Opgave 1.1.5

Opgave 1.1.5a

> Deriving the Taylor series at 0 for the function $f(x) = \ln(x+1)$

> $f := x \to \ln(x+1)$

$$f := x \to \ln(x+1) \quad (1.1.1)$$

> $f'(x)$

$$\frac{1}{x+1} \quad (1.1.2)$$

> $f''(x)$

$$-\frac{1}{(x+1)^2} \quad (1.1.3)$$

> $\ln(x+1) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)(x-0)^k + E_n(x)$

> I taylor series er $f(x)$

$$= \ln(x+1) + \frac{1}{x+1}(x-0) - \frac{2}{2(x+1)^2}(x-0)^2 + \ldots + \frac{(-1)^{k-1}}{k!(x+1)^k}(x+0)^k$$

> $taylor(f(x), x, 10)$

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8 + \frac{1}{9}x^9 + O(x^{10}) \quad (1.1.4)$$

> $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$

$$\ln(x+1) = \ln(x+1) \quad (1.1.5)$$

> k starter fra 1 fordi summeringen har k alene under brøkstregen. Vi kan se at maple også udregner summeringen til at være $\ln(x+1)$.

> Vi finder nu et udtryk for restmængden, når c=0:

> $E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x)^{n+1}$

$$E_n(x) = \frac{D^{(n+1)}(f)(\xi)(x)^{n+1}}{(n+1)!} \quad (1.1.6)$$

> $seq(diff(\ln(x+1), [x\$n]), n = 0..7)$

$$\ln(x+1), \frac{1}{x+1}, -\frac{1}{(x+1)^2}, \frac{2}{(x+1)^3}, -\frac{6}{(x+1)^4}, \frac{24}{(x+1)^5}, -\frac{120}{(x+1)^6}, \frac{720}{(x+1)^7} \quad (1.1.7)$$

> Heraf kan vi udlede at den $f^n(x) = \frac{(-1)^{n-1}}{(x+1)^n}(n-1)!$

> Nu kan vi omskrive vores rest fra før til:

> $\frac{1}{(n+1)!} \cdot \frac{(-1)^n}{(\xi+1)^{n+1}} \cdot (n!) \cdot (x)^{n+1}$

$$\frac{(-1)^n\, n!\, x^{n+1}}{(n+1)!\, (\xi+1)^{n+1}} \quad (1.1.8)$$

Vi kan forkorte dette en smule mere for at få et udtryk for E:

> $E_n(x) := \frac{(-1)^n \cdot x^{n+1}}{(n+1)\,(\xi+1)^{n+1}}$

$$E_n := x \mapsto \frac{(-1)^n x^{n+1}}{(n+1)\,(\xi+1)^{n+1}} \quad (1.1.9)$$

> Vi truncater vores serie til hhv 3 og 4 termer og giver herunder 2 udtryk for resten.

> $E_3(x) = \frac{(-1)^3 x^{3+1}}{(3+1)(\xi+1)^{3+1}}, (0 < \xi < x)$

Maple giver os af en eller anden en værdi for xi, så vi reducerer udtrykket manuelt:

> $E_3(x) =- \frac{x^4}{4(\xi+1)^4}, (0 < \xi < x)$

$$E_3(x) = -\frac{x^4}{4\,(\xi+1)^4}, 0 < \xi < x \quad (1.1.10)$$

> $E_4(x) = \frac{x^5}{5(\xi+1)^5}, (0 < \xi < x)$

$$E_4(x) = \frac{x^5}{5\,(\xi+1)^5}, 0 < \xi < x \quad (1.1.11)$$

Opgave 1.1.5b

> Vi skal forsøge at finde $\ln(1.5)$ ved hjælp af en taylor serie. Vores fejl margen skal være mindre end 10^{-8}.

> Vi kan se på vores formel fra 1.1.5a at fejlen er størst for mindst muligt xi. Vi sætter derfor xi=0.

> For at fejlen skal være mindre end 10^-8 har vi:

> $solve\left(\left\{n \geq 0, \frac{0.5^{n+1}}{n+1} < 10^{-8}\right\}, n\right)$

$$\{21.10887129 < n\} \quad (1.2.1)$$

> Her ser vi at der går minimum 22 termer til at få fejlen ned på mindre end 10^{-8}. Siden vi ikke kan have 21.11 termer.

Opgave 1.1.5c

> For at finde det mindste antal termer for $\ln(1.6)$ til en præcision på 10^{-10} tager vi altså:

> $solve\left(\left\{n \geq 0, \frac{0.6^{n+1}}{n+1} < 10^{-10}\right\}, n\right)$

$$\{36.95697913 < n\} \quad (1.3.1)$$

> **Her skal der bruges 37 termer for at få den ønskede præcision.**

▶ Opgave 1.1.34

▼ Opgave JH-1

▼ 1

> $\mathrm{trunc}(2.6)$

$$2 \quad (3.1.1)$$

> $\mathrm{trunc}(-2.4)$

$$-2 \quad (3.1.2)$$

> $\mathrm{round}(2.6)$

$$3 \quad (3.1.3)$$

> $\mathrm{round}(-2.4)$

$$-2 \quad (3.1.4)$$

> $\mathrm{floor}(2.6)$

$$2 \quad (3.1.5)$$

> $\mathrm{floor}(-2.4)$

$$-3 \quad (3.1.6)$$

> $\mathrm{ceil}(2.6)$

$$3 \quad (3.1.7)$$

> $\mathrm{ceil}(-2.4)$

$$-2 \quad (3.1.8)$$

> $\mathrm{round}(2.5)$

$$3 \quad (3.1.9)$$

> $\mathrm{round}(-2.5)$

$$-3 \quad (3.1.10)$$

▼ 2

> $\mathrm{trunc}(x) = \mathrm{floor}(x)$ **for** $x > 0$

> $\mathrm{trunc}(x) = \mathrm{ceil}(x)$ **for** $x < 0$

▼ 3

> $\mathrm{trunc}(x)$ *runder x ned til nærmeste heltal* **for** $x > 0$ *og runder x op til nærmeste heltal* **for** $x < 0$.
$\mathrm{round}(x)$ *runder x til nærmeste heltal* **for** *alle x.*
$\mathrm{floor}(x)$ *runder x ned til nærmeste heltal* **for** *alle x*
$\mathrm{ceil}(x)$ *runder x op til nærmeste heltal* **for** *alle x*

Opgave CP 1.2.1

> $formula(x, n, c) := \begin{cases} x_0 = 1 & x_1 = c \\ x_{n+1} = x_n + x_{n-1} & n \geq 1 \end{cases}$

$$formula := (x, n, c) \rightarrow piecewise\left(x_1 = c, x_0 = 1, 1 \leq n, x_{1+n} = x_n + x_{n-1}\right) \quad \textbf{(4.1)}$$

a

> $c := \frac{(1+\sqrt{5})}{2}; x_0 := 1; x_1 := c;$

for n **from** 1 **to** 29 **do**
$x_{n+1} := evalf(x_n + x_{n-1});$
end do

$$c := \frac{1}{2} + \frac{\sqrt{5}}{2}$$

$$x_0 := 1$$

$$x_1 := \frac{1}{2} + \frac{\sqrt{5}}{2}$$

$$x_2 := 2.618033988$$

$$x_3 := 4.236067976$$

$$x_4 := 6.854101964$$

$$x_5 := 11.09016994$$

$$x_6 := 17.94427190$$

$$x_7 := 29.03444184$$

$$x_8 := 46.97871374$$

$$x_9 := 76.01315558$$

$$x_{10} := 122.9918693$$

$$x_{11} := 199.0050249$$

$$x_{12} := 321.9968942$$

$$x_{13} := 521.0019191$$

$$x_{14} := 842.9988133$$

$$x_{15} := 1364.000732$$

$$x_{16} := 2206.999545$$

$$x_{17} := 3571.000277$$

$$x_{18} := 5777.999822$$

$$x_{19} := 9349.000099$$

$$x_{20} := 15126.99992$$

$$x_{21} := 24476.00002$$
$$x_{22} := 39602.99994$$
$$x_{23} := 64078.99996$$
$$x_{24} := 103681.9999$$
$$x_{25} := 167760.9999$$
$$x_{26} := 271442.9998$$
$$x_{27} := 439203.9997$$
$$x_{28} := 710646.9995$$
$$x_{29} := 1.149850999\ 10^{6}$$
$$x_{30} := 1.860497998\ 10^{6} \tag{4.1.1}$$

> **for** n **from** 1 **to** 30 **do** $x_n := evalf\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$; **end do**

$$x_{1} := 1.618033988$$
$$x_{2} := 2.618033986$$
$$x_{3} := 4.236067972$$
$$x_{4} := 6.854101954$$
$$x_{5} := 11.09016992$$
$$x_{6} := 17.94427186$$
$$x_{7} := 29.03444176$$
$$x_{8} := 46.97871359$$
$$x_{9} := 76.01315530$$
$$x_{10} := 122.9918688$$
$$x_{11} := 199.0050240$$
$$x_{12} := 321.9968926$$
$$x_{13} := 521.0019162$$
$$x_{14} := 842.9988083$$
$$x_{15} := 1364.000724$$
$$x_{16} := 2206.999531$$
$$x_{17} := 3571.000252$$
$$x_{18} := 5777.999779$$
$$x_{19} := 9349.000025$$
$$x_{20} := 15126.99979$$
$$x_{21} := 24475.99980$$
$$x_{22} := 39602.99957$$

$$x_{23} := 64078.99933$$
$$x_{24} := 103681.9988$$
$$x_{25} := 167760.9981$$
$$x_{26} := 271442.9967$$
$$x_{27} := 439203.9945$$
$$x_{28} := 710646.9908$$
$$x_{29} := 1.149850985\ 10^{6}$$
$$x_{30} := 1.860497974\ 10^{6} \qquad \textbf{(4.1.2)}$$

b

> $c := \frac{(1-\sqrt{5})}{2}; x_0 := 1; x_1 := c;$
> **for** n **from** 1 **to** 29 **do**
> $x_{n+1} := evalf(x_n + x_{n-1});$
> **end do**

$$c := \frac{1}{2} - \frac{\sqrt{5}}{2}$$
$$x_0 := 1$$
$$x_1 := \frac{1}{2} - \frac{\sqrt{5}}{2}$$
$$x_2 := 0.381966012$$
$$x_3 := -0.2360679760$$
$$x_4 := 0.1458980360$$
$$x_5 := -0.0901699400$$
$$x_6 := 0.0557280960$$
$$x_7 := -0.0344418440$$
$$x_8 := 0.0212862520$$
$$x_9 := -0.0131555920$$
$$x_{10} := 0.0081306600$$
$$x_{11} := -0.0050249320$$
$$x_{12} := 0.0031057280$$
$$x_{13} := -0.0019192040$$
$$x_{14} := 0.0011865240$$
$$x_{15} := -0.0007326800$$
$$x_{16} := 0.0004538440$$

$$x_{17} := -0.0002788360$$
$$x_{18} := 0.0001750080$$
$$x_{19} := -0.0001038280$$
$$x_{20} := 0.0000711800$$
$$x_{21} := -0.0000326480$$
$$x_{22} := 0.0000385320$$
$$x_{23} := 5.8840\ 10^{-6}$$
$$x_{24} := 0.0000444160$$
$$x_{25} := 0.0000503000$$
$$x_{26} := 0.0000947160$$
$$x_{27} := 0.0001450160$$
$$x_{28} := 0.0002397320$$
$$x_{29} := 0.0003847480$$
$$x_{30} := 0.0006244800 \tag{4.2.1}$$

> **for** n **from** 1 **to** 30 **do** $x_n := evalf\left(\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$;**end do**

$$x_1 := -0.6180339880$$
$$x_2 := 0.3819660103$$
$$x_3 := -0.2360679766$$
$$x_4 := 0.1458980330$$
$$x_5 := -0.09016994320$$
$$x_6 := 0.05572808959$$
$$x_7 := -0.03444185346$$
$$x_8 := 0.02128623605$$
$$x_9 := -0.01315561735$$
$$x_{10} := 0.008130618657$$
$$x_{11} := -0.005024998674$$
$$x_{12} := 0.003105619970$$
$$x_{13} := -0.001919378695$$
$$x_{14} := 0.001186241269$$
$$x_{15} := -0.0007331374225$$
$$x_{16} := 0.0004531038450$$
$$x_{17} := -0.0002800335763$$
$$x_{18} := 0.0001730702679$$

$$x_{19} := -0.0001069633079$$
$$x_{20} := 0.00006610695975$$
$$x_{21} := -0.00004085634797$$
$$x_{22} := 0.00002525061167$$
$$x_{23} := -0.00001560573623$$
$$x_{24} := 9.644875397\ 10^{-6}$$
$$x_{25} := -5.960860806\ 10^{-6}$$
$$x_{26} := 3.684014576\ 10^{-6}$$
$$x_{27} := -2.276846220\ 10^{-6}$$
$$x_{28} := 1.407168349\ 10^{-6}$$
$$x_{29} := -8.696778668\ 10^{-7}$$
$$x_{30} := 5.374904803\ 10^{-7} \qquad \textbf{(4.2.2)}$$

c

> $c := 1; x_0 := 1; x_1 := c;$
> **for** n **from** 1 **to** 29 **do**
> $x_{n+1} := x_n + x_{n-1};$
> **end do**

$$c := 1$$
$$x_0 := 1$$
$$x_1 := 1$$
$$x_2 := 2$$
$$x_3 := 3$$
$$x_4 := 5$$
$$x_5 := 8$$
$$x_6 := 13$$
$$x_7 := 21$$
$$x_8 := 34$$
$$x_9 := 55$$
$$x_{10} := 89$$
$$x_{11} := 144$$
$$x_{12} := 233$$
$$x_{13} := 377$$
$$x_{14} := 610$$
$$x_{15} := 987$$

$$x_{16} := 1597$$
$$x_{17} := 2584$$
$$x_{18} := 4181$$
$$x_{19} := 6765$$
$$x_{20} := 10946$$
$$x_{21} := 17711$$
$$x_{22} := 28657$$
$$x_{23} := 46368$$
$$x_{24} := 75025$$
$$x_{25} := 121393$$
$$x_{26} := 196418$$
$$x_{27} := 317811$$
$$x_{28} := 514229$$
$$x_{29} := 832040$$
$$x_{30} := 1346269 \quad \mathbf{(4.3.1)}$$

> **for** n **from** 1 **to** 30 **do** $x_n := \mathit{evalf}\left(\frac{1}{\mathrm{sqrt}(5)}\left(\frac{(1+\mathrm{sqrt}(5))}{2}\right)^{n+1} - \frac{1}{\mathrm{sqrt}(5)} \cdot \left(\left(\frac{1-\mathrm{sqrt}(5)}{2}\right)^{n+1}\right)\right)$; **end do**

$$x_1 := 0.9999999992$$
$$x_2 := 1.999999997$$
$$x_3 := 2.999999994$$
$$x_4 := 4.999999988$$
$$x_5 := 7.999999977$$
$$x_6 := 12.99999996$$
$$x_7 := 20.99999993$$
$$x_8 := 33.99999985$$
$$x_9 := 54.99999974$$
$$x_{10} := 88.99999953$$
$$x_{11} := 143.9999991$$
$$x_{12} := 232.9999986$$
$$x_{13} := 376.9999975$$
$$x_{14} := 609.9999959$$
$$x_{15} := 986.9999928$$
$$x_{16} := 1596.999987$$

$$x_{17} := 2583.999979$$
$$x_{18} := 4180.999962$$
$$x_{19} := 6764.999934$$
$$x_{20} := 10945.99989$$
$$x_{21} := 17710.99982$$
$$x_{22} := 28656.99969$$
$$x_{23} := 46367.99946$$
$$x_{24} := 75024.99912$$
$$x_{25} := 121392.9985$$
$$x_{26} := 196417.9975$$
$$x_{27} := 317810.9958$$
$$x_{28} := 514228.9932$$
$$x_{29} := 832039.9882$$
$$x_{30} := 1.346268980\ 10^{6} \quad \textbf{(4.3.2)}$$

> **For hvert sæt af resultater med samme c-værdi giver x_n i realiteten samme værdi. Men grundet approximationer i udregningerne bliver de endelige resultater fra maple lidt forskellige. Dette ses tydligst hvor $c=1$ hvor fejlen bliver mere tydelig, jo længere vi kommer i fibbonachi sekvensen.**

▶ Opgave 1.2.1

▼ Opgave JH-2

> *restart*;

> *with*(*LinearAlgebra*) :

▶ Ligegyldig fibonacci procedure

> **Vi begynder med en generel formel for de første led i fibonacci sekvensen:**

> $x[0]=1, x[1]=0, x[n+1]=x[n]+x[n-1]$ **for** $n=1, 2, 3, ...$

▼ 1

> **Dette kan omskrives til følgende for de enkelte værdier:**

> $x[0]=1$
$x[1]=0$
$x[2]=x[1]+x[0]$
$x[3]=x[2]+x[1]$
$x[4]=x[3]+x[2]$
$x[5]=x[4]+x[3]$
...

$x[n]=x[n-1]+x[n-2]$

> **Eller:**

> $x[0]=1$

$x[1]=0$

$x[2]-x[1]-x[0]=0$
$x[3]-x[2]-x[1]=0$
$x[4]-x[3]-x[2]=0$
$x[5]-x[4]-x[3]=0$
...
$x[n]-x[n-1]-x[n-2]=0$

> **Dette kan omskrives til et matricesystem i formen AX=B for x[2] til x[n+1]:**

>
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2

```
> N := 10 :
> matrixB := proc(val :: integer) :: Vector;
 local b, i;
 b := Vector(N);
 b[1] := 1;
if val ≥ 2 then
   for i from 2 to val do
   b[i] := 0;
 end do;
 end if;
 return b;
 end proc:
> B := matrixB(N);
```

(6.3.1)

$$B := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.3.1}$$

```
> matrixA := proc(val :: integer) :: Matrix;
  local a, i, j;
  a := Matrix(val, val);
  for i from 1 to val do
  for j from 1 to val do
        if i = j then a[i, j] := 1
        elif j > i then a[i, j] := 0
        elif j + 2 ≥ i then a[i, j] := -1
        else a[i, j] := 0
  end if
  end do;
  end do;
  return a;
  end proc:
> A := matrixA(N, N)
```

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \tag{6.3.2}$$

```
> X := LinearSolve(A, B);
```

$$X := \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ 13 \\ 21 \\ 34 \\ 55 \end{bmatrix} \quad \textbf{(6.3.3)}$$

3

> $N := 50:$

> $B := matrixB(N):$

> $A := matrixA(N):$

> $X := LinearSolve(A, B);$

$$X := \begin{bmatrix} \textit{1 .. 50 } \textit{Vector}_{column} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \quad \textbf{(6.4.1)}$$

> **for** n **from** 1 **to** N **do** $X[n]$; **end do**;

1
1
2
3
5
8
13
21
34
55
89
144
233
377
610
987
1597

2584
4181
6765
10946
17711
28657
46368
75025
121393
196418
317811
514229
832040
1346269
2178309
3524578
5702887
9227465
14930352
24157817
39088169
63245986
102334155
165580141
267914296
433494437
701408733
1134903170
1836311903
2971215073
4807526976
7778742049
12586269025 **(6.4.2)**

4

> $Xlist := convert(X, list)$

$Xlist := [1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169, 63245986, 102334155, 165580141, 267914296, 433494437, 701408733, 1134903170, 1836311903, 2971215073, 4807526976, 7778742049, 12586269025]$ **(6.5.1)**

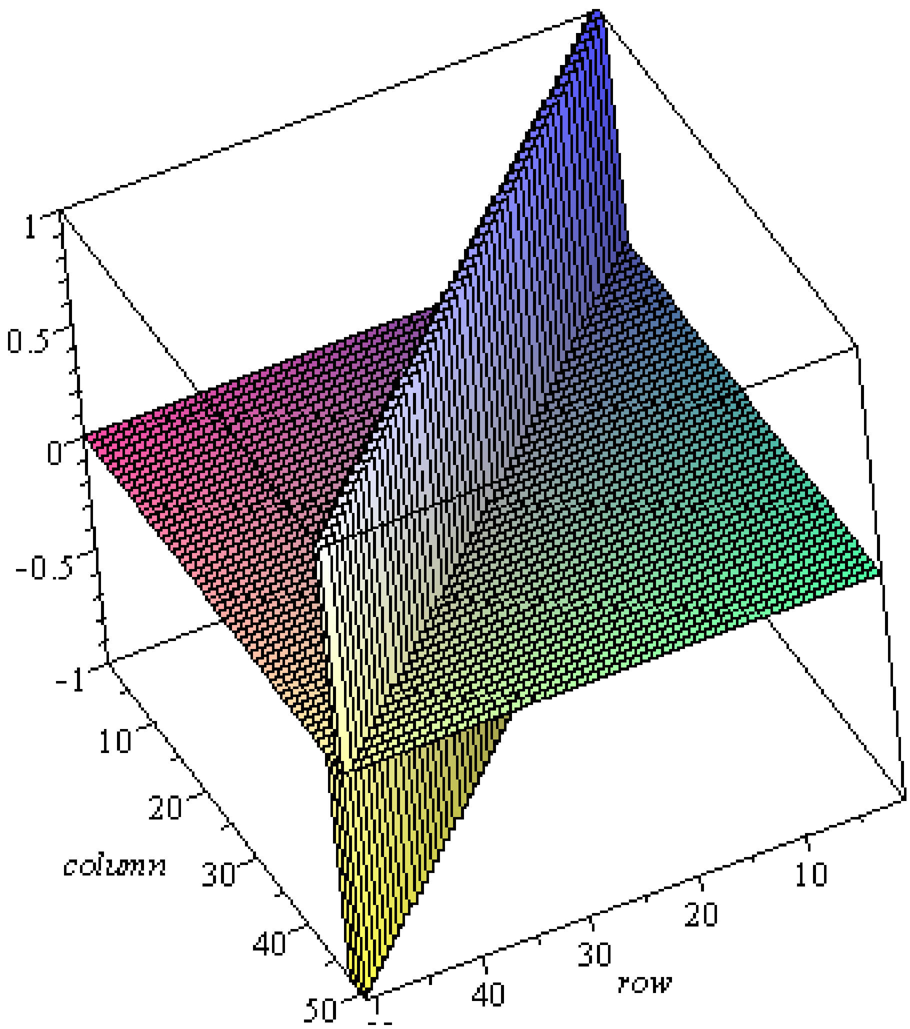
> $Xcomplete := [1, 0, op(Xlist)]$

$Xcomplete := [1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,$ **(6.5.2)**

2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169, 63245986, 102334155, 165580141, 267914296, 433494437, 701408733, 1134903170, 1836311903, 2971215073, 4807526976, 7778742049, 12586269025]

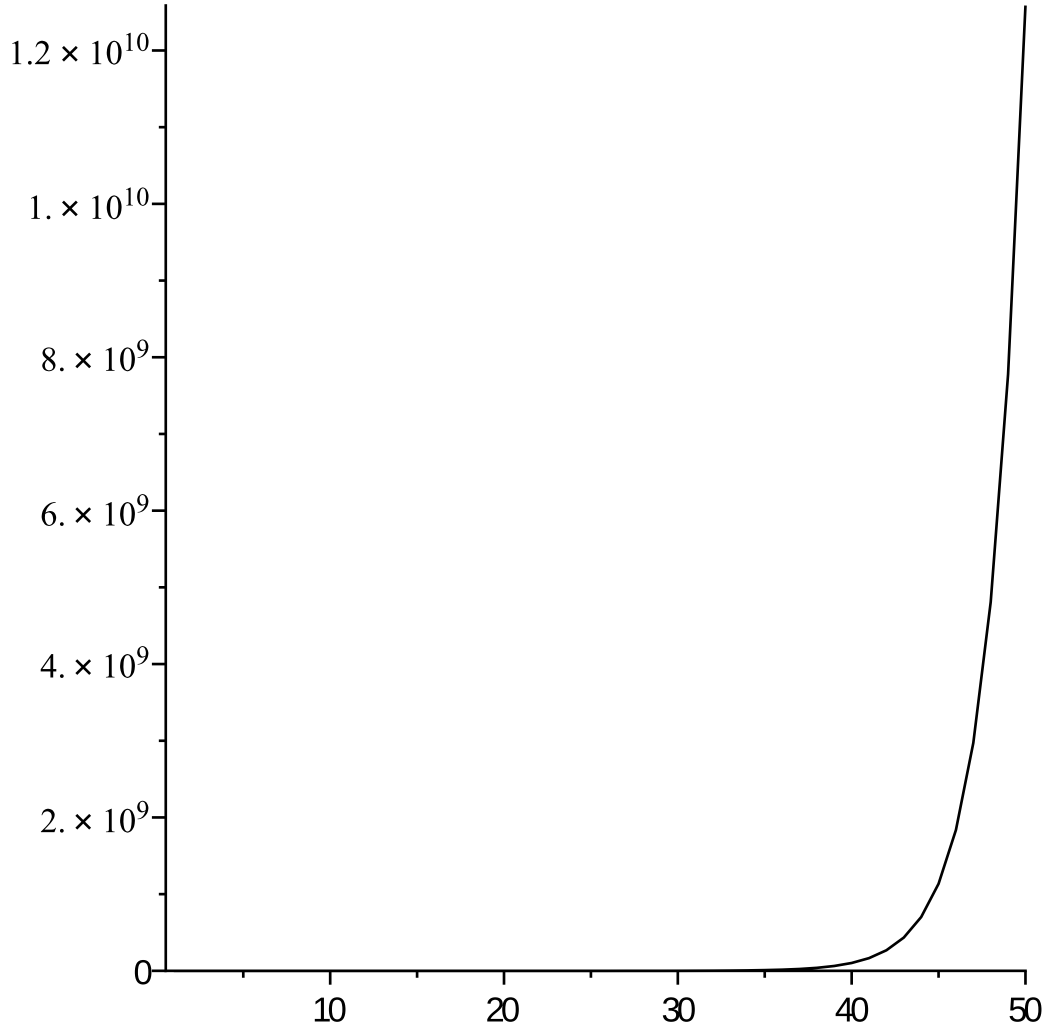
5

```
> with(plots) :
> matrixplot(A, heights = histogram, axes = boxed);
```



6

```
> listplot(X);
```



Opgave 1.2.4

a

> *restart*;

Vi har at:

> *taylor*(exp(tan(x)), x, 6)

$$1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 + \frac{37}{120}x^5 + O(x^6) \qquad \textbf{(7.1.1)}$$

> $\exp(\tan x) = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + .. \left(\mathrm{abs}(x) \leq \frac{\mathrm{pi}}{2}\right)$

> **Vi skal beholde 3 termer og estimere resten af udtrykket i lille oh notation.**

> **For lille oh har vi at:**

> $\lim_{x\to 0}\left[\frac{\frac{3x^3}{3!} + \frac{9x^4}{4!} + ..}{x^n}\right] = 0$

> For at tælleren skal dominere skal nævneren være højest x^2, for meget små værdier af x. Derfor er bedste estimation af resten = $oh(x^2)$.

b

> $restart;$

> $\ln(\tan(x)) = \ln(x) + \frac{x^2}{3} + \frac{7\,x^4}{90} + \frac{62\,x^6}{2835} + .. \left(0 < \operatorname{abs}(x) < \frac{\mathrm{pi}}{2}\right)$

> Vi beholder igen 3 termer af udtrykket men skal her estimere resten i bedste mulige Oh notation.

> Her ses det let, når vi beholder $\ln(x) + \frac{x^2}{3} + \frac{7\,x^4}{90}$, at resten estimeres til $Oh(x^6)$.

Opgave 1.2.8

> $\exp(h);\ (1 - h^4)^{-1};\ \cos(h);\ 1 + \sin(h^3);$

$$e^h$$
$$\frac{1}{1 - h^4}$$
$$\cos(h)$$
$$1 + \sin(h^3) \qquad \textbf{(8.1)}$$

> Disse 4 udtryk går alle mod 1 når h->0.

exp(h)

> For exp(h) har vi:

> $f(h) = c + Oh(h^{\alpha}) = c + oh(h^{\beta})$

> For at $f(h) = c + Oh(h^{\alpha})$ skal:

> $\operatorname{abs}(e^h) \leq C \cdot \operatorname{abs}(c + Oh(h^{\alpha}))$

$$e^{\Re(h)} \leq C\left|c + Oh(h^{\alpha})\right| \qquad \textbf{(8.1.1)}$$

> En estimation af resultatet kan finde vha. taylor serien til exp(h):

> $taylor(\exp(h), h, 4)$

$$1 + h + \frac{1}{2}\,h^2 + \frac{1}{6}\,h^3 + \mathrm{O}(h^4) \qquad \textbf{(8.1.2)}$$

> $taylor(\exp(h), h, 1)$

$$1 + \mathrm{O}(h) \qquad \textbf{(8.1.3)}$$

> Da vi ved at h->0 er de senere led af serien mindre betydende end det første. Vi har derfor: $c = 1$ og $\alpha = 1$.

> Nu for at finde beta:

> For at $f(h) = c + oh(h^{\beta})$ har vi:

> $\exp(h) - c = oh(h^{\beta})$

> $\lim_{h\to 0}\left[\frac{\exp(h)-c}{h^{\beta}}\right]=0$

> **For at dette holder skal tælleren bare være 0** $\Rightarrow c=1$**, og nævneren skal være forskellig fra 0. Derfor ses det let at bedste værdi for beta er** $\beta=0$**.**

(1-h^4)^-1

> *For* α :

> $\mathrm{abs}\left((1-h^4)^{-1}\right)\leq c+C\cdot\mathrm{abs}(h^{\alpha})$

$$\frac{1}{|-1+h^4|}\leq c+C\,|h^{\alpha}| \tag{8.2.1}$$

> $\lim_{h\to 0}\left[\frac{\mathrm{abs}\left((1-h^4)^{-1}-c\right)}{\mathrm{abs}(h^{\alpha})}\right]\leq C$

> **Vi kan bruge L'Hopitals regel, og for at gøre dette skal både nævner og tæller have grænseværdi gående mod 0, og for at** $(1-h^4)^{-1}-c\to 0$ **for** $h\to 0$ **skal** $c=1$**. Vi antager positivt alpha.**

> $\mathit{diff}\left((1-h^4)^{-1}-1,h\right)$

$$\frac{4\,h^3}{(1-h^4)^2} \tag{8.2.2}$$

> $\lim_{h\to 0}\left[\frac{(1-h^4)^{-1}-1}{h^{\alpha}}\right]=\lim_{h\to 0}\left[\frac{4\,h^3}{(1-h^4)^2\cdot\alpha\cdot h^{\alpha-1}}\right]$

> $h\to 0\Rightarrow(1-h^4)^2\to 1$, *derfor* :

> $=\lim_{h\to 0}\left[\frac{4\,h^3}{\alpha\cdot h^{\alpha-1}}\right]=\lim_{h\to 0}\left[\frac{4}{\alpha\cdot h^{\alpha-4}}\right]$

> **Hvis** $\alpha<4$ **går udtrykket mod** 0 **som det skal for lille oh. Hvis** $\alpha>4$ **går udtrykket mod** ∞**. Derfor er bedste værdi for** $\alpha=4$, $c=1$**.**

> **Ved samme analogi ses det let at bedste værdi for** $\beta=3$, $c=1$**.**

cos(h)

> $\lim_{h\to 0}\left[\frac{\mathrm{abs}(\cos(h)-c))}{\mathrm{abs}(h^{\alpha})}\right]\leq C$

For at bruge L'Hopital sætter vi $c=1$ *så* $\frac{(\cos(h)-1)}{h^{\alpha}}$

$\lim_{h\to 0}\left[-\frac{\sin(h)}{\alpha\cdot h^{\alpha-1}}\right]=\lim_{h\to 0}\left[-\frac{\cos(h)}{\alpha(\alpha-1)\,h^{\alpha-2}}\right]$

For alpha=2 får vi

$\lim_{h\to 0}\left[\frac{\mathrm{abs}(\cos(h)-c))}{\mathrm{abs}(h^{\alpha})}\right]=\frac{1}{2}$

Dette betyder at den bedste værdi for alpha=2 og c=1

> For beta: når beta=1 og c=1:

> $\lim_{h\to 0}\left[\frac{\cos(h)-c}{h^{\beta}}\right]=\lim_{h\to 0}\left[\frac{(\cos(h)-1)}{h}\right]=\lim_{h\to 0}\left[\left(-\frac{\sin(h)}{1}\right)\right]=0$

1+sin(h^3)

> $\lim_{h\to 0}\left[\frac{\operatorname{abs}(1+\sin(h^3)-c)}{\operatorname{abs}(h^{\alpha})}\right]\le C$

> c=1

> $seq(diff(h^{\alpha},[h\$n]),n=0..3)$

$$h^{\alpha},\ \frac{h^{\alpha}\alpha}{h},\ \frac{h^{\alpha}\alpha^2}{h^2}-\frac{h^{\alpha}\alpha}{h^2},\ \frac{h^{\alpha}\alpha^3}{h^3}-\frac{3\,h^{\alpha}\alpha^2}{h^3}+\frac{2\,h^{\alpha}\alpha}{h^3} \tag{8.4.1}$$

> $seq(diff(1+\sin(h^3)-1,[h\$n]),n=0..3)$

$$\sin(h^3),\ 3\cos(h^3)\,h^2,\ -9\sin(h^3)\,h^4+6\cos(h^3)\,h,\ -27\cos(h^3)\,h^6-54\sin(h^3)\,h^3+6\cos(h^3) \tag{8.4.2}$$

> $\lim_{h\to 0}\left[\frac{\operatorname{abs}(1+\sin(h^3)-1)}{\operatorname{abs}(h^{\alpha})}\right]=\lim_{h\to 0}\left[\frac{3\cos(h^3)h^2}{h^{\alpha-1}\alpha}\right]$

$=\lim_{h\to 0}\left[\frac{-9\sin(h^3)\,h^4+6\cos(h^3)\,h}{h^{\alpha-2}\alpha^2-h^{\alpha-2}\cdot\alpha}\right]$

$=\lim_{h\to 0}\left[\frac{1}{h^{\alpha-3}\cdot\alpha^3-3\cdot h^{\alpha-3}\cdot\alpha^2+2\cdot h^{\alpha-3}\cdot\alpha}(-9\sin(h^3)\,h^4+6\cos(h^3)\,h\right.$
$\left.-27\cos(h^3)\,h^6-54\sin(h^3)\,h^3+6\cos(h^3))\right]$

$=\lim_{h\to 0}\left[\frac{6}{(\alpha-2)\cdot(\alpha-1)\cdot\alpha\cdot h^{\alpha-3}}\right]$

> For alpha=3 og c=1

> $\lim_{h\to 0}\left[\frac{\operatorname{abs}(1+\sin(h^3)-c)}{\operatorname{abs}(h^{\alpha})}\right]=\frac{6}{6}=1$

> beta

> $\lim_{h\to 0}\left[\frac{\operatorname{abs}(1+\sin(h^3)-c)}{\operatorname{abs}(h^{\beta})}\right]=0$

> Vha. samme analyse som opgaverne ovenfor ser vi at beta=2 og c=1

> $\lim_{h\to 0}\left[\frac{1+\sin(h^3)-1)}{h^2}\right]=\lim_{h\to 0}\left[\frac{\sin(h^3)}{h^2}\right]=...=\lim_{h\to 0}\left[\frac{0}{2}\right]=0$

Opgave 1.3.1

> Vi har 3 sekvenser som følger:

> $[1,0,-2,-6,-14,-30,...];\ [1,1,1,1,...];\ [2,4,8,16,...]$

> Vi skal udtrykke første sekvens ud fra de 2 følgende. Først skriver vi sekvenserne op som ligninger, lad os kalde dem a,b og c:

> $a = [1, 0, -2, -6, -14, -30, ...] = 2 - 2^{n-1} = \alpha b + \beta c$

> $b = [1, 1, 1, 1, ...] = 1$

> $c = [2, 4, 8, 16, ...] = 2^n$

> $2 - 2^{n-1} = \alpha \cdot 1 + \beta \cdot 2^n$

> Hvis vi tager $\alpha = 2$ får vi:

> $-2^{n-1} = \beta \cdot 2^n$

> Vi sætter $\beta = -2^{-1} \Rightarrow rhs = -2^{n-1} = lhs$

> Så vores svar bliver $\alpha = 2$ og $\beta = -\frac{1}{2}$

Opgave 1.3.11

> Vi skal give en reel base af sekvenser for hver løsnings rum.

a

> $(4E^0 - 3E^2 + E^3)x = 0$

> Vi finder det karakteristiske polynomium:

> $p(\lambda) = \lambda^3 - 3\lambda^2 + 4$

$$p := \lambda \mapsto \lambda^3 - 3\lambda^2 + 4 \quad \textbf{(10.1.1)}$$

> $solve(\lambda^3 - 3\lambda^2 + 4, \lambda)$

$$-1, 2, 2 \quad \textbf{(10.1.2)}$$

> Vi har derfor følgende løsnings sekvenser:

> $x(-1) = [-1, 1, -1, 1, ...]$

> $x(2) = [2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, ...]$

> $x'(2) = [1, 4, 12, 32, ...]$

b

> $(3E^0 - 2E^1 + E^2)x = 0$

> Vi finder det karakteristiske polynomium:

> $p(\lambda) = \lambda^2 - 2\lambda + 3$

> $solve(\lambda^2 - 2\lambda + 3, \lambda)$

$$1 + I\sqrt{2}, 1 - I\sqrt{2} \quad \textbf{(10.2.1)}$$

> Her findes ingen reelle svar, så vores reelle løsningsmængder er tomme!

Opgave JH-3

```
> restart;
> with(LinearAlgebra) :
```

1.

```
> MyHilbert :=proc(num :: integer) :: matrix;
  local i, j, H;
  H := Matrix(num, num);
  for i from 1 to num do
  for j from 1 to num do
  H[i, j] := evalf(1 / (i + j-1));
  end do;
  end do;
  return H;
  end proc;
```

```
MyHilbert := proc(num::integer)::matrix;
   local i, j, H;
   H := Matrix(num, num);
   for i to num do for j to num do H[i, j] := evalf(1 / (i + j − 1)) end do end do;
   return H
end proc
```
(11.1.1)

2

```
> MyHilbert(5)
```

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix} \qquad \textbf{(11.2.1)}$$

3

```
> HilbertMatrix(5)
```

(11.3.1)

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix} \quad (11.3.1)$$

4

```
> RHS :=proc(n :: integer)
   local H, sumVector;
   H := MyHilbert(n);
   sumVector := Transpose(add(H[j], j=1..n));
  return sumVector;
  end proc;
RHS := proc(n::integer)
   local H, sumVector;
   H := MyHilbert(n);
   sumVector := LinearAlgebra:-Transpose(add(H[j], j=1..n));
   return sumVector
end proc
```

(11.4.1)

5

```
> RHS(5)
```

$$\begin{bmatrix} \frac{137}{60} \\ \frac{29}{20} \\ \frac{153}{140} \\ \frac{743}{840} \\ \frac{1879}{2520} \end{bmatrix} \quad (11.5.1)$$

6-7

```
> for N from 1 to 20 do
  if not Equal(LinearSolve(MyHilbert(N), RHS(N)), Vector((1..N), 1))
```

```
        then print("Maple error", N, i) end if;
 end do;
```

> **Ingen output betyder at maple må være fantastisk til at regne!**

8

```
> for N from 1 to 10 do
LinearSolve(MyHilbert(N), RHS(N));
 end do;
```

$$\begin{bmatrix} 1. \end{bmatrix}$$

$$\begin{bmatrix} 1.00000000000000 \\ 0.999999999999999 \end{bmatrix}$$

$$\begin{bmatrix} 0.999999999999998 \\ 1.00000000000001 \\ 0.999999999999990 \end{bmatrix}$$

$$\begin{bmatrix} 0.999999999999965 \\ 1.00000000000041 \\ 0.999999999999007 \\ 1.00000000000065 \end{bmatrix}$$

$$\begin{bmatrix} 0.999999999999782 \\ 1.00000000000390 \\ 0.999999999983675 \\ 1.00000000002408 \\ 0.999999999988432 \end{bmatrix}$$

$$\begin{bmatrix} 0.999999999999030 \\ 1.00000000002514 \\ 0.999999999840788 \\ 1.00000000039458 \\ 0.999999999580353 \\ 1.00000000016055 \end{bmatrix}$$

$$\begin{bmatrix} 0.999999999993690 \\ 1.00000000024045 \\ 0.999999997750873 \\ 1.00000000856675 \\ 0.999999984528270 \\ 1.00000001321731 \\ 0.999999995698972 \end{bmatrix}$$

$$\begin{bmatrix} 0.999999999952253 \\ 1.00000000247196 \\ 0.999999968463051 \\ 1.00000016785055 \\ 0.999999553720608 \\ 1.00000062535673 \\ 0.999999558410935 \\ 1.00000012380282 \end{bmatrix}$$

$$\begin{bmatrix} 0.999999999709672 \\ 1.00000001977483 \\ 0.999999668086864 \\ 1.00000235593724 \\ 0.999991393963274 \\ 1.00001751832324 \\ 0.999979928323834 \\ 1.00001210113191 \\ 0.999997014555486 \end{bmatrix} \tag{11.7.1}$$

$$\begin{bmatrix} 0.999999996546986 \\ 1.00000028144573 \\ 0.999994255531150 \\ 1.00005054615681 \\ 0.999765059045639 \\ 1.00063240973888 \\ 0.998980368636419 \\ 1.00097093997373 \\ 0.999496652652441 \\ 1.00010949191355 \end{bmatrix} \tag{11.7.1}$$

> **Jo flere gange estimationer af tal bliver lagt sammen, des større bliver den sammenlagte fejl. Det kan vi se her hvor vi bruger kommatal i stedet for præcise værdier.**

9

> *Digits* := 100

$$Digits := 100 \tag{11.8.1}$$

> **for** N **from** 1 **to** 10 **do**
> *LinearSolve*(*MyHilbert*(N), *RHS*(N));
> **end do**;

[
1.000000000000000000000000000000000000000000000000000000000000000000\
000000000000000000000000000000000]

[
[1.00000000000000000000000000000000000000000000000000000000000000000\
0000000000000000000000000000000000],
[1.00000000000000000000000000000000000000000000000000000000000000000\
0000000000000000000000000000000000]]

[
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> **Vi ser her at det tager lang lang tid før fejlen bliver så tydelig som det gjorde med 10 sig figs. Her er tallene stadig meget meget tæt på de "korrekte" resultater.**

10

> *interface*(*displayprecision* = 10);

$$-1 \tag{11.9.1}$$

```
> for N from 1 to 10 do
  LinearSolve(MyHilbert(N), RHS(N));
  end do;
```

$$\begin{bmatrix} 1.0000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000000 \\ 1.0000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \end{bmatrix} \tag{11.9.2}$$

>