$\mathcal{H}_2(t_f)$ Optimality Conditions for a Finite-time Horizon

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Abstract

In this paper we establish the interpolatory model reduction framework for optimal approximation of MIMO dynamical systems with respect to the $\mathcal{H}_2$ norm over a finite-time horizon under, denoted as the $\mathcal{H}_2(t_f)$ norm. Using the underlying inner product space, we derive the interpolatory first-order necessary optimality conditions for approximation in the $\mathcal{H}_2(t_f)$ norm. Then, we develop an algorithm, which yields a locally optimal reduced model that satisfies the established interpolation-based optimality conditions. We test the algorithm on various numerical examples to illustrate its performance.

Keywords: time-limited model reduction, interpolation, unstable system, $\mathcal{H}_2$-optimality, linear systems

1. Introduction

Simulation, design, and control of dynamical systems play an important role in numerous scientific and industrial tasks such as signal propagation in the nervous system, heat dissipation, prediction of major weather events etc. The need for detailed models due to the increasing demand for greater resolution leads to large-scale dynamical systems, posing tremendous computational difficulties when applied in numerical simulations. In order to overcome these challenges, we perform model reduction where we replace the large-scale dynamics with high-fidelity reduced representations.

Consider the linear dynamical system:

$$\begin{array}{rcl} \dot{\mathbf{x}}(t) & = & \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) & = & \mathbf{C}\mathbf{x}(t) \end{array} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{y}(t) = \int_0^t \mathbf{h}(t-\tau)\mathbf{u}(\tau)d\tau, \tag{1.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$ are constant matrices; the variable $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the internal variables, $\mathbf{u}(t) \in \mathbb{R}^m$ denotes the control inputs, and $\mathbf{y}(t) \in \mathbb{R}^p$ denotes the outputs; and $\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}$ is the impulse response of the full model. The length of the internal variable $\mathbf{x}(t)$, i.e., n, is called the order of the full model that we would like to reduce. Model reduction achieves this by replacing the original model with a lower dimensional one:

$$\begin{array}{rcl} \dot{\mathbf{x}}_r(t) & = & \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) & = & \mathbf{C}_r\mathbf{x}_r(t) \end{array} \quad \text{with} \quad \mathbf{x}_r(0) = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{y}_r(t) = \int_0^t \mathbf{h}_r(t-\tau)\mathbf{u}(\tau)d\tau, \tag{1.2}$$

where (1.1), $\mathbf{h}_r(t) = \mathbf{C}_r e^{\mathbf{A}_r t}\mathbf{B}_r$ is the impulse response of the reduced model, and $\mathbf{A}_r \in \mathbb{R}^{r\times r}$, $\mathbf{B}_r \in \mathbb{R}^{r\times m}$, and $\mathbf{C}_r \in \mathbb{R}^{p\times r}$ with $r \ll n$. The goal is that the output of the reduced model, $\mathbf{y}_r(t)$, approximates the true output, $\mathbf{y}(t)$, of the original system accurately in an appropriate norm.

For the linear dynamical systems we consider here, a plethora of methods exists for producing high-fidelity/ optimal reduced models, such as balanced truncation [**?** **?**] and its variants, optimal Hankel norm approximation [**?**], and the Iterative Rational Krylov Algorithm (IRKA) [**?**] and its variants. See [**?** **?** **?**] for further references. These methods usually focus on high-quality reduced model over an infinite time horizon. However, in various settings, we might either have access to simulations over a finite horizon or can only simulate the system under investigation for a finite horizon such as in the case unstable dynamical

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systems. Therefore, in those situations we are interested in the behavior of the dynamical system over a finite time interval $[0, t_f]$ where $t_f < \infty$, and we need the reduced model to be accurate only in the interval of interest.

Time-limited balanced truncation[? ? ? ?] and Proper Orthogonal Decomposition(POD)[?] are two common frameworks to create reduced models on a finite horizon. For time-limited balanced truncation, [?] establishes an upper bound for the $\mathcal{H}_\infty$ error between the full and reduced models, Redmann and Kürschner provide an $\mathcal{H}_2$ error bound in [?]. In this paper, we explore $\mathcal{H}_2$ optimal for model reduction over a finite time horizon. We establish $\mathcal{H}_2(t_f)$ optimality conditions over a finite time horizon and introduce an algorithm which yields better approximations of the large-scale dynamical systems compared to other model reduction methods.

The optimality requires a parametrization of the reduced model. We will work the time-domain representation of the dynamical system to derive the conditions for a finite time. Specifically, we represent the impulse response of the reduced dynamical system using the modal decomposition, i.e.,

$$\mathbf{h}_r(t) = \mathbf{C}_r e^{\mathbf{A}_r t}\mathbf{B}_r = \sum_{i=1}^{r} e^{\lambda_i t}\boldsymbol{\ell}_i \boldsymbol{r}_i^T. \tag{1.3}$$

where λ_i's are the eigenvalues of $\mathbf{A}_r$, and $\boldsymbol{\ell}_i \in \mathbb{R}^{p\times 1}, \boldsymbol{r}_i \in \mathbb{R}^{m\times 1}$. In other words, we are writing the impulse response as a sum of r rank-1 $p \times m$ matrices. For simplicity we assumed that λ_i's, the reduced order poles, are simple. The representation (3.1) is nothing but a state-space transformation on $\mathbf{h}_r(t) = \mathbf{C}_r e^{\mathbf{A}_r t}\mathbf{B}_r$ using the eigenvectors of $\mathbf{A}_r$. Using the time-domain representation of the impulse response of the reduced model, we derive interpolation-based optimality conditions in the $\mathcal{H}_2(t_f)$ norm and implement a model reduction algorithm which satisfies these optimality conditions. Establishing $\mathcal{H}_2(t_f)$ optimality conditions for model reduction over a finite horizon also enables us to reduce unstable systems optimally under the $\mathcal{H}_2(t_f)$ measure. Therefore, the advantages of time-limited model reduction under the $\mathcal{H}_2(t_f)$ norm are two fold: first, we can obtain a better approximation in the time interval of interest, and, second, we can obtain a locally optimal reduced order model even for unstable systems.

The rest of the paper is organizes as follows: In Section 2 we briefly review optimal $\mathcal{H}_2$ model reduction in the infinite horizon. The main results, including the new optimality conditions for finite horizon, are establised in Section 3 followed by numerical examples in Section 4. The papers ends with conclusions and future work in Section 5.

2. $\mathcal{H}_2$-Optimal Model Reduction: The Infinite Horizon Case

Interpolatory model reduction is a very effective approach used to approximate large-scale systems with smaller ones in a locally optimal manner under the $\mathcal{H}_2$ norm. Model reduction with respect to the $\mathcal{H}_2$ norm has been studied extensively; see [? ? ? ? ? ? ? ? ? ?] and the references therein. Before describing interpolation based reduced order modeling, we discuss the $\mathcal{H}_2$ error measure and its definitions in the time and frequency domain.

2.1. $\mathcal{H}_2$ Error Measure

When approximating a large scale dynamical system by a reduced order model, we need to compute the approximation error. The error analysis for linear dynamical systems may be conducted either in the frequency domain or in the time domain. Therefore, we define the $\mathcal{H}_2$ norm in each domain.

Definition 2.1. Define the $\mathcal{H}_2$ inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$ in the time domain as

$$\langle \mathbf{h}, \mathbf{g} \rangle_{\mathcal{H}_2} = \int_0^\infty tr((\mathbf{h}(t))^T \mathbf{g}(t))dt,$$

and, as a result,

$$\|\mathbf{h}\|_{\mathcal{H}_2}^2 = \int_0^\infty \|\mathbf{h}\|_F^2 \, dt.$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

The $\mathcal{H}_2$ norm of a dynamical system is the norm of the impulse response of the dynamical system e.g. the $\mathcal{H}_2$ norm of (1.1) is

$$\|\mathbf{h}\|_{\mathcal{H}_2}^2 = \int_0^\infty \|\mathbf{h}\|_F^2 \, dt$$

where $\mathbf{h} = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}$.

Prior to defining the $\mathcal{H}_2$ norm in the frequency domain, we need to obtain a frequency representation of the dynamical systems (1.1) and (1.2). Computing the Fourier transforms of $\mathbf{y}(t)$, $\mathbf{y}_r(t)$ and $\mathbf{u}(t)$, we have:

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s) \quad \text{where} \quad \mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$
$$\mathbf{Y}_r(s) = \mathbf{H}_r(s)\mathbf{U}(s) \quad \text{where} \quad \mathbf{H}_r(s) = \mathbf{C}_r(s\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r$$

$\mathbf{H}(s)$ and $\mathbf{H}_r(s)$ are the transfer functions associated with the full and reduced model respectively.

Definition 2.2. Define the $\mathcal{H}_2$ inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$ in the frequency domain as

$$\langle \mathbf{G}, \mathbf{H} \rangle_{\mathcal{H}_2} := \frac{1}{2\pi} \int_{-\infty}^{\infty} tr(\mathbf{G}(i\omega)^* \mathbf{H}(i\omega)) d\omega.$$

Now that we know what the transfer function looks like, and we have established the frequency domain $\mathcal{H}_2$ inner product, we define the $\mathcal{H}_2$ norm in the frequency domain:

$$\|\mathbf{H}\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathbf{H}(i\omega)\|_F^2 \, d\omega \right)^{1/2}.$$

We aim to approximate the output of the full model with the output of the reduced model. The following inequality shows that the transfer function error is an upper bound for the output error [?].

$$\|\mathbf{y} - \mathbf{y}_r\|_{L_\infty} \leq \|\mathbf{H} - \mathbf{H}_r\|_{\mathcal{H}_2} \|\mathbf{u}\|_{L_2} \tag{2.1}$$

Therefore, approximating the transfer function allows us to obtain a reduced order model whose output is very close to the original output. Hence, finding a locally $\mathcal{H}_2$ optimal reduced order transfer function that approximates the original transfer function is sufficient to guarantee that the outputs of the reduced system approximate the true outputs.

2.2. Interpolatory Model Reduction

When we reduce the order of a dynamical system we are essentially approximating the full order system with a reduced order model. Interpolation is a very effective and powerful approximation method. Our goal is to construct a locally optimal reduced model with respect to some norm, in our case, specifically, the $\mathcal{H}_2$ norm. In other words, if we have a full-order dynamical system $\mathbf{H}(s)$, we want to construct a reduced-order model $\mathbf{H}_r(s)$ such that

$$\|\mathbf{H} - \mathbf{H}_r\|_{\mathcal{H}_2} \leq \left\| \mathbf{H} - \hat{\mathbf{H}}_r \right\|_{\mathcal{H}_2},$$

where $\hat{\mathbf{H}}_r$ is any dynamical system of dimension r. The following theorem tells us that an $\mathcal{H}_2$ optimal reduced order model interpolates the original model at the poles of the reduced model [? ?].

Theorem 2.1 (Gugercin, Antoulas and Beattie, '08)**.** *Let $\mathbf{H}_r(s)$ be the best r^{th} order rational approximation of a stable linear model $\mathbf{H}$ with respect to the $\mathcal{H}_2$ norm. Then*

$$\boldsymbol{\ell}_k^T \mathbf{H}(-\lambda_k) = \boldsymbol{\ell}_k^T \mathbf{H}_r(-\lambda_k)$$
$$\mathbf{H}(-\lambda_k)\boldsymbol{r}_k = \mathbf{H}_r(-\lambda_k)\boldsymbol{r}_k$$
$$\boldsymbol{\ell}_k^T \mathbf{H}'(-\lambda_k)\boldsymbol{r}_k = \boldsymbol{\ell}_k^T \mathbf{H}_r'(-\lambda_k)\boldsymbol{r}_k$$

for $k = 1, 2, ..., r$ where λ_k denotes the poles of the reduced system and $\boldsymbol{\ell}_k, \boldsymbol{r}_k$ are the tangential directions.

The Iterative Rational Krylov Algorithm (IRKA) produces a reduced model that satisfies the first-order optimality conditions in the $\mathcal{H}_2$ norm. [**?**].

3. $\mathcal{H}_2(t_f)$ Optimal Model Reduction on a Finite Horizon

If we are interested only in the behavior of a system in a finite time interval, we can obtain better reduced models by approximating the original system over the interval of interest, instead of reducing the system over an infinite time interval. Also, if we are dealing with an unstable system i.e. some of the poles lie to the right of the imaginary axis, we run into difficulties because the solution might blow up as time gets larger and larger. For example, consider the impulse response of (1.1): $\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}$. If the eigenvalues of $\mathbf{A}$ which correspond to the poles of the full model have a positive real part, $\|\mathbf{h}\|_F$ approaches ∞. Thus, if our goal is to optimally reduce an unstable system, we need to consider a different error measure, i.e. an error measure that does not become infinitely large in spite of poles lying to right of the imaginary axis.

Next, we define a norm for a finite time horizon.

3.1. Error Measures on a Finite-time Horizon

We define the $\mathcal{H}_2(t_f)$ inner product and norm for a finite horizon as follows:

Definition 3.1. Define the finite-time $\mathcal{H}_2(t_f)$ inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_2(t_f)}$ as

$$\langle \mathbf{h}, \mathbf{g} \rangle_{\mathcal{H}_2(t_f)} = \int_0^{t_f} tr((\mathbf{h}(t))^T \mathbf{g}(t))dt,$$

and, as a result,

$$\|\mathbf{h}\|^2_{\mathcal{H}_2(t_f)} = \int_0^{t_f} \|\mathbf{h}(t)\|_F^2 \, dt.$$

Similar to the $\mathcal{H}_2$ norm, the $\mathcal{H}_2(t_f)$ norm can be computed as follows:

$$\|\mathbf{H}\|_{\mathcal{H}_2(t_f)} = \sqrt{\mathrm{tr}(\mathbf{C}\mathbf{P}(t_f)\mathbf{C}^T)}$$

where

$$\mathbf{P}(t_f) := \int_0^{t_f} e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t} dt \qquad\qquad \mathbf{Q}(t_f) := \int_0^{t_f} e^{\mathbf{A}^T t}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}t} dt$$

are the finite reachability and observability Gramians, respectively. We can show these finite Gramians are the solutions to the following Lyapunov equations:

$$\begin{aligned} \mathbf{A}^T\mathbf{Q}(t_f) + \mathbf{Q}(t_f)\mathbf{A} + \mathbf{C}^T\mathbf{C} - e^{\mathbf{A}^T t_f}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}t_f} &= 0 \\ \mathbf{A}\mathbf{P}(t_f) + \mathbf{P}(t_f)\mathbf{A}^T + \mathbf{B}\mathbf{B}^T - e^{\mathbf{A}t_f}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t_f} &= 0. \end{aligned}$$

3.2. Finite Horizon Interpolation Based $\mathcal{H}_2(t_f)$ Optimality Conditions

In this section we discuss interpolation based $\mathcal{H}_2(t_f)$ optimality conditions and how we obtained these conditions. In order to derive these $\mathcal{H}_2(t_f)$ optimality conditions we expand the impulse response of the reduced system as follows:

$$\mathbf{h}_r(t) = \sum_{i=1}^{r} e^{\lambda_i t} \boldsymbol{\ell}_i \boldsymbol{r}_i^T. \tag{3.1}$$

Using Definition 3.1 of the $\mathcal{H}_2(t_f)$ norm, we can measure the error between the full and reduced models. In the next lemma, we consider the norm of the error system.

Lemma 3.2. Let $\mathbf{h}(t)$ be the impulse response of the full order model and $\mathbf{h}_r(t)$ be the impulse response of the reduced order model. Then,

$$\|\mathbf{h}-\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)} = \|\mathbf{h}\|^2_{\mathcal{H}_2(t_f)} - 2Re\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)} + \|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)}. \tag{3.2}$$

Proof. The proof for this lemma is obvious if we use the properties of an inner product. □

Recall that our goal is to find an $\mathbf{h}_r$ that minimizes the error. In order to minimize the expression in (3.2), we need to differentiate with respect to $\mathbf{h}_r$. Since $\mathbf{h}_r$ is determined by the poles and the tangential directions of the reduced model, we write (3.2) in terms of these poles and tangential directions. The impulse response of the full system $\mathbf{h}$ is clearly constant with respect to the poles and residues of the reduced system; hence, when we differentiate (3.2) with respect to the poles and tangential direction of the reduced order model, the first term disappears. For this reason, in the following lemmas, we deal only with the last two terms in (3.2).

Lemma 3.3. Let $\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} = \sum_{j=1}^{n} \boldsymbol{c}_j e^{\rho_j t}\boldsymbol{b}_j^T$ and $\mathbf{h}_r(t) = \mathbf{C}_r e^{\mathbf{A}_r t}\mathbf{B}_r = \sum_{i=1}^{r} \boldsymbol{\ell}_i e^{\lambda_i t}\boldsymbol{r}_i^T$ where $\mathbf{C} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$, $\mathbf{C}_r \in \mathbb{R}^{m\times r}$, $\mathbf{B}_r \in \mathbb{R}^{r\times p}$, $\boldsymbol{c}_j, \boldsymbol{\ell}_i \in \mathbb{R}^m$, $\boldsymbol{b}_j, \boldsymbol{r}_i \in \mathbb{R}^p$, ρ_k-s are the poles of the full system, and λ_i-s are the poles of the reduced system. Then,

$$\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)} = \sum_{j=1}^{n} \boldsymbol{\ell}_k^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_k \frac{e^{(\lambda_k+\rho_j)t_f}-1}{\lambda_k+\rho_j} + \sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f}-1}{\lambda_i+\rho_j} \tag{3.3}$$

Proof. The proof follows from the definition of the $\mathcal{H}_2(t_f)$ inner product. Consider

$$\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)} = tr\left(\int_0^{t_f} \mathbf{h}_r^T \mathbf{h}\, dt\right).$$

If we write the impulse response of the reduced system as a sum of r rank-1 matrices and the impulse response of the full system as a sum of n rank-1 matrices, we obtain

$$\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)} = tr\left(\int_0^{t_f} \sum_{i=1}^{r} (\boldsymbol{\ell}_i e^{\lambda_i t}\boldsymbol{r}_i^T)^T \sum_{j=1}^{n} \boldsymbol{c}_j e^{\rho_j t}\boldsymbol{b}_j^T\, dt\right).$$

Using the properties of the trace, we get

$$\begin{aligned}\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)} &= tr\left(\sum_{j=1}^{n}\sum_{i=1}^{r} \boldsymbol{r}_i \boldsymbol{\ell}_i^T \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{(\lambda_i+\rho_j)t_f}-1}{\lambda_i+\rho_j}\right)\\ &= \sum_{j=1}^{n}\sum_{i=1}^{r} \boldsymbol{\ell}_i^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f}-1}{\lambda_i+\rho_j}.\end{aligned}$$

Finally, we rewrite the sum as

$$\langle \mathbf{h}, \mathbf{h}_r\rangle_{\mathcal{H}_2(t_f)} = \sum_{j=1}^{n} \boldsymbol{\ell}_k^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_k \frac{e^{(\lambda_k+\rho_j)t_f}-1}{\lambda_k+\rho_j} + \sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f}-1}{\lambda_i+\rho_j}.$$

□

Lemma 3.4. If $\mathbf{h}_r(t) = \sum_{i=1}^{r} \boldsymbol{\ell}_i e^{\lambda_i t} \boldsymbol{r}_i^T$ is the impulse response of the reduced model, then

$$\|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)} = \boldsymbol{\ell}_k^T \boldsymbol{\ell}_k \boldsymbol{r}_k^T \boldsymbol{r}_k \frac{e^{(2\lambda_k)t_f} - 1}{2\lambda_k} + \sum_{\substack{i=1\\i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_k \boldsymbol{r}_k^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_k)t_f} - 1}{\lambda_i + \lambda_k} + \sum_{\substack{j=1\\j\neq k}}^{r} \boldsymbol{\ell}_k^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_k \frac{e^{(\lambda_k+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j} + \sum_{\substack{j=1\\j\neq k}}^{r} \sum_{\substack{i=1\\i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j}. \tag{3.4}$$

Proof. Using the definition of the $\|\cdot\|_{\mathcal{H}_2}$ and writing the impulse response of the reduced model as a sum of r rank-1 matrices, we get

$$\|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)} = tr\left(\int_0^{t_f} \mathbf{h}_r^T \mathbf{h}_r dt\right) = tr\left(\int_0^{t_f} \sum_{i=1}^{r} (\boldsymbol{\ell}_i e^{\lambda_i t} \boldsymbol{r}_i^T)^T \sum_{j=1}^{r} (\boldsymbol{\ell}_j e^{\lambda_j t} \boldsymbol{r}_j^T) dt\right)$$

The properties of the trace yield

$$\|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)} = tr\left(\sum_{i=1}^{r} \sum_{j=1}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j}\right)$$

We can rewrite the sum as follows

$$\|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)} = \boldsymbol{\ell}_k^T \boldsymbol{\ell}_k \boldsymbol{r}_k^T \boldsymbol{r}_k \frac{e^{(2\lambda_k)t_f} - 1}{2\lambda_k} + \sum_{\substack{i=1\\i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_k \boldsymbol{r}_k^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_k)t_f} - 1}{\lambda_i + \lambda_k} + \sum_{\substack{j=1\\j\neq k}}^{r} \boldsymbol{\ell}_k^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_k \frac{e^{(\lambda_k+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j} + \sum_{\substack{j=1\\j\neq k}}^{r} \sum_{\substack{i=1\\i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j}$$

□

For an infinite horizon, Theorem 2.1 tells us that a locally $\mathcal{H}_2$ optimal reduced transfer function interpolates the transfer function of the full system at the mirror images of the poles of the reduced model. In the finite horizon case, we attain a similar result, even though the full and reduced transfer functions are not interpolants of each other at the mirror images of the poles. The following lemma is essential in proving the interpolation conditions for the finite horizon case.

Lemma 3.5. Let $\mathbf{G}(s) = -e^{-st_f}\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}e^{\mathbf{A}t_f}\mathbf{B} + \mathbf{H}(s)$. Then,

$$\mathbf{G}(-\lambda_j) = \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{(\lambda_k+\rho_j)t_f} - 1}{\lambda_k + \rho_j} \tag{3.5}$$

Proof. For this proof, we use the pole-residue expansion of the transfer functions, and write the other terms in a similar fashion. For instance

$$\begin{aligned}
\mathbf{G}(s) &= -e^{-st_f}\sum_{j=1}^{n}\boldsymbol{c}_j\boldsymbol{b}_j^T\frac{e^{\rho_j t_f}}{s-\rho_j}+\sum_{j=1}^{n}\boldsymbol{c}_j\boldsymbol{b}_j^T\frac{1}{s-\rho_j}\\
&= \sum_{j=1}^{n}\boldsymbol{c}_j\boldsymbol{b}_j^T\frac{-e^{-st_f}e^{\rho_j t_f}}{s-\rho_j}+\sum_{j=1}^{n}\boldsymbol{c}_j\boldsymbol{b}_j^T\frac{1}{s-\rho_j}\\
&= \sum_{j=1}^{n}\boldsymbol{c}_j\boldsymbol{b}_j^T\frac{-e^{(-s+\rho_j)t_f}+1}{s-\rho_j}\\
&= \sum_{j=1}^{n}\boldsymbol{c}_j\boldsymbol{b}_j^T\frac{e^{(-s+\rho_j)t_f}-1}{-s+\rho_j}.
\end{aligned}$$

Thus,

$$\mathbf{G}(-\lambda_k)=\sum_{j=1}^{n}\boldsymbol{c}_j\boldsymbol{b}_j^T\frac{e^{(\lambda_k+\rho_j)t_f}-1}{\lambda_k+\rho_j}$$

□

Writing the relevant terms of the error in (3.2) in terms of the poles and tangential directions of the reduced system enables us to differentiate the error. This representation of the error is essential in proving the following theorem, which establishes the necessary optimality conditions with respect to the $\mathcal{H}_2(t_f)$ norm.

Theorem 3.1. *Define*

$$\mathbf{G}(s)=-e^{-st_f}\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}e^{\mathbf{A}t_f}\mathbf{B}+\mathbf{H}(s).$$

Let $\mathbf{H}_r$ be the best r^{th} order approximation of $\mathbf{H}$ with respect to the $\mathcal{H}_2(t_f)$ norm. Define

$$\mathbf{G}_r(s)=-e^{-st_f}\mathbf{C}(s\mathbf{I}_r-\mathbf{A}_r)^{-1}e^{\mathbf{A}_r t_f}\mathbf{B}_r+\mathbf{H}_r(s).$$

Then

$$\begin{aligned}
\boldsymbol{\ell}_k^T\mathbf{G}(-\lambda_k) &= \boldsymbol{\ell}_k^T\mathbf{G}_r(-\lambda_k)\\
\mathbf{G}(-\lambda_k)\boldsymbol{r}_k &= \mathbf{G}_r(-\lambda_k)\boldsymbol{r}_k\\
\boldsymbol{\ell}_k^T\mathbf{G}'(-\lambda_k)\boldsymbol{r}_k &= \boldsymbol{\ell}_k^T\mathbf{G}_r'(-\lambda_k)\boldsymbol{r}_k
\end{aligned}$$

where λ_k for $k=1,2,...,r$ are the poles of the reduced system $\mathbf{H}_r$, and $\boldsymbol{\ell}_k,\boldsymbol{r}_k$ are the tangential directions.

Proof. Let $\mathbf{J}$ be the square $\mathcal{H}_2(t_f)$ error between the full and reduced models i.e.

$$\mathbf{J}=\|\mathbf{h}-\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)}.$$

From Lemma 3.3, and Lemma 3.4, we infer

$$\mathbf{J} = tr\Bigg(\int_0^{t_f} \mathbf{h}(t)^T \mathbf{h}(t) dt - 2 \left(\sum_{j=1}^{n} \boldsymbol{\ell}_k^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_k \frac{e^{(\lambda_k+\rho_j)t_f}-1}{\lambda_k+\rho_j} + \sum_{j=1}^{n} \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f}-1}{\lambda_i+\rho_j} \right)$$
$$+ \boldsymbol{\ell}_k^T \boldsymbol{\ell}_k \boldsymbol{r}_k^T \boldsymbol{r}_k \frac{e^{(2\lambda_k)t_f}-1}{2\lambda_k} + \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_k \boldsymbol{r}_k^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_k)t_f}-1}{\lambda_i+\lambda_k} \tag{3.6}$$
$$+ \sum_{\substack{j=1\\ j\neq k}}^{r} \boldsymbol{\ell}_k^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_k \frac{e^{(\lambda_k+\lambda_j)t_f}-1}{\lambda_i+\lambda_j} + \sum_{\substack{j=1\\ j\neq k}}^{r} \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_j)t_f}-1}{\lambda_i+\lambda_j} \Bigg)$$

Using (3.6) we obtain the gradient of the cost function by perturbing the cost functional with respect to the residue directions $\boldsymbol{\ell}_k \to \boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k$ and $\boldsymbol{r}_k \to \boldsymbol{r}_k + \Delta\boldsymbol{r}_k$ i.e.

$$\Delta\mathbf{J}_k = tr\Bigg(\int_0^{t_f} \mathbf{h}(t)^T \mathbf{h}(t) dt - 2\Bigg(\sum_{j=1}^{n} (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)^T \boldsymbol{c}_j \boldsymbol{b}_j^T (\boldsymbol{r}_k + \Delta\boldsymbol{r}_k) \frac{e^{(\lambda_k+\rho_j)t_f}-1}{\lambda_k+\rho_j}$$
$$+ \sum_{j=1}^{n} \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{c}_j \boldsymbol{b}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\rho_j)t_f}-1}{\lambda_i+\rho_j} \Bigg) + (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)^T (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)(\boldsymbol{r}_k + \Delta\boldsymbol{r}_k)^T (\boldsymbol{r}_k + \Delta\boldsymbol{r}_k) \frac{e^{(2\lambda_k)t_f}-1}{2\lambda_k}$$
$$+ \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)(\boldsymbol{r}_k + \Delta\boldsymbol{r}_k)^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_k)t_f}-1}{\lambda_i+\lambda_k} + \sum_{\substack{j=1\\ j\neq k}}^{r} (\boldsymbol{\ell}_k + \Delta\boldsymbol{\ell}_k)^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T (\boldsymbol{r}_k + \Delta\boldsymbol{r}_k) \frac{e^{(\lambda_k+\lambda_j)t_f}-1}{\lambda_i+\lambda_j}$$
$$+ \sum_{\substack{j=1\\ j\neq k}}^{r} \sum_{\substack{i=1\\ i\neq k}}^{r} \boldsymbol{\ell}_i^T \boldsymbol{\ell}_j \boldsymbol{r}_j^T \boldsymbol{r}_i \frac{e^{(\lambda_i+\lambda_j)t_f}-1}{\lambda_i+\lambda_j} \Bigg)$$

Using the properties of the trace and considering only the terms that are multiplied by $\Delta\boldsymbol{\ell}_k$ and $\Delta\boldsymbol{r}_k^T$ we obtain:

$$\frac{\partial J}{\partial \boldsymbol{r}_k} = -2\boldsymbol{\ell}_k^T \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{(\lambda_k+\rho_j)t_f}-1}{\lambda_k+\rho_j} + 2\boldsymbol{\ell}_k^T \sum_{j=1}^{n} \boldsymbol{\ell}_j \boldsymbol{r}_j^T \frac{e^{(\lambda_k+\lambda_j)t_f}-1}{\lambda_k+\lambda_j}$$
$$\frac{\partial J}{\partial \boldsymbol{\ell}_k} = -2 \left(\sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{e^{(\lambda_k+\rho_j)t_f}-1}{\lambda_k+\rho_j} \right) \boldsymbol{r}_k + 2 \left(\sum_{j=1}^{n} \boldsymbol{\ell}_j \boldsymbol{r}_j^T \frac{e^{(\lambda_k+\lambda_j)t_f}-1}{\lambda_k+\lambda_j} \right) \boldsymbol{r}_k$$

If we set $\dfrac{\partial J}{\partial \boldsymbol{r}_k} = 0$ and $\dfrac{\partial J}{\partial \boldsymbol{\ell}_k} = 0$, using Lemma 3.5, we obtain

$$\boldsymbol{\ell}_k^T \mathbf{G}(-\lambda_k) = \boldsymbol{\ell}_k^T \mathbf{G}_r(-\lambda_k)$$
$$\mathbf{G}(-\lambda_k)\boldsymbol{r}_k = \mathbf{G}_r(-\lambda_k)\boldsymbol{r}_k.$$

If we differentiate the error $\mathbf{J}$ with respect to the k-th pole λ_k, and set it equal to 0, we have

$$\begin{aligned}\frac{\partial \mathbf{J}}{\partial \lambda_k} &= -2\boldsymbol{\ell}_k^T \left(\sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{t_f(\lambda_k+\rho_j)e^{(\rho_j+\lambda_k)t_f} - e^{(\rho_j+\lambda_k)t_f} + 1}{(\lambda_k+\rho_j)^2} \right) \boldsymbol{r}_k \\ &+ 2\boldsymbol{\ell}_k^T \left(\sum_{i=1}^{r} \boldsymbol{\ell}_i \boldsymbol{r}_i^T \frac{t_f(\lambda_i+\lambda_j)e^{(\lambda_i+\lambda_k)t_f} - e^{(\lambda_i+\lambda_k)t_f} + 1}{(\lambda_i+\lambda_k)^2} \right) \boldsymbol{r}_k \\ &= 0.\end{aligned}$$

Thus,

$$\boldsymbol{\ell}_k^T \left(\sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{b}_j^T \frac{t_f(\lambda_k+\rho_j)e^{(\rho_j+\lambda_k)t_f} - e^{(\rho_j+\lambda_k)t_f} + 1}{(\lambda_k+\rho_j)^2} \right) \boldsymbol{r}_k = \boldsymbol{\ell}_k^T \left(\sum_{i=1}^{r} \boldsymbol{\ell}_i \boldsymbol{r}_i^T \frac{t_f(\lambda_i+\lambda_j)e^{(\lambda_i+\lambda_k)t_f} - e^{(\lambda_i+\lambda_k)t_f} + 1}{(\lambda_i+\lambda_k)^2} \right) \boldsymbol{r}_k$$

Consider

$$\begin{aligned}\mathbf{G}(s) &= -e^{-st_f}\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}e^{\mathbf{A}t_f}\mathbf{B} + \mathbf{H}(s) \\ &= -\sum_{i=1}^{n} \frac{\boldsymbol{c}_i \boldsymbol{b}_i^T e^{(-s+\rho_i)t_f}}{s-\rho_i} + \sum_{i=1}^{n} \frac{\boldsymbol{c}_i \boldsymbol{b}_i^T}{s-\rho_i}\end{aligned}$$

$$\begin{aligned}\mathbf{G}_r(s) &= -e^{-st_f}\mathbf{C}_r(s\mathbf{I}_r-\mathbf{A}_r)^{-1}e^{\mathbf{A}_r t_f}\mathbf{B}_r + \mathbf{H}_r(s) \\ &= -\sum_{i=1}^{r} \frac{\boldsymbol{\ell}_i \boldsymbol{r}_i e^{(-s+\lambda_i)t_f}}{s-\lambda_i} + \sum_{i=1}^{r} \frac{\boldsymbol{\ell}_i \boldsymbol{r}_i}{s-\lambda_i}\end{aligned}$$

If we differentiate $\mathbf{G}(s)$ with respect to s, we get

$$\begin{aligned}\mathbf{G}'(s) &= -\sum_{i=1}^{n} \boldsymbol{c}_i \boldsymbol{b}_i^T \frac{(-t_f)(s-\rho_i)e^{(-s+\rho_i)t_f} - e^{(-s+\rho_i)t_f}}{(s-\rho_i)^2} - \sum_{i=1}^{n} \frac{\boldsymbol{c}_i \boldsymbol{b}_i^T}{(s-\rho_i)^2} \\ &= \sum_{i=1}^{n} \boldsymbol{c}_i \boldsymbol{b}_i^T \frac{t_f(s-\rho_i)e^{(-s+\rho_i)t_f} + e^{(-s+\rho_i)t_f} - 1}{(s-\rho_i)^2}\end{aligned}$$

Similarly, if we differentiate $\mathbf{G}_r(s)$ with respect to s, we get

$$\mathbf{G}_r'(s) = \sum_{i=1}^{r} \boldsymbol{\ell}_i \boldsymbol{r}_i^T \frac{t_f(s-\lambda_i)e^{(-s+\lambda_i)t_f} + e^{(-s+\lambda_i)t_f} - 1}{(s-\lambda_i)^2}.$$

Therefore,

$$\boldsymbol{\ell}_k^T \mathbf{G}'(-\lambda_k)\boldsymbol{r}_k = \boldsymbol{\ell}_k^T \mathbf{G}_r'(-\lambda_k)\boldsymbol{r}_k$$

□

The following corollary deals with SISO systems i.e. we have a specific case of Theorem 3.1. In the SISO case, we obtain full interpolation, instead of the tangential interpolation obtained in the MIMO case.

Corollary 3.2. *Let* $\mathbf{G}(s) = -e^{-st_f}\boldsymbol{c}^T(s\mathbf{I}-\mathbf{A})^{-1}e^{\mathbf{A}t_f}\boldsymbol{b}+\mathbf{H}(s)$ *and* $\mathbf{G}_r(s) = -e^{-st_f}\boldsymbol{c}_r^T(s\mathbf{I}_r-\mathbf{A}_r)^{-1}e^{\mathbf{A}_r t_f}\boldsymbol{b}_r + \mathbf{H}_r(s)$ *where* $\mathbf{H}$ *and* $\mathbf{H}_r$ *are transfer functions of SISO models. If* $\mathbf{H}_r$ *is the best* r^{th} *order approximation of*

$\mathbf{H}$ *with respect to the* $\mathcal{H}_2(t_f)$ *norm, then*

$$\mathbf{G}(-\lambda_k) = \mathbf{G}_r(-\lambda_k)$$
$$\mathbf{G}'(-\lambda_k) = \mathbf{G}_r'(-\lambda_k)$$

where λ_k *for* $k = 1, 2, ..., r$ *are the poles of the reduced system.*

Proof. Since $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors instead of matrices, we can follow the proof of Theorem 3.1 and consider that $\boldsymbol{\ell}_k$ and $\boldsymbol{r}_k$ are scalars. ☐

If we know the poles of the reduced system, we can establish the necessary and sufficient optimality conditions for the residues. In other words, given the poles of a reduced system we can find the best residues so that we minimize the error between the full and reduced systems.

Corollary 3.3. *Let* $\mathbf{H} = \sum_{i=1}^{n} \psi_i e^{\rho_i t_f}$ *and* $\mathbf{H}_r = \sum_{i=1}^{r} \phi_i e^{\lambda_i t_f}$. *For a fixed set of eigenvalues* $\{\lambda_i\}_{i=1}^{r}$, $\mathbf{H}_r$ *is the best* r^{th} *order approximation of* $\mathbf{H}$ *with respect to the* $\mathcal{H}_2(t_f)$ *norm if and only if*

$$M\phi = z$$

where ϕ *is the vector of the residues* ϕ_i *of the reduced system,* $M_{i,j} = \dfrac{e^{(\lambda_i+\lambda_j)t_f} - 1}{\lambda_i + \lambda_j}$, *and* $z_j = e^{\lambda_j t_f}\mathbf{C}(-\lambda_j \mathbf{I} - \mathbf{A})^{-1}e^{\mathbf{A}t_f}\mathbf{B} - \mathbf{H}(-\lambda_j)$.

Proof. Note that if $\mathbf{H}$ and $\mathbf{H}_r$ are SISO systems, we can write the error $\mathbf{J}$ as follows

$$\mathbf{J} = \phi^T M\phi - 2\phi^T X + \int_0^{t_f} (\mathbf{h}(t))^2 dt.$$

where $X \in \mathbb{R}^{r\times 1}$ and $X_i = \sum_{k=1}^{n} \psi_k \dfrac{e^{(\rho_k+\lambda_i)t_f} - 1}{\lambda_i + \rho_k}$.

Using Lemma 3.4 we obtain

$$\phi^T M\phi = \|\mathbf{h}_r\|^2_{\mathcal{H}_2(t_f)}.$$

Since ϕ is arbitrary, we conclude M is positive definite. Thus, $\mathbf{J}$ is a quadratic function of the residues ϕ with a positive leading term. Hence, the critical point of the error squared function is the minimum. The rest follows from Lemma 3.5 and differentiating the error $\mathbf{J}$ with respect to the residues. ☐

The following proposition tells us that the $\mathcal{H}_2(t_f)$ norm of $\mathbf{G}$ and $\mathbf{H}$ is the same.

Proposition 3.6. Let $\mathbf{G}(s) = -e^{-st_f}\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}e^{\mathbf{A}t_f}\mathbf{B}+\mathbf{H}(s)$ where $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$ is the transfer function of (1.1). Then, $\|\mathbf{G}\|_{\mathcal{H}_2(t_f)} = \|\mathbf{H}\|_{\mathcal{H}_2(t_f)}$

Proof. Note

$$\begin{aligned}\mathbf{G}(s) &= -e^{-st_f}\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}e^{\mathbf{A}t_f}\mathbf{B} + \mathbf{H}(s)\\ &= -e^{-st_f}\sum_{i=1}^{n}\frac{\phi_i e^{\lambda_i t_f}}{s - \lambda_i} + \mathbf{H}(s)\end{aligned}$$

where λ_i's and ϕ_i's are the poles and the residues of the system, respectively. Consider the inverse Laplace transform of $\mathbf{G}(s)$:

$$\mathcal{L}^{-1}\{\mathbf{G}(s)\} = \mathbf{h}(t) + \sum_{i=1}^{n} u_{t_f}(t)\phi_i e^{\lambda_i t_f} e^{\lambda_i(t-t_f)} = \mathbf{h}(t) + \sum_{i=1}^{n} u_{t_f}(t)\phi_i e^{\lambda_i t}$$

where $u_{t_f}(t)$ is the unit step function

$$u_{t_f}(t) = \begin{cases} 0 & t < t_f \\ 1 & t \geq t_f \end{cases}.$$

Thus,

$$\|\mathbf{G}\|_{\mathcal{H}_2(t_f)} = \int_0^{t_f} \mathbf{h}(t) + \sum_{i=1}^{n} u_{t_f}(t)\phi_i e^{\lambda_i t} dt = \int_0^{t_f} \mathbf{h}(t)dt = \|\mathbf{H}\|_{\mathcal{H}_2(t_f)}$$

□

Remark 3.4. Melchior et al. introduced a method that constructs an optimal reduced model by minimizing the Frobenius norm of the error for linear time-varying systems on a finite horizon[**?**]. ♢

Remark 3.5. Goyal and Redmann established gramian-based optimality conditions independently of this paper. However, the presented algorithm in [**?**] approximately satisfies the optimality conditions presented in their paper. FHIRKAsatisfies the interpolation based conditions exactly. ♢

If we let $t_f \to \infty$, we retrieve the conditions for the infinite-horizon case, which is expected.

3.3. FHIRKA: A Descent Algorithm

Here we describe an algorithm which produces a reduced model that satisfies the necessary $\mathcal{H}_2(t_f)$ optimality conditions upon convergence. Since we are reducing the model over a finite horizon, we name this algorithm Finite Horizon IRKA (FHIRKA). In the numerical examples provided in the following chapter, we used only SISO systems, thus, we describe the algorithm within the SISO context.

Sketch of FHIRKA

- Pick an r-fold intial shift set that is closed under conjugation.
- while (not converged)
  - Find the optimal residues for the given shift set using Corollary 3.3
  - Update the shifts by minimizing the error $\mathbf{J}$.

As we see, FHIRKA is a descent algorithm. Once we are given an initial set of poles, we search for the optimal residues, meaning that the new reduced model is a better approximation than the previous ones. Then, keeping the newfound residues fixed, we search for optimal poles, guaranteeing that in each step we are obtaining a better approximation. Therefore, upon convergence, we achieve a local min for the error system. Once FHIRKA converges, using the obtained poles and residues we can construct the matrices $\mathbf{A}_r, \mathbf{B}_r$ and $\mathbf{C}_r$.

4. Numerical Results

In this section we compare our algorithm with the Proper Orthogonal Decomposition (POD), Time-Limited Balanced Truncation (TLBT), and a recently presented algorithm by Goyal and Redmann, which is based on the Sylvester equations associated with the system[**?**].

In our comparisons, we consider three models: a heat model of order $n = 197$, a model of data collected from the international space station (ISS) of order $n = 270$, and a toy unstable model of order $n = 402$. The unstable system has 400 stable poles and 2 unstable poles (positive real part). In all the examples shown below, we initially reduced the original model using POD, Goyal or TLBT. Afterwards, we used the obtained reduced model via POD, Goyal or TLBT to initialize FHIRKA.

When we reduce a heat model via FHIRKA, we use a trust-region algorithm to minimize the error $\mathbf{J}$, while when we reduce an ISS model or an unstable model, we use a quasi-newton algorithm to minimize $\mathbf{J}$. The graphs below show the $\mathcal{H}_2(t_f)$ approximation error for different values of r, the order of the reduced model.

As expected, FHIRKA yields a better approximation compared to the other algorithms for each model. We notice significant improvements over POD. It appears that Goyal and TLBT provide better initializations for FHIRKA compared to POD. POD yielded a worse initialization compared to TLBT and the Sylvester based algorithm for the ISS model. Nonetheless, we still noticed significant improvements after reducing via FHIRKA.

5. Conclusions and Future Work

We established $\mathcal{H}_2(t_f)$ optimality conditions on a finite horizon using two different approaches: the Gramian based approach, which was inspired by Wilson, and the interpolation based approach where we write the impulse response of the reduced system in terms of the poles and residues of the ROM. The Gramian-based optimality conditions that we derived appeared to be difficult to implement in practice. In the future, we hope to implement the Gramian-based conditions efficiently. When we obtain ROM via FHIRKA, the function $\mathbf{G}_r$ interpolates $\mathbf{G}$ at the mirror images of the poles of the reduced system $\mathbf{H}_r$. FHIRKA outperforms POD, IRKA and TLBT for the examples discussed in this paper. Numerical experiments were consistent with our theoretical results. For future research, we plan to improve our algorithm by cheaply approximating the exponential terms in $\mathbf{G}$ and finding more efficient ways to minimize the error $\mathbf{J}$. Another interesting future direction would be to show an equivalence between the Gramian-based approach and the kernel expansion approach.