Notes on Presheaves as Discrete Fibrations

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Contents

1 Presheaves - Objects probeable by a category of test objects

Remark – Presheaves as Probeable Objects. A general philosophy in mathematics is to study objects by "probing them with test objects" which we understand, where by "probing" we mean mapping these "test objects" into them. The "test objects" usually form a (small) category, which for this discussion let's call C. Our main example throughout will be $C =$"the category of triangles". A "presheaf over C" is exactly "an object that is probeable by test objects in C". The steps to formalising this statement are :

1. If X is an object "that can be mapped into by test objects in C", then we ought to have a set $X(c)$ for each c in C, which we think of as the set of maps $c \to X$. We will call these *naive morphisms* from c to X.

2. if we have a morphism $f : d \to c$ in C, then *restricting along* f should give a way of turning naive morphisms from c to X into naive morphisms from d to X.

$$
\begin{tikzcd}
d \arrow[r, "f"] \arrow[dr, "xf"'] & c \arrow[d, "x"] \\
& X
\end{tikzcd}
$$

In other words, for each morphism $f : d \to c$ in C, we want the data of a morphism of sets $X(c) \to X(d)$. Furthermore, we should have

$$
\begin{tikzcd}
e \arrow[r, "g"] \arrow[drr, "xfg=(xf)g"'] & d \arrow[r, "f"] & c \arrow[d, "x"] & c \arrow[l, "\mathbb{1}_c"'] \arrow[dl, "x=x\mathbb{1}_c"] \\
& & X &
\end{tikzcd}
$$

This gives exactly the definition of a presheaf over C as a functor $C^{op} \to \mathbf{Set}$.

3. The intuition that the morphisms of presheaves *should* be natural transformations of functors comes from the idea : "if $f : X \to Y$ is a morphism between objects probeable by test objects in C, then *composing along* f should give a way of turning naive morphisms $c \to X$ into naive morphisms $c \to Y$". Under this point of view, naturality is tautological.

Nice as the above interpretation is, this is *not* the definition of presheaves we will take. The following proposition gives an alternate definition, which can be seen as a generalisation of *covering spaces* from algebraic topology. Indeed, the classical theory of covering spaces can be seen as establishing conditions on the base topological space for which discrete fibrations over the (dual of the) fundamental groupoid correspond to actual topological spaces over the base space.

Proposition – "A bundle of sets is a bundle of sets"

Let C be a (small) category.

For $(p : X \to C)$ in $\mathbf{Cat}/C$, we say p is a *discrete fibration* when any of the following equivalent conditions are met :

1. any commutative square of categories of the following form has a *unique* diagonal making the diagram commute :

$$
\begin{tikzcd}
{[0]} \arrow[r] \arrow[d, "1"'] & X \arrow[d] \\
{[1]} \arrow[r] \arrow[ur, dashed] & C
\end{tikzcd}
$$

We use $[n]$ to mean the linear order $0 \leq \cdots \leq n$ viewed as a category.

The outter square is called a *lifting problem* and any such diagonal is called a *solution*.

2. the following is a 1-categorical pullback diagram of categories :

$$\begin{tikzcd} X^{[1]} \arrow[r, "\mathrm{ev}_1"] \arrow[d, "p^{[1]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ C^{[1]} \arrow[r, "\mathrm{ev}_1"'] & C \end{tikzcd}$$

Define the *category of discrete fibrations over C* to be the full subcategory Disc/C of $\mathbf{Cat}/C$ consisting of discrete fibrations. With abuse of notation, we sometimes simply write X to instead of $p : X \to C$.

We then have the following :

1. Let $\bullet/\mathbf{Set}$ be the category of (small) sets under the singleton $\bullet$, equivalently the category of pointed (small) sets. Then the forgetful functor $(\bullet/\mathbf{Set})^{op} \to \mathbf{Set}^{op}$ is a discrete fibration.
2. The (1-categorical) pullback of a discrete fibration is a discrete fibration. In particular, for any functor $F : C \to \mathbf{Set}^{op}$, we have

$$\begin{tikzcd} \Sigma F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\bullet/\mathbf{Set})^{op} \arrow[d] \\ C \arrow[r, "F"'] & \mathbf{Set}^{op} \end{tikzcd}$$

   where $\Sigma F \to C$ is a discrete fibration. The morphism $\Sigma F \to C$ can be explicitly defined :
   - the objects of the category ΣF are "naive morphisms $c \to F$" and the morphisms between two naive morphisms $c \to F$ and $d \to F$ are morphisms $f \in C(c, d)$ such that "the following triangle commutes" :

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[dr] & d \arrow[d] \\ & F \end{tikzcd}$$

   - the projection $\Sigma F \to C$ is given by taking the source of naive morphisms into F.

   The above construction defines a functor $\Sigma : (\mathbf{Set}^{op})^C \to \mathrm{Disc}/C$. We will call ΣF the *total space of F*.
3. We have a functor $\mathrm{Fib} : \mathrm{Disc}/C \to (\mathbf{Set}^{op})^C$ defined by
   - On objects, take $(p : X \to C)$ to the functor $C \to \mathbf{Set}^{op}$ defined by
     - on objects, c is sent to $p^{-1}(c)$, which is a discrete category, i.e. a set.
     - morphisms are defined by arrow-lifting definition of discrete fibrations. The uniqueness of the arrow-lifting gives functoriality.

– For a morphism of discrete fibrations over C

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & C \end{tikzcd}$$

fibers are mapped into fibers, thus giving a map of sets $f_c : \mathrm{Fib}(p)\, c \to \mathrm{Fib}(q)\, c$. Uniqueness of arrow-lifts for $(q : Y \to C)$ implies $(f_c)_{c \in C}$ defines a natural transformation $\mathrm{Fib}(p) \to \mathrm{Fib}(q)$.

For $(p : X \to C)$ in Disc/C, we call $\mathrm{Fib}(p)$ the *fiber functor of* p. If the morphism p is clear from the context, we will sometimes write $\mathrm{Fib}X$ instead.

4. The functors Fib, Σ define an equivalence of categories

$$\mathrm{Disc}/C \simeq (\mathbf{Set}^{op})^C \cong \mathbf{Set}^{C^{op}}$$

In other words, we say that $(\bullet/\mathbf{Set})^{op} \to \mathbf{Set}^{op}$ is the *universal discrete fibration* and that $\mathbf{Set}^{op}$ *classifies* discrete fibrations.

Proof. (*Equivalent definitions of discrete fibration*) $(2 \Rightarrow 1)$ by taking objects. $(1 \Rightarrow 2)$ The assumption says that we have the pullback on the level of objects. To show the pullback for the level of arrows, note that this is equivalent to showing the lifting problems of the following form have *unique* solutions :

$$\begin{tikzcd} {[1] \times \{1\}} \arrow[r] \arrow[d] & X \arrow[d] \\ {[1] \times [1]} \arrow[r] \arrow[ur, dashed] & C \end{tikzcd}$$

This can be thought of as a categorical version of the fact that in one can "lift squares by an edge along a covering space". Existence of the lift uses uniqueness of the arrow-lifts, and so does uniqueness.

(*1*) ok.

(2)(*Pullback of Discrete Fibrations*) Via the unique arrow-lifting definition of discrete fibrations, the proof is direct. Via the pullback definition of discrete fibrations, one can apply the pasting lemma for pullback squares.

(2)(Σ *defines a functor*) Let $h : F \to G$ be a natural transformation of functors $C \to \mathbf{Set}^{op}$. Then we the following commuting diagram of categories :

$$\begin{tikzcd} \Sigma F + \Sigma G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma h \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\bullet/\mathbf{Set})^{op} \arrow[d] \\ C + C \arrow[r] & C \times [1] \arrow[r, "h"'] & \mathbf{Set}^{op} \end{tikzcd}$$

where $C + C$ denotes the coproduct of C with itself and we have the left square is cartesian since the right and outter squares are cartesian (easy to check). One can check that the morphism $\Sigma F + \Sigma G \to \Sigma h$ exhibits $\Sigma F, \Sigma G$ as full subcategories of Σh. The unique-arrow lifting property of Σh then defines as functor $\Sigma F \to \Sigma G$ over C.

An alternate, more explicit construction is to use the explicit definition of $\Sigma F \to C$.

(3) ok.

(4) Completely formal. The functors Σ and Fib actually give an isomorphism of categories. $\square$

Definition – Presheaves over a Category

Let C be a (small) category. Then we will also call Disc/C the *category of presheaves over* C, denote it with $\mathbf{PSh}C$, and call its objects *presheaves* (*over* C). For c in C and X in $\mathbf{PSh}C$, we will also write $X(c)$ for the fiber of X over c.

Remark. The standard definition of the category of presheaves on a category C is the category of functors $C^{op} \to \mathbf{Set}$. Our perhaps unconventional choice of definition of presheaves as discrete fibrations has the following advantages :

– the proofs of all nice properties of $\mathbf{PSh}C$, i.e. topos-theoretic properties, are arguably more intuitive when phrased in terms of discrete fibrations.

– building intuition of presheaves as discrete fibrations makes the later generalisation to right fibrations of infinity categories obvious : right fibrations are exactly discrete fibrations where sets are replaced with "sets with equality as data". Specifically, right fibrations over an infinity category X correspond to functors $X \to \infty\mathbf{Grpd}^{op}$ to the infinity category of infinity groupoids.

Example.
Let I be a discrete category, i.e. a set. Then presheaves over I are the same data as a family of sets parameterised by I.

Example.
Let C be a groupoid and $p : X \to C$ a morphism of categories. Then p is a discrete fibration if and only if its opposite is.

There exists an example of a category C and a discrete fibration $X \to C$ such that $X^{op} \to C^{op}$ is not *a discrete fibration.*

2 Presheaf Categories closed under Slices - Bundles and Relativisation

Proposition

Let C be a (small) category. Let $(p : S \to C)$ in $\mathbf{PSh}C$. By composition with p, we get a functor $\mathbf{PSh}S \to \mathbf{PSh}C/S$. This yields an equivalence of categories

$$\mathbf{PSh}S \simeq \mathbf{PSh}C/S$$

Proof. By our definition of presheaves over C, this follows from the pasting lemma for pullback squares applied to a diagram of the form :

$$\begin{tikzcd} C^{[1]} \arrow[d] & S^{[1]} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner" very near start] & X^{[1]} \arrow[l] \arrow[d] \\ C & S \arrow[l] & X \arrow[l] \end{tikzcd}$$

This is one of the reasons for this choice of definition of $\mathbf{PSh}C$. $\square$

3 Yoneda's Lemma - Probeable objects are defined by how they can be probed

Proposition – "Test Objects are Clearly Probeable by Test Objects"
Let C be a (small) category. Then

1. taking over-categories defines a functor $C \to \mathbf{PSh}C$.
2. (Yoneda's lemma) Let $X \in \mathbf{PSh}C$ and $c \in C$. Given a morphism $\alpha : C/c \to X$ of presheaves, we have a naive-morphism of c to x by taking the "composition of α with $\mathbb{1}_c$", that is to say, the image of $\mathbb{1}_c$ under α. This defines the following bijection that is functorial in c and X :

   $$\mathbf{PSh}C(C/c, X) \cong X(c)$$

   In other words, "morphisms from c to X as C-probeable spaces are the same as naive-morphisms from c to X". In particular, the functor $C \to \mathbf{PSh}C$ is fully faithful. This is called the *Yoneda embedding*. Presheaves in its essential image are called *representable*.

   Since morphisms $x : C/c \to X$ correspond to objects in (the source category of) X, we also call these *points of* X. For general morphisms $y : Y \to X$ from other presheaves, we call them *generalised points of* X.
3. (Density Theorem / "Functional Extensionality for Presheaves") Let X in $\mathbf{PSh}C$. Consider the slice category $(C/\_)/X$ consisting of Yoneda embeddings over X. This has an obvious structure as a presheaf over C and by the Yoneda lemma has a morphism $(C/\_)/X \to X$ over C. In this way, we can convert X into a diagram of presheaves over C. The result is that X is the colimit of this diagram, written suggestively as $X = \varinjlim_{x \in X} x$.

Proof. (*1*) The functor $C \to \mathbf{PSh}C$ acts on morphisms "by composition".

(*2*) The hard part is realising that C/c has $\mathbb{1}_c$ as a final object.

(*3*) The point is that the data of a morphism out of $X \to Y$ is precisely "how to map the points and the morphisms", and this is equivalent to a cocone structure of Y over $(C/\_)/X$ by Yoneda's lemma.

$\square$

> *Example.*
> *The density theorem applied to the case of* $\mathbf{Set} = \mathbf{PSh}\bullet$ *recovers functional extensionality.*

4 Nerve and Realisation - Probing objects in other categories

Proposition

Let C be a (small) category and $F : C \to D$ a functor.

– (Nerve) We can now probe objects in D by objects in C via F. This is formalised by the *nerve functor* (with respect to F), which is defined as

$$\begin{aligned} &N_F : D \to \mathbf{PSh}C \\ &(N_F\, d)\, c := F/d \end{aligned}$$

This can equivalently be defined by the pullback diagram :

$$\begin{tikzcd} F/d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D/d \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

– (Realisation and the Adjunction) There is an adjunction

$$D(|\_|_F, \star) \cong \mathbf{PSh}C(\_, N_F \star)$$

if and only if for every presheaf X over C, the colimit

$$\varinjlim_{x \in X} Fc$$

exists in D, where we see each $x \in X$ as a morphism $x : C/c \to X$ as in the density theorem. In this case, $|\_|_F$ is in fact the left Kan extension of F along the Yoneda embedding $C \to \mathbf{PSh}C$.

The functor $|\_|_F$ is called *realisation* along F.

Proof. (*Adjunction*)($\Leftarrow$)

$$\begin{aligned} D(|X|_F, d) &\cong D(\varinjlim_{x \in X} Fc, d) \cong \varprojlim_{x \in X} D(Fc, d) \\ &\cong \varprojlim_{x \in X} \mathbf{PSh}C(C/c, N_F d) \cong \mathbf{PSh}C(\varinjlim_{x \in X} C/c, N_F d) \cong \mathbf{PSh}C(X, N_F d) \end{aligned}$$

($\Rightarrow$) Assuming we have such an adjunction, we have

$$D(|X|_F, d) \cong \mathbf{PSh}C(X, N_F d) \cong \mathbf{PSh}C(\varinjlim_{x \in X} C/c, N_F d) \cong \varprojlim_{x \in X} \mathbf{PSh}C(C/c, N_F d) \cong \varprojlim_{x \in X} D(Fc, d)$$

which shows $|X|_F$ has the structure of a colimit desired.

($|\_|_F$ *is the left Kan extension*) This comes down to the fact that the formula for the realisation is by colimits.

$\square$

5 Pullback , Shriek Pushforward, Pushforward - Reindexing, Dependent Sum, Dependent Products for Presheaves

Proposition

For every functor $u : C \to D$, we have an adjoint triple of functors

$$\begin{tikzcd} \mathbf{PSh}C \arrow[r, shift left=4, "u_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, shift right=4, "u_*"'] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{PSh}D \arrow[l, "u^*"'] \end{tikzcd}$$

where the three functors are defined as follows :

– (Shriek Pushforward / Dependent Sum) $u_!$ is given by realisation along the composition $C \to D \to \mathbf{PSh}D$. Recalling the formula for realisation, we have explicitly

$$u_! X := \varinjlim_{x \in X} D/uc$$

where we view $x \in X$ as $x : C/c \to X$ as in the density theorem.

We also write Σ_u for $u_!$.

In the functor perspective of presheaves, $u_! X$ is called the *left Kan extension of X along u*.

– (Pullback / Reindexing) u^* is given by the nerve functor along $C \to D \to \mathbf{PSh}D$. Explicitly, this is given by the pullback of presheaves along u.

In the perspective of presheaves as functors into $\mathbf{Set}^{op}$, u^* corresponds to precomposition with u.

– (Pushforward / Dependent Product) u_* is defined as the nerve functor along $u^* : D \to \mathbf{PSh}D \to \mathbf{PSh}C$. Explicitly,

$$u_* X := \mathbf{PSh}C(F/\_, X)$$

We also write Π_u for u_*.

In the functor perspective of presheaves, $u^* X$ is called the *right Kan extension of X along u*.

Hence, all presheaf categories have colimits and limits. Furthermore, they are computed fiberwise in the sense that for any diagram $X : I \to \mathbf{PSh}C$ and $c \in C$,

$$(\varinjlim_{i\in I} X)_c \cong \varinjlim_{i\in I}(X_i)_c$$
$$(\varprojlim_{i\in I} X)_c \cong \varprojlim_{i\in I}(X_i)_c$$

Proof. Let us first remark that the fact that limits of presheaves is computed fiberwise is an easy consequence of Yoneda's lemma :

$$(\varprojlim_{i\in I} X_i)_c \cong \mathbf{PSh}C(C/c, \varprojlim_{i\in I} X_i) \cong \varprojlim_{i\in I} \mathbf{PSh}C(C/c, X_i) \cong \varprojlim_{i\in I}(X_i)_c$$

On the other hand, the fact that colimits of presheaves can be computed fiberwise is saying that representables are *tiny*.

A surprising fact I noticed is that the existence of the adjoint triple for all $u : C \to D$ where C, D are (small) categories is actually equivalent to the cocompleteness of presheaf categories together with tininess of representables.

First, let us assume we have the adjoint triple for all $u : C \to D$. Let C be a (small) category. Then we can construct colimits and limits in $\mathbf{PSh}C$ by noticing that given a category I, a diagram $X : I \to \mathbf{PSh}C$ is the same as a functor $I \times C \to \mathbf{Set}^{op}$ by uncurrying, and hence is the same as the data of a presheaf $X \in \mathbf{PSh}(I \times C)$. It follows that colimits and limits of I-diagrams in $\mathbf{PSh}C$ are giving by the dependent sum and dependent product along the projection functor $\sigma : I \times C \to C$ since pulling back along σ corresponds exactly to taking constant diagrams $\mathbf{PSh}C \to (\mathbf{PSh}C)^I$ and we have the bijections,

$$\mathbf{PSh}C(\sigma_! X, \_) \cong \mathbf{PSh}(I \times C)(X, \sigma^* \_)$$
$$\mathbf{PSh}(I \times C)(\sigma^* \_, X) \cong \mathbf{PSh}C(\_, \sigma_* X)$$

Now for tininess of representables, the key observation is that for any object $c \in C$, pulling back along the corresponding functor $c : \bullet \to C$ is the same as taking fiber over c. But we know that $c^* : \mathbf{PSh}C \to \mathbf{PSh}\bullet = \mathbf{Set}$ has a right adjoint c_*, so c^* must preserve both colimits. The fact that colimits and limits are computed fiberwise can be nicely phrased (though be it not very useful) as some sort of base change result

$$c^* \sigma_! \cong \tau_! \gamma^*$$
$$c^* \sigma_* \cong \tau_* \gamma^*$$

where

$$\begin{tikzcd} I \arrow[r, "\gamma"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \times C \arrow[d, "\sigma"] \\ \bullet \arrow[r, "c"'] & C \end{tikzcd}$$

Now let's assume that $\mathbf{PSh}C$ is cocomplete and representables are tiny for all (small) C. Then by the nerve and realisation adjunction, we obtain the adjunction $u_! \dashv u^*$. We can also obtain the adjunction $u^* \dashv u_*$ by another application of the nerve-realisation adjunction provided we show that u^* satisfies the formula :

$$u^*X \cong \varinjlim_{x \in X} u^*D/d$$

By tininess of representables in $\mathbf{PSh}C$, it suffices to check the fibers of u^*X and $\varinjlim_{x\in X} u^*D/d$ match. But tininess of representables in $\mathbf{PSh}D$ give this :

$$(u^*X)_c = X_{uc} \cong \varinjlim_{x\in X} (D/d)_{uc} = \varinjlim_{x\in X} (u^*D/d)_c \cong (\varinjlim_{x\in X} u^*D/d)_c$$

Back to the proposition, by the above discussion it suffices to show cocompleteness of presheaf categories and tininess of representables. Well, given a diagram $X : I \to \mathbf{PSh}C$, we compromise the total space perspective and momentarily view presheaves as functors to define the presheaf

$$\tilde{X}(c) := \varinjlim_{i\in I} X_i(c)$$

following the expected formula. The action of morphisms is given by UPs of the colimits. It is also easy to show that there is a cocone over X with tip $\tilde{X}$ making $\tilde{X}$ the colimit of X. By construction, this makes all representables tiny. $\square$

> *Example.*
> *Let I be a discrete category, i.e. a set, and $X \in \mathbf{PSh}I$, seen as a family of sets $(X_i)_{i\in I}$ over I. Let $p : I \to \bullet$ be the canonical morphism of sets to the singleton. Show that $\Sigma_p X$ is the disjoint union $\Sigma_{i\in I} X_i$ and $\Pi_p X$ is the set of sections of X, i.e. $\Pi_{i\in I} X_i$.*

6 Presheaf Category as the Free Cocompletion

> **Proposition**
> Let $\mathrm{CoComplete}$ denote the category of cocomplete categories with cocontinuous functors as morphisms. Let C be a category. Then we have an equivalence of categories :
>
> $$\mathrm{CoComplete}(\mathbf{PSh}C, \_) \xrightarrow{\sim} \mathbf{Cat}(C, \_)$$

Proof. We've seen that $\mathbf{PSh}C$ is cocomplete. Now let $i : \mathrm{CoComplete}(\mathbf{PSh}C, \_) \to \mathbf{Cat}(C, \_)$ be obtained by precomposition with the Yoneda embedding $C \to \mathbf{PSh}C$. Full and faithfulness of i follows from the density theorem. For essential surjectivity, we know that any functor $F : C \to D$ where D is cocomplete extends along the Yoneda embedding to $|\_|_F : \mathbf{PSh}C \to D$ via the nerve-realisation adjunction. Since realisation is left adjoint to the nerve functor, it is cocontinuous. $\square$

7 Presheaf Categories are Cartesian Closed - Mapping Objects and Currying

Proposition

Let C be a (small) category. Then $\mathbf{PSh}C$ is cartesian closed. This means given two presheaves X, Y over C, we have a presheaf Y^X which "internalises" the morphisms from X to Y in the sense that we have an adjunction reflecting *currying* :

$$\mathbf{PSh}C(X \times (-), Y) \cong \mathbf{PSh}C(-, Y^X)$$

Proof. Given a presheaf X over C, apply the nerve-realisation adjunction to the functor $X \times (-) : \mathbf{PSh}C \to \mathbf{PSh}C$. ☐

8 Subobject Classifer - A universe of propositions

Proposition

Let C be a (small) category. For X in $\mathbf{PSh}C$, define a *subbundle of* p to be a full subcategory of X. Subbundles of a presheaf have an obvious structure as a presheaf. Define the category $\mathbf{Sub}$ over $\mathbf{PSh}C$ via the data :

- the source category has as objects subbundles of discrete bundles over C. For morphisms between two subbundles P, Q of respectively X, Y, they are morphisms $f \in \mathbf{PSh}C(X, Y)$ such that we have an induced pullback square :

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  In other words, that P is the "preimage" of Q.

- there is a functor from the above source category to $\mathbf{PSh}C$ by taking the ambient bundle of a subbundle.

Then $\mathbf{Sub}$ defines a presheaf over $\mathbf{PSh}C$ and is in fact representable by its pullback to along $C \to \mathbf{PSh}C$. We call its pullback to C the *universe of propositions* (of $\mathbf{PSh}C$) and denote it with Prop.

Proof. ($\mathbf{Sub} \in \mathbf{PSh}C$) We need to give a unique solution $\tilde{f}$ to the lifting problem

$$\begin{tikzcd} {[0]} \arrow[r] \arrow[d, "1"'] & \mathbf{Sub} \arrow[d, "t"] \\ {[1]} \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"'] & \mathbf{PSh}\mathcal{C} \end{tikzcd}$$

What we have is two presheaves X, Y over C, a morphism $f : X \to Y$ over C and a subbundle S of Y. A lift $\tilde{f}$ is exactly a functor $X \to Y$ that is equal to f together with a subbundle of X that is equal to the preimage of S under f. In other words, we have a lift and it is unique.

(*Representability of* $\mathbf{Sub}$ *by* Prop) The goal is to give a bijection

$$\mathbf{PSh}C(X, \mathrm{Prop}) \cong \mathbf{Sub}(X)$$

that is functorial in X. The idea is that Prop is the "universe of propositions" in $\mathbf{PSh}C$, so a morphism $X \to \mathrm{Prop}$ the same as a "predicate on X", which is exactly a "subset" of X, i.e. a subbundle. Thus, the forward map is defined as follows : given a $P \in \mathbf{PSh}C(X, \mathrm{Prop})$, give the subbundle $\{X \mid P\}$ of "points of X where P is true". In other words, $\{X \mid P\}$ is defined by the pullback diagram :

$$\begin{tikzcd}
\{X|P\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \top \arrow[d] \\
X \arrow[r, "P"'] & \mathrm{Prop}
\end{tikzcd}$$

The *universal subbundle* $\top \to \mathrm{Prop}$ is defined by the full subcategory of Prop consisting of inclusions of representables into themselves by identity, i.e. "$\{C/c \mid \top\}$" across all c in C.

We first show surjectivity. Let $S \to X$ be a subbundle of a presheaf over C. We need to give a morphism $P \in \mathbf{PSh}C(X, \mathrm{Prop})$ such that we have the pullback square :

$$\begin{tikzcd}
S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \top \arrow[d] \\
X \arrow[r, "P"'] & \mathrm{Prop}
\end{tikzcd}$$

Intuition : P is the predicate "$\cdot \in S$". Since X is the colimit of its elements, to give a morphism "$\cdot \in S$" : $X \to \mathrm{Prop}$ is the same as giving for each $x : C/c \to X$ a morphism $C/c \to \mathrm{Prop}$ that is functorial in the source c. The rest of the argument is contained in the following diagram together with the facts that limits are computed fiber-wise and that the correspondence we seek is trivial for the case of representables :

$$\begin{tikzcd}
x^{-1}S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[drr, bend right] & S \arrow[d] \arrow[dr, dashed] & \\
\widehat{c} \arrow[r, "x"'] \arrow[drr, bend right, "\cdot\in x^{-1}S"'] & X \arrow[dr, dashed, "\cdot\in S"'] & \top \arrow[d] \\
 & & \mathrm{Prop}
\end{tikzcd}$$

Now to show injectivity, let $P, Q : X \to \mathrm{Prop}$ such that $\{X \mid P\} = \{X \mid Q\}$. By "functional extensionality for presheaves", it suffices to show that for all points x of X we have $Px = Qx$. The rest again follows from the fact that what we want to show is a tautology for the case of representables.

$\square$

Example.
Already mentioned for motivation in the above proof, but for the case of $\mathbf{Set} = \mathbf{PSh}\bullet$*, the set* $\{\varnothing, \bullet\}$ *works as a subobject classifier. This is the categorical formalisation of the fact that in classical mathematics, "all propositions are either true or false".*

9 Presheaf toposes - A place to do mathematics with strict equality

TODO : write out how to interpret intuitionistic type theory in a presheaf topos.

Remark – Presheaf Categories as a "Place to Do Maths". What the properties in the previous sections show is that presheaf categories have the categorical properties needed to interpret all we need to do "maths", namely :

- finite products to talk about contexts with multiple variables
- finite equalizers to talk about when "elements" are equal
- exponential objects to talk about morphisms
- a universe of propositions, to talk about "predicates"

Abstracting these properties give the definition of an *elementary topos*. The canonical example is of course $\mathbf{Set} = \mathbf{PSh}\bullet$ the category of presheaves over "the abstract point".