Linear Algebra Working Group :: Day 3 Part 1

Note: All vector spaces will be finite-dimensional vector spaces over the field $\mathbb{R}$.

1 Duals and annihilators

Definition 1.1. Let V and W be vector spaces. The vector space of linear maps from V to W is the space:

$$\mathrm{Hom}(V, W) := \{T : V \to W \mid T \text{ is linear}\}$$

Exercise 1. Show that if V and W are vector spaces, then $\mathrm{Hom}(V, W)$ really is a vector space.

Exercise 2. Let V and W be finite-dimensional vector spaces. What is the dimension of $\mathrm{Hom}(V, W)$?

Exercise 3. Let V, A, B, C be vector spaces, let $F : A \to C$ and $G : B \to C$ be linear maps, and set:

$$A \times_C B := \{(a, b) \in A \times B \mid F(a) = G(b)\}.$$

Show that:

1. The space $\mathrm{Hom}(V, A \oplus B)$ is isomorphic to the space $\mathrm{Hom}(V, A) \oplus \mathrm{Hom}(V, B)$.
2. The space $\mathrm{Hom}\,(V, A \times_C B)$ is isomorphic to the space:

$$\mathrm{Hom}(V, A) \times_{\mathrm{Hom}(V,C)} \mathrm{Hom}(V, B) = \Big\{(S, T) \in \mathrm{Hom}(V, A) \times \mathrm{Hom}(V, B) \mid FS = GT\Big\}$$

Definition 1.2. Let V be a finite-dimensional vector space. The dual of V is the vector space:

$$V^* := \mathrm{Hom}(V, \mathbb{R}) = \{T : V \to \mathbb{R} \mid T \text{ is linear}\}$$

Elements of the dual V^* are sometimes called **covectors**.

Exercise 4. Let $\{e_1, ..., e_n\}$ be a basis of a finite-dimensional vector space V. Show that the set $\{e_1^*, ..., e_n^*\}$ defined by:

$$e_i^*(e_j) := \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is a basis of V^*. Conclude that V and V^* are isomorphic as vector spaces and that $\dim V = \dim V^*$.

Definition 1.3. For a finite-dimensional vector space V with a basis $\{e_1, ..., e_n\}$. The basis $\{e_1^*, ..., e_n^*\}$ of exercise 4 is called the **dual basis** of V dual to $\{e_i\}$.

Exercise 5. Describe the dual basis in the following cases:

1. Euclidean space $\mathbb{R}^n$ equipped with the standard basis vectors $\{e_1, ..., e_n\}$.
2. The space $\mathbb{R}[n]$ of polynomials of degree at most n equipped with the basis $\{1, t, t^2, ..., t^n\}$

Exercise 6. Let V and W be finite-dimensional vector spaces and let $T : V \to W$ be a linear map.

1. Show that for any covector $\rho : W \to \mathbb{R}$, the map:

$$T^*\rho : V \to \mathbb{R}, \qquad T^*\rho(v) := \rho(T(v))$$

is a covector.

2. Show that the map:

$$T^* : W^* \to V^*, \qquad \rho \mapsto T^*\rho$$

is a linear map.

Exercise 7. Show that if $T : U \to V$ and $S : V \to W$ are linear maps between finite-dimensional vector spaces then $(S \circ T)^* = T^* \circ S^*$

Exercise 8. Let V be a finite-dimensional vector space. Consider the double dual:

$$V^{**} := \mathrm{Hom}(\mathrm{Hom}(V, \mathbb{R}), \mathbb{R}).$$

1. Let $v \in V$ be a fixed vector. Prove that the evaluation map:

$$\mathrm{ev}_v : V^* \to \mathbb{R}, \qquad \mathrm{ev}_v(\rho) := \rho(v)$$

is an element of V^{**}. That is, prove that ev_v is a linear map.

2. Prove that the map:

$$\mathrm{ev} : V \to V^{**}, \qquad \mathrm{ev}(v) := \mathrm{ev}_v$$

is a linear isomorphism.

Definition 1.4. Let V be a vector space and let $U \subseteq V$ be a subspace. The **annihilator** of U is the subspace:

$$U^0 := \{\rho \in V^* \mid \rho|_U \equiv 0\}$$

Exercise 9. Let V be a vector space with a subspace $U \subseteq V$. Show that the annihilator U^0 really is a vector space.

Exercise 10. Let V be a vector space. What are the annihilators of $\{0\}$ and V respectively?

Exercise 11. Let V be a finite-dimensional vector space. Suppose $U \subseteq V$ is a subspace different from $\{0\}$. Show that $U^0 \neq V^*$.

Exercise 12. Let V be a finite-dimensional vector space and let $U \subseteq V$ be a subspace. Show that $\dim V = \dim U + \dim U^0$.

Exercise 13. Let V be a finite-dimensional vector space and let U and W be subspaces of V. Show that if $U \subseteq W$, then $W^0 \subseteq U^0$.

2 Bilinear Forms and Pairings

Definition 2.1. Let V be a finite dimensional vector space. A **bilinear form** on V is a bilinear map $\Omega : V \times V \to \mathbb{R}$. That is, for all $a, b \in \mathbb{R}$ and $u, v, w \in V$ we have:

$$\Omega(au + bv, w) = a\Omega(u, w) + b\Omega(v, w) \qquad \text{and} \qquad \Omega(w, au + bv) = a\Omega(w, u) + b\Omega(w, v).$$

Exercise 14. Show that the following are bilinear maps:

1. The standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

2. The determinant map $\det : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $(u, v) \mapsto \det \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$.

Exercise 15. Let V be a 2-dimensional vector space, let $\{e, f\}$ be a basis of V, and let $\{e^*, f^*\}$ be the corresponding dual basis of V. Show that the map:

$$\omega : V \times V \to \mathbb{R}, \qquad \omega(u, v) := \det \begin{pmatrix} e^*(u) & e^*(v) \\ f^*(u) & f^*(v) \end{pmatrix}$$

is a bilinear map on V. This generalizes the example of exercise 14-2.

Exercise 16. Let $T : V \to W$ be a linear map of finite-dimensional vector spaces. Show that for any bilinear map $\rho : W \times W \to \mathbb{R}$, the map:

$$T^*\rho : V \times \ V \to \mathbb{R}, \qquad (T^*\rho)(v, w) := \rho(T(v), T(w)),$$

is a bilinear map.

Exercise 17. Let $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear map. Show that there exists a matrix B such that for all $u, v \in \mathbb{R}^n$ we have:

$$\omega(u, v) = v^T B u$$

This is called the **matrix representative** of the bilinear form.

Exercise 18. Find the matrix representatives of the bilinear forms in exercises 14 and 15.

Definition 2.2. Let V be a finite dimensional vector space, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear form on V. The bilinear form is **symmetric** if for all $u, v \in V$ we have:

$$\Omega(u, v) = \Omega(v, u).$$

The bilinear form is **skew-symmetric** if for all $u, v \in V$ we have:

$$\Omega(u, v) = -\Omega(v, u).$$

Exercise 19. Classify the examples of exercises 14 and 15 into symmetric or skew-symmetric.

Definition 2.3. Let V be a finite dimensional vector space and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear form. The bilinear form Ω is **nondegenerate** if when given a vector $u \in V$ satisfying that:

$$\Omega(u, v) = 0, \qquad \text{for all } v \in V,$$

then the vector u must be the zero vector.

Exercise 20. Show that the bilinear forms of exercises 14 and 15 are nondegenerate.

Exercise 21. Let V be a finite dimensional vector space and let $\omega : V \times V \to \mathbb{R}$ be a bilinear form. Given a vector $u \in V$, define the map:

$$\widetilde{\omega}(u) : V \to \mathbb{R}, \qquad \widetilde{\omega}(u)(v) := \omega(u, v).$$

Show that the bilinear form ω is nondegenerate if and only if the map:

$$\widetilde{\omega} : V \to V^*, \qquad u \mapsto \widetilde{\omega}(u),$$

is an isomorphism.

Definition 2.4. Let V and W be finite dimensional vector spaces. A **pairing** is a bilinear map $\omega : V \times W \to \mathbb{R}$.

Exercise 22. Show that the evaluation map:

$$\mathrm{ev} : V \times V^* \to \mathbb{R}, \qquad \mathrm{ev}(v, \rho) := \rho(v),$$

is a pairing. Compare this with exercise 8.

Definition 2.5. Let V and W be finite dimensional vector spaces. A pairing $\omega : V \times W \to \mathbb{R}$ is **nondegenerate** if:

$$\begin{aligned} \omega(v, w) = 0 \ \text{ for all } w \in W \Rightarrow v = 0 \\ \omega(v, w) = 0 \ \text{ for all } v \in V \Rightarrow w = 0. \end{aligned}$$

Exercise 23. Show that if $\omega : V \times W \to \mathbb{R}$ is a nondegenerate pairing, then V is isomorphic to the dual W^*, and W is isomorphic to the dual V^*.

Exercise 24. Let V be a finite dimensional vector space, and let $\mathrm{ev} : V \times V^* \to \mathbb{R}$ be the evaluation map. Show that ev is a nondegenerate pairing.

3 Symplectic Linear Algebra

Definition 3.1. Let V be a finite-dimensional vector space. A **symplectic form** on V is a skew-symmetric nondegenerate bilinear form $\omega : V \times V \to \mathbb{R}$. A **symplectic vector space** is a pair (V, ω) consisting of a finite dimensional vector space and a symplectic form ω on V.

Exercise 25. Show that:

$$\omega_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \qquad \omega_0(u, v) := -\det(u, v),$$

is a symplectic form on $\mathbb{R}^2$. This is called the canonical symplectic form on $\mathbb{R}^2$ (the minus sign is just a convention).

Exercise 26. Let n be a nonnegative integer, and consider $\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n$. Let $e_i \in \mathbb{R}^{2n}$, for $i = 1, \ldots, n$ be the standard basis vectors for the first copy of $\mathbb{R}^n$, and let $f_i \in \mathbb{R}^{2n}$, for $i = 1, \ldots, n$ be the standard basis vectors for the second copy of $\mathbb{R}^n$. Let $e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*$ be the corresponding dual basis. Show that:

$$\omega_0 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}, \qquad \omega_0(u, v) := -\sum_{j=1}^{n} \det \begin{pmatrix} e_j^*(u) & e_j^*(v) \\ f_j^*(u) & f_j^*(v) \end{pmatrix}$$

is a symplectic form on $\mathbb{R}^{2n}$. This is called the **canonical symplectic form** on $\mathbb{R}^{2n}$. Explain geometrically what ω_0 is measuring for a given pair of vectors $u, v \in \mathbb{R}^{2n}$.

Definition 3.2. Let (V,ω) be a symplectic vector space, and let U be a linear subspace of V. The **symplectic orthogonal** to U is the subspace:

$$U^\omega := \{\, v \in V \mid \omega(v,\cdot)|_U \equiv 0 \,\}.$$

Exercise 27. Consider the symplectic vector space $(\mathbb{R}^2,\omega_0)$, where ω_0 is the canonical sympelctic form. Let $v \in \mathbb{R}^2$ be any nonzero vector, and let $U := \operatorname{span}\{v\}$. Compute the symplectic orthogonal U^ω.

Exercise 28. Consider the symplectic vector space $(\mathbb{R}^4,\omega_0)$, where ω_0 is the canonical symplectic form. Compute the symplectic orthogonal of the following subspaces:

1. $U := \operatorname{span}\{e_1, e_2\}$.
2. $U := \operatorname{span}\{e_1, f_1\}$.
3. $U := \operatorname{span}\{f_1, f_2\}$.

Remark 3.3. As you may have noticed, symplectic orthogonals don't need to complement the subspace in a symplectic vector space. That is if U is a linear subspace of a symplectic vector space (V,ω), it may happen that $U \cap U^\omega \neq \{0\}$. Nevertheless we have the following result.

Exercise 29. Let (V,ω) be a symplectic vector space, and let U be a linear subspace of V. Show that:

$$\dim(V) = \dim(U) + \dim(U^\omega).$$

(*Hint: Show that the isomorphism* $\widetilde{\omega} : V \to V^*$ *maps* U^ω *onto* U^0*, and use exercise 12*).

Exercise 30. Let (V,ω) be a symplectic vector space, and let U be a linear subspace. Show that $(U^\omega)^\omega = U$.

Remark 3.4. Having as symplectic form places significant constraints on the vector space as we now state.

Exercise 31 (Darboux Basis)**.** Let (V,ω) be a symplectic vector space. Show that there exists a nonnegative integer n and a basis $e_1,\ldots,e_n,f_1,\ldots,f_n$ of V such that:

$$\omega(e_i,f_j) = \delta_{i,j} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$
$$\omega(e_i,e_j) = \omega(f_i,f_j) = 0$$

for all $i,j = 1,\ldots,n$. In particular, every symplectic vector space is even-dimensional.
Hint: Mimick the idea of Gram-Schmidt as follows:

1. Start with a nonzero $e_1 \in V$.
2. Show that you can pick f such that $\omega(e_1,f) \neq 0$, and set $f_1 := \frac{f}{\omega(e_1,f)}$.
3. Set $W_1 := \operatorname{span}\{e_1,f_1\}$.
4. Show that $W_1 \cap W_1^\omega = \{0\}$, and thus $V = W_1 \oplus W_1^\omega$.

5. Show that the symplectic form restricts to a symplectic form on W_1^ω. Hence, we would be able to repeat the procedure on W_1^ω.

6. Use induction.

Exercise 32. Show that every even dimensional vector space has a symplectic form.

Definition 3.5. Let (V, ω) be a symplectic vector space, and let U be a linear subspace. The subspace U is:

1. **isotropic** if $U \subseteq U^\omega$.

2. **coisotropic** if $U^\omega \subseteq U$.

3. **Lagrangian** if $U = U^\omega$.

4. **symplectic** if $U \cap U^\omega = \{0\}$.

Exercise 33. Let (V, ω) be a symplectic vector space, and let U be a linear subspace. Show that:

1. U is isotropic if and only if the restriction $\omega|_{U \times U}$ is identically 0.

2. Show that if U is isotropic then $\dim U \leq \dim U^\omega$.

3. Show that U is Lagrangian if and only if it is maximally isotropic (i.e. if E is any isotropic subsapce containing U in U then $U = E$.)

Exercise 34. Let (V, ω) be a symplectic vector space, and let U be a linear subspace. Show that if $\dim U = \frac{1}{2}\dim(V)$ and U is isotropic, then U is Lagrangian. Show that if U is Lagrangian then $\dim U = \frac{1}{2}\dim(V)$.

Definition 3.6. Let (V, ω_V) and (W, ω_W) be a symplectic vector spaces. A linear map $T : V \to W$ is a **symplectomorphism** if $T^*\omega_W = \omega_V$. We say that (V, ω_V) and (W, ω_W) are **symplectomorphic** if there exists an invertible symplectomorphism $T : V \to W$.

Exercise 35. Show that the composition of symplectomorphisms is again a symplectomorphism.

Exercise 36. Let (V, ω_V) and (W, ω_W) be symplectic vector spaces of the same dimension. Then a symplectomorphism $T : V \to W$ has an inverse. (*Hint: Use the nondegeneracy of the symplectic form and consider the kernel.*)

Exercise 37. Show that the inverse of a symplectomorphism is also a symplectomorphism.

Exercise 38. Let (V, ω_V) and (W, ω_W) be symplectic vector spaces of the same dimension. Show that they must be symplectomorphic.

Exercise 39. Fix a nonnegative integer n, and set the matrix:

$$J := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. Let $\widehat{L} : V \to V$ be a linear transformation on a symplectic vector space (V, ω), and let L be its matrix representation with respect to the Darboux basis on V. Show that $\widehat{L}$ is a symplectomorphism if and only if the matrix representation L satisfies the matrix equation:

$$L^T J L = J. \tag{3.1}$$

The set of $2n \times 2n$ matrices L satisfying (3.1) is called the symplectic group $\mathsf{Sp}(2n)$. It is a very interesting group with many cool properties and applications.

References

[Hal58] P.R. Halmos. Finite-Dimensional Vector Sapces. Reprinting of the 1958 second edition. *Undergraduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1974.

[L] E. Lerman. Symplectic Geometry and Hamiltonian Mechanics. Unpublished notes.

[M13] A. McInerney. First Steps In Differential Geometry: Riemannian, Contact, Symplectic. *Undergraduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 2013.

[Rom08] S. Roman. Advanced Linear Algebra. Third Edition. *Graduate Texts in Mathematics, 135*. Springer, New York, 2008.