*Remark.* A scheme will be a "space modelled on $\mathbb{M}$ with an open cover by affine schemes". Since every "space" in $\mathbb{M}\mathbf{Set}$ is modelled on $\mathbb{M}$, we first define the notion of open covers for affine schemes, then define it for general $X \in \mathbb{M}\mathbf{Set}$.

We define closed subfunctors of affine schemes first, then define open ones as "complements" of closed.

> **Definition – Closed Subfunctors of Affine Schemes**
>
> Let $X \in \mathbf{Aff}$. For $I \in \operatorname{Ideal} \mathcal{O}(X)$, define the *vanishing of $I$*, $V(I) \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$, by
>
> $$A \in \mathbb{M}^{op} \mapsto \left\{ \varphi \in \mathbf{Aff}(\operatorname{Sp} A, X) \mid I \subseteq \ker \varphi^\flat \right\}$$
>
> In particular, for $I = (f)$, we write $V(f)$ instead of $V((f))$. A subfunctor $Z$ of $X$ is called *closed* when there exists $I \in \operatorname{Ideal} \mathcal{O}(X)$ such that $Z = V(I)$.

> **Proposition – Basic Facts about Closed Subfunctors**
>
> Let $X \in \mathbf{Aff}$. Then the following are true :
>
> – (Closed Subfunctors are Affine) For $I \in \operatorname{Ideal} \mathcal{O}(X)$, $V(I)$ is representable by $\mathcal{O}(X)/I$ and $I = \ker(\mathcal{O}(X) \to \mathcal{O}(V(I)))$. Hence for any closed subfunctor $Z$ of $X$, $Z$ is an affine scheme and $Z = V(I(Z))$ where $I(Z) := \ker(\mathcal{O}(X) \to \mathcal{O}(Z))$ is called the *ideal of definition of $Z$*.
>
> – (Ideals, Vanishing Adjunction) For $I, J \in \operatorname{Ideal} \mathcal{O}(X)$, $I \subseteq J$ if and only if $V(J) \subseteq V(I)$.
>
> – (Arbitrary Intersection of Closed) Let $\mathcal{X}$ be a collection of closed subfunctors of $X$. Then $\bigcap_{Z \in \mathcal{X}} Z = V(\sum_{Z \in \mathcal{X}} I(Z))$.
>
> – (Base Change / "Preimage of Closed are Closed") Let $\varphi \in \mathbf{Aff}(Y, X)$, $I \in \operatorname{Ideal} \mathcal{O}(X)$. Let the following be a pullback diagram :
>
> $$\begin{tikzcd} Y \arrow[r, "\varphi"] & X \\ \varphi^{-1}V(I) \arrow[u] \arrow[r] & V(I) \arrow[u] \end{tikzcd}$$
>
> Then $\varphi^{-1}V(I) = V(\mathcal{O}(Y)\varphi^\flat I)$.

> **Proposition – Open Subfunctors of Affine Schemes**
>
> Let $X \in \mathbf{Aff}$. Then the following are true :
>
> – (Open Subfunctors) For $I \in \operatorname{Ideal} \mathcal{O}(X)$, the following two subfunctors of $X$ are equal :
>
> – ("Complement of $V(I)$") Define $X \setminus V(I)$ by setting $(X \setminus V(I))(A)$ to be the set of $\alpha \in \mathbf{Aff}(\operatorname{Sp} A, X)$ such that $\alpha^{-1}V(I) = \operatorname{Sp} 0$, the "empty affine scheme".
>
> – ("Support of $I$") Define $D(I)$ by setting $D(I)(A)$ to be the set of $\alpha \in \mathbf{Aff}(\operatorname{Sp} A, X)$ such that $A = A\alpha^\flat I$.

A subfunctor $U$ of $X$ is called *open* when there exists $I \in \operatorname{Ideal} \mathcal{O}(X)$ such that $U = D(I) = X \setminus V(I)$.

We will use $\operatorname{Open} X$ to denote the full subcategory of $\mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$ consisting of open subfunctors of $X$.

- (Ideal, Open Adjunction) Let $I, J \in \operatorname{Ideal} \mathcal{O}(X)$. Then $D(I) \subseteq D(J)$ if and only if $\sqrt{I} \subseteq \sqrt{J}$. In particular, for $U \in \operatorname{Open} X$, there exists a unique $I_U \in \operatorname{Ideal} \mathcal{O}(X)$ such that $\sqrt{I_U} = I_U$ and $U = D(I_U)$.
- ("Fields given enough points for Opens") For any $Y \in \mathbb{M}\mathbf{Set}$, define $Y^{\mathrm{Pts}}$ to the restriction of $Y$ to the full subcategory $\mathrm{Pts}$ of $\mathbb{M}^{op}$ consisting of $K^{op}$ where $K$ is a field. We will call $Y^{\mathrm{Pts}}$ the *points of* $Y$.

  Then for $U, V$ opens of $X$, $U = V$ if and only if $U^{\mathrm{Pts}} = V^{\mathrm{Pts}}$.
- For $\mathcal{U} \subseteq \operatorname{Open} X$, define the *open union of* $\mathcal{U}$ to be

  $$\mathring{\bigcup}_{U \in \mathcal{U}} U := X \setminus \bigcap_{U \in \mathcal{U}} V(I_U) = D\left(\sum_{U \in \mathcal{U}} I_U\right)$$

  Then for $\mathcal{U} \subseteq \operatorname{Open} X$,

  - (Choice) for any $\{J_U\}_{U \in \mathcal{U}} \subseteq \operatorname{Ideal} \mathcal{O}(X)$ where each $U = D(J_U)$, we have $\mathring{\bigcup}_{U \in \mathcal{U}} U = D(\sum_{U \in \mathcal{U}} J_U)$.
  - ("Open Cover") TFAE :

    1. (Intuitive) $\mathring{\bigcup}_{U \in \mathcal{U}} U = X$.
    2. (Partition of Unity) $\mathcal{O}(X) = \sum_{U \in \mathcal{U}} J_U$ where $\{J_U\}_{U \in \mathcal{U}}$ is any collection of ideals of $\mathcal{O}(X)$ such that $U = D(J_U)$ for each $U$.
    3. (Surjective on Points) For all $x : \operatorname{Sp} K \to X$ where $K$ is a field, there exists $U \in \mathcal{U}$ such that $x$ factors through $U$.

    We say $\mathcal{U}$ *covers* $X$ when any (and thus all) of the above are true.
- (Base Change / "Preimage of Opens are Open") Let $\varphi \in \mathbf{Aff}(Y, X)$, $I \in \operatorname{Ideal} \mathcal{O}(X)$. Let the following be a pullback diagram :

  $$\begin{tikzcd} Y \arrow[r, "\varphi"] & X \\ \varphi^{-1} D(I) \arrow[u] \arrow[r] & D(I) \arrow[u] \end{tikzcd}$$

  Then $\varphi^{-1} D(I) = D(\varphi^\flat I)$. Hence the pullback of opens along morphisms of affine schemes remain open.

– (Fiber Product) Let $X, X_1 \in \mathbf{Aff}/Y$ where $Y \in \mathbf{Aff}$. Let $I, I_1$ be ideals of $\mathcal{O}(X)$, $\mathcal{O}(X_1)$ respectively. Then we have the pullback diagram :

$$\begin{array}{ccc} D(I) & \longrightarrow & Y \\ \uparrow & & \uparrow \\ D(I \otimes_{\mathcal{O}(Y)} I_1) & \longrightarrow & D(I_1) \end{array}$$

In particular, $X = X_1 = Y$ implies the intersection of two opens of $X$ is an open of $X$.

*Proof.*

(*Open Subfunctors*) $A \otimes_{\mathcal{O}(X)} \mathcal{O}(X)/I = A/A\alpha^\flat I$.

(*Ideal, Open Adjunction*) The reverse is straightforward. For forward implication, it suffices that $I \subseteq \sqrt{J}$. Suppose we have $f \in I \setminus \sqrt{J}$, in particular $\mathcal{O}(X) \neq 0$. Consider $\iota : \mathrm{Sp}(\mathcal{O}(X)/J)_f \to X$. Then $\iota \in D(I) \setminus D(J)$ since $(\mathcal{O}(X)/J)_f$ a non-zero ring by assumption. This is a contradiction.

(*Fields give enough points for opens*) Forwards is clear. Let $U = D(I), V = D(J)$. For reverse, suppose $D(I) \neq D(J)$. WLOG let $\alpha \in D(I)(A) \setminus D(J)(A)$. We're looking for a "point" $\mathrm{Sp}\, K \to D(I)$ that doesn't lift to $D(J)$. Well, since $A\alpha^\flat J \subsetneq A$, by Zorn's lemma there exists a map $\mathrm{ev}_x : A \to K$ where $K$ is a field and $J \subseteq \ker \mathrm{ev}_x$. Then $\alpha \circ x \in D(I)(K) \setminus D(J)(K)$ as desired.

(*Choice*) Straightforward by checking on points. (*Open Cover, Base Change, Fiber product*) Straightforward.

□

**Definition – Open Subfunctors, Open Covers**
Let $X \in \mathbb{M}\mathbf{Set}$. For $U \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$, $U$ is called *open* when for all $\alpha : \mathrm{Sp}\, A \to X$, the pullback $\alpha^{-1}U$ of $U$ along $\alpha$ is an open of $\mathrm{Sp}\, A$.

$$\begin{array}{ccc} \mathrm{Sp}\, A & \xrightarrow{\alpha} & X \\ \uparrow & & \uparrow \\ \alpha^{-1}U & \longrightarrow & U \end{array}$$

We will use $\mathrm{Open}\, X$ to denote the full subcategory of $\mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$ consisting of opens. When $X$ is affine, this agrees with our specialized notion for affine schemes.

**Proposition – Basic Facts about Open Subfunctors**
The following are true :

– ("Extensionality") Let $U, V \in \mathrm{Open}\, X$. Then $U = V$ if and only if $U^{\mathrm{Pts}} = V^{\mathrm{Pts}}$.

– (Composition) Let $V \in \mathrm{Open}\, U$, $U \in \mathrm{Open}\, X$, $X \in \mathbb{M}\mathbf{Set}$. Then $V \in \mathrm{Open}\, X$.

- (Base Change/"Preimage of Opens are Opens") Let $X \in \mathbb{M}\mathbf{Set}$, $U \in \operatorname{Open} X$ and $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$. Then the pullback $\varphi^{-1}U$ of $U$ along $\varphi$ is an open of $Y$.

$$\begin{tikzcd} Y \arrow[r, "\varphi"] & X \\ \varphi^{-1}U \arrow[u] \arrow[r] & U \arrow[u] \end{tikzcd}$$

- (Fiber Product) Let $U, U_1$ be opens of $X, X_1 \in \mathbb{M}\mathbf{Set}$ respectively. Then for any $X \to S, X_1 \to S$, the fiber product $U \times_S U_1$ is an open of $X \times_S X_1$. In particular, for $X = X_1 = S$, this proves the intersection of two opens of $X$ is again an open of $X$.
- ("Open Cover") Let $\mathcal{U} \subseteq \operatorname{Open} X$. TFAE :
  - ("Looks like Open Cover when Tested") For all $(A, \alpha) \in \operatorname{Sp} \downarrow X$, $\overset{\circ}{\bigcup}_{U \in \mathcal{U}} \alpha^{-1}U = \operatorname{Sp} A$.
  - ("Surjective on Points") For all $x : \operatorname{Sp} K \to X$ where $K$ is a field, there exists $U \in \mathcal{U}$ such that $x$ factors through $U$.

  We say $\mathcal{U}$ *covers* $X$ when any (and thus all) of the above are true.
- (Base Change of Open Cover) Let $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$ and $\mathcal{U} \subseteq \operatorname{Open} X$ cover $X$. Then $\varphi^{-1}\mathcal{U} := \left\{\varphi^{-1}U\right\}_{U \in \mathcal{U}}$ covers $Y$.

*Proof.* (Extensionality) $U = V$ if and only if for all $\alpha : \operatorname{Sp} A \to X$, $\alpha^{-1}U = \operatorname{Sp} A$ if and only if $\alpha^{-1}V = \operatorname{Sp} A$. This reduces to the affine global case. $\square$