Notes on Covering Spaces

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Contents

1 Definitions and Basic Results

Definition – Fundamental Groupoid, Fundamental Group

Let $B \in \mathbf{Top}$. Define the *fundamental groupoid of* B, $\Pi_1(B)$, to be following category :

- $\Pi_1(B)$ has B as its collection objects.
- For $b, b_1 \in B$, $\Pi_1(B)(b, b_1)$ consists of paths from b to b_1 up to path-homotopy.
- Composition is concatenation of paths up-to-homotopy. That this is well-defined and associative is an unenlightening exercise.
- For $b \in B$, $\mathbb{1}_b$ is the homotopy class of the constant path at b.

All morphisms of $\Pi_1 B$ are isomorphisms, i.e. $\Pi_1 B$ is indeed a groupoid. This gives a functor

$$\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}$$

For $b \in B$, use $\pi_1(B, b)$ to denote the group of automorphisms of b in $\Pi_1(B)$. This is called the *fundamental group of* B *at* b. This gives a commutative square of functors :

$$\begin{tikzcd}
\mathbf{Top}* \arrow[r, "\pi_1"] \arrow[d] & \mathbf{Grp} \arrow[d] \\
\mathbf{Top} \arrow[r, "\Pi_1"] & \mathbf{Grpd}
\end{tikzcd}$$

where $\mathbf{Top}*$ is pointed topological spaces.

Definition – Covering

Let $B \in \mathbf{Top}$. For $X \to B$ and $U \to B$ in $\mathbf{Top} \downarrow B$, we say U *trivialises* X when there exists discrete $X_U \in \mathbf{Top}$ satisfying the pullback diagram :

$$\begin{tikzcd} X & X_U \times U \arrow[l] \arrow[d] \\ B \arrow[from=1-1, d] & U \arrow[l, "\supseteq"'] \end{tikzcd}$$

where the right vertical morphism is projection in the U-component. In this case, we call X_U the *generic fibre over* U.

Define the *category of coverings of* B, $\mathbf{Cov}(B)$, as the full subcategory of $\mathbf{Top} \downarrow B$ consisting of $X \to B$ such that there exists a cover $\mathcal{U}$ of B consisting of opens that trivialise X.[a] Objects of $\mathbf{Cov}(B)$ are called *coverings* of B. When considering $\mathbf{Cov}(B)$, B is referred to as the *base space*.

[a] It is standard to require non-empty generic fibres, however I have seen no use of this in the theory so have chosen to not include it.

Remark – Big Picture of Covering Spaces. The invariant we seek to understand is $\Pi_1 B$. The actions of a group can tell a lot about the group. The main result of covering spaces is that given sufficiently nice base space B, covering spaces tell us everything about actions of the fundamental groupoid. Formally, we have an equivalence of categories :

$$\mathbf{Cov}(B) \simeq \mathbf{Set}^{\Pi_1 B}$$

This is theoretically good, but for computations we restrict attention to fundamental groups so we can have group theory at our disposal. Given a choice of $b \in B$, we have a restriction functor :

$$\mathbf{Set}^{\Pi_1 B} \to \mathbf{Set}^{\pi_1(B,b)}$$

Under additional assumptions on B, this is an equivalence and will yield :

$$\mathbf{Cov}(B) \simeq \mathbf{Set}^{\pi_1(B,b)}$$

This is called the *Galois theory of covering spaces*.

Proposition – Permenance Properties

The following are true :

- (Base Change) Consider a pullback diagram in $\mathbf{Top}$:

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l] \arrow[d] \\ B & Y \arrow[l] \end{tikzcd}$$

  Then $X \to B$ covering implies $Z \to Y$ covering. The converse is false.

- (Composition) Let $X \to Y \to B$ in **Top**. Then $X \to Y$ and $Y \to B$ coverings imply $X \to B$ is a covering. The converse is false.

Proof. (*Base Change*) We have the following commutative cube :

$$\begin{tikzcd}
& U \times X_U \arrow[dl, tail] \arrow[dd] & & V \times X_U \arrow[ll] \arrow[dl, tail] \arrow[dd] \\
X \arrow[dd] & & Z \arrow[ll] \arrow[dd] & \\
& U \arrow[dl, tail] & & V \arrow[ll] \arrow[dl, tail] \\
B & & Y \arrow[ll] &
\end{tikzcd}$$

The faces that are pullback squares are :

- front, by assumption.
- bottom, which gives V an open in Y.
- left, where U is a trivialising open of X over B and X_U is a discrete space, with $U \times X_U \to U$ being projection into first component.
- back, which follows easily. The projection $V \times X_U \to V$ is into the first component.

The left and back being pullback squares implies $V \times X_U$ is the pullback of X, V over B, and together with Z being the pullback of X, Y over B, this implies $V \times X_U$ is the pullback of Z, V over Y, i.e. isomorphic to the preimage of V under $Z \to Y$. This gives V as a trivialising open of Z over Y. We can hence obtain a trivialising cover for Z over Y.

(*Composition*) This is the diagram :

$$\begin{tikzcd}
X \arrow[d] & U \times X_U \times Y_U \arrow[l, tail] \arrow[d] \\
Y \arrow[d] & U \times Y_U \arrow[l, tail] \arrow[d] \\
B & U \arrow[l, tail]
\end{tikzcd}$$

The bottom square is the pullback square of X over B along a trivialising open U of Y over B. We can then shrink U such that each $U \to U \times X_U \to Y$ is (isomorphic to) a trivializing open of X over Y, hence the top square. Since X_U, Y_U are discrete, $X_U \times Y_U$ is discrete as well. The "rectangle" is a pullback diagram because the inner two are. Thus U is a trivialising open of X over B. This gives an open cover of B trivialising X.

□

2 Fundamental Theorem of Covering Spaces, Groupoids Version

Definition – Lift

Let $B \in \mathbf{Top}$, $X \to B$, $Y \to B$ in $\mathbf{Top}$. Then a *lift* of $Y \to B$ along $X \to B$ is a morphism $f \in \mathbf{Top} \downarrow B(Y, X)$.

Proposition – Monodromy Functor

Let $B \in \mathbf{Top}$ be locally path connected and $X \in \mathbf{Cov}(B)$. (It follows that X is locally path connected.) Then the *monodromy functor of* X is defined as the $\Pi_1(B)$-set :

$$\begin{aligned} \mathrm{Fib}_X : \Pi_1(B) &\to \mathbf{Set} \\ b &\mapsto \downarrow^{-1} b \\ [\gamma] &\mapsto \mathrm{Fib}_X([\gamma]) : \downarrow^{-1} s(\gamma) \to \downarrow^{-1} t(\gamma), x \mapsto \gamma_x(1) \end{aligned}$$

where γ_x is any lift of any representative of $[\gamma]$, such that $\gamma_x(0) = x$. This gives rise to a functor :

$$\mathrm{Fib} : \mathbf{Cov}(B) \to \mathbf{Set}^{\Pi_1 B}$$

Furthermore, for each $X \in \mathbf{Cov}(B)$, we can recover the fundamental groupoid of X via

$$\int_{\Pi_1(B)} \mathrm{Fib}_X \simeq \Pi_1(X)$$

where the former is the category of elements of Fib_X. Hence

- X is path-connected if and only if Fib_X is transitive, i.e. for all $(b, x), (b_1, x_1) \in \int_{\Pi_1(B)} \mathrm{Fib}_X$, there exists a morphism $[\gamma]$ of $\Pi_1(B)$ such that $\mathrm{Fib}_X([\gamma])(x) = x_1$.
- for all $(b, x) \in \int_{\Pi_1(B)} \mathrm{Fib}_X$, the induced group morphism $\pi_1(X, x) \to \pi_1(B, b)$ maps $\pi_1(X, x)$ isomorphically to $\mathrm{Stab}(x)$.

Proof. To define the $\Pi_1(B)$-action on fibres, we need to be able to lift paths uniquely *and* for path-homotopies to lift. This is exactly what the local trivisalisations of coverings allow us to achieve. We first prove paths lift uniquely.

Lemma (*Unique Path Lifting*).
Let $B \in \mathbf{Top}$, $X \in \mathbf{Cov}(B)$. Then X satisfies unique path lifting, *meaning for all commuting squares of the form :*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "0"'] & X \arrow[d] \\ I \arrow[r] \arrow[ur, dashed, "!"] & B \end{tikzcd}$$

there exists a unique morphism in the diagonal such that the whole diagram commutes. Such a diagonal *morphism is called a* lift *of the morphism* $I \to B$.

Proof. Let $\gamma : I \to B$ and $x \in X$ in the fibre over $\gamma(0)$. Then there exists a set $\mathcal{U}$ consisting of opens of B trivialising X such that $\gamma I \subseteq \bigcup \mathcal{U}$.

The idea is that each $U \in \mathcal{U}$ allows us to lift a part of γI and compactness of I allows for induction. Since I is compact, there exists a partition $\{0 = t_0 < \cdots < t_n = 1\}$ of I such that for each $t_i < t_n$, $\gamma[t_i, t_{i+1}]$ is in some $U_i \in \mathcal{U}$. Suppose by induction we have a unique lift $\overline{\gamma_{n-1}} : [0, t_{n-1}] \to X$ of $\gamma : [0, t_{n-1}] \to B$. Let $U_n \in \mathcal{U}$ with $\gamma[t_{n-1}, t_n] \subseteq U_n$. Let $s_n : U_n \to X$ be a section such that $\overline{\gamma_{n-1}}(t_{n-1}) \in s_n U_n$. The define a lift $\overline{\gamma} : I \to X$ by patching together $\overline{\gamma_{n-1}}$ and $s_n \circ \gamma|_{[t_{n-1}, t_n]}$. This lift is unique because any lift $\tilde{\gamma} : I \to X$ of γ must restrict to a lift of $\gamma : [0, t_{n-1}] \to B$, and thus $\tilde{\gamma}|_{[0, t_{n-1}]} = \overline{\gamma_{n-1}}$ by uniqueness of $\overline{\gamma_{n-1}}$ and finally $\tilde{\gamma}$ must also agree with $\overline{\gamma}$ on $[t_{n-1}, t_n]$ since $\tilde{\gamma}[t_{n-1}, t_n] \subseteq s_n U_n$ and s_n is a homeomorphism onto its image. ■

For lifting homotopies, we prove a more general lemma :

Lemma (Unique Homotopy Lifting).
Let $B \in \mathbf{Top}$ be locally path connected and $X \in \mathbf{Cov}(B)$. (It follows that X is also locally path connected.) Then X satisfies unique homotopy lifting with respect to locally connected spaces, *meaning for all commuting squares*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\mathbb{1}_Y \times 0"'] & X \arrow[d] \\ Y \times I \arrow[r] \arrow[ur, dashed, "!"] & B \end{tikzcd}$$

where Y is locally connected, there exists a unique morphism in the diagonal such that the diagram commutes. Such a morphism is called a lift *of the morphism* $Y \times I \to B$.

Proof. (*Existence*) (*Local Lifts*) Let $y \in Y$. We show the existence of an open neighbourhood U_y of y with a lift $\overline{H}_y : V_y \times I \to X$ of $H : U_y \times I \to B$. Let $\mathcal{U}$ be an open cover of B trivialising X. Since B is locally path connected, we can WLOG assume $\mathcal{U}$ consists of path connected opens. Now, for every $t \in I$, there exists $\varepsilon_t > 0$ and V_t open neighbourhood of y such that $HV_t \times [t - \varepsilon_t, t + \varepsilon_t] \subseteq U_t$ for some $U_t \in \mathcal{U}$. By compactness of I, there exists a partition $\{0 = t_0 < \cdots < t_n = 1\}$ of I and open neighbourhoods (V_i) of y such that $HV_i \times [t_i, t_{i+1}] \subseteq U_i$ for some $U_i \in \mathcal{U}$. We can now take $V_y = \bigcap_i V_i$ as a single open neighbourhood of y such that $HV_y \times [t_i, t_{i+1}] \subseteq U_i$ for some $U_i \in \mathcal{U}$. We construct a lift $\overline{H}_y$ inductively. By local connectedness of Y, we can WLOG assume V_y is connected, which we will use. Suppose by induction we have a lift $\tilde{H} : V_y \times [0, t_{n-1}] \to X$ of $H : V_y \times [0, t_{n-1}] \to X$. Let $U_n \in \mathcal{U}$ that covers $HV_y \times [t_{n-1}, t_n]$. By connectedness of V_y, $\tilde{H}V_y \times t_{n-1}$ lies within the image of a section $s_n : U_n \to X$. Hence we can define a lift $\overline{H} : V_y \times I \to X$ by patching together $\tilde{H}$ and $s_n \circ H|_{V_y \times [t_{n-1}, t_n]}$.

(*Global Lift*) We have an open cover $\mathcal{Y}$ of Y and for each $V \in \mathcal{Y}$ a lift $\overline{H}_V : V \times I \to X$ of $H : V \times I \to B$. For any two $V, W \in \mathcal{Y}$, $\overline{H}_V$ and $\overline{H}_W$ both restrict to lifts of $V \times I \cap W \times I = (V \cap W) \times I$. But these give lifts of paths starting in $V \cap W$ and lifts of paths are unique by the previous lemma, so $\overline{H}_V$

and $\overline{H}_W$ agree on $V \times I \cap W \times I$. Thus these lifts patch togther to give a global lift $\overline{H} : Y \times I \to X$ of H.

(*Uniqueness*) Let $\overline{H}, \overline{H}_1 : Y \times I \to X$ be lifts of $H : Y \times I \to B$. Again, these restrict to lifts of paths $y \times I \to B$, which are unique by the previous lemma so $\overline{H} = \overline{H}_1$.

$\blacksquare$

We can now define the $\Pi_1(B)$-action. Let $[\gamma]$ be a morphism in $\Pi_1(B)$. Choose a representative γ. Define

$$\mathrm{Fib}_X([\gamma]) : \downarrow^{-1} s([\gamma]) \to \downarrow^{-1} t([\gamma]) := x \mapsto \gamma_x(1)$$

where $\gamma_x : I \to X$ is the unique lift of γ with $\gamma_x(0) = x$. We now need to show this is independent of the choice of γ. Let γ^1 be another representative of $[\gamma]$. So we have a homotopy $H : I \times I \to B$ from γ to γ^1 that fixes endpoints. Since I is locally connected, we have a lift :

$$\begin{tikzcd}
I \arrow[r, "\gamma_x"] \arrow[d, "\mathbb{1}_Y \times 0"'] & X \arrow[d] \\
I \times I \arrow[r, "H"] \arrow[ur, dashed, "\overline{H}"] & B
\end{tikzcd}$$

We hope that $\overline{H}$ gives a path-homotopy. Well,

1. Restricted to $0 \times I$, $\overline{H}$ gives a lift of the constant point $s([\gamma])$. By uniqueness of path lifting, $\overline{H}$ must be constant along $0 \times I$. Similarly, $\overline{H}$ is the constant point $t([\gamma])$ along $1 \times I$.
2. Now restricted to $I \times 1$, $\overline{H}$ gives a lift of γ^1 starting at $s([\gamma])$. By uniquenss of path lifting, $\overline{H}$ must be γ^1_x along $I \times 1$.

Hence, $\overline{H}$ is indeed a homotopy from γ_x to γ^1_x fixing end points, i.e. $[\gamma_x] = [\gamma^1_x]$. In particular, $\gamma_x(1) = \gamma^1_X(1)$ so $\mathrm{Fib}_X([\gamma])(x)$ is well-defined.

(Fib) Let $f \in \mathbf{Cov}(Y, X)$. Then indeed for every morphism $[\gamma]$ in $\Pi_1(B)$, we have

$$\begin{tikzcd}
\mathrm{Fib}_X(s([\gamma])) \arrow[r, "f"] \arrow[d, "\mathrm{Fib}_X([\gamma])"] & \mathrm{Fib}_Y(s([\gamma])) \arrow[d, "\mathrm{Fib}_Y([\gamma])"] \\
\mathrm{Fib}_X(t([\gamma])) \arrow[r, "f"] & \mathrm{Fib}_Y(t([\gamma]))
\end{tikzcd}$$

since for every x in the fibre over the source of $[\gamma]$ and any lift γ_x of $[\gamma]$ starting at x, $f \circ \gamma_x$ is a lift of $[\gamma]$ starting at $f(x)$.

(*Furthermore*) We describe the functor $\int_{\Pi_1(B)} \mathrm{Fib}_X \to \Pi_1(X)$:

- for each object (b, x), map it to x.

– for each morphism $[\gamma] \in \int_{\Pi_1(B)} \mathrm{Fib}_X((b,x),(b_1,x_1))$, map it to $[\gamma_x]$ where γ_x is any lift of γ starting at x. We have seen this is well-defined and by the assumption of $\mathrm{Fib}_X([\gamma])(x) = x_1$, $[\gamma_x] \in \Pi_1(X)(x, x_1)$ indeed.

– Functoriality follows from uniqueness of path liftings.

The functor is clearly essentially surjective. Faithfulness comes from projecting paths back down $\Pi_1(X) \to \Pi_1(B)$. We have seen fullness.

(*Hence*) Straightforward. □

Remark. The monodromy functor gives one side of the equivalence $\mathbf{Cov}(B) \simeq \mathbf{Set}^{\Pi_1 B}$. For the quasi-inverse functor to exist, the extra condition on B is the following, which says that "there are enough opens U of B which are determined by their fundamental groupoids".

> *Lemma* (*Characterisation of Semi-Locally Simply Connected*).
> *Let $B \in \mathbf{Top}$ locally path connected. Given an open U of B and $b \in U$, TFAE :*
>
> 1. *U is path connected and the obvious morphism $\pi_1(U,b) \to \pi_1(B,b)$ is trivial.*
> 2. *The following being a bijection :*
> $$b \downarrow \Pi_1 U \to U$$
>
> *When U and b satisfies any (and thus both) of the above, call b a* centre *of U. If U satisfies the above* for some $b \in U$, *then call U a* centred *open.*[a] *Then TFAE :*
>
> 1. *There exists an open cover $\mathcal{U}$ of B consisting of U such that every $b \in U$ is a centre of U.*
> 2. *For every $b \in B$, there is a neighbourhood base of opens U with b as a centre.*
> 3. *There exists a cover $\mathcal{U}$ of B consisting of centred opens.*
>
> *We say B is* semi-locally simply connected *when it satisfies any (and thus all) of the above.*
>
> [a] I made up this terminology to avoid repeating long phrases in the proof.

Proof. ($1 \Leftrightarrow 2$ *for U and b*) U being path connected corresponds to $b \downarrow \Pi_1 U \to U$ being surjective. It suffices to prove $\pi_1(U,b) \to \pi_1(B,b)$ trivial if and only if $b \downarrow \Pi_1 U \to U$ injective. Forwards, given two morphisms $[\gamma], [\gamma_1]$ in $\Pi_1 U$ with source at b and same target, $[\gamma]^{-1}[\gamma_1] \in \pi_1(U,b)$. Triviality of $\pi_1(U,b) \to \pi_1(B,b)$ implies $[\gamma] = [\gamma_1]$ as morphisms in $\Pi_1 B$, in particular in $\Pi_1 U$. The converse is easy.

Now for equivalent conditions of B semi-locally simply connected. ($1 \Rightarrow 2$) Use local path connectedness of B and functoriality of $\pi_1(-,b)$. ($2 \Rightarrow 3$) Obvious. ($3 \Rightarrow 1$) Let $\mathcal{U}$ be an open cover of B such that for all $U \in \mathcal{U}$, there exists a centre b of U. It suffices to show for other $b_1 \in U$, $\pi_1(U,b_1) \to \pi_1(B,b_1)$ is also trivial. By assumption, there exists $[\gamma] \in \Pi_1 U(b,b_1)$, so we have a commutative square

$$\begin{tikzcd} \pi_1(U,b) \arrow[r, "1"] \arrow[d, "\cong"'] & \pi_1(X,b) \arrow[d, "\cong"] \\ \pi_1(U,b_1) \arrow[r] & \pi_1(X,b_1) \end{tikzcd}$$

where the vertical maps are "conjugation" by $[\gamma]$. This proves the bottom horizontal morphism is trivial.

$\square$

Proposition – Fundamental Theorem of Covering Spaces [a]

Let $B \in \mathbf{Top}$ be locally path connected and semi-locally simply connected. Consider the functor $\int_{\Pi_1 B} : \mathbf{Set}^{\Pi_1 B} \to \mathbf{Set} \downarrow B$ that sends X to its category of elements $\int_{\Pi_1 B} X$, which we then view as a set with a set-morphism down to B. Then

1. we can promote the $\int_{\Pi_1 B}$ to a functor $\mathbf{Set}^{\Pi_1 B} \to \mathbf{Cov}B$.
2. $\int_{\Pi_1 B}$ and Fib form an equivalence of categories

$$\mathbf{Cov}(B) \simeq \mathbf{Set}^{\Pi_1 B}$$

[a] As called on nLab page on covering space.

Proof. (1) Let $\rho \in \mathbf{Set}^{\Pi_1 B}$. By assumption, B has a topological base of centred opens. These will give the topology on $\int_{\Pi_1 B} \rho$ as well as the trivialising opens of the projection.

(1)(*Topology*) For $(b, x) \in \int_{\Pi_1 B} \rho$ and an open U of B centred at b, define

$$U[b, x] := \{\rho([\gamma](x) \mid [\gamma] \in b \downarrow \Pi_1 U\}$$

Then the set of these give a topological base because centred opens form a topological base for B. We topologise $\int_{\Pi_1 B} \rho$ using this topological base.

(1)(*Projection Topological*) To show $\int_{\Pi_1 B} \rho \to B$ is a morphism of topological spaces, since B has centred opens as a topological base, it suffices for each centred U, say at some b, to have preimage

$$\downarrow^{-1} U = \bigsqcup_{x \in \rho(b)} U[b, x]$$

The equality is straight forward from U being centred at b and disjoint union follows from $\Pi_1 U$ being a groupoid.

(1)(*Covering*) We saw already that $\downarrow^{-1} U = \bigsqcup_{x \in \rho(b)} U[b, x]$ as sets. Since $\int_{\Pi_1 B} \rho$ has topology generated by the $V[b, x]$'s, to show $U[b, x] \to U$ is a topological isomorphism, it suffices to show for all U with centres b and $x \in \rho(b)$, $U[b, x]$ maps to U bijectively. And it does because U is centred!

(1)(*Functorial*) Let $f \in \mathbf{Set}^{\Pi_1 B}(\rho, \rho_1)$. We need to show the induced $f_* : \int_{\Pi_1 B} \rho \to \int_{\Pi_1 B} \rho_1$ is a topological morphism over B. Suffices for basic opens $U[b, y]$ of $\int_{\Pi_1 B} \rho_1$ to have preimage

$$\downarrow^{-1} U[b, y] = \bigcup_{x \in f_b^{-1}(y)} U[b, x]$$

where $f_b : \rho(b) \to \rho_1(b)$. $\supseteq$ is clear. Let (b_1, x_1) be in the preimage of $U[b, y]$. By definition, there exists $[\gamma] \in \Pi_1 U(b, b_1)$ with $\rho[\gamma](y) = f_{b_1}(x_1)$. Then by naturality of f, we have

$$\begin{tikzcd}
\rho(b) \arrow[r, "f_b"] \arrow[d, "\rho[\gamma]"'] & \rho_1(b) \arrow[d, "\rho_1[\gamma]"] \\
\rho(b_1) \arrow[r, "f_{b_1}"] & \rho_1(b_1)
\end{tikzcd}$$

Since $\Pi_1 B$ is a groupoid, $\rho[\gamma]$ are isomorphisms of sets, so there exists $x \in \rho(b)$ with $\rho[\gamma](x) = x_1$ and $f_b(x) = y$, i.e. we have the other inclusion.

(2)$(\mathrm{Fib} \circ \int_{\Pi_1 B} \cong \mathbb{1})$ We know that $\mathrm{Fib}\left(\int_{\Pi_1 B} \rho\right)(b) = \rho(b)$ for $b \in \Pi_1 B$. So it suffices that for all morphisms $[\gamma]$ in $\Pi_1 B$,

$$\mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma] = \rho[\gamma]$$

which will actually show $\mathrm{Fib} \circ \int_{\Pi_1 B} = \mathbb{1}$. The left is topological and right is algebraic. We bridge from left to right by using compactness of paths to break $[\gamma]$ into finitely many pieces that lie within basic opens, where things are algebraic.

Let $[\gamma]$ be a morphism in $\Pi_1 B$. Since B is semi-locally simply connected, we have an open cover $\mathcal{U}$ of B such that every $b \in U$ is a centre of U. By compactness of $[0, 1]$, there exists morphisms $[\gamma_0], \ldots, [\gamma_n]$ in $\Pi_1 B$ such that $[\gamma] = [\gamma_n] \circ \cdots \circ [\gamma_0]$ and $[\gamma_i]$ is $\Pi_1 U_i$ for some $U_i \in \mathcal{U}$. Let $x \in \rho(s[\gamma]) = \rho(s[\gamma_0])$. By unique path lifting,

$$\left(\mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma_0]\right)(x) \in U_0\,[s[\gamma_0], x] \cap \rho(t[\gamma_0]) = U_0\,[t[\gamma_0], \rho[\gamma_0](x)] \cap \rho(t[\gamma_0]) = \{\rho[\gamma_0](x)\}$$

Thus $\left(\mathrm{Fib}_{\int_{\Pi_1 B} \rho}\right)[\gamma_0](x) = \rho[\gamma_0](x)$. The same goes for all i, giving

$$\mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma] = \mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma_n] \circ \cdots \circ \mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma_0] = \rho[\gamma_n] \circ \cdots \circ \rho[\gamma_0] = \rho[\gamma]$$

(2)$(\int_{\Pi_1 B} \circ \mathrm{Fib} \cong \mathbb{1})$ There's an obvious set isomorphism :

$$\int_{\Pi_1 B} \mathrm{Fib}_X \to X$$

We have a topological base of B consisting of centred opens, which *also* trivialise X, so X also has a topological base consisting of centred opens, which map isomorphically down to centred opens of B. Let us call such opens of X basic for duration of the rest of the proof. It suffices for preimage of basic opens of X to be open. Let U be a basic open of X and x a centre of U. Let $b \in B$ be the projection of x and U_0 the projection of U. The claim is that the preimage of U is $U_0[b, x]$. This is clear since

$$\begin{tikzcd}
U_0\,[b, x] \arrow[r] \arrow[dr, "\cong"'] & U \arrow[d, "\cong"'] \\
& U_0
\end{tikzcd}$$

□

3 Fundamental Theorem of Covering Spaces, Groups Version

Proposition – Decomposition to Path-Connected Components
Let $B \in \mathbf{Top}$ be locally path connected and semi-locally simply connected. Let $B = \bigsqcup_{B_i \in I} B_i$ where I is the set of path-connected components of B. Then

1. $\Pi_1 B \simeq \coprod_{B_i \in I}(\Pi_1 B_i)$.
2. we have a commutative square

$$\begin{tikzcd}
\mathbf{Cov}(B) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \Pi_1 B\mathbf{Set} \arrow[d, "\simeq"] \\
\prod_{B_i \in I} \mathbf{Cov}(B_i) \arrow[r, "\simeq"] & \prod_{B_i \in I} \Pi_1 B_i \mathbf{Set}
\end{tikzcd}$$

Proof. (1) ok. (2) The square commutes. By base change of covering space, we get the functor $\mathbf{Cov}(B) \to \prod_{B_i \in I} \mathbf{Cov}(B_i)$. For full and faithful, note that every $X \in \mathbf{Cov}(B)$ is isomorphic to $\coprod_{B_i \in I} X_i$ where X_i is the preimage of B_i. Essentially surjective is okay and this shows left vertical functor is equivalence. The other vertical equivalence and the bottom horizontal equivalence are formal.

□

Remark. The above justifies restricting to the case with the additional assumption of B path connected.

Proposition – Fundamental Theorem of Covering Spaces (Groups Version)
Let $B \in \mathbf{Top}$ be path connected. For $b \in B$, use $\pi_1(B, b)$ to refer to the single object category consisting of b and automorphisms in $\Pi_1 B$ as the only morphisms. Then for any $b \in B$, there's an equivalence of categories

$$\pi_1(B, b) \xrightarrow{\simeq} \Pi_1 B$$

which induces an equivalence of categories :

$$\Pi_1 B\mathbf{Set} \xrightarrow{\simeq} \pi_1(B, b)\mathbf{Set}$$

Thus, when B is path connected, locally path connected, semi-locally simply connected, we have equivalence of categories :

$$\begin{tikzcd}
\mathbf{Cov}(B) \arrow[r, "\mathrm{Fib}", "\simeq"'] \arrow[dr, "\mathrm{Fib}(b)"'] & \Pi_1 B\mathbf{Set} \arrow[d, "\simeq"] \\
& \pi_1(B, b)\mathbf{Set}
\end{tikzcd}$$

where

1. the equivalence restricts to an equivalence between path connected coverings of B and transitive $\pi_1(B, b)$-actions. We call $\mathrm{Fib}_X(b)$ a $\pi_1(B, b)$-*torsor*.

2. for $X \in \mathbf{Cov}(B)$ path connected and any $x \in \operatorname{Fib}_X(b)$, we have

$$\operatorname{Aut}_{\mathbf{Cov}(B)} X \cong \operatorname{Aut}_{\pi_1(B,b)} \operatorname{Fib}_X(b) \cong N\operatorname{Stab}(x)/\operatorname{Stab}(x)$$

In particular, $\operatorname{Aut}_{\mathbf{Cov}(B)}(X)$ acts transitively on $\operatorname{Fib}_X(b)$ if and only if $\operatorname{Stab}(x)$ is normal in $\pi_1(B,b)$.

Proof. The forgetful functor $\pi_1(B,b) \to \Pi_1 B$ is fully faithful and is essentially surjective because B is path connected. $\Pi_1 B\mathbf{Set} \simeq \pi_1(B,b)\mathbf{Set}$ is formal. For $\operatorname{Fib}(b) : \mathbf{Cov}(B) \to \pi_1(B,b)\mathbf{Set}$ being an equivalence, note that a composition of equivalences form an equivalence.

(*1*) Now let $X \in \mathbf{Cov}(B)$. X is path connected if and only if Fib_X is a transitive $\Pi_1 B$-action. This implies $\operatorname{Fib}_X(b)$ is a transitive $\pi_1(B,b)$-action. Conversely, B path connected implies that any two fibers $\operatorname{Fib}_X(b), \operatorname{Fib}_X(b_1)$ biject by some $\operatorname{Fib}_X[\gamma]$ with $[\gamma] \in \Pi_1 B(b, b_1)$ so $\operatorname{Fib}_X(b)$ being a transitive $\pi_1(B,b)$-action gives the desired result.

(*2*) Let $X \in \mathbf{Cov}(B)$ be path connected and $x \in \operatorname{Fib}_X(b)$. $\operatorname{Aut}_{\mathbf{Cov}(B)} X \cong \operatorname{Aut}_{\pi_1(B,b)} \operatorname{Fib}_X(b)$ comes from the equivalence of categories. For $\operatorname{Aut}_{\pi_1(B,b)} \operatorname{Fib}_X(b) \cong N\operatorname{Stab}(x)/\operatorname{Stab}(x)$ and the in particular, this is a general group theoretic fact :

Lemma (*Automorphisms of Non-empty Transitive Actions*).
Let $G \in \mathbf{Grp}$ and $X \in G\mathbf{Set}$ be transitive and $x \in X$. Then

1. $\operatorname{Aut}_{G\mathbf{Set}} X \cong N\operatorname{Stab}(x)/\operatorname{Stab}(x)$.
2. $\operatorname{Aut}_{G\mathbf{Set}} X$ *acts transitively on X if and only if $N\operatorname{Stab}(x) = G$.*

Proof. (*1*) Define :

$$N\operatorname{Stab}(x) \to \operatorname{Aut}_{G\mathbf{Set}} X, \sigma \mapsto (\sigma_0(x) \mapsto (\sigma_0 \circ \sigma)(x))$$

The group morphism is well-defined because $\operatorname{Stab}(x)$ is normal in $N\operatorname{Stab}(x)$. To show surjective, let $\varphi \in \operatorname{Aut}_{G\mathbf{Set}} X$. Since X is a transitive G-set, φ is determined by what it does to x. There exists $\sigma \in G$ with $\sigma(x) = \varphi(x)$. It suffices to show $\sigma \in N\operatorname{Stab}(x)$. This is true by a quick computation. It is clear that the kernel of the group morphism is $\operatorname{Stab}(x)$ so we have the desired group isomorphism.

(*2*) Assume $\operatorname{Aut}_{G\mathbf{Set}} X$ acts transitively on X. Let $\sigma \in G$. Then there exists $\sigma_0 \in N\operatorname{Stab}(x)$ such that $\sigma_0(x) = \sigma(x)$. It follows that $\sigma \in \sigma_0 \operatorname{Stab}(x) \subseteq N\operatorname{Stab}(x)$. The converse is clear. ■

□

Remark. The following is good for computing fundamental groups.

Proposition – Universal Cover

Let $B \in \mathbf{Top}$ be path connected, locally path connected, semi-locally simply connected. Let $b \in B$ and consider $\pi_1(B,b)$ with the obvious left action on itself. Then for $X \in \mathbf{Cov}(B)$, TFAE :

1. $\operatorname{Fib}_X(b)$ is a non-empty, transitive and free $\pi_1(B,b)$-action, in other words, $\operatorname{Fib}_X(b) \cong \pi_1(B,b)$ as $\pi_1(B,b)$-sets.

2. X is path connected and there exists $x \in \mathrm{Fib}_X(b)$ with $\pi_1(X, x)$ is trivial.

We call X *universal* when it satisfies any (and thus both) of the above. In this case,

$$\mathrm{Aut}_{\mathbf{Cov}(B)} X \cong \mathrm{Aut}_{\pi_1(B,b)} \pi_1(B, b) \cong \pi_1(B, b)$$

Such an X exists by the fundamental theorem.

Proof. $(1 \Leftrightarrow 2)$ $\pi_1(B, b)$ is a transitive $\pi_1(B, b)$-set if and only if X is path connected. For any $x \in \mathrm{Fib}_X(b)$, $\pi_1(X, x) \cong \mathrm{Stab}(x)$. The result follows.

□