## Definition - Fundamental Groupoid, Fundamental Group

Let  $B \in \mathbf{Top}$ . Define the *fundamental groupoid of* B,  $\Pi_1(B)$ , to be following category :

- $\Pi_1(B)$  has B as its collection objects.
- − For  $b, b_1 \in B$ ,  $\Pi_1(B)(b, b_1)$  consists of paths from b to  $b_1$  up to path-homotopy.
- Composition is concatenation of paths up-to-homotopy. That this is well-defined is an unenlightening exercise.
- Associativity is an unenlightening exercise.
- − For  $b \in B$ ,  $\mathbb{1}_b$  is the homotopy class of the constant path at b.

All morphisms of  $\Pi_1 B$  are isomorphisms, i.e.  $\Pi_1 B$  is indeed a groupoid. This gives a functor

$$\Pi_1: \mathbf{Top} \to \mathbf{Grpd}$$

For  $b \in B$ , use  $\pi_1(B,b)$  to denote the group of automorphisms of b in  $\Pi_1(B)$ . This is called the *fundamental group of* B *at* b. This gives a functor :

$$\pi_1: \mathbf{Top} * \to \mathbf{Grp}$$

## **Definition – Covering**

Let  $B \in \mathbf{Top}$ . For  $X \to B$  and  $U \to B$  in  $\mathbf{Top} \downarrow B$ , we say U trivialises X when there exists discrete  $X_U \in \mathbf{Top}$  satisfying the pullback diagram :

$$\begin{array}{ccc}
X &\longleftarrow & X_U \times U \\
\downarrow & & \downarrow \\
B &\longleftarrow & U
\end{array}$$

where the right vertical morphism is projection in the U-component. In this case, we call  $X_U$  the generic fibre over U.

Define the *category of coverings of* B,  $\mathbf{Cov}(B)$ , as the full subcategory of  $\mathbf{Top} \downarrow B$  consisting of  $X \to B$  such that there exists a cover  $\mathcal{U}$  of B consisting of opens that trivialise X. Objects of  $\mathbf{Cov}(B)$  are called *coverings* of B. When considering  $\mathbf{Cov}(B)$ , B is referred to as the *base space*.

Remark – Big Picture of Covering Spaces. The actions of a group can tell a lot about the group. The main result of covering spaces is that given sufficiently nice base space B, covering spaces tell us everything about actions of the fundamental groupoid, that is to say, we have an equivalence of categories :

$$\mathbf{Cov}(B) \simeq \mathbf{Set}^{\Pi_1 B}$$

<sup>&</sup>lt;sup>a</sup>It is standard to require non-empty generic fibres, however I have seen no use of this in the theory so have chosen to not include it.

## **Proposition – Permenance Properties**

The following are true:

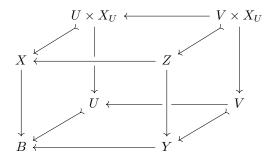
- (Base Change) Consider a pullback diagram in **Top**:

$$\begin{array}{ccc}
X &\longleftarrow & Z \\
\downarrow & & \downarrow \\
B &\longleftarrow & Y
\end{array}$$

Then  $X \to B$  covering implies  $Z \to Y$  covering. The converse is false.

– (Composition) Let  $X \to Y \to B$  in Top. Then  $X \to Y$  and  $Y \to B$  coverings imply  $X \to B$  is a covering. The converse is false.

*Proof.* (Base Change) We have the following commutative cube :

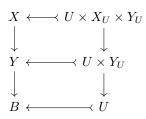


The faces that are pullback squares are:

- front, by assumption.
- bottom, which gives V an open in Y.
- left, where U is a trivialising open of X over B and  $X_U$  is a discrete space, with  $U \times X_U \to U$  being projection into first component.
- back, which follows easily. The projection  $V \times X_U \to V$  is into the first component.

The left and back being pullback squares implies  $V \times X_U$  is the pullback of X, V over B, and together with Z being the pullback of X, Y over B, this implies  $V \times X_U$  is the pullback of Z, V over Y, i.e. isomorphic to the preimage of V under  $Z \to Y$ . This gives V as a trivialising open of Z over Y. We can hence obtain a trivialising cover for Z over Y.

(Composition) This is the diagram:



The bottom square is the pullback square of X over B along a trivialising open U of Y over B. We can then shrink U such that each  $U \to U \times X_U \to Y$  is (isomorphic to) a trivializing open of X over Y, hence the top square. Since  $X_U, Y_U$  are discrete,  $X_U \times Y_U$  is discrete as well. The "rectangle" is a pullback diagram because the inner two are. Thus U is a trivialising open of X over B. This gives an open cover of B trivialising X.