

# Authors' Response to the Referee Comments

We thank the referees for their detailed reading of the manuscript and their valuable comments. We have incorporated the changes that the referees have suggested into the revised manuscript. We list below the referees' comments, followed by our responses, denoted by [AR](#).

Also in the revised manuscripts, all the revisions are highlighted by color blue for the referees to spot them easily.

## Referee #1

- My only concern is the fact that the term  $H_2$  is used on time-domain function, which might be inappropriate from a mathematical point of view?

[AR](#): We agree with the referee that the term  $H_2$  is associated with measure in the frequency domain. To clarify this point, we have added a footnote, Footnote 4 at the bottom of page 3, explaining the usage.

## Referee #2

- The introduction of the paper does not allow to position the problem accurately even though valuable elements of this positioning are given in section 3.3.

[AR](#): We thank the referee for pointing this out. We have added some of this discussion from Section 3.3 to Introduction now (the second to last paragraph in Introduction to help resolve this issue.

- Some particular application domains are cited in the beginning without justification.

[AR](#): We have added citations.

- What about the choice of the finite horizon  $t_f$  ?

[AR](#): The choice of the finite horizon is really problem and application dependent; however the interpolation conditions hold for any  $t_f$ . We added a sentence right before Remark 3.2 to state this fact.

- Justify the choice of simple poles for the reduced order model.

AR: The choice of simple poles was motivated by the infinite horizon  $\mathcal{H}_2$  optimal approximation problems and the corresponding interpolation, as in, for example, citations [3,6,7, 11,23] in the paper. Indeed, the interpolatory  $\mathcal{H}_2$  approximation has been extended some specific nonlinear dynamical systems as well, such as

- P. Benner and T. Breiten, Interpolation-based  $\mathcal{H}_2$ -model reduction of bilinear control systems, SIAM Journal on Matrix Analysis and Applications, 2012.
- G. Flagg and S. Gugercin, Multipoint Volterra Series Interpolation and  $\mathcal{H}_2$  Optimal Model Reduction of Bilinear Systems, SIAM Journal on Matrix Analysis and Applications, 2015.
- P. Benner , P. Goyal, and S. Gugercin,  $\mathcal{H}_2$ -quasi-optimal model order reduction for quadratic-bilinear control systems, SIAM Journal on Matrix Analysis and Applications, 2018.

These nonlinear extensions also use the same assumptions. It is certainly theoretically possible that an optimal  $\mathcal{H}_2$  reduced model can have repeated poles and interpolation conditions have been extended to this case as well:

- P. van Dooren, K. Gallivan, and P.-A. Absil,  $\mathcal{H}_2$ -Optimal Model Reduction with Higher-Order Poles, SIAM Journal on Matrix Analysis and Applications, 2010.

However, this is situation is rarely observed in practice and we chose to use simple poles. Also, we note that the full model can have repeated poles. The assumption is only on the reduced model.

- What is the reference of the existing result given by Theorem 2.1 ?

AR: We have added the citations immediately after the theorem.

- Notation are not fixed in the paper. For example  $H'(s)$  is the derivative of  $H$  with respect to  $s$ . This can be stated at the end of the introduction section (with all other notations...)

AR: We have added a clarification for this notation immediately after Theorem 2.1, which corresponds to the first occurrence of  $\mathbf{H}'(s)$  .

- Section 2.2 : the 5 introduction lines are very important and can be used to state the purpose of the paper.

AR: These five lines state the purpose of infinite horizon optimal model reduction even though what we consider here, i.e., optimal model reduction on a finite horizon, has some similar difficulties. Following the referee's earlier suggestion (the first comment), we have added some more discussion about our approach to the introduction.

- section 3 : the first 5 lines are a "useless repetition" and can be removed.

AR: We have removed these 5 lines and reorganized the beginning of section 3

- Section 3 : lines 6 to 10 can be given as a Remark (important remark!)

AR: We have followed the referee's recommendation and list these lines as Remark 3.3 now.

- Remark 3.2 can be given after the proof of the main result and can be a part of the discussion of the given result.

AR: We have moved the remark after the proof as the referee suggested.

- Lemma 3.3 : "Let  $G(s)$  and  $Gr(s)$  be as defined in (3.5) and (3.7)" instead of "Let  $G(s)$  and  $Gr(s)$  be as defined in (3.5) and (3.5)"

AR: We thank the referee for pointing out this issue. We have addressed it in the revised version.

- Page 8 : (3.18) is one equation no need to (3.19).

AR: We thank the referee for pointing out this issue. We have addressed it in the revised version.

- The computation of (3.18) is not clear. One can give more details allowing to obtain such result.

AR: We have added more details after Equation (3.18) better explaining these computations.

- Sentence after (3.19) : "...in the parentheses in (3.18) and (3.19)" instead of "...in the parentheses in (3.18) and (3.18)".

AR: We thank the referee for pointing out this issue. We have addressed it in the revised version.

- 3.3 section can be a (concise) part of the introduction of the paper.

AR: We thank the referee for the comment. We believe that section 3.3 addresses the importance of the result. We believe that explaining the result after it is given is more appropriate and it makes it more tangible for the reader.

- Section 4 : should state how to use the main result in constructing the reduced order model which is not the case in the current version.

AR: This comment was not very clear to us. Corollaries 4.1 and 4.2, which follow directly from the main result, explain how we are using the optimality conditions to construct the algorithm. We were not sure what else the referee would like to see explained.

- The numerical examples try to show the "supremacy" of the proposed result which is obviously not the case. Instead, it will be useful to explain clearly the aim of each numerical experimentation before giving a clear figure (with only one comparison aim at each time). The given figures are almost indecipherable !

AR: We thank the referee for pointing out the difficulties in reading the figures. We have not separated the figures into subplots, making the comparisons easier and hopefully making the figures more clear.

We were not sure why the referee claimed that the proposed method outperformed the other methods. For every model and for every reduced order we have tried, FHIRKA has improved whatever initialization we have used. In some cases the improvements were dramatic in some cases less so. However regardless it has produced a lower error. We have excluded some  $r$  values not because FHIRKA performed bad. It was simply because other methods have led to very poor approximations that should not be used as an initialization.

## Referee #3

- As stated in Remark 3.2, the optimality condition is not directly expressed by  $H(s)$  and  $H_r(s)$ , but by  $G(s)$  and  $G_r(s)$ , defined by (3.5) and (3.7). The reviewer cannot find a physical interpretation of  $G(s)$  and  $G_r(s)$ . The authors can comment on the interpretation.

AR: We thank the referee for this comment. We have addressed this issue on the revised version and added Remark 3.2 to precisely address this issue.

- In Numerical Simulation, the authors can add some time responses of the original and reduced-order models. For example, showing impulse responses of the models enables readers to understand "practical" importance of the proposed method.

AR: We have added a graph showing the time responses for the error for different methods; Figure 4 in the revised version. We have chosen to include the impulse error plot instead of the impulse responses to better illustrate the deviations. Due to the page limit stated by Automatica, we were able to include only one such figure. For a reference, we are including two additional graphs here.

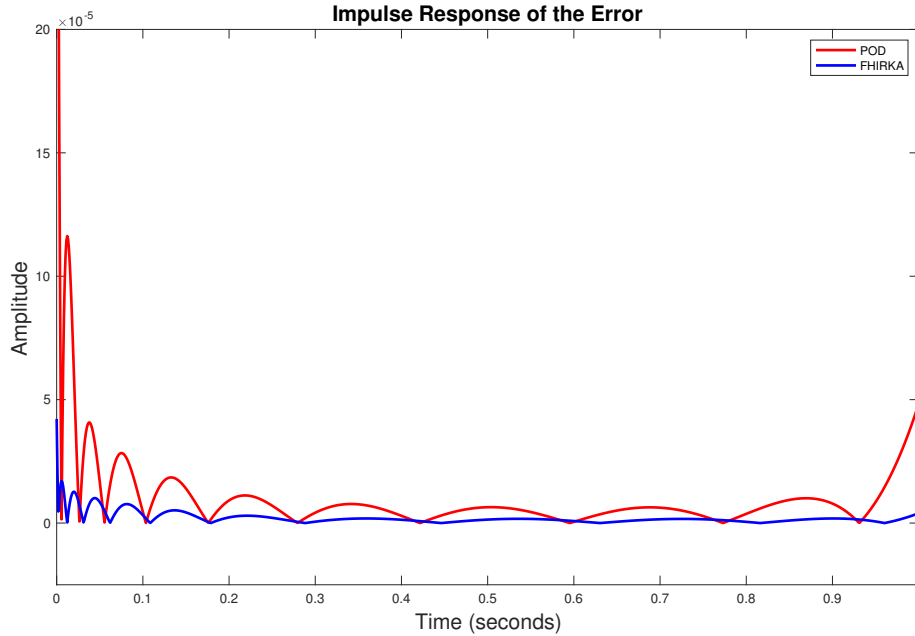


Figure 1: POD vs FHIRKA for the heat model

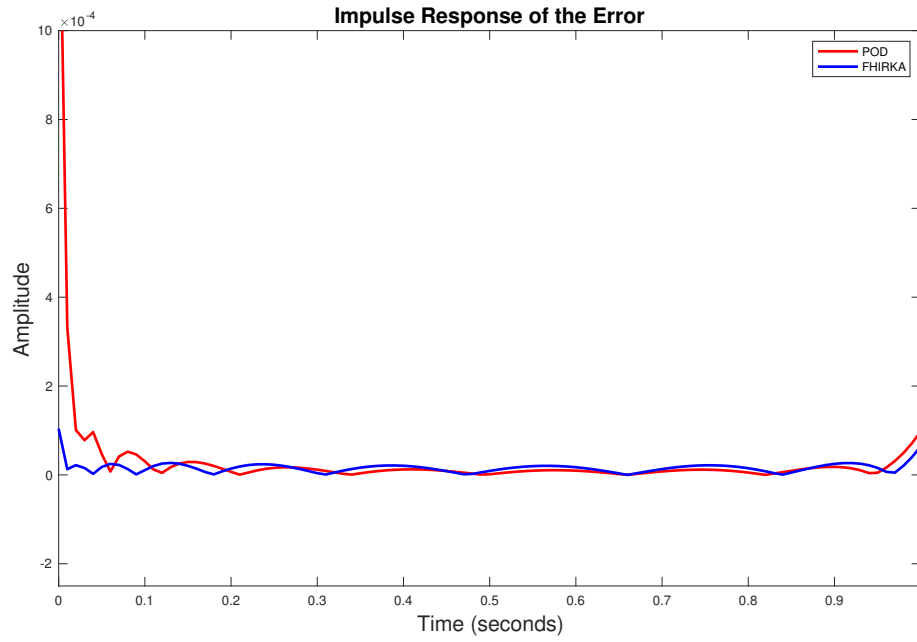


Figure 2: POD vs FHIRKA for the unstable model

- There are some typos in the manuscript. For example, the statement "G(s) and Gr(s) be as defined in (3.5) and (3.5)" in Lemma 3.3 should be corrected.

AR: We thank the referee for pointing out this issue. We have addressed it in the revised version.