1 New Keynesian Model with Sticky prices

1.1 Households

Utility function is

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln c_t - \frac{L_t^{\eta}}{\eta} \right]. \tag{1}$$

where c_t is consumption, L_t is labour supply, β is discount factor.

Budget constraint in real term are

$$c_t + d_t = w_t L_t + \frac{R_t d_{t-1}}{\pi_t} + T_t + \Pi_t \tag{2}$$

$$T = c + k - wL - Rk - \Pi \tag{3}$$

where R_t is nominal interest rate from $t-1$ to t, d_t is households lending, $\pi_t = \frac{P_t}{P_{t-1}}$ is inflation, w_t is real wage, Π_t is lump-sum profit received from retailers, T_t is public transfers. The profit terms can be expressed as

$$\Pi_t = \left(1 - \frac{1}{X_t}\right) Y_t.$$

Households' problem is to maximize lifetime utility s.t. the budget constraint,

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln c_t - \frac{L_t^{\eta}}{\eta} + \lambda_t \left(w_t L_t + \Pi_t + \frac{R_t d_{t-1}}{\pi_t} + T_t - c_t - d_t \right) \right].$$

FOCs are

$$\frac{1}{c_t} = \lambda_t,$$

$$-L_t^{\eta-1} + \lambda_t w_t = 0,$$

$$-\lambda_t + \beta E_t \lambda_{t+1} \frac{R_{t+1}}{\pi_{t+1}} = 0.$$

Solution is

$$\frac{1}{c_t} = \beta E_t \left[\frac{R_{t+1}}{\pi_{t+1} c_{t+1}} \right]. \tag{4}$$

$$c_t L_t^{\eta-1} = w_t. \tag{5}$$

1.2 Entrepreneurs

Competitive entrepreneurs, borrow capital, hire labour from households, produce intermediate output

$$Y_t = A_t k_t^{\alpha} L_t^{1-\alpha}.$$

They

$$\max \frac{P_t^w}{P_t} A_t k_t^{\alpha} L_t^{1-\alpha} - r_{k,t} k_t - w_t L_t,$$

where P_t^w is the price selling to retailers, and $r_{k,t}$ is marginal product of capital. We denote $X_t = \frac{P_t}{P_t^w}$ as price mark up.

FOCs are

$$\alpha \frac{A_t k_t^{\alpha-1} L_t^{1-\alpha}}{X_t} = r_{k,t}, \tag{6}$$

$$\frac{(1-\alpha) A_t k_t^{\alpha} L_t^{\alpha}}{X_t} = w_t \tag{7}$$

1.3 Capital accumulation

Capital accumulation with adjustment cost

$$k_{t+1} = (1-\delta) k_t + i_t.$$

Resource constraint

$$Y_t = c_t + i_t.$$

1.4 Retailers

$$\beta \widehat{\pi}_{t+1} = \widehat{\pi}_t + \frac{(1-\theta\beta)(1-\theta)}{\theta} \widehat{X}_t.$$

where $X = \frac{\varepsilon}{\varepsilon - 1}$ at the steady state.

1.5 System of equations

$$\frac{1}{c_t} = \beta E_t \left[\frac{R_{t+1}}{\pi_{t+1} c_{t+1}} \right]$$

$$c_t L_t^{\eta-1} = w_t$$

$$\alpha \frac{A_t k_t^{\alpha-1} L_t^{1-\alpha}}{X_t} = r_{k,t}$$

$$\frac{R_t}{\pi_t} = r_{k,t} + 1 - \delta$$

$$\frac{(1-\alpha) A_t k_t^{\alpha} L_t^{\alpha}}{X_t} = w_t$$

$$Y_t = A_t k_t^{\alpha} L_t^{1-\alpha}$$

$$k_{t+1} = (1-\delta)\, k_t + i_t$$

$$Y_t = c_t + i_t$$

2 Learning part

2.1 Consumption function

We start from intertemporal constraint (2)

$$c_t + d_t = w_t L_t + \frac{R_t d_{t-1}}{\pi_t} + T_t + \Pi_t.$$

It can be written as

$$\Lambda_t + \frac{R_t d_{t-1}}{\pi_t} = d_t, \tag{8}$$

where

$$\Lambda_t = w_t L_t + T_t + \Pi_t - c_t.$$

Rewite (8) one step ahead

$$\Lambda_{t+1} + \frac{R_{t+1} d_t}{\pi_{t+1}} = d_{t+1} \tag{9}$$

Substitute into (8)

$$\begin{aligned} \frac{R_{t+1}}{\pi_{t+1}}\Lambda_t + \frac{R_{t+1}}{\pi_{t+1}}\frac{R_t}{\pi_t}d_{t-1} + \Lambda_{t+1} + \frac{R_{t+1}d_t}{\pi_{t+1}} &= \frac{R_{t+1}}{\pi_{t+1}}d_t + d_{t+1} \\ \frac{R_{t+1}}{\pi_{t+1}}\Lambda_t + \Lambda_{t+1} + \frac{R_{t+1}}{\pi_{t+1}}\frac{R_t}{\pi_t}d_{t-1} &= d_{t+1} \end{aligned} \tag{10}$$

Rewrite (8) two steps ahead

$$\Lambda_{t+2} + \frac{R_{t+2}}{\pi_{t+2}} d_{t+1} = d_{t+2} \tag{11}$$

Substitute into (10)

$$\frac{R_{t+2}}{\pi_{t+2}}\frac{R_{t+1}}{\pi_{t+1}}\Lambda_t + \frac{R_{t+2}}{\pi_{t+2}}\Lambda_{t+1} + \Lambda_{t+2} + \frac{R_{t+2}}{\pi_{t+2}}\frac{R_{t+1}}{\pi_{t+1}}\frac{R_t}{\pi_t}d_{t-1} = d_{t+2} \tag{12}$$

So on so forth, we have

$$\Lambda_t + \frac{\pi_{t+1}}{R_{t+1}}\Lambda_{t+1} + ... + \left(\prod_{i=1}^{n}\frac{\pi_{t+i}}{R_{t+i}}\right)\Lambda_{t+j} + \frac{R_t}{\pi_t}d_{t-1} = \left(\sum_{i=1}^{j}\frac{\pi_{t+i}}{R_{t+i}}\right)d_{t+j}.$$

$$\Lambda_t + \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} \frac{\pi_{t+i}}{R_{t+i}} \right) \Lambda_{t+j} + \frac{R_t}{\pi_t} d_{t-1} = 0, \tag{13}$$

where

$$\Lambda_{t+j} = w_{t+j} L_{t+j} + T_{t+j} + \Pi_{t+j} - c_{t+j}. \tag{14}$$

Once we have labour supply (5) and Π_{t+j} in terms of output Y_t and price mark-up X_t, we can rearrange (14) to be

$$\Lambda_{t+j} = w_{t+j}^{\frac{\eta}{\eta-1}} c_{t+j}^{-\frac{1}{\eta-1}} + \left(1 - \frac{1}{X_{t+j}}\right) A_{t+j} k_{t+j}^{\alpha} w_{t+j}^{\frac{1-\alpha}{\eta-1}} c_{t+j}^{-\frac{1-\alpha}{\eta-1}} - c_{t+j} + T_{t+j}.$$

Linearize (13), more details and parameters are from another tex file 'linearization of consumption'.

$$\begin{aligned}
-\left[C_0 + \sum_{j=1}^{\infty} G_{cj}\right] \widehat{c}_t \quad = \quad & C_w E_{t-1} \widehat{w}_t + \frac{\theta}{(1-\theta\beta)(1-\theta)} C_X \left[\beta E_{t-1} \widehat{\pi}_{t+1} - \widehat{\pi}_t\right] + C_A \widehat{A}_t + C_k \widehat{k}_t + T E_{t-1} \widehat{T}_t ... \\
& + Rd \left[E_{t-1} \widehat{R}_t + \widehat{d}_{t-1} - \widehat{\pi}_t\right] + \Lambda \sum_{j=1}^{\infty} \left[\sum_{i=j}^{\infty} \left(\frac{1}{R}\right)^i \widehat{\pi}_{t+j}\right] - \Lambda \sum_{j=1}^{\infty} \left[\sum_{i=j}^{\infty} \left(\frac{1}{R}\right)^i \widehat{R}_{t+j}\right] ... \\
& + \sum_{j=1}^{\infty} G_{cj} \sum_{i=1}^{j} E_{t-1} \widehat{R}_{t+i} - \sum_{j=1}^{\infty} G_{cj} \sum_{i=1}^{j} E_{t-1} \widehat{\pi}_{t+i} + \sum_{j=1}^{\infty} G_{wj} E_{t-1} \widehat{w}_{t+j} + \frac{\theta}{(1-\theta\beta)(1-\theta} \\
& + \sum_{j=1}^{\infty} G_{Aj} \widehat{A}_{t+j} + \sum_{j=1}^{\infty} G_{kj} E_{t-1} \widehat{k}_{t+j} + \sum_{j=1}^{\infty} G_{Tj} E_{t-1} \widehat{T}_{t+j}
\end{aligned}$$

2.2 Learning process

For simplicity, we assume that retailers are fully rational so Phillips cure is always valid. Also, we make central bank monetary policy unknown by the households, so they make forecasts of interest rate by their PLM.

We assume at time $t-1$, we are at the steady state. We can get k_t from capital accumulation function and inflation π_t from Philips curve. Also, central bank set nominal interest rate to be R_t according toTaylor rule. At time t, the shock A_t comes. From the consumption function we derived above, households choose their consumption according to state variables k_t, A_t, and their updated forecasts of R_{t+i}, w_{t+i} and π_{t+i}, T_{t+i}. Once households pin down the consumption level c_t, labour supply is choosen according to the labour supply function. After rearrangements of system of equations, we can express L_t as a function of c_t, and state variables k_t, A_t. (we can express it in linearized form)

$$\begin{aligned}
c_t L_t^{\eta-1} \quad &= \quad \frac{(1-\alpha) k_t}{\alpha} L_t^{2\alpha-1} \left[\frac{R_t}{\pi_t} - 1 + \delta\right]. \\
L_t \quad &= \quad \left[\frac{(1-\alpha) k_t}{\alpha c_t} \left[\frac{R_t}{\pi_t} - 1 + \delta\right]\right]^{\frac{1}{\eta-2\alpha}}
\end{aligned}$$

As households choose labour supply L_t, wage w_t is derived and output Y_t is pinned down. We then know the capital in next period k_{t+1} and this learning procedure goes on.

Now lets linearize the above equation

$$(\eta-\alpha)L^{\eta-\alpha}\widehat{L}_t = \left[\frac{1-\alpha}{\alpha c}\right]\left(\frac{1}{\beta}\widehat{R}_t - \widehat{\pi}_t - \left(\frac{1}{\beta}-1-\delta\right)\widehat{c}_t\right)$$

now we can recall Taylor rule

$$\widehat{R}_t = \alpha_\pi \widehat{\pi}_t + \alpha_Y \widehat{Y}_t$$

from production function we have that

$$\widehat{Y}_t = \widehat{A}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{L}_t$$

plug it into Taylor rule

$$\widehat{R}_t = \alpha_\pi \widehat{\pi}_t + \alpha_Y\left(\widehat{A}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{L}_t\right)$$

finally, plug Taylor rule in the equation for $\widehat{L}_t$

$$(\eta-\alpha)L^{\eta-\alpha}\widehat{L}_t = \left[\frac{1-\alpha}{\alpha c}\right]\left(\frac{1}{\beta}\left(\alpha_\pi \widehat{\pi}_t + \alpha_Y\left(\widehat{A}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{L}_t\right)\right) - \widehat{\pi}_t - \left(\frac{1}{\beta}-1-\delta\right)\widehat{c}_t\right)$$

if we simplify the expression above we should get this guy

$$\widehat{L}_t = \frac{1}{(\eta-\alpha)L^{\eta-\alpha} - \left[\frac{1-\alpha}{\alpha c}\right]\frac{1}{\beta}\alpha_Y(1-\alpha)}\left[\frac{1-\alpha}{\alpha c}\right]\left(\frac{1}{\beta}\left(\alpha_\pi \widehat{\pi}_t + \alpha_Y\left(\widehat{A}_t + \alpha\widehat{k}_t\right)\right) - \widehat{\pi}_t - \left(\frac{1}{\beta}-1-\delta\right)\widehat{c}_t\right) \tag{15}$$

Now we can use the above expression to solve for $\widehat{L}_t$ since it is function only of the variables we know already, that is $\widehat{\pi}_t$, $\widehat{c}_t$, $\widehat{A}_t$ and $\widehat{k}_t$.