# 1 Compactness

**Theorem.** *Naive Characterisation of Compactness.*

*Let $X$ be a topological space, $C \subseteq X$. Then the following are equivalent :*

1. *$C$ compact.*
2. *For all sequences $\alpha$ in $C$, there exists $x \in C$ such that $x$ is a limit point of $\alpha$.*
3. *For all sequences $\alpha$ in $C$, there exists a sequence $\beta$ and $x \in C$, such that $\beta$ is a subsequence of $\alpha$ and $\beta$ converges to $x$.*

> **Definition.** Limit Point of a Filter.
>
> Let $X$ be a topological space, $F \in Fil(X)$, $x \in X$. Then $x$ is a *limit point of $F$* when for all $V \in F$, $x$ is a limit point of $V$.

**Theorem.** *Characterisation of Limit Points of a Filter.*

*Let $X$ be a topological space, $F \in Fil(X)$, $x \in X$. Then the following are equivalent :*

1. *$x$ is a limit point of $F$.*
2. *$\sqcup \{N(x), F\} \neq 2^X$.*
3. *There exists $G \neq 2^X \in Fil(X)$ such that $F \subseteq G$ and $G$ converges to $x$.*

*Proof.* $(1 \Leftrightarrow 2)$

$$\begin{aligned}\sqcup \{N(x), F\} \neq 2^X \Leftrightarrow \varnothing \notin \sqcup \{N(x), F\} \Leftrightarrow \forall U \in N(x), \forall V \in F, U \cap V \neq \varnothing. \\ \Leftrightarrow \forall V \in F, x \in \overline{V}. \Leftrightarrow \forall V \in F, x \text{ limit point of } V. \Leftrightarrow x \text{ limit point of } F\end{aligned}$$

$(2 \Leftrightarrow 3)$ Follows from definitions of convergence and join of filters. □

**Theorem.** *Characterisation of Compactness.*

*Let $X$ be a topological space, $C \subseteq X$. Then the following are equivalent :*

1. *$C$ compact.*
2. *For all $F \neq 2^C \in Fil(C)$, there exists $x \in C$ such that $x$ is a limit point of $F$.*
3. *For all $F \neq 2^C \in Fil(C)$, there exists $G \neq 2^C \in Fil(C)$ and $x \in C$ such that $F \subseteq G$ and $G$ converges to $x$.*

*Proof.* $(2 \Leftrightarrow 3)$ Follows from characterisation of limit points of a filter.

$(1 \Rightarrow 2)$ Let $F \neq 2^C \in Fil(C)$. Define $\overline{F} := \left\{\overline{V} \mid V \in F\right\}$, where closure is taken in $C$. Then $F \neq 2^C$ implies all finite intersections of sets in $\overline{F}$ are non-empty. Since $C$ is compact, we then have $\bigcap_{\overline{V} \in \overline{F}} \overline{V} \neq \varnothing$. Let $x \in \bigcap_{\overline{V} \in \overline{F}} \overline{V}$. Then for all $V \in F$, $x$ is a limit point of $V$.

(2 $\Rightarrow$ 1) Let $I$ be a set of closed subsets of $C$ such that for all finite $J \subseteq I$, $\bigcap_{V \in J} V \neq \varnothing$. Let

$$F_I := \left\{ W \subseteq C \mid \exists\, J \text{ fin} \subseteq I, \bigcap_{V \in J} V \subseteq W \right\}$$

Then $F_I \neq 2^C \in Fil(C)$. So there exists $x \in C$ such that $x$ is a limit point of $F_I$. Then for all $V \in I \subseteq F_I$, $V$ closed $\Rightarrow x \in V$, i.e. $\cap_{V \in I} V \neq \varnothing$. $\square$

> **Definition.** Maximal filters.
>
> Let $X$ be a set and $F \in Fil(X)$. Then $F$ is a *maximal* when $F \neq 2^X$ and for all $G \in Fil(X)$, $F \subseteq G \Rightarrow F = G$ or $G = 2^X$.

**Lemma.** *Existence of Maximal Filters.*

*Let $X$ be a set, $F \in Fil(X)$ and $F \neq 2^X$. Then there exists $G \in Fil(X)$ such that $F \subseteq G$ and $G$ is maximal.*

*Proof.* Standard application of Zorn's lemma. $\square$

**Corollary.** *Maximal Filters Characterisation of Compactness.*

*Let $X$ be a topological space. Then $X$ is compact $\Leftrightarrow$ for all $F \in Fil(X)$, $F$ maximal implies the existence of $x \in X$ such that $F$ converges to $x$.*

*Proof.* ($\Rightarrow$) Let $F \in Fil(X)$ be maximal. Then by the characterisation of compactness, there exists $G \in Fil(X)$ and $x \in X$ such that $F \subseteq G \neq 2^X$ and $G$ converges to $x$. By maximality of $F$, $F = G$.

($\Leftarrow$) By the characterisation of compactness, it suffices to show that for all $F \in Fil(X)$, $F \neq 2^X$ implies the existence of $G \in Fil(X)$ and $x \in X$ such that $F \subseteq G \neq 2^X$ and $G$ converges to $x$.

So let $F \in Fil(X)$ and $F \neq 2^X$. Then there exists $G \in Fil(X)$ such that $F \subseteq G$ and $G$ is maximal. There exists $x \in X$ such that $G$ converges to $x$ and $G \neq 2^X$ by definition. $\square$

**Theorem.** *Characterisation of Maximal filters.*

*Let $X$ be a set and $F \in Fil(X)$. Then the following are equivalent :*

1. *$F$ is maximal.*
2. *For all $A \subseteq X$, $A \in F$ or $X \setminus A \in F$.*

*Proof.* (1 $\Rightarrow$ 2) Let $A \subseteq X$, $\iota_A : A \to X$. Consider $\sqcup \{\iota_A \{A\}, F\}$. Note that $F \subseteq \sqcup \{\iota_A \{A\}, F\}$, so by maximality of $F$, $F = \sqcup \{\iota_A \{A\}, F\}$ or $\sqcup \{\iota_A \{A\}, F\} = 2^X$. In the first case, by definition of join and $\iota_A \{A\}$, $A \in F$. In the second case, $\varnothing \in \sqcup \{\iota_A \{A\}, F\}$, so there exists $U \supseteq A$ and $V \in F$ such that $U \cap V = \varnothing$. In particular, $V \subseteq X \setminus A$, which implies $X \setminus A \in F$.

(2 $\Rightarrow$ 1) Let $G \in Fil(X)$ and $F \subseteq G$. Suppose there exists $V \in G \setminus F$. Then $X \setminus V \in F \subseteq G$, which implies $\varnothing = V \cap (X \setminus V) \in G$, i.e. $G = 2^X$. $\square$

**Lemma.** *Image of Maximal filters.*

*Let $X, Y$ be sets, $f : X \to Y$, $F \in Fil(X)$, $F$ maximal filter. Then $fF$ is an maximal filter of $Y$.*

*Proof.* By the characterisation of maximal filters, it suffices for all $B \subseteq Y$, $B \in fF$ or $Y \setminus B \in fF$. Let $B \subseteq Y$. Then $f^{-1}B \in F$ or $X \setminus f^{-1}B \in F$. For the first case, $ff^{-1}B \subseteq B$ with $ff^{-1}B \in fF$, so $B \in fF$. In the latter case, $f(X \setminus f^{-1}B) \subseteq Y \setminus B$ with $f(X \setminus f^{-1}B) \in fF$, so $Y \setminus B \in fF$. $\square$

**Theorem.** *Product of Compact is Compact (Tychonoff).*

*Let $I$ be a set and for $i \in I$, $X_i$ a topological space. Then $\prod_{i \in I} X_i$ is compact $\Leftrightarrow$ for all $i \in I$, $X_i$ is compact.*

*Proof.* ($\Rightarrow$) Image of compact is compact under continuous maps.

($\Leftarrow$) By the maximal filters characterisation of compactness, it suffices that all maximal filters on $\prod_{i \in I} X_i$ converge. So let $F \in Fil\left(\prod_{i \in I} X_i\right)$ be maximal. Then for all $i \in I$, $\pi_i F \in Fil(X_i)$ is maximal. Since each $X_i$ is compact, there exists $x_i \in X_i$ such that $\pi_i F$ converges to $x_i$. By the axiom of choice, $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$. By characterisation of filters on products, $F$ converges to $x$. $\square$