Notes on Tychonoff's Theorem : 200cc

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Spring 2020

Contents

1 Basic Definitions and Galois Connection

Definition. Filter.

Let X be a set and $F \subseteq 2^X$. Then F is a *filter on* X when

1. $X \in F$.
2. For all $U \in F$ and $V \subseteq X$, $U \subseteq V \Rightarrow V \in F$.
3. For all $U, V \in F$, $U \cap V \in F$.

Let $Fil(X)$ denote the set of all filters on X.

F is called *proper* when $F \neq 2^X$, the largest filter in $Fil(X)$.

Definition. Image Filter, Preimage Filter.

Let $f : X \to Y$ be a map of sets, $F \in Fil(X)$, $G \in Fil(Y)$.

Then

$$fF := \{V \subseteq Y \mid \exists U \in F, fU \subseteq V\}$$
$$f^{-1}G := \left\{f^{-1}U \subseteq X \mid U \in G\right\}$$

Theorem. *Galois Connection of Filters.*

Let $f : X \to Y$ be a map of sets. Then

1. *For all $F_1, F_2 \in Fil(X)$, $F_1 \subseteq F_2 \Rightarrow fF_1 \subseteq fF_2$.*
2. *For all $G_1, G_2 \in Fil(Y)$, $G_1 \subseteq G_2 \Rightarrow f^{-1}G_1 \subseteq f^{-1}G_2$.*
3. *For $F \in Fil(X)$ and $G \in Fil(Y)$, $f^{-1}G \subseteq F \Leftrightarrow G \subseteq fF$.*

Proof. (3)($\Rightarrow$) Let $f^{-1}G \subseteq F$. Let $U \in G$. Then $ff^{-1}U \subseteq U$, where $f^{-1}U \in f^{-1}G \subseteq F$.

(3)($\Leftarrow$) Let $G \subseteq fF$. Let $U \in G$. Then there exists $V \in F$, $fV \subseteq U$. So $V \subseteq f^{-1}fV \subseteq f^{-1}U$ implies $f^{-1}U \in F$. $\square$

2 Continuity

Remark. Naive Characterisation of Continuity.

Let X, Y be topological spaces, $f : X \to Y$ a map of sets. Then the following are equivalent :

1. f continuous.
2. For all $x \in X$ and $\alpha : \mathbb{N} \to X$, α converges to $x \Rightarrow f \circ \alpha$ converges to $f(x)$.

Proposition. *Neighbourhood Filter.*

Let X be a topological space, $x \in X$, $Open(X)$ the set of opens of X.
Define the set of neighbourhoods of x *as*

$$N(x) := \{V \subseteq X \mid \exists U \in Open(X), x \in U \subseteq V\}$$

Then $N(x) \in Fil(X)$.

Proof. Clear. $\square$

Definition. Filter Converging to Point.

Let X be a topological space, $x \in X$, $F \in Fil(X)$. Then F *converges to* x when $N(x) \subseteq F$.

Remark. Motivation for Filter Convergence Definition.

Let X be a topological space, $x \in X$, $\alpha : \mathbb{N} \to X$. Define the *filter associated with* α as

$$F_\alpha := \{V \subseteq X \mid \exists\, n \in \mathbb{N}, \alpha\mathbb{N}_{\geq n} \subseteq V\}$$

Then α converges to x if and only if F_α converges to x.

Theorem. *Characterisation of Continuity.*

Let X, Y be topological spaces, $f : X \to Y$ a map of sets. Then the following are equivalent :

1. *f continuous.*

2. *For all $x \in X$ and $F \in Fil(X)$, F converges to $x \Rightarrow fF$ converges to $f(x)$.*
3. *For all $x \in X$, $fN(x)$ converges to $f(x)$.*

Proof. ($1 \Rightarrow 2$) Let $x \in X$, $F \in Fil(x)$, F converges to x. Let $V \in N(f(x))$. Then $f^{-1}V \in N(x) \subseteq F$ by continuity of f, and $ff^{-1}V \subseteq V$. So $N(f(x)) \subseteq fF$.

($2 \Rightarrow 3$) Clear.

($2 \Rightarrow 1$) Let $U \in Open(Y)$. Then for all $x \in f^{-1}U$, $fN(x)$ converges to $f(x)$ implies $U \in fN(x)$. So there exists $U_x \in N(x)$, $fU_x \subseteq U$. It follows that $f^{-1}U = \bigcup_{x \in f^{-1}U} U_x$, which is open. ☐

3 Constructions

Definition. Join of an arbitrary collection of Filters.

Let X be a set, $\mathbb{F} \subseteq Fil(X)$. Define the *join over* $\mathbb{F}$ as

$$\sqcup \mathbb{F} := \left\{ W \subseteq X \,|\, \exists I \text{ finite } \subseteq \bigcup_{F \in \mathbb{F}} F, \bigcap_{V \in I} V \subseteq W \right\}$$

This is the minimal filter containing all $F \in \mathbb{F}$.

Theorem. *Characterisation of Filters on Products.*

Let I be a set and for each $i \in I$, X_i a topological space, $x \in \prod_{i \in I} X_i$, $\pi_i : \prod_{i \in I} X_i \to X_i$ be the natural projection.

Then $N(x) = \sqcup \left\{ \pi_i^{-1} N(\pi_i(x)) \,|\, i \in I \right\}$. In particular, for all $F \in Fil\left(\prod_{i \in I} X_i\right)$, F converges to $x \Leftrightarrow$ for all $i \in I$, $\pi_i F$ converges to $\pi_i(x)$.

Proof. It suffices to show that $N(x)$ is the minimal filter containing all $\pi_i^{-1} N(\pi_i(x))$.

First note that for all $i \in I$, continuity of π_i implies $N(\pi_i(x)) \subseteq \pi_i N(x)$, which implies by the Galois connection of filters, $\pi_i^{-1} N(\pi_i(x)) \subseteq N(x)$.

Next, let $F \in Fil\left(\prod_{i \in I} X_i\right)$ such that all $\pi_i^{-1} N(\pi_i(x)) \subseteq F$. Let $W \in N(x)$. By definition, there exists $U \in Open(\prod_{i \in I} X_i)$, $x \in U \subseteq W$. Then by definition of the product topology, there exists finite $J \subseteq I$ and $(U_i) \in \prod_{i \in J} Open(X_i)$ such that $x \in \bigcap_{i \in J} \pi_i^{-1} U_i \subseteq U$. This implies for all $i \in J$, $\pi_i^{-1} U_i \in \pi_i^{-1} N(\pi_i(x))$. Since all $\pi_i^{-1} U_i \in N(x) \cap F$ and $\bigcap_{i \in J} \pi_i^{-1} U_i \subseteq W$, we have $W \in F$. This proves $N(x) = \sum_{i \in I} \pi_i^{-1} N(\pi_i(x))$.

Let F be a filter on the product. Then F converges to $x \Leftrightarrow N(x) \subseteq F \Leftrightarrow \forall\, i \in I, \pi_i^{-1} N(\pi_i(x)) \subseteq F. \Leftrightarrow \forall\, i \in I, N(\pi_i(x)) \subseteq \pi_i F \Leftrightarrow \forall\, i \in I, F$ converges to $\pi_i(x)$ by the Galois connection of filters. ☐

4 Closed Sets

Remark. Naive Characterisation of Closed Sets.

Let X be a topological space, $A \subseteq X$. Then for all $x \in X$, $x \in \overline{A} \Leftrightarrow x$ is a limit point of A.

Definition. Limit Points of a Subspace.

Let X be a topological space, $A \subseteq X$, $x \in X$. Let $\iota_A : A \to X$ be the natural inclusion, continuous by giving A the subspace topology. Then x is a *limit point of* A when there exists $F \in Fil(A)$ such that $F \neq 2^A$ and $\iota_A F$ converges to x.

Theorem. *Characterisation of Closed Sets.*

Let X be a topological space, $A \subseteq X$. Then for all $x \in X$, $x \in \overline{A} \Leftrightarrow x$ is a limit point of A.

Proof. Let $x \in X$. Then

$$\begin{aligned} x \in \overline{A} &\Leftrightarrow \forall C \in Closed(X), A \subseteq C \Rightarrow x \in C. \Leftrightarrow \forall U \in Open(X), x \in U \Rightarrow U \cap A \neq \varnothing. \\ &\Leftrightarrow \forall V \in N(x), V \cap A \neq \varnothing. \Leftrightarrow \varnothing \notin \iota_A^{-1} N(x). \Leftrightarrow \iota_A^{-1} N(x) \neq 2^A. \end{aligned}$$

So if $x \in \overline{A}$, by the Galois connection of filters, $\iota_A \iota_A^{-1} N$ converges to x. Conversely, if there exists $F \in Fil(A)$ such that $F \neq 2^A$ and $\iota_A F$ converges to x, then by the Galois connection of filters, $N(x) \subseteq \iota_A F \Rightarrow \iota_A^{-1} N(x) \subseteq \iota_A^{-1} \iota_A F \subseteq F \subsetneq 2^A$. ☐

5 Compactness

Remark. Naive Characterisation of Compactness.

Let X be a topological space. Then the following are equivalent :

1. X compact.
2. For all sequences α in X, there exists $x \in X$ such that x is a limit point of α.
3. For all sequences α in X, there exists a sequence β and $x \in X$, such that β is a subsequence of α and β converges to x.

Definition. Limit Point of a Filter.

Let X be a topological space, $F \in Fil(X)$, $x \in X$. Then x is a *limit point of* F when for all $V \in F$, x is a limit point of V.

Theorem. *Characterisation of Limit Points of a Filter.*

Let X be a topological space, $F \in Fil(X)$, $x \in X$. Then the following are equivalent :

1. *x is a limit point of F.*
2. $\sqcup \{N(x), F\} \neq 2^X$.
3. *There exists $G \in Fil(X)$ such that $F \subseteq G \neq 2^X$ and G converges to x.*

Proof. $(1 \Leftrightarrow 2)$

$$\sqcup \{N(x), F\} \neq 2^X \Leftrightarrow \varnothing \notin \sqcup \{N(x), F\} \Leftrightarrow \forall U \in N(x), \forall V \in F, U \cap V \neq \varnothing.$$
$$\Leftrightarrow \forall V \in F, x \in \overline{V}. \Leftrightarrow \forall V \in F, x \text{ limit point of } V. \Leftrightarrow x \text{ limit point of } F$$

$(2 \Leftrightarrow 3)$ Follows from definitions of convergence and join of filters. □

Theorem. *Characterisation of Compactness.*

Let X be a topological space. Then the following are equivalent :

1. *X compact.*
2. *For all $F \in Fil(X)$, $F \neq 2^X \Rightarrow$ there exists $x \in X$ such that x is a limit point of F.*
3. *For all $F \in Fil(X)$, $F \neq 2^X \Rightarrow$ there exists $G \in Fil(X)$ and $x \in X$ such that $F \subseteq G \neq 2^X$ and G converges to x.*

Proof. $(2 \Leftrightarrow 3)$ Follows from characterisation of limit points of a filter.

$(1 \Rightarrow 2)$ Let $F \neq 2^X \in Fil(X)$. Define $\overline{F} := \{\overline{V} \,|\, V \in F\}$. Then $F \neq 2^X$ implies all finite intersections of sets in $\overline{F}$ are non-empty. Since X is compact, we then have $\bigcap_{\overline{V} \in \overline{F}} \overline{V} \neq \varnothing$. Let $x \in \bigcap_{\overline{V} \in \overline{F}} \overline{V}$. Then for all $V \in F$, x is a limit point of V.

$(2 \Rightarrow 1)$ Let I be a set of closed subsets of X such that for all finite $J \subseteq I$, $\bigcap_{V \in J} V \neq \varnothing$. Let

$$F_I := \left\{ W \subseteq C \,|\, \exists\, J \text{ fin} \subseteq I, \bigcap_{V \in J} V \subseteq W \right\}$$

Then $F_I \neq 2^X \in Fil(X)$. So there exists $x \in X$ such that x is a limit point of F_I. Then for all $V \in I \subseteq F_I$, V closed $\Rightarrow x \in V$, i.e. $\cap_{V \in I} V \neq \varnothing$. □

Definition. Maximal filters.

Let X be a set and $F \in Fil(X)$. Then F is a *maximal* when $F \neq 2^X$ and for all $G \in Fil(X)$, $F \subseteq G \Rightarrow F = G$ or $G = 2^X$.

Lemma. *Existence of Maximal Filters.*

Let X be a set, $F \in Fil(X)$ and $F \neq 2^X$. Then there exists $G \in Fil(X)$ such that $F \subseteq G$ and G is maximal.

Proof. Standard application of Zorn's lemma. □

Corollary. *Maximal Filters Characterisation of Compactness.*

Let X be a topological space. Then X is compact $\Leftrightarrow$ for all $F \in Fil(X)$, F maximal implies the existence of $x \in X$ such that F converges to x.

Proof. (⇒) Let $F \in Fil(X)$ be maximal. Then by the characterisation of compactness, there exists $G \in Fil(X)$ and $x \in X$ such that $F \subseteq G \neq 2^X$ and G converges to x. By maximality of F, $F = G$.

(⇐) By the characterisation of compactness, it suffices to show that for all $F \in Fil(X)$, $F \neq 2^X$ implies the existence of $G \in Fil(X)$ and $x \in X$ such that $F \subseteq G \neq 2^X$ and G converges to x.

So let $F \in Fil(X)$ and $F \neq 2^X$. Then there exists $G \in Fil(X)$ such that $F \subseteq G$ and G is maximal. There exists $x \in X$ such that G converges to x and $G \neq 2^X$ by definition. □

Theorem. *Characterisation of Maximal filters.*

Let X be a set and $F \in Fil(X)$. Then the following are equivalent :

1. *F is maximal.*
2. *For all $A \subseteq X$, $A \in F$ or $X \setminus A \in F$.*

Proof. (1 ⇒ 2) Let $A \subseteq X$, $\iota_A : A \to X$. Consider $\sqcup \{\iota_A \{A\}, F\}$. Note that $F \subseteq \sqcup \{\iota_A \{A\}, F\}$, so by maximality of F, $F = \sqcup \{\iota_A \{A\}, F\}$ or $\sqcup \{\iota_A \{A\}, F\} = 2^X$. In the first case, by definition of join and $\iota_A \{A\}$, $A \in F$. In the second case, $\varnothing \in \sqcup \{\iota_A \{A\}, F\}$, so there exists $U \supseteq A$ and $V \in F$ such that $U \cap V = \varnothing$. In particular, $V \subseteq X \setminus A$, which implies $X \setminus A \in F$.

(2 ⇒ 1) Let $G \in Fil(X)$ and $F \subseteq G$. Suppose there exists $V \in G \setminus F$. Then $X \setminus V \in F \subseteq G$, which implies $\varnothing = V \cap (X \setminus V) \in G$, i.e. $G = 2^X$. □

Lemma. *Image of Maximal filters.*

Let X, Y be sets, $f : X \to Y$, $F \in Fil(X)$, F maximal filter. Then fF is an maximal filter of Y.

Proof. By the characterisation of maximal filters, it suffices for all $B \subseteq Y$, $B \in fF$ or $Y \setminus B \in fF$. Let $B \subseteq Y$. Then $f^{-1}B \in F$ or $X \setminus f^{-1}B \in F$. For the first case, $ff^{-1}B \subseteq B$ with $ff^{-1}B \in fF$, so $B \in fF$. In the latter case, $f(X \setminus f^{-1}B) \subseteq Y \setminus B$ with $f(X \setminus f^{-1}B) \in fF$, so $Y \setminus B \in fF$. □

Theorem. *Product of Compact is Compact (Tychonoff).*

Let I be a set and for $i \in I$, X_i a topological space. Then $\prod_{i \in I} X_i$ is compact ⇔ for all $i \in I$, X_i is compact.

Proof. (⇒) Image of compact is compact under continuous maps.

(⇐) By the maximal filters characterisation of compactness, it suffices that all maximal filters on $\prod_{i \in I} X_i$ converge. So let $F \in Fil\left(\prod_{i \in I} X_i\right)$ be maximal. Then for all $i \in I$, $\pi_i F \in Fil(X_i)$ is maximal. Since each X_i is compact, there exists $x_i \in X_i$ such that $\pi_i F$ converges to x_i. By the axiom of choice, $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$. By characterisation of filters on products, F converges to x. □