Algebraic Geometry : Functor of Points POV

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Contents

Goal : Develop all of basic scheme theory without a single mention of locally ringed spaces.

1 Schemes

1.1 $\mathbb{Z}$-Functors

Definition – $\mathbb{Z}$-Functor

Define $\mathbb{M} := \mathbb{Z}\mathbf{Alg}^{op}$. The *category of $\mathbb{Z}$-functors* $\mathbb{M}\mathbf{Set}$ is define to be the category of presheaves of sets on $\mathbb{M}$. For $X \in \mathbb{M}\mathbf{Set}$ and $A^{op} \in \mathbb{M}$, elements of $X(A^{op})$ are called *morphisms from A^{op} to X*.

Define $\mathrm{Sp} : \mathbb{M} \to \mathbb{M}\mathbf{Set}$ to be the fully faithful Yoneda embedding $A^{op} \mapsto \mathbb{M}(-, A^{op})$. For $A \in \mathbb{Z}\mathbf{Alg}$, $\mathrm{Sp}\, A$ is called the *spectrum of A*. The *category of affine schemes* is defined to be the essential image of Sp. We will denote it with $\mathbf{Aff}$.

Remark – Intuition of $\mathbb{Z}$-Functors and Spectrums. In differential geometry, given a morphism of smooth manifolds $\varphi : X \to Y$, one obtains a ring morphism $\varphi^\flat : C^\infty(Y) \to C^\infty(X)$ that is the pullback of global smooth functions. For algebraic geometry, here's how to think about it :

– $\mathbb{M}$ is the category of "test spaces". For smooth manifolds, this is open subsets of $\mathbb{R}^n$.

– A ring A is the "ring of global functions on A^{op}".

– A ring morphism $\varphi : A \to B$ is the "pullback of global functions along $\varphi^{op} : B^{op} \to A^{op}$".

– An $\mathbb{Z}$-functor $X \in \mathbb{M}\mathbf{Set}$ is something our "test spaces" in $\mathbb{M}$ can "map into". Essentially,

$$\text{“}X(A^{op}) = \mathrm{Mor}(A^{op}, X)\text{”}$$

Then given $\varphi^{op} : B^{op} \to A^{op}$, one should be able to turn "maps" $\alpha : A^{op} \to X$ into $\alpha \circ \varphi^{op} : B^{op} \to X$. This is precisely $X(A^{op}) \to X(B^{op})$ and functoriality simply expresses how pre-composition respects the identity morphisms and composition.

– The spectrum functor Sp realizes a "space" A^{op} in $\mathbb{M}$ as something our "test spaces" in $\mathbb{M}$ can map to. Then the intuition of "$X(A^{op}) = \mathrm{Mor}(A^{op}, X)$" is realised as $X(A^{op}) \cong \mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X)$. This is Yoneda's lemma.

– $\mathbf{Aff}$ formalizes what we mean by "test spaces in $\mathbb{M}$".

– $\mathbb{M}\mathbf{Set}$ is complete and cocomplete as a category (since $\mathbf{Set}$ is), i.e. it's the perfect playground for building more general "spaces" out of affine schemes.

Proposition – Yoneda

The following are true :

– ("Morphisms from A^{op} to $X \longleftrightarrow$ Morphisms from $\mathrm{Sp}\, A$ to X") For $X \in \mathbb{M}\mathbf{Set}$ and $A^{op} \in \mathbb{M}$,

$$\mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X) \xrightarrow{\sim} X(A^{op})$$

by $\alpha \mapsto \alpha_{A^{op}}(\mathbb{1}_{A^{op}})$. Furthermore, this is functorial in A^{op} and X.

– (Density of Representables / "The data of X is precisely how test spaces map into it") For $X \in \mathbb{M}\mathbf{Set}$, X is the colimit of the diagram $\mathrm{Sp} \downarrow X \to \mathbb{M}\mathbf{Set}$.

Proof. Straightforward. $\square$

Proposition – Affine Line

Let $n \in \mathbb{N}$. Define *affine n-space* to be $\mathbb{A}^n \in \mathbb{M}\mathbf{Set}$ sending $A^{op} \mapsto A^n$. Then

- $\mathbb{A}^n$ is representable by $\mathbb{Z}[T_1, \dots, T_n]^{op}$. Hence $\mathbb{A}^n \in \mathbf{Aff}$.
- for $n = 1$, $\mathbb{A}^1$ is a ring object in $\mathbb{M}\mathbf{Set}$. Hence for $X \in \mathbb{M}\mathbf{Set}$, $\mathcal{O}(X) := \mathbb{M}\mathbf{Set}(X, \mathbb{A}^1) \in \mathbb{Z}\mathbf{Alg}$. This is called the *ring of global functions on X* and gives a functor $\mathcal{O}(\star) : \mathbb{M}\mathbf{Set} \to \mathbb{Z}\mathbf{Alg}^{op}$. We call elements of $\mathcal{O}(X)$ *functions on X*.

  Concretely on morphisms $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$, the corresponding ring morphism $\varphi^\flat : \mathcal{O}(X) \to \mathcal{O}(Y)$ is given by $f \mapsto f \circ \varphi$, i.e. pulling back along φ.
- (Spec, Global Function Adjunction)

$$\mathbb{M}\mathbf{Set}(-, \operatorname{Sp} \star) \cong \mathbb{Z}\mathbf{Alg}(\star, \mathcal{O}(-))$$

- (Canonical Choice of Spectrum for Affine Schemes) for $X \in \mathbb{M}$, X is affine if and only if $\mathbb{1}^{\perp}_{\mathcal{O}(X)} : X \to \operatorname{Sp} \mathcal{O}(X)$ is an isomorphism of $\mathbb{Z}$-functors, where $\mathbb{1}^{\perp}_{\mathcal{O}(X)}$ is the adjunct of $\mathbb{1}_{\mathcal{O}(X)}$ under the previous adjunction.

Proof. (*Spec, Global Function Adjunction*) Follows from this chain of bijections functorial in A and X given by the density of representables :

$$\begin{aligned}\mathbb{M}\mathbf{Set}(X, \operatorname{Sp} A) &\cong \varprojlim_{(B,\beta) \in \operatorname{Sp}\downarrow X} \mathbb{M}\mathbf{Set}(\operatorname{Sp} B, \operatorname{Sp} A) \cong \varprojlim_{(B,\beta) \in \operatorname{Sp}\downarrow X} \mathbb{Z}\mathbf{Alg}(A, B) \cong \mathbb{Z}\mathbf{Alg}\left(A, \varprojlim_{(B,\beta) \in \operatorname{Sp}\downarrow X} B\right) \\ &\cong \mathbb{Z}\mathbf{Alg}\left(A, \varprojlim_{(B,\beta) \in \operatorname{Sp}\downarrow X} \mathbb{M}\mathbf{Set}(\operatorname{Sp} B, \mathbb{A}^1)\right) \cong \mathbb{Z}\mathbf{Alg}(A, \mathbb{M}\mathbf{Set}(X, \mathbb{A}^1)) = \mathbb{Z}\mathbf{Alg}(A, \mathcal{O}(X))\end{aligned}$$

(*Affine Schemes*) The reverse implication is clear. Let $X \xrightarrow{\sim} \operatorname{Sp} A$ be an isomorphism in $\mathbb{M}\mathbf{Set}$. The adjunction $\mathbb{M}\mathbf{Set}(X, \operatorname{Sp} A) \cong \mathbb{Z}\mathbf{Alg}(A, \mathcal{O}(X)) \cong \mathbb{M}\mathbf{Set}(\operatorname{Sp} \mathcal{O}(X), \operatorname{Sp} A)$ gives the commutative diagram :

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d] & \operatorname{Sp} A \\ \operatorname{Sp} \mathcal{O}(X) \arrow[ur, dashed] & \end{tikzcd}$$

where the dashed has the composition of the other two morphisms as an inverse, and hence an isomorphism.

□

Remark – Intuition of Affine n-Space. For a smooth manifold X, a smooth map $\varphi : X \to \mathbb{R}^n$ is equivalent to the data of n global smooth functions $f_1, \dots, f_n$ on X, i.e.

$$C^\infty\mathbf{Mfd}(X, \mathbb{R}^n) \cong C^\infty(X)^n$$

"$\mathbb{R}^n$ is the classifying space of n-tuples of global smooth functions." In the functorial POV of algebraic geometry, we take this as our definition of affine n-space.

Proposition – Categorical Properties of $\mathbb{Z}$-Functors

The following are true :

- (Completeness and Cocompleteness) $\mathbb{M}\mathbf{Set}$ has all (small) limits and colimits, all computed pointwise.

  In particular, a morphism $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$ is respectively a mono/epi/isomorphism if and only if for all $A \in \mathbb{M}^{op}$, $\varphi_A : Y(A) \to X(A)$ is injective/surjective/bijective.

- (Base Change) For $K \in \mathbf{CRing}$, define the *category of K-functors* to be the over-category $\mathbb{M}\mathbf{Set}/ \operatorname{Sp} K$. In particular, we call $\mathbf{Aff}/ \operatorname{Sp} K$ the *category of affine K-schemes*.

  Then we have for $\varphi \in \mathbb{M}\mathbf{Set}(\operatorname{Sp} L, \operatorname{Sp} K)$, we have the following adjunction

  $$(\mathbb{M}\mathbf{Set}/ \operatorname{Sp} L)\,(-, \operatorname{Sp} L \times_{\operatorname{Sp} K} (\star)) \cong (\mathbb{M}\mathbf{Set}/ \operatorname{Sp} K)\,(-, \star)$$

  Furthermore, this restricts to an adjunction between $\mathbf{Aff}/ \operatorname{Sp} L$ and $\mathbf{Aff}/ \operatorname{Sp} K$, i.e. the pullback of affine schemes is affine.

Proof. (*Base Change*) The first adjunction is categorical. For the restriction to affine schemes over K and L, note that for any K-algebra A,

$$\operatorname{Sp} L \times_{\operatorname{Sp} K} \operatorname{Sp} A \text{ “=” } \operatorname{Sp}\,(L \otimes_K A)$$

□

1.2 Covers and Open Subfunctors

Remark. A scheme will be a “space modelled on $\mathbb{M}$ with an open cover by affine schemes”. This section defines the notion of “open subfunctors” of a $\mathbb{Z}$-functor and an “open cover”. Since $\mathbb{Z}$-functors are meant to be modelled on $\mathbb{M}$, we first define everything for affine schemes.

Somehow, no one could eradicate the special role of fields as “points”. Perhaps it should be local rings instead. Something about points of the topos $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$ corresponding to local rings…

Definition – Points and Covers

[a] Define the *category of test points*, Pts, to be the full subcategory of $\mathbb{M}$ consisting of K^{op} where K is a field. For $X \in \mathbb{M}\mathbf{Set}$, a *point of* X is defined to be a morphism $x \in \mathbb{M}\mathbf{Set}(\operatorname{Sp} K, X)$ where $K^{op} \in \mathrm{Pts}$. For $\varphi \in \mathbb{M}\mathbf{Set}(X, Y)$, we will use $\varphi(x)$ to denote $\varphi \circ x$. For $f \in \mathcal{O}(X)$, we will use ev_x to denote the pullback $x^\flat : \mathcal{O}(X) \to \mathcal{O}(K)$.

For $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$, we will use $Y^{\mathrm{Pts}}, X^{\mathrm{Pts}}$ to denote the restriction of Y, X to Pts^{op} and $\varphi^{\mathrm{Pts}} : Y^{\mathrm{Pts}} \to X^{\mathrm{Pts}}$ the restricted morphism. Then we say φ is *surjective on points* when φ^{Pts} is an epimorphism (equivalently, component-wise surjective).

For a subset $\mathcal{U} \subseteq \mathbb{M}\mathbf{Set} \downarrow X$ for any $X \in \mathbb{M}\mathbf{Set}$, we say $\mathcal{U}$ *covers* X when $\coprod \mathcal{U} \to X$ is surjective on points. Equivalently, for all points $x : \operatorname{Sp} K \to X$ of X, there exists $U \in \mathcal{U}$ with a factoring

$$\begin{tikzcd} U \arrow[r] & X \\ & \operatorname{Sp} K \arrow[ul] \arrow[u, "x"'] \end{tikzcd}$$

[a]Pts and "surjective on points" is non-standard definition.

Proposition – Base Change of Cover
Let $\mathcal{U}$ be a cover of $X \in \mathbb{M}\mathbf{Set}$. Then for all $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$, the set $\varphi^{-1}\mathcal{U}$ of pullbacks of morphisms in $\mathcal{U}$ forms a cover of Y.

Proof. Follows from fiber product in $\mathbb{M}\mathbf{Set}$ being computed component-wise. $\square$

Proposition – Multiplicative Group Scheme
Consider the functor $\mathbb{G}^\times \in \mathbb{M}\mathbf{Set}$ defined by $A \in \mathbb{M}^{op} \mapsto A^\times$. Then

- $\mathbb{G}^\times$ is representable by the ring $\mathbb{Z}[T, T^{-1}]$ and hence affine.
- $\mathbb{G}^\times$ is a group object in $\mathbb{M}\mathbf{Set}$. In fact, for any $X \in \mathbb{M}\mathbf{Set}$, $\mathbb{M}\mathbf{Set}(X, \mathbb{G}^\times) = \mathcal{O}(X)^\times$.

Proof. UP of $\mathbb{Z}[T, T^{-1}]$ implies it represents $\mathbb{G}^\times$. The second property can be straightforwardly deduced either from the Spec-global functions adjunction or elementarily. $\square$

Proposition – Basic Opens of a $\mathbb{Z}$-Functor
Let $X \in \mathbb{M}\mathbf{Set}$ and $f \in \mathcal{O}(X)$. The *support of f*, X_f, is defined as the subfunctor of X sending $A \in \mathbb{M}^{op}$ to the set of $\alpha \in \mathbb{M}\mathbf{Set}(\operatorname{Sp} A, X)$ such that $\varphi^\flat(f) \in A^\times$.

Then X_f is the pullback of $\mathbb{G}^\times$ along $f : X \to \mathbb{A}^1$.

$$\begin{tikzcd} X_f \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{G}^\times \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

Subfunctors of X of the form X_f are called *basic opens*.

Proof. Easy. $\square$

Remark – Intuition of Multiplicative Group Scheme and Basic Opens. For a smooth manifold X,

$$C^\infty\mathbf{Mfd}(X, \mathbb{R}^\times) \cong C^\infty(X)^\times$$

"$\mathbb{R}^\times$ is classifying space for invertible global functions on X." One can thus think of $\mathbb{G}^\times$ as "$\mathbb{A}^1 \setminus \{0\}$". A basic open X_f is then just the preimage of "$\mathbb{A}^1 \setminus \{0\}$" under $f : X \to \mathbb{A}^1$.

Proposition – Opens of Affine Schemes
Let $X \in \mathbf{Aff}$. For $I \in \mathrm{Ideal}\,\mathcal{O}(X)$, define $D(I) \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$ by

$$A \in \mathbb{M}^{op} \mapsto \left\{\varphi \in \mathbf{Aff}(\mathrm{Sp}\,A, X) \mid A\varphi^\flat I = A\right\}$$

In particular, for $I = (f)$, we have $D(I) = X_f$. Sometimes, we use $D(f)$ to denote X_f.

Then

- (Basic Opens are Affine) for $f \in \mathcal{O}(X)$, $D(f)$ is representable by $\mathcal{O}(X)[f^{-1}]$.
- (Ideals to Opens) For $I, J \in \mathrm{Ideal}\,\mathcal{O}(X)$, $I \subseteq J$ implies $D(I) \subseteq D(J)$.

  This defines $D : \mathrm{Ideal}\,\mathcal{O}(X) \to \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$. The category $\mathrm{Open}\,X$ of *opens of* X is defined as the essential image of D. We call $U \in \mathrm{Open}\,X$ an *open of* X.
- (Intuitive Definition of Opens) for $I \in \mathrm{Ideal}\,\mathcal{O}(X)$, $\{D(f)\}_{f \in I}$ covers $D(I)$.
- (Partition of Unity) For $I \in \mathrm{Ideal}\,\mathcal{O}(X)$, $D(I)$ covers X if and only if there exists finite $I_0 \subseteq I$ such that $AI_0 = A$. Such $I_0 \subseteq A$ are called *partitions of unity*.
- (Base Change / "Preimage of Opens are Open") Let $\varphi \in \mathbf{Aff}(Y, X)$, $I \in \mathrm{Ideal}\,\mathcal{O}(X)$. Let the following be a pullback diagram :

$$\begin{tikzcd} Y \arrow[r, "\varphi"] & X \\ \varphi^{-1}D(I) \arrow[u] \arrow[r] & D(I) \arrow[u] \end{tikzcd}$$

  Then $\varphi^{-1}D(I) = D(\varphi^\flat I)$.
- (Fiber Product) Let $X, X_1 \in \mathbf{Aff}/Y$ where $Y \in \mathbf{Aff}$. Let I, I_1 be ideals of $\mathcal{O}(X)$, $\mathcal{O}(X_1)$ respectively. Then we have the pullback diagram :

$$\begin{tikzcd} D(I) \arrow[r] & Y \\ D(I \otimes_{\mathcal{O}(Y)} I_1) \arrow[u] \arrow[r] & D(I_1) \arrow[u] \end{tikzcd}$$

  In particular, $X = X_1 = Y$ implies the intersection of two opens of X is an open of X. The special case of $D(I) = D(f)$ and $D(I_1) = D(f_1)$ yields $D(f) \cap D(f_1) = D(ff_1)$.

Proof.

(*Basic Opens are Affine*) UP of $\mathcal{O}(X)[f^{-1}]$ as an $\mathcal{O}(X)$ algebra.

(*Intuitive Def of Opens*) Let $x : \mathrm{Sp}\,K \to X$ be a point of X. Then $Kx^\flat I = K$ if and only if there exists $f \in I$ with $f(x) \in K^\times$.

(*Partition of Unity*) Having finite $I_0 \subseteq I$ with $AI_0 = A$ is equivalent to $AI = A$. Clearly, $AI = A$ implies $D(I)$ covers $\operatorname{Sp} A$. Conversely, suppose $AI \subsetneq A$. $D(I)$ not covering $\operatorname{Sp} A$ is the same as it "missing a point of $\operatorname{Sp} A$", that is to say we are looking for a point $x : \operatorname{Sp} K \to \operatorname{Sp} A$ of $\operatorname{Sp} A$ that doesn't admit a lift across $D(I) \to \operatorname{Sp} A$. This is the same as $I \subseteq \ker \mathrm{ev}_x$. Well, $AI \subsetneq A$ implies by Zorn's lemma the existence of a map $\mathrm{ev}_x : A \to K$ where K is a field with the desired property. □

> *Counter Example* ($\bigcup_{f \in I} D(f) = D(I)$).
> *Consider the ring* $\mathbb{F}_2 \times \mathbb{F}_2$ *and elements* $(1,0), (0,1)$. *The ideal* I *generated by these is the whole ring. But* $D((1,0))(\mathbb{F}_2 \times \mathbb{F}_2) \cup D((0,1))(\mathbb{F}_2 \times \mathbb{F}_2) \subsetneq D(I)(\mathbb{F}_2 \times \mathbb{F}_2)$ *since the ring endomorphism* $(a,b) \mapsto (b,a)$ *doesn't map any of* $(1,0), (0,1)$ *to units. Thus* $D((1,0)) \cup D((0,1)) \subsetneq D(I)$.

Definition – Open Subfunctor

Let $X \in \mathbb{M}\mathbf{Set}$. For $U \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$, U is called *open* when for all $\varphi : \operatorname{Sp} A \to X$, the pullback $\varphi^{-1}U$ of U along φ is an open of $\operatorname{Sp} A$.

$$\begin{tikzcd} \operatorname{Sp} A \arrow[r, "\varphi"] & X \\ \varphi^{-1}U \arrow[r] \arrow[u] & U \arrow[u] \end{tikzcd}$$

We will use $\operatorname{Open} X$ to denote the full subcategory of opens of X in $\mathbf{Sub}\mathbb{M}\mathbf{Set}(X)$. When X is affine, this agrees with our specialized notion for affine schemes.

Proposition – Basic Facts about Open Subfunctors

The following are true :

- ("Extensionality") Let $U, V \in \operatorname{Open} X$. Then $U = V$ if and only if $U^{\mathrm{Pts}} = V^{\mathrm{Pts}}$.
- (Composition) Let $V \in \operatorname{Open} U$, $U \in \operatorname{Open} X$, $X \in \mathbb{M}\mathbf{Set}$. Then $V \in \operatorname{Open} X$.
- (Base Change/"Preimage of Opens are Opens") Let $X \in \mathbb{M}\mathbf{Set}$, $U \in \operatorname{Open} X$ and $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$. Then the pullback $\varphi^{-1}U$ of U along φ is an open of Y.

$$\begin{tikzcd} Y \arrow[r, "\varphi"] & X \\ \varphi^{-1}U \arrow[r] \arrow[u] & U \arrow[u] \end{tikzcd}$$

- (Fiber Product) Let U, U_1 be opens of $X, X_1 \in \mathbb{M}\mathbf{Set}$ respectively. Then for any $X \to S, X_1 \to S$, the fiber product $U \times_S U_1$ is an open of $X \times_S X_1$. In particular, for $X = X_1 = S$, this proves the intersection of two opens of X is again an open of X.

Proof. (Extensionality) Reduce to affine global case and use partition of unity. □

1.3 The Big and Small Zariski Site

Proposition – Big Zariski Site on $\mathbb{M}\mathbf{Set}$

For $X \in \mathbb{M}\mathbf{Set}$ and $\mathcal{U} \subseteq \mathbb{M}\mathbf{Set}/X$ a collection of morphisms into X, define $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ when "$\mathcal{U}$ is isomorphic to an open cover", meaning there exists $\{U_i\}_{i\in\mathcal{U}} \subseteq \mathrm{Open}\, X$ such that $\{U_i\}_{i\in\mathcal{U}}$ is a cover of X and for all $i \in \mathcal{U}$, $(i : s(i) \to X) \cong (U_i \to X)$ in $\mathbb{M}\mathbf{Set}/X$. Then the above defines a Grothendieck pretopology of $\mathbb{M}\mathbf{Set}$. Specifically :

- (Isomorphisms are Covers) For $X \in \mathbb{M}\mathbf{Set}$ and $\varphi \in \mathbb{M}\mathbf{Set}(U, X)$, φ iso implies $\{\varphi\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.
- (Pullback of Covers) For all $\varphi \in \mathbb{M}\mathbf{Set}(Y, X)$ and $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$, $\varphi^{-1}\mathcal{U} := \{Y \times_X s(i) \to Y\}_{i\in\mathcal{U}} \in \mathrm{Cov}_{\mathrm{Zar}}(Y)$.
- (Composite of Covers) Let $X \in \mathbb{M}\mathbf{Set}, \mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ and for each $i \in \mathcal{U}$, let $\mathcal{U}_i \in \mathrm{Cov}_{\mathrm{Zar}}(s(i))$. Then $\{s(j_i) \to s(i) \to X \mid i \in \mathcal{U}, j_i \in \mathcal{U}_i\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.

We will use $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$ to denote the site $\mathbb{M}\mathbf{Set}$ endowed with the topology generated by the above pretopology. We will call $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$ the *big Zariski site*. $\mathcal{X} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ are called *Zariski covers of* X.

Proof. Only slightly non-trivial part is pullback of covers. Use opens and covers preserved under base change. □

Remark – Intuition of Sheaves on $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$. For $X \in \mathbb{M}\mathbf{Set}$, if X is to be a "space" then for any other $Y \in \mathbb{M}\mathbf{Set}$ and open cover $\mathcal{U}$ of Y, the data of a morphism $Y \to X$ should be the same as a collection of morphisms $(U \to X)_{U\in\mathcal{U}}$ that agree on pairwise intersection. This is precisely what it means for $\mathbb{M}\mathbf{Set}(-, X)$ to be a sheaf on the site $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Remark. The following is a smaller site $\mathbf{Aff}_{\mathrm{Zar}}$ on affine schemes, where open covers consists only of basic opens. Since basic opens generate opens for affine schemes and affine schemes generate $\mathbb{M}\mathbf{Set}$ with compatible notion of opens, sheaves on $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$ will be the same as sheaves of $\mathbf{Aff}_{\mathrm{Zar}}$. This gives an easier check for when $X \in \mathbb{M}\mathbf{Set}$ is a sheaf on $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Proposition – Small Zariski Site on Aff

For $X \in \mathbf{Aff}$ and $\mathcal{U} \subseteq \mathbf{Aff}/X$, $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ when "$\mathcal{U}$ is isomorphic to a cover of X by basic opens", meaning there exists a cover $\{X_{f_\iota}\}_{\iota\in\mathcal{U}}$ where for all $\iota \in \mathcal{U}$, $(s(\iota) \to X) \cong (D(f_\iota) \to X)$ in $\mathbf{Aff}/X$. Then the above defines a Grothendieck pretopology on $\mathbf{Aff}$, specifically :

- (Isomorphisms are Covers) For all $X \in \mathbf{Aff}$ and $\iota \in \mathbf{Aff}(U, X)$, ι isomorphism implies $\{\iota\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.
- (Pullback of Covers) For all $\varphi \in \mathbf{Aff}(Y, X)$ and $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$, $\varphi^{-1}\mathcal{U} := \{Y \times_X s(\iota) \to Y \mid \iota \in \mathcal{U}\} \in \mathrm{Cov}_{\mathrm{Zar}}(Y)$.
- (Composite of Covers) Let $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ and for each $i \in \mathcal{U}$, let $\mathcal{U}_i \in \mathrm{Cov}_{\mathrm{Zar}}(s(i))$. Then $\{s(j_i) \to s(i) \to X \mid i \in \mathcal{U}, j_i \in \mathcal{U}_i\} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$.

We will use $\mathbf{Aff}_{\mathrm{Zar}}$ to denote the site $\mathbf{Aff}$ with the topology given by the above pretopology. We will call $\mathbf{Aff}_{\mathrm{Zar}}$ the *small Zariski site*. $\mathcal{X} \in \mathrm{Cov}_{\mathrm{Zar}}(X)$ will be called *basic Zariski covers of* X. [a]

[a] This is non-standard terminology, but helps avoid confusion between the topology on $\mathbf{Aff}$ just defined and the induced

topology from $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Proof. UP of tensor products and localization. $\square$

Proposition – Sheaves on Big and Small Zariski Site are the Same
Let $X \in \mathbb{M}\mathbf{Set}$. Then $\mathbb{M}\mathbf{Set}(-, X) \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$ if and only if $\mathbb{M}\mathbf{Set}(-, X) \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$.

Proof. Forward implication follows since the covers in $\mathbf{Aff}_{\mathrm{Zar}}$ are covers in $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$.

Now let $\mathbb{M}\mathbf{Set}(-, X) \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$. Let $U \in \mathbb{M}\mathbf{Set}$ and $\mathcal{U} \in \mathrm{Cov}_{\mathrm{Zar}}(U)$. Then for $(A, \alpha) \in \mathrm{Sp} \downarrow U$, the pullback $\alpha^{-1}\mathcal{U}$ of $\mathcal{U}$ is a cover of $\mathrm{Sp}\, A$ in the big Zariski site. The chain of isomorphisms to be justified is :

$$\begin{aligned}\mathbb{M}\mathbf{Set}(U, X) &\overset{(1)}{\cong} \varprojlim_{(A,\alpha)\in \mathrm{Sp}\downarrow U} \mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X) \overset{(2)}{\cong} \varprojlim_{(A,\alpha)\in \mathrm{Sp}\downarrow U} \varprojlim_{V,W\in\mathcal{U}} \mathbb{M}\mathbf{Set}(\alpha^{-1}V \cap \alpha^{-1}W, X) \\ &\overset{(3)}{\cong} \varprojlim_{V,W\in\mathcal{U}} \varprojlim_{(A,\alpha)\in \mathrm{Sp}\downarrow U} \mathbb{M}\mathbf{Set}(\alpha^{-1}V \cap \alpha^{-1}W, X) \overset{(4)}{\cong} \varprojlim_{V,W\in\mathcal{U}} \mathbb{M}\mathbf{Set}(V \cap W, X)\end{aligned}$$

(1) Density of representables. (3) Limits commute with limits.

(4) We know $\alpha^{-1}(V \cap W) = \alpha^{-1}V \cap \alpha^{-1}W$, so it suffices to prove the following.

Lemma. For $U \in \mathbb{M}\mathbf{Set}$ and $Z \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(U)$, we have $Z = \varprojlim_{(A,\alpha)\in\mathrm{Sp}\downarrow U} \alpha^{-1}Z$

Proof. The forgetful functor $\mathrm{Sp} \downarrow Z \to \mathrm{Sp} \downarrow U$ is a "section" of the pullback functor $\mathrm{Sp} \downarrow U \to \mathrm{Sp} \downarrow Z$, meaning for $(A, \alpha) \in \mathrm{Sp} \downarrow Z$, the following is a pullback diagram :

$$\begin{tikzcd} Z \arrow[r] & U \\ \mathrm{Sp}\, A \arrow[u, "\alpha"] \arrow[r, "\mathbb{1}"] & \mathrm{Sp}\, A \arrow[u] \end{tikzcd}$$

This implies pulling the diagram $\mathrm{Sp} \downarrow U$ back to $\mathrm{Sp} \downarrow Z$ only introduces duplicate objects with identity morphisms in between them. Hence $\varprojlim_{(A,\alpha)\in\mathrm{Sp}\downarrow U} \alpha^{-1}Z = \varprojlim_{(A_1,\alpha_1)\in\mathrm{Sp}\downarrow Z} \mathrm{Sp}\, A = Z$ by the density of representables. $\blacksquare$

(2) We need to show that $\mathbb{M}\mathbf{Set}(-, X)$ is a sheaf for $\mathbf{Aff}$ with covers from the big Zariski site $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$. The key is that basic opens cover opens for affine schemes.

Let $A \in \mathbb{M}^{op}$ and $\mathcal{U}$ be a $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$-cover of $\mathrm{Sp}\, A$. For each $i \in \mathcal{U}$, let $I_i \in \mathrm{Ideal}\, A$ with $i = D(I_i)$. Let $I := \bigsqcup_{i\in\mathcal{U}} I_i$. Then since $\{D(f)\}_{f\in I_i}$ is a $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$-cover of i for every $i \in \mathcal{U}$, $\{D(f)\}_{f\in I}$ is also a $\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}$-cover of $\mathrm{Sp}\, A$. We then have the commutative diagram :

$$\begin{array}{ccc}\mathbb{M}\mathbf{Set}(\operatorname{Sp} A, X) & \longrightarrow & \varprojlim_{i,j\in\mathcal{U}} \mathbb{M}\mathbf{Set}(i\cap j, X) \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle\sim} \\ \mathbb{M}\mathbf{Set}(\operatorname{Sp} A, X) & \xrightarrow{\sim} & \varprojlim_{f,g\in I} \mathbb{M}\mathbf{Set}(D(f)\cap D(g), X)\end{array}$$

where the horizontal isomorphism to due to $\mathbb{M}\mathbf{Set}(-, X)$ being a sheaf on $\mathbf{Aff}_{\mathrm{Zar}}$.

It remains to justify the vertical isomorphism. To do this, we apply the same argument as we're trying to do now, but on $i \cap j$. It's easy to see that $\{D(f)\cap D(g)\}_{f\in I_i, g\in I_j}$ covers $i\cap j$, so we get

$$\begin{aligned}\mathbb{M}\mathbf{Set}(i\cap j, X) &\cong \varprojlim_{(A_1,\alpha_1)\in \mathrm{Sp}\downarrow(i\cap j)} \mathbb{M}\mathbf{Set}(\operatorname{Sp} A_1, X) \cong \varprojlim_{(A_1,\alpha_1)\in \mathrm{Sp}\downarrow(i\cap j)} \varprojlim_{f\in I_i, g\in I_j} \mathbb{M}\mathbf{Set}(D(\alpha_1^\flat(f))\cap D(\alpha_1^\flat(g)), X) \\ &\cong \varprojlim_{f\in I_i, g\in I_j} \varprojlim_{(A_1,\alpha_1)\in \mathrm{Sp}\downarrow U} \mathbb{M}\mathbf{Set}(\alpha_1^{-1}(D(f)\cap D(g)), X) \overset{(4)}{\cong} \varprojlim_{f\in I_i, g\in I_j} \mathbb{M}\mathbf{Set}(D(f)\cap D(g), X)\end{aligned}$$

where (4) is as before.

$\square$

1.4 Schemes

> **Definition – Schemes**
>
> Let $X \in \mathbb{M}\mathbf{Set}$. Then X is called a *scheme* when we have :
>
> – ("Is a Space") $X \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$, equivalently $\mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$.
>
> – (Open cover by Affine schemes) there exists $\mathcal{U} \subseteq \operatorname{Open} X$ such that $\mathcal{U}$ covers X and every $U \in \mathcal{U}$ is affine.
>
> We use **Sch** to denote the full subcategory of schemes in $\mathbb{M}\mathbf{Set}$.

Remark – Intuition of Definition of Schemes. In the same way that smooth manifolds are spaces modeled on $\mathbb{R}^n$ that is locally $\mathbb{R}^n$, schemes are spaces modeled on $\mathbb{M}$ that is "locally $\mathbb{M}$". In particular, objects of $\mathbb{M}$ ought to be schemes.

> **Proposition – Affine Schemes are Schemes**
>
> Let $X \in \mathbf{Aff}$. Then $X \in \mathbf{Sch}$.

Proof. X is an affine open cover of itself, so it suffices to check the sheaf condition. Since $\mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}}) = \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$, it suffices to check that for $(A, \alpha) \in \operatorname{Sp}\downarrow X, \mathcal{U}$ a $\mathbf{Aff}_{\mathrm{Zar}}$-cover of $\operatorname{Sp} A$, we have

$$\mathbb{M}\mathbf{Set}(\operatorname{Sp} A, X) \xrightarrow{\sim} \varprojlim_{U,V\in\mathcal{U}} \mathbb{M}\mathbf{Set}(U\cap V, X)$$

Since $X = \operatorname{Sp}\mathcal{O}(X)$ and $\operatorname{Sp} : \mathbb{M}^{op} \to \mathbb{M}\mathbf{Set}$ is fully faithful, this is equivalent to

$$\mathbb{Z}\mathbf{Alg}(\mathcal{O}(X), \mathcal{O}(\operatorname{Sp} A)) \xrightarrow{\sim} \mathbb{Z}\mathbf{Alg}(\mathcal{O}(X), \varprojlim_{U,V\in\mathcal{U}} \mathcal{O}(U\cap V))$$

It thus suffices to show that

$$A = \mathcal{O}(\mathrm{Sp}\, A) \xrightarrow{\sim} \varprojlim_{U,V \in \mathcal{U}} \mathcal{O}(U \cap V)$$

where the isomorphism is from the restriction maps $\mathcal{O}(\mathrm{Sp}\, A) \to \mathcal{O}(U \cap V)$.

Let $(f_V \in \mathcal{O}(V))_{V \in \mathcal{U}}$ that agrees on pairwise intersections. Suppose for a moment, for any finite subcover $\mathcal{U}_0 \subseteq \mathcal{U}$, we have a unique $f_{\mathcal{U}_0} \in A$ that agree with f_V on $V \in \mathcal{U}_0$. WLOG $\mathcal{U} = \{D(f)\}_{f \in I}$ for some $I \subseteq A$. Then $\mathcal{U}$ covers $\mathrm{Sp}\, A$ implies $D(I)$ covers $\mathrm{Sp}\, A$, which implies $AI = A$, which gives a *finite* subset $I_0 \subseteq I$ where $AI_0 = A$. Hence, we do have a finite subcover $\mathcal{U}_0$ and such $f_{\mathcal{U}_0}$. Furthermore, for any $V \in \mathcal{U}$, $f_{\mathcal{U}_0} = f_{\mathcal{U}_0 \cup \{V\}}$ by uniqueness of $f_{\mathcal{U}_0}$ on $\mathcal{U}_0$ so

$$\downarrow^V f_{\mathcal{U}_0} = \downarrow^V f_{\mathcal{U}_0 \cup \{V\}} = f_V$$

So $f_{\mathcal{U}_0}$ actually agrees with f_V on all $V \in \mathcal{U}$. Furthermore, it is unique, again by uniqueness on $\mathcal{U}_0$. Thus, it suffices to do the case of $\mathcal{U}$ *finite*.

The naive idea is this : if each $f_V = g_V/h_V$ with $V = D(h_V)$, then "agreeing on intersections" *should* mean $g_V h_W = g_W h_V \in \mathcal{O}(U) = A$. We can then use a partition of unity $1 = \sum_{V \in \mathcal{U}} \lambda_V h_V$ to patch :

$$f_V = \frac{g_V}{h_V} = \frac{\sum_{W \in \mathcal{U}} \lambda_W g_V h_W}{h_V} = \frac{\sum_{W \in \mathcal{U}} \lambda_W g_W h_V}{h_V} = \frac{\sum_{W \in \mathcal{U}} \lambda_W g_W}{1}$$

So $f := \sum_{W \in \mathcal{U}} \lambda_W g_W \in A = \mathcal{O}_{\mathrm{Spec}\, A}(U)$ is the guy we want. This is even unique since if we have another such f_1, then $f/1 = f_1/1 \in \mathcal{O}(V) \cong A_{h_V}$ implies the existence of $N_V \in \mathbb{N}$ such that $(f - f_1)h_V^{N_V} = 0$. By *finiteness of* $\mathcal{U}$, we can pick a single $N \in \mathbb{N}$ with $(f - f_1)h_V^N = 0$ for all $V \in \mathcal{U}$. Then using another partition of unity $1 = \sum_{V \in \mathcal{U}} \mu_V h_V^N$, we have

$$f - f_1 = \sum_{V \in \mathcal{U}} \mu_V (f - f_1) h_V^N = 0$$

It thus suffices to write for all $V \in \mathcal{U}$, $f_V = g_V/h_V$ such that for all $W \in \mathcal{U}$, $g_V h_W = g_W h_V$.

Well, for each $V \in \mathcal{U}$, let $h_V \in A$ with $V = D(h_V)$. Then $f_V = g_V/h_V^{n_V}$. Since $D(h_V) = D(h_V^{n_V})$, WLOG $f_V = g_V/h_V$ with $V = D(h_V)$ Now, since f_V and f_W agree on $V \cap W = D(h_V h_W)$, we have $g_V h_W / h_V h_W = \downarrow^{V \cap W} g_V/h_V = \downarrow^{V \cap W} g_W/h_W = g_W h_V / h_V h_W$ and so the existence of $n(V, W) \in \mathbb{N}$ such that

$$(g_V h_W - g_W h_V)(h_V h_W)^{n(V,W)} = 0$$

Smashing it again with *finiteness of* $\mathcal{U}$, we can choose a single $N \in \mathbb{N}$ such that

$$(g_V h_W - g_W h_V)(h_V h_W)^N = 0$$

for *all* $V, W \in \mathcal{U}$. Then, since $g_V/h_V = g_V h_V^N / h_V^{N+1}$ and $D(h_V) = D(h_V^{N+1})$, we can WLOG $f_V = g_V/h_V$ with $V = D(h_V)$ and

$$g_V h_W - g_W h_V = 0$$

as desired, finishing the proof.

$\square$

Proposition – Opens Subschemes
Let $X \in \mathbf{Sch}$, $U \in \operatorname{Open} X$. Then $U \in \mathbf{Sch}$. We call U an *open subscheme of* X.

Proof. (*Sheaf*)

> *Lemma* (*Opens of Sheaves are Sheaves*). *Let* $U \in \operatorname{Open} X$ *where* $X \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$. *Then* $U \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$.

Proof. Given $Y \in \mathbb{M}\mathbf{Set}$, a compatible system $(\varphi_i)_{Y_i \in \mathcal{Y}}$ of morphisms from an open cover $\mathcal{Y}$ of Y to U glues uniquely to a morphism $\varphi : Y \to X$. Factoring φ through U is equivalent to $\varphi^{-1}U = Y$, which is true from $\varphi^{-1}U$ "containing" the cover $\mathcal{Y}$, and so is single open covering Y, and hence is equal to Y by extensionality of opens. ■

(*Affine Open Cover*) Let $\mathcal{U} \subseteq \operatorname{Open} X$, $\mathcal{U}$ consists of affine opens and covers X. Since opens and covers are preserved under base change, $\{U \cap V\}_{V \in \mathcal{U}}$ is an open cover of U. For each $V \in \mathcal{U}$, $U \cap V$ is also an open of V. By affineness of V, $U \cap V$ has a cover by basic opens V_f of V. The V_f are open in $U \cap V$ by base change and hence open in U by composition. This gives an affine open cover of $U \cap V$, and hence an affine open cover of U by taking the composite of these covers.

□

Proposition – Fiber Product of Schemes
Let $X, Y, S \in \mathbf{Sch}$ and $\varphi \in \mathbf{Sch}(X, S)$, $\psi \in \mathbf{Sch}(Y, S)$. Then the fiber product $X \times_S Y$ in $\mathbb{M}\mathbf{Set}$ is a scheme and is the fiber product of X, Y over S in $\mathbf{Sch}$.

1.5 Examples

Counter Example (*Surjective on Points implies Surjective*).
Consider $\operatorname{Sp} \mathbb{F}_2 \to \operatorname{Sp} \mathbb{F}_2[dT]$ *where* $dT^2 := 0$. $(\operatorname{Sp} \mathbb{F}_2)^{\mathrm{Pts}} \cong (\operatorname{Sp} \mathbb{F}_2[dT])^{\mathrm{Pts}}$ *but* $(\operatorname{Sp} \mathbb{F}_2[dT])(\mathbb{F}_2[dT])$ *bijects with* $\mathbb{F}_2$ *whilst* $(\operatorname{Sp} \mathbb{F}_2)(\mathbb{F}_2[dT])$ *is singleton.*

Example (*Local Rings*).

Example (*Affine Line with Two Origins*).
Define X *by the following pushout diagram in* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$.

$$\begin{tikzcd} \mathbb{G}^{\times} \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d, "\iota_b"] \\ \mathbb{A}^1 \arrow[r, "\iota_a"] & X \end{tikzcd}$$

In other words, X *is obtained by "gluing two affine lines along* $\mathbb{G}^{\times}$*". We prove that the two morphisms* $\mathbb{A}^1 \to X$ *form an open cover of* X *and hence* X *is a scheme.*

($\iota_a, \iota_b : \mathbb{A}^1 \to X$ *monomorphism in* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$) *First note that since* $\mathbb{M}\mathbf{Set}$ *and* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$ *are both have fiber products and sheafification is the free functor adjoint to the forgetful functor, monomorphisms in* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$ *are the same as monomorphisms in* $\mathbb{M}\mathbf{Set}$. *We shall make no distinct between the two from now on.*

Let X^{psh} *denote the pushout of* $\mathbb{A}^1 \leftarrow \mathbb{G}^\times \to \mathbb{A}^1$ *in* $\mathbb{M}\mathbf{Set}$ *so that* X *is the sheafification of* X^{psh} *with respect to the Zariski topology on* $\mathbf{Aff}$. *Specifically, we have*

$$X^{psh}(A) = A \sqcup_{A^\times} A = \{(i, f) \mid i \in \{a, b\}, f \in A\} / (i, f) \sim (j, g) := f = g \in A^\times$$

We will use $(a, -), (b, -)$ *to denote the two obvious "inclusions"* $\mathbb{A}^1 \to X^{psh}$. *A remark : by considering* $A = \mathbb{F}_2^2$, *one can show that* X^{psh} is not *a Zariski sheaf, which is why we must take the pushout in* $\mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$.

The morphism ι_a *is the composition* $\mathbb{A}^1 \to X^{psh} \to X$ *where the second morphism comes from sheafification and the first morphism is* $(a, -)$, *which is clearly mono. It thus suffices that* $X^{psh} \to X$ *is mono. Sheafification has the property that if* X^{psh} *is a separated presheaf, then this map is mono. So it suffices that* X^{psh} *is a separated presheaf, which means for all* $\alpha, \alpha_1 \in \mathbb{M}\mathbf{Set}(\mathrm{Sp}\, A, X^{psh})$ *and basic Zariski covers* $\mathcal{U}$ *of* $\mathrm{Sp}\, A$, α *and* α_1 *agreeing on every open in* $\mathcal{U}$ *implies* $\alpha = \alpha_1$.

Let the morphisms α, α *correspond to* $(i, f), (i_1, f_1) \in X^{psh}(A)$. *If* $i = i_1$, *then* f, f_1 *are just morphisms* $\mathrm{Sp}\, A \to \mathbb{A}^1$ *that agree on the basic Zariski cover* $\mathcal{U}$. *Since* $\mathbb{A}^1 \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$, *we obtain* $f = f_1$ *and hence* $\alpha = \alpha_1$. *Now let* $i \neq i_1$, *WLOG* $i = a$ *and* $i_1 = b$. *It follows that* f, f_1 *are morphisms from* $\mathrm{Sp}\, A \to \mathbb{G}^\times$ *that agree on* $\mathcal{U}$. *Since* $\mathbb{G}^\times \in \mathbf{Sh}(\mathbf{Aff}_{\mathrm{Zar}})$ *as well,* $f = f_1 \in A^\times$ *and so* $\alpha = \alpha_1$.

(*The images of* $\iota_a, \iota_b : \mathbb{A}^1 \to X$ *are open in* X) *Let* U_a *be the presheaf theoretic image of* $\iota_a : \mathbb{A}^1 \to X$. *We show* U_a *is an open of* X. *The argument for* ι_b *is analogous.*

We need to show that for all $(A, \alpha) \in \mathrm{Sp} \downarrow X$, $\alpha^{-1}U := \mathrm{Sp}\, A \times_X U$ *is open in* $\mathrm{Sp}\, A$. *Let* $(A, \alpha) \in \mathrm{Sp} \downarrow X$. *It is another property of sheafification that there now exists a basic Zariski cover* $\mathcal{V}$ *of* $\mathrm{Sp}\, A$ *together with morphisms* $\alpha_i : V_i \to X^{psh}$ *for every* $V_i \in \mathcal{V}$ *that are the "restriction of* α*", in the sense that the following diagram commutes :*

$$\begin{tikzcd} \mathrm{Sp}\, A \arrow[r, "\alpha"] & X \\ V_i \arrow[u] \arrow[r, "\alpha_i"] & X^{psh} \arrow[u] \end{tikzcd}$$

It should *suffice to prove that for all* $V_i \in \mathcal{V}$, $V_i \cap \alpha^{-1}U_a \in \mathrm{Open}\,\mathrm{Sp}\, A$, *since* $\mathcal{V}$ *is an open cover of* $\mathrm{Sp}\, A$ *and "a subset is open if and only if it is open when restricted to an open cover". We prove this lemma.*

Lemma. *Let $V \in \mathbb{M}\mathbf{Set}$, $\mathcal{V}$ a Zariski cover of V, $U \in \mathbf{Sub}\mathbb{M}\mathbf{Set}(V)$ such that $U \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$ and for all $V_i \in \mathcal{V}$, $V_i \cap U \in \operatorname{Open} V$. Then $U \in \operatorname{Open} V$.*

Proof. By definition of subfunctors being open, it suffices to do the case of $V \in \mathbf{Aff}$. For $V_i \in \mathcal{V}$, let $V_i \cap U = D(I_i)$ where $I_i \subseteq \mathcal{O}(V)$. The claim is that $U = D(\bigcup_{V_i \in \mathcal{V}} I_i)$. Well, for $A \in \mathbb{M}^{op}$ and $\alpha \in V(A)$, $\alpha \in U(A)$ if and only if $\alpha : \operatorname{Sp} A \to V$ factors through U. Since $\{D(I_i)\}_{V_i \in \mathcal{V}} \subseteq \operatorname{Open} V$, $\left\{\alpha^{-1} D(I_i)\right\}_{V_i \in \mathcal{V}} \subseteq \operatorname{Open} \operatorname{Sp} A$. Then $\alpha : \operatorname{Sp} A \to V$ factoring through U implies $\left\{\alpha^{-1} D(I_i)\right\}_{V_i \in \mathcal{V}}$ forms a Zariski cover of $\operatorname{Sp} A$. Conversely, if $\left\{\alpha^{-1} D(I_i)\right\}_{V_i \in \mathcal{V}}$ forms a Zariski cover of $\operatorname{Sp} A$, then $U \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$ implies $\alpha : \operatorname{Sp} A \to V$ factors through U by uniquely gluing $\alpha^{-1}(V_i \cap U) \to U$ together. Now $\left\{\alpha^{-1} D(I_i)\right\}_{V_i \in \mathcal{V}} = \{D(\alpha I_i)\}_{V_i \in \mathcal{V}}$ forms a Zariski cover of $\operatorname{Sp} A$ if and only if $A = A \bigcup_{V_i \in \mathcal{V}} \alpha I_i$, i.e. $\alpha \in D(\bigcup_{V_i} I_i)$. ■

Now, since $\mathbb{A}^1 \cong U_a$ in $\mathbb{M}\mathbf{Set}$, the fact that the three $\operatorname{Sp} A, X, \mathbb{A}^1$ are Zariski sheaves implies we indeed have $\alpha^{-1} U_a$ as a subfunctor of $\operatorname{Sp} A$ with $\alpha^{-1} U_a \in \mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$. So it does suffice that for all $V_i \in \mathcal{V}$, $V_i \cap \alpha^{-1} U_a$ is open in $\operatorname{Sp} A$. Note that again, pullbacks in $\mathbf{Sh}(\mathbb{M}\mathbf{Set}_{\mathrm{Zar}})$ and $\mathbb{M}\mathbf{Set}$ coincide, so we make no distinction between the two.

Let $V_i \in \mathcal{V}$. The intersection $V_i \cap \alpha^{-1} U_a$ is equal to the pullback $\alpha_i^{-1}\left(\uparrow^X_{X^{psh}}\right)^{-1} U_a$. We have $\left(\uparrow^X_{X^{psh}}\right)^{-1} U_a$ as the presheaf-theoretic image of $(a, -)$, which we will denote with $(a, \mathbb{A}^1)$. Suppose α_i corresponds to $(a, f) \in X^{psh}(\mathcal{O}(V_i))$. Then we that following pullback diagram,

$$\begin{array}{ccc} V_i & \xrightarrow{\alpha_i} & X^{psh} \\ {\scriptstyle \mathbb{1}_{V_i}}\uparrow & & \uparrow{\scriptstyle (a,-)} \\ V_i & \xrightarrow{f} & \mathbb{A}^1 \end{array}$$

i.e. $\alpha_i^{-1}(a, \mathbb{A}^1) = V_i$, which is open. In the other case that α_i corresponds to $(b, f) \in X^{psh}(\mathcal{O}(V_i))$, it is easily checked from the definition of X^{psh} that $\alpha_i^{-1}(a, \mathbb{A}^1) = (\operatorname{Sp} A)_f$, which is again open.

(U_a, U_b cover X) Let $x : \operatorname{Sp} K \to X$ be a point of X. Again, it is a property of sheafification that we obtain a basic Zariski cover $\mathcal{K}$ of $\operatorname{Sp} K$ with morphisms $\kappa_i : K_i \to X^{psh}$ for each $K_i \in \mathcal{K}$ such that

$$\begin{array}{ccc} \operatorname{Sp} K & \xrightarrow{x} & X \\ \uparrow & & \uparrow \\ K_i & \xrightarrow{\kappa_i} & X^{psh} \end{array}$$

But $\operatorname{Sp} K$ is local, so $K_i = \operatorname{Sp} K$ for some $K_i \in \mathcal{K}$. In other words, we reduced the problem to showing $(a, \mathbb{A}^1), (b, \mathbb{A}^1)$ cover X^{psh}. This is clear.

2 Properties of Schemes

2.1 Zariski-Local Properties of Schemes

Definition – Zariski-Local Properties

Let $P : \mathbf{Sch} \to \mathbf{Prop}$ be a predicate on schemes. Then P is called *Zariski-local* when for all $X \in \mathbf{Sch}$ and Zariski covers $\mathcal{X}$ of X, $P(X)$ is true if and only if for all $X_i \in \mathcal{X}$, $P(X_i)$ is true.

Definition – Affine-Local Properties

Let $P : \mathbf{Aff} \to \mathbf{Prop}$ be a predicate on affine schemes. Then P is called *affine-local* when for all $X \in \mathbf{Aff}$ and basic Zariski covers $\mathcal{X}$ of X, $P(X)$ is true if and only if for all $X_i \in \mathcal{U}$, $P(X_i)$ is true.

Proposition – Affine-Locality

Let $P : \mathbf{Aff} \to \mathbf{Prop}$ be affine-local. Define the predicate locally $P : \mathbf{Sch} \to \mathbf{Prop}$ by setting X is locally P when there exists an affine Zariski-cover $\mathcal{X}$ of X such that all $X_i \in \mathcal{X}$ satisfy P.

Then TFAE :

1. X is locally P
2. All opens U of X are locally P.
3. All affine opens U of X satisfy P.
4. There exists a Zariski cover $\mathcal{X}$ of X where all $X_i \in \mathcal{X}$ are locally P.

In particular, "locally P" is a Zariski-local property of schemes.

Proof.

$(1 \Rightarrow 2)$ Let $U \in \operatorname{Open} X$. Let $\mathcal{X}$ be an affine Zariski cover of X where all $X_i \in \mathcal{X}$ satisfy P. For each X_i, $X_i \cap U$ is an open of X_i and hence admits a Zariski covering $\mathcal{U}_i$ by basic opens of X_i. Since $P(X_i)$ is true, for every $U_{i,j} \in \mathcal{U}_i$, $P(U_{i,j})$ is true as well. Then note that $U_{i,j}$ are affine since X_i is and also open in U so the composite $\mathcal{U} := \bigcup_{X_i \in \mathcal{X}} \mathcal{U}_i$ gives an Zariski cover of U consisting of affines satisfying P.

$(2 \Rightarrow 3)$ Let $U \in \operatorname{Open} X$ be affine. We have an affine Zariski cover $\mathcal{U}$ of U consisting of opens satisfying P. Since P is affine-local, it suffices to find a Zariski cover of each $U_i \in \mathcal{U}$ by opens that are basic in *both* U and U_i. Well, we can certainly find a Zarisk cover $\mathcal{U}_i$ of U_i by basic opens of U. Then $\mathcal{U}_i \cup \{U\}$ is a basic Zariski cover of U, so its pullback is a basic Zariski cover of U_i. But its pullback is just $\mathcal{U}_i$ so $\mathcal{U}_i$ works.

$(3 \Rightarrow 4)$ By X being a scheme. $(4 \Rightarrow 1)$ Composites of open covers.

□

Proposition – Examples of Affine-Local Properties

The following predicates on $\mathbf{Aff}$ are affine-local :

1. For $X \in \mathbf{Aff}$, say X is *Noetherian* when $\mathcal{O}(X)$ is Noetherian.

2. For $X \in \mathbf{Aff}$, say X is *reduced* when $\mathcal{O}(X)$ has no nilpotent elements.

Definition – Globally

Let P be a affine-local property of affine schemes and $X \in \mathbf{Sch}$. We say X is *globally P*[a] when X is locally P and X is quasi-compact.

[a]This is non-standard terminology. A lot of affine-local properties are extended to schemes by quasi-compact + locally P. In this case, it is standard terminology to say simply say "X is P". However, this clashes with properties of schemes *not* coming from affine-local properties. The addition of the adverb "globally" is an attempt to highlight the fact that the property P is affine-local.

3 Properties of Morphisms

3.1 Permanence

Definition – Permanence

Let $P : \mathrm{Mor}(\mathbf{Sch}) \to \mathbf{Prop}$ be a predicate on morphisms of schemes. Then we say :

- P is *stable under composition* when for all $X \to Y \to Z$ in $\mathbf{Sch}$, $P(X \to Y)$ and $P(Y \to Z)$ implies $P(X \to Z)$.
- P is *stable under base change* when for all pullback diagrams

$$\begin{tikzcd} X \arrow[r] & S \\ X \times_S Y \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

  $P(X \to S)$ implies $P(X \times_S Y \to Y)$.
- P is *stable under fiber product* when for all pullback diagrams as the above, $P(X \to S)$ and $P(Y \to S)$ implies $P(X \times_S Y \to S)$.

3.2 Zariski-Local Properties of Morphisms of Schemes

Definition – Zariski-Local on Target, on Source

Let $P : \mathrm{Mor}(\mathbf{Sch}) \to \mathbf{Prop}$ be a predicate on morphisms of schemes. Then we say :

- P is *Zariski-local on target* when for all $\varphi \in \mathbf{Sch}(X, Y)$ and Zariski covers $\mathcal{Y}$ of Y, $P(\varphi : X \to Y)$ is true if and only if for all $Y_i \in \mathcal{Y}$, $P(\varphi^{-1}Y_i \to Y_i)$ is true.
- P is *Zariski-local on source* when for all $\varphi \in \mathbf{Sch}(X, Y)$ and Zariski covers $\mathcal{X}$ of X, $P(\varphi : X \to Y)$ is true if and only if for all $X_i \in \mathcal{X}$, $P(X_i \to X \to Y)$ is true.

Definition – Affine-Local Properties of Morphisms

Let $P : \mathrm{Mor}(\mathbf{Aff}) \to \mathbf{Prop}$ be a predicate on morphisms of affine schemes. Then we say :

- P is *affine-local on target* when for all $\varphi \in \mathbf{Aff}(X, Y)$ and basic Zariski covers $\mathcal{Y}$ of Y, $P(\varphi : X \to Y)$ is true if and only if for all $Y_i \in \mathcal{Y}$, $P(\varphi^{-1}Y_i \to Y_i)$ is true.
- P is *affine-local on source* when for all $\varphi \in \mathbf{Aff}(X, Y)$ and basic Zariski covers $\mathcal{X}$ of X, $P(\varphi : X \to Y)$ is true if and only if for all $X_i \in \mathcal{X}$, $P(X_i \to X \to Y)$ is true.
- P is *affine-local* when the following three things are true :
  - (Half of Affine-Local on Target) For $\varphi \in \mathbf{Aff}(X, Y)$ and $f \in \mathcal{O}(Y)$, $P(\varphi : X \to Y)$ implies $P(\varphi^{-1}Y_f \to Y_f)$.
  - (Half of Affine-Local on Source) For $\varphi \in \mathbf{Aff}(X, Y)$ and basic Zariski cover $\mathcal{X}$ of X, if $P(X_i \to X \to Y)$ for every $X_i \in \mathcal{X}$, then $P(X \to Y)$.

– ("Zig-Zag") For $\varphi \in \mathbf{Aff}(X, Y)$, $f \in \mathcal{O}(X)$ and $g \in \mathcal{O}(Y)$,

$$\begin{tikzcd} X_f \arrow[r] \arrow[drr] & X \arrow[d] & \\ & Y_g \arrow[r] & Y \end{tikzcd}$$

$P(X \to Y_g)$ implies $P(X_f \to Y)$.

Remark – Frustration with Affine-Local Properties. It is true for properties of morphisms of affine schemes P that P affine-local implies P affine-local on target and source. I was hoping for this to be an equivalence, however the "zig-zag" seems to contain the extra information that $P(\operatorname{Sp} B \to (\operatorname{Sp} A)_f)$ implies $P(\operatorname{Sp} B \to \operatorname{Sp} A)$.

Proposition – Affine-Locality for Morphisms

Let $P : \operatorname{Mor}(\mathbf{Aff}) \to \mathbf{Prop}$ be an affine local property of morphisms of affine schemes. For $\varphi \in \mathbf{Sch}(X, Y)$, we say φ is *locally* P when there exists a Zariski cover $\mathcal{X}$ of X such that for each $X_i \in \mathcal{X}$, there exists an affine open Y_i of Y with $X_i \subseteq \varphi^{-1}Y_i$ and $P(X_i \to Y_i)$ true. Then for any $\varphi \in \mathbf{Sch}(X, Y)$, TFAE :

1. φ is locally P.
2. For all opens U, V of X, Y respectively with $U \subseteq \varphi^{-1}V$, $U \to V$ is locally P.
3. For all affine opens U, V of X, Y respectively with $U \subseteq \varphi^{-1}V$, $P(U \to V)$.
4. There exists an affine Zariski cover $\mathcal{Y}$ of Y and for each $Y_i \in \mathcal{Y}$ an affine Zariski cover $\mathcal{X}_i$ of $\varphi^{-1}Y_i$ such that for all $X_{ij} \in \mathcal{X}_i$, $P(X_{ij} \to Y_i)$.
5. There exists a Zariski cover $\mathcal{Y}$ of Y and for each $Y_i \in \mathcal{Y}$ a Zariski cover $\mathcal{X}_i$ of $\varphi^{-1}Y_i$ such that for all $X_{ij} \in \mathcal{X}_i$, $X_{ij} \to Y_i$ is locally P.