# Notes on Filters in Topology

Ken Lee

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## Contents



## 1 Basic Definitions and Galois Connection

> **Definition – Filter**
>
> Let $X$ be a set and $F \subseteq 2^X$. Then $F$ is a *filter on* $X$ when
>
> 1. $X \in F$.
> 2. For all $U \in F$ and $V \subseteq X$, $U \subseteq V \Rightarrow V \in F$.
> 3. For all $U, V \in F$, $U \cap V \in F$.
>
> Let $Fil(X)$ denote the set of all filters on $X$.
>
> $F$ is called *proper* when $F \neq 2^X$, the largest filter in $Fil(X)$.

> **Definition – Image Filter, Preimage Filter**
>
> Let $f : X \to Y$ be a map of sets, $F \in Fil(X)$, $G \in Fil(Y)$.
>
> Then
>
> $$fF := \{V \subseteq Y \mid \exists U \in F, fU \subseteq V\}$$
> $$f^{-1}G := \{f^{-1}U \subseteq X \mid U \in G\}$$

**Theorem – Adjunction of Image and Preimage Filters**
Let $f : X \to Y$ be a map of sets. Then

1. For all $F_1, F_2 \in Fil(X)$, $F_1 \subseteq F_2 \Rightarrow fF_1 \subseteq fF_2$.
2. For all $G_1, G_2 \in Fil(Y)$, $G_1 \subseteq G_2 \Rightarrow f^{-1}G_1 \subseteq f^{-1}G_2$.
3. For $F \in Fil(X)$ and $G \in Fil(Y)$, $f^{-1}G \subseteq F \Leftrightarrow G \subseteq fF$.

*Proof.* (3)($\Rightarrow$) Let $f^{-1}G \subseteq F$. Let $U \in G$. Then $ff^{-1}U \subseteq U$, where $f^{-1}U \in f^{-1}G \subseteq F$.

(3)($\Leftarrow$) Let $G \subseteq fF$. Let $U \in G$. Then there exists $V \in F$, $fV \subseteq U$. So $V \subseteq f^{-1}fV \subseteq f^{-1}U$ implies $f^{-1}U \in F$. □

# 2 Continuity

*Remark – Naive Characterisation of Continuity.* Let $X, Y$ be topological spaces, $f : X \to Y$ a map of sets. Then the following are equivalent :

1. $f$ continuous.
2. For all $x \in X$ and $\alpha : \mathbb{N} \to X$, $\alpha$ converges to $x \Rightarrow f \circ \alpha$ converges to $f(x)$.

**Proposition – Neighbourhood Filter**
Let $X$ be a topological space, $x \in X$, $Open(X)$ the set of opens of $X$.
Define the set of *neighbourhoods of* $x$ as

$$N(x) := \{V \subseteq X \mid \exists U \in Open(X), x \in U \subseteq V\}$$

Then $N(x) \in Fil(X)$.

*Proof.* Clear. □

**Definition – Filter Converging to Point**
Let $X$ be a topological space, $x \in X$, $F \in Fil(X)$. Then $F$ *converges to* $x$ when $N(x) \subseteq F$.

*Remark.* Motivation for Filter Convergence Definition.

Let $X$ be a topological space, $x \in X$, $\alpha : \mathbb{N} \to X$. Define the *filter associated with* $\alpha$ as

$$F_\alpha := \{V \subseteq X \mid \exists n \in \mathbb{N}, \alpha\mathbb{N}_{\geq n} \subseteq V\}$$

Then $\alpha$ converges to $x$ if and only if $F_\alpha$ converges to $x$.

**Theorem – Characterisation of Continuity**
Let $X, Y$ be topological spaces, $f : X \to Y$ a map of sets. Then the following are equivalent :

1. $f$ continuous.
2. For all $x \in X$ and $F \in Fil(X)$, $F$ converges to $x \Rightarrow fF$ converges to $f(x)$.

3. For all $x \in X$, $fN(x)$ converges to $f(x)$.

*Proof.* $(1 \Rightarrow 2)$ Let $x \in X$, $F \in Fil(x)$, $F$ converges to $x$. Let $V \in N(f(x))$. Then $f^{-1}V \in N(x) \subseteq F$ by continuity of $f$, and $ff^{-1}V \subseteq V$. So $N(f(x)) \subseteq fF$.

$(2 \Rightarrow 3)$ Clear.

$(2 \Rightarrow 1)$ Let $U \in Open(Y)$. Then for all $x \in f^{-1}U$, $fN(x)$ converges to $f(x)$ implies $U \in fN(x)$. So there exists $U_x \in N(x)$, $fU_x \subseteq U$. It follows that $f^{-1}U = \bigcup_{x \in f^{-1}U} U_x$, which is open. □

# 3 Constructions

**Definition – Join of an arbitrary collection of Filters**

Let $X$ be a set, $\mathbb{F} \subseteq Fil(X)$. Define the *join over* $\mathbb{F}$ as

$$\sqcup \mathbb{F} := \left\{ W \subseteq X \mid \exists I \text{ finite } \subseteq \bigcup_{F \in \mathbb{F}} F, \bigcap_{V \in I} V \subseteq W \right\}$$

This is the minimal filter containing all $F \in \mathbb{F}$.

**Theorem – Characterisation of Filters on Products**

Let $I$ be a set and for each $i \in I$, $X_i$ a topological space, $x \in \prod_{i \in I} X_i$, $\pi_i : \prod_{i \in I} X_i \to X_i$ be the natural projection.

Then $N(x) = \sqcup \left\{ \pi_i^{-1} N(\pi_i(x)) \mid i \in I \right\}$. In particular, for all $F \in Fil\left(\prod_{i \in I} X_i\right)$, $F$ converges to $x \Leftrightarrow$ for all $i \in I$, $\pi_i F$ converges to $\pi_i(x)$.

*Proof.* It suffices to show that $N(x)$ is the minimal filter containing all $\pi_i^{-1} N(\pi_i(x))$.

First note that for all $i \in I$, continuity of $\pi_i$ implies $N(\pi_i(x)) \subseteq \pi_i N(x)$, which implies by the Galois connection of filters, $\pi_i^{-1} N(\pi_i(x)) \subseteq N(x)$.

Next, let $F \in Fil\left(\prod_{i \in I} X_i\right)$ such that all $\pi_i^{-1} N(\pi_i(x)) \subseteq F$. Let $W \in N(x)$. By definition, there exists $U \in Open(\prod_{i \in I} X_i)$, $x \in U \subseteq W$. Then by definition of the product topology, there exists finite $J \subseteq I$ and $(U_i) \in \prod_{i \in J} Open(X_i)$ such that $x \in \bigcap_{i \in J} \pi_i^{-1} U_i \subseteq U$. This implies for all $i \in J$, $\pi_i^{-1} U_i \in \pi_i^{-1} N(\pi_i(x))$. Since all $\pi_i^{-1} U_i \in N(x) \cap F$ and $\bigcap_{i \in J} \pi_i^{-1} U_i \subseteq W$, we have $W \in F$. This proves $N(x) = \sum_{i \in I} \pi_i^{-1} N(\pi_i(x))$.

Let $F$ be a filter on the product. Then $F$ converges to $x \Leftrightarrow N(x) \subseteq F \Leftrightarrow \forall\, i \in I, \pi_i^{-1} N(\pi_i(x)) \subseteq F. \Leftrightarrow \forall\, i \in I, N(\pi_i(x)) \subseteq \pi_i F \Leftrightarrow \forall\, i \in I, F$ converges to $\pi_i(x)$ by the Galois connection of filters. □

# 4 Closed Sets

*Remark – Naive Characterisation of Closed Sets.* Let $X$ be a topological space, $A \subseteq X$. Then for all $x \in X$, $x \in \overline{A} \Leftrightarrow x$ is a limit point of $A$.

**Definition – Limit Points of a Subspace**

Let $X$ be a topological space, $A \subseteq X$, $x \in X$. Let $\iota_A : A \to X$ be the natural inclusion, continuous by giving $A$ the subspace topology. Then $x$ is a *limit point of* $A$ when there exists $F \in Fil(A)$ such that $F \neq 2^A$ and $\iota_A F$ converges to $x$.

**Theorem – Characterisation of Closed Sets**

Let $X$ be a topological space, $A \subseteq X$. Then for all $x \in X$, $x \in \overline{A} \Leftrightarrow x$ is a limit point of $A$.

*Proof.* Let $x \in X$. Then

$$x \in \overline{A} \Leftrightarrow \forall C \in Closed(X), A \subseteq C \Rightarrow x \in C. \Leftrightarrow \forall U \in Open(X), x \in U \Rightarrow U \cap A \neq \varnothing.$$
$$\Leftrightarrow \forall V \in N(x), V \cap A \neq \varnothing. \Leftrightarrow \varnothing \notin \iota_A^{-1} N(x). \Leftrightarrow \iota_A^{-1} N(x) \neq 2^A.$$

So if $x \in \overline{A}$, by the Galois connection of filters, $\iota_A \iota_A^{-1} N$ converges to $x$. Conversely, if there exists $F \in Fil(A)$ such that $F \neq 2^A$ and $\iota_A F$ converges to $x$, then by the Galois connection of filters, $N(x) \subseteq \iota_A F \Rightarrow \iota_A^{-1} N(x) \subseteq \iota_A^{-1} \iota_A F \subseteq F \subsetneq 2^A$. □

# 5 Compactness

*Remark – Naive Characterisation of Compactness.* Let $X$ be a topological space. Then the following are equivalent :

1. $X$ compact.
2. For all sequences $\alpha$ in $X$, there exists $x \in X$ such that $x$ is a limit point of $\alpha$.
3. For all sequences $\alpha$ in $X$, there exists a sequence $\beta$ and $x \in X$, such that $\beta$ is a subsequence of $\alpha$ and $\beta$ converges to $x$.

**Definition – Limit Point of a Filter**

Let $X$ be a topological space, $F \in Fil(X)$, $x \in X$. Then $x$ is a *limit point of* $F$ when for all $V \in F$, $x$ is a limit point of $V$.

**Theorem – Characterisation of Limit Points of a Filter**

Let $X$ be a topological space, $F \in Fil(X)$, $x \in X$. Then the following are equivalent :

1. $x$ is a limit point of $F$.
2. $\sqcup \{N(x), F\} \neq 2^X$.
3. There exists $G \in Fil(X)$ such that $F \subseteq G \neq 2^X$ and $G$ converges to $x$.

*Proof.* $(1 \Leftrightarrow 2)$

$$\sqcup \{N(x), F\} \neq 2^X \Leftrightarrow \varnothing \notin \sqcup \{N(x), F\} \Leftrightarrow \forall U \in N(x), \forall V \in F, U \cap V \neq \varnothing.$$
$$\Leftrightarrow \forall V \in F, x \in \overline{V}. \Leftrightarrow \forall V \in F, x \text{ limit point of } V. \Leftrightarrow x \text{ limit point of } F$$

$(2 \Leftrightarrow 3)$ Follows from definitions of convergence and join of filters. $\square$

**Theorem – Characterisation of Compactness**

Let $X$ be a topological space. Then the following are equivalent :

1. $X$ compact.
2. For all $F \in Fil(X)$, $F \neq 2^X \Rightarrow$ there exists $x \in X$ such that $x$ is a limit point of $F$.
3. For all $F \in Fil(X)$, $F \neq 2^X \Rightarrow$ there exists $G \in Fil(X)$ and $x \in X$ such that $F \subseteq G \neq 2^X$ and $G$ converges to $x$.

*Proof.* $(2 \Leftrightarrow 3)$ Follows from characterisation of limit points of a filter.

$(1 \Rightarrow 2)$ Let $F \neq 2^X \in Fil(X)$. Define $\overline{F} := \{\overline{V} \mid V \in F\}$. Then $F \neq 2^X$ implies all finite intersections of sets in $\overline{F}$ are non-empty. Since $X$ is compact, we then have $\bigcap_{\overline{V} \in \overline{F}} \overline{V} \neq \varnothing$. Let $x \in \bigcap_{\overline{V} \in \overline{F}} \overline{V}$. Then for all $V \in F$, $x$ is a limit point of $V$.

$(2 \Rightarrow 1)$ Let $I$ be a set of closed subsets of $X$ such that for all finite $J \subseteq I$, $\bigcap_{V \in J} V \neq \varnothing$. Let

$$F_I := \left\{ W \subseteq C \mid \exists\, J \text{ fin} \subseteq I, \bigcap_{V \in J} V \subseteq W \right\}$$

Then $F_I \neq 2^X \in Fil(X)$. So there exists $x \in X$ such that $x$ is a limit point of $F_I$. Then for all $V \in I \subseteq F_I$, $V$ closed $\Rightarrow x \in V$, i.e. $\cap_{V \in I} V \neq \varnothing$. $\square$

**Definition – Maximal filters**

Let $X$ be a set and $F \in Fil(X)$. Then $F$ is a *maximal* when $F \neq 2^X$ and for all $G \in Fil(X)$, $F \subseteq G \Rightarrow F = G$ or $G = 2^X$.

*Lemma (Existence of Maximal Filters). Let $X$ be a set, $F \in Fil(X)$ and $F \neq 2^X$. Then there exists $G \in Fil(X)$ such that $F \subseteq G$ and $G$ is maximal.*

*Proof.* Standard application of Zorn's lemma. $\square$

**Corollary – Maximal Filters Characterisation of Compactness**

Let $X$ be a topological space. Then $X$ is compact $\Leftrightarrow$ for all $F \in Fil(X)$, $F$ maximal implies the existence of $x \in X$ such that $F$ converges to $x$.

*Proof.* $(\Rightarrow)$ Let $F \in Fil(X)$ be maximal. Then by the characterisation of compactness, there exists $G \in Fil(X)$ and $x \in X$ such that $F \subseteq G \neq 2^X$ and $G$ converges to $x$. By maximality of $F$, $F = G$.

$(\Leftarrow)$ By the characterisation of compactness, it suffices to show that for all $F \in Fil(X)$, $F \neq 2^X$ implies the existence of $G \in Fil(X)$ and $x \in X$ such that $F \subseteq G \neq 2^X$ and $G$ converges to $x$.

So let $F \in Fil(X)$ and $F \neq 2^X$. Then there exists $G \in Fil(X)$ such that $F \subseteq G$ and $G$ is maximal. There exists $x \in X$ such that $G$ converges to $x$ and $G \neq 2^X$ by definition. $\square$

**Theorem – Characterisation of Maximal filters**

Let $X$ be a set and $F \in Fil(X)$. Then the following are equivalent :

1. $F$ is maximal.
2. For all $A \subseteq X$, $A \in F$ or $X \setminus A \in F$.

*Proof.* ($1 \Rightarrow 2$) Let $A \subseteq X$, $\iota_A : A \to X$. Consider $\sqcup \{\iota_A \{A\}, F\}$. Note that $F \subseteq \sqcup \{\iota_A \{A\}, F\}$, so by maximality of $F$, $F = \sqcup \{\iota_A \{A\}, F\}$ or $\sqcup \{\iota_A \{A\}, F\} = 2^X$. In the first case, by definition of join and $\iota_A \{A\}$, $A \in F$. In the second case, $\varnothing \in \sqcup \{\iota_A \{A\}, F\}$, so there exists $U \supseteq A$ and $V \in F$ such that $U \cap V = \varnothing$. In particular, $V \subseteq X \setminus A$, which implies $X \setminus A \in F$.

($2 \Rightarrow 1$) Let $G \in Fil(X)$ and $F \subseteq G$. Suppose there exists $V \in G \setminus F$. Then $X \setminus V \in F \subseteq G$, which implies $\varnothing = V \cap (X \setminus V) \in G$, i.e. $G = 2^X$. □

*Lemma (Image of Maximal filters). Let $X, Y$ be sets, $f : X \to Y$, $F \in Fil(X)$, $F$ maximal filter. Then $fF$ is an maximal filter of $Y$.*

*Proof.* By the characterisation of maximal filters, it suffices for all $B \subseteq Y$, $B \in fF$ or $Y \setminus B \in fF$. Let $B \subseteq Y$. Then $f^{-1}B \in F$ or $X \setminus f^{-1}B \in F$. For the first case, $ff^{-1}B \subseteq B$ with $ff^{-1}B \in fF$, so $B \in fF$. In the latter case, $f(X \setminus f^{-1}B) \subseteq Y \setminus B$ with $f(X \setminus f^{-1}B) \in fF$, so $Y \setminus B \in fF$. □

**Theorem – Product of Compact is Compact (Tychonoff)**

Let $I$ be a set and for $i \in I$, $X_i$ a topological space. Then $\prod_{i \in I} X_i$ is compact $\Leftrightarrow$ for all $i \in I$, $X_i$ is compact.

*Proof.* ($\Rightarrow$) Image of compact is compact under continuous maps.

($\Leftarrow$) By the maximal filters characterisation of compactness, it suffices that all maximal filters on $\prod_{i \in I} X_i$ converge. So let $F \in Fil\left(\prod_{i \in I} X_i\right)$ be maximal. Then for all $i \in I$, $\pi_i F \in Fil(X_i)$ is maximal. Since each $X_i$ is compact, there exists $x_i \in X_i$ such that $\pi_i F$ converges to $x_i$. By the axiom of choice, $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$. By characterisation of filters on products, $F$ converges to $x$. □