Notes on Filters in Topology

Ken Lee

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Notes on some topological notions which are more nicely described using the language of filters.

Contents

1 Topological Space and Sequences

Definition – Topological Space

Let X be a set. A *topology on* X consists of the following data :

- (Set of Opens) a subset $\tau_X \subseteq \mathbf{SubSet}(X)$. Subsets U of X that are in τ_X are called *opens*.
- (Finite Intersection) For $I \subseteq \tau_X$ finite, $\bigcap_{U \in I} U \in \tau_X$. In particular, $X = \bigcap_{U \in \varnothing} U \in \tau_X$.
- (Arbitrary Union) For $I \subseteq \tau_X$, $\bigcup_{U \in I} U \in \tau_X$. In particular, $\varnothing = \bigcup_{U \in \varnothing} U \in \tau_X$.

A *topological space* is a set X together with a topology on it. We often write X instead of (X, τ_X) for a topological space.

For a sequence $a : \mathbb{N} \to X$ and $x \in X$, we write $a_n \to x$ and say a *converges to* x when for all $x \in U \subseteq X$ where U is open, there exists $N \in \mathbb{N}$ such that $a\mathbb{N}_{\geq N} \subseteq U$.

Remark – On the standard definition of a topological space.. The above definition is standard and can be motivated as the abstraction of opens in $\mathbb{R}^n$. However, this begs the question of why consider opens in $\mathbb{R}^n$. The idea of "getting close to a point" is arguably more intuitively captured by the notion of sequences converging. The view on *filters* that I adopt is that they are the generalisation of sequences for topological spaces.

$$\mathbb{R}^n \to \text{Topological Spaces}$$
$$\text{Sequences} \to \text{Filters}$$

Proposition – Filter of a Sequence

Let X be a set, $a : \mathbb{N} \to X$. For a subset $V \subseteq X$, we say *a converges to V* when there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \in V$. Let α be the set of subsets of X that a converges into. Then we have the following :

- (Universe) $X \in \alpha$.
- (Finite Intersection) For $U, V \in \alpha$, $U \cap V \in \alpha$.
- (Upwards Closed) For $U \subseteq V \subseteq X$, $U \in \alpha$ implies $V \in \alpha$.

α is called the *eventuality filter of a*. We will simply call it the *filter of a*.

Proof. Easy. □

Remark. For purposes of convergence, the only data of a sequence we really care about is which subsets they converge into. The filter of a sequence extracts this data from a sequence and one can think of a filter in general as abstracting this.

Definition – Filter

Let X be a set and $\alpha \subseteq 2^X$. Then α is a *filter on X* when

- (Universe) $X \in \alpha$.
- (Finite Intersection) For $U, V \in \alpha$, $U \cap V \in \alpha$.
- (Upwards Closed) For $U \subseteq V \subseteq X$, $U \in \alpha$ implies $V \in \alpha$.

For a filter α on X and $V \subseteq X$, we write $\alpha \to V$ and say *α converges into V* when $V \in \alpha$.

We will use $\mathrm{Fil}(X)$ denote the set of all filters on X. Define $\beta \to \alpha$ to mean $\beta \supseteq \alpha$, yielding a partial order $\to$ on $\mathrm{Fil}(X)$.

The minimal and maximal filters with respect to $\to$ are respectively the powerset of X and the filter $\{X\}$. We will denote these $\bot, \top$ and call them the *initial and terminal filter* respectively.

Remark – On the initial and terminal filter. Let X be a set and $\alpha \in \mathrm{Fil}(X)$. Then $\alpha = \bot$ if and only if $\alpha \to \varnothing$. You can thus think of $\bot$ as the "empty sequence".

On the other extreme, for any $A \subseteq X$, $\top \to A$ if and only if $A = X$. You can think of $\top$ as the "chaotic sequence", which "visits every subset of X but never converges into any".

Remark – On the Direction of Partial Order of Filters. If $b : \mathbb{N} \to X$ be a subsequence of a, then $\beta \to \alpha$ where α, β are filters of a, b respectively. So for general filters α, β, you can think of $\beta \to \alpha$ as saying "β is a subsequence of α". The next proposition says that for a point $x \in X$ where X is a topological space, there's is a "largest sequence converging to x".

Proposition – Neighbourhood Filter

Let X be a topological space, $x \in X$. For a subset $V \subseteq X$, we say V is a *neighbourhood of* x when there exists an open U of X such that $x \in U \subseteq V$. The *neighbourhood filter of* x, denoted $N(x)$, is then defined to be the set of neighbourhoods of x. We then have the following :

- $N(x)$ is a filter on X.
- For all sequences $a : \mathbb{N} \to X$, $a_n \to x$ if and only if $\alpha \supseteq N(x)$ where α is the filter of a.

Hence, for a filter α on X, we write $\alpha \to x$ and say α *converges to x* when $\alpha \supseteq N(x)$.

Proof. Easy. □

Remark. One may wonder if it is possible to give a topological space by choosing a neighbourhood filter at each point. The answer is yes.

Proposition – Topological Spaces by Neighbourhood Filters

Let X be a set. Define a *system of neighbourhood filters on* X by the following data :

- (The Neighbourhood Filters) A function $N : X \to \mathrm{Fil}(X)$
- (Centred) For each $x \in X$ and $U \in N(x)$, $x \in U$.
- (Mutual Neighbourhoods) For each $x \in X$ and $V \in N(x)$, there is a $U \subseteq V$ such that $x \in U$ and for all $y \in U$, $U \in N(y)$.

Let N be a system of neighbourhood filters on X and τ a topology in X. Consider the following constructions :

- Define the *topology associated to* N by declaring $U \subseteq X$ to be open when for all $x \in U$, $U \in N(x)$. Then this forms a topology on X.
- Define the *system of neighbourhood filters on* X *associated to* τ by assigning to each point x its neighbourhood filter as defined before. This is a system of neighbourhood filters on X.

Then the above two processes are inverses, yielding a bijection between systems of neighbourhood filters and topologies on X.

Proof. Easy. □

2 Continuity

Remark – Naive Characterisation of Continuity. Let X, Y be topological spaces, $f : X \to Y$ a map of sets. Suppose for a moment that we have the notion of what it means for f to be *continuous*. Then we would hope that the following are equivalent :

1. f is a continuous.
2. For all $x \in X$ and $\alpha : \mathbb{N} \to X$, $\alpha \to x \Rightarrow f \circ \alpha \to f(x)$.

This is essentially true; we need to replace α with filters. So clearly, we need a way of "taking the image of a filter". A nice thing about filters is that, whilst there is no canonical way of taking the *preimage of a sequence*, this is possible for filters.

Definition – Image Filter, Preimage Filter
Let $f : X \to Y$ be a map of sets, $\alpha \in \mathrm{Fil}(X)$, $\beta \in \mathrm{Fil}(Y)$. Then define the *image of* α and the *preimage of* β by

$$f\alpha := \{V \subseteq Y \mid \exists U \in \alpha, fU \subseteq V\}$$
$$f^{-1}\beta := \left\{f^{-1}U \subseteq X \mid U \in \beta\right\}$$

Proposition – Adjunction of Image and Preimage Filters
Let $f : X \to Y$ be a map of sets. Then

1. $f : \mathrm{Fil}(X) \to \mathrm{Fil}(Y)$ and $f^{-1} : \mathrm{Fil}(Y) \to \mathrm{Fil}(X)$ are order preserving.
2. For $\alpha \in \mathrm{Fil}(X)$ and $\beta \in \mathrm{Fil}(Y)$, $\alpha \to f^{-1}\beta \Leftrightarrow f\alpha \to \beta$.

Proof. Easy. □

Proposition – Characterisation of Continuity
Let X, Y be topological spaces, $f : X \to Y$ a map of sets. Then the following are equivalent :

1. (Standard) For all opens $U \subseteq Y$, $f^{-1}U$ is open in X.
2. (Filters) For all $x \in X$ and $\alpha \in \mathrm{Fil}(X)$, $\alpha \to x$ implies $f\alpha \to f(x)$.
3. (Neihbourhood Filters Suffice) For all $x \in X$, $fN(x) \to N(f(x))$.

We call f *continuous*, or a *morphism of topological spaces*, when it satisfies any (and thus all) of the above.

Proof. ($1 \Rightarrow 2$) Let $V \in N(f(x))$. Then $f^{-1}V \in N(x)$. Then $\alpha \to f^{-1}V$ implies $f\alpha \to ff^{-1}V$ implies $f\alpha \to V$.

($2 \Leftrightarrow 3$) Clear. ($3 \Rightarrow 1$) It suffices that for all $x \in f^{-1}U$, $f^{-1}U \in N(x)$. Well, for $x \in f^{-1}U$, $fN(x) \to N(f(x)) \to U$ implies $fN(x) \to U$, which implies $N(x) \to f^{-1}U$ as desired. □

Definition – Category of Topological Spaces
Define **Top**, the *category of topological spaces*, by setting :

- objects of **Top** to be topological spaces.
- for two $X, Y \in$ **Top**, the set of morphisms **Top**(X, Y) to be morphisms of topological spaces.
- for $X \in$ **Top**, $\mathbb{1}_X$ is automatically continuous.

– the composition of morphisms of topological spaces is easily seen to be another morphism of topological spaces.

3 Induced Topology

Remark – Motivation. Now that we have a category $\mathbf{Top}$, we hope to be able to construct new topological spaces from existing ones. For example, given $X, Y \in \mathbf{Top}$, we wish to give $X \times Y$ a suitable topology. Let $\downarrow_X \colon X \times Y \to X$, $\downarrow_X \colon X \times Y \to Y$ denote the two projections. Then we hope that for any filter α on $X \times Y$ and $p \in X \times Y$,

$$\alpha \to p \Leftrightarrow (\downarrow_X \alpha \to \downarrow_X p \text{ and } \downarrow_Y \alpha \to \downarrow_Y p)$$

But this is if and only if

$$\alpha \to \downarrow_X^{-1} N(\downarrow_X p) \text{ and } \downarrow_Y^{-1} N(\downarrow_Y p)$$

If we could "intersect" the two filters $\downarrow_X^{-1} N(\downarrow_X p), \downarrow_Y^{-1} N(\downarrow_Y p)$ then we can use that as the definition of $N(p)$ in $X \times Y$. This leads us to the following.

Definition – Meet of an arbitrary collection of Filters
Let X be a set, $F \subseteq \mathrm{Fil}(X)$. Define the *meet over* F as

$$\sqcap F := \left\{ W \subseteq X \mid \exists\, I \text{ finite } \subseteq \bigcup_{\alpha \in F} \alpha, \bigcap_{V \in I} V \subseteq W \right\}$$

This filter has the follow *universal property* : for all $\beta \in \mathrm{Fil}(X)$, $\beta \to \sqcap F$ if and only if for all $\alpha \in F$, $\beta \to \alpha$. In particular, $\sqcap\varnothing = \top$.

Proposition – Induced Topology
Let $(\downarrow_{X_i} \colon X \to X_i)_{i \in I}$ be a set of set-theoretic maps, where each $X_i \in \mathbf{Top}$. For $x \in X$, define

$$N(x) := \sqcap \left\{ \downarrow_{X_i}^{-1} N(\downarrow_{X_i} (x)) \mid i \in I \right\}$$

Then this defines a system of neighbourhood filters on X, making it a topological space with the following property : for all $\alpha \in \mathrm{Fil}\,(X)$, $\alpha \to x \Leftrightarrow$ for all $i \in I$, $\downarrow_{X_i} \alpha \to \downarrow_{X_i} x$.

In particular, for all set-theoretic $f : Y \to X$ where $Y \in \mathbf{Top}$, $f \in \mathbf{Top}(Y, X)$ if and only if for all $i \in I$, $\downarrow_{X_i} \circ f \in \mathbf{Top}(Y, X_i)$.

Proof. N is clearly centered and $N(x)$ is indeed a filter. To show existence of mutual neighbourhoods, let $W \in N(x)$. By definition, there exists $\{V_i\}_{i \in I_0}$ where $I_0 \subseteq I$ is finite and $V_i \in N(\downarrow_{X_i} x)$ with $W \supseteq \bigcap_{i \in I_0} \downarrow_{X_i}^{-1} V_i$. By definition of neighbourhood filters, we have $x \in U_i \subseteq V_i$ for some open U_i for every $i \in I_0$. It then follows that $x \in \bigcap_{i \in I_0} U_i \subseteq W$ and for all $y \in \bigcap_{i \in I_0} U_i$, $\bigcap_{i \in I_0} U_i \in N(y)$. The rest follows easily by design. $\square$

Remark. The above applies in particular to the arbitrary product of topological spaces, and making subsets into topological spaces. In these cases, the induced topology is respectively called the *product topology* and the *subspace* topology.

I hoped to dualize the above discussion by defining the *join* of an arbitrary collection of filters and use it to define the *co-induced topology* on X from a collection of maps $(X_i \to X)_{i \in I}$ where $X_i \in \mathbf{Top}$. In particular, it should be the case for a surjection $\pi : X_0 \to X$ where $X_0 \in \mathbf{Top}$, the co-induced topology is recovers the usual quotient topology. However, I was stuck on the following example :

$$\{0,1\} \overset{\subseteq}{\to} \{0,1,2\}$$

where all subsets of $\{0,1\}$ are open. I couldn't see an obvious way of defining the neighbourhood filter of 2, and could not find any sources on this.

4 Closed Sets

Remark – Naive Characterisation of Closed Sets. Let X be a topological space, $A \subseteq X$. Then the *closure of* A, $\overline{A}$, consists precisely of the limit points of A.

Definition – Limit Points of a Subset

Let X be a topological space, $A \subseteq X$, $x \in X$. Let $\uparrow_A : A \to X$ be the inclusion. Then x is a *limit point of* A when there exists $\alpha \in \mathrm{Fil}(A)$ such that $\perp \neq \alpha$ and $\uparrow_A \alpha$ converges to x.

Proposition – Characterisation of Closed Sets

Let X be a topological space, $A \subseteq X$. Then for all $x \in X$, TFAE :

- x is a limit point of A.
- $\uparrow_A^{-1} N(x) \neq \perp \in \mathrm{Fil}(A)$.
- $x \in \overline{A} := \bigcap_{A \subseteq C \text{ closed} \subseteq X} C$, the minimal closed subset of X containing A.

Proof. $(2 \Leftrightarrow 3)$ Easy.

$(1 \Leftrightarrow 2)$ $\exists \perp \neq \alpha \in \mathrm{Fil}(A), \uparrow_A \alpha \to x. \Leftrightarrow \exists \perp \neq \alpha \in \mathrm{Fil}(A), \alpha \to \uparrow_A^{-1} N(x). \Leftrightarrow \perp \neq \uparrow_A^{-1} N(x).$

□

5 Hausdorffness

Remark – Naive Characterisation of Hausdorffness. Let $X \in \mathbf{Top}$. Then points in X can be separated by opens if and only if for all sequences α in X and $x, x_1 \in X$, $\alpha \to x$ and $\alpha \to x_1$ implies $x = x_1$.

This does not work for $X \in \mathbf{Top}$ with more than one point and topology $\{\varnothing, X\}$.

Proposition – Characterisation of Hausdorff

Let $X \in \mathbf{Top}$. Then TFAE :

1. (Standard Definition) For $x, x_1 \in X$, $x \neq x_1$ implies the existence of $U \in N(x)$ and $U_1 \in N(x_1)$ with $U \cap U_1 = \varnothing$.
2. (Filters) For $x, x_1 \in X$ and $\alpha \in \mathrm{Fil}(X)$, $\perp \neq \alpha \to x$ and x_1 implies $x = x_1$.

3. (Neighbourhood Filter Suffices) For $x, x_1 \in X$, $N(x) \sqcap N(x_1) \neq \bot$ implies $x = x_1$.

Proof. ($1 \Leftrightarrow 3$) definition of meet of filters. ($2 \Leftrightarrow 3$) Easy. □

6 Compactness

Remark – Naive Characterisation of Compactness. Let X be a topological space. Then the following are equivalent :

1. X compact.
2. For all sequences α in X, there exists $x \in X$ such that x is a limit point of α.
3. For all sequences α in X, there exists a sequence β and $x \in X$, such that β is a subsequence of α and $\beta \to x$.

Note that given a chain of subsequences $\cdots \to \alpha_1 \to \alpha_0 \to \alpha$, it is not possible to have "α_∞". We will see that with filters, this is possible, i.e. we can have "sequences such that all subsequences are either empty or itself". Calling these "*minimal sequences*", we should have another equivalence :

4. For all "minimal sequences" α in X, there exists a point $x \in X$ such that $\alpha \to x$.

Definition – Limit Point of a Filter
Let X be a topological space, $\alpha \in \mathrm{Fil}(X)$, $x \in X$. Then x is a *limit point of* α when for all $V \in F$, x is a limit point of V.

Proposition – Characterisation of Limit Points of a Filter
Let X be a topological space, $\alpha \in \mathrm{Fil}(X)$, $x \in X$. Then TFAE :

1. x is a limit point of F.
2. (Converging Subfilter) There exists $\beta \in \mathrm{Fil}(X)$ such that $\bot \neq \beta \to \alpha$ and $\beta \to x$.
3. (Neighbourhood Filter Suffices) $\sqcap\{N(x), F\} \neq \bot$.

Proof. ($1 \Leftrightarrow 3$)

$$N(x) \sqcap \alpha \neq \bot \Leftrightarrow \varnothing \notin N(x) \sqcap \alpha \Leftrightarrow \forall U \in N(x), \forall V \in \alpha, U \cap V \neq \varnothing.$$
$$\Leftrightarrow \forall V \in \alpha, x \in \overline{V}. \Leftrightarrow x \text{ limit point of } \alpha$$

($2 \Leftrightarrow 3$) Follows from definitions of convergence and join of filters. □

Definition – Minimal Filters
Let X be a set and $\alpha \in \mathrm{Fil}(X)$. Then α is a *minimal* when $\bot \neq \alpha$ and for all $\beta \in \mathrm{Fil}(X)$, $\beta \to \alpha$

implies $\perp = \beta$ or $\beta = \alpha$. [a]

[a] In the literature, it is common to not use the dual partial order $\to$ as we have, resulting in these filters being *maximal* instead of minimal. Hence they are more commonly called *ultrafilters*.

Proposition – Characterisation of Minimal filters
Let X be a set and $\alpha \in \mathrm{Fil}(X)$. Then TFAE :

1. α is minimal.
2. ("α is decisive") For all $A \subseteq X$, $\alpha \to A$ or $\alpha \to X \setminus A$.

Proof. $(1 \Rightarrow 2)$ Let $A \subseteq X$, $\uparrow_A \colon A \to X$ the inclusion. Then $\uparrow_A \{A\} \sqcap \alpha \to \alpha$ so either $\perp =\uparrow_A \{A\} \sqcap \alpha$ or $\uparrow_A \{A\} \sqcap \alpha = \alpha$. In the first case, there exists $V \subseteq X$, $\alpha \to V$ with $A \cap V = \varnothing$. Hence $\alpha \to X \setminus A$. In the second case, $\alpha \to\uparrow_A \{A\}$ gives $\alpha \to A$.

$(2 \Rightarrow 1)$ Let $\beta \in \mathrm{Fil}(X)$ with $\beta \to \alpha$. Suppose $\beta \neq \alpha$, i.e. there exists $V \subseteq X$ such that $\beta \to V$ and $\alpha \not\to V$. Then $\alpha \to X \setminus V$ by assumption, whence $\beta \to V$ and $\beta \to X \setminus V$, i.e. $\perp = \beta$. $\square$

Proposition – Characterisation of Compactness
Let X be a topological space. Then TFAE :

1. (Standard Definition) for all $\mathcal{U} \subseteq \tau_X$, $X \subseteq \bigcup_{U \in \mathcal{U}} U$ implies there exists finite $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $X \subseteq \bigcup_{U \in \mathcal{U}_0} U$.
2. (Closed Variant of Standard Definition) for all sets I of closed subsets of X, if for all finite $I_0 \subseteq I$, $\bigcap_{V \in I_0} V \neq \varnothing$ then $\bigcap_{V \in I} V \neq \varnothing$.
3. (Limit Point of Filters) For all $\alpha \in \mathrm{Fil}(X)$, $\perp \neq \alpha$ implies there exists $x \in X$ such that x is a limit point of α.
4. (Convergent Subfilter) For all $\alpha \in \mathrm{Fil}(X)$, $\perp \neq \alpha$ implies there exists $\beta \in \mathrm{Fil}(X)$ and $x \in X$ such that $\perp \neq \beta \to \alpha$ and $\beta \to x$.
5. (Minimal Filters Convergent) For all minimal $\alpha \in \mathrm{Fil}(X)$, there exists $x \in X$, $\alpha \to x$.

X is called *compact* when any (and thus all) of the above are true.

Proof. $(1 \Leftrightarrow 2)$ Easy. $(2 \Rightarrow 3)$ Let $\perp \neq \alpha \in \mathrm{Fil}(X)$. Define $\overline{\alpha} := \left\{\overline{V} \mid \alpha \to V\right\}$. Then $\perp \neq \alpha$ implies all finite intersections of sets in $\overline{\alpha}$ are non-empty. Hence $\bigcap_{\overline{V} \in \overline{\alpha}} \overline{V} \neq \varnothing$, giving a limit point of α.

$(3 \Rightarrow 2)$ Let I be a set of closed subsets of X such that for all finite $J \subseteq I$, $\bigcap_{V \in J} V \neq \varnothing$. Take the "filter generated by I". Since every $V \in I$ is closed, limit points will be precisely elements of V in I.

$(3 \Leftrightarrow 4)$ By characterisation of limit points.

$(4 \Leftrightarrow 5)$ Forward is clear. For backwards, we use choice :

> *Lemma* (*Existence of Maximal Filters*). *Let X be a set, $\perp \neq \alpha_0 \in \mathrm{Fil}(X)$. Then there exists $\alpha \in \mathrm{Fil}(X)$ such that $\alpha \to \alpha_0$ and α is minimal.*

Proof. Standard application of Zorn's lemma. ■

□

Remark. The following is a (now) almost trivial proof of Tychonoff's theorem.

Proposition – Product of Compact is Compact (Tychonoff)
Let I be a set and for $i \in I$, X_i a topological space. Then $\prod_{i \in I} X_i$ is compact $\Leftrightarrow$ for all $i \in I$, X_i is compact.

Proof. ($\Rightarrow$)

Lemma (Image of compact is compact under continuous maps). Let $\pi \in \mathbf{Top}(X, Y)$. Then X compact implies πX compact (with subspace topology).

Proof. It follows from the induced topology that $\pi : X \to \pi X$ is continuous. Let $\alpha \in \mathrm{Fil}(\pi X)$ be minimal. Since $\pi : X \to \pi X$ is surjective, $\perp \neq \alpha$ implies $\perp \neq \varphi^{-1}\alpha$. By compactness of X, there exists $\beta \in \mathrm{Fil}(X)$ such that $\perp \neq \beta \to \pi^{-1}\alpha$ and a point $x \in X$ with $\beta \to x$. Then $\perp \neq \pi\beta \to \alpha$ and $\pi\beta \to \pi(x)$, where $\alpha = \pi\beta$ by minimality, giving a point in πX that α converges to as desired. ■

($\Leftarrow$) It suffices that all minimal filters on $\prod_{i \in I} X_i$ converge. So let $\alpha \in \mathrm{Fil}\left(\prod_{i \in I} X_i\right)$ be minimal. We hope that for all $i \in I$, $\pi_i F \in \mathrm{Fil}(X_i)$ is minimal so we can use compactness of the components. Indeed :

Lemma (Image of Minimal Filters). Let X, Y be sets, $f : X \to Y$, $\alpha \in \mathrm{Fil}(X)$ minimal. Then $f\alpha$ is a minimal filter of Y.

Proof. It suffices for all $B \subseteq Y$, $f\alpha \to B$ or $f\alpha \to Y \setminus B$. This is clear since any $B \subseteq Y$ partitions X by $f^{-1}B$ and $f^{-1}(Y \setminus B)$. ■

Then exists $x_i \in X_i$ such that $\downarrow_{X_i} \alpha$ converges to x_i where $\downarrow_{X_i}$ is projection into the i-th component. By the axiom of choice, $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$. Then by the induced topology, $\alpha \to x$. □