Definition – Topological Space

Let X be a set. A *topology on* X consists of the following data :

– (Set of Opens) a subset $\tau_X \subseteq \mathbf{SubSet}(X)$. Subsets U of X that are in τ_X are called *opens*.

– (Finite Intersection) For $I \subseteq \tau_X$ finite, $\bigcap_{U \in I} U \in \tau_X$. In particular, $X = \bigcap_{U \in \varnothing} U \in \tau_X$.

– (Arbitrary Union) For $I \subseteq \tau_X$, $\bigcup_{U \in I} U \in \tau_X$. In particular, $\varnothing = \bigcup_{U \in \varnothing} U \in \tau_X$.

A *topological space* is a set X together with a topology on it. We often write X instead of (X, τ_X) for a topological space.

For a sequence $a : \mathbb{N} \to X$ and $x \in X$, we write $a_n \to x$ and say a *converges to* x when for all $x \in U \subseteq X$ where U is open, there exists $N \in \mathbb{N}$ such that $a\mathbb{N}_{\geq N} \subseteq U$.

Remark – On the standard definition of a topological space.. The above definition is standard and can be motivated as the abstraction of opens in $\mathbb{R}^n$. However, this begs the question of why consider opens in $\mathbb{R}^n$. The idea of "getting close to a point" is arguably more intuitively captured by the notion of sequences converging. The view on *filters* that I adopt is that they are the generalisation of sequences for topological spaces.

$$\mathbb{R}^n \to \text{Topological Spaces}$$
$$\text{Sequences} \to \text{Filters}$$

Proposition – Filter of a Sequence

Let X be a set, $a : \mathbb{N} \to X$. For a subset $V \subseteq X$, we say *a converges to V* when there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \in V$. Let α be the set of subsets of X that a converges into. Then we have the following :

– (Universe) $X \in \alpha$.

– (Finite Intersection) For $U, V \in \alpha$, $U \cap V \in \alpha$.

– (Upwards Closed) For $U \subseteq V \subseteq X$, $U \in \alpha$ implies $V \in \alpha$.

α is called the *eventuality filter of a*. We will simply call it the *filter of a*.

Proof. Easy. □

Remark. For purposes of convergence, the only data of a sequence we really care about is which subsets they converge into. The filter of a sequence extracts this data from a sequence and one can think of a filter in general as abstracting this.

Definition – Filter

Let X be a set and $\alpha \subseteq 2^X$. Then α is a *filter on* X when

– (Universe) $X \in \alpha$.

– (Finite Intersection) For $U, V \in \alpha$, $U \cap V \in \alpha$.

– (Upwards Closed) For $U \subseteq V \subseteq X$, $U \in \alpha$ implies $V \in \alpha$.

For a filter α on X and $V \subseteq X$, we write $\alpha \to V$ and say α *converges into* V when $V \in \alpha$.

We will use $\mathrm{Fil}(X)$ denote the set of all filters on X. A filter α is called *proper* when $\alpha \neq \mathbf{SubSet}(X)$, the largest filter in $\mathrm{Fil}(X)$.

Remark – On the Direction of Partial Order of Filters. If $b : \mathbb{N} \to X$ be a subsequence of a, then $\beta \supseteq \alpha$ where α, β are filters of a, b respectively. So for general filters α, β, you can think of $\beta \supseteq \alpha$ as saying "β is a subsequence of α".

Now, for a point $x \in X$ where X is a topological space, there's is a "largest sequence converging to x".

Proposition – Neighbourhood Filter

Let X be a topological space, $x \in X$. For a subset $V \subseteq X$, we say V is a *neighbourhood of* x when there exists an open U of X such that $x \in U \subseteq V$. The *neighbourhood filter of* x, denoted $N(x)$, is then defined to be the set of neighbourhoods of x. We then have the following :

- $N(x)$ is a filter on X.
- For all sequences $a : \mathbb{N} \to X$, $a_n \to x$ if and only if $\alpha \supseteq N(x)$ where α is the filter of a.

Hence, for a filter α on X, we write $\alpha \to x$ and say α *converges to x* when $\alpha \supseteq N(x)$.

Proof. Easy. □

Remark. One may wonder if it is possible to give a topological space by choosing a neighbourhood filter at each point. The answer is yes.

Proposition – Topological Spaces by Neighbourhood Filters

Let X be a set. Define a *system of neighbourhood filters on* X by the following data :

- (The Neighbourhood Filters) A function $N : X \to \mathrm{Fil}(X)$
- (Centred) For each $x \in X$ and $U \in N(x)$, $x \in U$.
- (Mutual Neighbourhoods) For each $x \in X$, there is a $U \in N(x)$ such that for all $y \in U$, $U \in N(y)$.

Let N be a system of neighbourhood filters on X and τ a topology in X. Consider the following constructions :

- Define the *topology associated to* N by declaring $U \subseteq X$ to be open when for all $x \in U$, $U \in N(x)$. Then this forms a topology on X.
- Define the *system of neighbourhood filters on* X *associated to* τ by assigning to each point x its neighbourhood filter as defined before. This is a system of neighbourhood filters on X.

Then the above two processes are inverses, yielding a bijection between systems of neighbourhood filters and topologies on X.