1 Closed Sets

Theorem. *Naive Characterisation of Closed Sets.*

Let X be a topological space, $A \subseteq X$. Then for all $x \in X$, $x \in \overline{A} \Leftrightarrow x$ is a "limit point" of A.

> **Definition.** Limit Points of a Subspace.
>
> Let X be a topological space, $A \subseteq X$, $x \in X$. Let $\iota_A : A \to X$ be the natural inclusion, continuous by giving A the subspace topology. Then x is a *limit point of* A when there exists $F \in Fil(A)$ such that $F \neq 2^A$ and $\iota_A F$ converges to x.

Theorem. *Characterisation of Closed Sets.*

Let X be a topological space, $A \subseteq X$. Then for all $x \in X$, $x \in \overline{A} \Leftrightarrow x$ is a limit point of A.

Proof. Let $x \in X$. Then

$$\begin{aligned} x \in \overline{A} &\Leftrightarrow \forall C \in Closed(X), A \subseteq C \Rightarrow x \in C. \Leftrightarrow \forall U \in Open(X), x \in U \Rightarrow U \cap A \neq \varnothing. \\ &\Leftrightarrow \forall V \in N(x), V \cap A \neq \varnothing. \Leftrightarrow \varnothing \notin \iota_A^{-1} N(x). \Leftrightarrow \iota_A^{-1} N(x) \neq 2^A. \end{aligned}$$

So if $x \in \overline{A}$, by the Galois connection of filters, $\iota_A \iota_A^{-1} N$ converges to x. Conversely, if there exists $F \in Fil(A)$ such that $F \neq 2^A$ and $\iota_A F$ converges to x, then by the Galois connection of filters, $N(x) \subseteq \iota_A F \Rightarrow \iota_A^{-1} N(x) \subseteq \iota_A^{-1} \iota_A F \subseteq F \subsetneq 2^A$. $\square$