Proposition – Monodromy Functor[a]

Let $B \in \mathbf{Top}$ be locally path connected and $X \in \mathbf{Cov}(B)$. (It follows that X is locally path connected.) Then the *monodromy functor of* X is defined as the $\Pi_1(B)$-set :

$$\begin{aligned} \mathrm{Fib}_X : \Pi_1(B) &\to \mathbf{Set} \\ b &\mapsto \downarrow^{-1} b \\ [\gamma] &\mapsto \mathrm{Fib}_X([\gamma]) : \downarrow^{-1} s(\gamma) \to \downarrow^{-1} t(\gamma), x \mapsto \gamma_x(1) \end{aligned}$$

where γ_x is any lift of any representative of $[\gamma]$, such that $\gamma_x(0) = x$. This gives rise to a functor :

$$\mathrm{Fib} : \mathbf{Cov}(B) \to \mathbf{Set}^{\Pi_1 B}$$

Furthermore, for each $X \in \mathbf{Cov}(B)$, we can recover the fundamental groupoid of X via

$$\int_{\Pi_1(B)} \mathrm{Fib}_X \simeq \Pi_1(X)$$

where the former is the category of elements of Fib_X. Hence

- X is path-connected if and only if Fib_X is transitive, i.e. for all $(b, x), (b_1, x_1) \in \int_{\Pi_1(B)} \mathrm{Fib}_X$, there exists a morphism $[\gamma]$ of $\Pi_1(B)$ such that $\mathrm{Fib}_X([\gamma])(x) = x_1$.
- for all $(b, x) \in \int_{\Pi_1(B)} \mathrm{Fib}_X$, $\downarrow \pi_1(X, x) = \mathrm{Stab}(x) \subseteq \pi_1(B, b)$.

[a] The following doesn't require coverings be surjective?

Proof. To define the $\Pi_1(B)$-action on fibres, we need to be able to lift paths uniquely *and* for path-homotopies to lift. We first prove paths lift uniquely.

Lemma (Unique Path Lifting).
Let $B \in \mathbf{Top}$, $X \in \mathbf{Cov}(B)$. Then X satisfies unique path lifting, *meaning for all commuting squares of the form :*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "0"'] & X \arrow[d] \\ I \arrow[r] \arrow[ur, dashed, "!"] & B \end{tikzcd}$$

there exists a unique morphism in the diagonal such that the whole diagram commutes. Such a diagonal morphism is called a lift *of the morphism $I \to B$.*

Proof. Let $\gamma : I \to B$ and $x \in X$ in the fibre over $\gamma(0)$. Then there exists a set $\mathcal{U}$ consisting of opens of B trivialising X such that $\gamma I \subseteq \bigcup \mathcal{U}$.

The idea is that each $U \in \mathcal{U}$ allows us to lift a part of γI and compactness of I allows for induction. Since I is compact, there exists a partition $\{0 = t_0 < \cdots < t_n = 1\}$ of I such that for each $t_i < t_n$,

$\gamma[t_i, t_{i+1}]$ is in some $U_i \in \mathcal{U}$. Suppose by induction we have a unique lift $\overline{\gamma_{n-1}} : [0, t_{n-1}] \to X$ of $\gamma : [0, t_{n-1}] \to B$. Let $U_n \in \mathcal{U}$ with $\gamma[t_{n-1}, t_n] \subseteq U_n$. Let $s_n : U_n \to X$ be a section such that $\overline{\gamma_{n-1}}(t_{n-1}) \in s_n U_n$. The define a lift $\overline{\gamma} : I \to X$ by patching together $\overline{\gamma_{n-1}}$ and $s_n \circ \gamma|_{[t_{n-1}, t_n]}$. This lift is unique because any lift $\tilde{\gamma} : I \to X$ of γ must restrict to a lift of $\gamma : [0, t_{n-1}] \to B$, and thus $\tilde{\gamma}|_{[0,t_{n-1}]} = \overline{\gamma_{n-1}}$ by uniqueness of $\overline{\gamma_{n-1}}$ and finally $\tilde{\gamma}$ must also agree with $\overline{\gamma}$ on $[t_{n-1}, t_n]$ since $\tilde{\gamma}[t_{n-1}, t_n] \subseteq s_n U_n$ and s_n is a homeomorphism onto its image. ■

For lifting homotopies, we prove a more general lemma :

Lemma (*Unique Homotopy Lifting*).
*Let $B \in$ **Top** be locally path connected and $X \in$ **Cov**(B). (It follows that X is also locally path connected.) Then X satisfies* unique homotopy lifting with respect to locally connected spaces, *meaning for all commuting squares*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\mathbb{1}_Y \times 0"'] & X \arrow[d] \\ Y \times I \arrow[r] \arrow[ur, dashed, "!"] & B \end{tikzcd}$$

where Y is locally connected, there exists a unique morphism in the diagonal such that the diagram commutes. Such a morphism is called a lift *of the morphism $Y \times I \to B$.*

Proof. (*Existence*) (*Local Lifts*) Let $y \in Y$. We show the existence of an open neighbourhood U_y of y with a lift $\overline{H}_y : V_y \times I \to X$ of $H : U_y \times I \to B$. Let $\mathcal{U}$ be an open cover of B trivialising X. Since B is locally path connected, we can WLOG assume $\mathcal{U}$ consists of path connected opens. Now, for every $t \in I$, there exists $\varepsilon_t > 0$ and V_t open neighbourhood of y such that $HV_t \times [t - \varepsilon_t, t + \varepsilon_t] \subseteq U_t$ for some $U_t \in \mathcal{U}$. By compactness of I, there exists a partition $\{0 = t_0 < \cdots < t_n = 1\}$ of I and open neighbourhoods (V_i) of y such that $HV_i \times [t_i, t_{i+1}] \subseteq U_i$ for some $U_i \in \mathcal{U}$. We can now take $V_y = \bigcap_i V_i$ as a single open neighbourhood of y such that $HV_y \times [t_i, t_{i+1}] \subseteq U_i$ for some $U_i \in \mathcal{U}$. We construct a lift $\overline{H}_y$ inductively. By local connectedness of Y, we can WLOG assume V_y is connected, which we will use. Suppose by induction we have a lift $\tilde{H} : V_y \times [0, t_{n-1}] \to X$ of $H : V_y \times [0, t_{n-1}] \to X$. Let $U_n \in \mathcal{U}$ that covers $HV_y \times [t_{n-1}, t_n]$. By connectedness of V_y, $\tilde{H}V_y \times t_{n-1}$ lies within the image of a section $s_n : U_n \to X$. Hence we can define a lift $\overline{H} : V_y \times I \to X$ by patching together $\tilde{H}$ and $s_n \circ H|_{V_y \times [t_{n-1}, t_n]}$.

(*Global Lift*) We have an open cover $\mathcal{Y}$ of Y and for each $V \in \mathcal{Y}$ a lift $\overline{H}_V : V \times I \to X$ of $H : V \times I \to B$. For any two $V, W \in \mathcal{Y}$, $\overline{H}_V$ and $\overline{H}_W$ both restrict to lifts of $V \times I \cap W \times I = (V \cap W) \times I$. But these give lifts of paths starting in $V \cap W$ and lifts of paths are unique by the previous lemma, so $\overline{H}_V$ and $\overline{H}_W$ agree on $V \times I \cap W \times I$. Thus these lifts patch togther to give a global lift $\overline{H} : Y \times I \to X$ of H.

(*Uniqueness*) Let $\overline{H}, \overline{H}_1 : Y \times I \to X$ be lifts of $H : Y \times I \to B$. Again, these restrict to lifts of paths $y \times I \to B$, which are unique by the previous lemma so $\overline{H} = \overline{H}_1$.

■

We can now define the $\Pi_1(B)$-action. Let $[\gamma]$ be a morphism in $\Pi_1(B)$. Choose a representative γ. Define

$$\mathrm{Fib}_X([\gamma]) : \downarrow^{-1} s([\gamma]) \to \downarrow^{-1} t([\gamma]) := x \mapsto \gamma_x(1)$$

where $\gamma_x : I \to X$ is the unique lift of γ with $\gamma_x(0) = x$. We now need to show this is independent of the choice of γ. Let γ^1 be another representative of $[\gamma]$. So we have a homotopy $H : I \times I \to B$ from γ to γ^1 that fixes endpoints. Since I is locally connected, we have a lift :

$$\begin{tikzcd} I \arrow[r, "\gamma_x"] \arrow[d, "\mathbb{1}_Y \times 0"'] & X \arrow[d] \\ I \times I \arrow[r, "H"] \arrow[ur, dashed, "\overline{H}"] & B \end{tikzcd}$$

We hope that $\overline{H}$ gives a path-homotopy. Well,

1. Restricted to $0 \times I$, $\overline{H}$ gives a lift of the constant point $s([\gamma])$. By uniqueness of path lifting, $\overline{H}$ must be constant along $0 \times I$. Similarly, $\overline{H}$ is the constant point $t([\gamma])$ along $1 \times I$.
2. Now restricted to $I \times 1$, $\overline{H}$ gives a lift of γ^1 starting at $s([\gamma])$. By uniquenss of path lifting, $\overline{H}$ must be γ^1_x along $I \times 1$.

Hence, $\overline{H}$ is indeed a homotopy from γ_x to γ^1_x fixing end points, i.e. $[\gamma_x] = [\gamma^1_x]$. In particular, $\gamma_x(1) = \gamma^1_X(1)$ so $\mathrm{Fib}_X([\gamma])(x)$ is well-defined.

(Fib) Let $f \in \mathbf{Cov}(Y, X)$. Then indeed for every morphism $[\gamma]$ in $\Pi_1(B)$, we have

$$\begin{tikzcd} \mathrm{Fib}_X(s([\gamma])) \arrow[r, "f"] \arrow[d, "\mathrm{Fib}_X([\gamma])"] & \mathrm{Fib}_Y(s([\gamma])) \arrow[d, "\mathrm{Fib}_Y([\gamma])"] \\ \mathrm{Fib}_X(t([\gamma])) \arrow[r, "f"] & \mathrm{Fib}_Y(t([\gamma])) \end{tikzcd}$$

since for every x in the fibre over the source of $[\gamma]$ and any lift γ_x of $[\gamma]$ starting at x, $f \circ \gamma_x$ is a lift of $[\gamma]$ starting at $f(x)$.

(*Furthermore*) We describe the functor $\int_{\Pi_1(B)} \mathrm{Fib}_X \to \Pi_1(X)$:

- for each object (b, x), map it to x.
- for each morphism $[\gamma] \in \int_{\Pi_1(B)} \mathrm{Fib}_X((b, x), (b_1, x_1))$, map it to $[\gamma_x]$ where γ_x is any lift of γ starting at x. We have seen this is well-defined and by the assumption of $\mathrm{Fib}_X([\gamma])(x) = x_1$, $[\gamma_x] \in \Pi_1(X)(x, x_1)$ indeed.
- Functoriality follows from uniqueness of path liftings.

The functor is clearly essentially surjective. Faithfulness comes from projecting paths back down $\Pi_1(X) \to \Pi_1(B)$. We have seen fullness.

(*Hence*) clear. $\square$

Lemma (*Characterisation of Semi-Locally Simply Connected*).
Let $B \in \mathbf{Top}$ locally path connected. Given an open U of B and $b \in U$, TFAE :

1. *U is path connected and the obvious morphism $\pi_1(U, b) \to \pi_1(B, b)$ is trivial.*
2. *The following being a bijection :*
$$b \downarrow \Pi_1 U \to U$$

When U and b satisfies any (and thus both) of the above, call b a centre *of U. If U satisfies the above* for some $b \in U$, *then call U a* centred *open.*[a]

Then TFAE :

1. *There exists an open cover $\mathcal{U}$ of B consisting of U such that every $b \in U$ is a centre of U.*
2. *For every $b \in B$, there is a neighbourhood base of opens U with b as a centre.*
3. *There exists a cover $\mathcal{U}$ of B consisting of centred opens.*

We say B is semi-locally simply connected *when it satisfies any (and thus all) of the above.*

[a] I made up this terminology to avoid repeating long phrases in the proof.

Proof. ($1 \Leftrightarrow 2$ *for U and b*) U being path connected corresponds to $b \downarrow \Pi_1 U \to U$ being surjective. It suffices to prove $\pi_1(U, b) \to \pi_1(B, b)$ trivial if and only if $b \downarrow \Pi_1 U \to U$ injective. Forwards, given two morphisms $[\gamma], [\gamma_1]$ in $\Pi_1 U$ with source at b and same target, $[\gamma]^{-1}[\gamma_1] \in \pi_1(U, b)$. Triviality of $\pi_1(U, b) \to \pi_1(B, b)$ implies $[\gamma] = [\gamma_1]$ as morphisms in $\Pi_1 B$, in particular in $\Pi_1 U$. The converse is easy.

Now for equivalent conditions of B semi-locally simply connected. ($1 \Rightarrow 2$) Use local path connectedness of B and functoriality of $\pi_1(-, b)$. ($2 \Rightarrow 3$) Obvious. ($3 \Rightarrow 1$) Let $\mathcal{U}$ be an open cover of B such that for all $U \in \mathcal{U}$, there exists a centre b of U. It suffices to show for other $b_1 \in U$, $\pi_1(U, b_1) \to \pi_1(B, b_1)$ is also trivial. By assumption, there exists $[\gamma] \in \Pi_1 U(b, b_1)$, so we have a commutative square

$$\begin{tikzcd}
\pi_1(U, b) \arrow[r, "1"] \arrow[d, "\cong"'] & \pi_1(X, b) \arrow[d, "\cong"] \\
\pi_1(U, b_1) \arrow[r] & \pi_1(X, b_1)
\end{tikzcd}$$

where the vertical maps are "conjugation" by $[\gamma]$. This proves the bottom horizontal morphism is trivial.

□

Proposition – Fundamental Theorem of Covering Spaces [a]

Let $B \in \mathbf{Top}$ be locally path connected and semi-locally simply connected. Consider the functor $\int_{\Pi_1 B} : \mathbf{Set}^{\Pi_1 B} \to \mathbf{Set} \downarrow B$ that sends X to its category of elements $\int_{\Pi_1 B} X$, which we then view as a set with a set-morphism down to B. Then

1. we can promote the $\int_{\Pi_1 B}$ to a functor $\mathbf{Set}^{\Pi_1 B} \to \mathbf{Cov}B$.
2. $\int_{\Pi_1 B}$ and Fib form an equivalence of categories

$$\mathbf{Cov}(B) \simeq \mathbf{Set}^{\Pi_1 B}$$

[a] As called on nLab page on covering space.

Proof. (1) Let $\rho \in \mathbf{Set}^{\Pi_1 B}$. By assumption, B has a topological base of centred opens. These will give the topology on $\int_{\Pi_1 B} \rho$ as well as the trivialising opens of the projection.

(1)(*Topology*) For $(b, x) \in \int_{\Pi_1 B} \rho$ and an open U of B centred at b, define

$$U[b, x] := \{\rho([\gamma](x) \mid [\gamma] \in b \downarrow \Pi_1 U\}$$

Then the set of these give a topological base because centred opens form a topological base for B. We topologise $\int_{\Pi_1 B} \rho$ using this topological base.

(1)(*Projection Topological*) To show $\int_{\Pi_1 B} \rho \to B$ is a morphism of topological spaces, since B has centred opens as a topological base, it suffices for each centred U, say at some b, to have preimage

$$\downarrow^{-1} U = \bigsqcup_{x \in \rho(b)} U[b, x]$$

The equality is straight forward from U being centred at b and disjoint union follows from $\Pi_1 U$ being a groupoid.

(1)(*Covering*) We saw already that $\downarrow^{-1} U = \bigsqcup_{x \in \rho(b)} U[b, x]$ as sets. Since $\int_{\Pi_1 B} \rho$ has topology generated by the $V[b, x]$'s, to show $U[b, x] \to U$ is a topological isomorphism, it suffices to show for all U with centres b and $x \in \rho(b)$, $U[b, x]$ maps to U bijectively. And it does because U is centred!

(1)(*Functorial*) Let $f \in \mathbf{Set}^{\Pi_1 B}(\rho, \rho_1)$. We need to show the induced $f_* : \int_{\Pi_1 B} \rho \to \int_{\Pi_1 B} \rho_1$ is a topological morphism over B. Suffices for basic opens $U[b, y]$ of $\int_{\Pi_1 B} \rho_1$ to have preimage

$$\downarrow^{-1} U[b, y] = \bigcup_{x \in f_b^{-1}(y)} U[b, x]$$

where $f_b : \rho(b) \to \rho_1(b)$. $\supseteq$ is clear. Let (b_1, x_1) be in the preimage of $U[b, y]$. By definition, there exists $[\gamma] \in \Pi_1 U(b, b_1)$ with $\rho[\gamma](y) = f_{b_1}(x_1)$. Then by naturality of f, we have

$$\begin{tikzcd}
\rho(b) \arrow[r, "f_b"] \arrow[d, "\rho[\gamma]"'] & \rho_1(b) \arrow[d, "\rho_1[\gamma]"] \\
\rho(b_1) \arrow[r, "f_{b_1}"] & \rho_1(b_1)
\end{tikzcd}$$

Since $\Pi_1 B$ is a groupoid, $\rho[\gamma]$ are isomorphisms of sets, so there exists $x \in \rho(b)$ with $\rho[\gamma](x) = x_1$ and $f_b(x) = y$, i.e. we have the other inclusion.

(2) ($\mathrm{Fib} \circ \int_{\Pi_1 B} \cong \mathbb{1}$) We know that $\mathrm{Fib}\left(\int_{\Pi_1 B} \rho\right)(b) = \rho(b)$ for $b \in \Pi_1 B$. So it suffices that for all morphisms $[\gamma]$ in $\Pi_1 B$,

$$\mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma] = \rho[\gamma]$$

which will actually show $\mathrm{Fib} \circ \int_{\Pi_1 B} = \mathbb{1}$. The left is topological and right is algebraic. We bridge from left to right by using compactness of paths to break $[\gamma]$ into finitely many pieces that lie within basic opens, where things are algebraic.

Let $[\gamma]$ be a morphism in $\Pi_1 B$. Since B is semi-locally simply connected, we have an open cover $\mathcal{U}$ of B such that every $b \in U$ is a centre of U. By compactness of $[0,1]$, there exists morphisms $[\gamma_0], \dots, [\gamma_n]$ in $\Pi_1 B$ such that $[\gamma] = [\gamma_n] \circ \cdots \circ [\gamma_0]$ and $[\gamma_i]$ is $\Pi_1 U_i$ for some $U_i \in \mathcal{U}$. Let $x \in \rho(s[\gamma]) = \rho(s[\gamma_0])$. By unique path lifting,

$$\left(\mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma_0]\right)(x) \in U_0\,[s[\gamma_0], x] \cap \rho(t[\gamma_0]) = U_0\,[t[\gamma_0], \rho[\gamma_0](x)] \cap \rho(t[\gamma_0]) = \{\rho[\gamma_0](x)\}$$

Thus $\left(\mathrm{Fib}_{\int_{\Pi_1 B} \rho}\right)[\gamma_0](x) = \rho[\gamma_0](x)$. The same goes for all i, giving

$$\mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma] = \mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma_n] \circ \cdots \circ \mathrm{Fib}_{\int_{\Pi_1 B} \rho}[\gamma_0] = \rho[\gamma_n] \circ \cdots \circ \rho[\gamma_0] = \rho[\gamma]$$

(2) ($\int_{\Pi_1 B} \circ \mathrm{Fib} \cong \mathbb{1}$) There's an obvious set isomorphism :

$$\int_{\Pi_1 B} \mathrm{Fib}_X \to X$$

We have a topological base of B consisting of centred opens, which *also* trivialise X, so X also has a topological base consisting of centred opens, which map isomorphically down to centred opens of B. Let us call such opens of X basic for duration of the rest of the proof. It suffices for preimage of basic opens of X to be open. Let U be a basic open of X and x a centre of U. Let $b \in B$ be the projection of x and U_0 the projection of U. The claim is that the preimage of U is $U_0[b,x]$. This is clear since

$$\begin{tikzcd} U_0\,[b,x] \arrow[r] \arrow[dr, "\cong"'] & U \arrow[d, "\cong"] \\ & U_0 \end{tikzcd}$$

□