical - Pushforward along QCQS morphisms

Let $f : X \to \mathrm{Spec}\ A$ be schematic. Question : how to take global sections of $\mathcal{F} \in \mathrm{QCoh}\, X$?

In practice, take affine Zariski cover $\mathcal{U}$ of $X$ such that for $U_i, U_j \in \mathcal{U}$, $U_i \cap U_j$ is also affine. Then take equalizer in $\mathrm{QCoh}\, A$,

$$0 \to f_*\mathcal{F} \to \prod_{U_i \in \mathcal{U}} \mathcal{F}(U_i) \rightrightarrows \prod_{U_i, U_j \in \mathcal{U}} \mathcal{F}(U_i \cap U_j)$$

We can relax the condition of $U \cap V$ being affine to the existence of Zariski epimorphism $U_{ij} \to U_i \cap U_j$ with $U_{ij}$ affine. This is equivalent to requiring $X \to \mathrm{Spec}\ A$ being quasi-separated. With that, one can define $f_*\mathcal{F}$ by the equaliser diagram in $\mathrm{QCoh}\, A$,

$$0 \to f_*\mathcal{F} \to \prod_{U_i \in \mathcal{U}} \mathcal{F}(U_i) \rightrightarrows \prod_{U_i, U_j \in \mathcal{U}} \mathcal{F}(U_{ij})$$

This is enough to define a functor $f_* : \mathrm{QCoh}\, X \to \mathrm{QCoh}\, A$. We only needed $f$ quasi-separated so far.

Let's be more ambitious now and consider $f : X \to Y$ a morphisms between schemes. Goal : $f_* : \mathrm{QCoh}\, X \to \mathrm{QCoh}\, Y$.

The most basic way is for each $y : \mathrm{Spec}\, A \to Y$ try to produce a module $y^* f_* \mathcal{F} \in \mathrm{QCoh}\, A$ equipped with pullback compatibilities. Given $y : \mathrm{Spec}\, A \to Y$, one can form the pullback square :

$$\begin{tikzcd}
X_y \arrow[r, "\tilde{y}"] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\mathrm{Spec}\, A \arrow[r, "y"'] & Y
\end{tikzcd}$$

which suggests a way of forming $y^* f_* \mathcal{F}$ is by using $\tilde{f}_* \tilde{y}^* \mathcal{F}$ since we know how to deal with $\tilde{f}_*$.

We are met with another issue : $y^* f_* \mathcal{F}$ needs to effectively be functorial in $y$ but our current method of defining $\tilde{f}_*$ requires choosing an affine Zariski cover of $X_y$. This gives two ideas :

1. provide affine Zariski cover of $X_y$ functorial in $y$
2. redefine $\tilde{f}_*$ independently of the choice of affine Zariski cover of $X_y$.

(1) is somewhat undesireable since it still depends on a choice of functorial affine Zariski cover. We pursue idea (2), which brings us back to the case of $f : X \to \operatorname{Spec} A$.

Goal : define $f_* : \operatorname{QCoh} X \to \operatorname{QCoh} A$ independently of affine Zariski cover. Let's keep things simple by assuming more than quasi-separated-ness : we assume $X$ has affine diagonal so that finite intersection of affine opens is still affine. At this point, we've effectively rediscovered Cech cohomology and are trying to prove the independence of Cech cohomology w.r.t. an affine Zariski cover. Great. I finally self-motivated the colimit definition of Cech cohomology. TLDR of Cech cohomology :

1. define $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ for Zariski affine covers $\mathcal{U}$.
2. show if $\mathcal{V}$ refines $\mathcal{U}$, then $\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{V}, \mathcal{F})$ is quasi-isomorphism
3. define $\check{H}^\bullet(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^\bullet(\mathcal{U}, \mathcal{F})$

Note to self : $X$ has affine diagonal iff $f$ has affine diagonal since for any pair of affine opens $U, V$ of $X$, we have the pullback squares

$$\begin{tikzcd}
X \arrow[d, "\Delta"'] & X \arrow[l] \arrow[d, "\Delta_f"'] \arrow[dl, phantom, "\llcorner"] & U \cap V \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\
X \times X & X \times_Y X \arrow[l] & U \times V \arrow[l]
\end{tikzcd}$$

# 6 Smoothness

## 6.1 The start of calculus : Tangent vectors

Let $\operatorname{Spec} A$ be a physical system and $a_0$ a single state in it. Suppose $a_0$ actually lies in a line of states $a$,

$$\begin{tikzcd}
\mathrm{pt} \arrow[r, "a_0"] \arrow[d, "0"'] & \operatorname{Spec} A \\
\mathbb{A}^1 \arrow[ur, "a"'] &
\end{tikzcd}$$

Then given a measuring device $f$ on $\operatorname{Spec} A$, we get

$$f(a) = f_0 + f_1 t + f_2 t^2 + \cdots \in \mathcal{O}(\mathbb{A}^1)$$

where $t$ is the standard meausring device on $\mathbb{A}^1$. Note that $f_0 = f(a_0)$. This says the way the value of $f$ changes over the family of states $a$ is described as a polynomical expression in the values of $t$.

The idea of a *tangent vector at the state $a_0$ in the direction of $a$* is we want only the *first order change of $f$ along $a$*. With highschool intuition, we write

$$\left(\frac{\partial}{\partial a}\right)_{a_0} f = \frac{f(a) - f(a_0)}{t} \text{ ignore } t^2 \text{ and above}$$

But we see that this is actually formal! This is the coefficient $f_1$. This gives the first definition of a tangent vector.

**Definition – Deformation perspective of tangent vectors**
A tangent vector at $a_0$ is an extension

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "a_0"] \arrow[d, "0"'] & \operatorname{Spec} A \\ \operatorname{Spec} k[t]/(t^2) \arrow[ur, dashed, "a"'] & \end{tikzcd}$$

For such an extension, we will write

$$f(a) = f(a_0) + \left( \left( \frac{\partial}{\partial a} \right)_{a_0} f \right) t$$

More generally, a vector field along a map $a_0 : \operatorname{Spec} B \to \operatorname{Spec} A$ is an extension

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "a_0"] \arrow[d, "0"'] & \operatorname{Spec} A \\ \operatorname{Spec} B[t]/(t^2) \arrow[ur, dashed, "a"'] & \end{tikzcd}$$

Cons of this perspective :

– the $k$-vector space structure of tangent spaces is not clear. It is only clearly pointed. [1] At this point, this is too much work for something that should be trivial. This *is* however useful for defining tangent bundle of stacks because it is less obvious to generalise algebraic derivations.

Pros of this perspective :

– it is closest to highschool intuition : that of first order change.

– the definition of the tangent bundle of $\operatorname{Spec} A$ is intuitive. It is the space $T \operatorname{Spec} A$ such that

$$\operatorname{Aff}(\operatorname{Spec} \_, T \operatorname{Spec} A) \simeq \operatorname{Aff}(\operatorname{Spec}(\_[t]/(t^2)), \operatorname{Spec} A)$$

This exists a priori in $\operatorname{PSh} \operatorname{Aff}$.

[1] Actually, a $k$-vector space structure is present, but it is non-trivial to check. The key is

$$\frac{k[x]}{(x^2)} \times_k \frac{k[y]}{(y^2)} \simeq \frac{k[x,y]}{(x,y)^2}$$

Then the $k$-vector space structure on $(t)$ inside $k[t]/(t^2)$ makes $k[t]/(t^2)$ into a "*k-vector space object in k-algebras*". [2] Making this into spaces, we want $\operatorname{Spec} \frac{k[x,y]}{(x,y)^2}$ to be the pushout of $\operatorname{Spec} k[x]/(x^2) \leftarrow \operatorname{Spec} k \to k[y]/(y^2)$ at least in the category of $k$-schemes. One of showing this is to see that any map into a $k$-scheme must factor through an affine open and thus reduces to the affine case where it is true. This endows the $k$-valued points of the tangent bundle with a $k$-vector space structure. To get this for $B$-valued points where $B$ is any $k$-algebra, one must repeat the above for $B[t]/(t^2)$. The non-trivial part is showing $\operatorname{Spec} B[x,y]/(x,y)^2$ is the desired pushout in the category of schemes. This is [Stacks, Lemma 37.14.1].

– it is clear that tangent vectors pushforward; this is a direct consequence of composing maps.

– the relative definition is also clear. Consider the example of $\pi : \mathbb{A}^2 \to \mathbb{A}^1, (x, y) \mapsto x$ and finding a tangent vectors in $\mathbb{A}^2$ relative to $\pi$ at points. You will discover that the tangent vectors are forced to be *vertical*, i.e. lie within the fiber of $\pi$. This leads to the general definition : Given a map $X \to S$ of affine schemes, a vector field along a map $T \to X$ is a solution to the lifting problem :

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ T \times_k k[t]/(t)^2 \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

– Given a sequence of affine schemes $X \to Y \to S$, we get $T(X/S) \to T(Y/S)$ where $T(\_/S)$ denotes tangent bundle relative to $S$. There is also $T(X/Y)$ consisting of tangent vectors in $X$ which lie in the fibers of the map $X \to Y$. Since fibers of $X \to Y$ lie in fibers of $X \to S$, we obtain

$$T(X/Y) \subseteq T(X/S) \to T(Y/S)$$

Looking at the fibers as pointed sets, one can see that $T(X/Y)$ is in the "kernel" of $T(X/S) \to T(Y/S)$. Conversely, any tangent vector of $X$ relative $S$ which dies under projection to $Y$ must tautologically lie in $T(X/Y)$. So the above is "left exact", if only we are able to put these three spaces inside some abelian category. This is one of the pros of the next definition of tangent vectors.

One can see that the data of $a$ is equivalent to specifying the linear operator $\left(\frac{\partial}{\partial a}\right)_{a_0} \in k\mathbf{Mod}(A, k)$. Following one's nose, we see that such linear operators $\partial$ satisfy the Leibniz rule.

$$\partial(fg) = \partial(f)g(a_0) + f(a_0)\partial(g)$$

These are called *k-derivations*. Let $\mathrm{Der}_k(A, k) \subseteq k\mathbf{Mod}(A, k)$ denote the subset of $k$-derivations. One can show $\mathrm{Der}_k(A, k)$ bijects with the set tangent vectors in our first definition, giving a second definition of tangent vectors.

> **Definition – Pragmatic perspective on tangent vectors**
> A tangent vector at state $a_0$ is a $k$-derivation from $A$ to $k_{a_0}$.
>
> More generally, a vector field along $a_0 : \operatorname{Spec} B \to \operatorname{Spec} A$ is a $k$-derivation $A \to B$ where $B$ is an $A$-module via its $A$-algebra structure.

Pros of the this perspective :

– Computable! E.g. show that the tangent space of any point in $\mathbb{A}^n$ is an $n$-dimensional vector space.

– Makes obvious the $k$-vector space structure of tangent spaces. We can do even better. Given $a : \operatorname{Spec} B \to \operatorname{Spec} A$,

$$\mathrm{Aff}_A(\operatorname{Spec} B, T \operatorname{Spec} A) \simeq \mathrm{Der}_k(A, B)$$

shows that local sections of the tangent bundle $T \operatorname{Spec} A$ have an $A$-module structure.

– Let $X \to Y \to S$ be maps of affine schemes and $R \to A \to B$ be the corresponding maps of algebras of functions. Then the sequence of spaces

$$T(X/Y) \subseteq T(X/S) \to T(Y/S)$$

corresponds to the sequence

$$0 \to \mathrm{Der}_A(B, \_) \stackrel{(1))}{\to} \mathrm{Der}_R(B, \_) \stackrel{(2)}{\to} \mathrm{Der}_R(A, \_)$$

1. "fibers of $X \to Y$ lie in fibers of $X \to S$ so relative tangents of $X$ to $Y$ are also relative tangents of $X$ to $S$" is precisely the injection (1).
2. "relative tangents of $X \to Y$ die when pushforward to $Y$" is precisely the fact that at (2) the image lies in the kernel.
3. "relative tangents of $X \to S$ whose pushforward to $Y$ is zero are relative tangents of $X \to Y$" precisely says that at (2) the kernel lies in the image.

So far, we have seen how to differentiate $f \in A$ with respect to some state $a \in \mathrm{Spec}\, A$ using a tangent vector. In other words, we have the inclusion of $A$-modules

$$\mathrm{Der}_k(A, \kappa(a)) \to k\mathbf{Mod}(A, k)$$

But the duality of space and function via evaluation gives us

$$\begin{aligned} A &\to k\mathbf{Mod}(\mathrm{Der}_k(A, \kappa(a)), k) \\ f &\mapsto \left( \frac{\partial}{\partial v}\bigg|_a \mapsto \frac{\partial}{\partial v}\bigg|_a f \right) =: (df)_a \end{aligned}$$

In the example of $A = k[x]$, one sees that

$$(df)_a(X) = X(f) = f'(a)X(x) = f'(a)(dx)_a(X)$$

in other words

$$(df)_a = f'(a)(dx)_a$$

**Definition – Algebraic one forms as dual of vector fields**
Let $R \to A$ be a map of algebras. Then a *module of differentials of $A$ relative $R$* is an $A$-module $\Omega_{A/R}$ representing the functor on $A$-modules $\mathrm{Der}_R(A, \_)$. In other words, $\Omega_{A/R}$ is equipped with an isomorphism

$$A\mathbf{Mod}(\Omega_{A/R}, \_) \simeq \mathrm{Der}_R(A, \_)$$

This definition can be seen as the algebraic formulation of the idea that one forms are dual of tangent vectors. This is somewhat unsatisfying because it would be nice to have a way of thinking about one forms independent of tangent vectors.

Given a point $a : A \to k$, we have the following interpretation.

$$(df)_a := f - f(a) \bmod I(a)^2$$

To get one forms with domains larger than a point, we need to generalise the above formula to $a : A \to B$. This is an issue because there is no canonical copy of $B$ inside $A$ for general $B$ unlike $B = k$.

Let $I_\Delta$ be the kernel of $A \otimes A \to A, f \otimes g \mapsto fg$. Consider $I_\Delta$ as an $A$-module via the left component of $A$. If $f_0 \otimes g_0 + f_1 \otimes g_1$ is in $I_\Delta$, then

$$f_0 g_0 + f_1 g_1 = 0$$

It follows that

$$f_0 \otimes g_0 + f_1 \otimes g_1 = f_0(1 \otimes g_0 - g_0 \otimes 1) + f_1(1 \otimes g_1 - g_1 \otimes 1)$$

This generalises to $f_0 g_0 + \cdots + f_n g_n$ in $I_\Delta$ and hows that $I_\Delta$ is generated as a left $A$-module by elements of the form

$$1 \otimes f - f \otimes 1$$

For the example of $A = k[t]$ and $A \otimes A = k[x, y]$, elements like this look like

$$f(y) - f(x)$$

This inspires the choice of notation

$$\Delta f := 1 \otimes f - f \otimes 1$$

which in turn inspires the definition

$$df := \Delta f \bmod I_\Delta^2$$

Let's check this is $k$-linear. For $\lambda \in k$ and $f \in A$, we have

$$d(\lambda f) = 1 \otimes \lambda f - \lambda f \otimes 1 = \lambda(1 \otimes f - f \otimes 1) = \lambda(df)$$

So $k$-linearity of $d$ comes from the fact that $k$ is allowed to pass between the two sides in $A \otimes A$.

> **Definition – Algebraic one forms as linear change**
> For a map of algebras $R \to A$, let $I_\Delta$ be the kernel of multiplication $A \otimes_R A \to A$. Then define
>
> $$\Omega_{A/R} := I_\Delta / I_\Delta^2$$
>
> This is a left $A$-module via $A \otimes 1 \subseteq A \otimes A$. Define
>
> $$d : A \to \Omega_{A/R}, f \mapsto (1 \otimes f - f \otimes 1) \bmod I_\Delta^2$$
>
> This map is only $R$-linear.

Our computation earlier shows that $\Omega_{A/R}$ is generated as an $A$-module by the image of $d$.

Relation between the two definitions? Something something universal square zero extension.

# 7 Group actions

## 7.1 Fundamental vector fields

**Definition – Fundamental vector fields A.K.A. infinitesimal action**

Let $G$ be an algebraic group over $k$ acting on a $k$-space $X$. Let $\mathfrak{g}$ denote the Lie algebra of $G$. Then given $\delta \in \mathfrak{g} = G(k[\varepsilon])$, we obtain a global vector field $X_\delta$ on $X$ by

$$k[\varepsilon] \times_k X \overset{(\delta,\mathrm{id})}{\to} G \times X \overset{\mathrm{act}}{\to} X$$

$X_\delta$ is called the *fundamental vector field associated to* $\delta$.

We obtain a morphism of Lie algebras

$$\mathfrak{g} \to \Gamma(X, T_X)$$

In fact, for any affine open $U \subseteq X$, due to smoothness of $U$ we have that fundamental vector fields on $U$ must land inside $U$ :

$$\begin{tikzcd}
U \arrow[r, "\mathbb{1}"] \arrow[d] & U \arrow[d] \\
k[\epsilon] \times_k U \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

So we get for every affine open $U$,

$$\mathfrak{g} \to \Gamma(U, T_X)$$

## 7.2 Connections on $G$-torsors

Let's review the three definitions of connections on vector bundles in differential geometry. Let $\pi : E \to B$ be a rank $m$ vector bundle over an $n$-dimensional manifold $B$.

**Definition**

The *vertical bundle of* $E$ is defined as the kernel of vector bundles on $E$ :

$$0 \to VE \to TE \xrightarrow{D\pi} TB \times_B E \to 0$$

Let's see what this sequence looks like locally on $E$. Locally on $B$, we have an isomorphism of vector bundles over spaces

$$\begin{tikzcd}
E \arrow[r, "(x,a)", "\sim"'] \arrow[d] & \mathbb{R}^n \times \mathbb{R}^m \arrow[d] & (x,a) \arrow[d, mapsto] \\
B \arrow[r, "\sim", "x"'] & \mathbb{R}^n & x
\end{tikzcd}$$

Then the above SES looks like at each $p \in E$ with $b = \pi(p)$

$$0 \to \left\langle \frac{\partial}{\partial a^j}_p \right\rangle \to \left\langle \frac{\partial}{\partial a^j}_p, \frac{\partial}{\partial x^k}_p \right\rangle \to \left\langle \frac{\partial}{\partial x^k}_b \right\rangle \to 0$$

> **Definition – Family of Horizontal subspaces**
> A family of horizontal subspace is a splitting of the vertical bundle short exact sequence.

From linear algebra, giving a splitting $0 \to V \to W \to W/V \to 0$ is the same as giving a retraction of $V \to W$. Using this, let's describe a choice of family of horizontal subspaces in terms of concrete data. A retraction of $0 \to \left\langle \frac{\partial}{\partial a^j}_p \right\rangle \to \left\langle \frac{\partial}{\partial a^j}_p, \frac{\partial}{\partial x^k}_p \right\rangle$ is equivalent to giving cotangent vectors $\theta^1_p, \dots, \theta^m_p$ in $T^*_p E$ such that

$$\theta^i_p \frac{\partial}{\partial a^j}_p = \delta^i_j \frac{\partial}{\partial a^j}_p$$

This is equivalent to the condition

$$\theta^j = (da^j)_p + A^j_k(p)(dx^k)_p$$

> **Definition – Linear family of horizontal spaces**
> Let $r : TE \to VE$ be a family of horizontal subspaces. For $\lambda \in \mathbb{R}$ let $\lambda : E \to E$ be the fiberwise scaling by $\lambda$ map. This in particular induces $d\lambda : TE \to TE$. Then $r$ is called *linear* when for all $e \in E$ and scalars $\lambda$ we have
> $$r_{\lambda e}(d\lambda)_e = r_e$$

For some reason, differential geometers like working with vector bundles by their total spaces whilst algebraic geometers like working with them through their quasi-coherent sheaf of sections.

Let $X = \operatorname{Spec} A$ where $A$ is a commutative algebra over a field $k$. Let $G$ be an algebraic group over $k$. We assume we already know what a principal $G$-bundle $\pi : P \to X$ means. Considering the sequence of spaces $P \to X \to \operatorname{Spec} k$, we obtain the exact sequence

$$0 \to T_{P/X} \to T_P \to \pi^* T_X$$

Let us compute $T_{P/X}$. For this, I found it easiest to consider it as a space over $P$ and compute its points. Let $y : U \to P$ be a general point of $P$ where $U$ is an affine. Let $\tilde{U} := U \times_k \operatorname{Spec} k[\epsilon]$. Maps from $y$ to $T_{P/X}$ are lifts $\tilde{y}$ satisfying

$$\begin{tikzcd} U \arrow[r, "y"] \arrow[d, "\subseteq"'] & P \arrow[d, "\pi"] \\ \tilde{U} \arrow[r, "x=\pi(y)"'] \arrow[ur, dashed, "\tilde{y}"] & X \end{tikzcd}$$

Note that $\tilde{U}$ has a retraction to $U$ which gives rise to the zero tangent vector at $y$. By abuse of notation, we also use $y$ to denote the zero tangent vector $\tilde{U} \to P$ at $y$. Giving a general relative tangent vector at $y$ then amounts to a lift

$$\begin{tikzcd} & P \\ \tilde{U} \arrow[ur, "y"] \arrow[r, dashed, "(y,\tilde{y})"'] & P \times_X P \arrow[u, "\text{fst}"'] \end{tikzcd}$$

However by assumption $P \times_X P \simeq P \times G$, so one must have $\tilde{y} = gy$ for some unique $g \in G(\tilde{U}) = (\mathrm{Lie}\, G)(U)$. The above is all functorial in $y$ and hence we deduce that as spaces over $P$ we have

$$T_{P/X} \simeq P \times \mathrm{Lie}\, G$$

In fact, the reverse map is the map producing fundamental vector fields we previously saw.

Note that $P \to X$ is formally smooth. Writing everything as quasi-coherent sheaves on $P$ we then have the SES

$$0 \to \mathcal{O}_P \otimes \mathrm{Lie}\, G \to T_P \to \pi^* T_X \to 0$$

We can descend this SES to $X$ by modding out $G$. Formally, we have descent along $G$-bundles

$$(\mathrm{QCoh}\, P)^G \underset{\pi_*(\_^G)}{\overset{\pi^*}{\underset{\sim}{\leftrightarrows}}} \mathrm{QCoh}\, X$$

So we obtain the SES in $\mathrm{QCoh}\, X$,

$$0 \to \pi_*(\mathcal{O}_P \otimes \mathrm{Lie}\, G)^G \to \pi_*(T_P^G) \to T_X \to 0$$

A *connection on* $P$ is a splitting of this SES in $\mathrm{QCoh}\, X$. Computationally, one should give such data on $P$. One way of doing this is to give a $G$-equivariant retraction of $T_P$ to $\mathcal{O}_P \otimes \mathrm{Lie}\, G$. This amounts to giving a $G$-equivariant $(\mathrm{Lie}\, G)$-valued one form $\omega$ such that for any element $\delta \in \mathrm{Lie}\, G$,

$$\omega(X_\delta) = \delta$$

[Mic, Section 19.1]

Exercise (which I have not done) : boil this down for $G = \mathrm{GL}_n$.

Suppose one is given a connection on $P$ as a section $\nabla : T_X \to \pi_*(T_P^G)$. The connection is called *flat* if this is a morphism of Lie algebroids.

Relation of D-modules and differential equations : Let $(\mathcal{E}, \nabla)$ be a vector bundle on a smooth curve $X$ and $\nabla$ a connection. Then on an open $U$ of $X$ on which $\mathcal{E}$ admits trivialising sections $e_1, \dots, e_n$, we have for any section $s$ on $U$,

$$\nabla s = \nabla(s^i e_i) = ds^i \otimes e_i + s^i \nabla e_i = ds^i \otimes e_i + s^i A_i^j \otimes e_j = ds + As$$

Therefore looking for horizontal sections $\nabla s = 0$ is equivalent to solving the $n$ 1-dimensional ordinary differential equation $ds = -As$ and then pasting solutions together to all of $X$.

# 8 References


# References

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