1 Basic Definitions and Galois Connection

Definition. Filter.

Let X be a set and $F \subseteq 2^X$. Then F is a *filter on* X when

1. $X \in F$.
2. For all $U \in F$ and $V \subseteq X$, $U \subseteq V \Rightarrow V \in F$.
3. For all $U, V \in F$, $U \cap V \in F$.

Let $Fil(X)$ denote the set of all filters on X.

F is called *proper* when $F \neq 2^X$, the largest filter in $Fil(X)$.

Definition. Image Filter, Preimage Filter.

Let $f : X \to Y$ be a map of sets, $F \in Fil(X)$, $G \in Fil(Y)$.

Then

$$fF := \{V \subseteq Y \mid \exists U \in F, fU \subseteq V\}$$
$$f^{-1}G := \left\{f^{-1}U \subseteq X \mid U \in G\right\}$$

Theorem. *Galois Connection of Filters.*

Let $f : X \to Y$ be a map of sets. Then

1. *For all $F_1, F_2 \in Fil(X)$, $F_1 \subseteq F_2 \Rightarrow fF_1 \subseteq fF_2$.*
2. *For all $G_1, G_2 \in Fil(Y)$, $G_1 \subseteq G_2 \Rightarrow f^{-1}G_1 \subseteq f^{-1}G_2$.*
3. *For $F \in Fil(X)$ and $G \in Fil(Y)$, $f^{-1}G \subseteq F \Leftrightarrow G \subseteq fF$.*

Proof. (3)($\Rightarrow$) Let $f^{-1}G \subseteq F$. Let $U \in G$. Then $ff^{-1}U \subseteq U$, where $f^{-1}U \in f^{-1}G \subseteq F$.

(3)($\Leftarrow$) Let $G \subseteq fF$. Let $U \in G$. Then there exists $V \in F$, $fV \subseteq U$. So $V \subseteq f^{-1}fV \subseteq f^{-1}U$ implies $f^{-1}U \in F$. $\square$