**Definition – Fundamental Groupoid, Fundamental Group**

Let $B \in \mathbf{Top}$. Define the *fundamental groupoid of* $B$, $\Pi_1(B)$, to be following category :

- $\Pi_1(B)$ has $B$ as its collection objects.
- For $b, b_1 \in B$, $\Pi_1(B)(b, b_1)$ consists of paths from $b$ to $b_1$ up to path-homotopy.
- Composition is concatenation of paths up-to-homotopy. That this is well-defined is an unenlightening exercise.
- Associativity is an unenlightening exercise.
- For $b \in B$, $\mathbb{1}_b$ is the homotopy class of the constant path at $b$.

All morphisms of $\Pi_1 B$ are isomorphisms, i.e. $\Pi_1 B$ is indeed a groupoid. This gives a functor

$$\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}$$

For $b \in B$, use $\pi_1(B, b)$ to denote the group of automorphisms of $b$ in $\Pi_1(B)$. This is called the *fundamental group of* $B$ *at* $b$. This gives a functor :

$$\pi_1 : \mathbf{Top}* \to \mathbf{Grp}$$

**Definition – Covering**

Let $B \in \mathbf{Top}$. For $X \to B$ and $U \to B$ in $\mathbf{Top} \downarrow B$, we say $U$ *trivialises* $X$ when there exists discrete $X_U \in \mathbf{Top}$ satisfying the pullback diagram :

$$\begin{tikzcd}
X \arrow[d] & X_U \times U \arrow[l] \arrow[d] \\
B & U \arrow[l, "\supseteq"']
\end{tikzcd}$$

where the right vertical morphism is projection in the $U$-component. In this case, we call $X_U$ the *generic fibre over* $U$.

Define the *category of coverings of* $B$, $\mathbf{Cov}(B)$, as the full subcategory of $\mathbf{Top} \downarrow B$ consisting of $X \to B$ such that there exists a cover $\mathcal{U}$ of $B$ consisting of opens that trivialise $X$.[a] Objects of $\mathbf{Cov}(B)$ are called *coverings* of $B$. When considering $\mathbf{Cov}(B)$, $B$ is referred to as the *base space*.

[a] It is standard to require non-empty generic fibres, however I have seen no use of this in the theory so have chosen to not include it.

*Remark – Big Picture of Covering Spaces.* The actions of a group can tell a lot about the group. The main result of covering spaces is that given sufficiently nice base space $B$, covering spaces tell us everything about actions of the fundamental groupoid, that is to say, we have an equivalence of categories :

$$\mathbf{Cov}(B) \simeq \mathbf{Set}^{\Pi_1 B}$$

**Proposition – Permenance Properties**
The following are true :

- (Base Change) Consider a pullback diagram in **Top** :

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l] \arrow[d] \\ B & Y \arrow[l] \end{tikzcd}$$

  Then $X \to B$ covering implies $Z \to Y$ covering. The converse is false.
- (Composition) Let $X \to Y \to B$ in **Top**. Then $X \to Y$ and $Y \to B$ coverings imply $X \to B$ is a covering. The converse is false.

*Proof.* (*Base Change*) We have the following commutative cube :

$$\begin{tikzcd} & U \times X_U \arrow[dl, tail] \arrow[dd] & & V \times X_U \arrow[ll] \arrow[dl, tail] \arrow[dd] \\ X \arrow[dd] & & Z \arrow[ll] \arrow[dd] & \\ & U \arrow[dl, tail] & & V \arrow[ll] \arrow[dl, tail] \\ B & & Y \arrow[ll] & \end{tikzcd}$$

The faces that are pullback squares are :

- front, by assumption.
- bottom, which gives $V$ an open in $Y$.
- left, where $U$ is a trivialising open of $X$ over $B$ and $X_U$ is a discrete space, with $U \times X_U \to U$ being projection into first component.
- back, which follows easily. The projection $V \times X_U \to V$ is into the first component.

The left and back being pullback squares implies $V \times X_U$ is the pullback of $X, V$ over $B$, and together with $Z$ being the pullback of $X, Y$ over $B$, this implies $V \times X_U$ is the pullback of $Z, V$ over $Y$, i.e. isomorphic to the preimage of $V$ under $Z \to Y$. This gives $V$ as a trivialising open of $Z$ over $Y$. We can hence obtain a trivialising cover for $Z$ over $Y$.

(*Composition*) This is the diagram :

$$\begin{tikzcd}
X \arrow[d] & U \times X_U \times Y_U \arrow[l, tail] \arrow[d] \\
Y \arrow[d] & U \times Y_U \arrow[l, tail] \arrow[d] \\
B & U \arrow[l, tail]
\end{tikzcd}$$

The bottom square is the pullback square of $X$ over $B$ along a trivialising open $U$ of $Y$ over $B$. We can then shrink $U$ such that each $U \to U \times X_U \to Y$ is (isomorphic to) a trivializing open of $X$ over $Y$, hence the top square. Since $X_U, Y_U$ are discrete, $X_U \times Y_U$ is discrete as well. The "rectangle" is a pullback diagram because the inner two are. Thus $U$ is a trivialising open of $X$ over $B$. This gives an open cover of $B$ trivialising $X$.

$\square$