Algebraic Geometry : Functor of Points POV

Ken Lee

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Contents

Goal : Develop all of basic scheme theory without a single mention of locally ringed spaces.

1 Schemes

1.1 $\mathbb{Z}$-Functors

Definition – $\mathbb{Z}$-Functor

Define $\mathbb{M} := \mathbb{Z}\mathbf{Alg}^{op}$. The *category of $\mathbb{Z}$-functors* $\mathbb{M}\mathbf{Set}$ is define to be the category of presheaves of sets on $\mathbb{M}$. For $X \in \mathbb{M}\mathbf{Set}$ and $A^{op} \in \mathbb{M}$, elements of $X(A^{op})$ are called *morphisms from A^{op} to X*.

Define $\mathrm{Sp} : \mathbb{M} \to \mathbb{M}\mathbf{Set}$ to be the fully faithful Yoneda embedding $A^{op} \mapsto \mathbb{M}(-, A^{op})$. For $A \in \mathbb{Z}\mathbf{Alg}$, $\mathrm{Sp}\, A$ is called the *spectrum of A*. The *category of affine schemes* is defined to be the essential image of Sp. We will denote it with $\mathbf{Aff}$.

Remark – Intuition of $\mathbb{Z}$-Functors and Spectrums. In differential geometry, given a morphism of smooth manifolds $\varphi : X \to Y$, one obtains a ring morphism $\varphi^\flat : C^\infty(Y) \to C^\infty(X)$ that is the pullback of global smooth functions. For algebraic geometry, here's how to think about it :

– $\mathbb{M}$ is the category of "test spaces". For smooth manifolds, this is open subsets of $\mathbb{R}^n$.

– A ring A is the "ring of global functions on A^{op}".

– A ring morphism $\varphi : A \to B$ is the "pullback of global functions along $\varphi^{op} : B^{op} \to A^{op}$".

– An $\mathbb{Z}$-functor $X \in \mathbb{M}\mathbf{Set}$ is something our "test spaces" in $\mathbb{M}$ can "map into". Essentially,

$$\text{“}X(A^{op}) = \mathrm{Mor}(A^{op}, X)\text{”}$$

Then given $\varphi^{op} : B^{op} \to A^{op}$, one should be able to turn "maps" $\alpha : A^{op} \to X$ into $\alpha \circ \varphi^{op} : B^{op} \to X$. This is precisely $X(A^{op}) \to X(B^{op})$ and functoriality simply expresses how pre-composition respects the identity morphisms and composition.

– The spectrum functor Sp realizes a "space" A^{op} in $\mathbb{M}$ as something our "test spaces" in $\mathbb{M}$ can map to. Then the intuition of "$X(A^{op}) = \mathrm{Mor}(A^{op}, X)$" is realised as $X(A^{op}) \cong \mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X)$. This is Yoneda's lemma.

– $\mathbf{Aff}$ formalizes what we mean by "test spaces in $\mathbb{M}$".

– $\mathbb{M}\mathbf{Set}$ is complete and cocomplete as a category (since $\mathbf{Set}$ is), i.e. it's the perfect playground for building more general "spaces" out of affine schemes.

Proposition – Yoneda

The following are true :

– ("Morphisms from A^{op} to $X \longleftrightarrow$ Morphisms from $\mathrm{Sp}\, A$ to X") For $X \in \mathbb{M}\mathbf{Set}$ and $A^{op} \in \mathbb{M}$,

$$\mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X) \xrightarrow{\sim} X(A^{op})$$

by $\alpha \mapsto \alpha_{A^{op}}(\mathbb{1}_{A^{op}})$. Furthermore, this is functorial in A^{op} and X.

– (Density of Representables / "The data of X is precisely how test spaces map into it") For $X \in \mathbb{M}\mathbf{Set}$, X is the colimit of the diagram $\mathrm{Sp} \downarrow X \to \mathbb{M}\mathbf{Set}$.

Proof. Straightforward. $\square$

Proposition – Affine Line

Let $n \in \mathbb{N}$. Define *affine n-space* to be $\mathbb{A}^n \in \mathbb{M}\mathbf{Set}$ sending $A^{op} \mapsto A^n$. Then

- $\mathbb{A}^n$ is representable by $\mathbb{Z}[T_1, \dots, T_n]^{op}$. Hence $\mathbb{A}^n \in \mathbf{Aff}$.
- for $n = 1$, $\mathbb{A}^1$ is a ring object in $\mathbb{M}\mathbf{Set}$. Hence for $X \in \mathbb{M}\mathbf{Set}$, $\mathcal{O}(X) := \mathbb{M}\mathbf{Set}(X, \mathbb{A}^1) \in \mathbb{Z}\mathbf{Alg}$. This is called the *ring of global functions on X* and gives a functor $\mathcal{O}(\star) : \mathbb{M}\mathbf{Set} \to \mathbb{Z}\mathbf{Alg}^{op}$. We call elements of $\mathcal{O}(X)$ *functions on X*.

  Concretely on morphisms $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$, the corresponding ring morphism $\varphi^\flat : \mathcal{O}(X) \to \mathcal{O}(Y)$ is given by $f \mapsto f \circ \varphi$, i.e. pulling back along φ.
- (Spec, Global Function Adjunction)

$$\mathbb{M}\mathbf{Set}(-, \operatorname{Sp}\star) \cong \mathbb{Z}\mathbf{Alg}(\star, \mathcal{O}(-))$$

- (Canonical Choice of Spectrum for Affine Schemes) for $X \in \mathbb{M}$, X is affine if and only if $\mathbb{1}^{\perp}_{\mathcal{O}(X)} : X \to \operatorname{Sp}\mathcal{O}(X)$ is an isomorphism of $\mathbb{Z}$-functors, where $\mathbb{1}^{\perp}_{\mathcal{O}(X)}$ is the adjunct of $\mathbb{1}_{\mathcal{O}(X)}$ under the previous adjunction.

Proof. (*Spec, Global Function Adjunction*) Follows from this chain of bijections functorial in A and X given by the density of representables :

$$\begin{aligned}\mathbb{M}\mathbf{Set}(X, \operatorname{Sp} A) &\cong \varprojlim_{(B,\beta)\in \operatorname{Sp}\downarrow X} \mathbb{M}\mathbf{Set}(\operatorname{Sp} B, \operatorname{Sp} A) \cong \varprojlim_{(B,\beta)\in \operatorname{Sp}\downarrow X} \mathbb{Z}\mathbf{Alg}(A, B) \cong \mathbb{Z}\mathbf{Alg}\left(A, \varprojlim_{(B,\beta)\in \operatorname{Sp}\downarrow X} B\right) \\ &\cong \mathbb{Z}\mathbf{Alg}\left(A, \varprojlim_{(B,\beta)\in \operatorname{Sp}\downarrow X} \mathbb{M}\mathbf{Set}(\operatorname{Sp} B, \mathbb{A}^1)\right) \cong \mathbb{Z}\mathbf{Alg}(A, \mathbb{M}\mathbf{Set}(X, \mathbb{A}^1)) = \mathbb{Z}\mathbf{Alg}(A, \mathcal{O}(X))\end{aligned}$$

(*Affine Schemes*) The reverse implication is clear. Let $X \xrightarrow{\sim} \operatorname{Sp} A$ be an isomorphism in $\mathbb{M}\mathbf{Set}$. The adjunction $\mathbb{M}\mathbf{Set}(X, \operatorname{Sp} A) \cong \mathbb{Z}\mathbf{Alg}(A, \mathcal{O}(X)) \cong \mathbb{M}\mathbf{Set}(\operatorname{Sp}\mathcal{O}(X), \operatorname{Sp} A)$ gives the commutative diagram :

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d] & \operatorname{Sp} A \\ \operatorname{Sp}\mathcal{O}(X) \arrow[ur, dashed] & \end{tikzcd}$$

where the dashed has the composition of the other two morphisms as an inverse, and hence an isomorphism.

□

Remark – Intuition of Affine n-Space. For a smooth manifold X, a smooth map $\varphi : X \to \mathbb{R}^n$ is equivalent to the data of n global smooth functions $f_1, \dots, f_n$ on X, i.e.

$$C^\infty\mathbf{Mfd}(X, \mathbb{R}^n) \cong C^\infty(X)^n$$

"$\mathbb{R}^n$ is the classifying space of n-tuples of global smooth functions." In the functorial POV of algebraic geometry, we take this as our definition of affine n-space.

Proposition – Categorical Properties of $\mathbb{Z}$-Functors

The following are true :

- (Completeness and Cocompleteness) $\mathbb{M}\mathbf{Set}$ has all (small) limits and colimits, all computed pointwise.

  In particular, a morphism $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$ is respectively a mono/epi/isomorphism if and only if for all $A \in \mathbb{M}^{op}$, $\varphi_A : Y(A) \to X(A)$ is injective/surjective/bijective.

- (Base Change) For $K \in \mathbf{CRing}$, define the *category of K-functors* to be the over-category $\mathbb{M}\mathbf{Set}/\operatorname{Sp} K$. In particular, we call $\mathbf{Aff}/\operatorname{Sp} K$ the *category of affine K-schemes*.

  Then we have for $\varphi \in \mathbb{M}\mathbf{Set}(\operatorname{Sp} L, \operatorname{Sp} K)$, we have the following adjunction

  $$(\mathbb{M}\mathbf{Set}/\operatorname{Sp} L)\,(-, \operatorname{Sp} L \times_{\operatorname{Sp} K} (\star)) \cong (\mathbb{M}\mathbf{Set}/\operatorname{Sp} K)\,(-, \star)$$

  Furthermore, this restricts to an adjunction between $\mathbf{Aff}/\operatorname{Sp} L$ and $\mathbf{Aff}/\operatorname{Sp} K$, i.e. the pullback of affine schemes is affine.

Proof. (*Base Change*) The first adjunction is categorical. For the restriction to affine schemes over K and L, note that for any K-algebra A,

$$\operatorname{Sp} L \times_{\operatorname{Sp} K} \operatorname{Sp} A \text{ “=” } \operatorname{Sp}(L \otimes_K A)$$

□

1.2 Closed Subfunctors, Open Covers

Remark. A scheme will be a “space modelled on $\mathbb{M}$ with an open cover by affine schemes”. Since every “space” in $\mathbb{M}\mathbf{Set}$ is modelled on $\mathbb{M}$, we first define the notion of open covers for affine schemes, then define it for general $X \in \mathbb{M}\mathbf{Set}$.

We define closed subfunctors of affine schemes first, then define open ones as “complements” of closed.

Definition – Closed Subfunctors of Affine Schemes

Let $X \in \mathbf{Aff}$. For $I \in \operatorname{Ideal}\mathcal{O}(X)$, define the *vanishing of I*, $V(I) \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$, by

$$A \in \mathbb{M}^{op} \mapsto \left\{\varphi \in \mathbf{Aff}(\operatorname{Sp} A, X) \mid I \subseteq \ker \varphi^\flat\right\}$$

In particular, for $I = (f)$, we write $V(f)$ instead of $V((f))$. A subfunctor Z of X is called *closed* when there exists $I \in \operatorname{Ideal}\mathcal{O}(X)$ such that $Z = V(I)$.

Proposition – Basic Facts about Closed Subfunctors

Let $X \in \mathbf{Aff}$. Then the following are true :

- (Closed Subfunctors are Affine) For $I \in \operatorname{Ideal}\mathcal{O}(X)$, $V(I)$ is representable by $\mathcal{O}(X)/I$ and $I = \ker(\mathcal{O}(X) \to \mathcal{O}(V(I)))$. Hence for any closed subfunctor Z of X, Z is an affine scheme and $Z = V(I(Z))$ where $I(Z) := \ker(\mathcal{O}(X) \to \mathcal{O}(Z))$ is called the *ideal of definition of Z*.
- (Ideals, Vanishing Adjunction) For $I, J \in \operatorname{Ideal}\mathcal{O}(X)$, $I \subseteq J$ if and only if $V(J) \subseteq V(I)$.

- (Arbitrary Intersection of Closed) Let $\mathcal{X}$ be a collection of closed subfunctors of X. Then $\bigcap_{Z \in \mathcal{X}} Z = V(\sum_{Z \in \mathcal{X}} I(Z))$.
- (Base Change / "Preimage of Closed are Closed") Let $\varphi \in \mathbf{Aff}(Y, X)$, $I \in \operatorname{Ideal} \mathcal{O}(X)$. Let the following be a pullback diagram :

$$\begin{tikzcd} Y \arrow[r, "\varphi"] & X \\ \varphi^{-1}V(I) \arrow[u] \arrow[r] & V(I) \arrow[u] \end{tikzcd}$$

  Then $\varphi^{-1}V(I) = V(\mathcal{O}(Y)\varphi^\flat I)$.

Proposition – Open Subfunctors of Affine Schemes

Let $X \in \mathbf{Aff}$. Then the following are true :

- (Open Subfunctors) For $I \in \operatorname{Ideal} \mathcal{O}(X)$, the following two subfunctors of X are equal :
  - ("Complement of $V(I)$") Define $X \setminus V(I)$ by setting $(X \setminus V(I))\,(A)$ to be the set of $\alpha \in \mathbf{Aff}(\operatorname{Sp} A, X)$ such that $\alpha^{-1}V(I) = \operatorname{Sp} 0$, the "empty affine scheme".
  - ("Support of I") Define $D(I)$ by setting $D(I)(A)$ to be the set of $\alpha \in \mathbf{Aff}(\operatorname{Sp} A, X)$ such that $A = A\alpha^\flat I$.

  A subfunctor U of X is called *open* when there exists $I \in \operatorname{Ideal} \mathcal{O}(X)$ such that $U = D(I) = X \setminus V(I)$.

  We will use $\operatorname{Open} X$ to denote the full subcategory of $\mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$ consisting of open subfunctors of X.
- (Ideal, Open Adjunction) Let $I, J \in \operatorname{Ideal} \mathcal{O}(X)$. Then $D(I) \subseteq D(J)$ if and only if $\sqrt{I} \subseteq \sqrt{J}$. In particular, for $U \in \operatorname{Open} X$, there exists a unique $I_U \in \operatorname{Ideal} \mathcal{O}(X)$ such that $\sqrt{I_U} = I_U$ and $U = D(I_U)$.
- ("Fields given enough points for Opens") For any $Y \in \mathbb{M}\mathbf{Set}$, define Y^{Pts} to the restriction of Y to the full subcategory Pts of $\mathbb{M}^{op}$ consisting of K^{op} where K is a field. We will call Y^{Pts} the *points of* Y.

  Then for U, V opens of X, $U = V$ if and only if $U^{\mathrm{Pts}} = V^{\mathrm{Pts}}$.
- For $\mathcal{U} \subseteq \operatorname{Open} X$, define the *open union of* $\mathcal{U}$ to be

$$\mathring{\bigcup}_{U \in \mathcal{U}} U := X \setminus \bigcap_{U \in \mathcal{U}} V(I_U) = D\left(\sum_{U \in \mathcal{U}} I_U\right)$$

  Then for $\mathcal{U} \subseteq \operatorname{Open} X$,

– (Choice) for any $\{J_U\}_{U\in\mathcal{U}} \subseteq \operatorname{Ideal}\mathcal{O}(X)$ where each $U = D(J_U)$, we have $\mathring{\bigcup}_{U\in\mathcal{U}} U = D(\sum_{U\in\mathcal{U}} J_U)$.
– ("Open Cover") TFAE :

1. (Intuitive) $\mathring{\bigcup}_{U\in\mathcal{U}} U = X$.
2. (Partition of Unity) $\mathcal{O}(X) = \sum_{U\in\mathcal{U}} J_U$ where $\{J_U\}_{U\in\mathcal{U}}$ is any collection of ideals of $\mathcal{O}(X)$ such that $U = D(J_U)$ for each U.
3. (Surjective on Points) For all $x : \operatorname{Sp} K \to X$ where K is a field, there exists $U \in \mathcal{U}$ such that x factors through U.

We say $\mathcal{U}$ *covers* X when any (and thus all) of the above are true.

– (Base Change / "Preimage of Opens are Open") Let $\varphi \in \mathbf{Aff}(Y, X)$, $I \in \operatorname{Ideal}\mathcal{O}(X)$. Let the following be a pullback diagram :

$$\begin{tikzcd} Y \arrow[r, "\varphi"] & X \\ \varphi^{-1}D(I) \arrow[u] \arrow[r] & D(I) \arrow[u] \end{tikzcd}$$

Then $\varphi^{-1}D(I) = D(\varphi^\flat I)$. Hence the pullback of opens along morphisms of affine schemes remain open.

– (Fiber Product) Let $X, X_1 \in \mathbf{Aff}/Y$ where $Y \in \mathbf{Aff}$. Let I, I_1 be ideals of $\mathcal{O}(X)$, $\mathcal{O}(X_1)$ respectively. Then we have the pullback diagram :

$$\begin{tikzcd} D(I) \arrow[r] & Y \\ D(I \otimes_{\mathcal{O}(Y)} I_1) \arrow[u] \arrow[r] & D(I_1) \arrow[u] \end{tikzcd}$$

In particular, $X = X_1 = Y$ implies the intersection of two opens of X is an open of X.

Proof.

(*Open Subfunctors*) $A \otimes_{\mathcal{O}(X)} \mathcal{O}(X)/I = A/A\alpha^\flat I$.

(*Ideal, Open Adjunction*) The reverse is straightforward. For forward implication, it suffices that $I \subseteq \sqrt{J}$. Suppose we have $f \in I \setminus \sqrt{J}$, in particular $\mathcal{O}(X) \neq 0$. Consider $\iota : \operatorname{Sp}(\mathcal{O}(X)/J)_f \to X$. Then $\iota \in D(I) \setminus D(J)$ since $(\mathcal{O}(X)/J)_f$ a non-zero ring by assumption. This is a contradiction.

(*Fields give enough points for opens*) Forwards is clear. Let $U = D(I), V = D(J)$. For reverse, suppose $D(I) \neq D(J)$. WLOG let $\alpha \in D(I)(A) \setminus D(J)(A)$. We're looking for a "point" $\operatorname{Sp} K \to D(I)$ that doesn't lift to $D(J)$. Well, since $A\alpha^\flat J \subsetneq A$, by Zorn's lemma there exists a map $\operatorname{ev}_x : A \to K$ where K is a field and

$J \subseteq \ker \mathrm{ev}_x$. Then $\alpha \circ x \in D(I)(K) \setminus D(J)(K)$ as desired.

(*Choice*) Straightforward by checking on points. (*Open Cover, Base Change, Fiber product*) Straightforward.

□

Definition – Open Subfunctors, Open Covers

Let $X \in \mathbb{M}\mathbf{Set}$. For $U \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$, U is called *open* when for all $\alpha : \mathrm{Sp}\, A \to X$, the pullback $\alpha^{-1}U$ of U along α is an open of $\mathrm{Sp}\, A$.

$$\begin{array}{ccc} \mathrm{Sp}\, A & \xrightarrow{\alpha} & X \\ \uparrow & & \uparrow \\ \alpha^{-1}U & \longrightarrow & U \end{array}$$

We will use $\mathrm{Open}\, X$ to denote the full subcategory of $\mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$ consisting of opens. When X is affine, this agrees with our specialized notion for affine schemes.

Proposition – Basic Facts about Open Subfunctors

The following are true :

- ("Extensionality") Let $U, V \in \mathrm{Open}\, X$. Then $U = V$ if and only if $U^{\mathrm{Pts}} = V^{\mathrm{Pts}}$.
- (Composition) Let $V \in \mathrm{Open}\, U$, $U \in \mathrm{Open}\, X$, $X \in \mathbb{M}\mathbf{Set}$. Then $V \in \mathrm{Open}\, X$.
- (Base Change/"Preimage of Opens are Opens") Let $X \in \mathbb{M}\mathbf{Set}$, $U \in \mathrm{Open}\, X$ and $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$. Then the pullback $\varphi^{-1}U$ of U along φ is an open of Y.

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow & & \uparrow \\ \varphi^{-1}U & \longrightarrow & U \end{array}$$

- (Fiber Product) Let U, U_1 be opens of $X, X_1 \in \mathbb{M}\mathbf{Set}$ respectively. Then for any $X \to S, X_1 \to S$, the fiber product $U \times_S U_1$ is an open of $X \times_S X_1$. In particular, for $X = X_1 = S$, this proves the intersection of two opens of X is again an open of X.
- ("Open Cover") Let $\mathcal{U} \subseteq \mathrm{Open}\, X$. TFAE :
  - ("Looks like Open Cover when Tested") For all $(A, \alpha) \in \mathrm{Sp} \downarrow X$, $\mathring{\bigcup}_{U \in \mathcal{U}} \alpha^{-1}U = \mathrm{Sp}\, A$.
  - ("Surjective on Points") For all $x : \mathrm{Sp}\, K \to X$ where K is a field, there exists $U \in \mathcal{U}$ such that x factors through U.

  We say $\mathcal{U}$ *covers* X when any (and thus all) of the above are true.

– (Base Change of Open Cover) Let $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$ and $\mathcal{U} \subseteq \mathrm{Open}\, X$ cover X. Then $\varphi^{-1}U := \{\varphi^{-1}U\}_{U \in \mathcal{U}}$ covers Y.

Proof. (Extensionality) $U = V$ if and only if for all $\alpha : \mathrm{Sp}\, A \to X$, $\alpha^{-1}U = \mathrm{Sp}\, A$ if and only if $\alpha^{-1}V = \mathrm{Sp}\, A$. This reduces to the affine global case. □

1.3 The Big and Small Zariski Site

Proposition – Big Zariski Site on $\mathbb{M}\mathbf{Set}$

For $X \in \mathbb{M}\mathbf{Set}$ and $\mathcal{U} \subseteq \mathbb{M}\mathbf{Set}/X$ a collection of morphisms into X, define $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ when "$\mathcal{U}$ is isomorphic to an open cover", meaning there exists $\{U_i\}_{i \in \mathcal{U}} \subseteq \mathrm{Open}\, X$ such that $\{U_i\}_{i \in \mathcal{U}}$ is a cover of X and for all $i \in \mathcal{U}$, $(i : s(i) \to X) \cong (U_i \to X)$ in $\mathbb{M}\mathbf{Set}/X$. Then the above defines a Grothendieck pretopology of $\mathbb{M}\mathbf{Set}$. Specifically :

– (Isomorphisms are Covers) For $X \in \mathbb{M}\mathbf{Set}$ and $\varphi \in \mathbb{M}\mathbf{Set}(U, X)$, φ iso implies $\{\varphi\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.

– (Pullback of Covers) For all $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$ and $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$, $\varphi^{-1}\mathcal{U} := \{Y \times_X s(i) \to Y\}_{i \in \mathcal{U}} \in \mathrm{Cov}_{\mathrm{Zar}}(Y)$.

– (Composite of Covers) Let $X \in \mathbb{M}\mathbf{Set}, \mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ and for each $i \in \mathcal{U}$, let $\mathcal{U}_i \in \mathrm{Cov}_{\mathrm{Zar}}(s(i))$. Then $\{s(j_i) \to s(i) \to X \mid i \in \mathcal{U}, j_i \in \mathcal{U}_i\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.

We will use $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$ to denote the site $\mathbb{M}\mathbf{Set}$ endowed with the topology generated by the above pretopology. We will call $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$ the *big Zariski site*. $\mathcal{X} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ are called *Zariski covers of* X.

Proof. Only slightly non-trivial part is pullback of covers. Use opens and covers preserved under base change. □

Remark – Intuition of Sheaves on $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$. For $X \in \mathbb{M}\mathbf{Set}$, if X is to be a "space" then for any other $Y \in \mathbb{M}\mathbf{Set}$ and open cover $\mathcal{U}$ of Y, the data of a morphism $Y \to X$ should be the same as a collection of morphisms $(U \to X)_{U \in \mathcal{U}}$ that agree on pairwise intersection. This is precisely what it means for $\mathbb{M}\mathbf{Set}(-, X)$ to be a sheaf on the site $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Remark. The following is a smaller site $\mathbf{Aff}_{\mathrm{Zar}}$ on affine schemes, where open covers consists only of basic opens. Since basic opens generate opens for affine schemes and affine schemes generate $\mathbb{M}\mathbf{Set}$ with compatible notion of opens, sheaves on $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$ will be the same as sheaves of $\mathbf{Aff}_{\mathrm{Zar}}$. This gives an easier check for when $X \in \mathbb{M}\mathbf{Set}$ is a sheaf on $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Proposition – Small Zariski Site on Aff

For $X \in \mathbf{Aff}$ and $\mathcal{U} \subseteq \mathbf{Aff}/X, \mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ when "$\mathcal{U}$ is isomorphic to a cover of X by basic opens", meaning there exists a cover $\{X_{f_\iota}\}_{\iota \in \mathcal{U}}$ where for all $\iota \in \mathcal{U}$, $(s(\iota) \to X) \cong (D(f_\iota) \to X)$ in $\mathbf{Aff}/X$. Then the above defines a Grothendieck pretopology on $\mathbf{Aff}$, specifically :

– (Isomorphisms are Covers) For all $X \in \mathbf{Aff}$ and $\iota \in \mathbf{Aff}(U, X)$, ι isomorphism implies $\{\iota\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.

– (Pullback of Covers) For all $\varphi \in \mathbf{Aff}(Y, X)$ and $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$, $\varphi^{-1}\mathcal{U} := \{Y \times_X s(\iota) \to Y \mid \iota \in \mathcal{U}\} \in \mathrm{Cov}_{\mathrm{Zar}}(Y)$.

– (Composite of Covers) Let $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ and for each $i \in \mathcal{U}$, let $\mathcal{U}_i \in \mathrm{Cov}_{\mathrm{Zar}}(s(i))$. Then $\{s(j_i) \to s(i) \to X \mid i \in \mathcal{U}, j_i \in \mathcal{U}_i\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.

We will use $\mathbf{Aff}_{\mathrm{Zar}}$ to denote the site $\mathbf{Aff}$ with the topology given by the above pretopology. We will call $\mathbf{Aff}_{\mathrm{Zar}}$ the *small Zariski site*. $\mathcal{X} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ will be called *basic Zariski covers of* X. [a]

[a] This is non-standard terminology, but helps avoid confusion between the topology on $\mathbf{Aff}$ just defined and the induced topology from $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Proof. UP of tensor products and localization. □

Proposition – Sheaves on Big and Small Zariski Site are the Same

Let $X \in \mathbb{M}\mathbf{Set}$. Then $\mathbb{M}\mathbf{Set}(-, X) \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$ if and only if $\mathbb{M}\mathbf{Set}(-, X) \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$.

Proof. Forward implication follows since the covers in $\mathbf{Aff}_{\mathrm{Zar}}$ are covers in $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Now let $\mathbb{M}\mathbf{Set}(-, X) \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$. Let $U \in \mathbb{M}\mathbf{Set}$ and $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(U)$. Then for $(A, \alpha) \in \mathrm{Sp} \downarrow U$, the pullback $\alpha^{-1}\mathcal{U}$ of $\mathcal{U}$ is a cover of $\mathrm{Sp}\, A$ in the big Zariski site. The chain of isomorphisms to be justified is :

$$\begin{aligned}\mathbb{M}\mathbf{Set}(U, X) &\overset{(1)}{\cong} \varprojlim_{(A,\alpha)\in \mathrm{Sp}\downarrow U} \mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X) \overset{(2)}{\cong} \varprojlim_{(A,\alpha)\in \mathrm{Sp}\downarrow U} \varprojlim_{V,W\in\mathcal{U}} \mathbb{M}\mathbf{Set}(\alpha^{-1}V \cap \alpha^{-1}W, X) \\ &\overset{(3)}{\cong} \varprojlim_{V,W\in\mathcal{U}} \varprojlim_{(A,\alpha)\in \mathrm{Sp}\downarrow U} \mathbb{M}\mathbf{Set}(\alpha^{-1}V \cap \alpha^{-1}W, X) \overset{(4)}{\cong} \varprojlim_{V,W\in\mathcal{U}} \mathbb{M}\mathbf{Set}(V \cap W, X)\end{aligned}$$

(1) Density of representables. (3) Limits commute with limits.

(4) We know $\alpha^{-1}(V \cap W) = \alpha^{-1}V \cap \alpha^{-1}W$, so it suffices to prove the following.

Lemma. For $U \in \mathbb{M}\mathbf{Set}$ *and* $Z \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(U)$, *we have* $Z = \varprojlim_{(A,\alpha)\in\mathrm{Sp}\downarrow U} \alpha^{-1}Z$

Proof. The forgetful functor $\mathrm{Sp} \downarrow Z \to \mathrm{Sp} \downarrow U$ is a “section” of the pullback functor $\mathrm{Sp} \downarrow U \to \mathrm{Sp} \downarrow Z$, meaning for $(A, \alpha) \in \mathrm{Sp} \downarrow Z$, the following is a pullback diagram :

$$\begin{tikzcd} Z \arrow[r] & U \\ \mathrm{Sp}\, A \arrow[u, "\alpha"] \arrow[r, "\mathbb{1}"] & \mathrm{Sp}\, A \arrow[u] \end{tikzcd}$$

This implies pulling the diagram $\mathrm{Sp} \downarrow U$ back to $\mathrm{Sp} \downarrow Z$ only introduces duplicate objects with identity morphisms in between them. Hence $\varprojlim_{(A,\alpha)\in\mathrm{Sp}\downarrow U} \alpha^{-1}Z = \varprojlim_{(A_1,\alpha_1)\in\mathrm{Sp}\downarrow Z} \mathrm{Sp}\, A = Z$ by the density of representables. ■

(2) We need to show that $\mathbb{M}\mathbf{Set}(-, X)$ is a sheaf for $\mathbf{Aff}$ with covers from the big Zariski site $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$. The key is that basic opens cover opens for affine schemes.

Let $A \in \mathbb{M}^{op}$ and $\mathcal{U}$ be a $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$-cover of $\operatorname{Sp} A$. For each $i \in \mathcal{U}$, let $I_i \in \operatorname{Ideal} A$ with $i = D(I_i)$. Let $I := \bigsqcup_{i \in \mathcal{U}} I_i$. Then since $\{D(f)\}_{f \in I_i}$ is a $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$-cover of i for every $i \in \mathcal{U}$, $\{D(f)\}_{f \in I}$ is also a $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$-cover of $\operatorname{Sp} A$. We then have the commutative diagram :

$$\begin{tikzcd}
\mathbb{M}\mathbf{Set}(\operatorname{Sp} A, X) \arrow[r] \arrow[d, "\mathbb{1}"] & \varprojlim_{i,j \in \mathcal{U}} \mathbb{M}\mathbf{Set}(i \cap j, X) \arrow[d, "\sim"] \\
\mathbb{M}\mathbf{Set}(\operatorname{Sp} A, X) \arrow[r, "\sim"] & \varprojlim_{f,g \in I} \mathbb{M}\mathbf{Set}(D(f) \cap D(g), X)
\end{tikzcd}$$

where the horizontal isomorphism to due to $\mathbb{M}\mathbf{Set}(-, X)$ being a sheaf on $\mathbf{Aff}_{\mathrm{Zar}}$.

It remains to justify the vertical isomorphism. To do this, we apply the same argument as we're trying to do now, but on $i \cap j$. It's easy to see that $\{D(f) \cap D(g)\}_{f \in I_i, g \in I_j}$ covers $i \cap j$, so we get

$$\begin{aligned}
\mathbb{M}\mathbf{Set}(i \cap j, X) &\cong \varprojlim_{(A_1,\alpha_1) \in \operatorname{Sp}\downarrow(i \cap j)} \mathbb{M}\mathbf{Set}(\operatorname{Sp} A_1, X) \cong \varprojlim_{(A_1,\alpha_1) \in \operatorname{Sp}\downarrow(i \cap j)} \varprojlim_{f \in I_i, g \in I_j} \mathbb{M}\mathbf{Set}(D(\alpha_1^\flat(f)) \cap D(\alpha_1^\flat(g)), X) \\
&\cong \varprojlim_{f \in I_i, g \in I_j} \varprojlim_{(A_1,\alpha_1) \in \operatorname{Sp}\downarrow U} \mathbb{M}\mathbf{Set}(\alpha_1^{-1}\left(D(f) \cap D(g)\right), X) \overset{(4)}{\cong} \varprojlim_{f \in I_i, g \in I_j} \mathbb{M}\mathbf{Set}(D(f) \cap D(g), X)
\end{aligned}$$

where (4) is as before.

$\square$

1.4 Schemes

Definition – Schemes

Let $X \in \mathbb{M}\mathbf{Set}$. Then X is called a *scheme* when we have :

- ("Is a Space") $X \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$, equivalently $\mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$.
- (Open cover by Affine schemes) there exists $\mathcal{U} \subseteq \operatorname{Open} X$ such that $\mathcal{U}$ covers X and every $U \in \mathcal{U}$ is affine.

We use $\mathbf{Sch}$ to denote the full subcategory of schemes in $\mathbb{M}\mathbf{Set}$.

Remark – Intuition of Definition of Schemes. In the same way that smooth manifolds are spaces modeled on $\mathbb{R}^n$ that is locally $\mathbb{R}^n$, schemes are spaces modeled on $\mathbb{M}$ that is "locally $\mathbb{M}$". In particular, objects of $\mathbb{M}$ ought to be schemes.

Proposition – Affine Schemes are Schemes

Let $X \in \mathbf{Aff}$. Then $X \in \mathbf{Sch}$.

Proof. X is an affine open cover of itself, so it suffices to check the sheaf condition. Since $\mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}) = \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$, it suffices to check that for $(A, \alpha) \in \operatorname{Sp} \downarrow X, \mathcal{U}$ a $\mathbf{Aff}_{\mathrm{Zar}}$-cover of $\operatorname{Sp} A$, we have

$$\mathbb{M}\mathbf{Set}(\operatorname{Sp} A, X) \xrightarrow{\sim} \varprojlim_{U,V \in \mathcal{U}} \mathbb{M}\mathbf{Set}(U \cap V, X)$$

Since $X = \operatorname{Sp}\mathcal{O}(X)$ and $\operatorname{Sp} : \mathbb{M}^{op} \to \mathbb{M}\mathbf{Set}$ is fully faithful, this is equivalent to

$$\mathbb{Z}\mathbf{Alg}(\mathcal{O}(X), \mathcal{O}(\operatorname{Sp} A)) \xrightarrow{\sim} \mathbb{Z}\mathbf{Alg}(\mathcal{O}(X), \varprojlim_{U,V\in\mathcal{U}} \mathcal{O}(U\cap V))$$

It thus suffices to show that

$$A = \mathcal{O}(\operatorname{Sp} A) \xrightarrow{\sim} \varprojlim_{U,V\in\mathcal{U}} \mathcal{O}(U\cap V)$$

where the isomorphism is from the restriction maps $\mathcal{O}(\operatorname{Sp} A) \to \mathcal{O}(U\cap V)$.

Let $(f_V \in \mathcal{O}(V))_{V\in\mathcal{U}}$ that agrees on pairwise intersections. Suppose for a moment, for any finite subcover $\mathcal{U}_0 \subseteq \mathcal{U}$, we have a unique $f_{\mathcal{U}_0} \in A$ that agree with f_V on $V \in \mathcal{U}_0$. WLOG $\mathcal{U} = \{D(f)\}_{f\in I}$ for some $I \subseteq A$. Then $\mathcal{U}$ covers $\operatorname{Sp} A$ implies $D(I)$ covers $\operatorname{Sp} A$, which implies $AI = A$, which gives a *finite* subset $I_0 \subseteq I$ where $AI_0 = A$. Hence, we do have a finite subcover $\mathcal{U}_0$ and such $f_{\mathcal{U}_0}$. Furthermore, for any $V \in \mathcal{U}$, $f_{\mathcal{U}_0} = f_{\mathcal{U}_0\cup\{V\}}$ by uniqueness of $f_{\mathcal{U}_0}$ on $\mathcal{U}_0$ so

$$\downarrow^V f_{\mathcal{U}_0} = \downarrow^V f_{\mathcal{U}_0\cup\{V\}} = f_V$$

So $f_{\mathcal{U}_0}$ actually agrees with f_V on all $V \in \mathcal{U}$. Furthermore, it is unique, again by uniqueness on $\mathcal{U}_0$. Thus, it suffices to do the case of $\mathcal{U}$ *finite*.

The naive idea is this : if each $f_V = g_V/h_V$ with $V = D(h_V)$, then "agreeing on intersections" *should* mean $g_V h_W = g_W h_V \in \mathcal{O}(U) = A$. We can then use a partition of unity $1 = \sum_{V\in\mathcal{U}} \lambda_V h_V$ to patch :

$$f_V = \frac{g_V}{h_V} = \frac{\sum_{W\in\mathcal{U}} \lambda_W g_V h_W}{h_V} = \frac{\sum_{W\in\mathcal{U}} \lambda_W g_W h_V}{h_V} = \frac{\sum_{W\in\mathcal{U}} \lambda_W g_W}{1}$$

So $f := \sum_{W\in\mathcal{U}} \lambda_W g_W \in A = \mathcal{O}_{\operatorname{Spec} A}(U)$ is the guy we want. This is even unique since if we have another such f_1, then $f/1 = f_1/1 \in \mathcal{O}(V) \cong A_{h_V}$ implies the existence of $N_V \in \mathbb{N}$ such that $(f - f_1)h_V^{N_V} = 0$. By *finiteness of* $\mathcal{U}$, we can pick a single $N \in \mathbb{N}$ with $(f - f_1)h_V^N = 0$ for all $V \in \mathcal{U}$. Then using another partition of unity $1 = \sum_{V\in\mathcal{U}} \mu_V h_V^N$, we have

$$f - f_1 = \sum_{V\in\mathcal{U}} \mu_V (f - f_1) h_V^N = 0$$

It thus suffices to write for all $V \in \mathcal{U}$, $f_V = g_V/h_V$ such that for all $W \in \mathcal{U}$, $g_V h_W = g_W h_V$.

Well, for each $V \in \mathcal{U}$, let $h_V \in A$ with $V = D(h_V)$. Then $f_V = g_V/h_V^{n_V}$. Since $D(h_V) = D(h_V^{n_V})$, WLOG $f_V = g_V/h_V$ with $V = D(h_V)$ Now, since f_V and f_W agree on $V\cap W = D(h_V h_W)$, we have $g_V h_W/h_V h_W = \downarrow^{V\cap W} g_V/h_V = \downarrow^{V\cap W} g_W/h_W = g_W h_V/h_V h_W$ and so the existence of $n(V,W) \in \mathbb{N}$ such that

$$(g_V h_W - g_W h_V)(h_V h_W)^{n(V,W)} = 0$$

Smashing it again with *finiteness of* $\mathcal{U}$, we can choose a single $N \in \mathbb{N}$ such that

$$(g_V h_W - g_W h_V)(h_V h_W)^N = 0$$

for *all* $V, W \in \mathcal{U}$. Then, since $g_V/h_V = g_V h_V^N/h_V^{N+1}$ and $D(h_V) = D(h_V^{N+1})$, we can WLOG $f_V = g_V/h_V$ with $V = D(h_V)$ and

$$g_V h_W - g_W h_V = 0$$

as desired, finishing the proof.

$\square$

Proposition – Opens Subschemes
Let $X \in \mathbf{Sch}$, $U \in \operatorname{Open} X$. Then $U \in \mathbf{Sch}$. We call U an *open subscheme of* X.

Proof. (*Sheaf*)

Lemma (*Opens of Sheaves are Sheaves*). *Let* $U \in \operatorname{Open} X$ *where* $X \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$. *Then* $U \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$.

Proof. Given $Y \in \mathbb{M}\mathbf{Set}$, a compatible system $(\varphi_i)_{Y_i \in \mathcal{Y}}$ of morphisms from an open cover $\mathcal{Y}$ of Y to U glues uniquely to a morphism $\varphi : Y \to X$. Factoring φ through U is equivalent to $\varphi^{-1}U = Y$, which is true from $\varphi^{-1}U$ "containing" the cover $\mathcal{Y}$, and so is single open covering Y, and hence is equal to Y by extensionality of opens. ■

(*Affine Open Cover*) Let $\mathcal{U} \subseteq \operatorname{Open} X$, $\mathcal{U}$ consists of affine opens and covers X. Since opens and covers are preserved under base change, $\{U \cap V\}_{V \in \mathcal{U}}$ is an open cover of U. For each $V \in \mathcal{U}$, $U \cap V$ is also an open of V. By affineness of V, $U \cap V$ has a cover by basic opens V_f of V. The V_f are open in $U \cap V$ by base change and hence open in U by composition. This gives an affine open cover of $U \cap V$, and hence an affine open cover of U by taking the composite of these covers.

□

Proposition – Fiber Product of Schemes
Let $X, Y, S \in \mathbf{Sch}$ and $\varphi \in \mathbf{Sch}(X, S)$, $\psi \in \mathbf{Sch}(Y, S)$. Then the fiber product $X \times_S Y$ in $\mathbb{M}\mathbf{Set}$ is a scheme and is the fiber product of X, Y over S in $\mathbf{Sch}$.

1.5 Examples

Counter Example (*Surjective on Points implies Surjective*).
Consider $\operatorname{Sp}\mathbb{F}_2 \to \operatorname{Sp}\mathbb{F}_2[dT]$ *where* $dT^2 := 0$. $(\operatorname{Sp}\mathbb{F}_2)^{\mathrm{Pts}} \cong (\operatorname{Sp}\mathbb{F}_2[dT])^{\mathrm{Pts}}$ *but* $(\operatorname{Sp}\mathbb{F}_2[dT])(\mathbb{F}_2[dT])$ *bijects with* $\mathbb{F}_2$ *whilst* $(\operatorname{Sp}\mathbb{F}_2)(\mathbb{F}_2[dT])$ *is singleton.*

Example (*Local Rings*).

Example (*Affine Line with Two Origins*).
Define X *by the following pushout diagram in* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$.

$$\begin{tikzcd} \mathbb{G}^{\times} \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d, "\iota_b"] \\ \mathbb{A}^1 \arrow[r, "\iota_a"] & X \end{tikzcd}$$

In other words, X *is obtained by "gluing two affine lines along* $\mathbb{G}^{\times}$*". We prove that the two morphisms* $\mathbb{A}^1 \to X$ *form an open cover of* X *and hence* X *is a scheme.*

($\iota_a, \iota_b : \mathbb{A}^1 \to X$ *monomorphism in* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$) *First note that since* $\mathbb{M}\mathbf{Set}$ *and* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$ *are both have fiber products and sheafification is the free functor adjoint to the forgetful functor, monomorphisms in* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$ *are the same as monomorphisms in* $\mathbb{M}\mathbf{Set}$*. We shall make no distinct between the two from now on.*

Let X^{psh} *denote the pushout of* $\mathbb{A}^1 \leftarrow \mathbb{G}^\times \to \mathbb{A}^1$ *in* $\mathbb{M}\mathbf{Set}$ *so that* X *is the sheafification of* X^{psh} *with respect to the Zariski topology on* $\mathbf{Aff}$*. Specifically, we have*

$$X^{psh}(A) = A \sqcup_{A^\times} A = \{(i,f) \,|\, i \in \{a,b\}, f \in A\} / (i,f) \sim (j,g) := f = g \in A^\times$$

We will use $(a,-), (b,-)$ *to denote the two obvious "inclusions"* $\mathbb{A}^1 \to X^{psh}$*. A remark : by considering* $A = \mathbb{F}_2^2$*, one can show that* X^{psh} is not *a Zariski sheaf, which is why we must take the pushout in* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$*.*

The morphism ι_a *is the composition* $\mathbb{A}^1 \to X^{psh} \to X$ *where the second morphism comes from sheafification and the first morphism is* $(a,-)$*, which is clearly mono. It thus suffices that* $X^{psh} \to X$ *is mono. Sheafification has the property that if* X^{psh} *is a separated presheaf, then this map is mono. So it suffices that* X^{psh} *is a separated presheaf, which means for all* $\alpha, \alpha_1 \in \mathbb{M}\mathbf{Set}(\mathrm{Sp}\,A, X^{psh})$ *and basic Zariski covers* $\mathcal{U}$ *of* $\mathrm{Sp}\,A$*,* α *and* α_1 *agreeing on every open in* $\mathcal{U}$ *implies* $\alpha = \alpha_1$*.*

Let the morphisms α, α *correspond to* $(i,f), (i_1,f_1) \in X^{psh}(A)$*. If* $i = i_1$*, then* f, f_1 *are just morphisms* $\mathrm{Sp}\,A \to \mathbb{A}^1$ *that agree on the basic Zariski cover* $\mathcal{U}$*. Since* $\mathbb{A}^1 \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$*, we obtain* $f = f_1$ *and hence* $\alpha = \alpha_1$*. Now let* $i \neq i_1$*, WLOG* $i = a$ *and* $i_1 = b$*. It follows that* f, f_1 *are morphisms from* $\mathrm{Sp}\,A \to \mathbb{G}^\times$ *that agree on* $\mathcal{U}$*. Since* $\mathbb{G}^\times \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$ *as well,* $f = f_1 \in A^\times$ *and so* $\alpha = \alpha_1$*.*

(*The images of* $\iota_a, \iota_b : \mathbb{A}^1 \to X$ *are open in* X) *Let* U_a *be the presheaf theoretic image of* $\iota_a : \mathbb{A}^1 \to X$*. We show* U_a *is an open of* X*. The argument for* ι_b *is analogous.*

We need to show that for all $(A,\alpha) \in \mathrm{Sp} \downarrow X$*,* $\alpha^{-1}U := \mathrm{Sp}\,A \times_X U$ *is open in* $\mathrm{Sp}\,A$*. Let* $(A,\alpha) \in \mathrm{Sp} \downarrow X$*. It is another property of sheafification that there now exists a basic Zariski cover* $\mathcal{V}$ *of* $\mathrm{Sp}\,A$ *together with morphisms* $\alpha_i : V_i \to X^{psh}$ *for every* $V_i \in \mathcal{V}$ *that are the "restriction of* α*", in the sense that the following diagram commutes :*

$$\begin{tikzcd} \mathrm{Sp}\,A \arrow[r, "\alpha"] & X \\ V_i \arrow[u] \arrow[r, "\alpha_i"] & X^{psh} \arrow[u] \end{tikzcd}$$

It should *suffice to prove that for all* $V_i \in \mathcal{V}$*,* $V_i \cap \alpha^{-1}U_a \in \mathrm{Open}\,\mathrm{Sp}\,A$*, since* $\mathcal{V}$ *is an open cover of* $\mathrm{Sp}\,A$ *and "a subset is open if and only if it is open when restricted to an open cover". We prove this lemma.*

Lemma. Let $V \in \mathbb{M}\mathbf{Set}$, $\mathcal{V}$ a Zariski cover of V, $U \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(V)$ such that $U \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$ and for all $V_i \in \mathcal{V}$, $V_i \cap U \in \operatorname{Open} V$. Then $U \in \operatorname{Open} V$.

Proof. By definition of subfunctors being open, it suffices to do the case of $V \in \mathbf{Aff}$. For $V_i \in \mathcal{V}$, let $V_i \cap U = D(I_i)$ where $I_i \subseteq \mathcal{O}(V)$. The claim is that $U = D(\bigcup_{V_i \in \mathcal{V}} I_i)$. Well, for $A \in \mathbb{M}^{op}$ and $\alpha \in V(A)$, $\alpha \in U(A)$ if and only if $\alpha : \operatorname{Sp} A \to V$ factors through U. Since $\{D(I_i)\}_{V_i \in \mathcal{V}} \subseteq \operatorname{Open} V$, $\left\{\alpha^{-1} D(I_i)\right\}_{V_i \in \mathcal{V}} \subseteq \operatorname{Open} \operatorname{Sp} A$. Then $\alpha : \operatorname{Sp} A \to V$ factoring through U implies $\left\{\alpha^{-1} D(I_i)\right\}_{V_i \in \mathcal{V}}$ forms a Zariski cover of $\operatorname{Sp} A$. Conversely, if $\left\{\alpha^{-1} D(I_i)\right\}_{V_i \in \mathcal{V}}$ forms a Zariski cover of $\operatorname{Sp} A$, then $U \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$ implies $\alpha : \operatorname{Sp} A \to V$ factors through U by uniquely gluing $\alpha^{-1}(V_i \cap U) \to U$ together. Now $\left\{\alpha^{-1} D(I_i)\right\}_{V_i \in \mathcal{V}} = \{D(\alpha I_i)\}_{V_i \in \mathcal{V}}$ forms a Zariski cover of $\operatorname{Sp} A$ if and only if $A = A \bigcup_{V_i \in \mathcal{V}} \alpha I_i$, i.e. $\alpha \in D(\bigcup_{V_i} I_i)$. ∎

Now, since $\mathbb{A}^1 \cong U_a$ in $\mathbb{M}\mathbf{Set}$, the fact that the three $\operatorname{Sp} A, X, \mathbb{A}^1$ are Zariski sheaves implies we indeed have $\alpha^{-1} U_a$ as a subfunctor of $\operatorname{Sp} A$ with $\alpha^{-1} U_a \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$. So it does suffice that for all $V_i \in \mathcal{V}$, $V_i \cap \alpha^{-1} U_a$ is open in $\operatorname{Sp} A$. Note that again, pullbacks in $\mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$ and $\mathbb{M}\mathbf{Set}$ coincide, so we make no distinction between the two.

Let $V_i \in \mathcal{V}$. The intersection $V_i \cap \alpha^{-1} U_a$ is equal to the pullback $\alpha_i^{-1}\left(\uparrow^X_{X^{psh}}\right)^{-1} U_a$. We have $\left(\uparrow^X_{X^{psh}}\right)^{-1} U_a$ as the presheaf-theoretic image of $(a, -)$, which we will denote with $(a, \mathbb{A}^1)$. Suppose α_i corresponds to $(a, f) \in X^{psh}(\mathcal{O}(V_i))$. Then we that following pullback diagram,

$$\begin{tikzcd} V_i \arrow[r, "\alpha_i"] & X^{psh} \\ V_i \arrow[u, "\mathbb{1}_{V_i}"] \arrow[r, "f"] & \mathbb{A}^1 \arrow[u, "{(a,-)}"'] \end{tikzcd}$$

i.e. $\alpha_i^{-1}(a, \mathbb{A}^1) = V_i$, which is open. In the other case that α_i corresponds to $(b, f) \in X^{psh}(\mathcal{O}(V_i))$, it is easily checked from the definition of X^{psh} that $\alpha_i^{-1}(a, \mathbb{A}^1) = (\operatorname{Sp} A)_f$, which is again open.

(U_a, U_b cover X) Let $x : \operatorname{Sp} K \to X$ be a point of X. Again, it is a property of sheafification that we obtain a basic Zariski cover $\mathcal{K}$ of $\operatorname{Sp} K$ with morphisms $\kappa_i : K_i \to X^{psh}$ for each $K_i \in \mathcal{K}$ such that

$$\begin{tikzcd} \operatorname{Sp} K \arrow[r, "x"] & X \\ K_i \arrow[u] \arrow[r, "\kappa_i"] & X^{psh} \arrow[u] \end{tikzcd}$$

But $\operatorname{Sp} K$ is local, so $K_i = \operatorname{Sp} K$ for some $K_i \in \mathcal{K}$. In other words, we reduced the problem to showing $(a, \mathbb{A}^1), (b, \mathbb{A}^1)$ cover X^{psh}. This is clear.

2 Properties of Schemes

2.1 Zariski-Local Properties of Schemes

Definition – Zariski-Local Properties

Let $P : \mathbf{Sch} \to \mathbf{Prop}$ be a predicate on schemes. Then P is called *Zariski-local* when for all $X \in \mathbf{Sch}$ and Zariski covers $\mathcal{X}$ of X, $P(X)$ is true if and only if for all $X_i \in \mathcal{X}$, $P(X_i)$ is true.

Definition – Affine-Local Properties

Let $P : \mathbf{Aff} \to \mathbf{Prop}$ be a predicate on affine schemes. Then P is called *affine-local* when for all $X \in \mathbf{Aff}$ and basic Zariski covers $\mathcal{X}$ of X, $P(X)$ is true if and only if for all $X_i \in \mathcal{U}$, $P(X_i)$ is true.

Proposition – Affine-Locality

Let $P : \mathbf{Aff} \to \mathbf{Prop}$ be affine-local. Define the predicate locally $P : \mathbf{Sch} \to \mathbf{Prop}$ by setting X is locally P when there exists an affine Zariski-cover $\mathcal{X}$ of X such that all $X_i \in \mathcal{X}$ satisfy P.

Then TFAE :

1. X is locally P
2. All opens U of X are locally P.
3. All affine opens U of X satisfy P.
4. There exists a Zariski cover $\mathcal{X}$ of X where all $X_i \in \mathcal{X}$ are locally P.

In particular, "locally P" is a Zariski-local property of schemes.

Proof.

$(1 \Rightarrow 2)$ Let $U \in \mathrm{Open}\, X$. Let $\mathcal{X}$ be an affine Zariski cover of X where all $X_i \in \mathcal{X}$ satisfy P. For each X_i, $X_i \cap U$ is an open of X_i and hence admits a Zariski covering $\mathcal{U}_i$ by basic opens of X_i. Since $P(X_i)$ is true, for every $U_{i,j} \in \mathcal{U}_i$, $P(U_{i,j})$ is true as well. Then note that $U_{i,j}$ are affine since X_i is and also open in U so the composite $\mathcal{U} := \bigcup_{X_i \in \mathcal{X}} \mathcal{U}_i$ gives an Zariski cover of U consisting of affines satisfying P.

$(2 \Rightarrow 3)$ Let $U \in \mathrm{Open}\, X$ be affine. We have an affine Zariski cover $\mathcal{U}$ of U consisting of opens satisfying P. Since P is affine-local, it suffices to find a Zariski cover of each $U_i \in \mathcal{U}$ by opens that are basic in *both* U and U_i. Well, we can certainly find a Zarisk cover $\mathcal{U}_i$ of U_i by basic opens of U. Then $\mathcal{U}_i \cup \{U\}$ is a basic Zariski cover of U, so its pullback is a basic Zariski cover of U_i. But its pullback is just $\mathcal{U}_i$ so $\mathcal{U}_i$ works.

$(3 \Rightarrow 4)$ By X being a scheme. $(4 \Rightarrow 1)$ Composites of open covers.

□

Proposition – Examples of Affine-Local Properties

The following predicates on $\mathbf{Aff}$ are affine-local :

1. For $X \in \mathbf{Aff}$, say X is *Noetherian* when $\mathcal{O}(X)$ is Noetherian.

2. For $X \in \mathbf{Aff}$, say X is *reduced* when $\mathcal{O}(X)$ has no nilpotent elements.

Definition – Globally

Let P be a affine-local property of affine schemes and $X \in \mathbf{Sch}$. We say X is *globally* P[a] when X is locally P and X is quasi-compact.

[a]This is non-standard terminology. A lot of affine-local properties are extended to schemes by quasi-compact + locally P. In this case, it is standard terminology to say simply say "X is P". However, this clashes with properties of schemes *not* coming from affine-local properties. The addition of the adverb "globally" is an attempt to highlight the fact that the property P is affine-local.

3 Properties of Morphisms

3.1 Permanence

Definition – Permanence

Let $P : \mathrm{Mor}(\mathbf{Sch}) \to \mathbf{Prop}$ be a predicate on morphisms of schemes. Then we say :

- P is *stable under composition* when for all $X \to Y \to Z$ in **Sch**, $P(X \to Y)$ and $P(Y \to Z)$ implies $P(X \to Z)$.
- P is *stable under base change* when for all pullback diagrams

$$\begin{tikzcd} X \arrow[r] & S \\ X \times_S Y \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

  $P(X \to S)$ implies $P(X \times_S Y \to Y)$.
- P is *stable under fiber product* when for all pullback diagrams as the above, $P(X \to S)$ and $P(Y \to S)$ implies $P(X \times_S Y \to S)$.

3.2 Zariski-Local Properties of Morphisms of Schemes

Definition – Zariski-Local on Target, on Source

Let $P : \mathrm{Mor}(\mathbf{Sch}) \to \mathbf{Prop}$ be a predicate on morphisms of schemes. Then we say :

- P is *Zariski-local on target* when for all $\varphi \in \mathbf{Sch}(X, Y)$ and Zariski covers $\mathcal{Y}$ of Y, $P(\varphi : X \to Y)$ is true if and only if for all $Y_i \in \mathcal{Y}$, $P(\varphi^{-1}Y_i \to Y_i)$ is true.
- P is *Zariski-local on source* when for all $\varphi \in \mathbf{Sch}(X, Y)$ and Zariski covers $\mathcal{X}$ of X, $P(\varphi : X \to Y)$ is true if and only if for all $X_i \in \mathcal{X}$, $P(X_i \to X \to Y)$ is true.

Definition – Affine-Local Properties of Morphisms

Let $P : \mathrm{Mor}(\mathbf{Aff}) \to \mathbf{Prop}$ be a predicate on morphisms of affine schemes. Then we say :

- P is *affine-local on target* when for all $\varphi \in \mathbf{Aff}(X, Y)$ and basic Zariski covers $\mathcal{Y}$ of Y, $P(\varphi : X \to Y)$ is true if and only if for all $Y_i \in \mathcal{Y}$, $P(\varphi^{-1}Y_i \to Y_i)$ is true.
- P is *affine-local on source* when for all $\varphi \in \mathbf{Aff}(X, Y)$ and basic Zariski covers $\mathcal{X}$ of X, $P(\varphi : X \to Y)$ is true if and only if for all $X_i \in \mathcal{X}$, $P(X_i \to X \to Y)$ is true.
- P is *affine-local* when the following three things are true :
  - (Half of Affine-Local on Target) For $\varphi \in \mathbf{Aff}(X, Y)$ and $f \in \mathcal{O}(Y)$, $P(\varphi : X \to Y)$ implies $P(\varphi^{-1}Y_f \to Y_f)$.
  - (Half of Affine-Local on Source) For $\varphi \in \mathbf{Aff}(X, Y)$ and basic Zariski cover $\mathcal{X}$ of X, if $P(X_i \to X \to Y)$ for every $X_i \in \mathcal{X}$, then $P(X \to Y)$.

– ("Zig-Zag") For $\varphi \in \mathbf{Aff}(X, Y)$, $f \in \mathcal{O}(X)$ and $g \in \mathcal{O}(Y)$,

$$\begin{tikzcd} X_f \arrow[r] \arrow[drr] & X \arrow[d] & \\ & Y_g \arrow[r] & Y \end{tikzcd}$$

$P(X \to Y_g)$ implies $P(X_f \to Y)$.

Remark – Frustration with Affine-Local Properties. It is true for properties of morphisms of affine schemes P that P affine-local implies P affine-local on target and source. I was hoping for this to be an equivalence, however the "zig-zag" seems to contain the extra information that $P(\operatorname{Sp} B \to (\operatorname{Sp} A)_f)$ implies $P(\operatorname{Sp} B \to \operatorname{Sp} A)$.

Proposition – Affine-Locality for Morphisms

Let $P : \operatorname{Mor}(\mathbf{Aff}) \to \mathbf{Prop}$ be an affine local property of morphisms of affine schemes. For $\varphi \in \mathbf{Sch}(X, Y)$, we say φ is *locally* P when there exists a Zariski cover $\mathcal{X}$ of X such that for each $X_i \in \mathcal{X}$, there exists an affine open Y_i of Y with $X_i \subseteq \varphi^{-1}Y_i$ and $P(X_i \to Y_i)$ true. Then for any $\varphi \in \mathbf{Sch}(X, Y)$, TFAE :

1. φ is locally P.
2. For all opens U, V of X, Y respectively with $U \subseteq \varphi^{-1}V$, $U \to V$ is locally P.
3. For all affine opens U, V of X, Y respectively with $U \subseteq \varphi^{-1}V$, $P(U \to V)$.
4. There exists an affine Zariski cover $\mathcal{Y}$ of Y and for each $Y_i \in \mathcal{Y}$ an affine Zariski cover $\mathcal{X}_i$ of $\varphi^{-1}Y_i$ such that for all $X_{ij} \in \mathcal{X}_i$, $P(X_{ij} \to Y_i)$.
5. There exists a Zariski cover $\mathcal{Y}$ of Y and for each $Y_i \in \mathcal{Y}$ a Zariski cover $\mathcal{X}_i$ of $\varphi^{-1}Y_i$ such that for all $X_{ij} \in \mathcal{X}_i$, $X_{ij} \to Y_i$ is locally P.