$\mathcal{H}_2(t_f)$ Optimality Conditions for a Finite-time Horizon☆

Klajdi Sinani*, Serkan Gugercin

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

Abstract

In this paper we establish the interpolatory model reduction framework for optimal approximation of MIMO dynamical systems with respect to the $\mathcal{H}_2$ norm over a finite-time horizon, denoted as the $\mathcal{H}_2(t_f)$ norm. Using the underlying inner product space, we derive the interpolatory first-order necessary optimality conditions for approximation in the $\mathcal{H}_2(t_f)$ norm. Then, we develop an algorithm, which yields a locally optimal reduced model that satisfies the established interpolation-based optimality conditions. We test the algorithm on various numerical examples to illustrate its performance.

Keywords: time-limited model reduction, interpolation, unstable system, $\mathcal{H}_2$-optimality, linear systems

1. Introduction

Simulation, design, and control of dynamical systems play an important role in numerous scientific and industrial tasks such as signal propagation in the nervous system, heat dissipation, prediction of major weather events. The need for detailed models due to the increasing demand for greater resolution leads to large-scale dynamical systems, posing tremendous computational difficulties when applied in numerical simulations. In order to overcome these challenges, we perform model reduction where we replace the large-scale dynamics with high-fidelity reduced representations.

Consider the linear time-invariant dynamical system:

$$\begin{array}{rcl}\dot{\mathbf{x}}(t) & = & \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) & = & \mathbf{C}\mathbf{x}(t)\end{array} \quad \text{with } \mathbf{x}(0)=\mathbf{0}, \quad \iff \quad \mathbf{y}(t)=\int_0^t \mathbf{h}(t-\tau)\mathbf{u}(\tau)d\tau, \tag{1.1}$$

where $\mathbf{A}\in\mathbb{R}^{n\times n}$, $\mathbf{B}\in\mathbb{R}^{n\times m}$, and $\mathbf{C}\in\mathbb{R}^{p\times n}$ are constant matrices; the variable $\mathbf{x}(t)\in\mathbb{R}^n$ denotes the internal variables, $\mathbf{u}(t)\in\mathbb{R}^m$ denotes the control inputs, and $\mathbf{y}(t)\in\mathbb{R}^p$ denotes the outputs; and $\mathbf{h}(t)=\mathbf{C}e^{\mathbf{A}t}\mathbf{B}$ is the impulse response of the full model. The length of the internal variable $\mathbf{x}(t)$, i.e., n, is called the order of the full model that we would like to reduce. Model reduction achieves this by replacing the original model with a lower dimensional one:

$$\begin{array}{rcl}\dot{\mathbf{x}}_r(t) & = & \mathbf{A}_r\mathbf{x}_r(t)+\mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) & = & \mathbf{C}_r\mathbf{x}_r(t)\end{array} \quad \text{with } \mathbf{x}_r(0)=\mathbf{0}, \quad \iff \quad \mathbf{y}_r(t)=\int_0^t \mathbf{h}_r(t-\tau)\mathbf{u}(\tau)d\tau, \tag{1.2}$$

where as in (**??**), $\mathbf{h}_r(t)=\mathbf{C}_re^{\mathbf{A}_rt}\mathbf{B}_r$ is the impulse response of the reduced model, and $\mathbf{A}_r\in\mathbb{R}^{r\times r}$, $\mathbf{B}_r\in\mathbb{R}^{r\times m}$, and $\mathbf{C}_r\in\mathbb{R}^{p\times r}$ with $r\ll n$. The goal is that the output of the reduced model, $\mathbf{y}_r(t)$, approximates the true output, $\mathbf{y}(t)$, of the original system accurately in an appropriate norm.

For the linear dynamical systems we consider here, a plethora of methods exists for producing high-fidelity/optimal reduced models, such as balanced truncation [**? ?**] and its variants, optimal Hankel norm approximation [**?**], and the Iterative Rational Krylov Algorithm (IRKA) [**?**] and its variants; see [**? ?**] for further references. These methods usually focus on constructing high-quality reduced models over an infinite time horizon. However, in various settings, we might either have access to simulations over a finite horizon or can only simulate the system under investigation for a finite horizon such as in the case of unstable

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*Corresponding author

dynamical systems. Therefore, in those situations we are interested in the behavior of the dynamical system over a finite time interval $[0, t_f]$ where $t_f < \infty$, and we need the reduced model to be accurate only in the interval of interest.

Time-limited balanced truncation [? ? ? ?] and Proper Orthogonal Decomposition (POD) [?] are two common frameworks to create reduced models on a finite horizon. For time-limited balanced truncation, [?] establishes an upper bound for the $\mathcal{H}_\infty$ error between the full and reduced models, [?] provides an $\mathcal{H}_2$ error bound. In this paper, we explore $\mathcal{H}_2(t_f)$ optimality for model reduction over a finite time horizon. We establish $\mathcal{H}_2(t_f)$ optimality conditions over a finite time horizon and introduce an algorithm which yields better approximations of the large-scale dynamical systems compared to other model reduction methods.

The optimality requires a parametrization of the reduced model. We will work with the time-domain representation of the dynamical system to derive the optimality condition. Specifically, we represent the impulse response of the reduced dynamical system using the modal decomposition, i.e.,

$$\mathbf{h}_r(t) = \mathbf{C}_r e^{\mathbf{A}_r t}\mathbf{B}_r = \sum_{i=1}^{r} e^{\lambda_i t} \boldsymbol{\ell}_i \boldsymbol{r}_i^T. \tag{1.3}$$

where λ_i's are the eigenvalues of $\mathbf{A}_r$, and $\boldsymbol{\ell}_i \in \mathbb{C}^{p\times 1}, \boldsymbol{r}_i \in \mathbb{C}^{m\times 1}$. In other words, the impulse response is expressed as a sum of r rank-1 $p \times m$ matrices. To simplify the presentation, we assume that λ_i's, the reduced order poles, are simple. The representation (**??**) is nothing but a state-space transformation on $\mathbf{h}_r(t) = \mathbf{C}_r e^{\mathbf{A}_r t}\mathbf{B}_r$ using the eigenvectors of $\mathbf{A}_r$. Using the parametrization of the reduced model in (**??**), we derive interpolatory optimality conditions in the $\mathcal{H}_2(t_f)$ norm and implement a model reduction algorithm that satisfies these optimality conditions. The advantages of time-limited model reduction under the $\mathcal{H}_2(t_f)$ norm are two fold: first, one can obtain a better approximation in the time interval of interest, and, second, one can obtain a locally optimal reduced order model in the case of unstable systems as well.

The rest of the paper is organized as follows: In Section **??** we briefly review optimal $\mathcal{H}_2$ model reduction in the infinite horizon case. The main results, including the new optimality conditions for finite horizon, are established in Section **??** followed by numerical examples in Section **??**. The papers ends with conclusions and future work in Section **??**.

2. $\mathcal{H}_2$-Optimal Model Reduction: The Infinite Horizon Case

Model reduction with respect to the $\mathcal{H}_2$ norm in the infinite horizon case has been studied extensively; see, for examples, [? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ?] and the references therein. In this section, we briefly recall these results as they will help to highlight the similarities to and differences from the finite-horizon case that we are interested in.

2.1. $\mathcal{H}_2$ Norm and $\mathcal{H}_2$ Error Measure

The error analysis for model reduction of linear dynamical systems can be conducted either in the frequency domain or in the time domain. Therefore, we define the $\mathcal{H}_2$ norm in each domain.

Definition 2.1. Let $\mathbf{h}(t)$ and $\mathbf{g}(t)$ be the impulse responses of two asymptotically stable[1] linear dynamical systems with real state-space realizations. The $\mathcal{H}_2$ inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$ and the $\mathcal{H}_2$ norm $\|\cdot\|_{\mathcal{H}_2}$ are

$$\langle \mathbf{h}, \mathbf{g} \rangle_{\mathcal{H}_2} = \int_0^\infty \mathsf{Tr}\left((\mathbf{h}(t))^T \mathbf{g}(t) \right) dt \quad \text{and} \quad \|\mathbf{h}\|_{\mathcal{H}_2} = \sqrt{\int_0^\infty \|\mathbf{h}\|_F^2 \, dt},$$

respectively, where $\mathsf{Tr}(\cdot)$ denotes the trace and $\|\cdot\|_F$ denotes the Frobenius norm of a matrix.

[1] We will call $\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}$ asymptotically stable if all the eigenvalues of $\mathbf{A}$ have negative real parts. We will call $\mathbf{h}(t)$ stable when $\mathbf{A}$ has some semi-simple eigenvalues on the imaginary axis in addition to those with negative real parts. Otherwise, we call $\mathbf{h}(t)$ unstable.

To define the $\mathcal{H}_2$ norm in the frequency domain, let $\mathbf{Y}(s)$, $\mathbf{U}(s)$, and $\mathbf{H}(s)$ denote the Laplace transforms of the output $\mathbf{y}(t)$, the input $\mathbf{u}(t)$, and the impulse response $\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}$ in (**??**). Then, taking the Laplace transform of the convolution integral in (**??**), we obtain

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s), \quad \text{where} \quad \mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

is called the transfer function of (**??**). Let $\{\rho_1, \rho_2, \ldots, \rho_n\}$ denote the eigenvalues of $\mathbf{A}$, assumed simple. Then, similar to the parametrization of the reduced model $\mathbf{h}_r(t)$ in (**??**), the full-model impulse response can be equivalently written as

$$\mathbf{h}(t) = \sum_{i=1}^{n} e^{\rho_i t}\boldsymbol{c}_i\boldsymbol{b}_i^T \quad \text{with the transfer function} \quad \mathbf{H}(s) = \sum_{i=1}^{n} \frac{\boldsymbol{c}_i\boldsymbol{b}_i^T}{s - \rho_i}, \tag{2.1}$$

where $\boldsymbol{c}_i \in \mathbb{C}^p$ and $\boldsymbol{b}_i \in \mathbb{C}^m$, for $i = 1, \ldots, n$. This is called the pole-residue form where ρ_i's are the poles of the (rational) transfer function $\mathbf{H}(s)$ with the corresponding rank-1 residues $\boldsymbol{c}_i\boldsymbol{b}_i^T$.

Definition 2.2. Let $\mathbf{H}(s)$ and $\mathbf{G}(s)$ denote the transfer functions of two asymptotically stable dynamical systems with real state-space realizations. The $\mathcal{H}_2$ inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$ and the $\mathcal{H}_2$ norm $\| \cdot \|_{\mathcal{H}_2}$ are

$$\langle \mathbf{G}, \mathbf{H} \rangle_{\mathcal{H}_2} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathsf{Tr}(\overline{\mathbf{G}}(-\imath\omega)\mathbf{H}^T(\imath\omega))d\omega \qquad \text{and} \qquad \|\mathbf{H}\|_{\mathcal{H}_2} := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \|\mathbf{H}(i\omega)\|_F^2 \, d\omega},$$

respectively.

Similar to $\mathbf{H}(s)$, let $\mathbf{H}_r(s) = \mathbf{C}_r(s\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r$ denote the transfer function of the reduced model. Then, the relevance and importance of the $\mathcal{H}_2$ norm in the model reduction become clear by noting that

$$\|\mathbf{y} - \mathbf{y}_r\|_{L_\infty} \leq \|\mathbf{H} - \mathbf{H}_r\|_{\mathcal{H}_2} \|\mathbf{u}\|_{L_2}, \tag{2.2}$$

i.e., the L_∞ norm of the output error $\mathbf{y}(t) - \mathbf{y}_r(t)$ due to model reduction is bounded by the the $\mathcal{H}_2$ norm of error transfer function relative to the L_2 norm of the input $\mathbf{u}(t)$; see, e.g., [**?**], for a proof. Therefore, to guarantee that the reduced model output $\mathbf{y}_r(t)$ is close to the original one $\mathbf{y}(t)$, one might look for a reduced model that minimizes the $\mathcal{H}_2$ error norm.

2.2. Interpolatory Conditions for Optimal $\mathcal{H}_2$ Model Reduction

Given a reduced order r, the goal is to construct a reduced order model whose transfer function $\mathbf{H}_r(s)$ minimizes the $\mathcal{H}_2$ error norm $\|\mathbf{H} - \mathbf{H}_r\|_{\mathcal{H}_2}$. Since this is a non-convex optimization problem, the usual, numerically feasible, approach is to find a reduced model that satisfies the necessary conditions for $\mathcal{H}_2$ optimality. These conditions can be formulated in terms of Sylvester equations [**? ?**] or interpolation [**? ?**]. These two frameworks are equivalent [**?**]. In this paper, we will focus on the interpolation framework.

Theorem 2.1. *Let*

$$\mathbf{h}_r(t) = \sum_{k=1}^{r} e^{\lambda_k t}\boldsymbol{\ell}_k\boldsymbol{r}_k^T \qquad \Longleftrightarrow \qquad \mathbf{H}_r(s) = \sum_{k=1}^{r} \frac{\boldsymbol{\ell}_k\boldsymbol{r}_k^T}{s - \lambda_k}$$

be the best r^{th} order approximation of an asymptotically stable linear dynamical system $\mathbf{H}(s)$ with respect to the $\mathcal{H}_2$ norm. Then, for $k = 1, 2, ..., r$,

$$\boldsymbol{\ell}_k^T\mathbf{H}(-\lambda_k) = \boldsymbol{\ell}_k^T\mathbf{H}_r(-\lambda_k), \quad \mathbf{H}(-\lambda_k)\boldsymbol{r}_k = \mathbf{H}_r(-\lambda_k)\boldsymbol{r}_k, \quad \textit{and,} \quad \boldsymbol{\ell}_k^T\mathbf{H}'(-\lambda_k)\boldsymbol{r}_k = \boldsymbol{\ell}_k^T\mathbf{H}_r'(-\lambda_k)\boldsymbol{r}_k. \tag{2.3}$$

This result states that the transfer function of the optimal $\mathcal{H}_2$ approximation to $\mathbf{H}(s)$ is a (tangential) Hermite interpolant where the interpolation points are the mirror images of the reduced-order poles $\{\lambda_k\}$, and the tangental directions are given by the corresponding residues $\{\boldsymbol{\ell}_k\boldsymbol{r}_k^T\}$. Since the optimality conditions depend on the reduced-model to be computed, the solution requires a nonlinear iteration. The Iterative Rational Krylov Algorithm (IRKA) [**?**] and its variants such as [**? ? ? ?**] use these interpolation based optimality conditions to produce an interpolatory, locally $\mathcal{H}_2$ optimal reduced model. The next section will extend this framework to the finite-time interval case.

3. $\mathcal{H}_2(t_f)$ Optimal Model Reduction on a Finite Horizon

If we are interested in the behavior of a system only in a finite time interval (or access to a numerical simulation only on a finite time interval), we can obtain better reduced models by defining the measure of approximation quality over the interval of interest, instead of focussing on the infinite time horizon. Also, in case of an unstable system, since the impulse grows unbounded as $t \to \infty$, the regular $\mathcal{H}_2$ measure cannot be applied. Therefore, we will focus on model reduction on a finite time horizon.

We would like to point out that for an unstable dynamical system, if the matrix $\mathbf{A}$ does not have purely imaginary eigenvalues, one can work with the $\mathcal{L}_2$ norm by decomposing it into a stable and anti-stable system, and then obtain an interpolatory reduced model based on this measure. However, this solution requires destroying the causality of the underlying dynamics [**?**]. This is not the framework we are interested in here and we will work with a finite-time interval.

3.1. $\mathcal{H}_2(t_f)$ Norm on a Finite-time Horizon

It is immediately clear from the time-domain definition of the (infinite-horizon) $\mathcal{H}_2$ norm how to define the finite-horizon version:

Definition 3.1. Let $\mathbf{h}(t)$ and $\mathbf{g}(t)$ denote the impulse responses of two dynamical systems with real state-space realizations. For a finite-time horizon $[0, t_f]$, the $\mathcal{H}_2(t_f)$ inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_2(t_f)}$ and $\mathcal{H}_2(t_f)$ norm $\|\cdot\|_{\mathcal{H}_2(t_f)}$ are defined as

$$\langle \mathbf{h}, \mathbf{g} \rangle_{\mathcal{H}_2(t_f)} = \int_0^{t_f} \mathsf{Tr}((\mathbf{h}(t))^T \mathbf{g}(t)) dt \quad \text{and} \quad \|\mathbf{h}\|_{\mathcal{H}_2(t_f)} = \sqrt{\int_0^{t_f} \|\mathbf{h}(t)\|_F^2 \, dt},$$

respectively.

3.2. Finite Horizon Interpolation-based Conditions $\mathcal{H}_2(t_f)$ Optimal Model Reduction

The problem we are interested in is as follows: Given a dynamical system with impulse response $\mathbf{h}(t)$ (or equivalently with transfer function $\mathbf{H}(s)$) and a reduced order r, find the reduced model with the impulse response

$$\mathbf{h}_r(t) = \sum_{i=1}^{r} e^{\lambda_i t} \boldsymbol{\ell}_i \boldsymbol{r}_i^T. \tag{3.1}$$

such that $\|\mathbf{h} - \mathbf{h}_r\|_{\mathcal{H}_2(t_f)}$ is minimized. As in the regular $\mathcal{H}_2$ case, this is a non-convex optimization problem and we will focus on local minimizers. Using the parametrization (**??**), we we will derive interpolation-based necessary conditions for optimality. The main result is given by Theorem **??**. However, we need many supplementary results, Lemmas **??**-**??**, to reach this final conclusion.

It is immediately clear that, the $\mathcal{H}_2(t_f)$-error satisfies

$$\|\mathbf{h} - \mathbf{h}_r\|_{\mathcal{H}_2(t_f)}^2 = \|\mathbf{h}\|_{\mathcal{H}_2(t_f)}^2 - 2\langle \mathbf{h}, \mathbf{h}_r \rangle_{\mathcal{H}_2(t_f)} + \|\mathbf{h}_r\|_{\mathcal{H}_2(t_f)}^2, \tag{3.2}$$

where the inner product $\langle \mathbf{h}, \mathbf{h}_r \rangle_{\mathcal{H}_2(t_f)}$ is real since $\mathbf{h}(t)$ and $\mathbf{h}_r(t)$ are real. Finding the first-order necessary conditions for optimal $\mathcal{H}_2(t_f)$ model reduction will require computing the gradient of the error expression (**??**) with respect to the optimization variables. Since the reduced model, as described by the impulse response in $\mathbf{h}_r(t)$, is parametrized by the reduced order poles $\{\lambda_i\}$, and the residue directions $\{\boldsymbol{\ell}_i\}$ and $\{\boldsymbol{r}_i\}$, we will be computing the gradient of the error with respect to these variables. Since the first term in the error (**??**), i.e., $\|\mathbf{h}\|_{\mathcal{H}_2(t_f)}$, is a constant, we will be focusing on the remaining two terms only. First, in the next lemma, we will formulate these two last terms in terms of $\{\lambda_i\}$, $\{\boldsymbol{\ell}_i\}$ and $\{\boldsymbol{r}_i\}$.

Lemma 3.2. Let $\mathbf{h}(t) = \sum_{j=1}^{n} e^{\rho_j t}\boldsymbol{c}_j\boldsymbol{b}_j^T$ and $\mathbf{h}_r(t) = \sum_{i=1}^{r} e^{\lambda_i t}\boldsymbol{\ell}_i\boldsymbol{r}_i^T$ be, respectively, the impulse responses of the full and reduced models as described in (**??**) and (**??**). Then,

$$\langle \mathbf{h}, \mathbf{h}_r \rangle_{\mathcal{H}_2(t_f)} = \sum_{j=1}^{n}\sum_{i=1}^{r} \boldsymbol{\ell}_i^T\boldsymbol{c}_j\boldsymbol{b}_j^T\boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f}-1}{\lambda_i+\rho_j}. \tag{3.3}$$

and

$$\|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)} = \sum_{i=1}^{r}\sum_{j=1}^{r} \boldsymbol{\ell}_i^T\boldsymbol{\ell}_j\boldsymbol{r}_j^T\boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_j)t_f}-1}{\lambda_i+\lambda_j}. \tag{3.4}$$

Proof. The both results follow from the definition of the $\mathcal{H}_2(t_f)$ inner product. First consider

$$\langle \mathbf{h}, \mathbf{h}_r \rangle_{\mathcal{H}_2(t_f)} = \mathsf{Tr}\left(\int_0^{t_f} \mathbf{h}_r(t)^T\mathbf{h}(t)\,dt\right).$$

Plug $\mathbf{h}(t) = \sum_{j=1}^{n} \boldsymbol{c}_j\boldsymbol{b}_j^T e^{\rho_j t}$ and $\mathbf{h}_r(t) = \sum_{i=1}^{r} \boldsymbol{\ell}_i\boldsymbol{r}_i^T e^{\lambda_i t}$ into this formula to obtain

$$\langle \mathbf{h}, \mathbf{h}_r \rangle_{\mathcal{H}_2(t_f)} = \mathsf{Tr}\left(\int_0^{t_f} \sum_{i=1}^{r}(\boldsymbol{\ell}_i\boldsymbol{r}_i^T e^{\lambda_i t})^T \sum_{j=1}^{n} \boldsymbol{c}_j\boldsymbol{b}_j^T e^{\rho_j t}dt\right) = \mathsf{Tr}\left(\sum_{i=1}^{r}\sum_{j=1}^{n} \boldsymbol{r}_i\boldsymbol{\ell}_i^T\boldsymbol{c}_j\boldsymbol{b}_j^T \int_0^{t_f} e^{(\lambda_i+\rho_j)t}dt\right)$$

Computing the integral and using the fact that $\mathsf{Tr}(\mathbf{A}_1\mathbf{A}_2) = \mathsf{Tr}(\mathbf{A}_2\mathbf{A}_1)$ for two matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of appropriate sizes, we obtain

$$\langle \mathbf{h}, \mathbf{h}_r \rangle_{\mathcal{H}_2(t_f)} = \mathsf{Tr}\left(\sum_{j=1}^{n}\sum_{i=1}^{r} \boldsymbol{r}_i\boldsymbol{\ell}_i^T\boldsymbol{c}_j\boldsymbol{b}_j^T \frac{e^{(\lambda_i+\rho_j)t_f}-1}{\lambda_i+\rho_j}\right) = \sum_{j=1}^{n}\sum_{i=1}^{r} \boldsymbol{\ell}_i^T\boldsymbol{c}_j\boldsymbol{b}_j^T\boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f}-1}{\lambda_i+\rho_j},$$

which proves (**??**). Then, (**??**) follows directly by replacing $\mathbf{h}(t)$ with $\mathbf{h}_r(t)$ in this derivation. □

For infinite time horizon, Theorem **??** tells us that a locally $\mathcal{H}_2$ optimal reduced transfer function is a tangential Hermite interpolant of the original transfer function at the mirror images of the reduced poles. We will show that in the finite horizon case, even though Hermite tangential interpolation is still the necessary conditions for optimality, what is being interpolated and what the interpolant is are different.

Theorem 3.1. *Let $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$, with $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$, be the transfer function of the full-order model with the pole-residue representation $\mathbf{H}(s) = \sum_{i=1}^{n} \frac{\boldsymbol{c}_i\boldsymbol{b}_i^T}{s-\rho_i}$ as in (**??**), where $\boldsymbol{c}_i \in \mathbb{C}^p$, $\boldsymbol{b}_i \in \mathbb{C}^m$, and $\rho_i \in \mathbb{C}$ for $i = 1,\ldots,n$. For a finite-time horizon $[0,t_f]$, define*

$$\mathbf{G}(s) = -e^{-st_f}\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}e^{\mathbf{A}t_f}\mathbf{B} + \mathbf{H}(s). \tag{3.5}$$

Let

$$\mathbf{H}_r(s) = \mathbf{C}_r(s\mathbf{I}_r-\mathbf{A}_r)^{-1}\mathbf{B}_r = \sum_{i=1}^{r} \frac{\boldsymbol{\ell}_i\boldsymbol{r}_i^T}{s-\lambda_i} \tag{3.6}$$

be the transfer function of the best r^{th} order approximation of $\mathbf{H}(s)$ with respect to the $\mathcal{H}_2(t_f)$ norm where $\mathbf{A}_r \in \mathbb{R}^{r\times r}$, $\mathbf{B}_r \in \mathbb{R}^{r\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times r}$, $\boldsymbol{\ell}_i \in \mathbb{C}^p$, $\boldsymbol{r}_i \in \mathbb{C}^m$, and $\lambda_i \in \mathbb{C}$ for $i = 1,\ldots,r$. Define

$$\mathbf{G}_r(s) = -e^{-st_f}\mathbf{C}(s\mathbf{I}_r-\mathbf{A}_r)^{-1}e^{\mathbf{A}_r t_f}\mathbf{B}_r + \mathbf{H}_r(s). \tag{3.7}$$

Then, for $k = 1, 2, ..., r$,

$$\boldsymbol{\ell}_k^T \mathbf{G}(-\lambda_k) = \boldsymbol{\ell}_k^T \mathbf{G}_r(-\lambda_k), \tag{3.8}$$

$$\mathbf{G}(-\lambda_k)\boldsymbol{r}_k = \mathbf{G}_r(-\lambda_k)\boldsymbol{r}_k, \quad \textit{and} \tag{3.9}$$

$$\boldsymbol{\ell}_k^T \mathbf{G}'(-\lambda_k)\boldsymbol{r}_k = \boldsymbol{\ell}_k^T \mathbf{G}_r'(-\lambda_k)\boldsymbol{r}_k. \tag{3.10}$$

Remark 3.2. Before we prove Theorem **??**, we briefly comment on its implications. In the infinite-horizon case, if $\mathbf{H}_r(s)$ is the best $\mathcal{H}_2$ approximation to $\mathbf{H}(s)$, then $\mathbf{H}_r(s)$ interpolates $\mathbf{H}(s)$. However, in the finite-horizon case, the interpolant is $\mathbf{G}_r(s)$, which is a function of $\mathbf{H}_r(s)$, and the interpolated function is $\mathbf{G}(s)$, which is a function of $\mathbf{H}(s)$; thus $\mathbf{H}_r(s)$ does not interpolate $\mathbf{H}(s)$ in this case. However, the optimal interpolation points and directions are still determined by the reduced model $\mathbf{H}_r(s)$. The algorithmic implications of this result will be revisited in Section **??**. ♢

The following lemma will be used in the proof of Theorem **??**.

Lemma 3.3. Let $\mathbf{G}(s)$ and $\mathbf{G}_r(s)$ be as defined in (**??**) and (**??**), respectively. Then,

$$\mathbf{G}(-\lambda_k) = \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{(\lambda_k+\rho_j)t_f} - 1}{\lambda_k + \rho_j}, \tag{3.11}$$

$$\mathbf{G}'(-\lambda_k) = -\sum_{j=1}^{n} \boldsymbol{c}_i \boldsymbol{b}_i^T \frac{t_f(\lambda_k+\rho_j)e^{(\lambda_k+\rho_j)t_f} + e^{(\lambda_k+\rho_j)t_f} + 1}{(\lambda_k+\rho_j)^2}, \tag{3.12}$$

$$\mathbf{G}_r(-\lambda_k) = \sum_{j=1}^{n} \boldsymbol{\ell}_j \boldsymbol{r}_j^T \frac{e^{(\lambda_k+\lambda_j)t_f} - 1}{\lambda_k + \lambda_j}, \quad \text{and} \tag{3.13}$$

$$\mathbf{G}_r'(-\lambda_k) = -\sum_{j=1}^{n} \boldsymbol{\ell}_j \boldsymbol{r}_j^T \frac{t_f(\lambda_k+\lambda_j)e^{(\lambda_k+\lambda_i)t_f} + e^{(\lambda_k+\lambda_i)t_f} + 1}{(\lambda_k+\lambda_j)^2}. \tag{3.14}$$

Proof. Recall the definition of $\mathbf{G}(s) = -e^{-st_f}\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}e^{\mathbf{A}t_f}\mathbf{B} + \mathbf{H}(s)$. Note that we assume that the eigenvalues of $\mathbf{A}$ are simple. Therefore, $e^{\mathbf{A}t_f}$ is also diagonalizable by the eigenvectors of $\mathbf{A}$. Using this fact and the pole-zero residue decomposition of $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \sum_{j=1}^{n} \frac{\boldsymbol{c}_j \boldsymbol{b}_j^T}{s-\rho_j}$, we obtain

$$\begin{aligned}
\mathbf{G}(s) &= -e^{-st_f}\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}e^{\mathbf{A}t_f}\mathbf{B} + \mathbf{H}(s) \\
&= -e^{-st_f}\sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{\rho_j t_f}}{s-\rho_j} + \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{1}{s-\rho_j} = \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{-e^{(-s+\rho_j)t_f}}{s-\rho_j} + \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{1}{s-\rho_j} \\
&= \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{(-s+\rho_j)t_f} - 1}{-s+\rho_j}.
\end{aligned} \tag{3.15}$$

Thus,

$$\mathbf{G}(-\lambda_k) = \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{(\lambda_k+\rho_j)t_f} - 1}{\lambda_k + \rho_j},$$

which proves (**??**). To prove (**??**), we first differentiate (**??**) with respect to s to obtain

$$\mathbf{G}'(s) = \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{t_f(s-\rho_j)e^{(-s+\rho_j)t_f} + e^{(-s+\rho_j)t_f} - 1}{(s-\rho_j)^2}$$

Plugging in $s = -\lambda_k$ in this last expression yields the desired result (**??**). The proofs of (**??**) and (**??**) follow analogously. □

Proof of Theorem **??**. Let $\mathcal{J}$ denote the $\mathcal{H}_2(t_f)$ error norm square, i.e.,

$$\mathcal{J} = \|\mathbf{h} - \mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)} = \|\mathbf{h}\|^2_{\mathcal{H}_2(t_f)} - 2\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)} + \|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)}.$$

The expressions for $\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)}$ and $\|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)}$ in terms of the optimization variables $\{\lambda_k\}$, $\{\boldsymbol{r}_k\}$, and $\{\boldsymbol{\ell}_k\}$, for $k = 1, 2, \ldots, r$, were already derived in Lemma (**??**). To make the gradient computations with respect to the kth parameter more clear, we seperate the kth term from these expressions. For example, we write $\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)}$ in (**??**) as

$$\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)} = \sum_{j=1}^{n} \boldsymbol{\ell}_k^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_k \frac{e^{(\lambda_k+\rho_j)t_f} - 1}{\lambda_k + \rho_j} + \sum_{j=1}^{n} \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f} - 1}{\lambda_i + \rho_j}. \tag{3.16}$$

Following the same procedure for $\|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)}$, we obtain

$$\begin{aligned}
\mathcal{J} = \int_0^{t_f} \mathbf{h}(t)^T \mathbf{h}(t) dt &- 2\left(\sum_{j=1}^{n} \boldsymbol{\ell}_k^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_k \frac{e^{(\lambda_k+\rho_j)t_f} - 1}{\lambda_k + \rho_j} + \sum_{j=1}^{n} \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f} - 1}{\lambda_i + \rho_j} \right) \\
&+ \boldsymbol{\ell}_k^T \boldsymbol{\ell}_k \boldsymbol{r}_k^T \boldsymbol{r}_k \frac{e^{(2\lambda_k)t_f} - 1}{2\lambda_k} + \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_k \boldsymbol{r}_k^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_k)t_f} - 1}{\lambda_i + \lambda_k} \\
&+ \sum_{\substack{j=1\\ j\neq k}}^{r} \boldsymbol{\ell}_k^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_k \frac{e^{(\lambda_k+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j} + \sum_{\substack{j=1\\ j\neq k}}^{r} \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j}.
\end{aligned} \tag{3.17}$$

To compute the gradient of the cost function $\mathcal{J}$ we perturb the cost functional with respect to the residue directions, i.e., $\boldsymbol{\ell}_k \to \boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k$ and $\boldsymbol{r}_k \to \boldsymbol{r}_k + \Delta\boldsymbol{r}_k$:

$$\begin{aligned}
\widetilde{\mathcal{J}}_k = &\int_0^{t_f} \mathbf{h}(t)^T \mathbf{h}(t) dt - 2\Big(\sum_{j=1}^{n} (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)^T \boldsymbol{c}_j \boldsymbol{b}_j^T (\boldsymbol{r}_k + \Delta\boldsymbol{r}_k) \frac{e^{(\lambda_k+\rho_j)t_f} - 1}{\lambda_k + \rho_j} \\
&+ \sum_{j=1}^{n} \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f} - 1}{\lambda_i + \rho_j} \Big) + (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)^T (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)(\boldsymbol{r}_k + \Delta\boldsymbol{r}_k)^T (\boldsymbol{r}_k + \Delta\boldsymbol{r}_k) \frac{e^{(2\lambda_k)t_f} - 1}{2\lambda_k} \\
&+ \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)(\boldsymbol{r}_k + \Delta\boldsymbol{r}_k)^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_k)t_f} - 1}{\lambda_i + \lambda_k} + \sum_{\substack{j=1\\ j\neq k}}^{r} (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T (\boldsymbol{r}_k + \Delta\boldsymbol{r}_k) \frac{e^{(\lambda_k+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j} \\
&+ \sum_{\substack{j=1\\ j\neq k}}^{r} \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j}.
\end{aligned}$$

Then, collecting the terms that are multiplied by $\Delta\boldsymbol{\ell}_k$ and $\Delta\boldsymbol{r}_k$, we obtain

$$\nabla_{\boldsymbol{r}_k} \mathcal{J} = -2\boldsymbol{\ell}_k^T \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{(\lambda_k+\rho_j)t_f} - 1}{\lambda_k + \rho_j} + 2\boldsymbol{\ell}_k^T \sum_{j=1}^{n} \boldsymbol{\ell}_j \boldsymbol{r}_j^T \frac{e^{(\lambda_k+\lambda_j)t_f} - 1}{\lambda_k + \lambda_j}, \quad \text{and}$$

$$\nabla_{\boldsymbol{\ell}_k} \mathcal{J} = -2\left(\sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{(\lambda_k+\rho_j)t_f} - 1}{\lambda_k + \rho_j} \right) \boldsymbol{r}_k + 2\left(\sum_{j=1}^{n} \boldsymbol{\ell}_j \boldsymbol{r}_j^T \frac{e^{(\lambda_k+\lambda_j)t_f} - 1}{\lambda_k + \lambda_j} \right) \boldsymbol{r}_k.$$

Setting $\nabla_{\boldsymbol{r}_k}\mathcal{J} = 0$ and $\nabla_{\boldsymbol{\ell}_k}\mathcal{J} = 0$, and using Lemma **??**, mainly (**??**) and (**??**), yield

$$\boldsymbol{\ell}_k^T\mathbf{G}(-\lambda_k) = \boldsymbol{\ell}_k^T\mathbf{G}_r(-\lambda_k) \quad \text{and} \quad \mathbf{G}(-\lambda_k)\boldsymbol{r}_k = \mathbf{G}_r(-\lambda_k)\boldsymbol{r}_k,$$

which proves (**??**) and (**??**). To prove (**??**), we differentiate $\mathcal{J}$ in (**??**) with respect to the k-th pole λ_k: After some tedious manipulations, we obtain

$$\frac{\partial \mathcal{J}}{\partial \lambda_k} = -2\boldsymbol{\ell}_k^T\left(\sum_{j=1}^{n}\boldsymbol{c}_j\boldsymbol{b}_j^T\frac{t_f(\lambda_k+\rho_j)e^{(\rho_j+\lambda_k)t_f} - e^{(\rho_j+\lambda_k)t_f}+1}{(\lambda_k+\rho_j)^2}\right)\boldsymbol{r}_k \tag{3.18}$$

$$+\,2\boldsymbol{\ell}_k^T\left(\sum_{i=1}^{r}\boldsymbol{\ell}_i\boldsymbol{r}_i^T\frac{t_f(\lambda_i+\lambda_j)e^{(\lambda_i+\lambda_k)t_f} - e^{(\lambda_i+\lambda_k)t_f}+1}{(\lambda_i+\lambda_k)^2}\right)\boldsymbol{r}_k. \tag{3.19}$$

Lemma (**??**), specifically (**??**) and (**??**), show that the expressions in the parentheses in (**??**) and (**??**) are, respectively, $-\mathbf{G}'(-\lambda_k)$ and $-\mathbf{G}_r'(-\lambda_k)$. Thus, setting $\dfrac{\partial \mathcal{J}}{\partial \lambda_k} = 0$ yield

$$\boldsymbol{\ell}_k^T\mathbf{G}'(-\lambda_k)\boldsymbol{r}_k = \boldsymbol{\ell}_k^T\mathbf{G}_r'(-\lambda_k)\boldsymbol{r}_k,$$

which completes the proof.

□

3.3. Implication of the interpolatory $\mathcal{H}_2(t_f)$ optimality conditions

Theorem **??** extends the interpolatory infinite-horizon $\mathcal{H}_2$ optimality conditions (**??**) to the finite-horizon case. Note that in the case of asymptotically stable dynamical systems, if we let $t_f \to \infty$, we recover the infinite-horizon conditions (**??**).

The major difference from the regular $\mathcal{H}_2$ problem is that optimality no longer requires that the reduced model $\mathbf{H}_r(s)$ tangentially interpolate the full model $\mathbf{H}(s)$. Instead, the auxiliary reduced-order function $\mathbf{G}_r(s)$ should be a tangential Hermite interpolant to the auxiliary full-order function $\mathbf{G}(s)$. However, the optimal interpolation points and the optimal tangential directions still result from the pole-residue representation of the reduced-order transfer function $\mathbf{H}_r(s)$. This situation is similar to the interpolatory optimality conditions for the frequency-weighted $\mathcal{H}_2$-optimal model reduction problem in which one tries to minimize a weighted $\mathcal{H}_2$ norm in the frequency domain, i.e., find $\mathbf{H}_r(s)$ that minimizes $\|\mathbf{W}(\mathbf{H}-\mathbf{H}_r)\|_{\mathcal{H}_2}$ where $\mathbf{W}(s)$ represents a weighting function in the frequency domain. As [**?**] showed, the optimality in the frequency-weighted $\mathcal{H}_2$-norm requires that a function of $\mathbf{H}_r(s)$ tangentially interpolate a function of $\mathbf{H}(s)$. Despite this conceptual similarity, the resulting interpolation conditions are drastically different from what we obtained here as one would expect due to the different measures. For details, we refer the reader to [**?**].

As we pointed out in Section **??**, in addition to the interpolatory framework, one can represent the $\mathcal{H}_2$ optimality conditions in terms of Sylvester equations, leading to a projection framework for the reduced model. This means that given the full-model $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$, one constructs two bases $\mathbf{V},\mathbf{W} \in \mathbb{R}^{n\times r}$ with $\mathbf{V}^T\mathbf{W} = \mathbf{I}_r$ such that the reduced-order quantities are obtained via projection, i.e.,

$$\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \mathbf{B}_r = \mathbf{W}^T\mathbf{B}, \quad \text{and} \quad \mathbf{C}_r = \mathbf{C}\mathbf{V}. \tag{3.20}$$

In the infinite-horizon case, [**?**] showed that the optimal $\mathcal{H}_2$ reduced model is indeed guaranteed to be obtained via projection. Recently, Goyal and Redmann [**?**] have established the Sylvester-equation based optimality conditions for the time-limited $\mathcal{H}_2$ model reduction problem; i.e., they extended the Wilson framework to the time-limited (finite-horizon) $\mathcal{H}_2$ problem. Furthermore, they have developed a projection-based IRKA-type numerical algorithm to construct the reduced models. However, as the authors point out in [**?**], even though their algorithm yields high-fidelity reduced models in terms of the $\mathcal{H}_2(t_f)$ measure, the resulting reduced model satisfies the optimality conditions only approximately. This is not surprising in light of the optimality conditions we derived here. Since the optimality requires that $\mathbf{G}_r(s)$ should interpolate $\mathbf{G}(s)$ (as opposed to $\mathbf{H}_r(s)$ interpolating $\mathbf{H}(s)$), unlike in the infinite-horizon case, the reduced model in the finite-horizon case is *not* necessarily given by a projection as in (**??**). Therefore, a projection-based approach would satisfy the optimality conditions only approximately. This was also the case in [?] where even though a projection-based IRKA-like algorithm produced high-fidelity reduced models in the weighted norm, it satisfied the optimality conditions approximately.

The advantage of the interpolation framework and the parametrization (**??**) we consider here is that we do not require the reduced-model to be obtained via projection. By treating the poles and residues in (**??**) as the parameters and directly working with them, we can obtain a reduced model to satisfy the optimality conditions exactly. Even though the main focus of this paper is the theoretical interpolatory framework and a robust numerical algorithm will be fully considered in a future work, in the next section we will discuss a basic numerical framework one can develop using the interpolatory conditions.

Remark 3.3. The finite-horizon approximation problem for *discrete-time dynamical systems* has been considered in [?]. The derivation in [?], however, allows the reduced-model quantities to vary at every time- step, thus using a time-varying reduced model as opposed to the time-invariant formulation considered here and in [?]. Allowing time-varying quantities drastically simplifies the gradient computations, leading to a recurrence relations for optimality. Therefore, the model reduction problem for finite-horizon $\mathcal{H}_2$ approximation for *time-invariant* discrete-time dynamical systems is still an open question. $\diamondsuit$

4. Numerical considerations

In this section, we briefly discuss a numerical framework to construct a reduced model that satisfies the optimality conditions (**??**)-(**??**). To make the presentation and discussion concise, we will focus on the single-input/single-output (SISO) version only. The complete numerical framework for the general case with further details on the underlying optimization schemes will be discussed in a separate work.

4.1. A descent-type algorithm for the single-input/single-output case

Let $\mathbf{H}(s)$ and $\mathbf{H}_r(s)$ be SISO full- and reduced-model transfer functions, respectively, i.e.,

$$\mathbf{H}(s) = \boldsymbol{c}^T(s\mathbf{I}-\mathbf{A})^{-1}\boldsymbol{b} = \sum_{i=1}^{n} \frac{\psi_i}{s-\rho_i} \qquad \text{and} \qquad \mathbf{H}_r(s) = \boldsymbol{c}_r^T(s\mathbf{I}_r-\mathbf{A}_r)^{-1}\boldsymbol{b}_r = \sum_{i=1}^{r} \frac{\phi_i}{s-\lambda_i}, \tag{4.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$, $\mathbf{A}_r \in \mathbb{R}^{r\times r}$, and $\boldsymbol{b}_r, \boldsymbol{c}_r \in \mathbb{R}^r$. Note that the residues ψ_i and ϕ_i are scalar valued. The following result, which is an immediate consequence of Theorem **??**, summarizes the optimality conditions for SISO systems.

Corollary 4.1. *Given the SISO transfer functions $\mathbf{H}(s)$ and $\mathbf{H}_r(s)$ as defined in (**??**), define*

$$\mathbf{G}(s) = -e^{-st_f}\boldsymbol{c}^T(s\mathbf{I}-\mathbf{A})^{-1}e^{\mathbf{A}t_f}\boldsymbol{b} + \mathbf{H}(s) \quad \text{and} \quad \mathbf{G}_r(s) = -e^{-st_f}\boldsymbol{c}_r^T(s\mathbf{I}_r-\mathbf{A}_r)^{-1}e^{\mathbf{A}_r t_f}\boldsymbol{b}_r + \mathbf{H}_r(s).$$

If $\mathbf{H}_r$ is the best r^{th} order approximation of $\mathbf{H}$ with respect to the $\mathcal{H}_2(t_f)$ norm, then

$$\mathbf{G}(-\lambda_k) = \mathbf{G}_r(-\lambda_k), \qquad \text{and} \qquad \mathbf{G}'(-\lambda_k) = \mathbf{G}_r'(-\lambda_k) \tag{4.2}$$

*where λ_k for $k = 1, 2, ..., r$ are the poles of the reduced system $\mathbf{H}_r(s)$ as given in (**??**).*

As stated before, the $\mathcal{H}_2(t_f)$ minimization problem is a non-convex optimization problem and Corollary **??** gives the necessary conditions for optimality when both poles and residues are treated as variables. However, if the poles are fixed, we can establish the necessary and sufficient optimality conditions for the residues and find the global minimizer, the optimal residues, by solving a linear system.

Corollary 4.2. *Let $\mathbf{H}(s)$ and $\mathbf{H}_r(s)$ be as given in (**??**), and $\mathbf{G}(s)$ and $\mathbf{G}_r(s)$ as in (**??**). Assume the reduced poles $\{\lambda_i\}_{i=1}^r$ are fixed. Then, $\mathbf{H}_r(s)$ is the best r^{th} order approximation of $\mathbf{H}(s)$ with respect to the $\mathcal{H}_2(t_f)$ norm if and only if*

$$\mathbf{M}\boldsymbol{\phi} = \mathbf{z}, \quad \textit{or equivalently}, \quad \mathbf{G}(-\lambda_k) = \mathbf{G}_r(-\lambda_k), \quad \textit{for} \quad k = 1, 2, \ldots, r,$$

where $\boldsymbol{\phi} = [\phi_1 \ \phi_2 \ \cdots \ \phi_r]^T \in \mathbb{C}^r$ *is the vector of residues;* $\mathbf{z} \in \mathbb{C}^r$ *is the vector with entries*

$$\mathbf{z}_j = e^{\lambda_j t_f} \boldsymbol{c}^T (-\lambda_j \mathbf{I} - \mathbf{A})^{-1} e^{\mathbf{A} t_f} \boldsymbol{b} - \mathbf{H}(-\lambda_j), \quad \textit{for} \quad i = 1, 2, \ldots, r;$$

and $\mathbf{M} \in \mathbb{C}^{r \times r}$ *is the matrix with entries*

$$\mathbf{M}_{i,j} = \frac{e^{(\lambda_i + \lambda_j) t_f} - 1}{\lambda_i + \lambda_j}, \quad \textit{for} \quad i, j = 1, 2, \ldots, r.$$

Proof. Using the SISO counterparts for (**??**) and (**??**) and applying some algebraic manipulation, the cost functional $\mathcal{J}$ can be written as

$$\mathcal{J} = \|\mathbf{h}\|^2_{\mathcal{H}_2(t_f)} - 2\boldsymbol{\phi}^T \mathbf{w} + \boldsymbol{\phi}^T \mathbf{M} \boldsymbol{\phi},$$

where $\mathbf{w} \in \mathbb{C}^{r \times 1}$ has the entries $\mathbf{w}_i = \sum_{k=1}^{n} \psi_k \dfrac{e^{(\rho_k + \lambda_i) t_f} - 1}{\lambda_i + \rho_k}$ for $i = 1, 2, \ldots, r$. Note that $\mathbf{M}$ is positive definite, $\phi^T \mathbf{M} \phi = \|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)} > 0$, and the cost function is quadratic in $\boldsymbol{\phi}$. Thus the (global) minimizer is obtained by solving $\mathbf{M}\boldsymbol{\phi} = \mathbf{z}$, which corresponds to rewriting $\mathbf{G}(-\lambda_k) = \mathbf{G}_r(-\lambda_k)$ for $k = 1, 2, \ldots, r$ in a compact way. □

The result is analogous to the regular infinite-horizon $\mathcal{H}_2$ problem where the Lagrange optimality becomes necessary and sufficient once the poles are fixed [**? ?**]. What is important here is that once the reduced poles are fixed, the best residues can be computed directly by solving an $r \times r$ linear system $\mathbf{M}\boldsymbol{\phi} = \mathbf{z}$. This is the property that we will exploit in the numerical scheme next.

4.1.1. FHIRKA: A numerical algorithm for $\mathcal{H}_2(t_f)$ model reduction

Here we describe a numerical algorithm which produces a reduced model that satisfies the necessary $\mathcal{H}_2(t_f)$ optimality conditions upon convergence. Let $\boldsymbol{\lambda} \in \mathbf{C}^r$ denote the vector of reduced poles. Thus, the error $\mathcal{J}$ is a function of $\boldsymbol{\lambda}$ and $\boldsymbol{\phi}$. Since we explicitly know the gradients of the cost function with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{\phi}$ (and indeed we can compute the Hessians as well), one can (locally) minimize $\mathcal{J}$ using well established optimization tools. However, as Corollary **??** shows, we can easily compute the globally optimal $\boldsymbol{\phi}$ for fixed $\boldsymbol{\lambda}$. Therefore, we will treat the reduced poles $\boldsymbol{\lambda}$ as the optimization parameter, and once $\boldsymbol{\lambda}$ are updated at the *kth* step of an optimization algorithm, we find/update the corresponding optimal residues $\boldsymbol{\phi}$ based on Corollary **??**, and then repeat the process. Similar strategies have been successfully employed in the regular $\mathcal{H}_2$ optimal approximation problem as well; see [**? ?**]. In summary, we use a quasi-Newton type optimization as $\boldsymbol{\lambda}$ being the parameter and in each optimization step, we update the residues, $\boldsymbol{\phi}$, by solving the $r \times r$ linear system $\mathbf{M}\boldsymbol{\phi} = \mathbf{z}$ as in Corollary **??**. Since we are enforcing interpolation at every step of the algorithm, yet tackling the model reduction problem over a finite horizon, we name this algorithm Finite Horizon IRKA, denoted by FHIRKA. Unlike regular IRKA, FHIRKA is a descent algorithm, thus indeed mimics [**?**] more closely. Upon convergence, the locally optimal reduced model satisfies the first-order necessary conditions of Corollary **??**.

4.2. Numerical Results

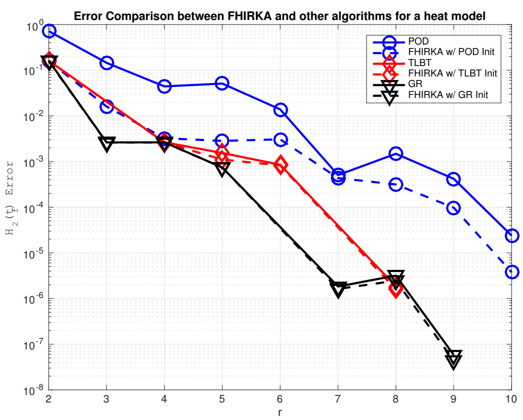
In this section we compare the proposed algorithm FHIRKA with Proper Orthogonal Decomposition (POD), Time-Limited Balanced Truncation (TLBT), and the recently introduced $\mathcal{H}_2(t_f)$-based algorithm by Goyal and Redmann (GR) [**?**], as briefly discussed in Section **??**.

We use three models: a heat model of order $n = 197$ [**?**], a model of the International Space Station 1R Module (ISS 1R) of order $n = 270$ [**?**], and a toy unstable model of order $n = 402$. The ISS 1R model has 3-inputs and 3-outputs. We focus on the SISO subsystem from the first-input to the first-output. We have created the unstable system such that it has 400 stable poles and 2 unstable poles (positive real part).

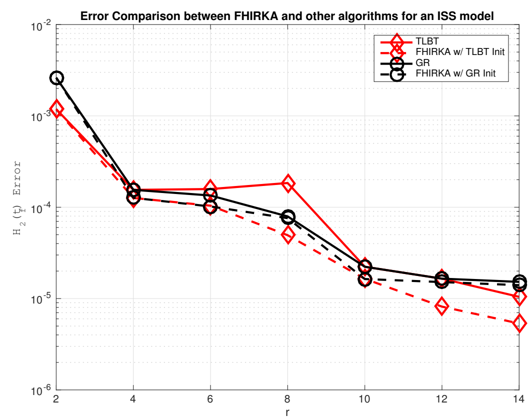
For all three models, we choose $t_f = 1$, first reduce the original model using POD, GR or TLBT, and then use the resulting reduced model to initialize FHIRKA. Thus, we are trying to investigate how these different initializations affect the final reduced model via FHIRKA and how much improvement one might expect. The results are shown in Figures **??**–**??**, where we show the $\mathcal{H}_2(t_f)$ approximation error for different values of r, the order of the reduced model. All three initializations are used for the heat model (Figure

??) where the order is reduced from $r = 2$ to $r = 10$ with increments of one. For some r values, certain initializations are excluded (e.g., the GR initialization for $r = 6$) since the algorithm either did not converge or produced poor approximations. However, this happened rarely. For the ISS model (Figure ??), we use TLBT and GR initializations since POD produced very poor approximations and thus is excluded. In this case, we reduce the order from $r = 2$ to $r = 14$ with increments of 2. For the unstable model (Figure ??), we use POD and GR initializations; for this model our implementation of TLBT produced poor results and is avoided. In this case, we reduce the order from $r = 2$ to $r = 12$ with increments of 2.

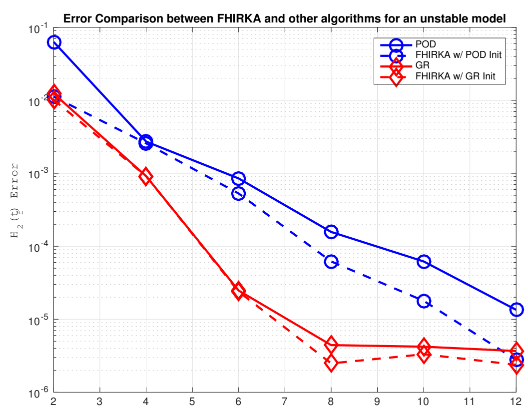
The first observation is that, since FHIRKA is a descent-method and drives the initialization to a local minimizer, it improves the accuracy of the reduced model for all three initializations as expected. The improvements could be dramatic. For example, FHIRKA is able to outperform POD as much as by an order of magnitude, see, for example, Figure ??, the $r = 4$ and $r = 5$ cases. While FHIRKA improves TLBT and GR initialization as well, the improvements for the heat model are not as significant. However, for the ISS model, FHIRKA is able to improve the TLBT performance as much as 50%; see, e.g., Figure ??, the $r = 8$ case. The best improvement of the GR initialization has occurred for the unstable model. For example, for $r = 8$, for the unstable model, FHIRKA improved the reduced model by more than 40%. Gains were significantly better for POD, especially for larger r values.



(a) FHIRKA and other algorithms for a heat model



(b) FHIRKA and other algorithms for the ISS model



(c) FHIRKA and other algorithms for the unstable model

Figure 1: Numerical results for three models

Overall, as expected, FHIRKA yields a better approximation compared to the other algorithms for each model. We find that GR provided the best initialization for FHIRKA. This is not surprising since GR produces a reduced-model that approximately satisfies the $\mathcal{H}_2(t_f)$ optimality conditions.

As we stated above, the numerical issues will be fully investigated in a future work where we will develop a robust interpolatory $\mathcal{H}_2(t_f)$-descent algorithm for MIMO systems. We will not only study better initialization techniques in terms of speed and accuracy, but also make the algorithm numerically more efficient by using approximation techniques for the matrix exponential $e^{\mathbf{A}t_f}$ appearing in the $\mathcal{H}_2(t_f)$ setting. We will also investigate the MIMO version of Corollary **??**. In the MIMO case, even for fixed poles, one cannot simply find the globally optimal residue directions by solving a linear system, since the problem is no longer quadratic in these variables. In the regular $\mathcal{H}_2$ case, finding the optimal residue directions for given poles required solving a nonlinear least-squares problem [**?**]. We anticipate a similar formulation here and will investigate the corresponding numerical implications.

5. Conclusions and Future Work

We established interpolatory $\mathcal{H}_2(t_f)$-optimality conditions for model reduction of MIMO dynamical systems over a finite horizon. Even though the optimal interpolation points and tangential directions are still determined by the reduced model, we showed that unlike the regular $\mathcal{H}_2$ problem, a modified reduced-transfer function should interpolate a modified full-order transfer function. For the special case of SISO models, we have studied a numerical algorithm and illustrated that it performs effectively.

As in the regular $\mathcal{H}_2$-problem, establishing equivalency between the Sylvester-equation based $\mathcal{H}_2(t_f)$-optimality conditions of [**?**] and the interpolation-based conditions we developed here will be an interesting direction to pursue. Furthermore, extensions to bilinear and quadratic-bilinear problems will be crucial.