1 Continuity

Remark – Naive Characterisation of Continuity. Let X, Y be topological spaces, $f : X \to Y$ a map of sets. Then the following are equivalent :

1. f continuous.
2. For all $x \in X$ and $\alpha : \mathbb{N} \to X$, α converges to $x \Rightarrow f \circ \alpha$ converges to $f(x)$.

Definition – Image Filter, Preimage Filter
Let $f : X \to Y$ be a map of sets, $F \in \text{Fil}(X)$, $G \in \text{Fil}(Y)$.

Then

$$fF := \{V \subseteq Y \mid \exists U \in F, fU \subseteq V\}$$
$$f^{-1}G := \left\{f^{-1}U \subseteq X \mid U \in G\right\}$$

Theorem – Adjunction of Image and Preimage Filters
Let $f : X \to Y$ be a map of sets. Then

1. For all $F_1, F_2 \in \text{Fil}(X)$, $F_1 \subseteq F_2 \Rightarrow fF_1 \subseteq fF_2$.
2. For all $G_1, G_2 \in \text{Fil}(Y)$, $G_1 \subseteq G_2 \Rightarrow f^{-1}G_1 \subseteq f^{-1}G_2$.
3. For $F \in \text{Fil}(X)$ and $G \in \text{Fil}(Y)$, $f^{-1}G \subseteq F \Leftrightarrow G \subseteq fF$.

Proof. (3)($\Rightarrow$) Let $f^{-1}G \subseteq F$. Let $U \in G$. Then $ff^{-1}U \subseteq U$, where $f^{-1}U \in f^{-1}G \subseteq F$.

(3)($\Leftarrow$) Let $G \subseteq fF$. Let $U \in G$. Then there exists $V \in F$, $fV \subseteq U$. So $V \subseteq f^{-1}fV \subseteq f^{-1}U$ implies $f^{-1}U \in F$. $\square$

Theorem – Characterisation of Continuity
Let X, Y be topological spaces, $f : X \to Y$ a map of sets. Then the following are equivalent :

1. f continuous.
2. For all $x \in X$ and $F \in Fil(X)$, F converges to $x \Rightarrow fF$ converges to $f(x)$.
3. For all $x \in X$, $fN(x)$ converges to $f(x)$.

Proof. ($1 \Rightarrow 2$) Let $x \in X$, $F \in Fil(x)$, F converges to x. Let $V \in N(f(x))$. Then $f^{-1}V \in N(x) \subseteq F$ by continuity of f, and $ff^{-1}V \subseteq V$. So $N(f(x)) \subseteq fF$.

($2 \Rightarrow 3$) Clear.

($2 \Rightarrow 1$) Let $U \in Open(Y)$. Then for all $x \in f^{-1}U$, $fN(x)$ converges to $f(x)$ implies $U \in fN(x)$. So there exists $U_x \in N(x)$, $fU_x \subseteq U$. It follows that $f^{-1}U = \bigcup_{x \in f^{-1}U} U_x$, which is open. $\square$