Definition 0.1. Normal Extensions. Let $\iota : K \to L$ be a finite extension. Then the following are equivalent:

1. For all irreducible polynomials $f \in K[X]$ that have root in L, L splits f.
2. For all elements $a \in L$, L splits $\min(a, K)$.
3. There exists generators $a_1, \dots, a_n \in L$ such that for all generators a_i, there exists a polynomial $f_i \in K[X]$ where a_i is a root and L splits f_i.
4. There exists some polynomial $f \in K[X]$ such that L is the splitting field of f.
5. For all K-extension $\iota_M : K \to M$ and K-extension morphisms $\phi, \psi : L \to M$, $\phi L = \psi L$.

If any of the above are true, $\iota : K \to L$ is called a **normal extension**. We also sometimes say L **is normal over** K.

Proof. ($1 \Rightarrow 2$) Since finite extensions are algebraic, the second statement is well-defined. By minimal polynomials are irreducible and have a root in L, we are done.

($2 \Rightarrow 3$) L is a finite extension so there exists $a_0, \dots, a_{n-1} \in L$ such that $L = K(a_0, \dots, a_{n-1})$. For each generator a_i, pick $f_i = \min(a, K)$.

($3 \Rightarrow 4$) Let $f = \prod_{i \in n} f_i$. L clearly splits f and since the roots of f include the generators $a_0, \dots, a_{n-1}$, L is generated by the roots of f.

($4 \Rightarrow 1$) Let g be an irreducible polynomial over K with a root $a \in L$. We do not know whether g splits in L, but we know there exists a finite extension $\iota_M : L \to M$ that splits $\bar{\iota} g$, i.e. contains all the roots of g. We will show in that in this larger field, all the roots of g are already in L.

Let a_i be any root of g in M. The proof is contained in the following diagram.

$$\begin{tikzcd}
& K(a) \arrow[r, "\subseteq"] & L \arrow[r, "\iota_M"] & \iota_M L \arrow[dd, "\subseteq"] \arrow[dr, "\subseteq"] & \\
K \arrow[ur, "\iota"] \arrow[dr, "\iota_M \circ \iota"'] & & & & M \\
& K(a_i) \arrow[uu, dashed, "\phi"] \arrow[rr, "\subseteq"] & & L(a_i) \arrow[uul, dashed, "\bar{\phi}"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

We will show that $\iota_M L = L(a_i)$, i.e. the root a_i is already in L. Note that by $\iota_M : L \to L(a_i)$ being an injective K-vector space morphism, it suffices to show $\dim_K L = \dim_K L(a_i)$.

We already have $\dim_K L \leq \dim_K L(a_i)$. For the other inequality, note that by definition, L splits f and L is generated by the roots of f. This clearly implies $L(a_i)$ splits $\overline{\iota_M \circ \iota} f$ and is generated by the roots of $\overline{\iota_M \circ \iota} f$, i.e. $L(a_i)$ is the splitting field of $\overline{\iota_M \circ \iota} f$. Since a and a_i are galois conjugates, there exists a K-extension morphism $\phi : K(a_i) \to K(a), a_i \mapsto a$. So then $\phi : K(a_i) \to L$ is a $K(a_i)$-extension that splits $\overline{\iota_M \circ \iota} f$. Hence by the minimal property of splitting fields, there exists a $K(a_i)$-extension morphism $\bar{\phi} : L(a_i) \to L$. This is clearly a K-extension morphism. In particular, $\bar{\phi}$ is an injective K-vector space morphism between finite dimensional K-vector spaces $L(a_i)$ and L. Hence $\dim_K L(a_i) \leq \dim_K L$.

($4 \Rightarrow 5$) This is image invariance of splitting fields.

($5 \Rightarrow 1$) We require the machinery of *normal closures*, which we will investigate below. $\square$

For now, we restrict the definition of normality to 1 to 4.

Example 0.2. Non-Normal Extensions.

The $\mathbb{Q}$-extension $\iota : \mathbb{Q} \to \mathbb{Q}(\sqrt[3]{2})$ is *not* normal, since

$$\bar{\iota}\min(\sqrt[3]{2}, \mathbb{Q}) = X^3 - 2 = (X - \sqrt[3]{2})(X^2 + \sqrt[3]{2}X + \sqrt[3]{2}^2)$$

does not factorise further. In a sense, the other roots $\sqrt[3]{2}\omega$, $\sqrt[3]{2}\omega^2$ are "missing" from the field $\mathbb{Q}(\sqrt[3]{2})$.

This can be fixed by adding elements to the extension until the extension is normal. The "smallest" such extension will be the *normal closure*.

Lemma 0.3. *Normality Lifts up.*

Let $K \overset{\iota_L}{\to} L \overset{\iota_M}{\to} M$ be extensions and f a polynomial over K. Then $\iota_M \circ \iota_L : K \to M$ is the splitting field of f implies $\iota_M : L \to M$ is the splitting field of $\bar{\iota}_L f$. Consequently, $\iota_M \circ \iota_L : K \to M$ normal implies $\iota_M : L \to M$ normal.

Proof. Suppose $\iota_M \circ \iota_L : K \to M$ is the splitting field of f. Clearly, M splits $\bar{\iota}_L f$. Let S_f be the roots of f in M. S_f is also the roots of $\bar{\iota}_L f$ in M. Then the smallest subfield of M containing $\iota_M L$ and S_f also contains $\iota_M(\iota_L K)$, i.e. $M = K(S_f) \subseteq L(S_f) \subseteq M$. Hence M is generated by S_f as an L-extension. $\square$

Definition 0.4. Normal Closure.

Let $K \overset{\iota_L}{\to} L \to\overset{\iota_N}{\to} N$ be extensions. Then (N, ι_N) is called a **normal closure of** (L, ι_L) when it is *a smallest* normal K-extension containing $\iota_L : K \to L$ in the sense that $\iota_N \circ \iota_L : K \to N$ is normal and for any $\iota_M : L \to M$ such that $\iota_M \circ \iota_L : K \to M$ is normal, there exists a L-extension morphism $\bar{\iota}_M : N \to M$. Diagrammatically,

$$\begin{tikzcd}
 & & N \arrow[dd, "\bar{\iota}_M"] \\
K \arrow[r, "\iota_L"] & L \arrow[ur, "\iota_N"] \arrow[dr, "\iota_M"'] & \\
 & & M
\end{tikzcd}$$

> Note that we have not proved normal closures to be unique yet, hence the emphasis on ”a normal closure” not ”the normal closure”.

Theorem 0.5. *Existence and Uniqueness of Normal Closure of Finite Extensions.*

Let $\iota_L : K \to L$ be a finite K-extension. Then there exists an L-extension $\iota_N : L \to N$ such that (N, ι_N) is a normal closure of (L, ι_L). Furthermore, N is a finite extension and unique up to isomorphism, i.e. any other normal closure of $\iota_L : K \to L$ is isomorphic to N as a K-extension. Thus we denote N as $N(L, \iota_L)$ and refer to anything isomorphic to it as the *normal closure of (L, ι). If the extension ι_L is clear, we write $N(L)$ instead.*

Proof. Since L is a finite dimensional K-vector space, by existence of a basis, let $a_0, \ldots, a_{n-1}$ be a finite set of generators of L, i.e. $L = K(a_0, \ldots, a_{n-1})$. Let $f = \prod_{i \in n} \min(a_i, K)$. There exists a K-extension $\iota_{\tilde{N}} : K \to \tilde{N}$ that splits f. Let N be the K-subextension of $\tilde{N}$ generated by the roots of f, i.e. the splitting field of f. Since for all generators a_i of L, N splits $\min(a_i, K)$, we have a K-extension $\iota_N : L \to N$ by embedding via conjugates. Clearly, $\iota_N \circ \iota_L : K \to N$ is normal. We will now show that it has the minimal property of normal closures.

Let $\iota_M : L \to M$ be an L-extension such that $\iota_M \circ \iota_L : K \to M$ is normal. By lifting normality of N, $\iota_N : L \to N$ is the splitting field of $\overline{\iota}_L f$. For a generator a_i of L, $\min(a_i, K) = \min(\iota_M(a_i), K)$. So normality of M over K implies M splits f, consequently splitting $\overline{\iota}_L f$. Hence by the minimal property of splitting fields, there exists an L-extension morphism $\overline{\iota}_M : N \to M$ such that $\overline{\iota}_M \circ \iota_N = \iota_M$. Thus, (N, ι_N) is a normal closure of $\iota_L : K \to L$.

Now let (M, ι_M) be another normal closure of (L, ι_L). Then by applying minimal property of normal closure twice, we have L-extension morphisms $\overline{\iota}_M : N \to M$ and $\overline{\iota}_N : M \to N$. These are also K-extension morphisms. Since N is a finite K-extension and $\overline{\iota}_N$ is an injective morphism of K-vector spaces from M to N, we have the dimension of M as finite and less than equal to that of N. Similarly, the dimension of N is less than equal to that of M. So $\dim_K N = \dim_K M$ and hence $\overline{\iota}_M : N \to M$ is actually a bijection. Thus N and M are isomorphic as L-extensions. $\square$

Remark. The following is a key property of normal extensions.

Theorem 0.6. *Two Embeddings into a Normal Extension Differ by an Automorphism.*

Let $K \overset{\iota_L}{\to} L \overset{\iota_N}{\to} N$ be extensions where $(N, \iota_N \circ \iota_L)$ is normal. Then for all K-extension morphisms $f : (L, \iota_L) \to (N, \iota_N \circ \iota_L)$, there exists a L-extension morphism $\bar{f} : (N, \iota_N) \to (N, f)$. Diagrammatically,

$$\begin{tikzcd}
 & & N \arrow[dd, "\bar{f}"] \\
K \arrow[r, "\iota_L"] & L \arrow[ur, "\iota_N"] \arrow[dr, "f"'] & \\
 & & N
\end{tikzcd}$$

This can be seen as if L embeds in N in two ways that agree on K, then there is a field automorphism of N bringing one L to the other, preserving K. So in a sense, two embeddings into a normal extension differ by an automorphism.

Proof. $(N, \iota_N \circ \iota_L)$ being normal implies there exists a finite set of generators $a_1, \ldots, a_n$ as a K-extension such that for all a_i, $(N, \iota_N \circ \iota_L)$ splits $\min(a_i, K)$. Then $N = K(a_1, \ldots, a_n) \subseteq L(a_1, \ldots, a_n) \subseteq N$ implies $a_1, \ldots, a_n$ are also generators of N as an L-extension. By definition of being a K-extension morphism, $f \circ \iota_L = \iota_N \circ \iota_L$. So $(N, f \circ \iota_L) = (N, \iota_N \circ \iota_L)$ as K-extensions. Namely, $(N, f \circ \iota_L)$ is normal, which implies for all generators a_i, $(N, f \circ \iota_L)$ splits $\min(a_i, K)$. Hence, (N, f) splits $\min(a_i, L)$. Thus, by embedding via conjugates, we have an L-extension morphism $\bar{f} : (N, \iota_N) \to (N, f)$. $\square$

We are now ready to prove $5 \Rightarrow 1$.

Theorem 0.7. *Image Invariance gives Splitting Irreducible Polynomials with Roots.*

Let $\iota_L : K \to L$ be a K-extension such that for all K-extensions $\iota_M : K \to M$ and K-extension morphisms $\phi, \psi : L \to M$, $\phi L = \psi L$. Then for all irreducible polynomials $f \in K[X]$ that have a root $a \in L$, L splits f.

Proof. Let f be an irreducible polynomial over K with a root $a \in L$. We do not know whether L contains all the roots of f, but we do know that L's normal closure definitely does. So let $\iota_N : L \to N(L)$ be the normal closure of L. Then f is an irreducible polynomial with $\iota_N(a)$ as a root. So by normality, $N(L)$ splits f. Let a_i be an arbitrary root of f in $N(L)$. We seek to show that the root a_i is already in $\iota_N L$. The following diagram is the situation.

$$\begin{tikzcd}
K \arrow[r, "\iota_L"] \arrow[dr, "\iota_N \circ \iota_L"'] & K(a) \arrow[r, "\subseteq"] \arrow[d, dashed, "\phi"] & L \arrow[r, "\iota_N"] & N(L/K) \arrow[loop right, dashed, "\bar{\phi}"] \\
& K(a_i) \arrow[urr, "\subseteq"'] & &
\end{tikzcd}$$

Since a_i is a galois conjugate of a, there exists a K-extension morphism $\phi : (K(a), \iota_L) \to (K(a_i), \iota_N \circ \iota_L)$ such that $a \mapsto a_i$. Then $\phi : (K(a), \iota_L) \to (N, \iota_N \circ \iota_L)$ is a K-extension morphism where $(N, \iota_N \circ \iota_L)$ is normal. Hence, there exists a $K(a)$-extension morphism $\bar{\phi} : (N, \iota_N) \to (N, \phi)$, since two embeddings into a normal extension differ by an automorphism. It follows that $\bar{\phi} \circ \iota_N : (L, \iota_L) \to (N, \iota_N \circ \iota_L)$ is a K-extension morphism. Since ι_N and $\bar{\phi} \circ \iota_N$ are both K-extension morphisms from L to N, $\iota_N L = \bar{\phi}(\iota_N L)$. But of course,

$$a_i = \phi(a) = \bar{\phi}(\iota_N(a)) \in \iota_N L$$

Thus, the root a_i is actually already in $\iota_N L$. This completes the proof. $\square$