**Definition 0.1.** Normal Extensions. Let  $\iota: K \to L$  be a finite extension. Then the following are equivalent:

- 1. For all irreducible polynomials  $f \in K[X]$  that have root in L, L splits f.
- 2. For all elements  $a \in L$ , L splits min(a, K).
- 3. There exists generators  $a_1, \ldots, a_n \in L$  such that for all generators  $a_i$ , there exists a polynomial  $f_i \in K[X]$  where  $a_i$  is a root and L splits  $f_i$ .
- 4. There exists some polynomial  $f \in K[X]$  such that L is the splitting field of f.
- 5. For all K-extension  $\iota_M: K \to M$  and K-extension morphisms  $\phi, \psi: L \to M, \phi L = \psi L$ .

If any of the above are true,  $\iota: K \to L$  is called a **normal extension**. We also sometimes say L is **normal over** K.

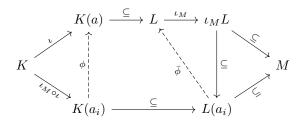
*Proof.*  $(1 \Rightarrow 2)$  Since finite extensions are algebraic, the second statement is well-defined. By minimal polynomials are irreducible and have a root in L, we are done.

 $(2 \Rightarrow 3)$  L is a finite extension so there exists  $a_0, \ldots, a_{n-1} \in L$  such that  $L = K(a_0, \ldots, a_{n-1})$ . For each generator  $a_i$ , pick  $f_i = \min(a, K)$ .

 $(3 \Rightarrow 4)$  Let  $f = \prod_{i \in n} f_i$ . L clearly splits f and since the roots of f include the generators  $a_0, \ldots, a_{n-1}, L$  is generated by the roots of f.

 $(4 \Rightarrow 1)$  Let g be an irreducible polynomial over K with a root  $a \in L$ . We do not know whether g splits in L, but we know there exists a finite extension  $\iota_M : L \to M$  that splits  $\bar{\iota}g$ , i.e. contains all the roots of g. We will show in that in this larger field, all the roots of g are already in L.

Let  $a_i$  be any root of g in M. The proof is contained in the following diagram.



We will show that  $\iota_M L = L(a_i)$ , i.e. the root  $a_i$  is already in L. Note that by  $\iota_M : L \to L(a_i)$  being an injective K-vector space morphism, it suffices to show  $\dim_K L = \dim_K L(a_i)$ .

We already have  $\dim_K L \leq \dim_K L(a_i)$ . For the other inequality, note that by definition, L splits f and L is generated by the roots of f. This clearly implies  $L(a_i)$  splits  $\overline{\iota_M \circ \iota} f$  and is generated by the roots of  $\overline{\iota_M \circ \iota} f$ , i.e.  $L(a_i)$  is the splitting field of  $\overline{\iota_M \circ \iota} f$ . Since a and  $a_i$  are galois conjugates, there exists a K-extension morphism  $\phi: K(a_i) \to K(a), a_i \mapsto a$ . So then  $\phi: K(a_i) \to L$  is a  $K(a_i)$ -extension that splits  $\overline{\iota_M \circ \iota} f$ . Hence by the minimal property of splitting fields, there exists a  $K(a_i)$ -extension morphism  $\overline{\phi}: L(a_i) \to L$ . This

is clearly a K-extension morphism. In particular,  $\bar{\phi}$  is an injective K-vector space morphism between finite dimensional K-vector spaces  $L(a_i)$  and L. Hence  $\dim_K L(a_i) \leq \dim_K L$ .

 $(4 \Rightarrow 5)$  This is image invariance of splitting fields.

 $(5 \Rightarrow 1)$  We require the machinery of normal closures, which we will investigate below.

For now, we restrict the definition of normality to 1 to 4.

## **Example 0.2.** Non-Normal Extensions.

The  $\mathbb{Q}$ -extension  $\iota: \mathbb{Q} \to \mathbb{Q}(\sqrt[3]{2})$  is not normal, since

$$\bar{\iota}\min(\sqrt[3]{2},\mathbb{Q}) = X^3 - 2 = (X - \sqrt[3]{2})(X^2 + \sqrt[3]{2}X + \sqrt[3]{2}^2)$$

does not factorise further. In a sense, the other roots  $\sqrt[3]{2}\omega$ ,  $\sqrt[3]{2}\omega^2$  are "missing" from the field  $\mathbb{Q}(\sqrt[3]{2})$ .

This can be fixed by adding elements to the extension until the extension is normal. The "smallest" such extension will be the *normal closure*.

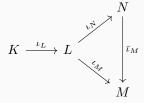
## Lemma 0.3. Normality Lifts up.

Let  $K \stackrel{\iota_L}{\to} L \stackrel{\iota_M}{\to} M$  be extensions and f a polynomial over K. Then  $\iota_M \circ \iota_L : K \to M$  is the splitting field of f implies  $\iota_M : L \to M$  is the splitting field of  $\bar{\iota}_L f$ . Consequently,  $\iota_M \circ \iota_L : K \to M$  normal implies  $\iota_M : L \to M$  normal.

*Proof.* Suppose  $\iota_M \circ \iota_L : K \to M$  is the splitting field of f. Clearly, M splits  $\bar{\iota}_L f$ . Let  $S_f$  be the roots of f in M.  $S_f$  is also the roots of  $\bar{\iota}_L f$  in M. Then the smallest subfield of M containing  $\iota_M L$  and  $S_f$  also contains  $\iota_M(\iota_L K)$ , i.e.  $M = K(S_f) \subseteq L(S_f) \subseteq M$ . Hence M is generated by  $S_f$  as an L-extension.  $\square$ 

## **Definition 0.4.** Normal Closure.

Let  $K \stackrel{\iota_L}{\longrightarrow} L \to \stackrel{\iota_N}{\longrightarrow} N$  be extensions. Then  $(N, \iota_N)$  is called a **normal closure of**  $(L, \iota_L)$  when it is a smallest normal K-extension containing  $\iota_L : K \to L$  in the sense that  $\iota_N \circ \iota_L : K \to N$  is normal and for any  $\iota_M : L \to M$  such that  $\iota_M \circ \iota_L : K \to M$  is normal, there exists a L-extension morphism  $\bar{\iota}_M : N \to M$ . Diagrammatically,



Note that we have not proved normal closures to be unique yet, hence the emphasis on "a normal closure" not "the normal closure".

## **Theorem 0.5.** Existence and Uniqueness of Normal Closure of Finite Extensions.

Let  $\iota_L: K \to L$  be a finite K-extension. Then there exists an L-extension  $\iota_N: L \to N$  such that  $(N, \iota_N)$  is a normal closure of  $(L, \iota_L)$ . Furthermore, N is a finite extension and unique up to isomorphism, i.e. any other normal closure of  $\iota_L: K \to L$  is isomorphic to N as a K-extension. Thus we denote N as  $N(L, \iota_L)$  and refer to anything isomorphic to it as the normal closure of  $(L, \iota)$ . If the extension  $\iota_L$  is clear, we write N(L) instead.

*Proof.* Since L is a finite dimensional K-vector space, by existence of a basis, let  $a_0, \ldots, a_{n-1}$  be a finite set of generators of L, i.e.  $L = K(a_0, \ldots, a_{n-1})$ . Let  $f = \prod_{i \in n} \min(a_i, K)$ . There exists a K-extension  $\iota_{\tilde{N}} : K \to \tilde{N}$  that splits f. Let N be the K-subextension of  $\tilde{N}$  generated by the roots of f, i.e. the splitting field of f. Since for all generators  $a_i$  of L, N splits  $\min(a_i, K)$ , we have a K-extension  $\iota_N : L \to N$  by embedding via conjugates. Clearly,  $\iota_N \circ \iota_L : K \to N$  is normal. We will now show that it has the minimal property of normal closures.

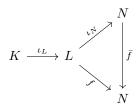
Let  $\iota_M:L\to M$  be an L-extension such that  $\iota_M\circ\iota_L:K\to M$  is normal. By lifting normality of N,  $\iota_N:L\to N$  is the splitting field of  $\bar\iota_L f$ . For a generator  $a_i$  of L,  $\min(a_i,K)=\min(\iota_M(a_i),K)$ . So normality of M over K implies M splits f, consequently splitting  $\bar\iota_L f$ . Hence by the minimal property of splitting fields, there exists an L-extension morphism  $\bar\iota_M:N\to M$  such that  $\bar\iota_M\circ\iota_N=\iota_M$ . Thus,  $(N,\iota_N)$  is a normal closure of  $\iota_L:K\to L$ .

Now let  $(M, \iota_M)$  be another normal closure of  $(L, \iota_L)$ . Then by applying minimal property of normal closure twice, we have L-extension morphisms  $\bar{\iota}_M: N \to M$  and  $\bar{\iota}_N: M \to N$ . These are also K-extension morphisms. Since N is a finite K-extension and  $\bar{\iota}_N$  is an injective morphism of K-vector spaces from M to N, we have the dimension of M as finite and less than equal to that of N. Similarly, the dimension of N is less than equal to that of M. So  $\dim_K N = \dim_K M$  and hence  $\bar{\iota}_M: N \to M$  is actually a bijection. Thus N and M are isomorphic as L-extensions.

**Remark.** The following is a key property of normal extensions.

**Theorem 0.6.** Two Embeddings into a Normal Extension Differ by an Automorphism.

Let  $K \xrightarrow{\iota_L} L \xrightarrow{\iota_N} N$  be extensions where  $(N, \iota_N \circ \iota_L)$  is normal. Then for all K-extension morphisms  $f: (L, \iota_L) \to (N, \iota_N \circ \iota_L)$ , there exists a L-extension morphism  $\bar{f}: (N, \iota_N) \to (N, f)$ . Diagrammatically,



This can be seen as if L embeds in N in two ways that agree on K, then there is a field automorphism of N bringing one L to the other, preserving K. So in a sense, two embeddings into a normal extension differ by an automorphism.

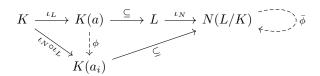
Proof.  $(N, \iota_N \circ \iota_L)$  being normal implies there exists a finite set of generators  $a_1, \ldots, a_n$  as a K-extension such that for all  $a_i$ ,  $(N, \iota_N \circ \iota_L)$  splits  $\min(a_i, K)$ . Then  $N = K(a_1, \ldots, a_n) \subseteq L(a_1, \ldots, a_n) \subseteq N$  implies  $a_1, \ldots, a_n$  are also generators of N as an L-extension. By definition of being a K-extension morphism,  $f \circ \iota_L = \iota_N \circ \iota_L$ . So  $(N, f \circ \iota_L) = (N, \iota_N \circ \iota_L)$  as K-extensions. Namely,  $(N, f \circ \iota_L)$  is normal, which implies for all generators  $a_i$ ,  $(N, f \circ \iota_L)$  splits  $\min(a_i, K)$ . Hence, (N, f) splits  $\min(a_i, L)$ . Thus, by embedding via conjugates, we have an L-extension morphism  $\bar{f}: (N, \iota_N) \to (N, f)$ .

We are now ready to prove  $5 \Rightarrow 1$ .

**Theorem 0.7.** Image Invariance gives Splitting Irreducible Polynomials with Roots.

Let  $\iota_L: K \to L$  be a K-extension such that for all K-extensions  $\iota_M: K \to M$  and K-extension morphisms  $\phi, \psi: L \to M$ ,  $\phi L = \psi L$ . Then for all irreducible polynomials  $f \in K[X]$  that have a root  $a \in L$ , L splits f.

*Proof.* Let f be an irreducible polynomial over K with a root  $a \in L$ . We do not know whether L contains all the roots of f, but we do know that L's normal closure definitely does. So let  $\iota_N : L \to N(L)$  be the normal closure of L. Then f is an irreducible polynomial with  $\iota_N(a)$  as a root. So by normality, N(L) splits f. Let  $a_i$  be an arbitrary root of f in N(L). We seek to show that the root  $a_i$  is already in  $\iota_N L$ . The following diagram is the situation.



Since  $a_i$  is a galois conjugate of a, there exists a K-extension morphism  $\phi: (K(a), \iota_L) \to (K(a_i), \iota_N \circ \iota_L)$  such that  $a \mapsto a_i$ . Then  $\phi: (K(a), \iota_L) \to (N, \iota_N \circ \iota_L)$  is a K-extension morphism where  $(N, \iota_N \circ \iota_L)$  is normal. Hence, there exists a K(a)-extension morphism  $\bar{\phi}: (N, \iota_N) \to (N, \phi)$ , since two embeddings into a normal extension differ by an automorphism. It follows that  $\bar{\phi} \circ \iota_N: (L, \iota_L) \to (N, \iota_N \circ \iota_L)$  is a K-extension morphism. Since  $\iota_N$  and  $\bar{\phi} \circ \iota_N$  are both K-extension morphisms from L to  $N, \iota_N L = \bar{\phi}(\iota_N L)$ . But of course,

$$a_i = \phi(a) = \bar{\phi}(\iota_N(a)) \in \iota_N L$$

Thus, the root  $a_i$  is actually already in  $\iota_N L$ . This completes the proof.