1 Constructions

Definition. Join of an arbitrary collection of Filters.

Let X be a set, $\mathbb{F} \subseteq Fil(X)$. Define the *join over* $\mathbb{F}$ as

$$\sqcup \mathbb{F} := \left\{ W \subseteq X \,|\, \exists\, I \text{ finite } \subseteq \bigcup_{F \in \mathbb{F}} F, \bigcap_{V \in I} V \subseteq W \right\}$$

This is the minimal filter containing all $F \in \mathbb{F}$.

Theorem. *Characterisation of Filters on Products.*

Let I be a set and for each $i \in I$, X_i a topological space, $x \in \prod_{i \in I} X_i$, $\pi_i : \prod_{i \in I} X_i \to X_i$ be the natural projection.

Then $N(x) = \sqcup \left\{ \pi_i^{-1} N(\pi_i(x)) \,|\, i \in I \right\}$. In particular, for all $F \in Fil\left(\prod_{i \in I} X_i\right)$, F converges to $x \Leftrightarrow$ for all $i \in I$, $\pi_i F$ converges to $\pi_i(x)$.

Proof. It suffices to show that $N(x)$ is the minimal filter containing all $\pi_i^{-1} N(\pi_i(x))$.

First note that for all $i \in I$, continuity of π_i implies $N(\pi_i(x)) \subseteq \pi_i N(x)$, which implies by the Galois connection of filters, $\pi_i^{-1} N(\pi_i(x)) \subseteq N(x)$.

Next, let $F \in Fil\left(\prod_{i \in I} X_i\right)$ such that all $\pi_i^{-1} N(\pi_i(x)) \subseteq F$. Let $W \in N(x)$. By definition, there exists $U \in Open(\prod_{i \in I} X_i)$, $x \in U \subseteq W$. Then by definition of the product topology, there exists finite $J \subseteq I$ and $(U_i) \in \prod_{i \in J} Open(X_i)$ such that $x \in \bigcap_{i \in J} \pi_i^{-1} U_i \subseteq U$. This implies for all $i \in J$, $\pi_i^{-1} U_i \in \pi_i^{-1} N(\pi_i(x))$. Since all $\pi_i^{-1} U_i \in N(x) \cap F$ and $\bigcap_{i \in J} \pi_i^{-1} U_i \subseteq W$, we have $W \in F$. This proves $N(x) = \sum_{i \in I} \pi_i^{-1} N(\pi_i(x))$.

Let F be a filter on the product. Then F converges to $x \Leftrightarrow N(x) \subseteq F \Leftrightarrow \forall\, i \in I, \pi_i^{-1} N(\pi_i(x)) \subseteq F. \Leftrightarrow \forall\, i \in I, N(\pi_i(x)) \subseteq \pi_i F \Leftrightarrow \forall\, i \in I, F$ converges to $\pi_i(x)$ by the Galois connection of filters. $\square$