ECI 249 HW 3

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```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import scipy.stats as stats
%matplotlib inline

# set LaTeX formatting
def mc(string):
    return '\multicolumn{1}{p{2.5cm}}{\centering %s}' % string.replace(', ', r' \\ ')
def _repr_latex_(self):
    lself = self
    lself = lself.rename(columns=dict(zip(self.columns.tolist(), [mc(name) for name in
     ↪  self.columns.tolist()])))
    return r'\begin{center}' + '\n%s\n' % lself.to_latex(index=True, escape=False) +
     ↪  r'\end{center}'

pd.DataFrame._repr_latex_ = _repr_latex_
```

1. Frat House Optimal Lifetime

Average time between house-burnings:

$$\mathbb{E}[T] = \sum_{t=1}^{\infty} t \cdot P(\text{burns in t-th year}) = \sum_{t=1}^{\infty} t \cdot (1-p)^{t-1}(p) = \frac{p}{1-p}\sum_{t=1}^{\infty} t(1-p)^t = \frac{p}{1-p}\frac{1-p}{(p)^2} = \frac{1}{p}$$

A. fire-proof frat house: $\mathbb{E}[T] = \frac{1}{0} = \infty$

B. fire-resistant frat house: $\mathbb{E}[T] = \frac{1}{0.05} = 20$ yrs

C. normal frat house: $\mathbb{E}[T] = \frac{1}{0.1} = 10$ yrs

a. Deterministic Approach

$$W = C + We^{-rT}$$

$$W = \frac{C}{1-e^{-rT}}$$

```python
costs = [1000000, 300000, 150000]
ps = [0, 0.05, 0.1]
Ts = [np.inf, 20, 10]

det_costs = []
for c, p, T in zip(costs, ps, Ts):
    evc = c/(1 - np.e**(-0.05 * T))
    det_costs.append(round(evc, 2))
```

b. Analytical Approach

Expected Value of Cost:

$$W = C + W \sum_{t=1}^{\infty} e^{-rt}(1-p)^{t-1}(p) = C + W \frac{p}{1-p} \sum_{t=1}^{\infty} (e^{-r}(1-p))^{t}$$

$$W = C + W \frac{p}{1-p} \frac{(e^{-r}(1-p))}{1-(e^{-r}(1-p))}, \quad \rightarrow \quad W = \frac{C}{\left(1 - \frac{p}{e^{r}-(1-p)}\right)}$$

```python
analytical_costs = []
for c, p, in zip(costs, ps):
    evc = c/(1-p/(np.e**0.05 - (1-p)))
    analytical_costs.append(round(evc, 2))
```

c. Monte Carlo Approach:

```python
class FratHouse:
    def __init__(self, cost, p, runs=10**5):
        # cost of frat house
        self.cost = cost
        # annual probability of fire
        self.p = p
        # number of years for each run
        self.sample_size = 10**3
        # discount rate
        self.r = 0.05
        # number of runs
        self.runs = runs
        self.mean_cost = self.mc_average()

    def monte_carlo(self):
        run_cost = self.cost
        samples = np.random.random(self.sample_size)
        for t, sample in enumerate(samples):
            if sample <= self.p:
                run_cost += self.cost*np.e**((-self.r)*(t+1))
        return run_cost

    def mc_average(self):
        run_costs = []
        for run in range(self.runs):
            run_cost = self.monte_carlo()
            run_costs.append(run_cost)
        mean_cost = np.mean(run_costs)
        return round(mean_cost, 2)

FireProof = FratHouse(1000000, 0)
FireResistant = FratHouse(300000, 0.05)
NormalHouse = FratHouse(150000, 0.1)
```

```
mc_costs = [FireProof.mean_cost, FireResistant.mean_cost, NormalHouse.mean_cost]
df = pd.DataFrame([det_costs, analytical_costs, mc_costs], columns=['Fireproof', 'Fire
↪ Resistant', 'Normal House'], index=['Deterministic Estimate', 'Analytical Solution',
↪ 'Monte Carlo'])
df
```

	Fireproof	Fire Resistant	Normal House
Deterministic Estimate	1000000.0	474593.01	381224.11
Analytical Solution	1000000.0	592562.50	442562.50
Monte Carlo	1000000.0	592927.89	442227.96

The average costs from the Monte Carlo simulations very closely approximate the analytical solution. From the average cost for each alternative listed above, it is clear that we want our frat houses to have as little fireproofing as possible.

2. Filtration Units

a. What is the probability distribution for removal after each number of filters?

Analytical approach:

total efficiency:

$$E = 1 - (1 - E_1)(1 - E_2)...(1 - E_n)$$

Where each $E_i = \min\left[\text{Lognormal}(0.9, 0.05), 1\right]$

Assuming the resulting distribution is approximately log-normally distributed, we can use the first-order second moment method to estimate the mean and variance.

$$\mu_E \approx 1 - (1 - \mu_{E_1})(1 - \mu_{E_2})...(1 - \mu_{E_n}) = 1 - (1 - 0.9)^n = 1 - 0.1^n$$

$$\sigma_E^2 \approx \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial g(\bar{\mu})}{\partial e_i}\frac{\partial g(\bar{\mu})}{\partial e_j}\text{Cov}(E_i, E_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}(1 - 0.9)^{2(n-1)}\text{Cov}(E_i, E_j)$$

The covariance matrix is zero except for diagonals since the E_i's are i.i.d. Therefore:

$$\sigma_E^2 \approx n\,(0.05)^2\,(0.1)^{2(n-1)} = \frac{n}{4}(0.1)^{2n}$$

So the standard deviation is

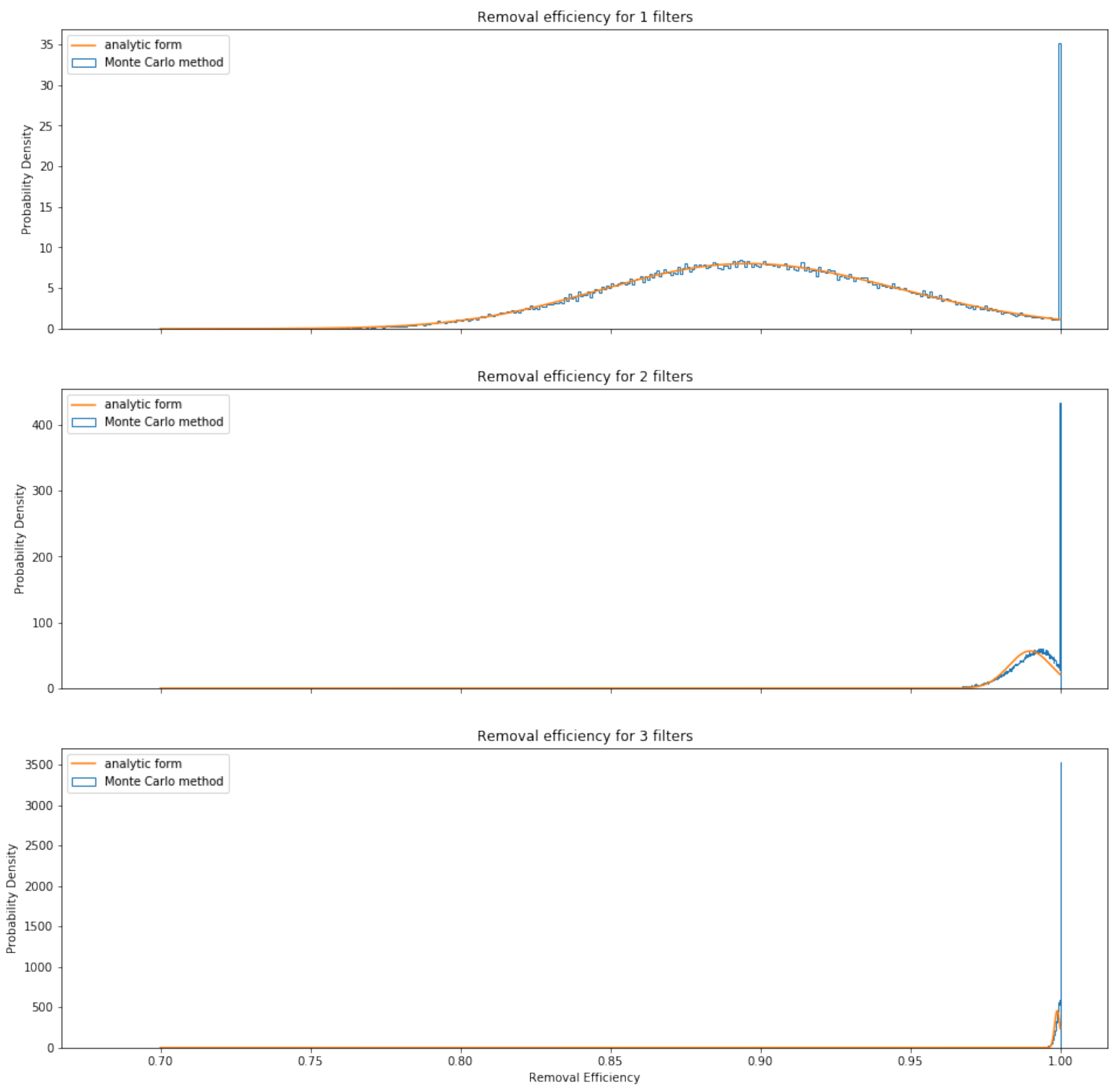
$$\sigma_E \approx \frac{\sqrt{n}}{2}0.1^n$$

Thus the distribution of overall filter efficiency is approximately $\min\left[\text{Lognormal}\left(1 - 0.1^n, \frac{\sqrt{n}}{2}0.1^n\right), 1\right]$

The corresponding pdfs are plotted below in comparison with the pdfs obtained via Monte Carlo simulation.

Monte Carlo approach:

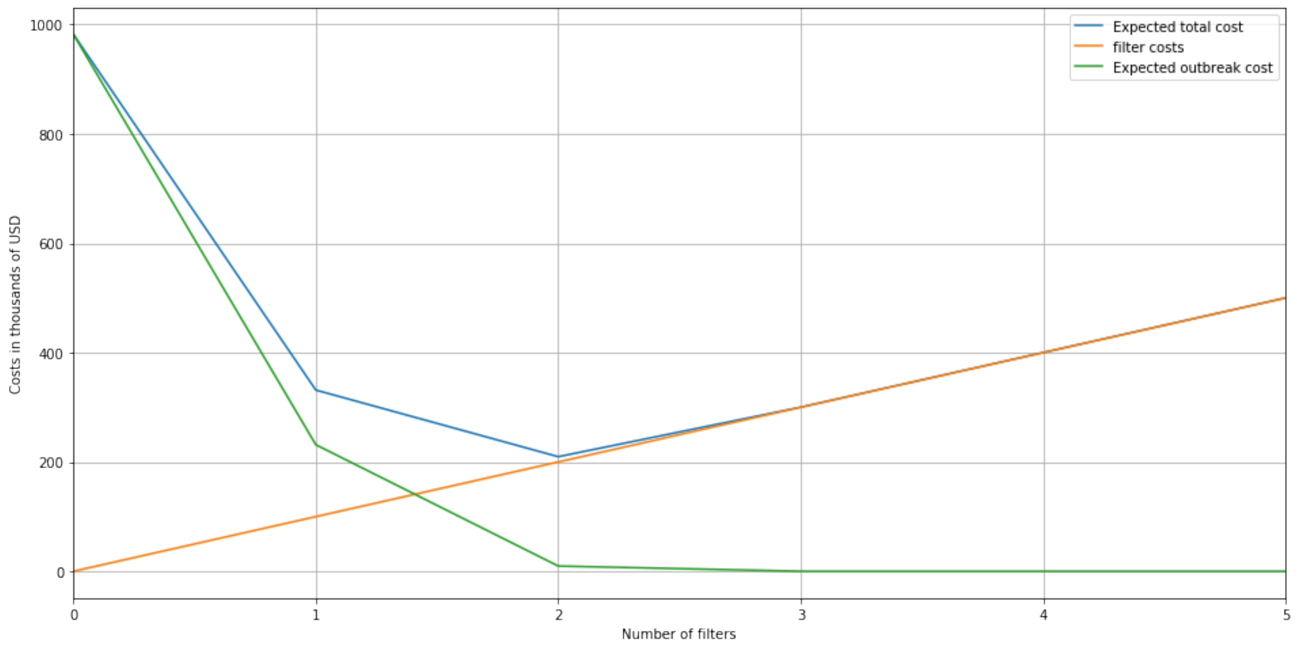
```python
class Filtration:
    def __init__(self):
        self.sample_size = 10**5
        # parameters mu and sigma for lognormal given mean and standard deviation
        self.mu_c = 1.455696
        self.sigma_c = 0.554513
        self.mu_e = -0.1069
        self.sigma_e = 0.05551276

    def apply_filters(self, num):
        c = np.random.lognormal(self.mu_c, self.sigma_c)
        if num == 0:
            return c
        else:
            for i in range(num):
                e = min(np.random.lognormal(self.mu_e, self.sigma_e), 1)
                c *= 1-e
            return c

    def get_eff(self, num):
        eff = 1
        for i in range(num):
            e = min(np.random.lognormal(self.mu_e, self.sigma_e), 1)
            eff *= 1-e
        eff = 1 - eff
        return eff

    def mc_cost(self, num_filters):
        mean_costs = []
        filter_costs = [n*10**5 for n in range(num_filters+1)]
        for n in range(num_filters+1):
            costs = []
            for run in range(self.sample_size):
                run_c = self.apply_filters(n)
                p_outbreak = np.clip(1/1.95*(run_c-0.05), 0, 1)
                run_cost = n*(10**5) + p_outbreak*(10**6)
                costs.append(run_cost)
            mean_cost = np.mean(costs)
            mean_costs.append(mean_cost)
        fig, ax = plt.subplots(figsize=(16, 8))
        ax.plot(mean_costs, label='Expected total cost')
        ax.plot(filter_costs, label='filter costs')
        ax.plot([(m - c) for m,c in zip(mean_costs, filter_costs)], label='Expected outbreak
         ↪  cost')
        ax.legend()
        ax.set_xlabel('Number of filters')
        ax.set_ylabel('Costs in thousands of USD')
        ticks = map(int, ax.get_yticks()*10**(-3))
        ax.set_yticklabels(ticks)
        ax.set_xlim(0, num_filters)
        plt.grid()
        plt.show()

    def mc_eff(self, num_filters):
        fig, ax = plt.subplots(num_filters, 1, figsize=(16, 16), sharex=True)
        for n in range(num_filters):
            es = []
            for run in range(self.sample_size):
                run_eff = self.get_eff(n+1)
                es.append(run_eff)
            ax[n].hist(es, bins=400, density=True, histtype='step', label='Monte Carlo
             ↪  method')
            x = np.linspace(0.7, 1, 1000)
            mean_n = 1-0.1**(n+1)
            var_n = (n+1)*1.0/4 * (0.1)**(2*(n+1))
            sigma_en = np.sqrt(np.log(var_n*1.0/mean_n**2 + 1))
            mu_en = np.log(1-0.1**(n+1)) - sigma_en**2*1.0/2
            ax[n].plot(x, stats.lognorm.pdf(x, s=sigma_en, scale=np.exp(mu_en)),
             ↪  label='analytic form')
            ax[n].set_ylabel('Probability Density')
            ax[n].set_title('Removal efficiency for %i filters' % (n+1))
            ax[n].legend(loc=2)
            #ax[n].grid()
        ax[num_filters-1].set_xlabel('Removal Efficiency')
        plt.show()


filt = Filtration()
filt.mc_eff(3)
```



Note the spikes where the efficiency is 1, indicating that the probability of removal efficiency being equal to 1 is the integral of the lognormal pdf from 1 to ∞.

b. How many filter stages should be employed?

```
filt.mc_cost(5)
```



From the above plot, we see that using 2 filter stages will minimize the expected total cost. Therefore, using 2 filters is the preferred alternative.

3. Probabilistic Peak Flood Flows

$$Q = \frac{k}{n} A R^{2/3} S^{1/2}$$

$$S = 0.01, \quad A = 500 \text{ ft}^2, \quad R = 30 \text{ ft}, \quad k = 1.49$$

$$F_N(n) = \frac{(n - 0.06)}{0.02}, \quad 0.06 \leq n \leq 0.08$$

$$F_Q(q) = P(\frac{k}{N} A R^{2/3} S^{1/2} \leq q) = P(N \geq \frac{k}{q} A R^{2/3} S^{1/2}) = 1 - F_N(\frac{k}{q} A R^{2/3} S^{1/2}) = 1 - \frac{\frac{k}{q} A R^{2/3} S^{1/2} - 0.06}{0.02}$$

$$F_Q(q) = 4 - \frac{\frac{k}{q} A R^{2/3} S^{1/2}}{0.02}, \quad 8991.12 \leq q \leq 11988.2$$

$$\boxed{f_Q(q) = \frac{\partial F_Q(q)}{\partial q} = \frac{\frac{k}{q^2} A R^{2/3} S^{1/2}}{0.02} = \frac{35964.5}{q^2}, \quad 8991.12 \leq q \leq 11988.2,} \quad \text{where } q \text{ is in cfs.}$$