Problem 2

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QUESTION 2A

a.

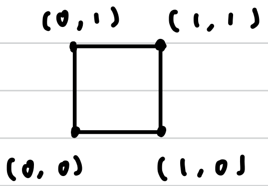
L_0 : $(0,1), (0,0)$

$a_0 = |\phi|$

$= 1$

$b_0 = |\{(0,0)\}|$

$= 1$



L_1 : $(0,0), (0,1), (1,1), (1,0)$

$a_1 = |\phi, \{(0,1)\}, \{(0,0)\}|$

$= 3$

$b_1 = |\{(1,0)\}, \{(1,0),(0,1)\}|$

$= 2$

$\therefore$ $a_0 = 1$, $b_0 = 1$, $a_1 = 3$, $b_1 = 2$

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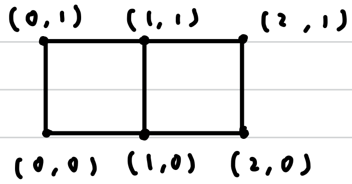
QUESTION 2B

b.



L_2 : (0,1), (0,0), (1,0), (1,1), (2,1), (2,0)

a_2 : ϕ, {(0,1)}, {(0,0)}, {(1,1)}, {(1,0)}, {(0,1), (1,0)}, {(0,0), (1,1)}

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b_2 : {(2,0)}, {(2,0), (1,1)}, {(2,0), (0,0)}, {(2,0), (0,1)}, {(0,0), (1,1), (2,0)}

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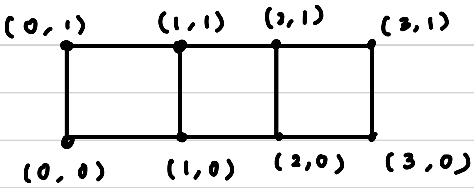

L_3 : (0,0), (0,1), (1,0), (1,1), (2,0), (2,1), (3,0), (3,1)

a_3 : ϕ, {(0,1)}, {(0,0)}, {(1,0)}, {(1,1)}, {(2,0)}, {(2,1)}, {(0,1), (1,0)}, {(0,1), (2,0)}, {(1,1), (0,0)}, {(1,1), (2,0)}, {(0,0), (2,1)}, {(1,0), (2,1)}, {(0,1), (1,0), (2,1)}, {(0,0), (1,1), (2,0)}, {(0,1), (2,1)}, {(0,0), (2,0)}

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b_3 : {(3,0)}, {(3,0), (2,1)}, {(3,0), (1,1)}, {(3,0), (0,1)}, {(3,0), (1,0)}, {(3,0), (0,0)}, {(1,0), (2,1), (3,0)}, {(3,0), (1,0), (0,1)}, {(3,0), (1,0), (0,1)}, {(3,0), (1,1), (0,0)}, {(3,0), (2,1), (1,0), (0,1)}, {(3,0), (2,1), (0,0)}

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a_0 : 1 $\quad$ b_0 : 1

a_1 : 3 $\quad$ b_1 : 2

a_2 : 7 $\quad$ b_2 : 5

a_3 : 17 $\quad$ b_3 : 12

- The recurrence relation for a_{n+1} is $a_{n+1} = a_n + 2b_n$
  We can do this by imagining that all a_n will be included to a_{n+1}. No changes are made since the independent set of a_n will always be within a_{n+1}. Since a new ladder step is created, b_n will also needs to be included. Since b_n only counts (n,0), it means that all the independent sets in b_n are not included in a_n (since a_n contains neither (n,0) nor (n,1)). By adding b_n to a_n, we have included the independent sets containing (n,0). But, all the independent sets in (n,1) is not included yet. By intuition, all independent sets containing (n,0) is equal to all independent sets contained in (n,1). Combining these together, we can get $a_{n+1} = a_n + 2b_n$ for $n \geq 0$

- The recurrence relation of $b_{n+1} = a_n + b_n$. This can be thought by having all sets in a_n added with $(n+1, 0)$ to satisfy b_{n+1}. This will definitely gives independent sets since a_n does not contain any pairs with $(n, 0)$ nor $(n, 1)$. b_n comes from the independent sets that can include pairs from the nth order of ladder. Since $(n, 0)$ and $(n+1, 0)$ cannot be an independent set, we can think of for every b_{n+1}, the sets will include the pair $(n+1, 0)$ added to all sets in b_n but y-axis will alternate between 0 and 1 each time. Hence, still giving the count of $b_{n+1} = a_n + b_n$ for $n \geq 0$

QUESTION 2C

Let $P(n)$ be $a_n \leq \sqrt{2}(\sqrt{2}+1)^n$ and $b_n \leq (\sqrt{2}+1)^n$

Base case : $n = 0$:

$a_0 = 1$ while $\sqrt{2}(\sqrt{2}+1)^0 = \sqrt{2}$ and

$b_0 = 1$ while $(\sqrt{2}+1)^0 = 1$,

so the inequality is true for $n = 0$

Inductive Step :

Induction Hypothesis :

Suppose that $a_k \leq \sqrt{2}(\sqrt{2}+1)^k$ and $b_k \leq (\sqrt{2}+1)^k$ is true for a particular number n, where $k \geq 0$

Body of Induction using IH :

Let's look at what happens at $k+1$

$a_{k+1} = a_k + 2b_k$ (to express it in terms of recursion)

$\leq \sqrt{2}(\sqrt{2}+1)^k + 2(\sqrt{2}+1)^k$ (by the Inductive Hypothesis, i.e., $a_k \leq \sqrt{2}(\sqrt{2}+1)^k$ and $b_k \leq (\sqrt{2}+1)^k$

$= (\sqrt{2}+2)(\sqrt{2}+1)^k$ (a slight rearrangement)

$= \sqrt{2}(1+\sqrt{2})(\sqrt{2}+1)^k$ (more arrangement)

$= \sqrt{2}(\sqrt{2}+1)^{k+1}$

and

$b_{k+1} = a_k + b_k$ (to express it in terms of recursion)

$\leq \sqrt{2}(\sqrt{2}+1)^k + (\sqrt{2}+1)^k$ (by the Inductive Hypothesis, i.e., $a_k \leq \sqrt{2}(\sqrt{2}+1)^k$ and $b_k \leq (\sqrt{2}+1)^k$

$= (\sqrt{2}+1)(\sqrt{2}+1)^k$

$= (\sqrt{2}+1)^{k+1}$ (a slight rearrangement)

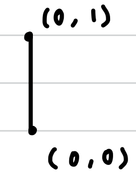
$\therefore$ This is just the equation for the theorem $n = k+1$ instead of k

So the inductive step is now complete

Conclusion :

Therefore, by the Principle of Mathematical Induction, the equation $a_n \leq \sqrt{2}(\sqrt{2}+1)^n$ and $b_n \leq (\sqrt{2}+1)^n$ holds for all $n \geq 0$

QUESTION 2D



C_0 : $\{(0,0)\}, \{(0,1)\}, \phi$
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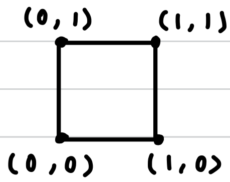

C_1 : $\phi, \{(0,1)\}, \{(1,1)\}, \{(0,0)\}, \{(1,0)\}, \{(1,1),(0,0)\}, \{(0,1),(1,0)\}$
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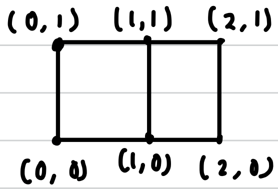

C_2 : $\phi, \{(0,1)\}, \{(1,1)\}, \{(2,1)\}, \{(0,0)\}, \{(1,0)\}, \{(2,0)\}, \{(0,1),(1,0)\},$
$\{(0,1),(2,0)\}, \{(0,1),(2,1)\}, \{(0,0),(1,1)\}, \{(0,0),(2,0)\}, \{(0,0),(2,1)\},$
$\{(1,1),(2,0)\}, \{(1,0),(2,1)\}, \{(0,0),(1,1),(2,0)\}, \{(0,1),(1,0),(2,1)\}$
: 17

a_0 : 1	b_0 : 1	C_0 : 3
a_1 : 3	b_1 : 2	C_1 : 7
a_2 : 7	b_2 : 5	C_2 : 17
a_3 : 17		

Based on the pattern, we can see that $C_n = a_{n+1}$. This is because a_{n+1} miss the last edge of the ladder, whereas C_n includes them.
We can think of C_n using multiple equation

- $C_n = a_{n+1}$

this leads to

$$C_n \leq \sqrt{2}(\sqrt{2}+1)^{n+1}$$

a_{n+1} can also be written as:

$$a_{n+1} = a_n + 2b_n$$

$$a_{n+1} \leq \sqrt{2}(\sqrt{2}+1)^n + 2(\sqrt{2}+1)^n$$

$$\therefore C_n \leq \sqrt{2}(\sqrt{2}+1)^n + 2\sqrt{2}$$

This leads to the same answer, which has been previously proven in 2c.

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To check a closed-form upper bound of C_n, we can compare both the equation given and the previously computed equation

$C_n \le \sqrt{2}(\sqrt{2}+1)^{n+1}$

when $(n=0)$

$C_0 \le \sqrt{2}(\sqrt{2}+1)^{0+1}$

$C_0 \le \sqrt{2}(\sqrt{2}+1)^{1}$

$3 \le 2+\sqrt{2}$

when $(n=1)$

$C_1 \le \sqrt{2}(\sqrt{2}+1)^{1+1}$

$C_1 \le \sqrt{2}(\sqrt{2}+1)^{2}$

$C_1 \le \sqrt{2}(2+2\sqrt{2}+1)$

$C_1 \le 2\sqrt{2}+4+\sqrt{2}$

$7 \le 3\sqrt{2}+4$

when $(n=2)$

$C_2 \le \sqrt{2}(\sqrt{2}+1)^{2+1}$

$C_2 \le \sqrt{2}(\sqrt{2}+1)^{3}$

$C_2 \le \sqrt{2}(2\sqrt{2}+6+3\sqrt{2}+1)$

$C_2 \le 4+6\sqrt{2}+6+\sqrt{2}$

$17 \le 10+7\sqrt{2}$

$C_n \le 3^{n+1}$

when $(n=0)$

$C_0 \le 3^{0+1}$

$C_0 \le 3^{1}$

$3 \le 3$

when $(n=1)$

$C_1 \le 3^{1+1}$

$C_1 \le 3^{2}$

$7 \le 9$

when $(n=2)$

$C_2 \le 3^{2+1}$

$C_2 \le 3^{3}$

$17 \le 27$

For C_0, 3^{n+1} gives 3 whereas $\sqrt{2}(\sqrt{2}+1)^{n+1}$ gives $2+\sqrt{2}$ which is a tad larger. But for C_n where $n \ge 1$, the value that $2(\sqrt{2}+1)^{n+1}$ grows way slower than 3^{n+1} (as shown from $C_1 \le 3\sqrt{2}+4 \le 9$ and $C_2 \le 10+7\sqrt{2} \le 27$).

This shows that a better closed-form upper bound for C_n is $\sqrt{2}(\sqrt{2}+1)^{n+1}$

$C_n \le \sqrt{2}(\sqrt{2}+1)^{n+1} \le 3^{n+1}$ for $n \ge 1$