1 Continuity

Remark. Naive Characterisation of Continuity.

Let X, Y be topological spaces, $f : X \to Y$ a map of sets. Then the following are equivalent :

1. f continuous.
2. For all $x \in X$ and $\alpha : \mathbb{N} \to X$, α converges to $x \Rightarrow f \circ \alpha$ converges to $f(x)$.

Proposition. *Neighbourhood Filter.*

Let X be a topological space, $x \in X$, $Open(X)$ the set of opens of X.
Define the set of neighbourhoods of x *as*

$$N(x) := \{V \subseteq X \mid \exists U \in Open(X), x \in U \subseteq V\}$$

Then $N(x) \in Fil(X)$.

Proof. Clear. □

> **Definition.** Filter Converging to Point.
>
> Let X be a topological space, $x \in X$, $F \in Fil(X)$. Then F *converges to* x when $N(x) \subseteq F$.

Remark. Motivation for Filter Convergence Definition.

Let X be a topological space, $x \in X$, $\alpha : \mathbb{N} \to X$. Define the *filter associated with* α as

$$F_\alpha := \{V \subseteq X \mid \exists n \in \mathbb{N}, \alpha\mathbb{N}_{\geq n} \subseteq V\}$$

Then α converges to x if and only if F_α converges to x.

Theorem. *Characterisation of Continuity.*

Let X, Y be topological spaces, $f : X \to Y$ a map of sets. Then the following are equivalent :

1. *f continuous.*
2. *For all $x \in X$ and $F \in Fil(X)$, F converges to $x \Rightarrow fF$ converges to $f(x)$.*
3. *For all $x \in X$, $fN(x)$ converges to $f(x)$.*

Proof. ($1 \Rightarrow 2$) Let $x \in X$, $F \in Fil(x)$, F converges to x. Let $V \in N(f(x))$. Then $f^{-1}V \in N(x) \subseteq F$ by continuity of f, and $ff^{-1}V \subseteq V$. So $N(f(x)) \subseteq fF$.

($2 \Rightarrow 3$) Clear.

($2 \Rightarrow 1$) Let $U \in Open(Y)$. Then for all $x \in f^{-1}U$, $fN(x)$ converges to $f(x)$ implies $U \in fN(x)$. So there exists $U_x \in N(x)$, $fU_x \subseteq U$. It follows that $f^{-1}U = \bigcup_{x \in f^{-1}U} U_x$, which is open. □