# 4.2.30

Find the LU-factorization of the matrix

> $A := Matrix([[3, 0, 1], [0, -1, 3], [1, 3, 0]])$

L is lower triangular and U is unit upper triangular.

$$A := \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \quad \textbf{(1.1)}$$

Herfra er det let at udlede U matricen:

> $U := Matrix\left(\left[\left[1, 0, \frac{1}{3}\right], [0, 1, -3], [0, 0, 1]\right]\right)$

$$U := \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(1.2)}$$

Herfra kan man ved et par lette ligninger finde L:

> $L := Matrix\left(\left[[3, 0, 0], [0, -1, 0], \left[1, 3, \frac{26}{3}\right]\right]\right)$

$$L := \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 3 & \frac{26}{3} \end{bmatrix} \quad \textbf{(1.3)}$$

> $L.U = A$

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \quad \textbf{(1.4)}$$

# doolittle procedure

> $doolittleLU := \mathbf{proc}(n, A :: Matrix)$

**local** $k, j, i, L, U$;

$L := Array(1 .. n, 1 .. n)$;

$U := Array(1 .. n, 1 .. n)$;

**for** $k$ **from** 1 **to** $n$ **do**

$L_{k,k} := 1$;

**for** $j$ **from** $k$ **to** $n$ **do**

$U_{k,j} := A[k][j] - add\left(L_{k,s} \cdot U_{s,j}, s = 1 .. k-1\right)$;

**end do**;

**for** $i$ **from** $k+1$ **to** $n$ **do**

$L_{i,k} := \frac{\left(A[i][k] - add\left(L_{i,s} \cdot U_{s,k}, s = 1 .. k-1\right)\right)}{U_{k,k}}$;

```
end do;
end do;
return L, U;
end proc:
```

> $B := \langle\langle 60, 30, 20\rangle | \langle 30, 20, 15\rangle | \langle 20, 15, 12\rangle\rangle$

$$B := \begin{bmatrix} 60 & 30 & 20 \\ 30 & 20 & 15 \\ 20 & 15 & 12 \end{bmatrix} \tag{2.1}$$

> $doolittleLU(3, B)$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{bmatrix}, \begin{bmatrix} 60 & 30 & 20 \\ 0 & 5 & 5 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \tag{2.2}$$

# 4.2.41

> $restart;$

> $A := Matrix([[2, 1, -2], [4, 2, -1], [6, 3, 11]])$

$$A := \begin{bmatrix} 2 & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & 11 \end{bmatrix} \tag{3.1}$$

> $doolittleLU(3, A)$

Error, (in doolittleLU) numeric exception: division by zero

Vi forsøger at køre metoden i hånden:

> $L := \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}:$

> $U := \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}:$

> $L.U = A$

$$\begin{bmatrix} d & e & f \\ a\,d & a\,e + g & a\,f + h \\ b\,d & b\,e + c\,g & b\,f + c\,h + i \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & 11 \end{bmatrix} \tag{3.2}$$

> $d := 2; e := 1; f := -2;$

$$d := 2$$
$$e := 1$$
$$f := -2 \tag{3.3}$$

> $a := \frac{4}{d}; g := 2 - a \cdot e; h := -1 - a \cdot f;$

$$a := 2$$
$$g := 0$$
$$h := 3 \tag{3.4}$$

> $b := \frac{6}{d}; c := \frac{(3 - b \cdot e)}{g}; i := 11 - b \cdot f - c \cdot h;$

$$b := 3$$

Error, numeric exception: division by zero

$$i := 17 - 3\,c \tag{3.5}$$

Men b*e = 3 så c = 0/0 dvs, c kan være hvad som helst. Derfor bliver vores løsning:

> $L := \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & x & 1 \end{bmatrix}$

$$L := \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & x & 1 \end{bmatrix} \tag{3.6}$$

> $U := \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 17 - 3\,x \end{bmatrix}$

$$U := \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 17 - 3\,x \end{bmatrix} \tag{3.7}$$

> $L.U$

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & 11 \end{bmatrix} \tag{3.8}$$

Heraf kan vi se at løsningen passer for alle værdier af x.

# cp 4.2.2

> $restart;$

Solve this system by the Cholesky method:

> $\begin{bmatrix} 0.05 & 0.07 & 0.06 & 0.05 \\ 0.07 & 0.10 & 0.08 & 0.07 \\ 0.06 & 0.08 & 0.10 & 0.09 \\ 0.05 & 0.07 & 0.09 & 0.10 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.23 \\ 0.32 \\ 0.33 \\ 0.31 \end{bmatrix}:$

## Cholesky algorithm

> $choleskyLU := \mathbf{proc}(n, A :: Matrix)$
$\mathbf{local}\ k, i, L;$
$L := Matrix(n, n);$
$\mathbf{for}\ k\ \mathbf{from}\ 1\ \mathbf{to}\ n\ \mathbf{do}$

$L_{k,k} := \left(A_{k,k} - add\left(L_{k,s}^2, s = 1..k-1\right)\right)^{\frac{1}{2}};$
$\mathbf{for}\ i\ \mathbf{from}\ k+1\ \mathbf{to}\ n\ \mathbf{do}$
$L_{i,k} := \frac{\left(A_{i,k} - add\left(L_{i,s} \cdot L_{k,s}, s = 1..k-1\right)\right)}{L_{k,k}};$

$\mathbf{end\ do};$
$\mathbf{end\ do};$
$\mathbf{return}\ L;$
$\mathbf{end\ proc}:$

> $A := \begin{bmatrix} 0.05 & 0.07 & 0.06 & 0.05 \\ 0.07 & 0.10 & 0.08 & 0.07 \\ 0.06 & 0.08 & 0.10 & 0.09 \\ 0.05 & 0.07 & 0.09 & 0.10 \end{bmatrix}:$

> $X := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}:$

> $B := \begin{bmatrix} 0.23 \\ 0.32 \\ 0.33 \\ 0.31 \end{bmatrix}:$

> $Digits := 20$

$$Digits := 20 \quad (4.1)$$

> $L := choleskyLU(4, A):$

> $L$

[[0.22360679774997896964, 0, 0, 0], **(4.2)**
[0.31304951684997055750, 0.044721359549995793906, 0, 0],
[0.26832815729997476357, −0.089442719099991587923, 0.14142135623730950484, 0],
[0.22360679774997896964, 0., 0.21213203435596425738, 0.070710678118654752263]]

> $LT := L^{\%T}:$

> $LT$

**(4.3)**

$$[[0.22360679774997896964, 0.31304951684997055750, 0.26832815729997476357, 0.22360679774997896964], [0, 0.044721359549995793906, -0.089442719099991587923, 0.], [0, 0, 0.14142135623730950484, 0.21213203435596425738], [0, 0, 0, 0.070710678118654752263]] \quad (4.3)$$

For at finde X matricen, skriver vi vores ligninger lidt om:
AX=B =>
L.LT.X=B , Lad LT.X = Y =>
L.Y=B , Vi skal altså finde Y ved forward substitution. Derefter kan vi finde X ved backward substitution.

## forward substitution

```
> forwardSub := proc(n, A :: Matrix, B) :: Matrix;
  local i, Y;
  Y := Vector(4);
  for i from 1 to n do
  Y[i] := (B[i] - add(A[i, j].Y[j], j = 1 .. i - 1)) / A[i, i];
  end do;
  return Y;
  end proc:
> Y := forwardSub(4, L, B);
```

$$Y := \begin{bmatrix} 1.0285912696499032604 \\ -0.044721359549995794398 \\ 0.35355339059327376188 \\ 0.070710678118654753183 \end{bmatrix} \quad (4.4)$$

## backward substitution

```
> backwardSub := proc(n, A, B) :: Matrix;
  local i, X;
  X := Vector(4);
  for i from n by -1 to 1 do
  X[i] := (B[i] - add(A[i, j].X[j], j = i + 1 .. n)) / A[i, i];
  end do;
  return X;
  end proc:
> X := backwardSub(4, LT, Y)
```

$$X := \begin{bmatrix} 1.0000000000000000868 \\ 0.99999999999999994763 \\ 0.99999999999999997808 \\ 1.0000000000000000130 \end{bmatrix} \quad (4.5)$$

Vi kan let se ud fra ligningen at resultatet burde være 1 for alle x-værdier. Men det lader til at der skal ret mange decimaler til før maple kan give resultatet så præcist.

# 4.3.1

Vi skal bruge maple til at udregne de 5 givne ligningssystemer ved brug af gauss elimination, og bagefter også med pivotering. Vi bruger vores doolittleLU funktion fra tidligere og skriver den om til at inkludere resultaterne. Dette siden doolittle laver gauss elimination, men samtidig giver en L matrix.

```
> restart :
> with(ArrayTools) :
```

## gauss elimination with L, U and answers

Vi har lavet en funktion gaussElim som bruger gauss elimination på A og B samtidig. Derefter udregner den vores tilsvarende L som doolittle algoritmen regnede den. Den finder til sidst svarene X[i] ved at løse ligningssystemet rekursivt med vores nye A (U) og B (B1).

```
> gaussElim := proc(n, A :: Matrix, B)
    local k, j, i, z, U, X, B1, L;
    U := Matrix(n, n);
    U := copy(A);
    B1 := Vector(n) ;
    B1 := copy(B);
    L := Matrix(n, n);
    X := Vector(n, symbol = x);
    for k from 1 to n - 1 do
    for i from k + 1 to n do
    z := U[i, k]/U[k, k];
    U[i, k] := 0;
    B1[i] := B1[i] - z·B1[k];
    for j from k + 1 to n do
    U[i, j] := U[i, j] - z·U[k, j];
    end do;
    end do;
    end do;
    for k from 1 to n do
    L[k, k] := 1;
    for i from k + 1 to n do
    L[i, k] := (A[i, k] - add(L[i, s]·U[s, k], s = 1 ..k - 1))/U[k, k];
    end do;
    end do;
    for i from n by -1 to 1 do
    X[i] := solve(U[i].X = B1[i], x[i]);
    end do;
    return L, U, B1, X;
    end proc:
```

## System a

$$> Aa := \begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix}:$$

$$> Ba := \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}:$$

$$> L, U, B1, X := gaussElim(3, Aa, Ba)$$

$$L, U, B1, X := \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & \frac{3}{2} & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & -4 \\ 0 & 4 & -8 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{5}{4} \\ -\frac{3}{4} \\ -\frac{1}{2} \end{bmatrix} \tag{5.2.1}$$

## System b

$$> Ab := \begin{bmatrix} 1 & 6 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}:$$

$$> Bb := \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}:$$

$$> L, U, B1, X := gaussElim(3, Ab, Bb)$$

$$L, U, B1, X := \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -\frac{2}{11} & 1 \end{bmatrix}, \begin{bmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ \frac{1}{11} \end{bmatrix}, \begin{bmatrix} \frac{3}{11} \\ \frac{5}{11} \\ \frac{1}{11} \end{bmatrix} \tag{5.3.1}$$

## System c

$$> Ac := \begin{bmatrix} -1 & 1 & 0 & -3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix}:$$

> $Bc := \begin{bmatrix} 4 \\ 0 \\ 3 \\ 1 \end{bmatrix}:$

> $L, U, B1, X := gaussElim(4, Ac, Bc)$

$$L, U, B1, X := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \tag{5.4.1}$$

## System d

> $Ad := \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 4 & 10 \\ 3 & -13 & 3 & 3 \\ -6 & 4 & 2 & -18 \end{bmatrix}:$

> $Bd := \begin{bmatrix} 0 \\ -10 \\ -39 \\ -16 \end{bmatrix}:$

> $L, U, B1, X := gaussElim(4, Ad, Bd)$

$$L, U, B1, X := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & -\frac{1}{2} & 2 & 1 \end{bmatrix}, \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 0 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -10 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} \tag{5.5.1}$$

## System e

> $Ae := \begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & -9 & 2 & 1 \\ 8 & 16 & 6 & 5 \\ 2 & 3 & 2 & 1 \end{bmatrix}:$

$$> Be := \begin{bmatrix} 2 \\ 14 \\ -3 \\ 0 \end{bmatrix}:$$

$$> L, U, B1, X := gaussElim(4, Ae, Be)$$

$$L, U, B1, X := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 8 & -\frac{16}{9} & 1 & 0 \\ 2 & -\frac{1}{3} & \frac{6}{31} & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -9 & -6 & -3 \\ 0 & 0 & -\frac{62}{3} & -\frac{25}{3} \\ 0 & 0 & 0 & -\frac{12}{31} \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -\frac{25}{3} \\ -\frac{12}{31} \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \textbf{(5.6.1)}$$

# gauss elimination with scaled row pivoting

```
> gaussElimPivot :=proc(n, A, B)
    local i, A1, p, P, S, k, j, z, tempmax, B1, X, L, U, tempi, temp, tempindex;
    A1 := Matrix(n, n);
    A1 := copy(A);
    B1 := Vector(n);
    B1 := copy(B);
    p := Vector(n);
    P := Matrix(n, n);
    S := Vector(n);
    X := Vector(n);
    L := Matrix(n, n);
    U := Matrix(n, n);
    for i to n do
        p[i] := i;
        S[i] := max(seq(abs(A1[i, j]), j = 1 ..n))
    end do;
    for k to n − 1 do
        tempmax := 0;
        tempindex := 1;
        for i from k to n do
            temp := abs(A1[i, k]) / S[i];
            if tempmax < temp then
                tempmax := temp;
                tempindex := i;
            end if
        end do;
        tempi := p[k];
        p[k] := p[tempindex];
        p[tempindex] := tempi;
        for i from k + 1 to n do
            z := A1[p[i], k]/A1[p[k], k];
            A1[p[i], k] := z;
            for j from k + 1 to n do
```

```
            A1[p[i], j] := A1[p[i], j] - z*A1[p[k], j]
          end do
        end do
      end do;
    for i from 1 to n do
    P[i, p[i]] := 1;
    end do;
  return p, P, A1;
  end proc:
```

> pa2, Pa2, Aa2 := gaussElimPivot(3, Aa, Ba);

$$pa2, Pa2, Aa2 := \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & 2 & -4 \\ 2 & 2 & 0 \\ \frac{3}{2} & 0 & 2 \end{bmatrix} \tag{5.7.1}$$

```
> gaussElimPSol :=proc(n, A, P, B)
      local X, B1, k, i;
      X := Vector(n);
      B1 := Vector(n);
      B1 := copy(B);
      for k to n - 1 do
          for i from k + 1 to n do
              B1[P[i]] := B1[P[i]] - A[P[i], k]*B1[P[k]]
          end do
      end do;
      for i from n by -1 to 1 do
          X[i] := (B1[P[i]] - add(A[P[i], j]*X[j], j = i + 1 ..n)) / A[P[i], i]
      end do;
      return X, B1;
  end proc:
```

> Xa2, Ba2 := gaussElimPSol(3, Aa2, pa2, Ba)

$$Xa2, Ba2 := \begin{bmatrix} \frac{5}{4} \\ -\frac{3}{4} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \\ -1 \end{bmatrix} \tag{5.7.2}$$

> L, U, B, X := gaussElim(3, Pa2.Aa, Pa2.Ba);

**(5.7.3)**

$$L, U, B, X := \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{5}{4} \\ -\frac{3}{4} \\ -\frac{1}{2} \end{bmatrix} \tag{5.7.3}$$

## System b

> $pb2, Pb2, Ab2 := gaussElimPivot(3, Ab, Bb);$

$$pb2, Pb2, Ab2 := \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{11}{4} & -\frac{11}{4} \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \tag{5.8.1}$$

> $Xb2, Bb2 := gaussElimPSol(3, Ab2, pb2, Bb);$

$$Xb2, Bb2 := \begin{bmatrix} \frac{3}{11} \\ \frac{5}{11} \\ \frac{1}{11} \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ 1 \\ 1 \end{bmatrix} \tag{5.8.2}$$

> $L, U, B, X := gaussElim(3, Pb2.Ab, Pb2.Bb);$

$$L, U, B, X := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{11}{4} & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{11}{4} \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{3}{11} \\ \frac{5}{11} \\ \frac{1}{11} \end{bmatrix} \tag{5.8.3}$$

## System c

> $pc2, Pc2, Ac2 := gaussElimPivot(4, Ac, Bc);$

$$pc2, Pc2, Ac2 := \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} & 1 & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{3} & 0 & \frac{8}{3} & \frac{1}{3} \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix} \tag{5.9.1}$$

> $Xc2, Bc2 := gaussElimPSol(4, Ac2, pc2, Bc);$

$$Xc2, Bc2 := \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{3} \\ 3 \\ 1 \end{bmatrix} \quad (5.9.2)$$

> $L, U, B, X := gaussElim(4, Pc2.Ac, Pc2.Bc);$

$$L, U, B, X := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 \\ -\frac{1}{3} & 1 & \frac{1}{2} & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \frac{8}{3} & \frac{1}{3} \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -\frac{1}{3} \\ \frac{3}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \quad (5.9.3)$$

## System d

> $pd2, Pd2, Ad2 := gaussElimPivot(4, Ad, Bd);$

$$pd2, Pd2, Ad2 := \begin{bmatrix} 1 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 6 & -2 & 2 & 4 \\ 2 & \frac{1}{3} & -\frac{2}{13} & -\frac{6}{13} \\ \frac{1}{2} & -12 & 2 & 1 \\ -1 & -\frac{1}{6} & \frac{13}{3} & -\frac{83}{6} \end{bmatrix} \quad (5.10.1)$$

> $Xd2, Bd2 := gaussElimPSol(4, Ad2, pd2, Bd);$

$$Xd2, Bd2 := \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{6}{13} \\ -39 \\ -\frac{45}{2} \end{bmatrix} \quad (5.10.2)$$

> $L, U, B, X := gaussElim(4, Pd2.Ad, Pd2.Bd);$

(5.10.3)

$$L, U, B, X := \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -1 & -\frac{1}{6} & 1 & 0 \\ 2 & \frac{1}{3} & -\frac{2}{13} & 1 \end{bmatrix}, \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -12 & 2 & 1 \\ 0 & 0 & \frac{13}{3} & -\frac{83}{6} \\ 0 & 0 & 0 & -\frac{6}{13} \end{bmatrix}, \begin{bmatrix} 0 \\ -39 \\ -\frac{45}{2} \\ -\frac{6}{13} \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} \quad \textbf{(5.10.3)}$$

## Systeme

> $pe2, Pe2, Ae2 := gaussElimPivot(4, Ae, Be);$

$$pe2, Pe2, Ae2 := \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{10} & \frac{6}{5} & \frac{3}{5} \\ 2 & -15 & -2 & -1 \\ 4 & -\frac{4}{15} & -\frac{19}{9} & 2 \\ 2 & 3 & 2 & 1 \end{bmatrix} \quad \textbf{(5.11.1)}$$

> $Xe2, Be2 := gaussElimPSol(4, Ae2, pe2, Be);$

$$Xe2, Be2 := \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{5} \\ 14 \\ 2 \\ 0 \end{bmatrix} \quad \textbf{(5.11.2)}$$

> $L, U, B, X := gaussElim(4, Pe2.Ae, Pe2.Be);$

$$L, U, B, X := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{10} & 1 & 0 \\ 4 & -\frac{4}{15} & -\frac{19}{9} & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & \frac{6}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 14 \\ \frac{3}{5} \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \textbf{(5.11.3)}$$

# 4.4.33

For any n x n matrix A, define:

> $restart;$

> $\|A\|_F := \left(sum\left(sum\left(a_{i,j}^2, j = 1..n\right), i = 1..n\right)\right)^{\frac{1}{2}}$

$$\|A\|_F := \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}^2} \quad \textbf{(6.1)}$$

Vi skal finde ud af om denne er en subordinate matrix norm. Vi kan forsøge med identitetsmatricen:

> $\|I\| = 1$

Men hvis vi lægger en 2x2 identitetsmatrice ind får vi:

> $(1^2 + 0^2 + 0^2 + 1^2)^{\frac{1}{2}} \neq 1$

$$\sqrt{2} \neq 1 \quad (6.2)$$

Derfor er denne form for matirx norm ikke en subordinate matrix norm.
Vi skal prøve det samme med den følgende formel:

> $\|A\| := \max_{1 \leq (i,j) \leq n}(\operatorname{abs}(a_{i,j}))$ :

Vi prøver igen først med identitetsmatricen, det ses let at den maksimale værdi af alle i identitetsmatricen = 1. Derfor passer denne norm for property (9).

> $\|I\| = 1$

For at bevise property (10) har vi:

> $\|AB\| \leq \|A\| \ \|B\|$

> $A := Matrix(2, [a, b, c, d])$

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (6.3)$$

> $B := Matrix(2, [e, f, g, h])$

$$B := \begin{bmatrix} e & f \\ g & h \end{bmatrix} \quad (6.4)$$

> $A.B$

$$\begin{bmatrix} a\,e + b\,g & a\,f + b\,h \\ c\,e + d\,g & c\,f + d\,h \end{bmatrix} \quad (6.5)$$

> $\|A.B\| = \max(|ae + bg|, |af + bh|, |ce + dg|, |cf + dh|)$ :

> $\|A\| \ \|B\| = \max(|a|, |b|, |c|, |d|) \cdot \max(|e|, |f|, |g|, |h|)$ :

Men hvis vi tager et eksempel som a,b,c,d,e,f,g,h = 2. Har vi allerede:

> $\|A.B\| = 4 + 4 = 8$

> $\|A\| \ \|B\| = 2 \cdot 2 = 4$

Derfor er denne heller ikke en subordinat matrix norm.

Nu skal vi i stedet forsøge at vise at disse to normer er matrix normer, selvom de ikke er subordinat matrix normer. Vi skal derfor bevise følgende:

> **( 1 )** $\|x\| > 0$ **if** $x \neq 0$, $x$ **in** $V$

> **( 2 )** $\|\lambda x\| = |\lambda| \cdot \|x\|$ **if** $\lambda$ **in** $\mathbb{R}$, $x$ **in** $V$

> **( 3 )** $\|x + y\| \leq \|x\| + \|y\|$ **if** $x, y$ **in** $V$

Vi begynder med den første matrix norm, $\|A\|_F$ :

(1) Eftersom alle værdier i vores matrix opløftes i anden potens vil vi for enhver matrix $\neq 0$ have en positiv matrix, eftersom alle reelle tal opløftet i anden potens bliver positivt.

(2) Denne ses let at det holder, hvis vi viser en 2x2 matrix:

> $((\lambda a)^2 + (\lambda b)^2 + (\lambda c)^2 + (\lambda d)^2)^{\frac{1}{2}} = \operatorname{abs}(\lambda) \cdot (a^2 + b^2 + c^2 + d^2)^{\frac{1}{2}}$ :

> $LHS = \left(\lambda^2 \cdot (a^2+b^2+c^2+d^2)\right)^{\frac{1}{2}}$

$$LHS = \sqrt{\lambda^2\,(a^2+b^2+c^2+d^2)} \tag{6.6}$$

> $LHS = \mathrm{abs}(\lambda) \cdot (a^2+b^2+c^2+d^2)^{\frac{1}{2}}$

$$LHS = |\lambda|\,\sqrt{a^2+b^2+c^2+d^2} \tag{6.7}$$

> $LHS = RHS:$

Det ses let at dette er tilfældet for alle n x n matricer.

(3) Vi definerer 2 matricer, X og Y:

> $\|X+Y\| := \left(sum\left(sum\left(\left(x_{i,j}+y_{i,j}\right)^2, j=1..n\right), i=1..n\right)\right)^{\frac{1}{2}}$

$$\left\|\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right\| := \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_{i,j}+y_{i,j}\right)^2} \tag{6.8}$$

> $\|X\| + \|Y\|: \left(sum\left(sum\left(x_{i,j}^2, j=1..n\right), i=1..n\right)\right)^{\frac{1}{2}} + \left(sum\left(sum\left(y_{i,j}^2, j=1..n\right), i=1..n\right)\right)^{\frac{1}{2}}$

$$\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_{i,j}^2} + \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} y_{i,j}^2} \tag{6.9}$$

Disse kan omskrives, lad A indeholde alle værdier (x_ij + y_ij) i 1 dimension, og m = n*n. Samtidig lader vi B indeholde b_i-værdier svarende til alle x_ij, og C indeholde c_i-værdier svarende til y_ij. Dvs, A=B+C.

> $LHS := \left(sum\left(a_i^2, i=1..m\right)\right)^{\frac{1}{2}}$

$$LHS := \sqrt{\sum_{i=1}^{m} a_i^2} \tag{6.10}$$

> $RHS := \left(sum\left(b_i^2, i=1..m\right)\right)^{\frac{1}{2}} + \left(sum\left(c_i^2, i=1..m\right)\right)^{\frac{1}{2}}$

$$RHS := \sqrt{\sum_{i=1}^{m} b_i^2} + \sqrt{\sum_{i=1}^{m} c_i^2} \tag{6.11}$$

Vi tager anden potens på begge sider:

> $LHS := LHS^2$

$$LHS := \sum_{i=1}^{m} a_i^2 \tag{6.12}$$

> $RHS^2$

$$\left(\sqrt{\sum_{i=1}^{m} b_i^2} + \sqrt{\sum_{i=1}^{m} c_i^2}\right)^2 \quad (6.13)$$

> $RHS := \sum_{i=1}^{m} b_i^2 + \sum_{i=1}^{m} c_i^2 + 2\sqrt{\sum_{i=1}^{m} b_i^2} \cdot \sqrt{\sum_{i=1}^{m} c_i^2} :$

> $LHS = \sum_{i=1}^{m} \left(b_i^2 + c_i^2\right) = \sum_{i=1}^{m} b_i^2 + \sum_{i=1}^{m} c_i^2 :$

Dermed har vi at $LHS \leq RHS$.

Nu skal vi se om det samme gælder for matrix normen som max funktion:

> $\|A\| := \max_{1 \leq (i,j) \leq n}\left(\mathrm{abs}\left(a_{i,j}\right)\right) :$

(1) Vi tager altid den nummeriske og største værdi af matricen A, derfor passer (1)

(2):

> $\|\lambda x\| = |\lambda| \cdot \|x\|$

> $LHS := \max_{1 \leq (i,j) \leq n}\left(\mathrm{abs}\left(\lambda \cdot a_{i,j}\right)\right) :$

> $RHS := |\lambda| \cdot \max_{1 \leq (i,j) \leq n}\left(\mathrm{abs}\left(a_{i,j}\right)\right) :$

Her ses det let at LHS=RHS for $\lambda \in \mathbb{R}$

(3): Denne del er triviel da:

> $LHS := \max\left(\mathrm{abs}\left(x_{i,j} + y_{i,j}\right)\right) :$

> $RHS := \max\left(\mathrm{abs}\left(x_{i,j}\right)\right) + \max\left(\mathrm{abs}\left(y_{i,j}\right)\right) :$

dermed er LHS=RHS.

# 4.4.40

Vi skal bruge maples inbyggede ConditionNumber fra LinearAlgebra til at udregne ConditionNumber for de givne matricer.

> $restart;$

> $with(LinearAlgebra) :$

> $A := Matrix(2, [a+1, a, a, a-1]);$

$$A := \begin{bmatrix} a+1 & a \\ a & a-1 \end{bmatrix} \quad (7.1)$$

> $B := Matrix(2, [0, 1, -2, 0]);$

$$B := \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \quad (7.2)$$

> $C := Matrix\left(2, \left[\alpha, 1, 1, 1\right]\right);$

$$C := \begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix} \quad (7.3)$$

## a

> $ConditionNumber(A)$

$$\max(|a| + |a+1|, |a-1| + |a|)^2 \tag{7.1.1}$$

For $a \geq 0$ får vi:

> $(a + a + 1)^2$

$$(2\,a + 1)^2 \tag{7.1.2}$$

For $a \leq 0$ får vi:

> $(\operatorname{abs}(a-1) + \operatorname{abs}(a))^2$

$$(|a-1| + |a|)^2 \tag{7.1.3}$$

Men da a er negativ kan vi gang hver absolutte værdi med -1:

> $(-a + 1 - a)^2 = (1 - 2\,a)^2$

Da a er negativ er dette det samme som for a>0. Derfor gælder det altid at konditions nummeret for A er:

> $(2 \cdot \operatorname{abs}(a) + 1)^2$

$$(2\,|a| + 1)^2 \tag{7.1.4}$$

## b

> $ConditionNumber(B)$

$$2 \tag{7.2.1}$$

Her bliver svaret serveret for os på et sølvfad!

## c

> $ConditionNumber(C)$

$$\max\left(\frac{2}{|\alpha-1|}, \frac{1}{|\alpha-1|} + \left|\frac{\alpha}{\alpha-1}\right|\right) \max(2, 1 + |\alpha|) \tag{7.3.1}$$

Det ses let at $\alpha \neq 1$ da vi ikke kan have division med 0. For $-1 < \alpha < 1$ får vi fat i venstresiden af begge udtryk for at få den maksimale værdi og for alle andre får vi fat i de 2 højresider. Dette kan også skrives op piecewise:

> $piecewise\left(\operatorname{abs}(\alpha) = 1, undefined, -1 < \alpha < 1, \frac{2}{\operatorname{abs}(\alpha-1)} \cdot 2, \left(\frac{1}{|\alpha-1|} + \left|\frac{\alpha}{\alpha-1}\right|\right) \cdot (1 + |\alpha|)\right)$

$$\begin{cases} undefined & |\alpha| = 1 \\ \dfrac{4}{|\alpha-1|} & -1 < \alpha < 1 \\ \left(\dfrac{1}{|\alpha-1|} + \left|\dfrac{\alpha}{\alpha-1}\right|\right)(1 + |\alpha|) & otherwise \end{cases} \tag{7.3.2}$$

>